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On risk minimizing portfolios and martingale measures in Lévy markets[1]

Sure Mataramvura

Department of Applied Mathematical Sciences, National University of Scince and Technology, Box AC 939 Ascot, Bulawayo, Zimbabwe
and
Department of Mathematic, University of Zimbabwe, Box MP 167, Mount Pleasant, Harare, Zimbabwe.
and
(**Secondary Address**)
Department of Mathematics, University of Oslo, P.O Box 1053 Blindern, N-0316, Norway.
e-mailsmataramvura@nust.ac.zw.

Abstract. Our aim in this paper is to find a market portfolio and equivalent martingale measure (EMM) that minimizes risk as defined in [1], but in the jump diffusion market. We use optimal control methods for the determination of explicit solutions for our controls.

Key words: coherent measure of risk, Lévy process, optimal control, HJB equation, admissible portfolio, martingale measure, maximum principle

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1 Introduction

The paper [1] defines market risk as an investor's "future net worth". In the sequel, the authors also define a measure of risk as a mapping from the set of all risks $\mathcal{G}$ into $\mathbb{R}$ and it is interpreted as the minimum extra cash an investor has

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to add to a given risky position which he will invest "prudently" to be allowed to proceed with his plans. Such strategies are common in markets which are not complete, where it is not enough to hold a self financing portfolio because one will not be able to hedge a given payoff. Our choice of the final net worth of a portfolio is motivated by the fact that a net worth which is always strictly negative, requires extra capital and thus the risk will not be zero.

For the choice of the final net worth considered in our paper, we aim primarily to find the explicit representations of the martingale measure that minimizes a given risk and the corresponding optimal portfolio. Similar considerations were done in [4] and also in [12]. In both cases, the authors consider the problem of risk minimization as a zero sum, two player stochastic differential game between the investor, who holds a portfolio of risky securities and a risk-free investment, and the "market". Such an idealized game was proved in both cases to be well posed. In the former, the authors consider the investor's efforts, as trying to hold a portfolio strategically with the aim of minimizing risk represented by the discounted net hedging loss. The measure of risk is the infimum, over all such *admissible* portfolios, of the discounted net loss or shortfall. On the other hand the market is choosing the volatility as its tool to counter the agent's objectives, that is, by trying to maximize over all volatility coefficients, the risk, for all admissible portfolios. The market chosen in both cases is a Gaussian market. It is well known that in both papers the market is complete. However in [4] the authors justify the inability to hedge a given payoff, as the agent's inability to pay the Black-Scholes price. In that case the portfolio will not perfectly hedge the given payoff. In the later, the considerations are the same but instead the market chooses the drift coefficient and the volatility, so that the market will have its control u as a pair $u = (\alpha, \sigma)$ where, α is the drift term and σ is the volatility term.

In our paper we consider the same predicament for the agent. Our market model consists of two assets, a stock and a bond. However the stock price dynamics is modelled by a stochastic process with jumps, which will make the market incomplete. As a result, a given portfolio, cannot hedge a given payoff perfectly. We also choose a risk which is coherent. We then aim to find the optimal portfolio and an optimal market price of risk which minimizes a chosen coherent risk. We use for the first part, the maximum principle to get a first result and then dynamic programming for the final result. We therefore consider our paper as an extension of the results in [4] and [12].

2 The market model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a given filtered probability space, satisfying,

1. the probability space $(\Omega, \mathcal{F}, P)$ is complete
2. the $\sigma-$algebra $\mathcal{F}_0$ contains all the P-null sets in $\mathcal{F}$
3. the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous in the sense that $\mathcal{F}_{t^+} = \cap_{s>t}\mathcal{F}_s$.

On the probability space, we define a pure jump process $\eta_t = \int_0^t \int_{\mathbb{R}} \gamma(s,z)\tilde{N}(st,dz)$, where, for later convenience, we shall assume that

$$\gamma(t,z) > -1; \qquad \int_{\mathbb{R}} \gamma^2(t,z)\nu(dz)dt < \infty \tag{1}$$

for almost all t, z, $dt \times d\nu(z)$, where $\nu(.) = E[N(1,.)]$ is a Lévy measure of η_t and $N(t,.)$ is a Poisson random measure of η_t. In this regard, $\tilde{N}(.,.)$ is the compensated Poisson random measure .

Let $\alpha = \alpha(t)$ be an adapted process, we then define the market model as follows:

$$\text{Asset1 (bond price)} \quad S_0(t) = 1 \quad \text{for a.a } (t,\omega) \in \ [0,T] \times \mathcal{F}_t \tag{2}$$

where T is some fixed time horizon.

$$\text{Asset2 (Stock Price)} \quad dS_1(t) = S_1(t^-)\left[\alpha(t)dt + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz)\right] \tag{3}$$

Let $\pi = \pi(t)$ be the proportion of wealth invested in the stock at time $t \in [0,T]$, so that $(1-\pi(t))$ of the wealth is invested in the bond.

