Dept. of Math. Univ. of Oslo
Pure Mathematics No. 5
ISSN 0806–2439 March 2004

DEFORMING SYZYGIES OF LIFTABLE MODULES AND GENERALISED KNÖRRER FUNCTORS

RUNAR ILE

Abstract. Several maps of deformation functors of modules are given which generalise the maps induced by the Knörrer functors. These maps become isomorphisms after introducing linear equations in the target functor. In this manner the obstruction ideal for one module occurs as obstruction ideals for other modules over other rings. In particular we obtain a map to the deformation functor of the maximal Cohen-Macaulay approximation over a quotient ring defined by a regular sequence.

1. Introduction

H. Knörrer introduced in 1987 a functor H which gives an equivalence between the stable category of maximal Cohen-Macaulay (MCM) modules of the hypersurface singularities f and $f+uv$; [18]. In the author's master's thesis he proved that H induces an isomorphism on deformation functors of MCM modules; [13]. The present article is the result of an attempt to generalise Knörrer's functor such that one obtains an interesting map of deformation functors of modules over different rings.

The main results have the following form (which we call the standard result). There is a natural map $\mathrm{Def}^{A_1}_{M_1} \to \mathrm{Def}^{A_2}_{M_2}$ of deformation functors of A_i-modules M_i such that when restricting the target functor to deformations with tangential directions coming from the source functor, an isomorphism is obtained. If source and target both have versal families, this corresponds to an embedding of the source versal deformation space in the target versal deformation space, where the embedding is defined by "linear" equations in the ambient space.

The main vehicle for producing the association $(A_1, M_1) \mapsto (A_2, M_2)$ is the syzygy. The syzygy operation is not a functor of modules, however it gives well defined maps for the Ext^i if $i > 0$ and also a well defined map of deformation functors. In Theorem 1 there is a surjection $A_2 \twoheadrightarrow A_1$ and M_2 is an iterated syzygy module of M_1 as A_2-module. The number of iterations equals the length of the regular sequence $I \subset A_2$ which defines A_1. The last condition is that there should exist a (non-flat) *lifting* of M_1 to A_2/I^2. As an application of Theorem 1 the standard result is obtained in Corollary 3 where M_2 is the maximal Cohen-Macaulay approximation of M_1 as A_2-module, and M_1 is a MCM A_1-module.

Theorem 2 introduces a flat A_1-algebra B with surjections $A_2 \twoheadrightarrow B \twoheadrightarrow A_1$. Then M_2 is the n^{th} syzygy of $M_1 \otimes_{A_1} B$ as A_2-module where $n = \mathrm{pdim}_{A_2} B$. The existence of a lifting is implied by the condition that the equations which define A_2 should be perturbations of the equations which define A_1 such that the parameter-monomials

2000 *Mathematics Subject Classification.* Primary 14B12, 13D02; Secondary 13D10, 13C14.
Key words and phrases. Versal deformation space, obstruction class, maximal Cohen-Macaulay approximation,
Acknowledgement. The author is grateful for partial financial support from RCN's Strategic University Program in Pure Mathematics at the Dept. of Mathematics, University of Oslo (No 154077/420). This article is partly based on the author's 2001 Dr. Phil. Thesis at the same institution.

only occur with minimal degree at least two, as in $f(\underline{x}) \mapsto F = f(\underline{x}) + uv \cdot g(\underline{x})$. In the proof a free resolution is constructed from the "sum" tensor product of "Eisenbud systems", and the degree $\geqslant 2$-assumption implies that many differentials vanish at the central fibre. Theorem 2 gives the standard result in a fairly wide class of situations beyond Theorem 1, in particular it covers Knörrer's H-functor for which the result holds without the tangential restriction, see Theorem 3.

As an application of the standard results we show that if M_1 is smoothable so is M_2, see Corollary 5.

Some definitions used throughout the article: *A local k-algebra A* is (possibly the Henselisation of) a local k-algebra essentially of finite type where k is a field. *An A-module M* is (usually) a finitely generated A-module. For a Noetherian k-algebra A, let A_S be the Henselisation of $A \otimes_k S$ in the ideal $A \otimes_k \mathfrak{m}_S$ where S is an object in the category Hens_k of local (in particular Noetherian), Henselian k-algebras with residue field k. A *deformation* of M to S is an A_S-module M_S , flat as S-module, together with an A_S-linear map $\pi : M_S \to M$ inducing an isomorphism $\pi \otimes_{A_S} k : M_S \otimes_S k \xrightarrow{\simeq} M$. The *deformation functor* $\mathrm{Def}^A_M : \mathsf{Hens}_k \to \mathsf{Sets}$ associates to S the set of equivalence classes of deformations M_S of M to S. Two deformations are equivalent if they are isomorphic over M, i.e the isomorphism is compatible with the πs. Maps are induced by tensorisation.

Some references on deformation theory of modules: In [26, 2.4] H. von Essen shows that the existence of a *versal family* (R, M_R) for Def^A_M in the case A is a local k-algebra and M is an A-module which is locally free on the complement of the closed point, follows from R. Elkik's [7, Thm. 3] and M. Artin's [2, 3.3], see also [17, 2.6]. A formally versal formal family exists quit generally if $\mathrm{Ext}^1_A(M, M)$ has finite k-dimension, see [23]. A. Siqveland gives the degeneracy diagram of torsion free rank 1-modules on the $\mathbf{E_6}$-singularity by explicit calculation of the Massey products in [25], and extends his result to the compactified Jacobian in [24], elaborating A. Laudal's setup in [19]. The author develops a change of rings formalism for the obstruction theory in [12], and proves a non-trivial dimension estimate in the case of rank 1 MCM modules on hypersurface singularities in [15]. A. Ishii gives a stratification of the versal deformation space of a reflexive module on a rational surface singularity in [17] by proving representability of certain moduli functors of (semi-)full sheaves on the resolution of the singularity. T. S. Gustavsen and the author find these moduli spaces in the case of a cone over a rational normal curve, see [10], and in [11] they prove irreducibility of the versal deformation spaces in the case of rational double points.

Most of the results in this article will suitably adapted hold for the graded case as well.

2. Deforming higher syzygies of a liftable module

Theorem 1 gives the standard result for the n^{th} iterated syzygy of a module liftable along a regular sequence of length n. In Lemma 4 cohomology conditions are given which imply that the syzygy map gives an isomorphism of deformation functors.

The following two lemmas and definitions are vital prerequisites for the rest of the article.

Lemma 1. *Suppose A is a local k-algebra and M a finitely generated A-module, then there is a map*

$$\mathrm{Def}^A_M \longrightarrow \mathrm{Def}^A_{\Omega_A M}$$

defined by sending $\pi : M_R \to M$ *to* $\Omega_A\pi : \Omega_A M_R \to \Omega_A M$. *The map is functorial for isomorphisms in* M *and in particular independent of the choice of minimal resolution.*

Proof. Fix a minimal A-free resolution F of M, in particular $\Omega_A M \subseteq F_0$. If $\pi : M_S \to M$ is a deformation of M to S, let F^S be a minimal A_S-free resolution of M_S. Then there is lifting $\pi. : F^S \to F$ of π, by S-flatness of M_S one has an isomorphism $\pi.\otimes_S k : F^S \otimes_S k \xrightarrow{\simeq} F$, and in particular $\pi.$ is surjective. Define $[\Omega\pi : \Omega M_S \to \Omega M] \in \mathrm{Def}^A_M(S)$ as the equivalence class of π_0 restricted to $\Omega M_S := \Omega_{A_S} M_S$. Remark that ΩM_S is S-flat since $\mathrm{Tor}^i_S(\Omega M_S, -) \cong \mathrm{Tor}^{i+1}_S(M_S, -) = 0$ for $i > 0$. Any other lifting of π has the form $\pi_0' = \pi_0 + d_1 h$ where $h : F^S_0 \to F_1$. Let $\tilde{h} : F^S_0 \to F^S_1$ be a lifting of h. Then $[\Omega\pi] = [\Omega\pi']$ since $\pi_0(\mathrm{id} + d^S_1\tilde{h}) = \pi_0 + (\pi_0 d^S_1)\tilde{h} = \pi_0 + d_1(\pi_1\tilde{h}) = \pi_0 + d_1 h = \pi_0'$.

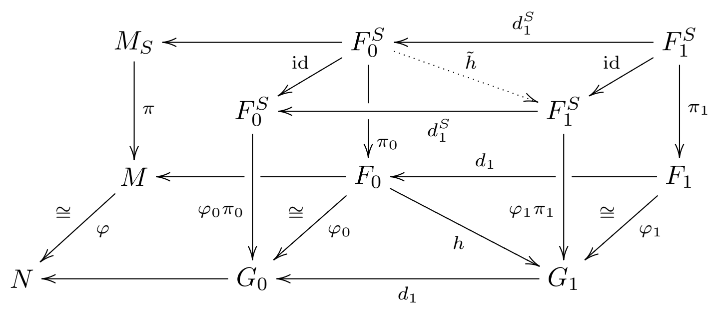
If $\rho. : G^S \to F$ is another A_S-free resolution of $\pi : M_S \to M$, then there is a lifting $\varphi^S_. : F^S \to G^S$ of the identity of F, i.e. $\rho^S_. \varphi^S_. = \pi^S_.$. Remark that $\varphi^S_.$ is an isomorphism. Then φ^S_0 induces an isomorphism $\Omega^F M_S \xrightarrow{\simeq} \Omega^G M_S$ over ΩM and $[\Omega^F M_S] = [\Omega^G M_S] \in \mathrm{Def}^A_{\Omega M}$. It follows that the syzygy operation on deformations factors through the equivalence: If $\varphi^S : M_S \to M'_S$ with $\pi'\varphi^S = \pi$, and $\varepsilon : F^S \to M_S$ is a resolution of M_S above F, then $\varphi^S\varepsilon : F^S \to M'_S$ is a resolution of M'_S above F and id_{F^S} is a comparison map over F commuting with φ^S.

