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The calculus of variations for processes with independent increments

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Abstract

The purpose of this paper is to construct the calculus of variations for general zero mean processes with independent increments and, in particular for Lévy processes. The calculus based on the operators D and δ, is such that for the Gaussian processes they coincide with the Malliavin derivative and Skorohod integral, respectively. We introduce the family of polynomials which contains the Sheffer set of polynomials. By using these polynomials it is proved that the operators D and δ are equal respectively to the annihilation and the creation operators on the Fock space representation of $L^2(\Omega)$.

Key words and phrases: Lévy processes, processes with independent increments, Malliavin calculus, Skorohod integral, multiple integral, orthogonal polynomials, chaos expansion.

1 Introduction

The stochastic calculus of variations developed by Malliavin [15] is a powerful tool in the studying the smoothness of the densities of the solutions of stochastic differential equations. Some years ago it was shown how this calculus could be used in finance. This discovery led to an increase in the interest in the Malliavin calculus.

In the Brownian setup the calculus of variations has a complete form and it is based on the operators D and δ which are called Malliavin derivative and Skorohod integral, respectively (see the elegant presentation in [16]). There are two different ways to define the operator D which turn out to be equivalent for the Gaussian case: one as a weak derivative in canonical space and the other one through the chaos decomposition of $L^2(\Omega)$.

In the Poisson case the definition of D is quite different. The small perturbations of the trajectories lead to a certain difference operator (see, e.g., [18]). For the extension of the definition of D for pure jump Lévy processes and for the combined Brownian motion and Poisson process case the reader referred to [1, 4, 21]; see also [22] for the one dimensional Lévy processes and [5] for Lévy stochastic measures.

Alternatively, the operator D can be defined by its action on the chaos representation of L^2-functionals. But, in general, a Lévy process has no chaotic representation property in the sense that Brownian motion, Poisson process or so-called normal martingales have (see [14]). There are two different chaotic expansions introduced in [9] and [17]. By using these expansions two types of Malliavin operators for some classes of Lévy processes have been studied in the papers [12, 13, 2, 7, 20, 6]. The relationship between them has been shown in [2]. It worth mentioning here that most of the papers cited above deal with pure jump Lévy process or combination of Brownian motion and Poisson process. The general Lévy processes, satisfying certain conditions, were considered in [2] and [22], see also [14] for the normal martingale case.

The purpose of the present paper is to construct the stochastic calculus of variations for zero mean processes with independent increments, in particular for general Lévy processes without drift. In the presentation of the stochastic calculus of variations we have chosen the framework of an arbitrary family of infinitely divisible random variables. The Gaussian part of this family can be described in the terms of the σ-finite measure μ defined on the measurable space $(T, \mathcal{A})$, while the non Gaussian part can be described the σ-finite measure ν on the other measurable space $(T \times X_0, \mathcal{B})$. In Section 2 we combine these measures into the measure π and obtain the analog of the Wiener space for the infinitely divisible distributions. We define a system of generalized orthogonal polynomials, which include, in particular, the Sheffer system of polynomials, and obtain a chaos decomposition in the term of these polynomials.

Section 3 deals with multiple integrals with respect to L^2-valued measure with independent values. In this section we establish the relationship between multiple integrals and generalized orthogonal polynomials.

In Section 4 we define the operator D and show that its action on the chaos representation of L^2-functionals coincides, in particular, with derivatives considered in the papers [2, 6, 7, 13, 14, 16, 20] for certain classes of processes.

In the last section we introduce the operator δ which is adjoint of the operator D. Then we show that this operator can be considered as Skorohod integral in the Gaussian case (see [28]) and the extended stochastic integral defined by Kabanov (see [11]) in the pure discontinuous case.

2 The chaos decomposition

This section describes the basic framework that will be used in the paper. The general context consists of a probability space $(\Omega, \mathcal{F}, P)$ and a closed subspace $\mathcal{P}_1$ of $L^2(\Omega, \mathcal{F}, P)$ whose elements are zero mean infinitely divisible random variables. We will assume, that $\mathcal{P}_1$ is isometric to the separable space $L^2(T \times X, \mathcal{G}, \pi)$, where π is a σ-finite measure without atoms. In this case the elements of $\mathcal{P}_1$ can be interpreted as stochastic integrals of functions in $L^2(T \times X, \mathcal{G}, \pi)$ with respect to a random measure with independent values on disjoint sets.

Suppose that μ and ν are σ-finite measures without atoms on the measurable spaces $(T, \mathcal{A})$ and $(T \times X_0, \mathcal{B})$ respectively. Define a new measure $\pi(dtdx) = \mu(dt)\delta_\Delta(dx) + \nu(dtdx \cap X_0)$ on a measurable space $(T \times X, \mathcal{G})$, where $X = X_0 \cup \{\Delta\}$, $\mathcal{G} = \sigma(\mathcal{A} \times \{\Delta\}, \mathcal{B})$ and $\delta_\Delta(dx)$ is the measure which gives mass one to the point Δ. We assume that the Hilbert space $H = L^2(T \times X, \mathcal{G}, \pi)$ is separable. The scalar product and the norm will be denoted by

$\langle\cdot;\cdot\rangle_H$ and $||\cdot||_H$ respectively.

Definition 2.1 *We say that a stochastic process $L = \{L(h), h \in H\}$ defined in a complete probability space $(\Omega, \mathcal{F}, P)$ is an isonormal Lévy processes (or a Lévy processes on H) if the following conditions are satisfied.*

1. *The mapping $h \to L(h)$ is linear.*
2. $\mathsf{E}e^{izL(h)} = \exp(\Psi(z,h))$, *where*

$$\Psi(z,h) = \int_{T\times X}\left(e^{izh(t,x)} - 1 - izh(t,x) - \frac{1}{2}z^2h^2(t,x)\mathbf{1}_\Delta(x)\right)\pi(dtdx).$$

In what follows we will always assume that $\mathcal{F}$ is generated by L, i.e., $\mathcal{F} = \sigma\{L(h), h \in H\}$.

Remark 2.2

1. *Using the definition of measure π one can obtain the following representation for $\Psi(z,h)$*

$$\Psi(z,h) = -\frac{1}{2}z^2\int_T h^2(t,\Delta)\mu(dt) + \int_{T\times X_0}\left(e^{izh(t,x)} - 1 - izh(t,x)\right)\nu(dtdx).$$

Therefore the random variable $L(h)$ has an infinitely divisible distribution with Lévy measure νh^{-1} (see e.g., [24, Def. 8.2, p. 38]).

2. *It is easy to show that $\mathsf{E}L(h) = 0$, $\mathsf{E}(L(h)L(g)) = \langle h;g\rangle_H$ for all $h,g \in H$ and the mapping $h \to L(h)$ is continuous. Moreover, if $h \in H \cap L^\infty(T\times X_0, \mathcal{B}, \nu)$, then $\mathsf{E}|L(h)|^k < \infty$ for all $k \geq 1$ (see, e.g., [24, Th. 25.3, p. 159]).*
3. *If measure ν is zero then L is an isonormal Gaussian process (see, e.g., [16, Def. 1.1.1, p. 4])*
4. *By Kolmogorov's theorem, on the Hilbert space H we can always construct a probability space and a stochastic process $\{L(h)\}$ verifying the above conditions.*

Example 2.3 *Suppose that $T = \mathbb{R}_+ \times \{1,\dots,d\}$ and the measure μ is the product of the Lebesgue measure times the uniform measure, which gives mass one to each point $1,\dots,d$. Let $X_0 = \mathbb{R}^d \setminus \{0\}$ and the measure β satisfying $\int_{X_0}(|x|^2 \wedge 1)\beta(dx) < \infty$ be defined on the Borel σ-algebra $\mathcal{B}(X_0)$. Denote by $\mathcal{T}$ the trivial σ-algebra of the set $\{1,\dots,d\}$, e.g., $\mathcal{T} = \{\emptyset, \{1,\dots,d\}\}$. Let α be a measure on $\mathcal{T}$ such that $\alpha(\{1,\dots,d\}) = 1$. Assume that the σ-algebra $\mathcal{B}$ is the product of the Lebesgue σ-algebra $\mathcal{L}$ times the trivial σ-algebra $\mathcal{T}$ times the Borel σ-algebra $\mathcal{B}(X_0)$, and the measure ν is the product of the Lebesgue measure times the measure α times the measure β. Set $\Delta = 0$. In this case we have that $B_t^i = L(\mathbf{1}_{[0;t]\times\{i\}\times\{0\}})$, $t \geq 0$, $i = 1,\dots,d$ is a standard d-dimensional Brownian motion. Furthermore, the random measure $\widetilde{N}(dtdx)$ on $\mathcal{L}\otimes\mathcal{B}(X_0)$, defined by $\widetilde{N}(dtdx) = L(\mathbf{1}_{dtdx}\mathbf{1}_{\mathbb{R}^d\setminus\{0\}})$, is a compensated Poisson measure with the characteristic measure $dt\beta(dx)$, and for any $h \in H$, the random variable $L(h)$ can be represented as the stochastic integral $L(h) = \sum_{i=1}^d\int_0^\infty h^i(t,0)dB_t^i + \int_0^\infty\int_{\mathbb{R}^d\setminus\{0\}} h(t,x)\widetilde{N}(dtdx)$.*

Denote by $\overline{x} = (x_1, x_2, \dots, x_n, \dots)$ a sequence of real numbers.
Define a function $F(z, \overline{x})$ by

$$F(z, \overline{x}) = \exp\left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{z^k}{k} x_k\right). \tag{2.1}$$

If $R(\overline{x}) = (\limsup |x_k|^{1/k})^{-1} > 0$ then the series in (2.1) converges for all $|z| < R(\overline{x})$. So the function $F(z, \overline{x})$ is analytic for $|z| < R(\overline{x})$.