If $X^{(\pi)}(t) = X(t)$ is the corresponding wealth process, then

$$dX(t) = X(t^-)\left[\pi(t)\alpha(t)dt + \int_{\mathbb{R}} \pi(t)\gamma(t,z)\tilde{N}(dt,dz)\right] \tag{4}$$
$$X(0) = x, \ t \in [0,T], (T > 0)$$

whose solution is

$$X(t) = X(0)\exp(\int_0^t \alpha(s)\pi(s)ds + \int_0^t \int_{\mathbb{R}} \ln(1+\pi(s)\gamma(s,z))\tilde{N}(ds,dz) + \int_0^t \int_{\mathbb{R}} [\ln(1+\pi(s)\gamma(s,z)) - \pi(s)\gamma(s,z)]\nu(dz)ds) \tag{5}$$

We assume that $\pi(.)\gamma(.,z) > -1$ and $X(0) = x > 0$

Fix $\beta \in (0,1)$, then $(X(t))^\beta = X^\beta(t)$ gives

$$X^\beta(t) = X^\beta(0)\exp(\int_0 \beta\alpha(s)\pi(s)ds + \beta\int_0^t \int_{\mathbb{R}} \ln(1+\pi(s)\gamma(s,z))\tilde{N}(ds,dz) + \beta\int_0^t \int_{\mathbb{R}} [\ln(1+\pi(s)\gamma(s,z)) - \pi(s)\gamma(s,z)]\nu(dz)ds) \tag{6}$$

By the Itô formula (see Definition 3.2), we get

$$\begin{aligned} d(X^\beta(t)) = X^\beta(t)[(\beta\alpha(t)\pi(t) &+ \int_{\mathbb{R}} \{\exp(\beta\ln(1+\pi(t)\gamma(t,z))) - 1 - \beta\pi(t)\gamma(t,z)\}\nu(dz))dt \\ &+ \int_{\mathbb{R}} \{\exp(\beta\ln(1+\pi(t)\gamma(t,z))) - 1\}\tilde{N}(dt,dz)] \end{aligned} \tag{7}$$

Note the $X^{\beta}(t)$ is a martingale if

$$\beta\alpha(t)\pi(t) + \int_{\mathbb{R}} (\exp(\beta \ln(1 + \pi(t)\gamma(t,z))) - 1 - \beta\pi(t)\gamma(t,z))\nu(dz) = 0$$

Let $\theta(t) = \theta(t,z)$ be another adapted process that satisfies

$$\int_{\mathbb{R}} \gamma(t,z)\theta(t)\nu(dz) = \alpha(t) \tag{8}$$

and

$$1 - \int_{\mathbb{R}} \theta(t)\nu(dz) < 0 \tag{9}$$

Define a process $Z_\theta(t)$ by

$$\begin{aligned} Z_\theta(t) = \exp\Bigg(& \int_0^t \int_{\mathbb{R}} \ln(1 - \theta(s,z))\tilde{N}(ds,dz) \\ & + \int_0^t \int_{\mathbb{R}} \{\ln(1 - \theta(s,z)) + \theta(s,z)\}\nu(dz)ds \Bigg) \end{aligned} \tag{10}$$

Then, since for, $T < \infty$, we have,
$E_P\left[\exp\left(\int_0^t \int_{\mathbb{R}} \ln(1 - \theta(s,z))\tilde{N}(ds,dz)\right)\right]$
$= \exp\left(\int_0^t \int_{\mathbb{R}} -\{\ln(1 - \theta(s,z)) + \theta(s,z)\}\nu(dz)ds\right)$, then $E[Z_\theta(T)] = \exp(0) = 1$.

Next, we define a measure $Q_\theta(T,\omega) = Q_\theta(\omega) = Q(\omega)$, by

$$dQ(\omega) = Z_\theta(T)dP(\omega) \tag{11}$$

then Q is equivalent to P and $S_1(t)$ is a (local) martingale with respect to Q . Let $\mathcal{M}$ be the set of all equivalent martingale measures.

3 The stochastic control problem and measure of risk

Let $U = [0,1]$ be a Borel set. We set $\mathcal{U}[0,T]$ as the set of all controls (portfolios) for the agent, that is $\mathcal{U}[0,T] = \{\pi : [0,T] \times \Omega \to U\}$.

