A PDE REPRESENTATION OF THE DENSITY OF
THE MINIMAL ENTROPY MARTINGALE MEASURE
IN STOCHASTIC VOLATILITY MARKETS

FRED ESPEN BENTH AND KENNETH HVISTENDAHL KARLSEN

ABSTRACT. Under general conditions stated in Rheinländer [30], we prove that in a stochastic volatility market the Radon-Nikodym density of the minimal entropy martingale measure can be expressed in terms of the solution of a semilinear PDE. The semilinear PDE is suggested by the dynamic programming approach to the utility indifference pricing problem of contingent claims. We apply our PDE approach to the Stein-Stein and Heston stochastic volatility models.

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions, and $T < \infty$ is the time horizon. Let B and W be two independent Brownian motions defined on this filtered probability space, and suppose the risky asset S evolves according to the following general stochastic volatility model:

$$(1.1) \qquad dS_t = \mu(Y_t)S_t \, dt + \sigma(Y_t)S_t \, dB_t,$$

where the stochastic volatility is driven by the process

$$(1.2) \qquad dY_t = \alpha(Y_t) \, dt + \beta(Y_t) \, dB_t + \delta(Y_t) \, dW_t.$$

We assume that the parameter functions μ, σ, α, β, and δ are Borel measurable functions on $\mathbb{R}$ such that unique strong solutions of the stochastic differential equations (1.1)-(1.2) exist. Without loss of generality, we suppose that the rate of return from a risk-free investment is zero.

In a complete market (the Black-Scholes model) any contingent claim can be perfectly replicated and its arbitrage free price is given in terms of an expectation value with respect to the unique equivalent martingale (risk neutral) measure Q. On the other hand, in an incomplete market a claim cannot be perfectly replicated and there exists a continuum of equivalent martingale measures Q and, correspondingly, arbitrage free prices. Consequently, to fix the price a contingent claim one needs to select an appropriate equivalent martingale measure. Over the years several approaches to incomplete markets have been suggested in the literature. We refer to [27] for a general overview of the superhedging, mean-variance hedging, and shortfall risk minimization approaches, but see also [18, 6, 17, 24, 28] (to mention just a few) for more specific applications to stochastic volatility models. Herein we are interested in the minimal entropy martingale measure [16, 15, 30, 17] and the utility indifference pricing approach [20, 12, 2, 7, 11, 31, 8, 13, 3]. In a general semimartingale context, the relationship between the minimal entropy martingale measure and the utility indifference pricing problem is by now well known and comes from a fundamental duality result [13, 22] (see also [5, 15, 34, 32, 26, 33]). In [3] (see also [31, 13]) many properties of the utility indifference price of a contingent claim is derived from this duality result. In utility indifference pricing one considers the difference between the maximum utility from final wealth

when there is no contingent claim liability and when there is such a liability, and then define the price of the claim as the unique cash increment which offsets the difference.

Our stochastic volatility market (1.1)-(1.2) is incomplete as soon as $\delta \neq 0$, and we are interested in using the minimal entropy martingale measure for pricing contingent claims. A martingale measure is a probability measure Q on $(\Omega, \mathcal{F})$ such that $Q \ll P$ and S is a local Q-martingale. We denote by $\mathcal{M}$ the set of martingale measures and by $\mathcal{M}_e$ the set of martingale measures that are equivalent to P. Let Q be a probability measure on $(\Omega, \mathcal{F})$. The relative entropy, or Kullback-Leibler distance, $H(Q,P)$ of Q with respect to P is defined as

$$H(Q,P) = \begin{cases} \mathbb{E}\left[\dfrac{dQ}{dP} \ln\left(\dfrac{dQ}{dP}\right)\right], & Q \ll P, \\ +\infty, & \text{otherwise,} \end{cases}$$

where we understand $\mathbb{E}$ as the expectation operator under P. We look for a probability measure Q_{ME} that minimizes the relative entropy with respect to P in the class $\mathcal{M}$. We call Q_{ME} a minimal entropy martingale measure. More precisely, we call Q_{ME} a minimal entropy martingale measure (MEMM henceforth) if

$$H(Q_{\mathrm{ME}}, P) = \min_{Q \in \mathcal{M}} H(Q, P).$$

In [16] it is proved that if there exists a $Q \in \mathcal{M}_e$ with $H(Q,P) < \infty$, then Q_{ME} exists, is unique, and is equivalent to P (i.e., $Q_{\mathrm{ME}} \in \mathcal{M}_e$).

Recently Rheinländer [30] presented a martingale duality method for finding the MEMM in a general continuous semimartingale model. He illustrated his method on the Stein-Stein stochastic volatility model. The objective of the present paper is to show that one can determine the MEMM via the solution of a semilinear partial differential equation (PDE henceforth), and thereby providing an alternative to the duality approach developed in [30], at least for stochastic volatility models of the form (1.1)-(1.2). We illustrate our PDE approach with explicit calculations of the MEMM Q_{ME} for the Stein-Stein and Heston stochastic volatility models.

In [4] a system of reaction diffusion equations is derived for determining the MEMM in the case of a financial market modeled as a system of interacting Itô and point processes. Moreover, the existence and uniqueness of solutions to this system is proved. In an incomplete financial market driven by continuous semimartingales, the work [25] proves that the density of the MEMM can be expressed in terms of a value process that is the unique solution to a semimartingale backward stochastic differential equation.

We now detail our PDE approach a bit more. Suppose there exists a unique classical solution $v = v(t,y)$ of the semilinear PDE

$$(1.3) \qquad -v_t - \frac{1}{2}a^2(y)v_{yy} + F(y, v_y) = 0, \qquad (t,y) \in [0,T) \times \mathbb{R},$$

with terminal condition

$$(1.4) \qquad v(T,y) = 0, \qquad y \in \mathbb{R},$$

where

$$a^2(y) = \beta^2(y) + \delta^2(y)$$

and the nonlinear function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined as

$$F(y,p) = \frac{1}{2}\delta^2(y)p^2 - \left\{\alpha(y) - \frac{\mu(y)\beta(y)}{\sigma(y)}\right\}p - \frac{1}{2}\frac{\mu^2(y)}{\sigma^2(y)}.$$

The term "classical solution" means that $v(t,y)$ is once continuously differentiable in t and twice continuously differentiable in y for $(t,y) \in [0,T) \times \mathbb{R}$ and continuous in t and y for $(t,y) \in [0,T] \times \mathbb{R}$, i.e.,

$$v \in C^{1,2}([0,T) \times \mathbb{R}) \cap C([0,T] \times \mathbb{R}),$$

and v satisfies (1.3)-(1.4) in the usual pointwise sense.

Suppose furthermore that

$$(1.5) \qquad \int_0^T \frac{\mu^2(Y_t)}{\sigma^2(Y_t)}\,dt < \infty, \quad \int_0^T \delta^2(Y_t)v_y^2(t,Y_t)\,dt < \infty, \quad P - a.s.$$

Define for $0 \leq t \leq T$ the stochastic process Z_t by

$$
\begin{aligned}
Z_t = \exp\Bigg(&- \int_0^t \frac{\mu(Y_s)}{\sigma(Y_s)} \, dB_s - \int_0^t \delta(Y_s) v_y(s, Y_s) \, dW_s \\
&- \frac{1}{2} \int_0^t \left\{ \frac{\mu^2(Y_s)}{\sigma^2(Y_s)} + \delta^2(Y_s) v_y^2(s, Y_s) \right\} ds \Bigg),
\end{aligned}
$$

(1.6)

which is well-defined by the assumptions in (1.5). The purpose of this paper is to verify under natural additional assumptions on v and the parameters of the diffusion dynamics (1.1)-(1.2) that Z_T is the Radon-Nikodym derivative of the MEMM Q_{ME}. The argument applies a verification result of Rheinländer [30]. Furthermore, we state sufficient conditions for the well-posedness of (1.3)-(1.4). When the volatility dynamics Y_t has linear growth on the drift and additive noise, we prove the existence of a unique quadratically growing classical solution $v(t, y)$ of (1.3)-(1.4) with v_y having linear growth. The uniqueness of such a (viscosity) solution follows from [10]. With these properties at hand, we identify the density of the MEMM Q_{ME} as Z_T given in (1.6).

This paper is organized as follows: In Section 2 we motivate the semilinear PDE (1.3) by essentially solving the utility indifference pricing problem of contingent claims by the dynamic programming approach. In Section 3, under certain conditions (Conditions A and B), we prove rigorously that Z_T in (1.6) is the density of the MEMM Q_{ME}. In Section 4 we discuss the well-posedness of semilinear PDE terminal value problems like (1.3)-(1.4) coming from stochastic volatility models with additive noise and at most a linearly growing drift. We prove the existence of a quadratically growing classical solution with linearly growing derivative. We use this to identify the density of the MEMM Q_{ME} for this class of volatility models (i.e., we verify Conditions A and B), which includes the Stein-Stein model as a special case. Finally, Section 5 is devoted to verifying the conditions needed in Section 3 for the Heston volatility model, which does not have a linearly growing drift, i.e., it does not fit into the framework in Section 4.

2. Formal derivation of the semilinear PDE

We want to determine the utility indifference price [20, 12, 2, 7, 11, 31, 8, 13, 3] of a European type contingent claim in the stochastic volatility market (1.1)-(1.2). We consider the utility indifference price from the perspective of an issuer. Hodges and Neuberger [20] were the first to introduce preferences to determine a "fair" price of a contingent claim under proportional transaction costs.

The utility indifference price will be derived by solving two utility maximization problems. We shall use the dynamic programming (or Bellman) method to solve the two stochastic control problems. Provided that the value functions are sufficiently regular, it is well known that the associated Hamilton-Jacobi-Bellman (HJB henceforth) equations can be derived using the dynamic programming principle. It is often difficult to show that the value function in question is sufficiently smooth so as to solve the dynamic programming equation in the classical sense. The by now standard approach is to weaken the concept of solution and prove instead that the value function is a viscosity solution of the dynamic programming equation. Herein we will not be concerned with these issues. Instead we will simply assume that all functions involved are sufficiently regular to make sense to the subsequent calculations. After all, in this section we just want to explain from where we got the semilinear PDE (1.3). Later we will prove rigorously that (1.3) can be related to the MEMM Q_{ME}, which is the main purpose of this paper. We refer to [14] for an introduction to the dynamic programming method and the theory of viscosity solutions.