Given an A-module isomorphism $\varphi : M \xrightarrow{\simeq} N$ and minimal A-free resolutions $F \to M$ and $G \to N$. Then there is a map of complexes $\varphi. : F \to G$ above φ. Let $\Omega\varphi$ be the map $\Omega M \to \Omega N$ induced by $\varphi_0 : F_0 \to G_0$. Let $(\Omega\varphi)_* : \mathrm{Def}^A_{\Omega M} \to \mathrm{Def}^A_{\Omega N}$ be given by $\pi_0 : \Omega M_S \to \Omega M \mapsto \varphi_0\pi_0 : \Omega M_S \to \Omega N$. The diagram

(1)
$$\begin{array}{ccc} \mathrm{Def}_M & \longrightarrow & \mathrm{Def}_{\Omega M} \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle (\Omega\varphi)_*} \\ \mathrm{Def}_N & \longrightarrow & \mathrm{Def}_{\Omega N} \end{array}$$

commutes: If $\pi. : F^S \to F$ lifts π, we choose $\varphi.\pi.$ as the lifting of $\varphi_*(\pi)$. Then $\Omega(\varphi_0\pi_0) = (\varphi\pi)_0 = \varphi_0\pi_0 = (\varphi_0)_*\pi_0 = (\Omega\varphi)_*(\Omega\pi)$. Moreover; $(\Omega\varphi)_*$ is unique, independent of the choice of chain map $\varphi.$. If $\varphi_0' = \varphi_0 + d_1 h$ and $\tilde{h} : F^S_0 \to F^S_1$ lifts h, then one calculates $\varphi_0\pi_0(\mathrm{id} + d^S_1\tilde{h}) = (\varphi_0 + d_1 h)\pi_0$ and thus $(\varphi_0)_*\pi_{0|\Omega M_S} \to \Omega N$ and $(\varphi_0 + d_1 h)_*\pi_{0|\Omega M_S} \to \Omega N$ are equivalent liftings of ΩN.

The situation is summarised in the following diagram:



In particular we have proved that $\Omega : \mathrm{Def}^A_M \to \mathrm{Def}^A_{\Omega M}$ is independent up to a canonical isomorphism of the chosen resolution F of M. □

Lemma 2. *Suppose* $C \to A$ *is a map of* k*-algebras and* N *is a finitely generated* C*-module, let* $M = N \otimes_C A$. *If* $\mathrm{Tor}^C_1(N, A) = 0$ *then there is a natural map* $\mathrm{Def}^C_N \to \mathrm{Def}^A_M$ *given by* $[N_S] \mapsto [N_S \otimes_C A]$.

Proof. The map respects the equivalence relation, we have to show that $M_S := N_S \otimes_C A$ is S-flat. By the local criterion of flatness, cf. [20, 22.3], it is sufficient to show that $\mathrm{Tor}_1^S(M_S, k) = 0$. We claim that $\mathrm{Tor}_1^C(N_S, A) = 0$ which follows by inspecting the commutative diagram (for $i = 1$)

$$\begin{array}{ccccccc} \mathrm{Tor}_i^C(N_S, A)\otimes_S S^e & \longrightarrow & \mathrm{Tor}_i^C(N_S, A) & \longrightarrow & \mathrm{Tor}_i^C(N_S, A)\otimes_S k & \longrightarrow & 0 \\ \downarrow\cong & & \downarrow = & & \downarrow \overline{\varphi}_i & & \\ \mathrm{Tor}_i^C(N_S\otimes_S S^e, A) & \longrightarrow & \mathrm{Tor}_i^C(N_S, A) & \longrightarrow & \mathrm{Tor}_i^C(N, A) & & \end{array} \tag{2}$$

with exact upper row obtained from an S-free presentation of the residue field k; $S^e \to S \to k \to 0$. We find that $\overline{\varphi}_i$ is surjective if and only if it is an isomorphism. In particular; $\mathrm{Tor}_i^C(N_S, A) = 0$ if $\mathrm{Tor}_i^C(N, A) = 0$.

If $F^S \twoheadrightarrow N_S$ is a C_S-free resolution of N_S, then $F^S\otimes_C A \twoheadrightarrow M_S$ is an A_S-free complex without homology in degree less than or equal to one since $\mathrm{Tor}_1^C(N_S, A) = 0$. Thus $\mathrm{Tor}_1^S(M_S, k) = \mathrm{H}_1(F^S\otimes_C A\otimes_S k) \cong \mathrm{H}_1(F\otimes_C A) = 0$ since $F = F^S\otimes_S k$ is a C-free resolution of N, and the assumption. □

Definition 1. Suppose $C \to A$ is a map of rings with kernel I and M is an A module. Then M has a *lifting* to C if there is a C-module N and a C-linear map $\pi : N \to M$ such that $\mathrm{Tor}_1^C(N, A) = 0$ and $\pi\otimes A : N\otimes_C A \to M$ is an isomorphism.

Recall that if we restrict attention to deformations M_S with $\mathfrak{m}_S^2 = 0$ then there is a universal family $M_1 \in \mathrm{Def}_M^A(H_1)$ where $H_1 = k[\mathrm{Ext}_A^1(M, M)^*] = k \oplus \mathrm{Ext}_A^1(M, M)^*$ and the Zariski tangent space $\mathrm{Def}_M^A(k[\varepsilon]) \cong \mathrm{Ext}_A^1(\mathrm{M}, \mathrm{M})$ is naturally a k-vector space (if $\mathrm{Ext}_A^1(M, M)$ is of countable and not finite k-dimension, one has to introduce a topology on the vector space, take the continuous dual, and the universal tangential family becomes a pro-object, cf. [12]). The universal tangential family is given by the universal extension

$$M_1 : \quad 0 \longrightarrow M\otimes_k \mathrm{Ext}_A^1(M, M)^* \longrightarrow M_1 \xrightarrow{\pi_1} M \longrightarrow 0\,. \tag{3}$$

Since $S = k\oplus\mathfrak{m}_S$, M_S is the pushout induced by the k-linear map $\mathrm{Ext}_A^1(M, M)^* \to \mathfrak{m}_S$ corresponding to $H_1 \to S$. One has $\mathrm{Def}_M^A(S) \cong \mathrm{Ext}_A^1(M, M)\otimes_k\mathfrak{m}_S$ canonically, see [23, 2.10].

Definition 2. Let V be a k-sub-vector space in $\mathrm{Def}_M^A(k[\varepsilon]) \cong \mathrm{Ext}_A^1(M, M)$. Then $\mathrm{Def}_{(M,V)}^A$ is the sub-functor of Def_M^A of deformations M_S such that $[M_S\otimes_{A_S} A_{S_1}] \in V\otimes_k\mathfrak{m}_{S_1}$ where $S_1 = S/\mathfrak{m}_S^2$.

It follows that $[M_S\otimes_{A_S} A_{S'}] \in V\otimes_k\mathfrak{m}_{S'}$ for all $S \to S'$ with $\mathfrak{m}_{S'}^2 = 0$. Remark that $\mathrm{Def}_{(M,V)}^A$ satisfies condition (H_2) and therefore also condition (H_1) in [23, 2.11].

Theorem 1. *Let $\pi : C \to A$ be a surjective map of local k-algebras. Set $I = \ker\pi$, and assume I is generated by a regular sequence of length n and M is a finitely generated A-module which has a lifting to C/I^2. Then there is an isomorphism of deformation functors*

$$\sigma : \mathrm{Def}_M^A \xrightarrow{\cong} \mathrm{Def}_{(\Omega_C^n M, V)}^C$$

where $V = \mathrm{im}\,\mathrm{Def}_M^A(k[\varepsilon])$.

Example 1. If L is any A-module, set $N = \Omega_C^n L$. Then $M = N\otimes_C A$ satisfies the conditions of Theorem 1 since $\mathrm{Tor}_1^C(N, A) = \mathrm{Tor}_{n+1}^C(L, A) = 0$ implies that $\mathrm{Tor}_1^{C/I^2}(N\otimes_C C/I^2, A) = 0$.

With the notation in Definition 1 we furthermore have:

Lemma 3. *Suppose $C \to A$ is surjective. There exists an obstruction class*

$$\mathrm{o}(C/I^2, M) \in \operatorname{Ext}^2_A(M, M \otimes_A I/I^2) \tag{4}$$

such that $\mathrm{o}(C/I^2, M) = 0$ if and only if M has a lifting to C/I^2. If $C \to A$ in addition is a map of local rings and I is generated by a regular sequence, then for any $n \geqslant 0$ there is an n^{th} syzygy map

$$\operatorname{Ext}^2_A(M, M \otimes_A I/I^2) \longrightarrow \operatorname{Ext}^2_A(\Omega^n_A M, \Omega^n_A M \otimes_A I/I^2) \tag{5}$$

which takes $\mathrm{o}(C/I^2, M)$ to $\mathrm{o}(C/I^2, \Omega^n_A M)$, and in particular

$$\mathrm{o}(C/I^2, M) = 0 \quad \Longrightarrow \quad \mathrm{o}(C/I^2, \Omega^n_A M) = 0\,.$$

Proof. [16] contains the first part, in [12, Thm. 1] a representative in the Yoneda complex is given, only the construction is repeated here. If $F \twoheadrightarrow M$ is an A-free resolution of M with differential d, lift F and d to a map $\widetilde{d}$ of a graded module $\widetilde{F}$ which in each degree is C/I^2-free such that $(\widetilde{F}, \widetilde{d}) \otimes_{C/I^2} A = (F, d)$. Since there is short exact sequence of graded modules commuting with "differentials"; $0 \to F \otimes_A I/I^2 \xrightarrow{\iota} \widetilde{F} \xrightarrow{\pi} F \to 0$, we get that $(\widetilde{d})^2$ is induced by a map $\sigma \in \operatorname{Hom}^2_A(F, F \otimes_A I/I^2)$ which is a cocycle: $\iota\partial(\sigma)\pi = \iota(d \otimes I)\sigma\pi - \iota\sigma d\pi = \widetilde{d}\iota(\sigma\pi) - (\iota\sigma\pi)\widetilde{d} = \widetilde{d}(\widetilde{d}^2) - (\widetilde{d}^2)\widetilde{d} = 0$. Via the map $\eta : F \otimes I/I^2 \twoheadrightarrow M \otimes I/I^2$ we get a class $\mathrm{o}(C/I^2, M) = [\eta\sigma_2] \in \mathrm{H}^2 \operatorname{Hom}_A(F, M \otimes_A I/I^2) = \operatorname{Ext}^2_A(M, M \otimes_A I/I^2)$ which is independent of the choices involved.

For all $i > 0$ there are quite generally natural maps

$$\omega^i : \operatorname{Ext}^i_A(M, M) \to \operatorname{Ext}^i_A(\Omega_A M, \Omega_A M) \tag{6}$$

obtained by composing the connecting map $\operatorname{Ext}^i_A(M, M) \to \operatorname{Ext}^{i+1}_A(M, \Omega_A M)$ with the inverse of the connecting isomorphism $\operatorname{Ext}^i_A(\Omega_A M, \Omega_A M) \xrightarrow{\simeq} \operatorname{Ext}^{i+1}_A(M, \Omega_A M)$ which are obtained by applying $\operatorname{Hom}_A(M, -)$ and $\operatorname{Hom}_A(-, M)$ to the defining sequence $0 \to \Omega_A M \to F_0 \to M \to 0$. If I is generated by a regular sequence then I/I^2 is A-free of finite rank and $\operatorname{Ext}^2_A(M, M \otimes_A I/I^2) \cong \operatorname{Ext}^2_A(M, M) \otimes_A I/I^2$. The map in the lemma is ω^2 iterated n times tensored with I/I^2. In the Yoneda complex this is simply to chop off the n first maps, clearly σ_{n+2} composed with $F_n \otimes_A I/I^2 \twoheadrightarrow \Omega^n_A M \otimes_A I/I^2$ represents $\mathrm{o}(C/I^2, \Omega^n_A M)$. □

Remark 1. Let M_S be a deformation of M in $\operatorname{Def}^A_M(S)$ and $\pi : R \to S$ a *small* surjection (i.e. $\mathfrak{m}_R \cdot \ker \pi = 0$), then there is a an obstruction class $\mathrm{o}_A(\pi, M_S) \in \operatorname{Ext}^2_A(M, M) \otimes_k \ker \pi$ which vanish if and only if there exists a deformation M_R of M to R such that $M_R \otimes_R S$ is equivalent to M_S, cf. Theorem 1 and Remark 4 in [12]. Since $- \otimes_k \ker \pi$ may be taken outside the Ext^2, it follows analogously to the argument in Lemma 3 that $\omega^2 \otimes \mathrm{id}_{\ker \pi}(\mathrm{o}_A(\pi, M_S)) = \mathrm{o}_A(\pi, \Omega_{A_S} M_S) \in \operatorname{Ext}^2_A(\Omega_A M, \Omega_A M) \otimes_k \ker \pi$.