We assume that the owner of the portfolio π, should not be able to exercise his decision $\pi(t)$ before the time t really comes. As a result, we demand that $\pi(t)$ should be $\{\mathcal{F}_t\}_{t\geq 0}$ adapted.

Definition 3.1 *A stochastic control for the agent $\pi(.) \in \mathcal{U}[0,T]$ is called feasible for (6) if*

1. *$\pi(.)$ is $\{\mathcal{F}_t\}_{t\geq 0}$ adapted and*
2. *$X(t)$ given by (6) is the unique solution of (7)*

The set of all admissible controls for the agent shall be denoted by $\mathcal{U}_{feas}[0,T]$. The pair $(X(.),\pi(.))$, is called a feasible pair.

The following definition can be found in [10]:

Definition 3.2 (Itô formula) *Let $X(t)$ be an Itô - Lévy diffusion given by*

$$dX(t) = \alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz) \tag{12}$$

where α, β and $\gamma(.,.)$ are adapted real valued functions and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function. Then the process $Y(t) = f(t,X(t))$ is again an Itô-Lévy process and the Itô formula of $X(t)$ is denoted $dX(t)$ and is given by

$dX(t) = \frac{\partial f}{\partial t}(t,X(t))dt + \frac{\partial f}{\partial x}(t,X(t))dX^{(c)}(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X(t)).\beta^2(t)dt + \int_{\mathbb{R}}\{f(t,X(t^-)+\gamma(t,z)) - f(t,X(t^-))\}N(dt,dz)$

where $X^{(c)}$ is the continuous part of $X(t)$, obtained by removing the jumps from $X(t)$

Let $Y_\theta(t) = X^\beta(t)Z_\theta(t)$, so that , by Itô's formula we have

$dY_\theta(t) = Z_\theta(t)dX^\beta(t) + X^\beta(t)dZ_\theta(t) + < X^\beta, Z >_t$ where the last part of this expression is the cross variation of X^β and Z, $dZ(t) = -Z(t)\int_{\mathbb{R}}\theta(t,z)\tilde{N}(dt,dz)$ which was given before in its Poisson integral form and $X^\beta(t)$ was given in (6). We then get

$$dY_\theta(t) = Y_\theta(t)[(\tilde{\alpha}(t) - \int_{\mathbb{R}}\tilde{\gamma}(t,z)\theta(t,z)\nu(dz))dt$$

$$+\int_{\mathbb{R}}(\tilde{\gamma}(t,x) - \theta(t,z) - \gamma(t,z)\theta(t,z))\tilde{N}(dt,dz)] \quad ,Y_\theta(0) > 0$$

where $\tilde{\alpha}(t) = \beta\alpha(t)\pi(t) + \int_{\mathbb{R}}\{\exp(\beta\ln(1+\pi(t)\gamma(t,z))) - 1 - \beta\pi(t)\gamma(t,z)\}\nu(dz)$ and

$\tilde{\gamma}(t,z) = \exp(\beta\ln(1+\pi(t)\gamma(t,z))) - 1$

Now we assume that the coefficients of $Y_\theta(t)$ satisfy the existence and uniqueness properties of a Lévy stochastic differential equation, that is, the property of Lipschitz continuity and the *at most linear growth* property. Our controlled stochastic process $Y_\theta(t)$ is not required to satisfy some terminal conditions, but, in [4], they consider a portfolio which is admissible in the sense that, the value process $X(t)$ is bounded below by an adapted process. In our case, our wealth process is naturally bounded by A(t) = 0 for almost all t.

Definition 3.3 *A stochastic control for the agent (market) is called admissible if it is feasible, the stochastic differential equation given by (13) admits a unique solution and the bequest function in the cost function of a generalized stochastic control problem (which we shall give later) is in $L^1_{\mathcal{F}_T}(\Omega,\mathbb{R})$.*

The set of all admissible controls for the agent (market) is denoted by $\mathcal{U}^a_{adm}$, $(\mathcal{U}^m_{adm})$.

With respect to the controls $\pi(.)$ and $\theta(.,.)$, a general cost function is of the form

$$J(u(.)) = E\left[\int_0^T f(t,Y_\theta(t),u(t))dt + h(Y_\theta(T))\right]$$

for some functions f and g and for $u(.) = (\pi(.), \theta(.))$. In our case we shall consider $f \equiv 0$ and $h(x) = -x^\delta$, where $\delta \in (0,1)$. It is therefore certain that we are considering a utility optimization problem. An optimal control problem with $h = 0$ is called a Lagrange problem, while if $f = 0$ it is called a Mayer's problem. In the case that $f \neq 0$ and $h \neq 0$, the problem is called a Bolza problem. Therefore our problem is a Mayer's problem.

