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Variational Solutions of Semilinear Wave Equations Driven by Fractional Brownian Noise

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Abstract

This paper focuses on variational solutions of the Cauchy problem for a non-linear wave equation with space-time fractional Brownian noise driving force of Hurst index $H \in (1/2, 1)$ and random initial data. It is shown that this problem has a unique solution which depends continuously on the random initial data.

Keywords and phrases: Fractional Brownian processes, stochastic partial differential equations, variational solution, wave equations, .

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1 Introduction

Gaussian processes with independent increments and a certain self-similarity property were first studied by [Kolmogorov 1940a] and [Kolmogorov 1940b] in which they were called "Wiener spirals". They were later renamed as fractional Brownian motion in [Mandelbrot and Van Ness 1968] where a representation in terms of a stochastic integral with respect to a standard Brownian motion was given. For an encyclopedic review of the intrinsic properties of the process see the forthcoming book [Biagini *et al.*]. These processes has now found applications in such diverse fields as finance, see e.g. [Barndorff-Nielsen 1995] and the references therein, climatology and hydrology [Pelletier and Turcotte 1997], temperature modelling [Brody *et al.* 2002] and traffic networks [Leland *et al.* 1994] to name a few.

In many applications of these processes, the mathematical model is a differential equation in time, possibly also depending on spatial coordinates, in which case the model is a stochastic partial differential equation perturbed by fractional Brownian noise in some sense. An elliptic equation is treated [Hu *et al.* 2000] in a white noise setting but more often parabolic equations are on the menu. Some papers are [Nualart 2004] and [Nualart b]. To the best of authors knowledge, the only two papers dealing with hyperbolic equations are [Erraoui 2003] which considers a 1-dimensional wave equation without diffusion term, and [Duncan 2001], on a classical linear wave equation, both with additive space/time noise.

In general, hyperbolic equations are known for their notorious difficulty due to the fact that the fundamental solution is not smoothing, as in the parabolic case. Moreover, it is not even a function in dimensions greater than two but a distribution. In case the noise is not fractional but Brownian, some works exist, see e.g. [Millet 2001] for an equation appearing in relativistic quantum mechanics, and an effort has been made to extend the work on martingale measures in [Walsh 1984] to allow for distributional fundamental solutions which are then applicable to wave equations, see [Dalang 1999]. However, since a fractional Brownian process is never a martingale that approach is not applicable here.

The chosen methodin this paper is a variational one, using finite-dimensional Galerkin approximations to generate a sequence of functions, converging in a suitable space to a solution of the original equation. This paves the way for a numerical treatment which, however, is lacking in the present paper, in which focus is on existence, uniqueness, and continuity with respect to input data.

The purpose of this paper is to study stochastic hyperbolic equations with initial values of the form

$$\frac{\partial^2 u}{\partial t^2} = \mathscr{L}u + b\cdot Du + f(u) + \sigma(u)\frac{dB^H}{dt}, \qquad (u(0), u_t'(0)) = (g,h) \quad (1.1)$$

on a bounded domain U and finite time horizon $I = [0,T]$, and Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \partial U \times I. \quad (1.2)$$

The random force, B^H, is a vector valued fractional Brownian process.

Existence will be proved in a variational setting to this Cauchy problem. Continuous dependence on initial data will also be shown.

Section 2 is devoted to some preliminary result. In Section 3 the fractional Brownian noise is described. The variational setting is defined in Section 4 and then, in Section 5, the equation is properly formulated. In Section 6 a unique solution to the Galerkin approximated problem is shown to exist and the existence of a solution to the original equation is the goal of Section 7. In Section 8 we prove uniqueness and continuity with respect to initial data.

2 Preliminaries

Some parameters in the equation are assumed to have a Lipschitz continuous variable and the following trivial Lemma is needed.