Proof of Theorem 1. A deformation of M as A-module is also a deformation of M as C-module, hence there is map $\operatorname{Def}^A_M \to \operatorname{Def}^C_M$. By Lemma 1 there is a map $\operatorname{Def}^C_M \to \operatorname{Def}^C_{\Omega^n_C M}$, and by Lemma 2 there is a map $\operatorname{Def}^C_{\Omega^n_C M} \to \operatorname{Def}^A_{\Omega^n_C M \otimes_C A}$ since $\operatorname{Tor}^C_1(\Omega^n_C M, A) = \operatorname{Tor}^C_{n+1}(M, A) = 0$. The composition $\operatorname{Def}^A_M \to \operatorname{Def}^C_{\Omega^n_C M}$ factors through $\sigma : \operatorname{Def}^A_M \to \operatorname{Def}^C_{(\Omega^n_C M, V)}$ via the inclusion. By [4, 3.6] $\Omega^n_C M \otimes_C A$ containes M as a direct summand if M is liftable to C/I^2 with the additional assumption that $\operatorname{Tor}^{C/I^2}_i(N, A) = 0$ for all $i > 0$. However we claim that $\operatorname{Tor}^{C/I^2}_1(N, A) = 0 \Rightarrow \operatorname{Tor}^{C/I^2}_i(N, A) = 0$ for all $i > 0$. From the proof of Lemma 3 we see that since I/I^2 is A-free we have $\mathrm{o}(C/I^2, M) = [\sigma] \in \mathrm{H}^2 \operatorname{Hom}_A(F, F) \otimes_A I/I^2 = \operatorname{Ext}^2_A(M, M) \otimes_A I/I^2$. Since $\mathrm{o}(C/I^2, M) = 0$, there is a $\tau \in \operatorname{Hom}^1_A(F, F) \otimes_A I/I^2$ with $\partial\tau = \sigma$. Adjusting $\tilde{d}$ with τ gives a differential $\tilde{d}'$ on $\widetilde{F}$, i.e. $(\tilde{d}')^2 = 0$, hence

$0 \to F \otimes_A I/I^2 \xrightarrow{\iota} \widetilde{F} \xrightarrow{\pi} F \to 0$ is a short exact sequence of complexes and by the long exact homology sequence, $(\widetilde{F}, \tilde{d}')$ is a resolution of N. Tensoring $(\widetilde{F}, \tilde{d}')$ by A gives F and hence $\operatorname{Tor}_i^{C/I^2}(N, A) = 0$ for all $i > 0$. We have obtained a natural map

$$\tau : \operatorname{Def}_M^A \to \operatorname{Def}^A_{(M \oplus Y, V')}; \quad M_S \mapsto \tau M_S = \Omega^n_{C_S} M_S \otimes_{C_S} A_S \tag{7}$$

where $\Omega^n_C M \otimes_C A \cong M \oplus Y$ for some finitely generated A-module Y, and $V' = \operatorname{im}(\operatorname{id}, \eta^1)$ where

$$(\operatorname{id}, \eta^i) : \operatorname{Ext}^i_A(M, M) \hookrightarrow \operatorname{Ext}^i_A(M \oplus Y, M \oplus Y) \quad i > 0 \tag{8}$$

is the composition of $\operatorname{Ext}^i_A(M, M) \to \operatorname{Ext}^i_C(M, M)$, the n^{th} iterate $(\omega^i)^n$ of (6), and the natural map $\operatorname{Ext}^i_C(\Omega, \Omega) \to \operatorname{Ext}^i_A(\overline{\Omega}, \overline{\Omega})$ obtained by tensorisation and the collapse of the spectral sequence $\mathrm{E}_2^{pq} = \operatorname{Ext}^p_A(\operatorname{Tor}^C_q(\Omega, A), \overline{\Omega}) \Rightarrow \operatorname{Ext}^{p+q}_C(\Omega, \overline{\Omega})$ (where $\Omega = \Omega^n_C M$ and $\overline{\Omega} = \Omega \otimes_C A$).

For formal smoothness of σ, let σ also denote the natural map $\operatorname{Ext}^i_A(M, M) \to \operatorname{Ext}^i_C(\Omega, \Omega)$ (for $i > 0$). From $\operatorname{Tor}^C_i(\sigma M, A) = 0$ for all $i > 0$, it follows that $(\operatorname{id}, \eta^2)(\mathrm{o}_A(\pi, M_S)) = \mathrm{o}_C(\pi, \sigma M_S) \otimes_C A = \mathrm{o}_A(\pi, \tau M_S)$. Since $(\operatorname{id}, \eta^2)$ is injective, $\mathrm{o}_C(\pi, \sigma M_S) = 0 \Rightarrow \mathrm{o}_A(\pi, M_S) = 0$.

Given a deformation L_S in $\operatorname{Def}^C_{(\Omega^n_C M, V)}(S)$ there in particular exists a deformation M_i of M to $S_i = S/\mathfrak{m}_S^{i+1}$ and an isomorphism $\varphi_i : \sigma M_i \xrightarrow{\simeq} L_i$ for all $i > 0$. We show that the isomorphisms φ_i can be chosen compatible. Suppose compatibility is achieved up to φ_{i-1}. The "difference" between L_i and the via $\sigma M_i \to \sigma M_{i-1}$ composed with φ_{i-1} induced deformation σM_i of L_{i-1} is an element $\sigma(\xi) \in \operatorname{Ext}^1_A(\sigma M, \sigma M) \otimes_k J$ where $J = \ker(S_i \to S_{i-1})$, as follows by the definition of $\operatorname{Def}^C_{(\Omega^n_C M, V)}$, see [23, 2.17] and [12, Thm. 1]. Then we "add" $\xi \in \operatorname{Ext}^1_A(M, M) \otimes_k J$ to the deformation M_i of M_{i-1} to obtain a deformation M'_i such that the induced deformation $\sigma M'_i$ of L_{i-1} is equivalent to L_i, i.e. there exists an isomorphism $\varphi'_i : \sigma M'_i \xrightarrow{\simeq} L_i$ compatible with φ_{i-1}. By induction and [20, 22.1] we get an $\hat{S}$-flat $\hat{A}_{\hat{S}} := A \hat{\otimes}_k \hat{S}$-module $\hat{M}_{\hat{S}}$ and an isomorphism $\hat{\varphi} : \Omega^n_{\hat{C}_{\hat{S}}} \hat{M}_{\hat{S}} \xrightarrow{\simeq} \hat{L}_{\hat{S}}$.

Let $L = L_S \otimes_{C_S} A_S$, and let $\hat{L} = L \otimes_{A_S} \hat{A}_S$ be the completion of L. Via the isomorphism induced from $\hat{\varphi}$ and the splitting $\hat{M}_{\hat{S}} \oplus Y = \Omega^n_{\hat{C}_{\hat{S}}} \hat{M}_{\hat{S}} \otimes_{\hat{C}_{\hat{S}}} \hat{A}_{\hat{S}}$, there is a map $L \to \hat{M}_{\hat{S}}$. Let M_S be defined as the image of L under this map. Then M_S is a finitely generated A_S-module, and the completion of M_S is $\hat{M}_{\hat{S}}$. From [20, 7.11] it follows that there exists a map $\varphi_S : \sigma M_S \to L_S$ inducing φ_1. By [20, 22.5] φ is injective and $\operatorname{coker} \varphi$ is S-flat. Since $\varphi \otimes_S k$ is an isomorphism, it follows that $\operatorname{coker} \varphi = 0$ and φ is an isomorphism. Hence σM_S is equivalent to the deformation L_S and σ is surjective.

To get injectivity of σ we prove injectivity of τ. Assume $\varphi : \tau M_S \xrightarrow{\simeq} \tau M'_S$. Restricting φ to the direct summand M_S and composing with the projection $\tau M'_S \to M'_S$ gives a map $\psi : M_S \to M'_S$ compatible with the structure maps to M. By [20, 22.5] ψ is an isomorphism as above, hence τ is injective and so is σ. □

Remark 2. One similarly shows that τ in (7) is an isomorphism. Moreover; we have maps

$$\operatorname{Def}^A_M \xrightarrow{\alpha} \operatorname{Def}^C_{(M, V_1)} \to \operatorname{Def}^C_{(\Omega^n_C M, V_2)} \to \operatorname{Def}^A_{(\Omega^n_C M \otimes_C A, V_3)} \xrightarrow{\beta} \operatorname{Def}^A_M \tag{9}$$

(where the V_i are the images of $\operatorname{Def}^A_M(k[\varepsilon])$) which all except β exist without the condition $\mathrm{o}(A/I^2, M) = 0$ in Theorem 1. Let M and M' be A-modules and $A = C/I$ any quotient ring. In [12] an obstruction theory for Def^A_M as a sub-functor of

Def^C_M is given. Let M_S be a deformation of M as A-module. If the obstruction class o_C for deforming M_S along a small surjection $R \to S$ as C-module is zero, there exists a secondary class o_I which vanish if and only if there is a deformation of M_S as A-module, see [12, Thm. 1]. Moreover, there is a change of rings spectral sequence $\mathrm{E}_2^{pq} = \mathrm{Ext}^p_A(M, \mathrm{Ext}^q_C(B, M')) \Rightarrow \mathrm{Ext}^{p+q}_C(M, M')$ with d_2-differential $\mathrm{Hom}_A(M, \mathrm{Ext}^1_C(A, M')) \xrightarrow{d_2} \mathrm{Ext}^2_A(M, M')$ induced by cupping with $\mathrm{o}(C/I^2, M) \in \mathrm{Ext}^2_A(M, M \otimes_B I/I^2)$ via the isomorphism $\mathrm{Hom}_A(M \otimes_A I/I^2, M') \cong \mathrm{Hom}_A(M, \mathrm{Ext}^1_C(A, M'))$, see [12, Prop. 3]. In [12, Thm. 4] it is shown that o_I is in the image of d_2, hence is zero if $\mathrm{o}(C/I^2, M) = 0$. It follows that α in (9) is an isomorphism in this case.