Consider the cost function $J(u(.)) = E_Q[-X^\beta(T)]$.

Definition 3.4 *The measure of risk r is a mapping from the set of all random variables Z to $\mathbb{R}$ and is given by $r(Z) = \sup_{Q\in\mathcal{M}} E[-Z]$.*

The measure of risk is considered as the amount of money that the agent is prepared to pay in order to face the worst possible damage that arises from being unable to hedge a given payoff, which in our case is $F(\omega) = 0$ in a market with zero interest rates. Note that, a more general formulation is given in [4] with $F(\omega) = C$ and with a continuous compounding rate of interest ρ. It is with interest that the authors consider the case of a complete market where the agent is unable to pay the market price of a given liability (which is given by the Black-Scholes formula in the case of call and put options). We admit that it has already been proven that in the case of complete markets, any price paid other than the Black Scholes price will result in the creation of an arbitrage. In any case, if an agent has the money determined by the measure of risk, why not add to the amount he is prepared to pay so that the discounted shortfall becomes small. If this argument is continued, then the agent will manage to pay the Black-Scholes price. In our case we consider an incomplete market setup, and as such the terminal value of a portfolio can be less than the payoff.

Note that, the function h mentioned before is in this case
$h(X(T)) = -X^\beta(T)$.

We want to find θ and π such that

$$\inf_{\pi\in\mathcal{U}^a_{adm}} \left(\sup_{Q\in\mathcal{M}} E_Q[-X^\beta(T)] \right) = -\sup_{\pi\in\mathcal{U}^a_{adn}} \left(\inf_{\theta\in\mathcal{U}^m_{adm}} E[Z_\theta(T)X^\beta(T)] \right) \tag{13}$$

$$= -\sup_{\pi\in\mathcal{U}^a_{adn}} \left(\inf_{\theta\in\mathcal{U}^m_{adm}} E[Y_\theta(T)] \right) = \inf_{\pi\in\mathcal{U}^a_{adm}} \left(\sup_{\theta\in\mathcal{U}^m_{adm}} E[-Y_\theta(T)] \right) \tag{14}$$

Definition 3.5 *The min-max quantity*
$\overline{V}(x) = \inf_{\pi\in\mathcal{U}^a_{adm}} \left(\sup_{\theta\in\mathcal{U}^m_{adm}} E[-Y(T)] \right)$
is called the upper measure of risk, while the max-min quantity
$\underline{V}(x) = \sup_{\theta\in\mathcal{U}^m_{adm}} \left(\inf_{\pi\in\mathcal{U}^a_{adm}} E[-Y(T)] \right)$ *is called the lower measure of risk.*

The lower measure of risk represents (see [4]) the maximal risk, from the point of view of an agent faced with the worst possible scenario θ. In the same way, the upper-measure of risk, is viewed by a regulator, for example, insurer, as an attempt by the agent, of containing the worst that can happen. These two are thus lower (max-min) and upper (min-max) values of a fictious two player, zero sum, stochastic differential game between the agent and the market. We justify below that in the case of our model, the game has value.

Note that in our particular case, our measure of risk is coherent as defined in [1]. Moreover, we have the following:

Proposition 3.1 *Let the market be as defined before and let* $\overline{V}(x)$ *and* $\underline{V}(x)$ *be the upper measure of risk and the lower measure of risk as defined before. Then* $\overline{V}(x) = \underline{V}(x)$

Proof: We note that, for $\theta(t,z) = 0$ then P = Q. Then by the remarks in [4], the equality holds. □

As a result of the above proposition, we now look for a saddle point $(\hat{\pi}(.), \hat{\theta}(.))$ of the game which is a solution of

$\inf_{\pi \in \mathcal{U}^a_{adm}} \left(\sup_{\theta \in \mathcal{U}^m_{adm}} E[-Y_\theta(T)] \right)$

We solve this problem by first considering the problem of finding $\hat{\theta}$ that solves

$$\Lambda^{\hat{\theta}}(x) = \sup_{\theta \in \mathcal{U}^m_{adm}} E[-Y(T)] \tag{15}$$

We use the maximum principle for this part. Recall that the maximum principle is a set of necessary and sufficient conditions for the existence of an optimal control $\hat{\pi}$ which states that any optimal control along with the optimal state trajectory must solve the Hamiltonian systems which is a two-point boundary value problem plus a maximum condition of a function called a Hamiltonian. The case of control problems without jumps was discussed fully in [13]. We give here, as in [10] the modified versions of the conditions for the case of the general one dimensional Lévy market.

