DEPT. OF MATH. UNIV. OF OSLO
PURE MATHEMATICS No. 24
ISSN 0806–2439 AUGUST 2004

Explicit Representation of Solutions of Forward Stochastic Differential Equations

Thilo Meyer-Brandis[1] and Frank Proske[1]

Abstract

In this paper we present a method to derive explicit representations of strong solutions of forward stochastic differential equations driven by a Brownian motion. These representations open new perspectives in the study of important topics like large time behaviour or the flow property of solutions of such equations.

Key words and phrases: anticipative stochastic differential equations, white noise analysis

AMS 2000 classification: 60H7, 60H10, 60H40

1. Framework

1.1. **Basic concepts of Gaussian white noise analysis.** In this section we briefly recollect some concepts of Gaussian white noise analysis. In Section 2 we will employ this theory to provide explicit solution formulas of forward stochastic differntial equations driven by a Brownian motion. For general backround information about white noise theory the reader is referred to the books of [HKPS], [Ku] and [O].

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space on $\mathbb{R}$ and denote by $\mathcal{S}'(\mathbb{R})$ its dual, i.e. the space of tempered distribution. Then the Bochner-Minlos theorem implies the existence of a unique probability measure μ on the Borel sets of $\mathcal{S}'(\mathbb{R})$, satisfying

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle\omega,\phi\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R})}} \tag{1.1.1}$$

for all $\phi \in \mathcal{S}(\mathbb{R})$, where $\langle\omega,\phi\rangle = \omega(\phi)$ is the action of $\omega \in \mathcal{S}'(\mathbb{R})$ on $\phi \in \mathcal{S}(\mathbb{R})$. The measure μ on $\Omega = \mathcal{S}(\mathbb{R})$ is called *(Gaussian)white noise probability measure.*

[1]Centre of Mathematics for Applications (CMA), Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, N-0316 Oslo, Norway.

E-mail address: meyerbr@math.uio.no, proske@math.uio.no

In the following we consider a Brownian motion B_t defined on the *white noise probability space*

$$(\Omega, \mathcal{F}, P) = \left(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu\right).$$

Further we denote by $\mathcal{J}$ the set of all multi-indices $\alpha = (\alpha_1, \alpha_2, ...)$ with finitely many non-zero entries $\alpha_i \in \mathbb{N}_0$. Let $Index(\alpha) = \max\{i : \alpha_i \neq 0\}$ and $|\alpha| = \sum_i \alpha_i$ for $\alpha \in \mathcal{J}$. One can construct orthogonal $L^2(\mu)$ basis $\{H_\alpha(\omega)\}_{\alpha\in\mathcal{J}}$, given by

$$H_\alpha(\omega) = \prod_{j\geq 1} h_{\alpha_j}(\langle \omega, \xi_j \rangle), \tag{1.1.2}$$

where $\langle \omega, \cdot \rangle = \omega(\cdot)$ and where ξ_j resp. $h_j, j = 1, 2, ...$ are the Hermite functions resp. Hermite polynomials. constitutes an orthogonal basis of $L^2(\mu)$. So every $F \in L^2(\mu)$ can be written as

$$F = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha(\omega) \tag{1.1.3}$$

for a unique sequence of real numbers $(c_\alpha)_{\alpha\in\mathcal{J}}$, where

$$\|F\|^2_{L^2(\mu)} = \sum_{\alpha\in\mathcal{J}} \alpha! c_\alpha^2, \tag{1.1.4}$$

with $\alpha! := \alpha_1!\alpha_2!...$, if $\alpha = (\alpha_1, \alpha_2, ...) \in \mathcal{J}$. The *Hida test function space* $(\mathcal{S})$ can be described as the space of all $f = \sum_{\alpha\in\mathcal{J}} c_\alpha K_\alpha \in L^2(\mu)$ such that the growth condition

$$\|f\|^2_{0,k} := \sum_{\gamma\in\mathcal{J}^m} \alpha! c_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty \tag{1.1.5}$$

holds for all $k \in \mathbb{N}_0$ with weight $(2\mathbb{N})^{k\alpha} = (2\cdot 1)^{k\alpha_1}(2\cdot 2)^{k\alpha_2}...(2\cdot l)^{k\alpha_l}$, if $Index(\alpha) = l$. The space $(\mathcal{S})$ is endowed with projective topology, based on the family of norms $(\|\cdot\|_{0,k})_{k\in\mathbb{N}_0}$ in (1.1.5). The *Hida distribution space*, denoted by $(\mathcal{S})^*$ is defined as the topological dual of $(\mathcal{S})$. Thus we obtain the following Gel'fand triple

$$(\mathcal{S}) \hookrightarrow L^2(\mu) \hookrightarrow (\mathcal{S})^*. \tag{1.1.6}$$