Lemma 2.1. *Let $0 \in U \subset \mathbb{R}^{\mathrm{d}}$ and $V \subset \mathbb{R}^{\mathrm{m}}$. Assume $f : U \times V \mapsto \mathbb{R}$ be Lipschitz continuous in U with Lipschitz constant $L(v)$ at $v \in V$. Then*

$$|f(u,v)| \leq L(v)|u| + |f(0,v)|.$$

Proof: $|f(u,v)| \leq |f(u,v) - f(0,v)| + |f(0,v)| \leq L(v)|u| + |f(0,v)|$. □

We will on several occassions want to estimate integral averages and for these it can be shown (see [Folland 1984]) that, if $y \in I$ then

$$\sup\left\{ \frac{1}{|J|}\int_J |f|\,dx \;\middle|\; J \text{ is an interval containing } y \right\} \leq 2Hf(y), \quad (2.1)$$

where Hf is the maximal function of f. The Hardy-Littlewood-Wiener maximal theorem (see e.g. [Ziemer 1989]) gives continuity of H in any L^p, $p > 1$:

$$\|Hf\|_p \leq C_p\|f\|_p, \quad \text{for any } p \in (1,\infty].$$

If f depends on several variables and the maximal function is constructed in the k'th variable only we write $H_k(f(x_1,\ldots,x_d))(y)$ for its value at $(x_1,\ldots,x_{k-1},y,x_{k+1},\ldots,x_d)$.

3 The infinite-dimensional noise

The infinite-dimensional noise is the time derivative of the following $L^2(U)$-valued process

$$B^H(x,t) \quad = \quad \sum_{j=1}^{\infty}\sqrt{\lambda_j}e_j(x)\beta_j^H(t)$$

where $\{e_j\}_1^\infty$ is an orthonormal basis of $L^2(U)$ and $\{\beta_j^H\}_{j=1}^\infty$ is a sequence of independent, zero mean fractional Brownian motions on $\mathbb{R}$ with covariace given by

$$r(t,s) \quad = \quad \mathbf{E}\beta^H(t)\beta^H(s) \quad = \quad \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$$

and Hurst index $H \in (1/2,1)$. The noise is white in in time and correlated in space which is in agreement with the suspicion that, in many real-world processes, the correlation in time is often of a much smaller magnitude than the spatial correlation, see [Biswas and Ahmed 1985] and [Miller 1990]. The process is convergent a.s. in $L^2(U)$:

$$\mathbf{E}\|B^H(\cdot,t)\|_2^2 \quad = \quad \mathbf{E}\int_U |B^H(x,t)|^2\,dx \quad = \quad \sum_{j=1}^{\infty}\lambda_j\mathbf{E}|\beta_j^H(t)|^2 = |t|^{2H}\sum_{j=1}^{\infty}\lambda_j < \infty,$$

i.e. we are imposing the condition that the sequence $\{\sqrt{\lambda_j}\}_{j=1}^\infty \in \ell^2$ which is the same as saying that the covariance is a nuclear operator (or trace class) and that its eigenvectors and eigenvalues are $\{e_j\}_1^\infty$ and $\{\lambda_j\}_1^\infty$ respectively. Hence $\{B^H(.,t)\}_{t\in I}$ is a centered, Gaussian process with covariance given by

$$\mathbf{E}\left[\left\langle B^H(\cdot,t),\, u\right\rangle_{L^2(U)}\left\langle B^H(\cdot,s),\, v\right\rangle_{L^2(U)}\right] \quad = \quad r(t,s)\int_U\int_U u(x)\kappa(x,y)v(y)\,dx\,dy$$

where $\kappa \in L^2(U\times U)$.

We will use the following hypothesis regarding the continuity properties of the covariance operator:

$$\text{(C)} \qquad \sum_{j=1}^{\infty}\sqrt{\lambda_j}\|e_j\|_\infty \quad < \quad \infty.$$

3.1 The pathwise integral with respect to β^H

For the sake of estimating stochastic integrals with respect to a fractional Brownian motion, β^H, we will fix a constant, $\alpha \in (1-H,1)$. The following space will be needed.

Definition 3.1. *Denote by $W^{\alpha,1}(I)$ the Banach space of measurable functions $f : I \mapsto \mathbb{R}^d$ such that*

$$\|f\|_{W^{\alpha,1}(I)} \quad = \quad \int_I \frac{|f(\tau)|}{\tau^\alpha}\,d\tau + \int_I\int_0^\tau \frac{|f(\tau)-f(\theta)|}{|\tau-\theta|^{1+\alpha}}\,d\theta\,d\tau \;<\; \infty.$$