3.1 The maximum principle

Suppose that the controlled jump diffusion in $\mathbb{R}$ is given by

$$\begin{aligned} dX(t) &= b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB(t) \\ &\quad + \int_{\mathbb{R}} \gamma(t, X(t^-), u(t^-), z)\tilde{N}(dt, dz) \end{aligned} \tag{16}$$

Define , for some fixed investment time horizon T, the performance criterion by

$$J(u) = E\left[\int_0^T f(t, X(t), u(t))dt + g(X(T))\right]$$

where f and g are real valued continuous functions and g is C^1.

Suppose that $E\left[\int_0^T f^-(t, X(t), u(t))dt + g^-(X(T))\right] < \infty$ for all admissible controls u.

Consider the problem of finding $u^* \in \mathcal{A}$ such that

$$J(u^*) = \sup_{u \in \mathcal{A}} J(u)$$

where $\mathcal{A}$ is the set of all admissible controls.

In order to solve the stochastic control above, we may use the maximum principle which consists, first of defining the Hamiltonian H by

$$\begin{aligned} H(t,x,u,p,q,r) &= f(t,x,u) + b(t,x,u)p + \sigma(t,x,u)q \\ &+ \int_{\mathbb{R}} \gamma(t,x,u,z) r(t,z)\nu(dz) \end{aligned} \tag{17}$$

where p, q and r are some unknown processes to be determined.

Next we setup the adjoint equation in the unknown processes $p(t), q(t)$ and $r(t,z)$ as the backward stochastic differential equation (BSDE)

$$dp(t) = -\frac{\partial}{\partial x} H(t, X(t), u(t), q(t), r(t,.))dt + q(t)dB(t) + \int_{\mathbb{R}} r(t^-, z)\tilde{N}(dt, dz) \tag{18}$$

$$p(T) = \frac{\partial}{\partial x} g(X(T)), \qquad t < T \tag{19}$$

where we assume that

$E\left[\int_0^T \{\sigma^2(t, X(t), u(t)) + \int_{\mathbb{R}} |\gamma(t, X(t), u(t), z)|^2 \nu(dz)\} dt\right] < \infty$ for all $u \in \mathcal{A}$.

We then have the following result from [10].

Theorem 3.1 (A sufficient maximum principle) *Let $\hat{u} \in \mathcal{A}$ and let $\hat{X} = X^{(\hat{u})}$ be the corresponding solution of the controlled equation (16). Suppose that there exists a solution $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z))$ of the corresponding adjoint equation (18) and (19) satisfying*

$$E\left[\int_0^T \{\hat{q}^2(t) + \int_{\mathbb{R}} |r(t,z)|^2 \nu(dz)\} dt\right] < \infty.$$

Moreover, suppose that

$$H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) = \sup_{u \in \mathcal{A}} H(t, \hat{X}(t), u, \hat{p}(t), \hat{r}(t,.))$$

for all t; that g is a concave function of x and that

$\hat{H}(x) = \max_{u \in \mathcal{A}} H(t, x, u, \hat{p}(t), \hat{r}(t,.))$ *exists and is concave in x for all $t \in [0,T]$. Then $\hat{u}$ is an optimal control.*

This theorem was proved in [10] for the multidimensional case. What is important to our case is that we can establish the first order adjoint equations for the Itô - Lévy case as an extension of the Itô diffusion case which was well treated in [13].

The only disadvantage of the maximum principle is that one has to check on the smoothness properties of $h(.)$ and concave or convex properties of the Hamiltonian. In our case one can easily check that both conditions are satisfied. However even in the case that the Hamiltonian is not concave, one can establish the second order adjoint equation which makes the *new* Hamiltonian concave.

3.1.1 Adjoint equation and first result

The Hamiltonian for our optimal stochastic control problem is

$$\begin{aligned} H(t,y,\theta,p,r) &= (\tilde{\alpha}(t) - \int_{\mathbb{R}} \tilde{\gamma}(t,z)\theta\nu(dz))yp \\ &+ y\int_{\mathbb{R}} (\tilde{\gamma}(t,z) - \theta - \theta\tilde{\gamma}(t,z))r(t,z)\nu(dz) \end{aligned} \tag{20}$$

and this can be written

$$\begin{aligned} H(t,y,\theta,p,r) = \theta[-yp\int_{\mathbb{R}} \tilde{\gamma}(t,z)\nu(dz) - y\int_{\mathbb{R}} (\tilde{\gamma}(t,z)+1)r(t,z)\nu(dz)] \\ +\tilde{\alpha}(t)yp + y\int_{\mathbb{R}} \tilde{\gamma}(t,z)r(t,z)\nu(dz) \end{aligned} \tag{21}$$