If A and C are *algebraic* k-algebras (i.e. the Henselisations of local k-algebras) with residue field k, then one can show that A as an A_A-module gives a versal family for Def^A_k. If C has the same embedding dimension as A, then $\mathrm{Def}^C_{(k,V_1)} = \mathrm{Def}^C_k$, so by Theorem 1 and sequence (9) one has maps $A \to C \to A$ such that the composition is id_A. If C is smooth and A is not, this cannot happen. One can show directly that $\mathrm{o}(C/I^2, k) \neq 0$, see Lemma 7.

The following result gives modules of different depths and dimensions which have isomorphic deformation functors.

Lemma 4. *Let M be a finitely generated A-module where A is an algebraic k-algebra. If $\mathrm{Ext}^i_A(M, A) = 0$ for all $0 < i < g$, and $g \geqslant 3$, then*

$$\mathrm{Def}^A_M \xrightarrow{\simeq} \mathrm{Def}^A_{\Omega M} \xrightarrow{\simeq} \dots \xrightarrow{\simeq} \mathrm{Def}^A_{\Omega^{g-2} M}\,. \tag{10}$$

In particular

$$\mathrm{Def}^A_k \xrightarrow{\simeq} \mathrm{Def}^A_{\mathfrak{m}} \xrightarrow{\simeq} \dots \xrightarrow{\simeq} \mathrm{Def}^A_{\Omega^{d-2} k} \tag{11}$$

where $d = \operatorname{depth} A$. If A is the A_A-module defined via the multiplication map $A_A \xrightarrow{m} A$ then $(A, \Omega^i_{A_A} A)$ is a (mini-)versal family for $\mathrm{Def}^A_{\Omega^i k}$ for all $0 \leqslant i \leqslant d-2$.

Proof. Assume $\mathrm{Ext}^1_A(M, A) = \mathrm{Ext}^2_A(M, A) = 0$, we show that $\mathrm{Def}^A_M \to \mathrm{Def}^A_{\Omega_A M}$ in Lemma 1 is an isomorphism. For surjectivity, let $(\Omega M)_S \in \mathrm{Def}^A_{\Omega_A M}(S)$ and choose a minimal A_S-free resolution $\dots \to F_2^S \to F_1^S \twoheadrightarrow (\Omega M)_S$, then a minimal A-free resolution $\dots \to F_2 \to F_1 \xrightarrow{d_1} F_0 \twoheadrightarrow M$ is obtained by extending $F^S \otimes_S k$. By dualisation of the syzygy of $(\Omega M)_S$ one obtains a map $\varphi : (\Omega(\Omega M)_S)^\vee \to (\Omega^2 M)^\vee$. The cokernel of $F_1^\vee \to (\Omega^2 M)^\vee$ is $\mathrm{Ext}^2_A(M, A) = 0$, and so φ is surjective which is equivalent to $\varphi \otimes_S k$ being an isomorphism by an argument as in (2). Since the map $\varphi_1 : \operatorname{coker}((F_1^S)^\vee \xrightarrow{\rho_1} \Omega(\Omega M)_S)^\vee) = \mathrm{Ext}^1_{A_S}((\Omega M)_S, A_S) \to \mathrm{Ext}^1_A(\Omega M, A) = 0$ is surjective, it is an isomorphism, and ρ_1 is surjective. Then it follows that $((\Omega M)_S)^\vee \to (\Omega M)^\vee$ is surjective since $\varphi \otimes_S k$ is injective. We can therefore lift the map $F_0^\vee \to (\Omega M)^\vee$ to a map $\rho_0 : F_0^S \to (\Omega M)_S)^\vee$ where F_0^S is A_S-free of the same rank as F_0. Let σ be the composition of ρ_0 with the natural inclusion $((\Omega M)_S)^\vee \hookrightarrow (F_1^S)^\vee$. Define $d_1^S := \sigma^\vee$ and $M_S := \operatorname{coker} d_1^S$. Then $\dots \to F_1^S \xrightarrow{d_1^S} F_0^S \twoheadrightarrow M_S$ gives an A_S-free resolution of M_S which lifts $F \twoheadrightarrow M$ since the natural map $\mathrm{H}_1(F^S) \otimes_S k \to \mathrm{H}_1(F) = 0$ is an isomorphism if it is surjective. In particular $\mathrm{Tor}^S_1(M_S, k) = 0$ and so M_S is S-flat and a deformation of M. We have $\Omega M_S = (\Omega M)_S$.

For the injectivity, let $\psi : \Omega M_S \to \Omega M'_S$ be an isomorphism of deformations. Dualisation of the inclusions in F_0^S gives surjective maps since $\mathrm{Ext}^1_A(M, A) = 0$. There is a lifting $\tau : (F_0^S)^\vee \to (F_0^S)^\vee$ of $\psi^\vee$ with $\tau \otimes_S k = \mathrm{id}_{F_0}$. Let $\psi_0 := \tau^\vee$, then ψ_0 induces an isomorphism $M_S \to M'_S$ of deformations since it is compatible with ψ.

For the final statement one checks that A as A_A-module is a (mini-)versal family for Def_k^A, cf. [12, Ex. 4]. ☐

3. Duality and maximal Cohen-Macaulay approximation

Various dualities induce isomorphisms of deformation functors which together with Theorem 1 relates the deformation functors of a MCM A-module and its maximal Cohen-Macaulay approximation as C-module in Corollary 3.

Lemma 5. *Let M_S and N_S be S-flat deformations of finitely generated A-modules M and N, for a local k-algebra A. Fix an $n \geqslant 0$. If $\mathrm{Ext}^i_A(M,N) = 0$ for $i = n-1, n+1$, then the N_S-dual $M^\nu_S := \mathrm{Ext}^n_{A_S}(M_S, N_S)$ is a deformation of $\mathrm{Ext}^n_A(M,N)$ to S. In particular $M_S \mapsto M^\nu_S$ gives a map of deformation functors*

$$\mathrm{Def}^A_M \longrightarrow \mathrm{Def}^A_{M^\nu}\,. \tag{12}$$

If $\mathrm{Ext}^i_A(M,N) = 0$ for $0 \leqslant i < n$ and for $i = n+1$ and $\mathrm{Ext}^i_A(M^\nu,N) = 0$ for $i = n-1, n+1$, there is a natural map to the double dual; $c_S : M_S \to (M_S)^{\nu\nu}$. If $c : M \to M^{\nu\nu}$ is an isomorphism, then c_S is an isomorphism too, (12) *is an isomorphism and $\mathrm{Def}^A_{M^\nu} \to \mathrm{Def}^A_M$ is the inverse.*

Proof. The first part is a special case of [1, 1.9]. Since the composition π^ν : $\mathrm{Ext}^n_{A_S}(M_S,N_S) \to \mathrm{Ext}^n_{A_S}(M_S,N_S)\otimes_S k \to \mathrm{Ext}^n_{A_S}(M_S,N) \xrightarrow{\simeq} \mathrm{Ext}^n_A(M,N)$ is functorial in the map $\pi : M_S \to M$, and since M^ν_S is S-flat, the map $\mathrm{Def}^A_M \to \mathrm{Def}^A_{M^\nu}$ is well defined.

For the second part; choose minimal A_S-free resolutions $F \to M_S$ and $G \to M^\nu_S$. We use the notation $M^{\nu_0}_S := \mathrm{Hom}_{A_S}(M_S,N_S)$. Since $0 \to F^{\nu_0}_0 \to \ldots \to F^{\nu_0}_{n-1} \to (\Omega^n M_S)^{\nu_0} \to \mathrm{Ext}^n_{A_S}(M_S,N_S) \to 0$ is exact, there is a lifting of the identity map to a map of complexes τ with $\tau_0 : G_0 \to (\Omega^n M_S)^{\nu_0}$ and $\tau_i : G_i \to F^{\nu_0}_{n-i}$ for $0 < i \leqslant n$. Dualising in N_S and (pre-)composing with the natural map $F \to F^{\nu_0\nu_0}$ gives a map of exact sequences where c_S is the 0^{th}-cohomology:

$$\begin{array}{ccccccccccccc}
0 & \longleftarrow & M_S & \longleftarrow & F_0 & \longleftarrow & F_1 & \longleftarrow \cdots \longleftarrow & F_{n-1} & \longleftarrow & \Omega^n_{A_S} M & \longleftarrow & 0 \\
 & & \big\downarrow{\scriptstyle c_S} & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longleftarrow & M^{\nu\nu}_S & \longleftarrow & (\Omega^n_{A_S} M^\nu)^{\nu_0} & \longleftarrow & G^{\nu_0}_{n-1} & \longleftarrow \cdots \longleftarrow & G^{\nu_0}_1 & \longleftarrow & G^{\nu_0}_0 & \longleftarrow & 0\,.
\end{array} \tag{13}$$

If c is an isomorphism, $\mathrm{coker}(c_S)\otimes_S k = 0$, i.e. $\mathrm{coker}\, c_S = 0$ and since $M^{\nu\nu}_S$ is S-flat c_S too has to be an isomorphism. ☐

Definition 3 ([8, 9])**.** Let A be a local Noetherian ring and K and M finitely generated A-modules. Set $\mathrm{G}_K\text{-}\dim M = 0$ if M is K*-reflexive*, i.e. $M \to M^{\nu_0\nu_0}$ is an isomorphism, where $M^{\nu_0} = \mathrm{Hom}_A(M,K)$, and $\mathrm{Ext}^i_A(M,K) = 0 = \mathrm{Ext}^i_A(M^{\nu_0},K)$ for all $i > 0$. K is called *suitable* if $\mathrm{G}_K\text{-}\dim A = 0$, and then let

$$\mathrm{G}_K\text{-}\dim M = \inf\{n \,|\, 0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0\}$$

where the sequence is exact and $\mathrm{G}_K\text{-}\dim P_i = 0$ for all i.

An A-module M is G_K*-perfect* if $\mathrm{grade}\, M = \mathrm{G}_K\text{-}\dim M$ where $\mathrm{grade}\, M = \inf\{i \,|\, \mathrm{Ext}^i_A(M,K) \neq 0\}$.