If $\{u_t\}_{t\in I}$ is a process with trajectories in $W^{\alpha,1}(I)$, then its pathwise integral with respect to fractional Brownian motion, β^H exists (see [Zähle 1998]), and we have the estimate

$$\left|\int_I u(t)\,d\beta^H(t)\right| \le G\|u\|_{\alpha,1}, \tag{3.1}$$

where G is a random variable only depending on β and having finite moments of all orders. The estimate is a result from [Nualart b] and we will use it frequently. Since we will be dealing with infinitely many fractional Brownian motions, G_j will be the random variable associated with β_j^H via (3.1). An often encountered random variable in the following is

$$\widehat{G}_\infty \quad = \quad \sum_{j=1}^{\infty}\sqrt{\lambda_j}\|e_j\|_\infty\, G_j$$

which is a.s. finite because of condition (C) and since the G_j's are independent and identically distributed with a finite moment.

4 The variational setting

We will consider variational solutions and shall therefore assume given a sequence of supposedly easily computable functions, the "elements", $\{w_n\}_{n=1}^\infty$ with each w_n belonging to $H_0^1(U)$ and such that

$$\{w_n\}_{n=1}^\infty \text{ is an orthonormal basis in } L^2(U)$$

together with

$$\{w_n\}_{n=1}^\infty \text{ is an orthogonal basis in } H_0^1(U).$$

5 The equation

The equation is

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= \mathscr{L}u(x,t) + b(x)\cdot Du(x,t) + f(x,t,u(x,t)) + \sigma(x,t,u)\frac{dB^H}{dt}(x,t), \\ u(\cdot,0) &= g(\cdot), \\ \frac{\partial u}{\partial t}(\cdot,0) &= h(\cdot), \end{aligned} \tag{5.1}$$

with Dirichlet boundary condition

$$u(x,t) = 0 \qquad (x,t)\in\partial U\times I. \tag{5.2}$$

Here $U\subset\mathbb{R}^{\mathrm{d}}$ is open and bounded with a C^1 boundary (so that Gauss divergence theorem is applicable)) and $I=[0,T]$ for some finite T. The operator $\mathscr{L}=\mathscr{L}(x)$, the diffusion term, is a second order differential operator in divergence form defined by

$$\mathscr{L}u \quad = \quad \sum_{k,l=1}^{d}\frac{\partial}{\partial x_k}\left(a_{k,l}(x)\frac{\partial u}{\partial x_l}\right)$$

The matrix $A=\{a_{k,l}\}$ satisfies the conditions

$$(\Delta)\begin{cases} a_{k,l}=a_{l,k} & \text{symmetry} \\ a_0|\xi|^2 \le \sum_{k,l=1}^{d} a_{k,l}(x)\xi_k\xi_l & \text{uniform ellipticity} \\ \sum_{k,l=1}^{d} a_{k,l}(x)\xi_k\xi_l \le A_0|\xi|^2 & \text{boundedness} \end{cases}$$

where $0<a_0\le A_0<\infty$. Du denotes the gradient of u. The transport term satisfies

$$(T) \quad b\in L^\infty(U).$$

The drift term f is Lipschitz continuous in the u-variable with integrability conditions on the Lipschitz coefficient L_f and on $f_0=f(\cdot,\cdot,0)$:

$$(D) \quad |f(x,t,u)-f(x,t,v)|\le L_f(x,t)|u-v|, \quad L_f\in L^1(I;L^\infty(U))$$

$$(D0) \qquad f_0\in L^1(I;L^2(U)).$$

The diffusion coefficient is of the form

$$\sigma(x,t,u) \;=\; \int_0^t \rho(x,t,\xi,u(x,\xi))\,d\xi, \tag{5.3}$$

where the restictions on ρ are Lipschitz continuity in the u variable together with integrability conditions on the Lipschitz coefficient L_ρ and $\rho_0=\rho(\cdot,\cdot,\cdot,0)$

$$(\delta 1)\begin{cases} |\rho(x,t,\xi,u)-\rho(x,t,\xi,v)| \;\le\; L_\rho(x,t,\xi)|u-v| \\ \int_0^T\int_\xi^T \frac{\|L_\rho(\cdot,t,\xi)\|_\infty}{(t-\xi)^\alpha}\,dt\,d\xi \;<\; \infty \\ \int_0^T\int_\xi^T \frac{\|\rho_0(\cdot,t,\xi)\|_2}{(t-\xi)^\alpha}\,dt\,d\xi \;<\; \infty. \end{cases}$$