The Hida space $(\mathcal{S})^*$ enjoys the nice property to accommodate the (singular) white noise W_t of B_t, that is

$$W_t = \frac{d}{dt} B_t \in (\mathcal{S})^* \tag{1.1.7}$$

for all t. On $(\mathcal{S})^*$ a multiplication of distributions can be introduced by means of the *Wick product* $\diamond$, given by

$$(H_\alpha \diamond H_\beta)(\omega) = (H_{\alpha+\beta})(\omega), \ \alpha, \beta \in \mathcal{J} \tag{1.1.8}$$

The product is linearly extensible to the whole space. Since $(\mathcal{S})^*$ forms a topological algebra with respect to the Wick product, it is possible e.g. to define the Wick version of the exponential function exp by

$$\exp^{\diamond} X := \sum_{n\geq 0} \frac{1}{n!} X^{\diamond n} \tag{1.1.9}$$

for $X \in (\mathcal{S})^*$, where the Wick powers in (1.1.9) are defined as

$$X^{\diamond n} = X \diamond X \diamond ... \diamond X \qquad \text{(n times)}.$$

The *Hermite transform* $\mathcal{H}$ can be used to give a unique characterization of Hida distributions (see characterization Theorem 2.6.11 in [HØUZ]). The construction of $\mathcal{H}$ rests on the expansion along the basis $\{H_\alpha(\omega)\}_{\alpha\in\mathcal{J}}$ in (1.1.2). The *Hermite transform* of $X(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \in (\mathcal{S})^*$, indicated by $\mathcal{H}X$, is defined by

$$\mathcal{H}X(z) = \sum_\alpha c_\alpha z^\alpha \in \mathbb{C} \quad \text{(when convergent)}, \tag{1.1.10}$$

where $z = (z_1, z_2, ...) \in \mathbb{C}^{\mathbb{N}}$, i.e. in the space of $\mathbb{C}-$valued sequences, and where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}$... It can be shown that $\mathcal{H}X(z)$ in (1.1.10) converges on the infinite dimensional neighbourhood

$$\mathbb{K}_q(R) := \left\{ (\eta_1, \eta_2, ...) \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha\neq 0} |\eta^\alpha| \, (2\mathbb{N})^{q\alpha} < R^2 \right\} \tag{1.1.11}$$

for some $0 < q, R < \infty$. Since the Hermite transform maps the algebra $(\mathcal{S})^*$ into the algebra of power series in infinitely many complex variables, homomorphically, we find above all that

$$\mathcal{H}(X \diamond Y)(z) = \mathcal{H}(X)(z) \cdot \mathcal{H}(Y)(z) \tag{1.1.12}$$

holds on some $\mathbb{K}_q(R)$.

Finally we remark that obviously the above described white noise theory can be established on any time interval $[0, T]$ instead of the complete time line $\mathbb{R}$ (which is used in the next section).

1.2. **Forward integrals, anticipative Girsanov theorem.** We need some

concepts and results from Malliavin calculus and the theory of forward integrals to establish explicit representations of strong solutions of forward stochastic differential equations. First we recapitulate the definition of the forward integral for Brownian motion. Then we state an Itô-formula for forward processes. We conclude this section with a version of an anticipative Girsanov theorem.

Definition 1.1. *Let $\phi(t, \omega)$ be a measurable process (not necessarely adapted). Then the forward stochastic integral of ϕ is defined as*

$$\int_0^\infty \phi(t, \omega) d^- B(t) = \lim_{\varepsilon \longrightarrow 0} \int_0^\infty \phi(t, \omega) \frac{B(t+\varepsilon) - B(t)}{\varepsilon} dt$$

if the convergence is in probability.

Definition 1.2. *A (1-dimensional)* ***forward process*** *$X(t)$ is defined as a process of the form*

$$X(t) = x + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)d^- B(s); \ t > 0,$$

where $u(s,\omega)$ and $v(s,\omega)$ measurable processes (not necessarely adapted) such that

$$\int_0^t |u(s,\omega)|\, ds < \infty \ \text{a.s. for all } t > 0$$

and the Itô forward integral

$$\int_0^t v(s,\omega)d^- B(s)$$

exists for all $t > 0$.

For more information about Itô forward integrals consult e.g. [RV].