We obtain the following corollary of Lemma 5:

Corollary 1. *If M is G_K-perfect of $\mathrm{G}_K\text{-}\dim M = n$, then $\mathrm{Ext}^n_{A_S}(M_S, K\otimes_A A_S)$ is a deformation of $\mathrm{Ext}^n_A(M,K)$ and* (12) *is an isomorphism.*

In particular; if A is a Cohen-Macaulay and $K = \omega$ is a dualising module for A, then for any Cohen-Macaulay A-module M of codimension n, $\mathrm{Ext}^n_{A_S}(M_S, \omega\otimes_A A_S)$ is a deformation of the codimension n Cohen-Macaulay A-module $\mathrm{Ext}^n_A(M,\omega)$ and (12) *is an isomorphism.*

Proof. Generally $\mathrm{G}_K\text{-}\dim M < \infty$ implies that $\mathrm{G}_K\text{-}\dim M = \sup\{i \mid \mathrm{Ext}^i_A(M,K) \neq 0\}$; cf. [9]. Moreover, since M is G_K-perfect of $\mathrm{G}_K\text{-}\dim M = n$ one has that $\mathrm{Ext}^n_A(M,K)$ is G_K-perfect of $\mathrm{G}_K\text{-}\dim M^\nu = n$ and $M \xrightarrow{\simeq} M^{\nu\nu}$ is an isomorphism; cf. [9]. Hence the strongest conditions in Lemma 5 are satisfied for M. If A is a Cohen-Macaulay ring, then M is G_K-perfect if and only if $\mathrm{G}_K\text{-}\dim M < \infty$ and M is a Cohen-Macaulay module, and then $\mathrm{G}_K\text{-}\dim M = \operatorname{depth} A - \operatorname{depth} M$; cf. [9]. We have $\mathrm{G}_K\text{-}\dim M < \infty$ for all modules M if (and only if) $K = \omega$. □

In the case $\mathrm{G}_A\text{-}\dim M = 0$ there is a *Tate resolution* which is a minimal complex F of free A-modules which is exact and with $M = \operatorname{coker}(F_1 \to F_0)$. It is constructed by splicing a minimal resolution of M with the dual of a minimal resolution of $M^\vee$. Define $\Omega^n_A M = \operatorname{coker}(F_{n+1} \to F_n)$ for all $n \in \mathbb{Z}$.

Corollary 2. *Suppose M is a finitely generated A-module. If $\mathrm{G}_A\text{-}\dim M = 0$ then $\mathrm{Def}^A_M \cong \mathrm{Def}^A_{\Omega^n_A M}$ for all $n \in \mathbb{Z}$.*

Proof. Since $\mathrm{G}_A\text{-}\dim A = 0$, we have $\mathrm{G}_A\text{-}\dim \Omega^n_A M = 0$ for all $n \in \mathbb{Z}$. The result follows immediately from Lemma 4. □

Example 2. If $X = \operatorname{Spec} A$ is a normal surface singularity and M is reflexive on X one does not in general have that $\mathrm{Def}^A_M \cong \mathrm{Def}^A_{M^\vee}$ unless A is Gorenstein. If X is the cone over the rational normal curve of degree m, i.e. A is the Henselisation of $k[u^m, u^{m-1}v, \dots, v^m]$ with indecomposable reflexive modules $M_i = \langle u^i, u^{i-1}v, \dots, v^i\rangle$, then $M^\vee_{m-1} \cong M_1$, but the minimal stratum in a filtration of the versal base space ([17]) of M^r_{m-1} is an isolated singularity of dimension $(r-1)m$ while M^r_1 is infinitesimally rigid, see [10]. In fact $\mathrm{G}_A\text{-}\dim M = 0 \Rightarrow M$ is free. Since $\mathrm{Ext}^1_A(M_i, M_j) = 0$ for $i \leqslant j+1 \leqslant m-1$, if M only has such M_i as direct summands, we have from Lemma 5 a map $\mathrm{Def}^A_M \to \mathrm{Def}^A_{M^\nu}$ where $N = M_j$, and if $i \leqslant j \leqslant m-2$ this map is an isomorphism.

Definition 4. Suppose A is a local Cohen-Macaulay ring with a dualising module ω, then a *maximal Cohen-Macaulay approximation* of an A-module M is an exact sequence $0 \to Y^A_M \to X^A_M \to M \to 0$ of finitely generated A-modules with $\operatorname{injdim} Y^A_M < \infty$ and X^A_M a maximal Cohen-Macaulay module.

This is a particular instance of the categorical concept of MCM approximation introduced by M. Auslander and R. O. Buchweitz, and by Theorem A in [3] there exists MCM approximations. If M is a Cohen-Macaulay module (so is G_ω-perfect) then the sequence $0 \to Y \to (\Omega^n_A \mathrm{Ext}^n_A(M,\omega))^{\nu_0} \to M \to 0$, obtained from the bottom left of (13) via the isomorphism $M \xrightarrow{\simeq} \mathrm{Ext}^n_A(\mathrm{Ext}^n_A(M,\omega),\omega)$ with $n = \operatorname{codim} M$, is a minimal MCM approximation of M.

Corollary 3. *Suppose $\pi : C \to A$ is a surjective map of local k-algebras where C is Cohen-Macaulay and M is a finitely generated A-module which is Cohen-Macaulay of codimension n as C-module. Let $M^\nu = \mathrm{Ext}^n_C(M, \omega_C)$, then there is an isomorphism*

$$\mathrm{Def}^A_M \xrightarrow{\simeq} \mathrm{Def}^A_{M^\nu} \quad \textit{and a natural map} \quad \mathrm{Def}^A_{M^\nu} \longrightarrow \mathrm{Def}^C_{X^C_M}\,. \tag{14}$$

If $I = \ker \pi$ is generated by a regular sequence of length n and $\mathrm{o}(C/I^2, M) = 0$, then there are isomorphisms of deformation functors

$$\mathrm{Def}^C_{(X^C_{M^\nu}, V)} \xleftarrow{\simeq} \mathrm{Def}^A_M \cong \mathrm{Def}^A_{M^\nu} \xrightarrow{\simeq} \mathrm{Def}^C_{(X^C_M, V')} \tag{15}$$

where V and V' are the images of $\mathrm{Def}^A_M(k[\varepsilon])$.

Proof. Since $M_S^\nu = \mathrm{Ext}^n_{C_S}(M_S, \omega_C \otimes_C C_S)$ is an A_S-module as C_S-module for any deformation M_S of M as A-module, the isomorphism $\mathrm{Def}^C_M \to \mathrm{Def}^C_{M^\nu}$ obtained in Corollary 1 induce an isomorphism $\mathrm{Def}^A_M \to \mathrm{Def}^A_{M^\nu}$ via the natural change of rings inclusions.

Remark that $X^C_{M^\nu} = (\Omega^n_C M)^{\nu_0}$, and we have maps $\mathrm{Def}^A_{M^\nu} \cong \mathrm{Def}^A_M \to \mathrm{Def}^C_{\Omega^n_C M} \cong \mathrm{Def}^C_{\Omega^n_C(M)^{\nu_0}}$ obtained in Corollary 1 and Lemma 1. The final statement follows from the last isomorphism and Theorem 1. □

4. Generalised Knörrer functors

It is not hard to provide general examples of A and C in Theorem 1 such that the conditions are satisfied for all A-modules M. We will however in Theorem 2 give a class of examples only partially covered by Theorem 1, and which also generalises both of Knörrer's functors, which are discussed at the end of the section.

Definition 5. If $I(\rho)$ is the ideal generated by the maximal minors of the $a \times b$-matrix ρ with entries from the maximal ideal of a local ring R, then $I(\rho)$ is *determinental* if $\operatorname{depth} I(\rho) = |a - b| + 1$, the maximal possible value.

Let P be a local k-algebra with residue field k, and let Q and R be the localisations of the polynomial rings $P[\underline{u}]$ and $P[\underline{u}, \underline{v}]$ respectively, where $\underline{u} = \{u_1, \dots, u_p\}$ and $\underline{v} = \{v_1, \dots, v_q\}$ are indeterminants. Let (f_i) and (F_i) be b elements from $\mathfrak{m}_P$ and $\mathfrak{m}_R$ respectively. Set $h_i = F_i - f_i \in R$. Moreover, let $\psi = (g_{ij})$ be an $l \times m$-matrix ($l \leqslant m$) with $g_{ij} \in Q$, let $\overline{g}_{ij}$ be the image of g_{ij} under the natural map $Q \to Q \otimes_P k = Q_0 \cong k[\underline{u}]_\mathfrak{m}$ and put $\psi_0 = (\overline{g}_{ij})$.

With this notation we have:

Theorem 2. *Assume $(\underline{f})$ is a regular sequence and $I(\psi_0)$ is a determinental ideal, and let $A = P/(\underline{f})$, $B = Q/((\underline{f}) + I(\psi))$ and $C = R/(\underline{F})$. For any finitely generated A-module M, let $M' = M \otimes_A B$ which is a C-module via the natural surjective map $C \to B$.*

If $h_{ij} \in (\underline{v})(\underline{u}, \underline{v})R$ and $g_{ij} \in (\underline{u})Q$ for all i, j, and $n = q + m - l + 1$ ($n = q$ if ψ is empty), then there is an isomorphism of deformation functors

$$\sigma : \mathrm{Def}^A_M \xrightarrow{\simeq} \mathrm{Def}^C_{(\Omega^n_C M', V)}$$

where $V = \operatorname{im} \mathrm{Def}^A_M(k[\varepsilon])$.

If M is a maximal Cohen-Macaulay A-module, then $\Omega^n_C M'$ is a maximal Cohen-Macaulay C-module.

The proof will employ a construction of D. Eisenbud which to an R-free resolution L of M gives an A-free resolution of M if A is a quotient ring of R by an ideal generated by a regular sequence.

A 'sum' tensor product of Eisenbud systems.

Definition 6 (D. Eisenbud)**.** Let R be a commutative ring and $J = (f_1, \dots, f_n)$ a sequence of elements in R. An *Eisenbud system* relative to J on an R-complex $L = (L, d^L)$ is a system of R-linear endomorphisms $\{s_\alpha\}$ of L as graded R-module of degree $2|\alpha| - 1 \geqslant 1$, where α is an n-multi index, satisfying

$$s_\alpha d^L + d^L s_\alpha = - \sum_{\beta_1 + \beta_2 = \alpha} s_{\beta_1} s_{\beta_2} \tag{16}$$

for $|\alpha| > 1$ and $s_i d + d s_i$ is multiplication by f_i on L, see [5].

If L is an R-free resolution of an $A = R/J$-module M, there exists an Eisenbud system on L. Let $S = R[t_1, \dots, t_n]$ and let $\mathbb{D} = \mathrm{Hom}_{\text{grad. } R\text{-alg.}}(S, R)$ (where $\deg t_i = -2$) be the divided power algebra. It has generators $\tau^{(\alpha)}$ which are dual to the t^α and t_i acts on $\mathbb{D}$ by subtracting the i-th index in α by 1 if possible, or else $t_i \cdot \tau^{(\alpha)} = 0$. If we put $s_0 = d^L$ and $d = \sum_\alpha t^\alpha \otimes s_\alpha$ then $\mathbb{D} \otimes L \otimes A = (\mathbb{D} \otimes_R L \otimes_R A, d)$ is a complex of A-free modules, and if $(f_1, \dots, f_n)$ is a regular sequence then $\mathbb{D} \otimes L \otimes A$ is an A-free resolution of M, see [5, 7.2].