Consider an expansion in powers of z of the function $F(z, \overline{x})$

$$F(z, \overline{x}) = \sum_{n=0}^{\infty} z^n P_n(\overline{x}).$$

Using this development, one can easily show the following equalities:

$$(n+1) P_{n+1}(\overline{x}) = \sum_{k=0}^{n} (-1)^k x_{k+1} P_{n-k}(\overline{x}), \quad n \geq 0, \tag{2.2}$$

$$\frac{\partial}{\partial x_l} P_n(\overline{x}) = \begin{cases} 0, & \text{if } \; l > n, \\ (-1)^{l+1} \frac{1}{l} P_{n-l}(\overline{x}), & \text{if } \; l \leq n. \end{cases} \tag{2.3}$$

Indeed, (2.2) and (2.3) follow from $\frac{\partial F}{\partial z} = \sum_{k=0}^{\infty} (-1)^k z^k x_{k+1} F$, respectively, and $\frac{\partial F}{\partial x_l} = (-1)^{l+1} \frac{F}{l} z^l$. From (2.3) it follows that P_n depends only on finite number of variables, namely $x_1, x_2, \dots, x_n$. Since $P_0 \equiv 1$, then (2.2) implies that $P_n(x_1, x_2, \dots, x_n)$ is a polynomial with the highest-order term $\frac{x_1^n}{n!}$. The first polynomials are $P_1(x_1) = x_1$ and $P_2(x_1, x_2) = \frac{1}{2}(x_1^2 - x_2)$.

Using the equality $F(z, \overline{x} + \overline{y}) = F(z, \overline{x}) F(z, \overline{y})$, where $\overline{y} = (y_1, y_2, \dots, y_n, \dots)$ and $\overline{x} + \overline{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n, \dots)$ it is easy to show that

$$P_n(\overline{x} + \overline{y}) = \sum_{k=0}^{n} P_k(\overline{x}) P_{n-k}(\overline{y}). \tag{2.4}$$

If $\overline{u}(y) = (y, y^2, y^3, \dots, y^n, \dots)$ then $F(z, \overline{u}(y)) = 1 + zy$ for $|zy| < 1$. Hence $P_1(\overline{u}(y)) = y$ and $P_n(\overline{u}(y)) = 0$ for all $n \geq 2$. Furthermore, equation (2.4) implies that

$$P_n(\overline{x} + \overline{u}(y)) - P_n(\overline{x}) = y P_{n-1}(\overline{x}). \tag{2.5}$$

We will call polynomials P_n generalized orthogonal polynomials. In particular, from P_n we can obtain the classical orthogonal polynomials (see, e.g. [25]). Moreover, the Sheffer polynomials [26, 27] with generator function $\exp(A(z)x)B(z)$, $x \in \mathbb{R}$, where B and A are analytic functions and $B(0) = 1$, can be obtained by using function $F(z, \overline{x})$ for appropriative values of $\overline{x}$.

For example, if $\overline{x} = (x, 0, 0, \dots, 0, \dots)$, then $F(z, \overline{x}) = e^{zx}$ and $P_n(\overline{x}) = \frac{x^n}{n!}$.

If $\overline{x} = (x, \lambda, 0, \dots, 0, \dots)$, then

$$F(z, \overline{x}) = \exp(zx - \frac{z^2}{2} \lambda) = \sum_{n=0}^{\infty} H_n(x, \lambda) z^n, \tag{2.6}$$

where $H_n(x, \lambda)$ are the Hermite polynomials. So $P_n(x, \lambda, 0, \dots, 0) = H_n(x, \lambda)$.

If $\overline{x} = (x - t, x, x, \dots, x, \dots)$, then

$$F(z, \overline{x}) = (1+z)^x e^{-tz} = \sum_{n=0}^{\infty} C_n(x,t) \frac{z^n}{n!}, \tag{2.7}$$

where C_n are the Charlier polynomials. Hence $n! P_n(x-t, x, \dots, x) = C_n(x,t)$.

Other classical orthogonal polynomials can be obtained in the same way.

For $h \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)$ let $\overline{x}(h) = (x_k(h))_{k=1}^\infty$ denote the sequence of the random variables, such that $x_1(h) = L(h)$, $x_2(h) = L(h^2 \mathbf{1}_{X_0}) + ||h||_H^2$, $x_k(h) = L(h^k \mathbf{1}_{X_0}) + \int_{T \times X_0} h^k(t,x) \nu(dtdx)$, $k = 3, 4, \dots$.

The relationship between generalized orthogonal polynomials and isonormal Lévy process is given by the following result.

Lemma 2.4 *Let h and $g \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)$. Then for all $n, m \geq$ we have $P_n(\overline{x}(h))$ and $P_m(\overline{x}(g)) \in L^2(\Omega)$, and*

$$\mathsf{E}(P_n(\overline{x}(h)) P_m(\overline{x}(g))) = \begin{cases} 0, & \text{if } \ n \neq m, \\ \frac{1}{n!} \left(\mathsf{E}(L(h)L(g)) \right)^n, & \text{if } \ n = m. \end{cases}$$

Proof. Since $h, g \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)$ and P_n, P_m are the polynomials, then by Remark 2.2 $P_n(\overline{x}(h))$ and $P_m(\overline{x}(g)) \in L^2(\Omega)$.

Denote by $\phi(z, \overline{x})$ the power of the exponent in the formula (2.1), i.e.

$$\phi(z, \overline{x}) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{z^k}{k} x_k.$$

Since

$$\frac{1}{R} = \limsup_{k \to \infty} ||x_k(h)||_{L^2(\Omega)}^{1/k}$$

$$\leq \lim_{k \to \infty} \left(\left(\int_{T \times X_0} h^{2k}(t,x) \nu(dtdx) \right)^{1/2} + \int_{T \times X_0} |h(t,x)|^k \nu(dtdx) \right)^{1/k} \leq ||h||_{L^\infty}.$$

Then the series

$$\sum_{k=1}^{\infty} \frac{|z|^k}{k} ||x_k(h)||_{L^2(\Omega)}$$

converges if $|z| < 1/||h||_{L^\infty} \leq R$, which implies that $\phi(z, \overline{x}(h)) \in L^2(\Omega)$ for all $|z| < 1/||h||_{L^\infty}$.

Let's note that for all $|z| < 1/||h||_{L^\infty}$ we have $\ln(1 + zh\mathbf{1}_{X_0}) \in H$. Indeed, by using Taylor's formula, we get

$$(\ln(1 + zh\mathbf{1}_{X_0}))^2 \leq \frac{z^2 h^2}{(1 - |z| \, ||h||_{L^\infty})^2}.$$

In the same way one can obtain the following inequality

$$|\ln(1 + zh) - zh| \leq \frac{z^2 h^2}{(1 - |z| \, ||h||_{L^\infty})^2},$$

which implies that $\ln(1 + zh(t, x)) - zh(t, x)$ is integrable with respect to measure $\nu(dtdx)$ for all $|z| < 1/\,||h||_{L^\infty}$.

So by using the linearity and the continuity of the mapping $h \to L(h)$ we have for all $|z| < 1/\,||h||_{L^\infty}$

$$\phi(z, \overline{x}(h)) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^k}{k}\left(L(h^k\mathbf{1}_{X_0}) + \int_{T\times X_0} h^k(t,x)\nu(dtdx)\right)$$

$$+zL(h\mathbf{1}_\Delta) - \frac{z^2}{2}\int_T h^2(t,\Delta)\mu(dt) = L(\ln(1 + zh\mathbf{1}_{X_0}))$$

$$+\int_{T\times X_0}(\ln(1 + zh(t,x)) - zh(t,x))\nu(dtdx) + zL(h\mathbf{1}_\Delta) - \frac{z^2}{2}\int_T h^2(t,\Delta)\mu(dt). \quad (2.8)$$

This random variable has an infinitely divisible distribution. By Theorem 25.17 in [24, p. 165] $F(z, \overline{x}(h)) = \exp(\phi(z, \overline{x}(h))) \in L^2(\Omega)$ if and only if

$$\int_{|\ln(1+zh(t,x))|>1}\exp(2\ln(1 + zh(t,x)))\nu(dtdx) < \infty.$$

But for all $|z| < 1/\,||h||_{L^\infty}$ we have

$$\int_{|\ln(1+zh(t,x))|>1}\exp(2\ln(1 + zh(t,x)))\nu(dtdx) = \int_{1+zh(t,x)<e^{-1}}(1 + zh(t,x))^2\nu(dtdx)$$

$$\le \int_{1-e^{-1}<|zh(t,x)|}(1 + |zh(t,x)|)^2\nu(dtdx) \le \frac{(2-e^{-1})^2}{(1-e^{-1})^2}z^2\,||h||_H^2 < \infty.$$

So $F(z, \overline{x}(h)) \in L^2(\Omega)$ if $|z| < 1/\,||h||_{L^\infty}$.