The investor can place her money in the risky asset S_t given by (1.1)-(1.2), or in a bond yielding a sure rate of return 0, that is, a bond with dynamics $R_t = 1$. If the investor allocates a fraction π_t of her wealth X_t at time t in the risky asset, it follows from the self-financing hypothesis that

(2.1)
$$
dX_t = \pi_t \mu(Y_t) X_t \, dt + \pi_t \sigma(Y_t) X_t \, dB_t.
$$

The control π is called *admissible* if it is an adapted stochastic process for which there exists a wealth process X_t^π solving the stochastic differential equation (2.1). We denote the set of all such controls by $\mathcal{A}_t$, where the subscript t indicates that we start the wealth dynamics at time t.

Note that we are considering a space of admissible controls allowing for negative wealth. Also, we assume that the investor has full information about the volatility since the controls are assumed adapted to $\mathcal{F}_t$, and not only to the filtration generated by the asset price $\mathcal{F}_t^S$. Considering Markov controls, this entails that the investor will allocate a fraction $\pi \equiv \pi(t, x, y)$ into the risky asset when the wealth is $X_t = x$ and volatility $Y_t = y$.

The goal of the investor is to find the investment strategy that maximizes her final utility. We will also consider the (same) investor who first issues a claim and then maximizes her final utility. The claim is of the European type, and we assume it has payoff at time T given by $g(S_T)$, where g is a bounded and measurable function. We shall only deal with Markovian claims. We choose an exponential utility function $U(x) = 1 - \exp(-\gamma x)$. The index of risk $-U''(x)/U'(x)$ for the exponential utility function is γ, so that the parameter γ reflects the investor's aversion towards risk. With the utility function of exponential type, for which the index of risk is independent of the investor's wealth, we are able to separate the value functions' dependence of wealth and volatility. This will lead to a price of the claim that is independent of the investor's initial wealth.

The utility indifference price approach to the problem of pricing a European type contingent claim goes as follows. First consider the stochastic control problem of maximizing the expected utility from final wealth. The resulting value function is in this case

$$(2.2) \qquad V^0(t, x, y) = \sup_{\pi \in \mathcal{A}_t} \mathbb{E}\left[1 - \exp\left(-\gamma X_T\right) \middle| X_t = x, Y_t = y\right].$$

In the second stochastic control problem, we suppose that a claim has been issued. The final wealth then becomes

$$X_T - g(S_T),$$

and the value function is therefore

$$(2.3) \qquad V(t, x, s, y) = \sup_{\pi \in \mathcal{A}_t} \mathbb{E}\left[1 - \exp\left(-\gamma(X_T - g(S_T))\right) \middle| X_t = x, Y_t = y, S_t = s\right].$$

By selling the claim at time t when the stock price is s and the volatility is given by y, the issuer will charge a premium $\Lambda^{(\gamma)}(t, y, s)$ and then optimally invest in the market. The premium is fixed such that the investor is indifferent between investing at her own account or issuing the claim and then investing. Hence $\Lambda^{(\gamma)}(t, y, s)$ should satisfy

$$(2.4) \qquad V^0(t, x, y) = V(t, x + \Lambda^{(\gamma)}(t, y, s), y, s).$$

The *utility indifference price* is defined as the unique solution $\Lambda^{(\gamma)}(t, y, s)$ of the algebraic equation (2.4). In a complete market, the utility indifference price coincides with the Black-Scholes price, see, e.g., [12]. In the present incomplete stochastic volatility market the utility indifference approach still gives us a unique price for each fixed value of the risk aversion parameter γ. By formally solving the two stochastic control problems (2.2) and (2.3) by the dynamic programming method, we will be able to determine a semilinear PDE satisfied by $\Lambda^{(\gamma)}(t, y, s)$.

The HJB equation for the value function (2.2) without a claim issued reads

$$(2.5) \qquad V_t^0 + \max_{\pi \in \mathbb{R}} \left\{ \mu(y)\pi x V_x^0 + \frac{1}{2}\sigma^2(y)\pi^2 x^2 V_{xx}^0 + \sigma(y)\beta(y)\pi x V_{xy}^0 \right\}$$
$$+ \mathcal{L}_Y V^0 = 0, \qquad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R},$$

with terminal data

$$(2.6) \qquad V^0(T, x, y) = 1 - \exp(-\gamma x), \qquad (x, y) \in \mathbb{R} \times \mathbb{R},$$

where

$$(2.7) \qquad \mathcal{L}_Y V^0 = \alpha(y)V_y^0 + \frac{1}{2}\left\{\beta^2(y) + \delta^2(y)\right\} V_{yy}^0.$$

The first order condition for an optimal investment strategy is

$$\mu(y)x V_x^0 + \sigma^2(y)\pi x^2 V_{xx}^0 + \sigma(y)\beta(y)x V_{xy}^0 = 0,$$

and the solution π^* of this equation is

$$\pi^* = -\frac{\mu(y)V_x^0 + \sigma(y)\beta(y)V_{xy}^0}{\sigma^2(y)xV_{xx}^0}.$$

Inserting π^* into the HJB equation (2.5) yields the nonlinear PDE

$$(2.8) \quad V_t^0 - \frac{\mu^2(y)(V_x^0)^2}{2\sigma^2(y)V_{xx}^0} - \frac{\beta^2(y)(V_{xy}^0)^2}{2V_{xx}^0} - \frac{\mu(y)\beta(y)V_{xy}^0 V_x^0}{\sigma(y)V_{xx}^0}$$
$$+ \mathcal{L}_Y V^0 = 0, \qquad (t,x,y) \in [0,T) \times \mathbb{R} \times \mathbb{R}.$$

We reduce the state space by one dimension by making the ansatz

$$(2.9) \qquad V^0(t,x,y) = 1 - \exp\big(-\gamma x - v(t,y)\big).$$

This logarithmic transform simplifies the nonlinearities in (2.8) considerably, and it is not hard to see that $v(t,y)$ satisfies the semilinear PDE

$$(2.10) \qquad -v_t - \frac{\mu^2(y)}{2\sigma^2(y)} - \mathcal{L}_Y v + \frac{\mu(y)\beta(y)}{\sigma(y)}v_y + \frac{1}{2}\delta^2(y)\big(v_y\big)^2 = 0, \qquad (t,y) \in [0,T) \times \mathbb{R},$$

with terminal data

$$(2.11) \qquad v(T,y) = 0, \qquad y \in \mathbb{R}.$$

In passing we note that the terminal value problem (2.10)-(2.11) coincides with (1.3)-(1.4).

It seems appropriate here to mention that the idea of using a logarithmic transformation to reduce the nonlinearity in the HJB equation goes back to Fleming, see [14]. Under the assumption of power utilities, this idea was used to solve rather general multi-dimensional stochastic volatility models in [29], see also [36] for a power transformation that reduces the HJB equation to a linear PDE (this works for the one-dimensional case with constant correlation).

The HJB equation for the value function (2.2) when the investor has issued a claim with payoff function $g(s)$ at time T reads

$$(2.12) \quad V_t + \max_{\pi \in \mathbb{R}}\left\{\mu(y)\pi x V_x + \frac{1}{2}\sigma^2(y)\pi^2 x^2 V_{xx} + \sigma(y)\beta(y)\pi x V_{xy} + \sigma^2(y)\pi x s V_{xs}\right\}$$
$$+ \mathcal{L}_S V + \mathcal{L}_Y V + \sigma(y)\beta(y)s V_{ys} = 0, \qquad (t,x,y,s) \in [0,T) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+,$$

with terminal data

$$(2.13) \qquad V(T,x,y,s) = 1 - \exp\big(-\gamma(x - g(s))\big),$$

where $\mathcal{L}_Y$ is defined in (2.7) and

$$\mathcal{L}_S V = \mu(y)s V_s + \frac{1}{2}\sigma^2(y)s^2 V_{ss}.$$

From the first order condition we can derive the following expression for the optimal investment strategy π^*:

$$\pi^* = -\frac{\mu(y)V_x + \sigma(y)\beta(y)V_{xy} + \sigma^2(y)s V_{xs}}{\sigma^2(y)xV_x}.$$

Inserting π^* into the HJB equation (2.12) yields the nonlinear PDE

$$V_t - \frac{\mu^2(y)V_x^2}{2\sigma^2(y)V_{xx}} - \frac{\beta^2(y)V_{xy}^2}{2V_{xx}} - \frac{\sigma^2(y)s^2 V_{xs}^2}{2V_{xx}}$$
$$(2.14) \qquad - \frac{\mu(y)\beta(y)V_x V_{xy}}{\sigma(y)V_{xx}} - \frac{\mu(y)s V_x V_{xs}}{V_{xx}} - \frac{\sigma(y)\beta(y)s V_{xy}V_{xs}}{V_{xx}}$$
$$+ \mathcal{L}_S V + \mathcal{L}_Y V + \sigma(y)\beta(y)s V_{ys} = 0, \qquad (t,x,y,s) \in [0,T) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+.$$

We now make the ansatz

$$(2.15) \qquad V(t,x,y,s) = 1 - \exp\big(-\gamma x + \gamma f(t,y,s) - v(t,y)\big),$$

for some function $f(t, y, s)$ to be determined and with $v(t, y)$ solving (2.10)-(2.11). With this representation the utility indifference price for the claim will be given by $f(t, y, s)$. Indeed, by the representations (2.9), (2.15) and the definition of the utility indifference price (2.4),

$$1 - \exp\big(-\gamma(x + \Lambda^{(\gamma)}(t, y, s)) + \gamma f(t, y, s) - v(t, y)\big) = 1 - \exp\big(-\gamma x - v(t, y)\big),$$

which implies that

$$f(t, y, s) = \Lambda^{(\gamma)}(t, y, s).$$

From here on we will use $\Lambda^{(\gamma)}(t, y, s)$ instead of $f(t, y, s)$ in the ansatz (2.15). We can derive a PDE for $\Lambda^{(\gamma)}$. Plugging (2.15) into (2.14) and using the PDE (2.10) for v, we derive the following semilinear PDE for $\Lambda^{(\gamma)}$:

$$(2.16) \quad \begin{aligned} &\Lambda_t^{(\gamma)} + \frac{1}{2}\sigma^2(y)s^2\Lambda_{ss}^{(\gamma)} + \mathcal{L}_Y\Lambda^{(\gamma)} + \sigma(y)\beta(y)s\Lambda_{ys}^{(\gamma)} \\ &\quad - \left(\delta^2(y)v_y + \frac{\mu(y)\beta(y)}{\sigma(y)}\right)\Lambda_y^{(\gamma)} + \frac{1}{2}\gamma\delta^2(y)\big(\Lambda_y^{(\gamma)}\big)^2 = 0, \qquad (t, y, s) \in [0, T) \times \mathbb{R} \times \mathbb{R}_+. \end{aligned}$$