We will need some additional smoothness in the t variable of ρ:

$$(\delta 2)\begin{cases} |\partial_2\rho(x,t,\theta,u)-\partial_2\rho(x,t,\theta,v)| \;\le\; L_{\partial_2\rho}(x,t,\theta)|u-v| \\ \int_I \|L_{\partial_2\rho}(\cdot,\cdot,\theta)\|_{L^2(I;L^\infty(U))}\,d\theta \;<\; \infty \\ \int_0^T \|\partial_2\rho_0(\cdot,\cdot,\xi)\|_{L^2(U\times I)}\,d\xi \;<\; \infty. \end{cases}$$

This gives the flexibility of treating additive noise as well as the case when the noise acts on the "time averaged" solution in the sense of by (5.3). Note however that it does not include the case of multiplicative noise and the proof in this article does not generalize as such to cover it.

By (Δ), the matrix norm of A is bounded by

$$\|A(x)\| \;\le\; A_0. \tag{5.4}$$

The initial condition will be the following: g and h are random fields on U such that

$$(I) \quad \|g\|_{L^2(U)} \quad \text{and} \quad \|g\|_{L^2(U)} \quad \text{are finite a.s.}$$

Notational convention: When a structural parameter has been estimated in its proper norm, the actual space in which the norm is defined will not always be written out. E.g., $\|L_f\|$ instaed of $\|L_f\|_{L^1(I;L^\infty(U))}$ etc.

Multiplying (5.1) with $v \in H_0^1(U)$ and integrating one gets the relation

$$\begin{aligned}
\int_U u'(x,\tau)v(x)\,dx &= \int_D h(x)v(x)\,dx \\
&+ \int_U \left\langle \int_0^\tau Du(x,\theta)\,d\theta,\ v(x)b(x) \right\rangle_{\mathbb{R}^d} dx \\
&- \int_U \left\langle \int_0^\tau Du(x,\theta)\,d\theta,\ A(x)Dv(x) \right\rangle_{\mathbb{R}^d} dx \\
&+ \int_0^\tau \int_U f(x,\theta,u(x,\theta))v(x)\,dx\,d\theta \\
&+ \int_0^\tau \int_U \sigma(x,\theta,u)v(x)\,B^H(dx,d\theta). \qquad (5.5)
\end{aligned}$$

where the stochastic integral is defined as

$$\int_0^\tau \int_U \sigma(x,\theta,u)\,B^H(dx,d\theta) = \sum_{j=1}^\infty \sqrt{\lambda_j} \int_0^\tau \langle \sigma(\cdot,\theta,u), ve_j\rangle_2\, d\beta_j^H(\theta).$$

While this equation does not contain any distributional terms we will integrate once more to involve all initial conditions in the equation. Integrating on $[0,t]$ produces the following equation

$$\begin{aligned}
\langle u(\cdot,t),v\rangle_2 &= t\langle h,v\rangle_2 + \langle g,v\rangle_2 \\
&+ \int_0^t \left\langle \int_0^\tau Du(\cdot,\theta)\,d\theta,\ vb \right\rangle_2 d\tau \\
&- \int_0^t \left\langle \int_0^\tau Du(\cdot,\theta)\,d\theta,\ ADv \right\rangle_2 d\tau \\
&+ \int_0^t \int_0^\tau \langle f(\cdot,\theta,u(\cdot,\theta)),v\rangle_2\,d\theta\,d\tau \\
&+ \sum_{j=1}^\infty \sqrt{\lambda_j} \int_0^t \int_0^\tau \langle \sigma(\cdot,\theta,u), ve_j\rangle_2\, \beta_j^H(d\theta)\,d\tau \qquad (5.6)
\end{aligned}$$

and we need not specify the initial conditions. In view of (5.6) it can be considered natural to adopt the following solution concept:

Definition 5.1. *An $L^2(U)$-valued random field $u(t)$, $t \in I$, is a variational solution to (5.1) if*

(1) $u \in L^\infty(I;L^2(U))$ *a.s.*

(2) $\int_0^\cdot Du(\cdot,\theta)\,d\theta \in L^\infty(I;(L^2(U))^{\otimes d})$ *a.s.*

(3) *The integral relation (5.6) holds a.s. for every $v \in H_0^1(U)$ and every $t \in I$.*

It should be mentioned here that condition (2) is to be understood in distribution sense. No classical differentiability properties are required of the solution. It is only the (distributional) integral of the distribution Du that is (a.s.) supposed to be in $L^\infty(I;(L^2(U))^{\otimes d})$.