Hence for $|z| < 1/\,||h||_{L^\infty}$ and $|y| < 1/\,||g||_{L^\infty}$ we get from (2.8)

$$\mathsf{E}(F(z,\overline{x}(h))F(y,\overline{x}(g))) = \mathsf{E}\exp(\phi(z,\overline{x}(h)) + \phi(y,\overline{x}(g)))$$

$$= \mathsf{E}\exp(L(\ln[(1 + zh\mathbf{1}_{X_0})(1 + yg\mathbf{1}_{X_0})])$$

$$+\int_{T\times X_0}(\ln[(1 + zh(t,x))(1 + yg(t,x))] - zh(t,x) - yg(t,x))\nu(dtdx)$$

$$+L(zh\mathbf{1}_\Delta + yg\mathbf{1}_\Delta) - \frac{1}{2}\int_T(z^2h^2(t,\Delta) + y^2g^2(t,\Delta))\mu(dt))$$

$$= \exp(\int_{T\times X_0}(e^{\ln[(1+zh(t,x))(1+yg(t,x))]} - 1 - \ln[(1 + zh(t,x))(1 + yg(t,x))])\nu(dtdx)$$

$$+\int_{T\times X_0}(\ln[(1 + zh(t,x))(1 + yg(t,x))] - zh(t,x) - yg(t,x))\nu(dtdx)$$

$$+\frac{1}{2}\int_T((zh(t,\Delta) + yg(t,\Delta))^2 - z^2h^2(t,\Delta) - y^2g^2(t,\Delta))\mu(dt))$$

$$= \exp(zy\int_{T\times X}h(t,x)g(t,x)\pi(dtdx)) = \exp(zy\mathsf{E}(L(h)L(g))),$$

where we have used Theorem 25.17 from [24, p. 165] to calculate the expectation.

Taking the $(n+m)$th partial derivative $\frac{\partial^{n+m}}{\partial z^n \partial y^m}$ at $z=y=0$ in both sides of the above equality yields

$$\mathsf{E}(n!m!P_n(\overline{x}(h))P_m(\overline{x}(g))) = \begin{cases} 0, & \text{if } \ n \neq m, \\ n!\left(\mathsf{E}(L(h)L(g))\right)^n, & \text{if } \ n = m. \end{cases}$$

□

Lemma 2.5 *The random variables $\{e^{L(h)},\ h \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)\}$ form a total subset of $L^2(\Omega, \mathcal{F}, P)$.*

PROOF. We claim that $e^{L(h)} \in L^2(\Omega)$ if $h \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)$. In fact, the random variable $L(h)$ has an infinitely divisible distribution with Lévy measure $\nu h^{-1}(dy)$ (see e.g., [24, Def. 8.2, p. 38]). Hence by Theorem 25.17 in [24, p. 165] the variable $e^{L(h)} \in L^2(\Omega)$ if and only if $\int_{|y|>1} e^y \nu h^{-1}(dy) < \infty$. But $\int_{|y|>1} e^y \nu h^{-1}(dy) = \int_{|h(t,x)|>1} e^{h(t,x)} \nu(dtdx) \leq e^{||h||_{L^\infty}} \int_{|h(t,x)|>1} \nu(dtdx) \leq e^{||h||_{L^\infty}} ||h||_H^2 < \infty$, and we have that $e^{L(h)} \in L^2(\Omega)$.

Let $\xi \in L^2(\Omega)$ be such that $\mathsf{E}(\xi e^{L(h)}) = 0$ for all $h \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)$. The linearity of the mapping $h \to L(h)$ implies

$$\mathsf{E}\left(\xi \exp \sum_{k=1}^{n} z_k L(h_k)\right) = 0 \tag{2.9}$$

for any $z_1, \ldots, z_n \in \mathbb{R},\ h_1, \ldots, h_n \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu),\ n \geq 1$. Suppose that $n \geq 1$ and $h_1, \ldots, h_n \in H \cap L^\infty(T \times X, \mathcal{G}, \pi)$ are fixed. Then (2.9) says that Laplace transform of the signed measure

$$\tau(B) = \mathsf{E}(\xi \mathbf{1}_B(L(h_1), \ldots, L(h_n))),$$

where B is a Borel subset of $\mathbb{R}^n$, is identically zero on $\mathbb{R}^n$. Consequently, this measure is zero, which implies $\mathsf{E}(\xi \mathbf{1}_G) = 0$ for any $G \in \mathcal{F}$. So $\xi = 0$, completing the proof of the lemma.
□

For each $n \geq 1$ we will denote by $\mathcal{P}_n$ the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $\{P_n(\overline{x}(h)), h \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)\}$. $\mathcal{P}_0$ will be the set of constants. For $n = 1$, $\mathcal{P}_1$ coincides with the set of random variables $\{L(h), h \in H\}$. From Lemma 2.4 we obtain that $\mathcal{P}_n$ and $\mathcal{P}_m$ are orthogonal whenever $n \neq m$. We will call the space $\mathcal{P}_n$ chaos of order n.

Theorem 2.6 *The space $L^2(\Omega, \mathcal{F}, P)$ can be decomposed into the infinite orthogonal sum of the subspaces $\mathcal{P}_n$:*

$$L^2(\Omega, \mathcal{F}, P) = \bigoplus_{n=0}^{\infty} \mathcal{P}_n.$$

PROOF. Let $\xi \in L^2(\Omega, \mathcal{F}, P)$ such that ξ is orthogonal to all $\mathcal{P}_n$, $n \geq 0$. We have to show that $\xi = 0$. For all $h \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)$ we get $\mathsf{E}(\xi P_n(\overline{x}(h))) = 0$. Since from the proof of Lemma 2.4 we have that $F(z, \overline{x}(h)) \in L^2(\Omega)$ for all $z < 1/\,||h||_{L^\infty}$, then $\mathsf{E}(\xi F(z, \overline{x}(h))) = 0$ for $z < 1/\,||h||_{L^\infty}$. Using equality (2.8) we obtain

$$0 = \mathsf{E}(\xi F(z, \overline{x}(h))) = \mathsf{E}(\xi e^{\phi(z, \overline{x}(h))}) = \mathsf{E}(\xi \exp(L(\ln(1 + zh\mathbf{1}_{X_0}))$$

$$+\int_{T\times X_0}(\ln(1+zh(t,x))-zh(t,x))\nu(dtdx)+L(zh\mathbf{1}_\Delta)-\frac{1}{2}\int_T z^2h^2(t,\Delta)\mu(dt))).$$

Thus for any $z<1/\left\|h\right\|_{L^\infty}$

$$\mathsf{E}(\xi\exp(L(\ln(1+zh\mathbf{1}_{X_0}))+L(zh\mathbf{1}_\Delta)))=0. \tag{2.10}$$

Since $\mathsf{E}(\xi F(z,\overline{x}(h)))$ is an analytic function for $z<1/\left\|h\right\|_{L^\infty}$, then $\mathsf{E}(\xi\exp(L(\ln(1+zh\mathbf{1}_{X_0}))+L(zh\mathbf{1}_\Delta)))$ has an analytic extension to $z\in[0;1]$ if $h\mathbf{1}_{X_0}>-1$. For any $g\in H\cap L^\infty(T\times X_0,\mathcal{B},\nu)$ we have $(e^g-1)\in H\cap L^\infty(T\times X_0,\mathcal{B},\nu)$ and $(e^g-1)\mathbf{1}_{X_0}>-1$. Putting in (2.10) $h=(e^g-1)\mathbf{1}_{X_0}+g\mathbf{1}_\Delta$ and $z=1$ we deduce that $\mathsf{E}(\xi e^{L(g)})=0$ for all $g\in H\cap L^\infty(T\times X_0,\mathcal{B},\nu)$. By Lemma 2.5 we get $\xi=0$, which completes the proof of the theorem. □

3 Multiple integrals

Since separable Hilbert space H has the form $H=L^2(T\times X,\mathcal{G},\pi)$, where π is a σ-finite measure without atoms, then the process L is characterized by the family of random variables $\{L(A),A\in\mathcal{G},\pi(A)<\infty\}$, where $L(A)=L(\mathbf{1}_A)$. We can consider $L(A)$ as a $L^2(\Omega,\mathcal{F},P)$-valued measure on the parametric space $(T\times X,\mathcal{G})$, which takes independent values on any family of disjoint subsets of $T\times X$. In that sense $L(h)$ can be considered as the stochastic integral of the function $h\in H$ with respect to L. The purpose of the section is to show that the nth chaos $\mathcal{P}_n$ is generated by multiple stochastic integrals with respect to L. The construction of multiple stochastic integral provided by Itô in [10]. We briefly recall some basic facts about about them.

Set $\mathcal{G}_0=\{A\in\mathcal{G}:\ \pi(A)<\infty\}$. For any $m\geq 1$ we denote by $\mathcal{E}_m$ the set of all linear combinations of the following functions $f\in L^2((T\times X)^m,\mathcal{G}^m,\pi^m)$

$$f(t_1,x_1,\ldots,t_m,x_m)=\mathbf{1}_{A_1\times\cdots\times A_m}(t_1,x_1,\ldots,t_m,x_m), \tag{3.1}$$

where $A_1,\ldots A_m$ are pairwise-disjoint sets in $\mathcal{G}_0$. The fact that measure π without atoms implies that $\mathcal{E}_m$ is dense in $L^2((T\times X)^m)$ (see, e.g. [9, Th. 2.1] or [16, p. 8-9]).

We define the multiple integral of the mth order

$$I_m(f)=L(A_1)\cdots L(A_m),$$

for the functions f of the form (3.1), then $I_m(s)$ for all functions s in $\mathcal{E}_m$ by linearity and finally $I_m(g)$ for all functions g in $L^2((T\times X)^m)$ by continuity.

It was shown in [9, 10] that the definition is possible and the following properties hold:

1. I_m is linear.
2. $I_m(f)=I_m(\widetilde{f})$, where $\widetilde{f}$ denotes the symmetrization of f, which is defined by

$$\widetilde{f}(t_1,x_1,\ldots,t_m,x_m)=\frac{1}{m!}\sum_\sigma f(t_{\sigma(1)},x_{\sigma(1)},\ldots,t_{\sigma(m)},x_{\sigma(m)}),$$

σ running over all permutations of $\{1,\ldots,m\}$.