Also, since (2.13) holds, $\Lambda^{(\gamma)}$ obeys the terminal condition

$$(2.17) \qquad\qquad \Lambda^{(\gamma)}(T, y, s) = g(s), \qquad (y, s) \in \mathbb{R} \times \mathbb{R}_+.$$

Example 2.1. Consider the case when $\delta = 0$, i.e., the complete case. The PDE (2.16) then becomes

$$(2.18) \quad \Lambda_t^{(\gamma)} + \frac{1}{2}\sigma^2(y)s^2\Lambda_{ss}^{(\gamma)} + \left(\alpha(y) - \frac{\mu(y)\beta(y)}{\sigma(y)}\right)\Lambda_y^{(\gamma)} + \frac{1}{2}\beta^2(y)\Lambda_{yy}^{(\gamma)} + \sigma(y)\beta(y)s\Lambda_{ys}^{(\gamma)} = 0.$$

Introduce the Girsanov transformation of B_t given by

$$dB_t = -\frac{\mu(Y_t)}{\sigma(Y_t)}\,dt + d\widetilde{B}_t,$$

where $\widetilde{B}$ is a Brownian motion under Q (at least when the Novikov condition for $\mu(Y_t)/\sigma(Y_t)$ holds). The unique arbitrage free price of the contingent claim is

$$\mathbb{E}_Q\left[g(S_T)\,\big|\,Y_t = y, S_t = s\right],$$

and it is not hard to see that this expression solves (2.18)-(2.17). Thus, in the complete case, the arbitrage free price coincides with the utility indifference price (as we have already mentioned).

Let us suppose that the zero risk aversion limit

$$\Lambda(t, y, s) := \lim_{\gamma \to 0}\Lambda^{(\gamma)}(t, y, s)$$

exists and that $\Lambda_y^{(\gamma)}(t, y, s)$ remains uniformly bounded as $\gamma \to 0$, for each fixed (t, y, s). Then, from (2.16), Λ satsfies the linear Black-Schole type PDE problem

$$(2.19) \quad \begin{cases} \Lambda_t + \dfrac{1}{2}\sigma^2(y)s^2\Lambda_{ss} + \mathcal{L}_Y\Lambda \\ \qquad - \left(\delta^2(y)v_y + \dfrac{\mu(y)\beta(y)}{\sigma(y)}\right)\Lambda_y + \sigma(y)\beta(y)s\Lambda_{ys} = 0, \qquad (t, y, s) \in [0, T) \times \mathbb{R} \times \mathbb{R}_+, \\ \Lambda(T, y, s) = g(s), \qquad (y, s) \in \mathbb{R} \times \mathbb{R}_+. \end{cases}$$

The Feynman-Kac formula yields the following representation for Λ:

$$(2.20) \qquad\qquad \Lambda(t, y, s) = \mathbb{E}\left[g\left(\widetilde{S}_T\right)\,\big|\,\widetilde{Y}_t = y, \widetilde{S}_t = s\right],$$

where the stochastic processes $\widetilde{S}$ and $\widetilde{Y}$ are given by

$$d\widetilde{S}_t = \sigma\left(\widetilde{Y}_t\right)\widetilde{S}_t\,dB_t,$$

$$d\widetilde{Y}_t = \left(\alpha\left(\widetilde{Y}_t\right) - \delta^2\left(\widetilde{Y}_t\right)v_y\left(t, \widetilde{Y}_t\right) - \frac{\mu(\widetilde{Y}_t)\beta\left(\widetilde{Y}_t\right)}{\sigma\left(\widetilde{Y}_t\right)}\right)dt + \beta\left(\widetilde{Y}_t\right)dB_t + \delta\left(\widetilde{Y}_t\right)dW_t.$$

Formally, we rewrite (2.20) in terms of the original processes under an equivalent (martingale) measure. Introduce the Girsanov transformations

$$dB_t = d\widetilde{B}_t - \frac{\mu(Y_t)}{\sigma(Y_t)} \, dt, \qquad dW_t = d\widetilde{W}_t - \delta(Y_t)v_y(t, Y_t) \, dt,$$

where $\widetilde{B}$ and $\widetilde{W}$ are two independent Brownian motions under the equivalent martingale measure Q, whose Radon-Nikodym derivative is $dQ/dP = Z_T$ with

(2.21)
$$\begin{aligned} Z_T = \exp\bigg(&- \int_0^T \frac{\mu(Y_t)}{\sigma(Y_t)} \, dB_t - \int_0^T \delta(Y_t)v_y(t, Y_t) \, dW_t \\ &- \frac{1}{2} \int_0^T \frac{\mu^2(Y_t)}{\sigma^2(Y_t)} + \delta^2(Y_t)v_y^2(t, Y_t) \, dt \bigg). \end{aligned}$$

Then (2.20) takes the form

(2.22)
$$\Lambda(t, y, s) = \mathbb{E}_Q\left[g(S_T) \,\middle|\, Y_t = y, S_t = s \right].$$

This Girsanov tranform is valid as long as Z_T is a martingale.

Note that the right-hand side in (2.21) with v solving (2.10)-(2.11) coincides with the right-hand side in (1.6) with v solving (1.3)-(1.4). In the next section we prove (under certain conditions) that Q defined by (2.21) coincides with the MEMM Q_{ME} and Λ defined in (2.22) is hence just the arbitrage free price under Q_{ME} (the so-called minimal entropy price).

Finally, let us mention that from general (duality) theory (see, e.g., [31, 13, 3]), and without any reference to PDEs, it is known that the zero risk aversion asymptotic of the utility indifference price with exponential utility coincides with the minimal entropy price. It is thus natural to propose Z_T given in (2.21) as the candidate density for the minimal entropy measure. Although this is not of our concern here, we mention that the $\gamma \to \infty$ asymptotic of the utility indifference price coincides with the price of the cheapest superhedging strategy, which is known to be too expensive in general for any practical purposes.

3. Identification of the MEMM

We want to prove that Z_T given in (1.6) is the density of the MEMM Q_{ME}. To this end, we need to verify that Z_T is a martingale (not only a local martingale) defining a probability measure with finite relative entropy, which moreover is minimal among all probability measures of finite relative entropy. We will do this by verifying certain conditions stated in Rheinländer [30].

First we show that SZ is a local P-martingale (thus S is a local Q-martingale), where S solves (1.1). This follows easily by applying Itô's formula on the product SZ:

$$\begin{aligned} d(SZ)_t &= S_t dZ_t + Z_t dS_t + d[SZ]_t \\ &= \left(\sigma(Y_t) - \frac{\mu(Y_t)}{\sigma(Y_t)} \right) (SZ)_t \, dB_t - \delta(Y_t)v_y(t, Y_t)(SZ)_t \, dW_t. \end{aligned}$$

If Z is a martingale, and not merely a local martingale, Z_T will be the density of a martingale measure. We will give conditions for this to be true.

Following the notation in Rheinländer [30], we introduce the processes

$$K_t = \int_0^t \frac{\mu^2(Y_s)}{\sigma^2(Y_s)} \, ds,$$

$$L_t = - \int_0^t \delta(Y_s)v_y(s, Y_s) \, dW_s.$$

Recall that the quadratic variation process of L_t, denoted by $[L]_t$, is given as

$$[L]_t = \int_0^t \delta^2(Y_s)v_y^2(s, Y_s) \, ds.$$

Define for each natural number n the stopping time

$$\tau_n = \inf\left\{ t > 0 \,\middle|\, \max(K_t, [L]_t) \geq n \right\},$$

and let $T_n = \min(\tau_n, T)$. Set $Z_t^n := Z_{t \wedge T_n}$. Novikov's criterion now implies that Z_T^n is the density of a probability measure Q_n. The following theorem is taken from Rheinländer [30]:

Theorem 3.1 ([30]). *The following assertions are equivalent:*

(1) $\sup_n \mathbb{E}_{Q_n}\left[K_{T_n} + [L]_{T_n}\right] < \infty.$

(2) $\sup_n H(Q_n, P) < \infty.$

(3) Z_T *is the density of a probability measure* Q *with* $H(Q, P) < \infty.$

We introduce the following condition:

Condition A. Let the functions $\mu(y)$, $\sigma(y)$, $\delta(y)$, and $v(t, y)$ be such that assertion (1), or equivalently assertion (2), in Theorem 3.1 holds.

Under Condition A we are ensured that Z_T is the density of a probability measure Q with finite relative entropy, so that Q is in fact a martingale measure with finite relative entropy.

Our next task is the find conditions such that this measure has minimal entropy. To succeed with this, we will first rewrite the expression (1.6) for Z_T as

$$(3.1) \qquad \exp\left(c + \int_0^T \eta_t \, dS_t\right),$$

for a constant c and some adapted process η_t, and then we will identify Z_T as the density of the MEMM Q_{ME} by verifying the condition in the following proposition due to Rheinländer [30]:

Proposition 3.2 ([30]). *Let S be a locally bounded semimartingale. If $\overline{Q} \in \mathcal{M}_{\mathrm{e}}$ has finite relative entropy and the Radon-Nikodym derivative $\dfrac{dQ}{dP}$ is of the form (3.1) with $\displaystyle\int_0^T \eta_t^2 \, d[S]_t$ belonging to the Orlicz space $L^1_{\exp}(P)$ generated by the Young function $\exp(\cdot)$. Then $\int \eta \, dS$ is a true Q-martingale for all $Q \in \mathcal{M}_{\mathrm{e}}$ with finite relative entropy, and therefore $\overline{Q}$ coincides with the MEMM.*

To apply Proposition 3.2 we shall need the following condition:

Condition B. There exists a positive constant ε such that

$$\exp\left(\varepsilon \int_0^T \left\{\frac{\mu^2(Y_t)}{\sigma^2(Y_t)} + \beta^2(Y_t) v_y^2(t, Y_t)\right\} dt\right) \in L^1(P).$$

Condition B will be a sufficient condition to ensure that Z_T is the density of the MEMM Q_{ME}.