The Hamiltonian given above is linear in θ so that H can only be optimized with respect to θ if

$$p(t)\int_{\mathbb{R}} \tilde{\gamma}(t,z)\nu(dz) + \int_{\mathbb{R}} (\tilde{\gamma}(t,z)+1)r(t,z)\nu(dz) = 0 \tag{22}$$

Therefore the first order adjoint equation, after considering (22) is

$$\begin{aligned} & dp(t) = (\tilde{\alpha}(t)p(t) + \int_{\mathbb{R}} \tilde{\gamma}(t,z)r(t,z)\nu(dz))p(t)dt + \int_{\mathbb{R}} r(t,z)\tilde{N}(dt,dz) \\ & p(T) = 1 \end{aligned} \tag{23}$$

We try $p(t) = m(t)Y_\theta(t)$ where m is a C^1 function. We get

$$\begin{aligned} dp(t) &= [m'(t)Y_\theta(t) + m(t)Y_\theta(t)(\tilde{\alpha}(t) - \int_{\mathbb{R}} \tilde{\gamma}(t,z)\theta(t,z)\nu(dz)]dt \\ &+ m(t)Y_\theta(t)\int_{\mathbb{R}} (\tilde{\gamma}(t,z) - \theta(t,z) - \tilde{\gamma}(t,z)\theta(t,z))\tilde{N}(dt,dz) \end{aligned} \tag{24}$$

Comparing (24) and (23), we get the following system of equations:

$$\begin{aligned} m'(t)Y_\theta(t) + m(t)Y_\theta(t)(\tilde{\alpha}(t) - \int_{\mathbb{R}} \tilde{\gamma}(t,z)\theta(t,z)\nu(dz) \\ = (\tilde{\alpha}(t)p(t) + \int_{\mathbb{R}} \tilde{\gamma}(t,z)r(t,z)\nu(dz))p(t) \end{aligned} \tag{25}$$

$$m(t)Y_\theta(t)(\tilde{\gamma}(t,z) - \theta(t,z) - \tilde{\gamma}(t,z)\theta(t,z)) = \hat{r}(t,z) \tag{26}$$

Substituting (26) into (22), we get

$$\int_{\mathbb{R}} \tilde{\gamma}(t,z)\nu(dz) - \int_{\mathbb{R}} (\tilde{\gamma}(t,z)+1)(\tilde{\gamma}(t,z) - \theta - \theta\tilde{\gamma}(t,z)\nu(dz)) = 0 \tag{27}$$

Solving we get

$$\hat{\theta}(t,z) = \frac{\int_{\mathbb{R}} \tilde{\gamma}(t,z)(\tilde{\gamma}(t,z)+2)\nu(dz)}{\int_{\mathbb{R}} (\tilde{\gamma}(t,z)+1)^2\nu(dz)} = 1 - \int_{\mathbb{R}} \frac{\nu(dz)}{(1+\tilde{\gamma}(t,z))^2\nu(dz)} \tag{28}$$

where $\tilde{\gamma}(t,z) = e^{\beta \ln(1+\pi(t)\gamma(t,z))} - 1 = (1+\pi(t)\gamma(t,z))^{\beta} - 1$. Therefore $\Lambda^{\hat{\theta}} = E[-Y_{\hat{\theta}}]$ where $Y_{\hat{\theta}}(t)$ is actually $Y_{\theta}(t)$ given by (13) with θ substituted with $\hat{\theta}$.

We now want to find $\hat{\pi} = \hat{\pi}(t)$ such that $\tilde{J}^{\hat{\pi}} = \inf_{\pi \in \mathcal{U}^a_{adm}} E[-Y_{\hat{\theta}}(T) = -\sup_{\pi \in \mathcal{U}^a_{adm}} E[Y_{\hat{\theta}}(T)]$.

4 Dynamic programming

To solve the second part of our problem, we use the method of dynamic programming by using the Hamilton Jacobi Bellman (HJB) equation for jump diffusions.

Just like the maximum principle, dynamic programming is another mathematical technique for making a sequence of interrelated decisions which can be applied to optimal control problems, which are special cases of the more general optimization problems. Even with its weaknesses, one can always resort to the viscosity solutions to the HJB. A detailed treatment for the non-jump case can be found in [13]. Here, we provide, thanks to [10], the dynamic programming framework for the general Itô-Lévy one dimensional case and then use this to solve the second part of our problem, that is, that of finding $\hat{\pi}$.