3.
$$\mathsf{E}(I_m(f)I_p(g)) = \begin{cases} 0, & \text{if } \; p \neq m, \\ m!\langle \widetilde{f}; \widetilde{g}\rangle_{L^2((T\times X)^m)}, & \text{if } \; p = m. \end{cases}$$

We refer to [9, 10, 16] for details.

If $f \in L^2((T \times X)^p)$ and $g \in L^2((T \times X)^q)$ are symmetric functions the contraction of the indices of f and g is denoted by $f \otimes_1 g$ and is defined by

$$(f \otimes_1 g)(t_1, x_1, \dots, t_{p+q-2}, x_{p+q-2})$$

$$= \int_{T\times X} f(t_1, x_1, \dots, t_{p-1}, x_{p-1}, s, z) g(t_p, x_p, \dots, t_{p+q-2}, x_{p+q-2}, s, z)\pi(dsdz).$$

Notice that $f \otimes_1 g \in L^2((T \times X)^{p+q-2})$.

The following, so called product formula, will be useful in the sequel. It was initially derived by Itô [9] for Gaussian case and by Kabanov [11] for Poisson case, then extended by Russo and Vallois [23] to products of two multiple stochastic integrals with respect to a normal martingale.

Proposition 3.1 *Let $f \in L^2((T \times X)^p)$ be a symmetric function and let $g \in L^2(T \times X)$ such that $fg\mathbf{1}_{X_0} \in L^2((T \times X)^p)$. Then*

$$I_p(f)I_1(g) = I_{p+1}(f \otimes g) + pI_{p-1}(f \otimes_1 g) + pI_p(fg\mathbf{1}_{X_0}). \tag{3.2}$$

Proof. The proof of the proposition can be obtained as slight modification of the proof of Proposition 1.1.2 in [16]. □

The next result gives the relationship between generalized orthogonal polynomials and multiple stochastic integrals.

Theorem 3.2 *Let P_n be the nth generalized orthogonal polynomial, and $\overline{x}(h) = (x_k(h))_{k=1}^{\infty}$, where $x_1(h) = L(h)$, $x_2(h) = L(h^2\mathbf{1}_{X_0}) + ||h||_H^2$, $x_k(h) = L(h^k\mathbf{1}_{X_0}) + \int_{T\times X_0} h^k(t, x)\nu(dtdx)$, $k = 3, 4, \dots$ and $h \in \cap_{p\geq 2} L^p(T \times X_0, \mathcal{B}, \nu) \cap H$. Then it holds that*

$$n!P_n(\overline{x}(h)) = I_n(h^{\otimes n}), \tag{3.3}$$

where $h^{\otimes n}(t_1, x_1, \dots, t_n, x_n) = h(t_1, x_1) \cdots h(t_n, x_n)$.

Proof. We will prove the theorem by induction on n. For $n = 1$ it is immediate. Assume it holds for $1, 2, \dots, n$. Using the product formula (3.2) and recursive relation for generalized orthogonal polynomials (2.2), we have

$$I_{n+1}(h^{\otimes(n+1)}) = I_n(h^{\otimes n})I_1(h) - nI_{n-1}\left(h^{\otimes(n-1)} \int_{T\times X} h^2(t, x)\pi(dtdx)\right)$$

$$-nI_n(h^{\otimes(n-1)} \otimes (h^2\mathbf{1}_{X_0})) = n!P_n(\overline{x}(h))L(h) - n!\,||h||_H^2\, P_{n-1}(\overline{x}(h))$$

$$-nI_{n-1}(h^{\otimes(n-1)})I_1(h^2\mathbf{1}_{X_0}) + n(n-1)I_{n-2}(h^{\otimes(n-2)}) \int_{T\times X_0} h^3(t, x)\nu(dtdx)$$

$$+n(n-1)I_{n-1}(h^{\otimes(n-2)}\otimes(h^3\mathbf{1}_{X_0}))=n!\sum_{k=0}^{1}(-1)^{k+1}x_{k+1}(h)P_{n-k}(\overline{x}(h))$$

$$+n!P_{n-2}(\overline{x}(h))\int_{T\times X_0}h^3(t,x)\nu(dtdx)+n(n-1)I_{n-1}(h^{\otimes(n-2)}\otimes(h^3\mathbf{1}_{X_0}))=\ldots$$

$$=n!\sum_{k=0}^{n-1}(-1)^{k+1}x_{k+1}(h)P_{n-k}(\overline{x}(h))+n!(-1)^nP_0(\overline{x}(h))\int_{T\times X_0}h^{n+1}(t,x)\nu(dtdx)$$

$$+n!(-1)^nI_1(h^{n+1})=n!\sum_{k=0}^{n}(-1)^{k+1}x_{k+1}(h)P_{n-k}(\overline{x}(h))=(n+1)!P_{n+1}(\overline{x}(h)),$$

which completes the proof of the theorem. □

From this theorem and Theorem 2.6 we deduce the following classical result of Itô.

Corollary 3.3 ([10]) *Any square integrable random variable $\xi\in L^2(\Omega,\mathcal{F},P)$ can be expanded into a series of multiple stochastic integrals:*

$$\xi=\sum_{k=0}^{\infty}I_k(f_k).$$

Here $f_0=\mathsf{E}\xi$, and I_0 is the identity mapping on the constants. Furthermore, this representation is unique provided the functions $f_k\in L^2((T\times X)^k)$ are symmetric.

Assuming $T=\mathbb{R}_+$, $X_0=\mathbb{R}\setminus\{0\}$, $X=\mathbb{R}$, $\mu(dt)=dt$ and $\nu(dtdx)=dt\beta(dx)$, where the measure β such that $\int_{\mathbb{R}\setminus\{0\}}(x^2\wedge 1)\beta(dx)<\infty$, we have that for any symmetric function $f_n\in L^2((\mathbb{R}_+\times\mathbb{R})^n)$ the multiple stochastic integral $I_n(f_n)$ with respect to the process $\{L(h),h\in H\}$ coincides with an iterated integral with respect to L^2-valued measure $L(dtdx)$ generated by $L(h)$:

$$I_n(f_n)=n!\int_0^\infty\int_{\mathbb{R}}\cdots\int_0^{t_2-}\int_{\mathbb{R}}f_n(t_1,x_1,\ldots t_n,x_n)L(dt_1dx_1)\cdots L(dt_ndx_n).$$

This equality can be shown for elementary processes $f_n\in\mathcal{E}_n$ and in the general case the equality will follow by the density arguments, taking into account that the iterated stochastic integral verifies the same isometry property as the multiple stochastic integral.

Consider the process $L^h_t=L(\mathbf{1}_{[0;t]}h)$, $h\in H$. It is easy to show by definition that L^h_t has independent increments. Let $\mathcal{F}^h_t=\sigma\{L^h_s,s\le t\}\vee\mathcal{N}$, $t\ge 0$ be a σ-algebra generated by L^h_s and the family $\mathcal{N}$ of P-null sets of $\mathcal{F}$. Then L^h_t is a martingale with respect to $\{\mathcal{F}^h_t\}_{t\ge0}$. Since $L(dtdx)$ is a L^2-valued measure with independent values on any family of the disjoint subsets of $\mathbb{R}_+\times\mathbb{R}$, then $\{I_n(h^{\otimes n}\mathbf{1}^{\otimes n}_{[0;t]}),t\ge0\}$ is a square integrable martingale with respect to $\{\mathcal{F}^h_t\}_{t\ge0}$ for any $h\in H$. Hence, it follows from the equation (3.3) that $P_n(\overline{x}(\mathbf{1}_{[0;t]}h))=I_n(h^{\otimes n}\mathbf{1}^{\otimes n}_{[0;t]})$ is a square integrable martingale with respect to $\{\mathcal{F}^h_t\}_{t\ge0}$ for any $h\in\cap_{p\ge2}L^p(\mathbb{R}_+\times(\mathbb{R}\setminus\{0\}),\mathcal{B},\nu)\cap H$. So we obtain the following result.

Proposition 3.4 *Let P_n be the nth generalized orthogonal polynomial, $L^h_t=L(\mathbf{1}_{[0;t]}h)=\int_0^t h(s,0)dB_s+\int_0^t\int_{\mathbb{R}\setminus\{0\}}h(s,x)\widetilde{N}(dsdx)$, $t\ge0$ where $h\in\cap_{p\ge2}L^p(\mathbb{R}_+\times(\mathbb{R}\setminus\{0\}),\mathcal{B},\nu)\cap H$,*

$B_t = L(\mathbf{1}_{[0;t]}\mathbf{1}_{\{0\}})$ and $\widetilde{N}(dsdx) = L(\mathbf{1}_{\mathbb{R}\setminus\{0\}}\mathbf{1}_{dsdx})$. Set $\mathcal{F}_t^h = \sigma\{L_s^h, s \le t\}$, $t \ge 0$ and $\overline{x}(h) = (x_k(t,h))_{k=1}^{\infty}$ such that

$$x_1(t,h) = \int_0^t h(s,0)dB_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} h(s,x)\widetilde{N}(dsdx),$$

$$x_2(t,h) = \int_0^t \int_{\mathbb{R}\setminus\{0\}} h^2(s,x)\widetilde{N}(dsdx) + \int_0^t h^2(s,0)\mu(ds) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} h^2(s,x)\nu(dsdx),$$

$$x_k(t,h) = \int_0^t \int_{\mathbb{R}\setminus\{0\}} h^k(s,x)\widetilde{N}(dsdx) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} h^k(s,x)\nu(dsdx),\ k = 3, 4, \ldots$$

Then $P_n(\overline{x}(t,h))$ is a square integrable martingale with respect to $\{\mathcal{F}_t^h\}_{t\ge 0}$.