We are now in a position to prove the following result:

Theorem 3.3. *Assume that (1.5) and Conditions A and B hold. Then Z_T in (1.6) is the density of the MEMM Q_{ME}.*

Proof. Using that

$$S_t^{-1} dS_t = \mu(Y_t) \, dt + \sigma(Y_t) \, dB_t$$

we find

$$(3.2) \qquad \begin{aligned} Z_T = \exp\Bigg(&-\int_0^T \frac{\mu(Y_t)}{\sigma^2(Y_t)} S_t^{-1} \, dS_t - \int_0^T \delta(Y_t) v_y(t, Y_t) \, dW_t \\ &+ \frac{1}{2} \int_0^T \frac{\mu^2(Y_t)}{\sigma^2(Y_t)} - \delta^2(Y_t) v_y^2(t, Y_t) \, dt \Bigg). \end{aligned}$$

Let us derive an expression for the term $\displaystyle\int_0^T \delta(Y_t) v_y(t, Y_t) \, dW_t$. By Itô's formula it holds (since we assume $v \in C^{1,2}$)

$$dv(t, Y_t) = v_t \, dt + \mathcal{L}_Y v \, dt + v_y \left\{\beta(Y_y) \, dB_t + \delta(Y_t) \, dW_t\right\},$$

where $\mathcal{L}_Y$ is defined in (2.7). Integrating and appealing to the PDE (1.3) satisfied by v,

$$v(T, Y_T) = v(0, y) + \int_0^T \left(v_t(t, Y_t) + \mathcal{L}_Y v(t, Y_t) \right) dt + \int_0^T \beta(Y_t) v_y(t, Y_t) \, dB_t$$

$$+ \int_0^T \delta(Y_t) v_y(t, Y_t) \, dW_t$$

$$= v(0, y) + \int_0^T \left(-\frac{1}{2} \frac{\mu^2(Y_t)}{\sigma^2(Y_t)} + \frac{1}{2} \delta^2(Y_t) v_y^2(t, Y_t) + \frac{\mu(Y_t)\beta(Y_t)}{\sigma(Y_t)} v_y(t, Y_t) \right) dt$$

$$+ \int_0^T \beta(Y_t) v_y(t, Y_t) \, dB_t + \int_0^T \delta(Y_t) v_y(t, Y_t) \, dW_t.$$

Since $v(T, y) = 0$ for all y, we obtain the relation

$$\int_0^T \delta(Y_t) v_y(t, Y_t) \, dW_t = -v(0, y) + \frac{1}{2} \int_0^T \frac{\mu^2(Y_t)}{\sigma^2(Y_t)} \, dt - \frac{1}{2} \int_0^T \delta^2(Y_t) v_y^2(t, Y_t) \, dt$$

$$- \int_0^T \frac{\mu(Y_t)\beta(Y_t)}{\sigma(Y_t)} v_y(t, Y_t) \, dt - \int_0^T \beta(Y_t) v_y(t, Y_t) \, dB_t.$$

Inserted into the expression (3.2) for Z_T this yields

$$Z_T = \exp\left(-\int_0^T \frac{\mu(Y_t)}{\sigma^2(Y_t)} S_t^{-1} \, dS_t + v(0, y) + \int_0^T \beta(Y_t) v_y(t, Y_t) \, dB_t \right.$$

$$\left. + \int_0^T \frac{\mu(Y_t)\beta(Y_t)}{\sigma(Y_t)} v_y(t, Y_t) \, dt \right).$$

Using $\sigma^{-1}(Y_t) S_t^{-1} \, dS_t = \mu(Y_t) \sigma^{-1}(Y_t) \, dt + dB_t$ gives

$$(3.3) \qquad Z_T = \exp\left(\int_0^T \left\{ \frac{\beta(Y_t)}{\sigma(Y_t)} v_y(t, Y_t) - \frac{\mu(Y_t)}{\sigma^2(Y_t)} \right\} S_t^{-1} \, dS_t + v(0, y) \right),$$

which shows that Z_T can be written in the form (3.1) with

$$c = v(0, y), \qquad \eta_t = \left\{ \frac{\beta(Y_t)}{\sigma(Y_t)} v_y(t, Y_t) - \frac{\mu(Y_t)}{\sigma^2(Y_t)} \right\} S_t^{-1}.$$

We know already that Z_T is the density of a martingale measure Q with finite relative entropy. From [16] it then follows that the MEMM Q_{ME} exists and is moreover unique. Since $d[S]_t = \sigma^2(Y_t) S_t^2 \, dt$, we must have for some $\varepsilon > 0$ that

$$\exp\left(\varepsilon \int_0^T \left\{ \beta(Y_t) v_y(t, Y_t) - \frac{\mu(Y_t)}{\sigma(Y_t)} \right\}^2 \right) \in L^1(P).$$

Condition B ensures this. Consequently,

$$\exp\left(\varepsilon \int_0^T \eta_t^2 \, d[S]_t \right) \in L^1(P)$$

for some $\varepsilon > 0$. Proposition 3.2 now implies that Z_T is the density of the MEMM Q_{ME}. $\qquad \square$

As long as Condition A is satisfied, it follows from Girsanov's theorem that the processes $\widetilde{B}$ and $\widetilde{W}$, defined via

$$(3.4) \qquad \begin{aligned} dB_t &= d\widetilde{B}_t - \frac{\mu(Y_t)}{\sigma(Y_t)} \, dt, \\ dW_t &= d\widetilde{W}_t - \delta(Y_t) v_y(t, Y_t) \, dt, \end{aligned}$$

are two independent Brownian motions under the martingale measure that has Z_T as its density. Furthermore, observe that $\widetilde{B}_{t \wedge T_n}$ and $\widetilde{W}_{t \wedge T_n}$ are two independent (stopped) Brownian motions under Q_n.

4. Application to a class of stochastic volatility models

We consider the following class of stochastic volatility models:

$$(4.1) \qquad\qquad dY_t = \alpha(Y_t)\, dt + \beta\, dU_t, \qquad \beta > 0,$$

where $U_t := \rho B_t + \sqrt{1-\rho^2}\, W_t$ and $-1 < \rho < 1$ for a constant ρ. Note that U_t is a Brownian motion correlated with B_t (see (1.1)) with correlation coefficient ρ. Before continuing, we introduce the following (growth) assumptions on the asset and volatility coefficients in (1.1) and (4.1):

$$(4.2) \qquad\qquad \begin{cases} |\alpha(y)| \le C\,|y|, \quad |\alpha'(y)| \le C, & y \in \mathbb{R}, \\ \left| \frac{\mu(y)}{\sigma(y)} \right| \le C\,|y|, \quad \left| \left(\frac{\mu(y)}{\sigma(y)} \right)' \right| \le C, & y \in \mathbb{R}. \end{cases}$$

We prove that the conditions in (4.2) are sufficient for the existence of a unique quadratically growing classical solution $v = v(t,y)$, with a linearly growing derivative $v_y(t,y)$, of the semilinear PDE

$$(4.3) \qquad\qquad -v_t - \frac{1}{2}\beta^2 v_{yy} + F(y, v_y) = 0, \qquad (t,y) \in [0,T) \times \mathbb{R},$$

with terminal condition

$$(4.4) \qquad\qquad v(T,y) = 0, \qquad y \in \mathbb{R}.$$

The nonlinear function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$F(y,p) = \frac{1}{2}\delta^2 p^2 - \left\{ \alpha(y) - \frac{\mu(y)\beta\rho}{\sigma(y)} \right\} p - \frac{1}{2}\frac{\mu^2(y)}{\sigma^2(y)}$$

and $\delta^2 = \beta^2(1-\rho^2) > 0$ is a constant. Note that the problem (4.3)-(4.4) corresponds to (1.3)-(1.4) with $\delta(y) = \beta\sqrt{1-\rho^2}$ and $\beta(y) = \beta\rho$ for all y. Furthermore, we will prove that the linear growth of the derivative v_y implies that Conditions A and B hold. Hence, under the conditions stated in (4.2) on the asset price model (1.1) and the volatility model (4.1), we have the existence of the MEMM Q_{ME} with density as in (1.6), or equivalently (3.3). The Stein-Stein volatility model is covered by the theory in this section, and for this model we will see that an explicit solution of (4.3)-(4.4) can be found.

4.1. Well-posedness of the semilinear PDE.
The existence of a classical solution to (4.3)-(4.4) cannot be found directly in the literature [23] since $p \mapsto F(y,p)$ is not globally Lipschitz continuous (our solutions grow quadratically in y on $\mathbb{R}$). Here we will reiterate the approach taken in [14, 29] by considering a certain sequence of approximating PDEs which are the HJB-equations of certain stochastic control problems for which the existence of smooth solutions is well-known.