Theorem 1.3. *(Itô formula for forward integrals) Let*

$$X(t) = \int_0^t u(s)ds + \int_0^t v(s)d^- B(s)$$

be a forward process. Further let $f \in C^2(\mathbb{R})$ and define

$$Y(t) = f(X(t)).$$

Then $Y(t)$ is also a forward process and can be represented as

$$Y(t) = \int_0^t f'(X(s))d^- X(s) + \frac{1}{2}\int_0^t f''(X(s))v^2(s)d^- B(s).$$

Proof. The proof is based on the same arguments as in the non-anticipating case (see e.g. [HØ]) ∎

Next we restate the anticipative Girsanov theorem as presented in [N]. This theorem, which will be an essential in Section 2, takes place in the context of an abstract Wiener space, i.e. a quartuple $(\Omega, \mathcal{F}, \mu, H)$ such that Ω is a separable Banach space, μ is a Gaussian measure with full support, $\mathcal{F}$ is the completion of the Borel σ-field with respect to μ, and H is a separable Hilbert space that is continuously and densely ambedded in Ω through $i : H \hookrightarrow \Omega$. Notice that our white noise space (nummer) is an abstract Wiener space with $H = L^2(\mathbb{R}) \subset \Omega$.

Definition 1.4. *A random variable F is (a.s.)* H-continuously differentiable *if for (almost) all $\omega \in \Omega$ the mapping $h \to F(\omega + i(h))$ is continuously differentiable in H.*

It can be proven that H-continuously differentiability implies Malliavin differentiability. With this notion of differentiability given, we will now concentrate on a version of an anticipative Girsanov theorem which goes back to ([K], Theorem 6.4). Note however that there exist other versions of

Girsanov's theorems involving different conditions which could be employed instead if needed.

Theorem 1.5. *Let u be an H-valued random variable, i.e. a stochastic process, that is H-continuously differentiable, and denote by T the transformation $T : \Omega \to \Omega$ given through $T(\omega) = \omega + i(u(\omega))$. Suppose that T is bijective and* $\det_2(I + Du) \neq 0$ *a.s.. Then there exists a probability Q equivalent to μ such that $Q \circ T^{-1} = \mu$, given by*

$$\frac{dQ}{d\hat{\mu}} = \eta(u) := |det_2(I + D\,u)| \exp\left\{ -\delta(u) - \frac{1}{2}\|u\|_H^2 \right\}.$$

Remark 1.6. *Here the notation* $\det_2$ *is used for the Carleman-Fredholm determinant (see e.g. Appendix A.4 in* [N]*), Du denotes the Malliavin derivative of u, , and $\delta(u)$ the Skorohod integral of u.*

Remark 1.7. *Instead of assuming T bijective we could require $E[\eta(u)] = 1$. In case u is adapted $det_2(I + D\,u) = 1$ and $\eta(u)$ then reduces to the familiar Girsanov exponential martingale.*

2. Explicit Representation of a Forward Diffusion

In the sequel we fix a time interval $[0, T]$ and operate on the corresponding Gaussian white noise space $(\Omega, \mathcal{F}, \mu)$ defined in Section 1.1 with associated Brownian motion B_t.We denote by $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu})$ a copy of the initial white noise space with Brownian motion $\hat{B}_t$. The object of interest is the following forward stochastic differential equation (from here on denoted by FSDE)

$$Y_t = Y_0 + \int_0^t b(\omega, s, Y_s)ds + \int_0^t \sigma(Y_s)d^- B_s,\ 0 \le t \le T, \tag{2.1}$$

where Y_0 is a random variable, and $b(\omega, s, x) : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$, $\sigma(x) : \mathbb{R} \to \mathbb{R}$ are measurable (possibly anticipating) mappings. We remark that the choice of a time homogeneous difffusion coefficient $\sigma(x)$ instead of $\sigma(s, x)$ is only done due to notational convenience. Further, we impose $\sigma(x) > 0$ and $\sigma(x)$ continously differentiable.

In this section we suppose there exists a square integrable $\mathcal{F}$-measurable solution $Y_t \in L^2(\mu)$ for all $t \in [0, T]$ of equation (2.1). For sufficient conditions for the existence of a solution see for example [OP] (for Lévy process driven FSDE's see also [ØZ]). The objective of this section will then be to give an explicit expression for Y_t. To this end we adopt the methodology from [LP] where adapted SDE's are treated (see also [MP] for Lévy process driven SDE's) to the anticipative situation given in equation (2.1). First, we reduce equation (2.1) to an equation with difffusion coefficient $\sigma(x) = 1$. If we define the strictly increasing function $\Lambda : \mathbb{R} \to \mathbb{R}$ by