Definition 4.1 *Let*
$dX(t) = b(X(t))dt + \sigma(X(t))dB(t) + \int_{\mathbb{R}} \gamma(X(T^-),z)\tilde{N}(dt,dz) \quad ; X(0) = x \in \mathbb{R}$
be a Lévy-Itô diffusion. Then the generator of $X(t)$ on $C_0^2(\mathbb{R})$ is A , given by

$$A\phi(x) = b(x)\frac{\partial \phi}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 \phi}{\partial x^2} + \int_{\mathbb{R}} \{\phi(x+\gamma(x,z)) - \phi(x) - \frac{\partial \phi}{\partial x}(x).\gamma(x,z)\}\nu(dz) \tag{29}$$

for all $\phi : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$ such that the sums and integrals in (29) exist.

Now we consider the controlled diffusion $X(t) = X^{(u)}(t)$ given by
$dX(t) = b(X(t),u(t))dt + \sigma(Y(t),u(t))dB(t) + \int_{\mathbb{R}} \gamma(X(t^-),u(t^-),z)\tilde{N}(dt,dz)$; $X(0) = x \in \mathbb{R}$ for functions b, σ and γ satisfying the necessary conditions for the above equation to have a unique strong solution.

Let $S \in \mathbb{R}$ be a fixed domain and consider the stopping time
$\tau_S \triangleq \inf\{t > 0 : X^{(u)}(t) \notin S\}$. Let $f : S \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be given functions. Then we consider the performance criterion $J = J^{(u)}(x)$ of the form

$$J^{(u)}(x) = E^x \left[\int_0^{\tau_S} f(X(t),u(t))dt + g(X(\tau_S)) \right].$$

We assume that $E^x \left[\int_0^{\tau_S} f^{-1}(X(t),u(t))dt + g^{-1}(X(\tau_S)) \right] < \infty$.

The stochastic control problem consists of finding the optimal control $u^* \in \mathcal{A}$ and the value function $\Phi(x)$ such that

$$\Phi(x) = \sup_{u \in \mathcal{A}} J^{(u)}(x) = J^{(u^*)}(x).$$

The dynamic programming approach to solve the above problem is summarized by the HJB conditions as given in the theorem below, whose proof is omitted and can be found in [10].

Theorem 4.1 (HJB for optimal control of the jump diffusions) *(a) Suppose that $\phi \in C^2(S) \cap C(\overline{S})$ satisfies the following*

1. $A\phi(x) + f(x, v) \leq 0$ *for all* $y \in S, v \in \mathcal{A}$
2. $\lim_{t \to \tau_S^-} \phi(X(t)) = g(X(\tau_S))$ *a.s for all* $u \in \mathcal{A}$.
3. $E^x\big[\int_0^{\tau_S} \{\sigma(X(t))\frac{\partial}{\partial x}\phi(X(t))|^2 + \int_{\mathbb{R}} |\phi(X(t)+\gamma(X(t), u(t), z)) -\phi(X(t))|^2\nu(dz)\}dt\big] < \infty$
4. $\{\phi^{-1}(X(\tau))_{\tau \leq \tau_S}\}$ *is uniformly integrable for all* $u \in \mathcal{A}$ *and* $x \in S$. *Then* $\phi(x) \geq \Phi(x)$ *for all* $x \in S$. *(b) Moreover, suppose that for each* $x \in S$, *there exists* $v = \hat{u}(x) \in \mathcal{A}$ *such that*
5. $A^{\hat{u}(x)}\phi(x) + f(x, \hat{u}(x)) = 0$ *and*
6. $\{\phi(X^{(\hat{u})}(\tau))\}$ *is uniformly integrable*

Suppose that $u^*(t) := \hat{u}(X(t^-)) \in \mathcal{A}$. *Then* u^* *is an optimal control and* $\phi(x) = \Phi(x) = J^{(u^*)}(x)$ *for all* $x \in S$.

The previous theorem gives both the necessary and sufficient conditions for the existence of the optimal control u^* for the stochastic control problem described before.

We therefore apply the theorem for our particular case.

4.1 Optimal portfolio: the main result

Recall that

$$dY_\theta(t) = Y_\theta(t)[(\tilde{\alpha}(t) - \int_{\mathbb{R}} \tilde{\gamma}(t,z)\theta(t,z)\nu(dz))dt + \int_{\mathbb{R}} (\tilde{\gamma}(t,x) - \theta(t,z) - \tilde{\gamma}(t,z)\theta(t,z))\tilde{N}(dt,dz)] \tag{30}$$

where $\tilde{\alpha}(t) = \beta\alpha(t)\pi(t) + \int_{\mathbb{R}}\{\exp(\beta\ln(1+\pi(t)\gamma(t,z))) - 1 - \beta\pi(t)\gamma(t,z)\}\nu(dz)$ and