Introduce the function $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$L(y,q) = \max_{p \in \mathbb{R}} \left\{ -qp - F(y,p) \right\}.$$

One can easily check that

$$L(y,q) = \frac{1}{2\delta^2} \left(q - \left\{ \alpha(y) - \frac{\mu(y)\beta\rho}{\sigma(y)} \right\} \right)^2 + \frac{1}{2}\frac{\mu^2(y)}{\sigma^2(y)}.$$

One can also easily check that the following duality relation holds:

$$F(y,p) = \max_{q \in \mathbb{R}} \left\{ -qp - L(y,q) \right\}.$$

Consider the auxiliary function $F^k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined for each natural number k by

$$F^k(y,p) = \max_{|q| \le k} \left\{ -qp - L(y,q) \right\}.$$

We have that $L \in C^1(\mathbb{R} \times \mathbb{R})$ and L, L_y satisfy a polynomial growth condition in y. More precisely, from

$$\left| \alpha(y) - \frac{\mu(y)\beta\rho}{\sigma(y)} \right| \le C\,|y|, \qquad y \in \mathbb{R},$$

we find

$$(4.5) \qquad |L(y,q)| \leq C\left(q^2 + y^2\right), \qquad (y,q) \in \mathbb{R} \times \mathbb{R}.$$

Moreover, since

$$L_y(y,q) = -\frac{1}{\delta^2}\left(q - \left\{\alpha(y) - \frac{\mu(y)\beta\rho}{\sigma(y)}\right\}\right)\left\{\alpha'(y) - \left(\frac{\mu(y)}{\sigma(y)}\right)'\beta\right\} + \frac{1}{2}\left(\frac{\mu^2(y)}{\sigma^2(y)}\right)',$$

it follows from (4.2) that

$$(4.6) \qquad |L_y(y,q)| \leq C\left(|q| + |y|\right), \qquad (y,q) \in \mathbb{R} \times \mathbb{R}.$$

In particular, $|L|$ and $|L_y|$ are of polynomial growth in y uniformly in q when $|q| \leq k$. Hence by classical theory [14, Theorem 4.3 on p. 169] there exists a unique polynomially growing solution

$$v^k \in C^{1,2}([0,T) \times \mathbb{R}) \cap C([0,T] \times \mathbb{R}),$$

of the semilinear PDE

$$(4.7) \qquad -v_t^k - \frac{1}{2}\beta^2 v_{yy}^k + F^k(y, v_y^k) = 0, \qquad (t,y) \in [0,T) \times \mathbb{R},$$

with terminal condition $v^k(T,y) = 0$ for all $y \in \mathbb{R}$.

We need to derive estimates on v_t^k and v_y^k that are independent of k.

Lemma 4.1. *There exists a constant C that is independent of k such that*

$$\left|v_y^k(t,y)\right| \leq C\left(1 + |y|\right), \qquad (t,y) \in [0,T) \times \mathbb{R}.$$

Proof. From standard theory [14], v^k can be represented as the solution of the stochastic control problem

$$(4.8) \qquad v^k(t,y) = \inf_{q \in \mathcal{A}_t^k} \mathbb{E}\left[\int_t^T L\left(\widehat{Y}_s, q_s\right) ds \,\big|\, \widehat{Y}_t = y\right],$$

where $\mathcal{A}_t^k$ denotes the set of adapted control processes that are bounded by k, and

$$(4.9) \qquad d\widehat{Y}_t = q_t\, dt + \beta\, dU_t.$$

Furthermore, an optimal control for (4.8) is Markov and is given in feedback form by

$$(4.10) \qquad q_k^\star(t,y) = \arg\min_{|q| \leq k}\left\{q v_y^k(t,y) + L(y,q)\right\}.$$

Consequently,

$$(4.11) \qquad v^k(t,y) = \mathbb{E}\left[\int_t^T L\left(\widehat{Y}_s^\star, q_k^\star\left(s, \widehat{Y}_s^\star\right)\right) ds \,\big|\, \widehat{Y}_t^\star = y\right],$$

where $\widehat{Y}^\star$ solves (4.9) with the control $q_t = q_k^\star\left(t, \widehat{Y}_t^\star\right)$.

From standard theory [14, Lemma 11.4 on p. 209], we have

$$(4.12) \qquad v_y^k(t,y) = \mathbb{E}\left[\int_t^T L_y\left(\widehat{Y}_s^\star, q_k^\star\left(s, \widehat{Y}_s^\star\right)\right) ds \,\big|\, \widehat{Y}_t^\star = y\right].$$

Furthermore, using the definition of $L(y,q)$ and its derivative $L_y(y,q)$, it is straightforward to show that

$$|L_y(y,q)| \leq C\sqrt{L(y,q)},$$

for some constant C. Thus, by the Cauchy-Schwarz inequality and the fact that $L(y,0) \leq cy^2$ for a constant c independent of k,

$$\left|v_y^k(t,y)\right| \leq \mathbb{E}\left[\int_t^T \left|L_y\left(\widehat{Y}_s^\star, q_k^\star\left(s, \widehat{Y}_s^\star\right)\right)\right| ds \,\big|\, \widehat{Y}_t^\star = y\right]$$

$$\leq (T-t)^{1/2} \mathbb{E} \left[\int_t^T \left| L_y \left(\widehat{Y}_s^\star, q_k^\star \left(s, \widehat{Y}_s^\star \right) \right) \right|^2 ds \, \middle| \, \widehat{Y}_t^\star = y \right]^{1/2}$$

$$\leq C \mathbb{E} \left[\int_t^T L \left(\widehat{Y}_s^\star, q_k^\star \left(s, \widehat{Y}_s^\star \right) \right) ds \, \middle| \, \widehat{Y}_t^\star = y \right]^{1/2}$$

$$\leq C \mathbb{E} \left[\int_t^T L \left(\widehat{Y}_s, 0 \right) ds \, \middle| \, \widehat{Y}_t = y \right]^{1/2}$$

$$\leq C \mathbb{E} \left[\int_t^T \widehat{Y}_s^2 \, ds \, \middle| \, \widehat{Y}_t = y \right]^{1/2}$$

$$\leq C \left(1 + |y| \right),$$

where we have used that for the control $q = 0$, $\widehat{Y}_s = y + \beta(U_s - U_t)$ for $s \geq t$. The constant C changes from line to line in the above the estimation process, but is always independent of k. □

Lemma 4.2. *There exists a constant C that is independent of k such that*

$$\left| v_t^k(t, y) \right| \leq C \left(1 + |y|^2 \right), \qquad (t, y) \in [0, T) \times \mathbb{R}.$$

Proof. Let

$$\widehat{q}_k(t, y) := \underset{q \in \mathbb{R}}{\arg\min} \left\{ q v_y^k(t, y) + L(y, q) \right\}.$$

We easily check that

$$\widehat{q}_k(t, y) = \widetilde{\alpha}(y) - \delta^2 v^k(y, y),$$

where $\widetilde{\alpha}(y) = \alpha(y) - \frac{\mu(y)\beta\rho}{\sigma(y)}$. From the growth conditions on α, μ/σ and the estimate on $v_y^k(t, y)$ in Lemma 4.1, there exists a constant C independent of k such that

(4.13) $$|\widehat{q}_k(t, y)| \leq C(1 + |y|).$$

Observe that $q_k^\star(t, y) = k \vee \widehat{q}_k(t, y) \wedge (-k)$, where $q_k^\star$ is defined in (4.10). Therefore

(4.14) $$|q_k^\star(t, y)| \leq C(1 + |y|),$$

where C appears in (4.13). Of course $q_k^\star$ is bounded by k, but it is important for us later to have a bound of $q_k^\star$ which is independent of k. A change of time in (4.11) yields

$$v^k(t, y) = \mathbb{E} \left[\int_0^{T-t} L \left(\widehat{Y}_{t+s}^\star, q_k^\star \left(t+s, \widehat{Y}_{t+s}^\star \right) \right) ds \, \middle| \, \widehat{Y}_t^\star = y \right].$$

If $\overline{U}_s := U_{t+s}$, then $\overline{U}$ becomes a Brownian motion since it is a time change of U. Assume that $\overline{q}_t$ belongs to the set of adapted stochastic processes bounded on $[0, T-t]$ bounded by k, which we denote by $\overline{\mathcal{A}}^k$. Consider the optimal control problem with criterion function

$$\mathbb{E} \left[\int_0^{T-t} L \left(\overline{Y}_s, \overline{q}_s \right) ds \, \middle| \, \overline{Y}_0 = y \right],$$

and dynamics

$$d\overline{Y}_s = \overline{q}_s \, ds + \beta \, d\overline{U}_s.$$

¿From standard theory [14], there exists an optimal feedback control $\overline{q}_k^\star$ solving this problem, and it holds that $\overline{Y}_s^\star = \widehat{Y}_s^\star$ in distribution since $\overline{q}_k^\star = q_k^\star$. Thus,

$$v^k(t, y) = \mathbb{E} \left[\int_0^{T-t} L \left(\overline{Y}_s^\star, \overline{q}^\star \left(s, \overline{Y}_s^\star \right) \right) ds \, \middle| \, \overline{Y}_0 = y \right].$$

Now following the estimation procedure in [14, Proof of Lemma 8.2 on p. 192] and using (4.5), we get

$$
(4.15) \qquad
\begin{aligned}
\left| v_t^k(t,y) \right| &\le C \mathbb{E}\left[\left(\overline{q}_k^\star \left(T-t, \overline{Y}_{T-t} \right) \right)^2 + \overline{Y}_{T-t}^2 \big| \overline{Y}_0 = y \right] \\
&\le C \left(1 + \mathbb{E}\left[\overline{Y}_{T-t}^2 \big| \overline{Y}_0 = y \right] \right),
\end{aligned}
$$

where (4.14) was used to derive the second inequality.

Let us estimate $\mathbb{E}\left[\overline{Y}_t^2 \right]$ for a $t \in [0,T]$. Appealing to Itô's formula and (4.14), we find for the stopping times $s_n = \tau_n \wedge t$, where $\tau_n = \inf\{ t \ge 0 \text{ and } |\overline{Y}_t| \ge n \}$,

$$
\mathbb{E}\left[\overline{Y}_{s_n}^2 \right] \le y^2 + (\beta^2 + 2C)T + 2C \int_0^t \mathbb{E}\left[\overline{Y}_s^2 \right] ds.
$$

Since $s_n \uparrow t$ when $n \to \infty$, we get from Fatou's lemma and Gronwall's inequality that

$$
(4.16) \qquad \mathbb{E}\left[\overline{Y}_t^2 \right] \le C(1 + |y|^2).
$$

The constant C may have changed from line to line in the above estimation process, but is always independent of k. The lemma follows now from (4.15) and (4.16). $\qquad\square$

Theorem 4.3. *Suppose the conditions in* (4.2) *hold. Then there exists a unique classical solution*

$$
(4.17) \qquad v \in C^{1,2}([0,T) \times \mathbb{R}) \cap C([0,T] \times \mathbb{R})
$$

of the terminal value problem (4.3)-(4.4). *Moreover,* $v(t,y)$ *is at most quadratically growing in* y *while the derivative* $v_y(t,y)$ *is at most linearly growing in* y.