Definition 7. If $\mathcal{E} = (L, \{s_\alpha(\underline{f})\})$ and $\mathcal{E}' = (L', \{s_\alpha(\underline{g})\})$ are Eisenbud systems for the sequences $(f_1, \dots, f_n)$ and $(g_1, \dots, g_n)$ in R, then their *sum* tensor product is the Eisenbud system $\mathcal{E} \otimes \mathcal{E}' = (L \otimes_R L', \{s_\alpha(\underline{f}) \otimes 1 \pm 1 \otimes s_\alpha(\underline{g})\})$ for the sequence $(f_1 + g_1, \dots, f_n + g_n)$.

Proof of Theorem 2. Suppose we have surjections $C \to B$ and $B \to A$, and a flat splitting $A \to B$, (all maps of local k-algebras) and a finitely generated A-module M. Define σ by the composition $\mathrm{Def}^A_M \to \mathrm{Def}^B_{M'} \to \mathrm{Def}^C_{M'} \to \mathrm{Def}^C_\Omega$ (where $\Omega = \Omega^n_C M')$ of maps defined in Lemma 2, by change of rings, and in Lemma 1 respectively. Moreover; there is a map $\mathrm{Def}^C_\Omega \to \mathrm{Def}^B_{\overline{\Omega}}$ where $\overline{\Omega} = \Omega^n_C M' \otimes_C B$ by Lemma 2 if $n \geqslant \mathrm{pdim}_C B$. Since a deformation of a B-module N is also a deformation of N as A-module, there is a map $\mathrm{Def}^B_{\overline{\Omega}} \to \mathrm{Def}^A_{\overline{\Omega}}$. By the splitting of B as A-module $\Omega^n_C M' \otimes_C A$ becomes a direct summand of $\Omega^n_C M' \otimes_C B$. We claim, under the additional conditions, that M is a direct summand of $\Omega^n_C M' \otimes_C A$. We define $\tau : \mathrm{Def}^A_M \to \mathrm{Def}^A_{(\overline{\Omega}, V')}$ where $V' = \mathrm{im}\, \mathrm{Def}^A_M(k[\varepsilon])$ by $M_S \mapsto \Omega^n_{C_S} M'_S \otimes_{C_S} B_S$ considered as (possibly non-finitely generated) A_S-module. That σ is an isomorphism now follows analogously to the argument in Theorem 1: Define (id, η^i) for $i > 0$ to be the composition of the natural maps $\mathrm{Ext}^i_A(M, M) \to \mathrm{Ext}^i_C(M', M') \to \mathrm{Ext}^i_C(\Omega, \Omega) \to \mathrm{Ext}^i_B(\overline{\Omega}, \overline{\Omega}) \to \mathrm{Ext}^i_A(\overline{\Omega}, \overline{\Omega}) = \mathrm{Ext}^i_A(M \oplus Y, M \oplus Y)$. In particular the (id, η^i) are injective. Considering the obstruction classes as 4-term exact sequences (see the proof of Lemma 7) one can show that $\mathrm{o}_C(\pi, \sigma M_S) \otimes_C B \mapsto \mathrm{o}_A(\pi, \tau M_S)$, so $\mathrm{o}_C(\pi, \sigma M_S) = 0 \Rightarrow \mathrm{o}_A(\pi, M_S) = 0$ and formal smoothness follows for σ. Given a $L_S \in \mathrm{Def}^C_{(\Omega, V)}(S)$, then there is an $A \hat{\otimes}_k \hat{S}$-module $\hat{M}_{\hat{S}}$ and an isomorphism $\hat{\varphi} : \Omega^n_{\hat{C}_{\hat{S}}} \hat{M}'_{\hat{S}} \xrightarrow{\simeq} \hat{L}_{\hat{S}}$. Let $L^B = L_S \otimes_{C_S} B_S$ and L^A the A_S-linear direct summand of L_B induced by the splitting of A in B. The image of the map $L^A \to \hat{M}_{\hat{S}}$, defined by the splitting and $\hat{\varphi}$, defines M_S. We obtain an isomorphism $\sigma M_S \cong L_S$ compatible with $\hat{\varphi} \mod \mathfrak{m}_S^2$ by [20, 7.11]. Hence σM_S is equivalent to the deformation L_S and σ is surjective. For the injectivity of σ, see the proof of Theorem 1.

For B to be A-flat it is sufficient that $Q/I(\psi)$ is P-flat. Since $I(\psi_0)$ is determinental, the Eagon-Northcott complex $\mathcal{F}(\psi_0)$ (cf. [6, A2.6]) by assumption gives a Q_0-free resolution of $Q_0/I(\psi_0)$. There is a natural map $\mathrm{H}_i(\mathcal{F}(\psi)) \otimes_P k \to \mathrm{H}_i(\mathcal{F}(\psi_0))$ which is surjective if and only if it is an isomorphism (as in (2)). Hence $\mathcal{F}(\psi)$ is a Q-free resolution of $Q/I(\psi)$ of length $m - l + 1$ and similarly the Koszul complex $K(\underline{F})$ gives an R-free resolution of C. We have $\mathrm{Tor}^P_i(Q/I(\psi), k) \cong \mathrm{Tor}^Q_i(Q/I(\psi), Q_0) = \mathrm{H}_i(\mathcal{F}(\psi) \otimes_Q Q_0) = \mathrm{H}_i(\mathcal{F}(\psi_0)) = 0$ for $i > 0$ by assumption, and we conclude by the local criterion of flatness.

If $C_0 = C \otimes_{k[\underline{v}]} k$, we have surjections $C \to C_0 \to B$, we will show that $\mathrm{pdim}_C C_0 = q$ and $\mathrm{pdim}_{C_0} B = m - l + 1$. There is a change of rings spectral sequence $\mathrm{E}^{ij}_2 = \mathrm{Ext}^i_{C_0}(B, \mathrm{Ext}^j_C(C_0, -)) \Rightarrow \mathrm{Ext}^{i+j}_C(B, -)$. If $i > m - l + 1$ or $j > q$, then $\mathrm{E}^{ij}_\infty = 0$, and thus $\mathrm{pdim}_C B \leqslant q + m - l + 1$. We have $\mathrm{Tor}^{k[\underline{v}]}_i(C, k) \cong \mathrm{Tor}^R_i(C, Q) \cong \mathrm{H}_i(K(\underline{F}) \otimes_R Q) \cong \mathrm{H}_i(K(\underline{f})) \otimes_P Q = 0$ for $i > 0$ by assumption, hence $\underline{v}$ is a C-regular sequence and $\mathrm{pdim}_C C_0 = q$. Since $Q/I(\psi)$ is P-flat, $\mathcal{F}(\psi) \otimes_P A$ gives an C_0-free resolution of B and the length of $\mathcal{F}(\psi)$ is $m - l + 1$.

If M is a MCM A-module (and A is Cohen-Macaulay), then $M_0 = M \otimes_A C_0$ is a MCM C_0-module. We have that $\mathcal{F}(\overline{\psi}) = \mathcal{F}(\psi) \otimes_Q C_0$ gives a C_0-free resolution of B and $\mathrm{H}_i(\mathcal{F}(\overline{\psi}) \otimes_{C_0} M_0) \cong \mathrm{Tor}_i^{C_0}(B, M_0) \cong \mathrm{Tor}_i^A(B, M) = 0$ for $i > 0$ since B is A-flat. We get an "M_0"-resolution of M' of length $m - l + 1$. By [6, 18.6] $\operatorname{depth} M' \geqslant \operatorname{depth} M_0 - (m - l + 1) = \operatorname{depth} C_0 - \operatorname{pdim}_{C_0} B = \operatorname{depth} B = \dim B$ since B is Cohen-Macaulay, so M' is a MCM B-module, and $\Omega_C^n M'$ is a MCM C-module since $n \geqslant \operatorname{pdim}_C B$.

For the claim; let $\mathcal{E}$ be an Eisenbud system on a minimal P-free resolution L of M for the regular sequence $(f_1, \dots, f_b)$ and $\mathcal{E}'$ an Eisenbud system on the R-free Koszul resolution $K(\underline{v})$ of Q for the sequence $(h_1, \dots, h_b)$. Remark that we may assume $s_\alpha(\underline{h}) = 0$ for $|\alpha| > 1$. The tensor product of these complexes with the resolution $\mathcal{F}(\psi)$ gives an R-free complex with $\mathrm{H}_0 = M[\underline{u}, \underline{v}]_{\mathfrak{m}} \otimes_R Q \otimes_R R/I(\psi) \cong M'$ and $\mathrm{H}_i = \mathrm{Tor}_i^R(M \otimes_P R, Q/I(\psi)) \cong \mathrm{Tor}_i^P(M, Q/I(\psi)) = 0$ for $i > 0$, hence an R-free resolution of M'. The tensor product of the Eisenbud systems yields an Eisenbud system for $(F_1, \dots, F_b)$, hence we obtain a C-free resolution $(\mathcal{L}, d)$ of M'. Assuming $(\mathbb{D} \otimes L \otimes A, d)$ is a minimal A-free resolution of M, we have

$$\begin{aligned}(\Omega_C^n M') \otimes_C A &= \operatorname{coker} d_{n+1} \otimes_C A = \operatorname{coker}(d_{n+1} \otimes_C A) \\ &= \operatorname{coker}(\bigoplus_{i=0}^{n} (\sum t^a \otimes s_a(\underline{f}))_{i+1} \otimes 1 \otimes 1 \\ &= \bigoplus_{i=0}^{n} \Omega_A^i(M) \otimes_A G_{n-i}\,,\end{aligned}$$

where $G_{n-i} = \bigoplus_{j=0}^{n-i} (\bigwedge^{n-i-j} A^q) \otimes_A A^{\operatorname{rk} \mathcal{F}(\psi)_j}$, since, by assumption, $h_i \in (\underline{v})(\underline{u}, \underline{v})R$, so we may assume $I_1(s_i(\underline{h})) \subseteq (\underline{u}, \underline{v})R$ and thus that the $\mathbb{D} \otimes K(\underline{v}) \otimes C$- and $\mathcal{F}(\psi)$-differentials vanish when applying $- \otimes_C A$. Non-minimality of $(\mathbb{D} \otimes L \otimes A, d)$ will only give certain extra free addends in $\operatorname{coker}(d_{n+1} \otimes_C A)$, the conclusion is still valid. $\square$

Remark 3. Any k-algebra B resolved by a finite functorial complex like $\mathcal{F}$ may be used to obtain results similar to Theorem 2.