$\tilde{\gamma}(t,z) = \exp(\beta\ln(1+\pi(t)\gamma(t,z))) - 1$

Let $W(t) = \begin{pmatrix} s+t \\ Y_\theta(t) \end{pmatrix}$, $t \geq 0$; $W(0^-) = w = \begin{pmatrix} s \\ y \end{pmatrix}$, Then the generator $\mathcal{A}^{(\pi)}$ of the controlled process $W(t)$ is

$\mathcal{A}^{(\pi)}\phi(w) = \frac{\partial\phi}{\partial s} + (\tilde{\alpha} - \int_{\mathbb{R}} \tilde{\gamma}\theta\nu(dz))\frac{\partial\phi}{\partial y}$
$+ \int_{\mathbb{R}}\{\phi(s, y + \tilde{\gamma} - \hat{\theta} - \tilde{\gamma}\hat{\theta}) - \phi(s,y) - (\tilde{\gamma} - \hat{\theta} - \tilde{\gamma}\hat{\theta})\frac{\partial\phi}{\partial y}\}\nu(dz)$

We "try" $\phi(s,y) = k(s) + h(s)y$ where k and h are C^1.

Substituting, we get

$\mathcal{A}^{(\pi)}\phi(w) = k'(s) + h'(s)y + (\tilde{\alpha} - \int_{\mathbb{R}} \tilde{\gamma}\hat{\theta}\nu(dz))h(s)$
$= k'(s) + h'(s)y + (\beta\alpha\pi + \int_{\mathbb{R}}\{(1+\pi\gamma)^\beta - 1 - \beta\pi\gamma\}\nu(dz))h(s)$
$- h(s)\int_{\mathbb{R}} \hat{\theta}[(1+\pi\gamma)^\beta - 1]\nu(dz)$.

Let $p(\pi) = k'(s) + h'(s)y + (\beta\alpha\pi + \int_{\mathbb{R}}\{(1+\pi\gamma)^\beta - 1 - \beta\pi\gamma\}\nu(dz))h(s)$
$- h(s)\int_{\mathbb{R}} \hat{\theta}[(1+\pi\gamma)^\beta - 1]\nu(dz)$, then

$\frac{\partial p}{\partial \pi} = h(s)\left[\beta\gamma - \int_{\mathbb{R}} \beta\gamma\nu(dz) + \int_{\mathbb{R}} \beta\gamma(1+\pi\gamma)^{\beta-1}(1-\hat{\theta})\nu(dz)\right]$.

Then $\frac{\partial p}{\partial \pi} = 0 \Rightarrow \beta\gamma - \int_{\mathbb{R}} \beta\gamma\nu(dz) + \int_{\mathbb{R}} \beta\gamma(1+\pi\gamma)^{\beta-1}(1-\hat{\theta})\nu(dz) = 0$

$$\Rightarrow 1 - \int_{\mathbb{R}} \nu(dz) + (1-\hat{\theta}) \int_{\mathbb{R}} (1+\pi\gamma)^{\beta-1}\nu(dz) = 0 \qquad (31)$$

Let $m(\pi) = 1 - \int_{\mathbb{R}} \nu(dz) + (1-\hat{\theta}) \int_{\mathbb{R}} (1+\pi\gamma)^{\beta-1}\nu(dz)$, then there exists at least one value $\tilde{\pi}$ such that $m(\tilde{\pi}) > 0$. A good example could be $\tilde{\pi} = \frac{1}{\gamma}$ for $\hat{\theta} < \frac{1}{2}$

Moreover, $m \to 1 - \int_{\mathbb{R}} \nu(dz) < 0$ as $\pi \to \infty$. Furthermore, by (28), we have that $\hat{\theta}(t, z) < 1$, for all t, z, so that $1 - \hat{\theta}(t, z) \geq 0$.

Now, $m'(\pi) = (1-\hat{\theta})(\beta - 1) \int_{\mathbb{R}} (1+\pi\gamma)^{\beta-2}\nu(dz)$ which implies that $m'(\pi)$ is always negative, and thus $m(\pi)$ is monotonic decreasing. Therefore there exists a unique solution π of (31).

We call this solution $\hat{\pi}$ and have proved the following:

Theorem 4.2 *Let $\hat{\pi}$ be given by (31) and let $\hat{\theta}(t)$ be given by (28), then the risk is minimized by the portfolio $\hat{\pi}(t)$ and the equivalent martingale measure Q given by $\hat{\theta}(t)$.*

4.2 Conclusion

Our results show that the optimal portfolio $\hat{\pi}$ which represents the proportion of wealth invested in the stock, is unique. The mini-max problem was simplified thanks to the maximum principle and dynamic programming.

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