Proof. We just sketch the existence proof, which is standard and relies on the k-independent estimates obtained in Lemmas 4.1 and 4.2 for the classical solution v^k of (4.7) as well as well standard regularizing properties of the heat equation. First of all, Lemmas 4.1 and 4.2 and the Ascoli-Arzela theorem imply immediately that for a subsequence v^k converges locally uniformly to a limit function v which is continuous and has quadratic growth. We must prove similar convergence results for the derivatives v_t^k, v_y^k, and v_{yy}^k. To this end, let us write (4.7) in the form of a nonhomogeneous heat equation

$$
(4.18) \qquad -v_t^k - \frac{1}{2}\beta^2 v_{yy}^k = f^k(t,y), \qquad f^k := -F^k(y, v_y^k).
$$

Thanks to Lemmas 4.1and 4.2, the function f^k is locally uniformly (in k) bounded. Classical regularity theory for the heat equation (see, e.g., [23]) implies then that v_y^k, and thus also f^k, is locally uniformly (in k) Hölder continuous. The Hölder regularity of f^k implies, again via standard regularity theory for the heat equation [23], that v_t^k and v_{yy}^k are locally uniformly (in k) Hölder continuous. From this and the Ascoli-Arzelá theorem, it is not hard to prove that along subsequences v_t^k, v_y^k, and v_{yy}^k converge locally uniformly to v_t, v_y, and v_{yy}, respectively. Moreover, v satisfies (4.17), has a derivative v_y that grows at most linearly, and solves the terminal value problem (4.3)-(4.4).

The uniqueness assertion of the theorem follows from [10]. $\qquad\square$

Remark. We mention that to identify Z_T in (1.6) as the density of the MEMM Q_{ME} we only need to know that there exists at least one solution of the type provided by Theorem 4.3 (uniqueness is strictly speaking not needed for the identification process).

4.2. **Verification of Conditions A and B.** Now we prove that Conditions A and B hold under the assumptions stated in (4.2) for the asset and volatility dynamics (1.1)-(4.1). We split the proof into two propositions.

Proposition 4.4. *Assume that the conditions stated in* (4.2) *hold. Then Condition A holds for the model* (1.1)-(4.1).

Proof. We will prove that condition (1) in Theorem 3.1 holds, which is the case if we can prove that $\sup_n \mathbb{E}_{Q_n} \left[\int_0^{T_n} Y_s^2 \, ds \right] < \infty$, where Q_n and T_n are defined just before the Theorem 3.1.

Define the process $\widetilde{U}_t^n := \rho \widetilde{B}_{t \wedge T_n} + \sqrt{1 - \rho^2} \, \widetilde{W}_{t \wedge T_n}$, and note from the discussion (3.4) at the end of Section 3 that $\widehat{U}^n$ is a Brownian motion under Q_n. For $t \leq T_n$, it holds

$$dY_t = \widetilde{\alpha}(t, Y_t) \, dt + \beta \, d\widetilde{U}_t^n,$$

with

$$\widetilde{\alpha}(t, y) = \alpha(y) - \frac{\mu(y)\beta\rho}{\sigma(y)} - \beta^2 (1 - \rho^2) v_y^2(t, y).$$

From the assumptions (4.2) and the linear growth of v_y we find

(4.19) $$|\widetilde{\alpha}(t, y)| \leq K \left(1 + |y| \right),$$

for some positive constant K.

Itô's formula yields for $t \leq T_n$

$$Y_t^2 = y^2 + \beta^2 t + 2 \int_0^t Y_s \, \widetilde{\alpha}(s, Y_s) \, ds + 2 \int_0^t Y_s \, d\widetilde{U}_s^n,$$

and hence,

$$\int_0^{T_n} Y_t^2 \, dt = y^2 T_n + \frac{1}{2} \beta^2 T_n^2 + 2 \int_0^{T_n} \int_0^t Y_s \, \widetilde{\alpha}(s, Y_s) \, ds \, dt + 2 \int_0^{T_n} \int_0^t Y_s \, d\widetilde{U}_s^n \, dt.$$

Taking the expectation with respect to Q_n and using $T_n \leq T$,

$$\mathbb{E}_{Q_n} \left[\int_0^{T_n} Y_t^2 \, dt \right] \leq y^2 T + \frac{1}{2} \beta^2 T^2 + 2\mathbb{E}_{Q_n} \left[\int_0^{T_n} \int_0^t Y_s \, \widetilde{\alpha}(s, Y_s) \, ds \, dt \right]$$

$$+ 2\mathbb{E}_{Q_n} \left[\int_0^{T_n} \int_0^t Y_s \, d\widetilde{U}_s^n \, dt \right].$$

Appealing to the Cauchy-Schwarz inequality we can estimate the last term on the right-hand side as follows:

$$\mathbb{E}_{Q_n} \left[\int_0^{T_n} \int_0^t Y_s \, d\widetilde{U}_s^n \, dt \right] = \mathbb{E}_{Q_n} \left[\int_0^\infty 1_{t \leq T_n} \int_0^t Y_s \, d\widetilde{U}_s^n \, dt \right]$$

$$\leq \mathbb{E}_{Q_n} \left[\int_0^\infty 1_{t \leq T_n} \, dt \right]^{1/2} \mathbb{E}_{Q_n} \left[\int_0^\infty 1_{t \leq T_n} \left(\int_0^t Y_s \, d\widetilde{U}_s^n \right)^2 dt \right]^{1/2}$$

$$\leq \sqrt{T} \mathbb{E}_{Q_n} \left[\int_0^{T_n} \left(\int_0^t Y_s \, d\widetilde{U}_s^n \right)^2 dt \right]^{1/2}$$

$$\leq \sqrt{T} \mathbb{E}_{Q_n} \left[\int_0^T \left(\int_0^t Y_s \, d\widetilde{U}_s^n \right)^2 dt \right]^{1/2}.$$

From the Itô isometry and the inequality $2ab \leq a^2 + b^2$, there exists a positive constant C such that

$$\mathbb{E}_{Q_n} \left[\int_0^{T_n} \int_0^t Y_s \, d\widetilde{U}_s^n \, dt \right] \leq \frac{\sqrt{T}}{2} \left(1 + \left(\int_0^T \mathbb{E}_{Q_n} \left[\int_0^t Y_s^2 \, ds \right] dt \right)^2 \right)^{1/2}$$

$$\leq C \left(1 + \int_0^T \mathbb{E}_{Q_n} \left[\int_0^t Y_s^2 \, ds \right] dt \right).$$

Since $T_n \uparrow T$ a.s. when $n \to \infty$, it follows from Fatou's lemma together with the linear growth (4.19) of the coefficient $\widetilde{\alpha}$ that

$$
\mathbb{E}_{Q_n} \left[\int_0^T Y_s^2 \, ds \right] \leq \liminf_{n \to \infty} \mathbb{E}_{Q_n} \left[\int_0^{T_n} Y_s^2 \, ds \right]
$$

$$
\leq y^2 T + \frac{1}{2} \beta^2 T^2 + 2C \liminf_{n \to \infty} \mathbb{E}_{Q_n} \left[\int_0^{T_n} \int_0^t |Y_s| \left(1 + |Y_s| \right) ds \, dt \right]
$$

$$
+ C \left(1 + \int_0^T \mathbb{E}_{Q_n} \left[\int_0^t Y_s^2 \, ds \right] dt \right)
$$

$$
\leq K \liminf_{n \to \infty} \left(1 + \int_0^T \mathbb{E}_{Q_n} \left[\int_0^t Y_s^2 \, ds \right] dt \right)
$$

$$
\leq C \left(1 + \int_0^T \sup_n \mathbb{E}_{Q_n} \left[\int_0^t Y_s^2 \, ds \right] dt \right),
$$

where the constant C has possibly changed from line to line in the estimation process. Hence

$$
\sup_n \mathbb{E}_{Q_n} \left[\int_0^T Y_s^2 \, ds \right] \leq C \left(1 + \int_0^T \sup_n \mathbb{E}_{Q_n} \left[\int_0^t Y_s^2 \, ds \right] dt \right),
$$

and from Gronwall's inequality it follows that

$$
\sup_n \mathbb{E}_{Q_n} \left[\int_0^T Y_s^2 \, ds \right] \leq C \mathrm{e}^{CT}.
$$

But then we have

$$
\sup_n \mathbb{E}_{Q_n} \left[\int_0^{T_n} Y_s^2 \, ds \right] \leq \sup_n \mathbb{E}_{Q_n} \left[\int_0^T Y_s^2 \, ds \right] \leq C \mathrm{e}^{CT},
$$

and the proposition is proved. $\qquad\square$

Before we prove that Condition B holds, let us state the following lemma which yields an explicit bound on the moments of Y_t starting from zero.