The extra conditions ensure that $\mathrm{o}(C/I^2, M) = 0$ so that in particular Theorem 2 (with $p = 0$) gives examples of $C \to A$ satisfying the conditions in Theorem 1 for all A-modules. Let $I = (\underline{u}, \underline{v})C = \ker(C \to A)$ and let $(L, d^L) \twoheadrightarrow M$ be a P-free resolution of M. The regular sequence $(\underline{f})$ defines a 0-homotopic multiplication map $m : E_1 \otimes_P L_0 \to L_0$ where $E_1 \cong P^b$, let $s : E_1 \otimes_P L_0 \to L_1$ be a lifting of m, i.e. $d_1^L s = m$. Then

$$F^P : \quad F_0 \xleftarrow{d_1^L} F_1 \xleftarrow{(s, d_2^L)} E_1 \otimes F_0 \oplus F_2$$

gives, after applying $- \otimes_P A$, an A-free 2-presentation $(\mathcal{L}, d)$ of M, see [12, Lem. 3]. There is a natural map $P \to C$, and $F^P \otimes_P C/I^2$ gives a lifting of $\mathcal{L}$ to C/I^2 as graded module. By [12, Prop. 3] (see also Lemma 3) we have that $\mathrm{o}(C/I^2, M)$ is induced from $d_1 s$. However $d_1 s = (\underline{f}) \equiv (\underline{F}) \equiv 0 \mod I^2$ since $h_{ij} \in I^2$.

Observe that $\mathrm{o}(C/I^2, M')$ may be non-zero even though $\mathrm{o}(C/I^2, M) = 0$, which for instance is the case with Knörrer's H-functor, see below.

Lemma 6. *Let $\pi : C \to A$ be a surjective map of local k-algebras. Assume that $I = \ker \pi$ is generated by a regular sequence and that $n \geqslant \operatorname{pdim}_C A$. Let M be any finitely generated A-module. Then there is an isomorphism of deformation functors*

$$\sigma : \mathrm{Def}^C_{\Omega_C^n M} \xrightarrow{\simeq} \mathrm{Def}^A_{(\Omega_C^n M \otimes_C A, V)} \tag{17}$$

where $V = \operatorname{im} \mathrm{Def}^C_{\Omega_C^n M}(k[\varepsilon])$.

Proof. The map σ is the one given in Lemma 2. If N is a C-module and the length of the regular sequence is r, one has that $\Omega^r_C(N\otimes_C A) \cong \bigoplus_{j=0}^r \bigwedge^{r-j} C^r \otimes_C \Omega^j_C N$ if and only if $I\cdot\operatorname{Ext}^i_C(N,-) = 0$ for all $i > 0$, by [21, 2.2]. Set $N = \Omega^n_C M$, then we have $\operatorname{Ext}^i_C(N,-) = \operatorname{Ext}^{i+n}_C(M,-)$ (for $i > 0$) which certainly is annihilated by I. Define $\tau : \operatorname{Def}^C_N \to \operatorname{Def}^C_{(\Omega^r_C \overline{N}, V')}$, where $\overline{N} = N\otimes_C A$ and $V' = \operatorname{im}\operatorname{Def}^C_N(k[\varepsilon])$, by $N_S \mapsto \Omega^r_{C_S}\overline{N_S}$. The proof is concluded by proving surjectivity and injectivity of σ as in the proof of Theorem 1. □

The Knörrer functors.

Definition 8 (D. Eisenbud)**.** If f is a regular element in a ring P, then a *matrix factorisation* of f is a pair of linear maps (ρ,σ) of free P-modules $L_0 \xrightarrow{\sigma} L_1 \xrightarrow{\rho} L_0$ of finite rank such that $\rho\sigma = f\cdot \mathrm{id}_{L_0}$ and $\sigma\rho = f\cdot \mathrm{id}_{L_1}$.

A matrix factorisation is a special case of an Eisenbud system, see Definition 6. If $A = P/(f)$ one obtains an A-free resolution $\dots \xrightarrow{\overline{\rho}} \overline{L}_0 \xrightarrow{\overline{\sigma}} \overline{L}_1 \xrightarrow{\overline{\rho}} \overline{L}_0 \twoheadrightarrow M$ where $\overline{\rho} = \rho\otimes_P A$ etc. If P is a regular ring, then M is a MCM module, and any MCM A-module is (if P is local) given by a matrix factorisation of f. See [5].

Definition 9 (H. Knörrer)**.** With notation as before Theorem 2, let $F = f + v^2$ (i.e. $h = v^2$, $p = 0$, $q = 1$), then the G-functor in [18] takes the matrix factorisation (ρ,σ) of f over P to the matrix factorisation of F (in block matrix notation)

$$G(\rho,\sigma) = \left(\begin{bmatrix} \rho & v\cdot \mathrm{id} \\ -v\cdot \mathrm{id} & \sigma \end{bmatrix}, \begin{bmatrix} \sigma & -v\cdot \mathrm{id} \\ v\cdot \mathrm{id} & \rho \end{bmatrix}\right) = (\Sigma, \Sigma')$$

over R.

If $M = \operatorname{coker}\rho$ then M is an $A = P/(f)$-module. Let $G(M) = \operatorname{coker}\Sigma$ which is a $C = R/(F)$-module.

Corollary 4. *There are isomorphisms of deformation functors*

$$\operatorname{Def}^A_M \xrightarrow{\simeq} \operatorname{Def}^C_{(G(M),V)} \quad \text{and} \quad \operatorname{Def}^C_{G(M)} \xrightarrow{\simeq} \operatorname{Def}^A_{(M\oplus\Omega_A M, V')} \tag{18}$$

where $V = \operatorname{im}\operatorname{Def}^A_M(k[\varepsilon])$ *and* $V' = \operatorname{im}\operatorname{Def}^C_{G(M)}(k[\varepsilon])$.

Proof. Let $L' = L\otimes_P R\otimes_R C$. One checks that

$$M \twoheadleftarrow L'_0 \xleftarrow{[v\cdot \mathrm{id},\, \overline{\rho}]} L'_0\oplus L'_1 \xleftarrow{\overline{\Sigma}} L'_1\oplus L'_0 \xleftarrow{\overline{\Sigma}'} \dots \tag{19}$$

gives a C-free resolution of M. Hence $G(M) = \Omega_C M$ and the conclusions follow from Theorem 1 and Lemma 6 since $G(M)\otimes_C A = M\oplus\Omega_A M$. □

Example 3. Let P and R be the Henselisations of $k[x]$ and $k[x,v]$ respectively, and $A = A_n = P/(f)$ where $f = x^{n+1}$, so that $C = R/(F)$ where $F = f{+}v^2$. Let $M = k$, the residue field of A, then $G(k) = \mathfrak{m}_C$ and $G(k)\otimes_C A = k \oplus \mathfrak{m}_A$. Consider the first three maps of deformation functors in (9), but without restricting to the images of $\operatorname{Def}^A_k(k[\varepsilon])$. In fact we have $\operatorname{Def}^C_k \xrightarrow{\simeq} \operatorname{Def}^C_{\mathfrak{m}_C}$. By the general identification $\operatorname{Def}^A_M(k[\varepsilon]) \cong \operatorname{Ext}^1_A(M,M)$, one calculates $\operatorname{Def}^A_k(k[\varepsilon]) = \langle \xi_{11}\rangle$, $\operatorname{Def}^C_{G(k)}(k[\varepsilon]) = \langle \eta_1, \eta_2\rangle$, and $\operatorname{Def}^A_{k\oplus\mathfrak{m}_A}(k[\varepsilon]) = \langle \xi_{ij}\rangle_{1\leqslant i,j\leqslant 2}$. The maps give $\xi_{11} \mapsto \omega^1_C(\xi_{11}) = \eta_1 \mapsto \xi_{11}{+}\xi_{22}$ where $\xi_{22} = \omega^1_A(\xi_{11})$, and $\eta_2 \mapsto \xi_{12}{+}\xi_{21}$. Let the variables t_{ij} and s_i at the cotangent spaces correspond to the k-duals of the ξ_{ij} and the η_i, then we know by Lemma 4 that $S_1 = k[t_{11}]^{\mathrm{h}}/(t_{11}^{n+1})$ and $S_2 = k[s_1,s_2]^{\mathrm{h}}/(s_1^{n+1}{+}s_2^2)$ are the versal deformation rings of Def^A_k and $\operatorname{Def}^C_{G(k)}$ respectively. The obstruction calculus, involving cup and Massey products (see [19, 25, 12]), gives the obstruction ideal.

It terminates after $n+1$ steps in this case, and we obtain the versal deformation space S_3 for $\mathrm{Def}^A_{k\oplus\mathfrak{m}_A}$ as

$$S_3 = k[t_{11}, t_{12}, t_{21}, t_{22}]^{\mathrm{h}}/\begin{pmatrix} t_{11}^{n+1}+t_{12}t_{21}\,, & t_{11}t_{12}-t_{12}t_{22}\,, \\ t_{21}t_{11}-t_{22}t_{21}\,, & t_{22}^{n+1}+t_{21}t_{12} \end{pmatrix} \tag{20}$$

where the equations are valied even without assuming that the t_{ij} commute. The choice of liftings of the dual maps of the Zariski tangent spaces of the functors to the deformation rings given by $t_{11}, t_{22} \mapsto s_1 \mapsto t_{11}$, and $t_{12}, t_{21} \mapsto s_2 \mapsto 0$ is respected by the equations. However, there is no map $S_1 \to S_3$ such that the composition $S_1 \to S_3 \to S_2 \to S_1$ is the identity! Hence there cannot be any "projection" map $\mathrm{Def}^A_{k\oplus\mathfrak{m}_A} \to \mathrm{Def}^A_k$ for which the natural $\mathrm{Def}^A_k \to \mathrm{Def}^A_{k\oplus\mathfrak{m}_A}$ is a section.

Using an existence result of A. Ishii in [17], T. S. Gustavsen and the author in [11] show that the versal deformation space of a (not necessarily indecomposable) reflexive module on a rational double point is irreducible, and by Theorem 3 below this result is extended to MCM modules on the simple singularities of even dimension. So for instance is $\operatorname{Spec} S_3$ irreducible. However, the above example shows that for the A_n-singularities with n odd in odd dimension, there are (indecomposable) MCM modules which have versal deformation spaces with two components.

Definition 10 (H. Knörrer)**.** With notation as before Theorem 2, let $F = f + uv$ (i.e. $h = uv$, $p = q = 1$), then the H-functor in [18] takes the matrix factorisation (ρ, σ) of f over P to the matrix factorisation of F (in block matrix notation)

$$H(\rho,\sigma) = \Big(\begin{bmatrix} \rho & u\cdot \mathrm{id} \\ -v\cdot \mathrm{id} & \sigma \end{bmatrix}, \begin{bmatrix} \sigma & -u\cdot \mathrm{id} \\ v\cdot \mathrm{id} & \rho \end{bmatrix}\Big) = (\Phi, \Phi')$$

over R.

Let $A = P/(f)$ and $C = R/(F)$, then $M = \operatorname{coker}\rho$ is an A-module and $H(M) = \operatorname{coker}\Phi$ is a C-module. Knörrer's main result is that H induces an equivalence between the stable category of MCM A-modules and the stable category of MCM C-modules in the case P is complete and regular, see [18, 3.1].