Remark 3.5 *If the function $h(t,x) = x$ and the measure ν has the form $\nu(dtdx) = dt\beta(dx)$, where the measure β such that $\int_{\mathbb{R}\setminus\{0\}} |x|^k\beta(dx) < \infty$ for all $k \ge 2$, then the martingales $\int_0^t \int_{\mathbb{R}\setminus\{0\}} x^k\widetilde{N}(dsdx)$, $k \ge 2$ are so-called Teugels martingales and $\int_0^t \int_{\mathbb{R}\setminus\{0\}} x^k\widetilde{N}(dsdx) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} x^k\nu(dsdx)$ are power jump processes (see, e.g., [25]).*

Example 3.6 *If the measures μ is the Lebesgue measure and the measure ν is equal to zero, then for $h = \mathbf{1}_{[0;t]}$ we have that $L_t^h = B_t$ is a Brownian motion and $x_1(t,h) = B_t$, $x_2(t,h) = t$ and $x_k(t,h) = 0$ for all $k \ge 3$. Hence $P_n(\overline{x}(t,h)) = H_n(B_t,t)$ is a martingale, where $H_n(y,z)$ is the nth Hermite polynomial (2.6) (see, e.g., [8, 25]).*

Example 3.7 *If the measure μ is equal to zero and the measure ν is the product of the Lebesgue measure times the delta-measure, which gives mass one to the point 1, then for $h(s,x) = x\mathbf{1}_{[0;t]}(s)$ we have $L_t^h = P_t - t$, where P_t is a Poisson process. Hence $x_1(t,h) = P_t - t$ and $x_k(t,h) = P_t$ for all $k \ge 2$. Then $P_n(\overline{x}(t,h)) = C_n(P_t,t)$ is a martingale, where $C_n(y,z)$ is the nth Charlier polynomial (2.7) (see, e.g., [19, 25]).*

4 The derivative operator

In this section we introduce the operator D. Then we will show that it is equal to the Malliavin derivatives in the Gaussian case (see, e.g., [16]) and to the difference operator defined in [18, 21] in the Poisson case. We will also proof that the derivatives operators defined via the chaos decomposition in [2, 3, 13, 14, 20, 22] for certain Lévy processes coincide with the operator D.

We denote by $C_b^\infty(\mathbb{R}^n)$ the set of all infinitely continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all of its partial derivatives are bounded.

Let $\mathcal{S}$ denote the class of smooth random variables such that a random variable $\xi \in \mathcal{S}$ has the form

$$\xi = f(L(h_1), \ldots, L(h_n)), \tag{4.1}$$

where f belongs to $C_b^\infty(\mathbb{R}^n)$, $h_1, \ldots, h_n$ are in H, and $n \ge 1$.

Lemma 4.1 *The set $\mathcal{S}$ is dense in $L^p(\Omega)$, for any $p \ge 1$.*

Proof. Let $\{h_k\}_{k=1}^{\infty}$ be a dense subset of H. Define $\mathcal{F}_n = \sigma(L(h_1), \ldots, L(h_n))$. Then $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and $\mathcal{F}$ is the smallest σ-algebra containing all the $\mathcal{F}_n$'s. Choose a $g \in L^p(\Omega)$. Then

$$g = \mathsf{E}(g|\mathcal{F}) = \lim_{n\to\infty} \mathsf{E}(g|\mathcal{F}_n).$$

By the Doob-Dynkin Lemma we have that for each n, there exist a Borel measurable function $g_n : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathsf{E}(g|\mathcal{F}_n) = g_n(L(h_1), \ldots, L(h_n)).$$

Each such g_n can be approximated by functions $f_m^{(n)}$ where $f_m^{(n)} \in C_b^{\infty}(\mathbb{R}^n)$ such that $||f_m^{(n)}(L(h_1), \ldots, L(h_n)) - g_n(L(h_1), \ldots, L(h_n))||_{L^p(\Omega)}$ converges to zero as $m \to \infty$. Since $f_m^{(n)}(L(h_1), \ldots, L(h_n)) \in \mathcal{S}$ we have the statement of the lemma. □

Definition 4.2 *The stochastic derivative of a smooth random variable ξ of the form (4.1) is the H-valued random variable $D\xi = \{D_{t,x}\xi,\ (t,x) \in T \times X\}$ given by*

$$D_{t,x}\xi = \sum_{k=1}^{n} \frac{\partial f}{\partial y_k}(L(h_1), \ldots, L(h_n)) h_k(t,x) \mathbf{1}_{\Delta}(x)$$
$$+ \Big(f(L(h_1) + h_1(t,x), \ldots, L(h_n) + h_n(t,x)) - f(L(h_1), \ldots, L(h_n)) \Big) \mathbf{1}_{X_0}(x). \tag{4.2}$$

We will consider $D\xi$ as an element of $L^2(T \times X \times \Omega) \cong L^2(\Omega; H)$; namely, $D\xi$ is a random process indexed by the parameter space $T \times X$.

Remark 4.3
1. *If the measure ν is zero or $h_k,\ k = 1, \ldots, n$ from (4.1) such that $h_k(t,x) = 0,\ k = 1, \ldots, n$ when $x \neq \Delta$ then $D\xi$ coincides with the Malliavin derivative (see, for example, [16, Def. 1.2.1, p. 24]).*
2. *If the measure μ is zero or $h_k,\ k = 1, \ldots, n$ from (4.1) such that $h_k(t,x) = 0,\ k = 1, \ldots, n$ when $x = \Delta$ then $D\xi$ coincides with the difference operator defined in [21].*
3. *If $T = \mathbb{R}_+$, the measure μ is the Lebesgue measure and X is a metric space and the measure ν is the product of the Lebesgue measure times the measure β satisfying $\int_M (|x|^2 \wedge 1)\beta(dx)$, then D is the operator ∇^- from [22].*

Lemma 4.4 *Suppose that ξ is smooth functional of the form (4.1) and $h \in H$. Then*

$$\mathsf{E}(\langle D\xi; h\rangle_H) = \mathsf{E}(\xi L(h)). \tag{4.3}$$

Proof. The proof will be done in several steps.

Step 1. Suppose first that

$$\xi = e^{iz_1 L(h_1)} \cdots e^{iz_n L(h_n)}.$$

Then $\xi \in \mathcal{S}$ and

$$\mathsf{E}(\xi L(h)) = \frac{1}{i}\frac{d}{dz}\left(\mathsf{E} \exp\left(i \sum_{k=1}^{n} z_k L(h_k) + izL(h) \right) \right)\Bigg|_{z=0}$$

$$= \frac{1}{i}\frac{d}{dz}\exp\left(-\frac{1}{2}\int_T\left(\sum_{k=1}^n z_k h_k(t,\Delta)+zh(t,\Delta)\right)^2\mu(dt)\right.$$

$$+\int_{T\times X_0}\left(\exp(i\sum_{k=1}^n z_k h_k(t,x)+izh(t,x))-1\right.$$

$$\left.\left.-i\left(\sum_{k=1}^n z_k h_k(t,x)+zh(t,x)\right)\right)\nu(dtdx)\right)\Bigg|_{z=0}$$

$$=\left(\int_{T\times X_0}h(t,x)\left(\exp(i\sum_{k=1}^n z_k h_k(t,x))-1\right)\nu(dtdx)\right.$$

$$\left.+i\int_T h(t,\Delta)\sum_{k=1}^n z_k h_k(t,\Delta)\mu(dt)\right)\exp\left(-\frac{1}{2}\int_T\left(\sum_{k=1}^n z_k h_k(t,\Delta)\right)^2\mu(dt)\right.$$

$$\left.+\int_{T\times X_0}\left(\exp(i\sum_{k=1}^n z_k h_k(t,x))-1-i\sum_{k=1}^n z_k h_k(t,x)\right)\nu(dtdx)\right)$$

$$=\mathsf{E}(\xi)\left(\int_{T\times X_0}h(t,x)\left(\exp(i\sum_{k=1}^n z_k h_k(t,x))-1\right)\nu(dtdx)\right.$$

$$\left.+i\int_T h(t,\Delta)\sum_{k=1}^n z_k h_k(t,\Delta)\mu(dt)\right).$$

On the other hand

$$\mathsf{E}(\langle D\xi,h\rangle_H)=\mathsf{E}\int_{T\times X}D_{t,x}\xi h(t,x)\pi(dtdx)$$

$$=\mathsf{E}\int_{T\times X_0}\left(\exp(i\sum_{k=1}^n z_k(L(h_k)+h_k(t,x)))-\exp(i\sum_{k=1}^n z_k L(h_k))\right)h(t,x)\nu(dtdx)$$

$$+\mathsf{E}\int_T i\sum_{j=1}^n z_j\exp(i\sum_{k=1}^n z_k L(h_k))h_j(t,\Delta)\mu(dt)$$

$$=\mathsf{E}(\xi)\left(\int_{T\times X_0}h(t,x)\left(\exp(i\sum_{k=1}^n z_k h_k(t,x))-1\right)\nu(dtdx)\right.$$

$$\left.+i\int_T h(t,\Delta)\sum_{k=1}^n z_k h_k(t,\Delta)\mu(dt)\right).$$

Hence we have (4.3). By linearity we deduce that (4.3) also holds for smooth variables of the form (4.1), where the function f is a trigonometric polynomial.