Lemma 4.5. *Let $Y_0 = 0$. Suppose $|\alpha(y)| \leq C |y|$ for all y, then*

$$
\mathbb{E} \left[Y_t^{2n} \right] \leq 2^{-2n} \frac{(2n)!}{n!} \left(\frac{\beta^2}{C} \right)^n \left(e^{2Ct} - 1 \right).
$$

Proof. The proof goes by induction. Let $n = 1$. From Itô's formula we find

$$
Y_t^2 = \beta^2 t + 2 \int_0^t Y_s \alpha(Y_s) \, ds + 2\beta \int_0^t Y_s \, dU_s.
$$

Introduce the stopping times $s_n = t \wedge e_n$, where e_n is the first exit time for Y_t from the ball with radius n and center in 0. We have that $s_n \uparrow t$ when $n \to \infty$ and Fatou's lemma yields

$$
\mathbb{E} \left[Y_t^2 \right] \leq \liminf_{n \to \infty} \mathbb{E} \left[Y_{s_n}^2 \right] \leq \beta^2 t + 2C \int_0^t \mathbb{E} \left[Y_s^2 \right] ds.
$$

From Gronwall's inequality we therefore obtain

$$
\mathbb{E} \left[Y_t^2 \right] \leq \frac{\beta^2}{2C} \left(\mathrm{e}^{2Ct} - 1 \right),
$$

and the assertion holds for $n = 1$. Assume that it holds for n. Following the argumentation for $n = 1$ we estimate

$$
\mathbb{E} \left[Y_t^{2n+2} \right] \leq (2n+2)C \int_0^t \mathbb{E} \left[Y_s^{2n+2} \right] ds + \frac{1}{2} (2n+2)(2n+1) \beta^2 \int_0^t \mathbb{E} \left[Y_s^{2n} \right] ds
$$

$$\leq (2n+2)C \int_0^t \mathbb{E}\left[Y_s^{2n+2}\right] ds + 2^{-(2n+1)} \frac{(2n)!}{n!} \frac{\beta^{2n+2}}{C^n} \int_0^t \left(\mathrm{e}^{2Cs} - 1\right)^n ds,$$

where we have used the induction hypothesis. Appealing to Gronwall's inequality once more gives

$$\mathbb{E}\left[Y_t^{2n+2}\right] \leq 2^{-(2n+1)} \frac{(2n+2)!}{n!} \frac{\beta^{2n+2}}{C^n} \mathrm{e}^{(2n+2)Ct} \int_0^t \mathrm{e}^{-(2n+2)Cs} \left(\mathrm{e}^{2Cs} - 1\right)^n ds$$

$$\leq 2^{-(2n+2)} \frac{(2n+2)!}{(n+1)!} \frac{\beta^{2n+2}}{C^{n+1}} \left(\mathrm{e}^{2Ct} - 1\right)^{n+1}.$$

This concludes the proof of the lemma.   □

Proposition 4.6. *Assume that the conditions stated in* (4.2) *hold. Then Condition B holds for the model* (1.1)-(4.1).

Proof. To prove Condition B, first observe that from the assumptions on $\mu(y)$, $\sigma(y)$, $\delta(y)$ in (4.2) and the linear growth on v_y (see Theorem 4.3) we have

$$\frac{\mu^2(Y_t)}{\sigma^2(Y_t)} + \delta^2(Y_t)v_y^2(t, Y_t) \leq C\left(1 + Y_t^2\right).$$

Before proceeding further, let us show that

$$|Y_t(y)| \leq |Y_t(0)| + |y|\,\mathrm{e}^{Ct},$$

where $Y_t(y)$ is the process Y_t starting in y at time zero, and C is the growth rate of α. From [21], we find that (since α is assumed to be differentiable)

$$\frac{\partial}{\partial y}Y_t(y) = 1 + \int_0^t \alpha'(Y_s(y))\frac{\partial}{\partial y}Y_s(y)\,ds,$$

and thus

$$\frac{\partial}{\partial y}Y_t(y) = \exp\left(\int_0^t \alpha'(Y_s(y))\,ds\right).$$

Since $|\alpha'(y)| \leq C$, the desired bound follows by using the fundamental theorem of calculus. It is therefore sufficient to prove that $\exp\left(\varepsilon \int_0^T Y_s^2\,ds\right) \in L^1(P)$ for an $\varepsilon > 0$ when Y is starting at zero. Using Hölder's inequality (with $p = n/(n-1)$ and $q = 1/n$) and Lemma 4.5 we find

$$\mathbb{E}\left[\exp\left(\varepsilon \int_0^T Y_s^2(0)\,ds\right)\right] = 1 + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!}\mathbb{E}\left[\left(\int_0^T Y_s^2(0)\,ds\right)^n\right]$$

$$\leq 1 + T^{-1}\sum_{n=1}^{\infty} \frac{(\varepsilon T)^n}{n!}\mathbb{E}\left[\int_0^T Y_s^{2n}\,ds\right]$$

$$\leq 1 + \sum_{n=1}^{\infty}(\varepsilon k)^n \frac{(2n)!}{(n!)^2},$$

for some constant k depending on β, C, and T. By choosing ε sufficiently small (that is, $\varepsilon < 1/4k$), this series becomes convergent. Hence, Condition B holds.   □

4.3. **The Stein-Stein volatility model.** We apply our general results to a version of the Stein-Stein stochastic volatility model [35]. Assume $\dfrac{\mu(y)}{\sigma(y)} = \xi y$ for some constant ξ different than zero, and let the volatility Y_t follow an Ornstein-Uhlenbeck process

$$(4.20) \qquad\qquad dY_t = \left(m - \alpha Y_t\right)dt + \beta\rho\,dB_t + \beta\sqrt{1-\rho^2}\,dW_t,$$

where m, α, β are positive constants and $-1 < \rho < 1$.

The parameters of this model satisfy the conditions in (4.2), and thus we have that the density of the MEMM Q_{ME} for the Stein-Stein model is given as (see (3.3))

$$Z_T = \exp\left(\int_0^T \left\{\frac{\beta\rho}{\sigma(Y_t)}v_y(t, Y_t) - \frac{\xi Y_t}{\sigma(Y_t)}\right\} S_t^{-1}\, dS_t + v(0, y)\right).$$

where v is the solution to the semilinear PDE (1.3), which in the present context reads

$$(4.21) \qquad -v_t - \frac{1}{2}\beta^2 v_{yy} + \frac{1}{2}\beta^2(1 - \rho^2)(v_y)^2 - (m - (\alpha + \xi\beta\rho)y)\,v_y - \frac{1}{2}\xi^2 y^2 = 0.$$

A solution to (4.21) is derived in the next lemma.

Lemma 4.7. *The solution v to (4.21), satisfying the terminal condition $v(T, y) = 0$ for all y, is given by*

$$(4.22) \qquad v(t, y) = a(t)y^2 + b(t)y + c(t),$$

where

$$a(t) = a_1 \tanh\big(a_2(T - t) + a_3\big) + a_4,$$

$$b(t) = -2m \int_t^T a(s) e^{-(\alpha+\xi\beta\rho)(s-t)+2\beta^2(1-\rho^2)\int_t^s a(u)\, du}\, ds,$$

$$c(t) = \beta^2 \int_t^T a(s)\, ds + m \int_t^T b(s)\, ds - \frac{1}{2}\beta^2(1 - \rho^2) \int_t^T b^2(s)\, ds,$$

and

$$a_1 = \frac{\sqrt{\xi^2\beta^2 + 2\xi\alpha\beta\rho + \alpha^2}}{2\beta^2(1 - \rho^2)}, \qquad\qquad a_2 = \sqrt{\xi^2\beta^2 + 2\xi\alpha\beta\rho + \alpha^2},$$

$$a_3 = \frac{1}{2}\ln\left(\frac{a_2 + \alpha + \xi\beta\rho}{a_2 - (\alpha + \xi\beta\rho)}\right), \qquad\qquad a_4 = -\frac{\alpha + \xi\beta\rho}{2\beta^2(1 - \rho^2)}.$$

Proof. Inserting the expression (4.22) into the PDE (4.21), we derive the following differential equations for the coefficients $a(t), b(t), c(t)$:

$$a'(t) = -\frac{1}{2}\xi^2 + 2(\alpha + \xi\beta\rho)a(t) + 2\beta^2(1 - \rho^2)a^2(t),$$

$$b'(t) = \big((\alpha + \xi\beta\rho) - 2\beta^2(1 - \rho^2)a(t)\big)\,b(t) - 2ma(t),$$

$$c'(t) = -\beta^2 a(t) - mb(t) + \frac{1}{2}\beta^2(1 - \rho^2)b^2(t).$$

The terminal conditions are $a(T) = b(T) = c(T) = 0$ since $v(T, y) = 0$ for all $y \in \mathbb{R}$. One easily verifies that $a(t), b(t), c(t)$ given in the lemma are solutions, and hence the proof is complete. $\square$

Observe that $a_2 + \alpha + \xi\beta\rho/a_2 - (\alpha + \xi\beta\rho)$ is positive since $|\rho| < 1$ and therefore

$$\sqrt{\xi^2\beta^2 + 2\xi\beta\alpha\rho + \alpha^2} > |\alpha + \xi\beta\rho|.$$

Thus, a_3 is well-defined for all possible choices of the parameters. Note furthermore that $a'(t) < 0$, and hence $a(t)$ is non-negative and monotonically non-decreasing for $t < T$ since $a(T) = 0$.

Since $v_y(t, y) = 2a(t)y + b(t)$, we find that the density of the MEMM Q_{ME} takes the form

$$(4.23) \qquad Z_T = \exp\left(v(0, y) + \int_0^T \left\{\frac{(2\beta\rho a(t) - \xi)Y_t}{\sigma(Y_t)} + \frac{\beta\rho b(t)}{\sigma(Y_t)}\right\} S_t^{-1}\, dS_t\right).$$

An explicit density for the Stein-Stein model was derived and analyzed in Rheinländer [30] with $\mu(y) = \mu y^2$ and $\sigma(y) = \sigma y$, for μ, σ positive constants. Letting $\xi = \mu/\sigma$ we see that Rheinländer's result is covered by our more general volatility model.

5. Application to the Heston volatility model

In the previous section we utilized our PDE framework to derive the density of the MEMM Q_{ME} for a class of stochastic volatility models that included the Stein-Stein model. Another popular volatility model is provided by Heston [19], which assumes that the volatility Y_t follows the process

$$(5.1) \qquad dY_t = \left(\lambda Y_t^{-1} - \alpha Y_t\right) dt + \beta\rho\, dB_t + \beta\sqrt{1-\rho^2}\, dW_t,$$

where λ, α, and β are positive constants and $-1 < \rho < 1$. We generalize slightly the Heston model and suppose that the relation between the expected return and variance is $\mu(y)/\sigma(y) = \xi y$ for a constant $\xi \neq 0$. Unfortunately, the volatility dynamics Y_t does not satisfy the assumptions in (4.2). However, as we will demonstrate, one can still derive a solution to the associated semilinear PDE for v and identify the density of the MEMM.

The associated semilinear PDE (1.3) is

$$(5.2) \qquad -v_t - \frac{1}{2}\beta^2 v_{yy} + \frac{1}{2}\beta^2(1-\rho^2) - \left(\lambda y^{-1} - (\alpha + \xi\beta\rho)y\right)v_y - \frac{1}{2}\xi^2 y^2 = 0.$$

The following lemma provides us with the solution to the PDE (5.2).