One should check that all terms in (5.6) are well defined and finite in the chosen function space and this is the topic of the two following Lemmas. The first deals with all terms but the stochastic integral which has been given a separate Lemma due to its somewhat special treatment.

Lemma 5.2. *The first five terms appearing in (5.6) are well defined and finite a.s.*

Proof: For the transport term we have, by Hölder's inequality and (T),

$$\left|\left\langle \int_0^\tau Du(\cdot,\theta)\,d\theta,\ vb\right\rangle_2\right| \quad \leq \quad \|v\|_2\,\|b\|_\infty \left\|\int_0^\tau Du(\cdot,\theta)\,d\theta\right\|_2$$

which belongs to $L^\infty(I)$ by definition. Estimating the diffusion term gives, by (5.4)

$$\left|\left\langle \int_0^\tau Du(\cdot,\theta)\,d\theta,\ ADv\right\rangle_2\right| \quad \leq \quad A_0\|v\|_2 \left\|\int_0^\tau Du(\cdot,\theta)\,d\theta\right\|_2$$

which is in $L^\infty(I)$ according to definition. As for the drift term we use Lemma 2.1 and Hölder's inequality to get

$$\begin{aligned} |\langle f(\cdot,\theta,u(\cdot,\theta)),\ v\rangle_2| &\leq \big\langle L_f(\cdot,\theta)|u(\cdot,\theta)|+|f_0(\cdot,\theta)|,\ |v|\big\rangle_2 \\ &\leq \big(\|L_f(\cdot,\theta)\|_\infty\|u(\cdot,\theta)\|_2+\|f_0(\cdot,\theta)\|_2\big)\,\|v\|_2 \end{aligned} \tag{5.7}$$

which, again by Hölder's inequality, gives the bound

$$\left|\int_0^\tau \langle f(\cdot,\theta,u(\cdot,\theta)),\ v\rangle_2\,d\theta\right| \leq \quad \Big(\|L_f\|\,\|u\|_{L^\infty(I;L^2(U))}+\|f_0\|\Big)\,\|v\|_2. \qquad \square$$

To prove the same result for the last term we first show a more general estimate which will turn out to be useful later on.

Lemma 5.3. *Let $u,v\in L^\infty(I;L^2(U))$. Then there is a constant C such that the following bound on the stochastic integral is valid:*

$$\begin{aligned} &\left|\sum_{j=1}^\infty \sqrt{\lambda_j}\int_0^t\int_0^\tau \langle\sigma(\cdot,\theta,u),\ v(\cdot,\tau)e_j\rangle_2\,\beta_j^H(d\theta)\,d\tau\right| \\ &\leq \quad C\widehat{G}_\infty \int_0^t\int_\theta^t \|v(\cdot,\tau)\|_2\,d\tau\bigg[\|\partial_2\rho_0(\cdot,\cdot,\theta)\|_{L^2(I\times U)} + \int_\theta^t \frac{\|\rho_0(\cdot,\eta,\theta)\|_2}{(\eta-\theta)^\alpha}\,d\eta \\ &\qquad + \|u(\cdot,\theta)\|_2\left(\|L_{\partial_2\rho}(\cdot,\cdot,\theta)\|_{L^2(I;L^\infty(U))} + \int_\theta^t \frac{\|L_\rho(\cdot,\eta,\theta)\|_\infty}{(\eta-\theta)^\alpha}\,d\eta\right)\bigg]\,d\theta \end{aligned} \tag{5.8}$$

Proof: Using the notation $\sigma(\theta,\tau)=\langle\sigma(\cdot,\theta,u),\ v(\cdot,\tau)e_j\rangle_2$, (3.1) gives the following bound on the left hand side of (5.8)

$$\sum_{j=1}^\infty \sqrt{\lambda_j}G_j\int_0^t\left(\int_0^\tau \frac{|\sigma(\theta,\tau)|}{\theta^\alpha}\,d\theta+\int_0^\tau\int_0^\theta \frac{|\sigma(\theta,\tau)-\sigma(y,\tau)|}{|\theta-y|^{1+\alpha}}\,dy\,d\theta\right)d\tau \tag{5.9}$$