$$\Lambda(y) := \begin{cases} \int_x^y \frac{1}{\sigma(u)} du, & y > x \\ -\int_y^x \frac{1}{\sigma(u)} du, & y \le x \end{cases},$$

we get with the help of Itô's Lemma for forward processes (see Theorem 1.3) that $Z_t := \Lambda(Y_t)$ fulfills

$$Z_t = \Lambda(Y_0) + \int_0^t b^*(\omega, s, Z_s)dt + B_t, \tag{2.2}$$

where

$$b^*(\omega, s, y) = \left(\frac{b(\omega, s, \Lambda^{-1}(y))}{\sigma(\Lambda^{-1}(y))}\right) - \frac{1}{2}\sigma'(\Lambda^{-1}(y)).$$

Now, the essential tool to treat the anticipative situation is the use of the anticipative Girsanov theorem (see Theorem 1.5). This requires to make the following assumptions (for definitions see Section 1)

A1: The process u defined through $u(\omega, s) := b^*(\omega, s, Z_s(\omega))$ is H-continuously differentiable.
A2: $\det_2(I + D\,u) \neq 0$ for a.a. ω.
A3: The transformation $T : \Omega \to \Omega$ given through $T(\omega) = \omega + u(\omega, \cdot)$ is bijective, where $u(\omega, \cdot) \in L^2([0, T]) \subset \Omega$.

Notice (see Remark 1.7) that instead of assumption A3 we could have assumed that $E[\eta(u)] = 1$, where

$$\eta(u) = |\det_2(I + D\,u)| \exp\left\{-\int_0^T b^*(\omega, s, Z_s)\delta B_s - \frac{1}{2}\int_0^T b^*(\omega, s, Z_s)^2 ds\right\},$$

and where δB_s denotes the Skorohod integral. We then get the following representation of the solution Y_t.

Theorem 2.1. *Let ρ be a Borel measurable function from $\mathbb{R}$ to $\mathbb{R}$ such that $\rho(Y_t) \in L^2(\mu)$ for all $t \in [0, T]$. Given A1-A3, the solution Y_t of equation (2.1) takes the explicit form*

$$\rho(Y_t) = E_{\widehat{\mu}}\left[\rho\left(\Lambda^{-1}\left(\Lambda(\hat{Y}_0) + \widehat{B}_t\right)\right) M^\diamond\right] \tag{2.3}$$

where

$$M^\diamond = |det_2(I + D\,u)| \exp^\diamond\left\{\int_0^T \left(W_s(\omega) + b^*(\hat{\omega}, s, \Lambda(\hat{Y}_0) + \widehat{B}_s)\right)\delta\widehat{B}_t - \frac{1}{2}\int_0^T \left(W_s(\omega) + b^*(\hat{\omega}, s, \Lambda(\hat{Y}_0) + \widehat{B}_s)\right)^{\diamond 2} ds\right\}.$$

Here the Wick product $\diamond$ is with respect to ω and the integrals occuring in (2.3) are Bochner integrals on the Hida distribution space.

Proof. Assume first that $\sigma(x) = 1$ in (2.1). From Theorem 2.7.10 in [HØUZ] we get that for $X \in L^2(\mu)$ the Hermite transform can be expressed as

$$\mathcal{H}X(z) = E\left[X \cdot \exp\left\{\int_0^T \phi_z(t)dB_t - \frac{1}{2}\int_0^T \phi_z(t)^2 dt\right\}\right],$$

where $\phi_z(t) = \mathcal{H}(W_t)(z) = \sum_k z_k \xi_k(t),\ z \in \mathbb{C}_c^{\mathbb{N}}$. So taking the Hermite transform of $\rho(Y_t)$ gives

$$\begin{aligned} \mathcal{H}(\rho(Y_t))(z) &= E_\mu \left[\rho(Y_t) \cdot \exp\left\{ \int_0^T \phi_z(s) dB_s - \frac{1}{2} \int_0^T \phi_z(s)^2 ds \right\} \right] \\ &= E_{\widehat{\mu}} \left[\rho(\hat{Y}_t) \cdot \exp\left\{ \int_0^T \phi_z(s) d\widehat{B}_s - \frac{1}{2} \int_0^T \phi_z(s)^2 ds \right\} \right]. \end{aligned} \tag{2.4}$$