Lemma 5.1. *The solution v to (5.2), satisfying the terminal condition $v(T,y) = 0$ for all y, is given by*

$$(5.3) \qquad v(t,y) = a(t)y^2 + b(t),$$

where $a(t)$ is given in Lemma 4.7 and $b(t) = (2\lambda + \beta^2)\displaystyle\int_t^T a(s)\, ds.$

Proof. Inserting the expression (5.3) into (5.2) leads to

$$a'(t) = -\frac{1}{2}\xi^2 + 2(\alpha + \xi\beta\rho)a(t) + 2\beta^2(1-\rho^2)a^2(t),$$
$$b'(t) = -(2\lambda + \beta^2)a(t),$$

with terminal conditions $a(T) = 0$ and $b(T) = 0$. Note that $a(t)$ solves the same differential equation as for the Stein-Stein model, see the proof of Lemma 4.7. The lemma now follows. $\square$

The candidate density for the MEMM Q_{ME} becomes

$$(5.4) \qquad Z_T = \exp\left(v(0,y) + \int_0^T \left\{\frac{(2\beta\rho a(t) - \xi)Y_t}{\sigma(Y_t)}\right\} S_t^{-1}\, dS_t\right).$$

Note the similarity with the MEMM for the Stein-Stein model when $m = 0$. The arguments needed to verify that (5.4) is the density for the minimal entropy martingale measure in the Heston model are, however, slightly different. We verify Conditions A and B under an extra assumption on the parameters in the Heston model.

Proposition 5.2. *Assume $\alpha > \beta\xi$. Then Conditions A and B hold for the Heston model (1.1)-(5.1). Consequently, (5.4) is the density of the MEMM Q_{ME} for the Heston model (5.1).*

Proof. Since $K_t = \xi^2 \displaystyle\int_0^t Y_s^2\, ds$ and

$$[L]_t = 4\beta^2(1-\rho^2)\int_0^t a^2(s)Y_s^2\, ds \leq 4a^2(0)\beta^2(1-\rho^2)\int_0^t Y_s^2\, ds,$$

we need to prove that $\displaystyle\sup_n \mathbb{E}_{Q_n}\left[\int_0^{T_n} Y_s^2\, ds\right] < \infty$ to verify Condition A. Recall that the process $\widetilde{U}_t^n = \rho\widetilde{B}_{t\wedge T_n} + \sqrt{1-\rho^2}\widetilde{W}_{t\wedge T_n}$ is a Brownian motion under Q_n, see (3.4). Moreover, we find for $t \leq T_n$ that

$$dX_t = \left(2\lambda + \beta^2 - \left(2\alpha + 2\beta\xi\rho + 4\beta^2(1-\rho^2)a(t)\right)X_t\right)dt + 2\beta\sqrt{X_t}\, d\widetilde{U}_t^n.$$

Integrating from zero to T_n yields,

$$\int_0^{T_n} \left(2\alpha + 2\beta\xi\rho + 4\beta^2(1-\rho^2)a(t)\right)Y_t^2 \, dt \leq x + (2\lambda + \beta^2)T + 2\beta \int_0^{T_n} Y_t \, d\widetilde{U}_t^n,$$

since $X_{T_n} \geq 0$. Using that $\widetilde{U}_t^n$ is a Q_n-Brownian motion, we find $\mathbb{E}_{Q_n}\left[\int_0^{T_n} Y_t \, d\widetilde{U}_t^n\right] = 0$ since

$$\mathbb{E}_{Q_n}\left[\left(\int_0^{T_n} \sqrt{X_t} \, d\widetilde{U}_t^n\right)^2\right] = \mathbb{E}_{Q_n}\left[\int_0^{T_n} Y_t^2 \, dt\right] = \mathbb{E}_{Q_n}\left[K_{T_n}\right] \leq n.$$

Hence we get

$$\mathbb{E}_{Q_n}\left[\int_0^{T_n} \left\{2\alpha + 2\beta\xi\rho + 4\beta^2(1-\rho^2)a(t)\right\}Y_t^2 \, dt\right] \leq x + (2\lambda + \beta^2)T.$$

From $a(t) \geq 0$, it holds that $2\alpha + 2\beta\xi\rho \leq 2\alpha + 2\beta\xi\rho + 4\beta^2(1-\rho^2)a(t)$, and by the assumption $\alpha > \beta\xi$ we find,

$$\mathbb{E}_{Q_n}\left[\int_0^{T_n} Y_t^2 \, dt\right] \leq \frac{x + (2\lambda + \beta^2)T}{\alpha + \beta\xi\rho}.$$

This proves that Condition A holds.

To verify Condition B, we proceed as follows:

$$\frac{\mu^2(Y_t)}{\sigma^2(Y_t)} + \delta^2(Y_t)v_y^2(t,Y_t) = \xi^2 Y_t^2 + \beta^2(1-\rho^2)4a^2(t)Y_t^2 \leq kY_t^2,$$

with $k = \xi^2 + 4\beta^2(1-\rho^2)a^2(0)$. Apply Hölder's inequality to reach

$$(5.5) \qquad \mathbb{E}\left[\exp\left(\varepsilon k \int_0^T Y_t^2 \, dt\right)\right] \leq T^{-1} \sum_{n=0}^{\infty} \frac{(\varepsilon kT)^n}{n!} \int_0^T \mathbb{E}\left[Y_t^{2n}\right] \, dt.$$

Note that $X_t := Y_t^2$ satisfies (via Itô's formula) the Cox-Ingersoll-Ross (CIR) process [9]

$$(5.6) \qquad dX_t = 2\alpha\left(\frac{2\lambda + \beta^2}{2\alpha} - X_t\right) dt + 2\beta\sqrt{X_t} \, dU_t,$$

where $dU_t := \rho dB_t + \sqrt{1-\rho^2}dW_t$ is a Brownian motion. From (5.6) we know that Y_t^2 is a CIR process which is non-central χ^2-distributed. Following (partly) the notation in [9], the density of the probability distribution of Y_t^2 starting in y^2 at time zero is

$$f(z;t) = c_t \mathrm{e}^{-u-v}\left(\frac{v}{u}\right)^{q/2} I_q\left(2\sqrt{uv}\right),$$

where I_q is the modified Bessel function of the first kind of order q and

$$c_t = \frac{4\alpha}{4\beta^2\left(1 - \mathrm{e}^{-2\alpha t}\right)}, \qquad\qquad u = c_t y^2 \mathrm{e}^{-2\alpha t},$$

$$v = c_t z, \qquad\qquad q = \frac{2\lambda + \beta^2}{2\beta^2} - 1.$$

After rewriting slightly, we find

$$f(z;t) = k_t c_t \mathrm{e}^{-c_t z} z^{q/2} I_q\left(2c_t y \mathrm{e}^{-\alpha t}\sqrt{z}\right).$$

Here, $k_t = \mathrm{e}^{-c_t y^2 \mathrm{e}^{-2\alpha t}}(y^2 \mathrm{e}^{-2\alpha t})^{-q/2}$, which can bounded by a constant C on $[0,T]$. Hence, we find (using formula 9.6.18 on page 376 in [1] and the Fubini-Tonelli theorem)

$$\mathbb{E}\left[X_t^n\right] \leq C c_t \int_0^{\infty} z^{n+\frac{q}{2}} \mathrm{e}^{-c_t z} I_q\left(2c_t y \mathrm{e}^{-\alpha t}\sqrt{z}\right) \, dz$$

$$= C c_t \int_0^{\infty} z^{n+q} \mathrm{e}^{-c_t z} \int_0^{\pi} \sin^{2q}(\theta)\mathrm{e}^{2c_t y \mathrm{e}^{-\alpha t}\cos(\theta)\sqrt{z}} \, d\theta \, dz$$

$$= Cc_t \int_0^\pi \int_0^\infty z^{n+q} \mathrm{e}^{-c_t z + 2c_t y \mathrm{e}^{-\alpha t}\sqrt{z}\cos(\theta)} \, dz \sin^{2q}(\theta) \, d\theta$$

$$\leq Cc_t \int_0^\infty z^{n+q} \mathrm{e}^{-c_t z + 2c_t y \sqrt{z}} \, dz.$$

Note that the constant C has changed from line to line in the estimation process. It is straightforward to show that

$$\mathrm{e}^{-c_t z + 2c_t y \sqrt{z}} = \mathrm{e}^{c_t y^2} \mathrm{e}^{-c_t(\sqrt{z}-y)^2},$$

where $\mathrm{e}^{-c_t(\sqrt{z}-y)^2}$ has a maximum point for $z = y^2$ and behaves like $\mathrm{e}^{-c_t z}$ for z large. We have

$$\mathbb{E}\left[X_t^n\right] \leq Cc_t \int_0^\infty z^{n+q} \mathrm{e}^{-c_t z} \, dz$$

$$= Cc_t^{-(n+q)} \Gamma(n+q+1)$$

$$\leq C \left(\frac{\beta^2}{\alpha}\right)^{n+q} \Gamma(n+q+1).$$

In view of (5.5), choosing ε sufficiently small gives

$$\mathbb{E}\left[\exp\left(\varepsilon k \int_0^T Y_t^2 \, dt\right)\right] \leq T^{-1} \sum_{n=0}^\infty \frac{(\varepsilon k T)^n}{n!} \int_0^T \mathbb{E}\left[X_t^n\right] \, dt < \infty.$$

It follows that Condition B is fulfilled, and the proposition is proved. $\qquad\square$

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(Fred Espen Benth)
CENTRE OF MATHEMATICS FOR APPLICATIONS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OSLO
P.O. BOX 1053, BLINDERN
N–0316 OSLO, NORWAY
AND
AGDER UNIVERSITY COLLEGE
DEPARTMENT OF ECONOMICS AND BUSINESS ADMINISTRATION
SERVICEBOKS 422
N-4604 KRISTIANSAND, NORWAY
   E-mail address: `fredb@math.uio.no`
   URL: `http://www.math.uio.no/~fredb/`

(Kenneth Hvistendahl Karlsen)
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BERGEN
JOHS. BRUNSGT. 12
N–5008 BERGEN, NORWAY
AND
CENTRE OF MATHEMATICS FOR APPLICATIONS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OSLO
P.O. BOX 1053, BLINDERN
N–0316 OSLO, NORWAY
   E-mail address: `kennethk@mi.uib.no`
   URL: `http://www.mi.uib.no/~kennethk/`