Step 2. Assume that ξ of the form (4.1) such that $f\in C_b^\infty(\mathbb{R}^n)$ is periodic on every variable function. Then there is a sequence of trigonometric polynomials g_m such that

$g_m \to f$ and $\partial g_m/\partial x_k \to \partial f/\partial x_k$ for every $k = 1, \dots n$ uniformly on $\mathbb{R}^n$ as $m \to \infty$. Denote $\eta_m = g_m(L(h_1), \dots, L(h_n))$. Then $\eta_m \in \mathcal{S}$, and by Step 1 we get

$$\mathsf{E}(\eta_m L(h)) = \mathsf{E}(\langle D\eta_m; h\rangle_H). \tag{4.4}$$

Since $\eta_m \to \xi$ in $L^2(\Omega)$ and $D\eta_m \to D\xi$ in $L^2(T \times X \times \Omega)$ then letting $m \to \infty$ in (4.4) we obtain (4.3).

Step 3. Assume that ξ of the form (4.1). Consider the sequence $\{\chi_m, m = 1, 2, \dots\}$ of functions, such that $\chi_m \in C^\infty(\mathbb{R}^n)$, $0 \le \chi_m \le 1$, $\chi_m(x) = 1$ if $|x| \le m$, $\chi(x) = 0$, if $|x| > m+1$ and $|\nabla \chi_m| \le 2$. Define g_m as a periodic extension on all variables of the function $f\chi_m$. Then $\zeta_m = g_m(L(h_1), \dots, L(h_n))$ is smooth variable such that $|\zeta_m| \le ||f||_{L^\infty}$ and $|D\zeta_m| \le ||\nabla f||_{L^\infty} \sum_{i=1}^n |h_i|$. Hence by the dominated convergence theorem $\zeta_m \to \xi$ in $L^2(\Omega)$ and $D\zeta_m \to D\xi$ in $L^2(T \times X \times \Omega)$ as $m \to \infty$. Since by Step 2 (4.3) is true for ζ_m then letting $m \to \infty$ complete the proof of the lemma. □

Applying this lemma to the product of two smooth functionals we obtain the "integration by parts" formula.

Lemma 4.5 *Suppose ξ and η are the smooth functionals and $h \in H$, then*

$$\mathsf{E}(\xi\eta L(h)) = \mathsf{E}(\xi\langle D\eta; h\rangle_H) + \mathsf{E}(\eta\langle D\xi; h\rangle_H) + \mathsf{E}(\langle D\eta; h\mathbf{1}_{X_0} D\xi\rangle_H). \tag{4.5}$$

As a consequence of the above lemma we obtain the following result.

Lemma 4.6 *The expression of the derivative $D\xi$ given in (4.2) does not depend on the particular representation of ξ in (4.1).*

Proof. Let $\xi = f(L(h_1), \dots, L(h_n)) = 0$. We have to show that $D\xi = 0$. From Lemma 4.5 we get for any $\eta \in \mathcal{S}$ and $h \in H$

$$0 = \mathsf{E}(\xi\eta L(h)) = \mathsf{E}(\xi\langle D\eta; h\rangle_H) + \mathsf{E}(\eta\langle D\xi; h\rangle_H) + \mathsf{E}(\langle D\eta; h\mathbf{1}_{X_0} D\xi\rangle_H).$$

Hence

$$\mathsf{E}(\eta\langle D\xi; h\rangle_H) + \mathsf{E}(\langle D\eta; h\mathbf{1}_{X_0} D\xi\rangle_H) = 0. \tag{4.6}$$

Replacing η by ξ in (4.6) we obtain

$$\int_{T\times X_0} \mathsf{E}(D_{t,x}\xi)^2 h(t,x)\nu(dtdx) = 0.$$

Hence $D_{t,x}\xi = 0$ for $\nu \times P$-a.a. $(t, x, \omega) \in T \times X_0 \times \Omega$.

Substituting this expression into (4.6) we have for all $h \in H$ and $\eta \in S$

$$\int_T \mathsf{E}(\eta D_{t,\Delta}\xi) h(t, \Delta)\mu(dt) = 0.$$

Since by Lemma 4.1 the set $\mathcal{S}$ is dense in $L^2(\Omega)$ then $D_{t,\Delta}\xi = 0$ for $\mu \times P$-a.a. $(t, \omega) \in T \times \Omega$, which implies the desired result. □

Operating in the same way we obtain the following lemma.

Lemma 4.7 *The operator D is closable as an operator from $L^p(\Omega)$ to $L^p(\Omega; H)$, for any $p \geq 1$.*

PROOF. Let $\{\xi_n, n \geq 1\}$ be a sequence of smooth random variables such that $\mathsf{E}|\xi_n|^p \to 0$ and $D\xi_n$ converges to ζ in $L^p(\Omega; H)$. Then from Lemma 4.5 it follows that for any $h \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)$ and $\eta \in \mathcal{S}$ we have

$$\mathsf{E}(\xi_n \eta L(h)) = \mathsf{E}(\xi_n \langle D\eta; h\rangle_H) + \mathsf{E}(\eta\langle D\xi_n; h\rangle_H) + \mathsf{E}(\langle D\xi_n; h\mathbf{1}_{X_0} D\eta\rangle_H).$$

Taking the limit as $n \to \infty$, since η, $\mathbf{1}_{X_0} D\eta$ and $\langle D\eta; h\rangle_H$ are bounded, and $h \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)$ we obtain

$$\mathsf{E}(\eta\langle\zeta; h\rangle_H) + \mathsf{E}(\langle\zeta; h\mathbf{1}_{X_0} D\eta\rangle_H) = 0. \tag{4.7}$$

If $h(t,x) = 0$ for $x \neq \Delta$, then (4.7) implies, that

$$\int_T \mathsf{E}(\eta\zeta_{t,\Delta}) h(t,\Delta)\mu(dt) = 0.$$

Thus $\zeta_{t,\Delta} = 0$ $\mu \times P$-a.a. $(t,\omega) \in T \times \Omega$. Substituting this expression into (4.7) we have for any $h \in H$

$$\mathsf{E}\int_{T\times X_0} (\zeta_{t,x} h(t,x)(\eta + D_{t,x}\eta))\nu(dtdx) = 0. \tag{4.8}$$

Let $\phi_n \in C_b^\infty(\mathbb{R})$ such that $0 \leq \phi_n(x) \leq e^x$ and $\phi_n \to e^x$ for all $x \in \mathbb{R}$. Putting in (4.8) $\eta = \phi_n(L(g))$ and $h(t,x) = u(t,x)e^{-g(t,x)}$, where $u \in H$ and $g \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)$ and then letting $n \to \infty$ we get

$$\int_{T\times X_0} \mathsf{E}(e^{L(g)}\zeta_{t,x}) u(t,x)\nu(dtdx) = 0.$$

Since by Lemma 2.5 the set of the random variables $\{e^{L(g)}, g \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)\}$ is a total family in $L^2(\Omega)$ it follows that $\zeta_{t,x} = 0$ for $\pi \times P$-a.a. $(t,x,\omega) \in T \times X \times \Omega$ completing the proof of the lemma. □

We will denote the closure of D again D and its domain in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$.

In the same way we can introduce iterated derivatives for smooth random variables. If ξ is a smooth random variable and k is an integer, we set

$$D^k_{t_1,x_1,\ldots,t_k,x_k}\xi = D_{t_1,x_1}\cdots D_{t_k,x_k}\xi.$$

By induction one can prove that this operator is closable as an operator from $L^p(\Omega)$ to $L^p((T \times X)^k \times \Omega)$ for all $p \geq 1$. We will denote its closure by D^k and its domain in $L^p(\Omega)$ by $\mathbb{D}^{k,p}$.

Now we will state the chain rule.

Proposition 4.8 *Suppose $p \geq 1$ is fixed and $\xi = (\xi^1, \ldots, \xi^m)$ is a random vector whose components belong to the space $\mathbb{D}^{1,p}$. Let $\phi \in C^1(\mathbb{R}^m)$ be a function with bounded partial derivatives, such that $\phi(\xi) \in L^p(\Omega)$. Then $\phi(\xi) \in \mathbb{D}^{1,p}$ and*

$$D_{t,x}\phi(\xi) = \begin{cases} \sum_{k=1}^m \frac{\partial\phi}{\partial x_k}(\xi) D_{t,\Delta}\xi^k, & \textit{if } x = \Delta, \\ \phi(\xi^1 + D_{t,x}\xi^1, \ldots, \xi^m + D_{t,x}\xi^m) - \phi(\xi^1, \ldots, \xi^m), & \textit{if } x \neq \Delta. \end{cases} \tag{4.9}$$

Proof. The proof can be easily obtain by approximation ξ by smooth random variables and the function ϕ by smooth functions with compact support. $\square$

Applying the above proposition we obtain, that $L(h) \in \mathbb{D}^{1,2}$ for all $h \in H$ and $D_{t,x}L(h) = h(t,x)$.

By using the same arguments one can show the following result.