By (δ1) and Lemma 2.1

$$\begin{aligned} |\sigma(x,\theta,u)| &\leq \int_0^\theta |\rho(x,\theta,\xi,u(x,\xi))|\,d\xi \\ &\leq \int_0^\theta \big(L_\rho(x,\theta,\xi)|u(x,\xi)|+|\rho_0(x,\theta,\xi)|\big)\,d\xi. \end{aligned}$$

Fubini's theorem together with Hölder's inequality gives

$$
\begin{aligned}
|\sigma(\theta,\tau)| &= \|e_j\|_\infty \left\langle \int_0^\theta \left(L_\rho(x,\theta,\xi)|u(x,\xi)| + |\rho_0(x,\theta,\xi)|\right) d\xi,\ |v(\cdot,\tau)| \right\rangle_2 \\
&\leq \|e_j\|_\infty \, \|v(\cdot,\tau)\|_2 \int_0^\theta \left(\|L_\rho(\cdot,\theta,\xi)\|_\infty \, \|u(\cdot,\xi)\|_2 + \|\rho_0(\cdot,\theta,\xi)\|_2\right) d\xi
\end{aligned}
$$

Plugging this estimate into the first term in (5.9) and changing the order of integration we have

$$
\begin{aligned}
&\widehat{G}_\infty \int_0^t \int_0^\tau \frac{1}{\theta^\alpha} \|v(\cdot,\tau)\|_2 \int_0^\theta \left(\|L_\rho(\cdot,\theta,\xi)\|_\infty \, \|u(\cdot,\xi)\|_2 + \|\rho_0(\cdot,\theta,\xi)\|_2\right) d\xi \, d\theta \, d\tau \\
&= \widehat{G}_\infty \int_0^t \int_\xi^t \left(\|u(\cdot,\xi)\|_2 \, \frac{\|L_\rho(\cdot,\theta,\xi)\|_\infty}{\theta^\alpha} + \frac{\|\rho_0(\cdot,\theta,\xi)\|_2}{\theta^\alpha} \right) \int_\theta^t \|v(\cdot,\tau)\|_2 \, d\tau \, d\theta \, d\xi \\
&\leq \widehat{G}_\infty \int_0^t \left(\|u(\cdot,\xi)\|_2 \int_\xi^t \frac{\|L_\rho(\cdot,\theta,\xi)\|_\infty}{\theta^\alpha} \, d\theta + \int_\xi^t \frac{\|\rho_0(\cdot,\theta,\xi)\|_2}{\theta^\alpha} \, d\theta \right) \\
&\qquad\qquad \times \int_\xi^t \|v(\cdot,\tau)\|_2 \, d\tau \, d\xi \qquad (5.10)
\end{aligned}
$$

To estimate the second integral, let $0 \leq y \leq \theta \leq T$ and consider the following difference, using the triangle inequality, $(\delta 1)$ and $(\delta 2)$,

$$
\begin{aligned}
&|\sigma(x,\theta,u) - \sigma(x,y,u)| \\
&= \left| \int_0^\theta \rho(x,\theta,\xi,u(\cdot,\xi)) \, d\xi - \int_0^y \rho(x,y,\xi,u(\cdot,\xi)) \, d\xi \right| \\
&\leq \left| \int_0^y [\rho(x,\theta,\xi,u(x,\xi)) - \rho(x,y,\xi,u(x,\xi))] \, d\xi \right| + \left| \int_y^\theta \rho(x,\theta,\xi,u(x,\xi)) \, d\xi \right| \\
&= \left| \int_0^y \int_y^\theta \partial_2 \rho(x,\lambda,\xi,u(x,\xi)) \, d\lambda \, d\xi \right| + \left| \int_y^\theta \rho(x,\theta,\xi,u(x,\xi)) \, d\xi \right| \\
&\leq \int_0^y \int_y^\theta \left(L_{\partial_2\rho}(x,\lambda,\xi)|u(x,\xi)| + |\partial_2\rho_0(x,\lambda,\xi)|\right) d\lambda \, d\xi \\
&\qquad + \int_y^\theta \left(L_\rho(x,\theta,\xi)|u(x,\xi)| + |\rho_0(x,\theta,\xi)|\right) d\xi
\end{aligned}
$$