Notice that applying Hölders inequality and Lemma 3.1 in [LP] yields that

$$E_{\widehat{\mu}} \left[\rho(\hat{Y}_t) \cdot \exp^\diamond \left\{ \int_0^T W_s(\omega) d\widehat{B}_s - \frac{1}{2} \int_0^T W_s(\omega)^{\diamond 2} ds \right\} \right]$$

is a well defined element in S^*. So we can extract the Hermite transform in (2.4) and get by means of the characterization theorem (Theorem 2.6.11 in [HØUZ]) that

$$\rho(Y_t) = E_{\widehat{\mu}} \left[\rho(\hat{Y}_t) \cdot \exp^\diamond \left\{ \int_0^T W_s(\omega) d\widehat{B}_s - \frac{1}{2} \int_0^T W_s(\omega)^{\diamond 2} ds \right\} \right].$$

But now we get by A1-A3 and the anticipative Girsanov theorem (see framework) that the law of $Z_t = \Lambda(Y_t)$ under the probability Q defined by

$$\frac{dQ}{d\hat{\mu}} = \eta(u)$$

is equal to the law of $\Lambda(\hat{Y}_0) + \widehat{B}_t$ under the probability $\hat{\mu}$. So setting $Y_t = \Lambda^{-1}(Z_t)$ we get

$$\begin{aligned} \rho(Y_t) &= E_{\widehat{\mu}} \left[\rho(\hat{Y}_t) \cdot \exp^\diamond \left\{ \int_0^T W_s(\omega) d\widehat{B}_s - \frac{1}{2} \int_0^T W_s(\omega)^{\diamond 2} ds \right\} \cdot \eta(u) \cdot \eta^{-1}(u) \right] \\ &= E_{\widehat{\mu}} \left[\rho \left(\Lambda^{-1} \left(\Lambda(\hat{Y}_0) + \widehat{B}_t \right) \right) M^\diamond \right]. \end{aligned}$$

∎

Remark 2.2. *Let u in Theorem 2.1 be of the form $u = k_s(Z, \omega_T)$. Then under certain assumptions on k (see* [BF]*) the Carleman-Fredholm determinant can be evaluated explicitly and we get*

$$\eta(u) = \left| 1 + \int_0^T k_s^{!}(Z, \omega_T) ds \right| \exp\left\{ - \int_0^T k_s(Z, \omega_T) \delta B_s - \frac{1}{2} \int_0^T k_s^2(Z, \omega_T) ds \right\},$$

where $k_s^{!}(Z, y) = \frac{d}{dy} k_s(Z, y)$.

References

[BF] Buckdahn, R., Föllmer, H.: A conditional approach to the anticipating Girsanov transformation. Probab. Theory Rel. Fields 95, 311-330 (1993).

[HKPS] Hida, T., Kuo, H.-H., Potthoff, J., Streit, J.: White Noise. An Infinite Dimensional Approach. Kluwer (1993).

[HØUZ] Holden, H., Øksendal, B., Ubøe, J., Zhang T.-S.: Stochastic Partial Differential Equations- A Modeling, White Noise Functional Approach. Birkhäuser, Boston 1996.

[HØ] Hu, Y., Øksendal, B.: Optimal smooth portfolio selection for an insider. Preprint series, University of Oslo (2003).

[Ku] Kuo, H. H.: White Noise Distribution Theory. Prob. and Stoch., Series, Boca Raton, FL: CRC Press (1996).

[K] Kusuoka, S.: The non-linear transformation of Gaussian measure on Banach space and its absolute continuity, J. Fac. Sci. Univ. Tokyo IA 29 (1982), pp 567-597.

[LP] Lanconelli, A., Proske, F.: On explicit strong solutions of Itô-SDE's and the Donsker delta function of a diffusion. Preprint series, University of Oslo, No.27 (2003).

[MP] Meyer-Brandis, T., Proske, F.:On the Existence and Explicit Representability of Strong Solutions of Lévy Noise Driven SDE's with Irregular Coefficients, Preprint series, University of Oslo, No.7 (2004).

[N] Nualart, D.:The Malliavin Calculus and Related Topics, Springer, Berlin Heidelberg New York (1995).

[O] Obata, N., White Noise Calculus and Fock Space, LNM, 1577, Berlin: Springer-Verlag, (1994).

[OP] Ocone,D., Pardoux, E.: A generalized Itô-Ventzell formula. Application to a class of anticipating stochastic differential equations. Ann. Inst. Henri Poincaré, Probabilités et Statistiques 25:1, (1989), 39-71.

[ØZ] Øksendal, B., Zhang T.-S.: The Itô-Ventzell Formula and Forward Stochastic Differential Equations Driven by Poisson Random Measures, Preprint series, University of Oslo, No.21 (2004).

[RV] Russo, F., Vallois, P.: Forward, backward and symmetric stochastic integration. Probab. Theory Related Fields 97, No. 3, 403-421 (1993).