Lemma 4.9 *It holds that $P_n(\overline{x}(h)) \in \mathbb{D}^{1,p}$ for all $p \geq 1$, $h \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)$, $n = 1, 2, \dots$ and*

$$D_{t,x}P_n(\overline{x}(h)) = P_{n-1}(\overline{x}(h))h(t,x). \tag{4.10}$$

Proof. As in the proof of Proposition 4.8 one can obtain that $P_n(\overline{x}(h)) \in \mathbb{D}^{1,p}$ for all $p \geq 1$, $h \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)$, $n = 1, 2, \dots$ and (4.9) holds. Then the definition of $\overline{x}(h)$ and equality (2.3) imply

$$D_{t,\Delta}P_n(\overline{x}(h)) = \frac{\partial P_n}{\partial x_1}(\overline{x}(h))h(t,\Delta) = P_{n-1}(\overline{x}(h))h(t,\Delta).$$

It follows from the relationships (4.9) and (2.5) that for $x \neq \Delta$ we have

$$D_{t,x}P_n(\overline{x}(h)) = P_n(\overline{x}(h) + \overline{u}(h(t,x))) - P_n(\overline{x}(h)) = h(t,x)P_{n-1}(\overline{x}(h)),$$

where $\overline{u}(y) = (y, y^2, \dots, y^k, \dots)$. The proof is complete. $\square$

The product rule can be proved in the same manner.

Proposition 4.10 *Let $\xi \in \mathbb{D}^{1,p}$, $p \geq 1$ and η is a smooth variable from $\mathcal{S}$. Then $\xi\eta \in \mathbb{D}^{1,p}$ and*

$$D(\xi\eta) = \xi D\eta + \eta D\xi + D\xi D\eta \mathbf{1}_{X_0}. \tag{4.11}$$

Proof. The equation (4.11) holds if ξ and η are smooth variables. Then, the general case follows by a limit argument, using the fact that D is closed. $\square$

The following result shows the action of the operator D via the chaos decomposition.

Proposition 4.11 *Let $\xi \in L^2(\Omega)$ with a development*

$$\xi = \sum_{k=0}^{\infty} I_k(f_k), \tag{4.12}$$

where $f_k \in L^2((T \times X)^k)$ is symmetric. Then $\xi \in \mathbb{D}^{1,2}$ if and only if

$$\sum_{k=1}^{\infty} kk! \, ||f_k||^2_{L^2((T\times X)^k)} < \infty \tag{4.13}$$

and in this case we have

$$D_{t,x}\xi = \sum_{k=1}^{\infty} kI_{k-1}(f_k(\cdot, t, x)) \tag{4.14}$$

and $\mathsf{E}\int_{T\times X}(D_{t,x}\xi)^2\pi(dtdx)$ coincides with the sum of the series (4.13).

PROOF. The proof will be done in several steps.

Step 1. Suppose first that

$$\xi = P_k(\overline{x}(h)) = \frac{1}{k!} I_k(h^{\otimes n}), \tag{4.15}$$

with $h \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)$. Then by Lemma 4.9 $\xi \in \mathbb{D}^{1,2}$ and by equality (4.10) we get

$$D_{t,x} P_k(\overline{x}(h)) = P_{k-1}(\overline{x}(h)) h(t,x).$$

Hence for all $(t,x) \in T \times X$ we have

$$D_{t,x}\xi = k I_{k-1}(f_k(\cdot, t, x)). \tag{4.16}$$

Equality (4.16) holds for any linear combination of random variables of the form (4.15). Since formula (4.16) implies that $\mathsf{E}\,||D\xi||_H^2 = k\mathsf{E}\xi^2$ then it follows that $\mathcal{P}_k$ is included in $\mathbb{D}^{1,2}$.

Step 2. Let $\xi \in L^2(\Omega)$ has an expansion (4.12). Suppose that (4.13) holds. Define

$$\xi_n = \sum_{k=0}^{n} I_k(f_k).$$

Then the sequence ξ_n converges to ξ in $L^2(\Omega)$, and by Step 1 we have $\xi_n \in \mathbb{D}^{1,2}$ and $D_{t,x} = \sum_{k=1}^{n} k I_{k-1}(f_k(\cdot, t, x))$. It follows from (4.13) that $D_{t,x}\xi_n$ converges in $L^2(\Omega; H)$ to the right-hand side of (4.14). Therefore $\xi \in \mathbb{D}^{1,2}$ and (4.14) holds.

Step 3. Suppose $\xi \in \mathbb{D}^{1,2}$. Note that formula (4.5) holds for $\xi \in \mathbb{D}^{1,2}$ and $\eta \in \mathbb{D}^{1,p}$ for some $p > 2$ if $h \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)$. Since by Proposition 4.9 $\eta = P_m(\overline{x}(g)) \in \mathbb{D}^{1,p}$ for all $p \geq 1$ and $g \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)$, then we have

$$\lim_{n\to\infty} (\mathsf{E}\langle D\xi_n; \eta h\rangle_H + \mathsf{E}\langle D\xi_n; D\eta h \mathbf{1}_{X_0}\rangle_H) = \lim_{n\to\infty} \mathsf{E}(\xi_n \eta L(h) - \xi_n \langle D\eta; h\rangle_H)$$
$$= \mathsf{E}(\xi\eta L(h) - \xi\langle D\eta; h\rangle_H) = \mathsf{E}\langle D\xi; \eta h\rangle_H + \mathsf{E}\langle D\xi; D\eta h \mathbf{1}_{X_0}\rangle_H.$$

It follows from equation (4.9) that $\eta + \mathbf{1}_{X_0} D\eta = P_m(\overline{x}(g)) + \mathbf{1}_{X_0} g P_{m-1}(\overline{x}(g))$. Then for all $m = 1, 2, \ldots$ we obtain

$$\lim_{n\to\infty} (\mathsf{E}\langle D\xi_n; P_m(\overline{x}(g))h\rangle_H + \mathsf{E}\langle D\xi_n; P_{m-1}(\overline{x}(g)) g h \mathbf{1}_{X_0}\rangle_H)$$
$$= \mathsf{E}\langle D\xi; P_m(\overline{x}(g))h\rangle_H + \mathsf{E}\langle D\xi; P_{m-1}(\overline{x}(g)) g h \mathbf{1}_{X_0}\rangle_H.$$

Since $P_0 = 1$ and $\lim_{n\to\infty} \mathsf{E}\langle D\xi_n; P_0(\overline{x}(g))h\rangle_H = \mathsf{E}\langle D\xi; P_0(\overline{x}(g))h\rangle_H$ for all $h \in H \cap L^\infty(T \times X_0, \mathcal{B}, \nu)$, then we deduce by induction that

$$\lim_{n\to\infty} \mathsf{E}\langle D\xi_n; P_m(\overline{x}(g))h\rangle_H = \mathsf{E}\langle D\xi; P_m(\overline{x}(g))h\rangle_H.$$

For $n > m$ the expectation $\mathsf{E}\langle D\xi_n; P_m(\overline{x}(g))h\rangle_H$ is equal to

$$\mathsf{E}\left((m+1) I_m \left(\int_{T\times X} f_{m+1}(\cdot, t, x) h(t,x) \pi(dtdx)\right) P_m(\overline{x}(g))\right).$$

Hence the projection of $\langle D\xi_n; h\rangle_H$ on the mth chaos is equal to

$$(m+1) I_m \left(\int_{T\times X} f_{m+1}(\cdot, t, x) h(t,x) \pi(dtdx)\right).$$

If $\{e_i, i = 1, 2, \dots\}$ is an orthonormal basis of H then

$$\sum_{k=1}^{\infty} kk!\,||f_k||^2_{L^2((T\times X)^k)} = \mathsf{E}\sum_{k=0}^{\infty}\sum_{i=1}^{\infty}\left((k+1)I_k\left(\int_{T\times X} f_{k+1}(\cdot,t,x)e_i(t,x)\pi(dtdx)\right)\right)^2$$

$$= \sum_{i=1}^{\infty}\mathsf{E}\langle D\xi; e_i\rangle_h^2 = ||D\xi||^2_{L^2(\Omega;H)} < \infty,$$

which completes the proof of the proposition. □

Remark 4.12 *This proposition implies that the operator D is an annihilation operator on the Fock space on Hilbert space H.*

The equations (4.14) can be considered as a definition of the operator D. This approach was developed for pure jump Lévy process, the particular case of Poisson processes, the case of general Lévy process with no drift and the case of certain class of martingales in [2, 3, 13, 14, 20, 22].

Applying the lemma above one can easily obtain the action of the operator D^k via chaos expansion.

Proposition 4.13 *Let $\xi \in \mathbb{D}^{m,2}$ with a development (4.12). Then*

$$D^m_{t_1,x_1,\dots,t_m,x_m}\xi = \sum_{k=m}^{\infty} k(k-1)\cdots(k-m+1)I_{k-m}(f_k(\cdot,t_1,x_1,\dots,t_m,x_m))$$

and

$$\mathsf{E}(||D^m\xi||^2_{L^2((T\times X)^m)}) = \sum_{k=m}^{\infty}\frac{k!^2}{(k-m)!}\,||f_k||^2_{L^2((T\times X)^k)}\,. \tag{4.17}$$

Moreover $\xi \in \mathbb{D}^{m,2}$ if and only if the series in the right-hand side of (4.17) converges.

The following result is an evident modification of Proposition 1.2.5 from [16, p. 32] and it shows how to compute the derivative of a conditional expectation with respect to a σ-algebra generated by stochastic integrals. Let $A \in \mathcal{G}$. We will denote by $\mathcal{F}_A$ the complete σ-algebra generated by the random variables $\{L(B), B \subset A, B \in \mathcal{G}_0\}$.

Proposition 4.14 *Suppose that $\xi \in \mathbb{D}^{1,2}$, and $A \in \mathcal{G}$. Then $\mathsf{E}(\xi|\mathcal{F}_A) \in \mathbb{D}^{1,2}$ and we have*

$$D_{t,x}(\mathsf{E}(\xi|\mathcal{F}_A)) = \mathsf{E}(D_{t,x}\xi|\mathcal{F}_A)\mathbf{1}_A(t,x)$$

a.e. in $T \times X \times \Omega$.