From Hölder's inequality and Fubini's theorem we then get

$$
\begin{aligned}
&|\sigma(\theta,\tau) - \sigma(y,\tau)| \\
&\leq \|e_j\|_\infty \left\langle \int_0^y \int_y^\theta \left(L_{\partial_2\rho}(\cdot,\lambda,\xi)|u(x,\xi)| + |\partial_2\rho_0(\cdot,\lambda,\xi)|\right) d\lambda \, d\xi \right. \\
&\qquad \left. + \int_y^\theta \left(L_\rho(\cdot,\theta,\xi)|u(\cdot,\xi)| + |\rho_0(\cdot,\theta,\xi)|\right) d\xi,\ |v(\cdot,\tau)| \right\rangle_2 \\
&\leq \|e_j\|_\infty \, \|v(\cdot,\tau)\|_2 \left(\int_0^y \int_y^\theta \left(\|L_{\partial_2\rho}(\cdot,\lambda,\xi)\|_\infty \, \|u(\cdot,\xi)\|_2 + \|\partial_2\rho_0(\cdot,\lambda,\xi)\|_2\right) d\lambda \, d\xi \right. \\
&\qquad \left. + \int_y^\theta \left(\|L_\rho(\cdot,\theta,\xi)\|_\infty \, \|u(\cdot,\xi)\|_2 + \|\rho_0(\cdot,\theta,\xi)\|_2\right) d\xi \right). \qquad (5.11)
\end{aligned}
$$

Let us put the two terms in (5.11) into (5.9) one at the time. Starting with the first one and changing the order of integration

$$
\begin{aligned}
&\widehat{G}_\infty \int_0^t \int_0^\tau \int_0^\theta \|v(\cdot,\tau)\|_2 \, \frac{1}{(\theta-y)^{1+\alpha}} \left(\int_0^y \|u(\cdot,\xi)\|_2 \int_y^\theta \|L_{\partial_2\rho}(\cdot,\lambda,\xi)\|_\infty \, d\lambda \, d\xi \right.\\
&\qquad \left. + \int_0^y \int_y^\theta \|\partial_2\rho_0(\cdot,\lambda,\xi)\|_2 \, d\lambda \, d\xi \right) dy \, d\theta \, d\tau \\
&\leq \quad \widehat{G}_\infty \left\{ \int_0^t \int_\xi^t \|v(\cdot,\tau)\|_2 \, d\tau \left[\|u(\cdot,\xi)\|_2 \int_\xi^t \int_\xi^\theta \frac{1}{(\theta-y)^{1+\alpha}} \right.\right.\\
&\qquad \left.\left. \times \int_y^\theta \left(\|L_{\partial_2\rho}(\cdot,\lambda,\xi)\|_\infty + \|\partial_2\rho_0(\cdot,\lambda,\xi)\|_2 \right) d\lambda \, dy \, d\theta \right] d\xi \right\} \qquad (5.12)
\end{aligned}
$$

Estimating the structural terms using the maximal and Hölder's inequality

$$
\begin{aligned}
&\int_\xi^t \int_\xi^\theta \frac{dy\,d\theta}{(\theta-y)^{1+\alpha}} \int_y^\theta \|L_{\partial_2\rho}(\cdot,\lambda,\xi)\|_\infty \, d\lambda \\
&\leq \quad C \int_\xi^t \int_\xi^\theta \frac{H_2(\|L_{\partial_2\rho}(\cdot,\lambda,\xi)\|_\infty)(\theta)}{(\theta-y)^\alpha} \, dy \, d\theta \\
&= \quad C \int_\xi^t (\theta-\xi)^{1-\alpha} H_2(\|L_{\partial_2\rho}(\cdot,\lambda,\xi)\|_\infty)(\theta) \, d\theta \\
&\leq \quad C \, \|L_{\partial_2\rho}(\cdot,\cdot,\xi)\|_{L^2(I;L^\infty(U))} \qquad (5.13)
\end{aligned}
$$

and exactly the same calculation on the $\partial_2\rho_0$ term gives similarly

$$
\int_\xi^t \int_\xi^\theta \frac{dy\,d\theta}{(\theta-y)^{1+\alpha}} \int_y^\theta \|\partial_2\rho_0(\cdot,\lambda,\xi)\|_2 \, d\lambda \quad \leq \quad C \, \|\partial_2\rho_0(\cdot,\cdot,\xi)\|_{L^2(I\times U)} \qquad (5.14)
$$