With this notation we have:

Theorem 3. *If (ρ, σ) is a matrix factorisation of f, $M = \operatorname{coker}\rho$ and H is the Knörrer functor, then*

$$\mathrm{Def}^A_M \cong \mathrm{Def}^C_{H(M)}\,. \tag{21}$$

Proof. Let $L' = L\otimes_P R \otimes_R C$. One checks that

$$M' \twoheadleftarrow L'_0 \xleftarrow{[v\cdot \mathrm{id},\, \overline{\rho}]} L'_0 \oplus L'_1 \xleftarrow{\overline{\Phi}} L'_1 \oplus L'_0 \xleftarrow{\overline{\Phi}'} \dots \tag{22}$$

gives a C-free resolution of $M' = M\otimes_A B$ (with C-module structure induced from the natural surjection $C \to B$). Hence $H(M) = \Omega_C M'$ and the conclusion follows by Theorem 2 if we can prove the tangential isomorphism $\mathrm{Def}^A_M(k[\varepsilon]) \xrightarrow{\simeq} \mathrm{Def}^C_{H(M)}(k[\varepsilon])$. Since σ is injective we have $\mathrm{Ext}^1_A(M, A) = 0$, which implies that

$$\begin{aligned} \mathrm{Ext}^1_A(M, M) &\cong \underline{\mathrm{Hom}}_A(\Omega_A M, M) \\ &\cong \underline{\mathrm{Hom}}_C(H(\Omega_C M), H(M)) \quad \text{by [18, 3.1]} \\ &\cong \underline{\mathrm{Hom}}_C(\Omega_C H(M), H(M)) \quad \text{by [18, 3.5]} \\ &\cong \mathrm{Ext}^1_C(H(M), H(M)) \end{aligned}$$

where $\underline{\mathrm{Hom}}$ is the quotient of stable maps. □

Remark 4. Notice that H is also well defined for matrix factorisations over non-local rings. If we restrict attention to functors of Art_k then the conclusion in Theorem 3 is still valied. The argument in Theorem 2 can be followed for the syzygy *defined* as $H(M)$ only using the obstruction theory. For the tangential result one explicitly constructs a chain homotopy from $H(\xi_A)$ to a given cocyle ξ_C with $[\xi_C] \in \operatorname{Ext}^1_C(H(M), H(M))$ where $\xi_A = \xi_C \otimes_C A$, proving surjectivity, as was done in [14, 7.4.18]. The result was first proved in 1990 by the author in his master's thesis, see [13, 2.5.4]. A version for P regular, i.e. for maximal Cohen-Macaulay modules, appeared in 1996, see [22, 3.16]. The obvious generalisation of H is obtained if we in Theorem 2 assume that ψ is empty. Indeed the initial motivation for this work was to understand what was behind Theorem 3 and thereby possibly obtain generalisations of it.

Remark 5. There is a short exact sequence

$$0 \to M' \longrightarrow H(M) \otimes_C B \longrightarrow (\Omega_A M)' \to 0\,. \tag{23}$$

The exact sequences arising from applying $\operatorname{Hom}_B(\overline{H(M)}, -)$ and $\operatorname{Hom}_B(-, \overline{H(M)})$ splits into short exact sequences since the connecting maps may be shown to be zero. E.g.

$$0 \to \operatorname{Ext}^1_A(\Omega_A M, M) \to \operatorname{Ext}^1_B(\overline{H(M)}, \overline{H(M)}) \to \operatorname{Ext}^1_A(M, M) \to 0 \tag{24}$$

which in particular shows that we cannot expect isomorphism in Lemma 6 without restricting to the image of the tangent space.

5. Smoothing of MCM modules, lifting of the residue field

After defining smoothable modules, we show that if the "source" module is smoothable, then so is the "target" module in several of the maps of deformation functors considered above. Finally we show that the drop in embedding dimension observed in Theorem 2 is equivalent to liftability of the residue field.

Assume that Def^A_M has a versal family (R, M_R) which we fix, where we assume that the Zariski tangent space is of minimal dimension at the central point (see the introduction for the existence of versal families). Since Def^A_M is locally of finite presentation, there exists a *germ* representing (R, M_R), i.e. an affine k-pointed k-scheme R^{ft} of finite type and an R^{ft}-flat family of modules $M_{R^{\mathrm{ft}}}$, finitely generated as $A_{R^{\mathrm{ft}}} = A \otimes_k R^{\mathrm{ft}}$-module, such that the Henselisation at the k-point gives (R, M_R). Let N be any A-module and suppose k is an algebraically closed field.

Definition 11. Let $\operatorname{Loc}(N)$ be the set of k-points $t \in \operatorname{Spec} R^{\mathrm{ft}}$ such that the pullback M_t of $M_{R^{\mathrm{ft}}}$ to t is isomorphic to N. Then M *locally deforms to* N, denoted $M \dashrightarrow N$, if the Zariski closure $\overline{\operatorname{Loc}}(N)$ strictly contains the central k-point corresponding to M. We say that M is *smoothable* if M locally deforms to a free module.

It follows that the relation $\dashrightarrow$ is independent of choice of germ, and is *transitive* by openness of versality, see [17, 2.13]. Results on $\dashrightarrow$ are found in [10] and in [11].

Corollary 5. *Let* $\operatorname{Def}^{A_1}_{M_1} \to \operatorname{Def}^{A_2}_{(M_2,V)}$ *be the map of deformation functors of either Theorem 1, the second half of Corollary 3, Theorem 2 or Lemma 6, with corresponding conditions. Assume furthermore that there exists a versal family for* $\operatorname{Def}^{A_i}_{M_i}$, $i = 1, 2$. *If* M_1 *is smoothable, then* M_2 *is smoothable.*

Proof. Assume first that we are in the situation of Theorem 1. We may assume that there is a $C \otimes_k R^{\mathrm{ft}}$-projective resolution $F_{R^{\mathrm{ft}}} \twoheadrightarrow M_{R^{\mathrm{ft}}}$ which after Henselisation at the central k-point gives a C_R-free resolution $F_R \twoheadrightarrow M_R$ which is minimal for the $n+2$ first terms. Suppose $M_t \cong A^r$, then $\Omega^n_C M_t \cong C^r$ by assumption on n and I. So if $F(t) = F_{R^{\mathrm{ft}}} \otimes_{R^{\mathrm{ft}}} k(t)$ and $R(t)$ is the Henselisation of R^{ft} at t, then $\Omega^n_{C_{R(t)}} M_{R(t)} \otimes_{R(t)} k(t)$ is a direct summand of $\operatorname{coker}(F_{n+1}(t) \to F_n(t)) \cong C\text{-free} \oplus \Omega^n_C M_t$, and hence $\Omega^n_{C_{R(t)}} M_{R(t)} \otimes_{R(t)} k(t)$ is C-free.

In the situation of Theorem 2, $\mathrm{Def}^A_M \to \mathrm{Def}^B_{(M',V')}$ is an isomorphism where $V' = \operatorname{im}\mathrm{Def}^A_M(k[\varepsilon])$, and so $M_R \otimes_{A_R} B_R$ gives a versal family of $\mathrm{Def}^B_{(M',V')}$ and if $M_t \cong A^r$ then $M'_t \cong B^r$. By the assumption on n we get that $\Omega^n_C M'$ is smoothable as above.

In the situation of Lemma 6, $((\Omega^n_C M)_{R^{\mathrm{ft}}} \otimes_C A) \otimes_{R^{\mathrm{ft}}} k(t) \cong (\Omega^n_C M)_t \otimes_C A$ is A-free if $(\Omega^n_C M)_t$ is C-free. □

In Theorem 2 with $p = 0$ we have that the drop of embedding dimension equals the drop of dimension; $\operatorname{edim} C - \operatorname{edim} A = \dim C - \dim A$. This is not coincidental.

Lemma 7. *Suppose $\pi : C \to A$ is a surjective map of local rings and assume k is the residue field of A. If $I = \ker \pi$ is generated by a regular sequence of length n, then*

$$\mathrm{o}(C/I^2, k) = 0 \iff \operatorname{edim} C = \operatorname{edim} A + n. \tag{25}$$

Proof. ($\Leftarrow$): Let $L \twoheadrightarrow k$ be a minimal C-free resolution of $k = C/(x_1, \dots, x_e)$, $e = \operatorname{edim} C$. Choose an Eisenbud system $\{s_\alpha\}$ relative to $I = (h_1, \dots, h_n)$ on L, and let $F = (\mathbb{D} \otimes_C L \otimes_C A, d)$, as given after Definition 6. Then $\mathrm{o}(C/I^2, k) \in \mathrm{Ext}^2_A(k, k) \otimes_k I/\mathfrak{m}I$ is represented in the complex $\mathrm{Hom}_A(F, k) \otimes_k I/\mathfrak{m}I$ by the cocycle η given as $F_2 = \overline{L}^n_0[2] \oplus \overline{L}_2 \xrightarrow{(\mathrm{id}[2],0)} \overline{L}^n_0 = A^n$ composed with $A^n \twoheadrightarrow I/\mathfrak{m}I$, see [12, Prop. 3]. We have $s_i : L_0 \to L_1$ with $(x_1, \dots, x_e)s_i = h_i$. We may assume that $h_i = x_i + g_i$, $g_i \in \mathfrak{m}^2$ for $i = 1, \dots, n$. Hence $s_i = e_i + \delta_i$ where $I(\delta_i) \subseteq \mathfrak{m}$, and η is the coboundary induced from $[\mathrm{id} \,|\, 0] : L_1 \to L^n_0$ composed with $L^n_0 \twoheadrightarrow I/\mathfrak{m}I$.

($\Rightarrow$): Applying $-\otimes_C A$ to the short exact sequence $0 \to \mathfrak{m}_C \to C \to k \to 0$ gives a 4-term exact sequence $0 \to \mathrm{Tor}^C_1(k, A) \to \mathfrak{m}_C \otimes_C A \to A \to k \to 0$. It represents $\mathrm{o}(C/I^2, k)$, cf. [4, 3.5]. The connecting $\mathrm{Ext}^1_A(\mathfrak{m}_C, k) \to \mathrm{Ext}^2_A(k, k)$ is an isomorphism, so $\mathrm{o}(C/I^2, k) = 0$ implies that

$$0 \to \mathrm{Tor}^C_1(k, A) \longrightarrow \mathfrak{m}_C \otimes_C A \longrightarrow \mathfrak{m}_A \longrightarrow 0 \tag{26}$$

splits. Since $\mathrm{Tor}^C_1(k, A) \cong I/\mathfrak{m}I$, we have, after applying $-\otimes_A k$ to (26), a splitting $\mathfrak{m}_C/\mathfrak{m}^2_C = \mathfrak{m}_A/\mathfrak{m}^2_A \oplus I/\mathfrak{m}I$. □

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University of Oslo, Dept. of Mathematics, 0316 Oslo, NORWAY
E-mail address: ile@math.uio.no