Remark 4.15 *In particular, if ξ is $\mathcal{F}_A$-measurable and belongs to $\mathbb{D}^{1,2}$, then $D_{t,x}\xi = 0$ a.e. in $A^c \times \Omega$.*

5 The Skorohod integral

In this section we consider the adjoint of the operator D, and we will show that it coincides with the Skorohod integral [28] in the Gaussian case and with extended stochastic integrals introduced by Kabanov [11] in the pure jump Lévy case. See also [2, 3, 13, 22]. So it can be considered as a generalization of the stochastic integral. We will call it Skorohod integral and will establish the expression of it in terms of the chaos expansion as well as prove some of its properties.

We recall that the derivative operator D is a closed and unbounded operator defined on the dense subset $\mathbb{D}^{1,2}$ of $L^2(\Omega)$ with values in $L^2(\Omega; H)$.

Definition 5.1 *We denote by δ the adjoint of the operator D and will call it Skorohod integral.*

The operator δ is closed unbounded operator on $L^2(\Omega; H)$ with values in $L^2(\Omega)$ defined on Dom δ, where Dom δ is the set of processes $u \in L^2(\Omega; H)$ such that

$$\left| \mathsf{E} \int_{T \times X} D_{t,x} \xi u(t,x) \pi(dtdx) \right| \le c \, ||\xi||_{L^2(\Omega)}$$

for all $\xi \in \mathbb{D}^{1,2}$, where c is some constant depending on u.

If $u \in$ Dom δ, then $\delta(u)$ is the element of $L^2(\Omega)$ such that

$$\mathsf{E}(\xi \delta(u)) = \mathsf{E} \int_{T \times X} D_{t,x} \xi u(t,x) \pi(dtdx) \tag{5.1}$$

for any $\xi \in \mathbb{D}^{1,2}$.

The following proposition shows the behavior of δ in terms of the chaos expansion.

Proposition 5.2 *Let $u \in L^2(\Omega; H)$ with the expansion*

$$u(t,x) = \sum_{k=0}^{\infty} I_k(f_k(\cdot, t, x)). \tag{5.2}$$

Then $u \in$ Dom δ if and only if the series

$$\delta(u) = \sum_{k=0}^{\infty} I_{k+1}(\tilde{f}_k) \tag{5.3}$$

converges in $L^2(\Omega)$.

Recall that $\tilde{f}_k$ is a symmetrization of f_k in all its variables is given by

$$\tilde{f}_k(t_1, x_1, \dots, t_k, x_k, t, x) = \frac{1}{k+1} (f_k(t_1, x_1, \dots, t_k, x_k, t, x)$$

$$+ \sum_{i=1}^{k} f_k(t_1, x_1, \dots, t_{i-1}, x_{i-1}, t, x, t_{i+1}, x_{i+1}, \dots, t_i, x_i)).$$

Proof. The proof is the same as in the Gaussian case (see, e.g., [16, Prop. 1.3.1, p. 36]).

□

Remark 5.3 *It follows from Proposition 5.2 that the operator δ coincides with Skorohod integral in the Gaussian case and with extended stochastic integral introduced by Kabanov for pure jump Lévy processes (see, e.g., [28, 11, 16, 2, 3, 13, 22]).*

It follows from proposition above that Dom δ is the subspace of $L^2(\Omega)$ formed by the processes that satisfy the following condition:

$$\sum_{k=1}^{\infty}(k+1)!||\tilde{f}_k||^2_{L^2((T\times X)^{k+1})} < \infty. \tag{5.4}$$

If $u \in$ Dom δ, then the sum of the series (5.4) is equal to $\mathsf{E}\delta(u)^2$.

Note that the Skorohod integral is a linear operator and has zero mean, e.g., $\mathsf{E}(\delta(u)) = 0$ if $u \in$ Dom δ. The following statements prove some properties of δ.

Proposition 5.4 *Suppose that u is a Skorohod integrable process. Let $\xi \in \mathbb{D}^{1,2}$ such that $\mathsf{E}(\int_{T\times X}(\xi^2 + (D_{t,x}\xi)^2\mathbf{1}_{X_0})u(t,x)^2\pi(dtdx)) < \infty$. Then it holds that*

$$\delta((\xi + \mathbf{1}_{X_0}D\xi)u) = \xi\delta(u) - \int_{T\times X}(D_{t,x}\xi)u(t,x)\pi(dtdx), \tag{5.5}$$

provided that one of the two sides of the equality (5.5) exists.

Proof. Let $\eta \in \mathcal{S}$ be a smooth random variables. Then by the product rule (4.11) and by the duality relation (5.1), we get

$$\mathsf{E}(\int_{T\times X}(D_{t,x}\eta)(\xi + \mathbf{1}_{X_0}(x)D_{t,x}\xi)u(t,x)\pi(dtdx)) = \int_{T\times X}\mathsf{E}(u(t,x)(D_{t,x}(\xi\eta) - \eta D_{t,x}\xi))\pi(dtdx)$$

$$= \mathsf{E}\left(\eta(\xi\delta(u) - \int_{T\times X}(D_{t,x}\xi)u(t,x)\pi(dtdx))\right),$$

and the result follows. □

As in the Gaussian case in order to prove some other properties of Skorohod integral we will define a class of processes contained in Dom δ (see [16]).

Definition 5.5 *Let $\mathbb{L}^{1,2}$ denote the class of processes $u \in L^2(T\times X\times\Omega)$ such that $u(t,x) \in \mathbb{D}^{1,2}$ for almost all (t,x), and there exists a measurable version of the multi-process $D_{t,x}u(s,y)$ satisfying $\mathsf{E}\int_{T\times X}\int_{T\times X}(D_{t,x}u(s,y))^2\pi(dtdx)\pi(dsdy) < \infty$.*

If the process u has the expansion (5.2), then $u \in \mathbb{L}^{1,2}$ if and only if the series

$$\int_{T\times X}\int_{T\times X}\mathsf{E}\left(\sum_{k=1}^{\infty}kI_{k-1}(f_k(\cdot,t,x,s,y))\right)^2\pi(dtdx)\pi(dsdy) = \sum_{k=1}^{\infty}kk!\,||f_k||^2_{L^2((T\times X)^{k+1})}$$

converges.

Since $||\tilde{f}_k||_{L^2((T\times X)^{k+1})} \le ||f_k||_{L^2((T\times X)^{k+1})}$ then from (5.4) we deduce that $\mathbb{L}^{1,2} \subset$ Dom δ.

The proofs of the following propositions are the same as in the Gaussian case (see, for instance [16, pp. 38 - 40]).

Proposition 5.6 *Suppose that $u \in \mathbb{L}^{1,2}$ and for almost all $(t,x) \in T \times X$ the two-parameter process $\{D_{t,x}u(s,y), (s,y) \in T \times X\}$ is Skorohod integrable, and there exists a version of the process $\{\delta(D_{t,x}u(\cdot,\cdot)), (t,x) \in T \times X\}$ which belongs to $L^2(T \times X \times \Omega)$. Then $\delta(u) \in \mathbb{D}^{1,2}$ and we have*

$$D_{t,x}\delta(u) = u(t,x) + \delta(D_{t,x}u(\cdot,\cdot)). \tag{5.6}$$

Proposition 5.7 *Suppose that $u \in \mathbb{L}^{1,2}$ and $v \in \mathbb{L}^{1,2}$. Then we have*

$$\mathsf{E}(\delta(u)\delta(v)) = \int_{T\times X} \mathsf{E}(u(t,x)v(t,x))\pi(dtdx)$$
$$+ \int_{T\times X}\int_{T\times X} \mathsf{E}(D_{s,y}u(t,x)D_{t,x}v(s,y))\pi(dtdx)\pi(dsdy). \tag{5.7}$$

Now we will show that the operator δ is an extension of the Itô integral. Let $B = \{B^i(t); t \geq 0, i = 1, \dots, d\}$ be a d-dimensional Brownian motion, and $\widetilde{N}(dtdx)$ is a compensated Poisson measure on the Borel σ-algebra of $\mathbb{R}^d \setminus \{0\}$, with characteristic measure $\nu(dtdx) = dt\beta(dx)$, where the measure β such that $\int_{\mathbb{R}^d\setminus\{0\}}(|x|^2 \wedge 1)\beta(dx) < \infty$. For each $t \geq 0$ we will denote by $\mathcal{F}_t$ the σ-algebra generated by the random variables $\{B^i(s),\ \widetilde{N}((0;s] \times A);\ 0 \leq s \leq t,\ i = 1, \dots, d,\ A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),\ \beta(A) < \infty\}$ and the null sets of $\mathcal{F}$. Suppose that T, X_0 and the measures μ and ν as in Example 2.3. We denote by L_p^2 the subset of $L^2(\Omega; H)$ formed by $(\mathcal{F}_t)$-predictable processes.

Proposition 5.8 $L_p^2 \subset \mathrm{Dom}\,\delta$, *and the restriction of the operator δ to the space L_p^2 coincides with the usual stochastic integral, that is*

$$\delta(u) = \sum_{i=1}^{d} \int_0^\infty u^i(t,0)dB^i(t) + \int_0^\infty \int_{\mathbb{R}^d\setminus\{0\}} u(t,x)\widetilde{N}(dtdx).$$

Proof. The proof follows along the same line as the proof for the Gaussian case (see, e.g., [16, Prop. 1.3.4, pp. 41-42]) and, therefore, is omitted. □

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