Plugging the second integral term in (5.11) into (5.9) and changing the order of integration gives the bound

$$
\begin{aligned}
&\widehat{G}_\infty \int_0^t \int_0^\tau \int_0^\theta \|v(\cdot,\tau)\|_2 \, \frac{1}{(\theta-y)^{1+\alpha}} \\
&\qquad \times \left(\int_y^\theta \|u(\cdot,\xi)\|_2 \, \|L_\rho(\cdot,\theta,\xi)\|_\infty \, d\xi + \int_y^\theta \|\rho_0(\cdot,\theta,\xi)\|_2 \, d\xi \right) dy \, d\theta \, d\tau \\
&\leq \quad \widehat{G}_\infty \int_0^t \int_\xi^t \|v(\cdot,\tau)\|_2 \, d\tau \left(\|u(\cdot,\xi)\|_2 \int_\xi^t \int_0^\xi \frac{\|L_\rho(\cdot,\theta,\xi)\|_\infty}{(\theta-y)^{1+\alpha}} \, dy \, d\theta \right.\\
&\qquad \left. + \int_\xi^t \int_0^\xi \frac{\|\rho_0(\cdot,\theta,\xi)\|_2}{(\theta-y)^{1+\alpha}} \, dy \, d\theta \right) d\xi \\
&\leq \quad C\widehat{G}_\infty \int_0^t \int_\xi^t \|v(\cdot,\tau)\|_2 \, d\tau \left(\|u(\cdot,\xi)\|_2 \int_\xi^t \frac{\|L_\rho(\cdot,\theta,\xi)\|_\infty}{(\theta-\xi)^\alpha} \, d\theta \right.\\
&\qquad \left. + \int_\xi^t \frac{\|\rho_0(\cdot,\theta,\xi)\|_2}{(\theta-\xi)^\alpha} \, d\theta \right) d\xi. \qquad (5.15)
\end{aligned}
$$

Putting (5.13) and (5.14) into (5.12) and adding the result with (5.15) results in the bound

$$
\begin{aligned}
&C\widehat{G}_\infty \int_0^t \int_\xi^t \|v(\cdot,\tau)\|_2 \, d\tau \left[\|\partial_2\rho_0(\cdot,\cdot,\xi)\|_{L^2(I\times U)} + \int_\xi^t \frac{\|\rho_0(\cdot,\theta,\xi)\|_2}{(\theta-\xi)^\alpha} \, d\theta \right.\\
&\qquad \left. + \|u(\cdot,\xi)\|_2 \left(\|L_{\partial_2\rho}(\cdot,\cdot,\xi)\|_{L^2(I;L^\infty(U))} + \int_\xi^t \frac{\|L_\rho(\cdot,\theta,\xi)\|_\infty}{(\theta-\xi)^\alpha} \, d\theta \right) \right] d\xi
\end{aligned}
$$

which actually is the final bound since the terms in (5.10) are dominated by this expression. □

It is now a simple task to show that the last term in (5.6) is bounded.

Corollary 5.4. *Let $u \in L^\infty(I;L^2(U))$ and $v \in L^2(U)$. Then there is a constant C such that the following bound on the stochastic integral is valid:*

$$\left| \sum_{j=1}^{\infty} \sqrt{\lambda_j} \int_0^t \int_0^\tau \langle \sigma(\cdot,\theta,u),\ v e_j\rangle_2\, \beta_j^H(d\theta)\, d\tau \right| \\ \leq \quad C\widehat{G}_\infty\, \|v\|_2 \left[\left(\|L_\rho\| + \|L_{\partial_2\rho}\| \right) \|u\|_{L^\infty(I;L^2(U))} \ + \ \|\partial_2 \rho_0\| \ + \ \|\rho_0\| \right].$$

Hence, equation (5.6) is well defined.

We will later on need a variant of Lemma 5.3 in the special case of $u = v$.

Corollary 5.5. *Let $u \in L^\infty(I;L^2(U))$. Then there is a constant C such that the following bound on the stochastic integral is valid:*

$$\left| \sum_{j=1}^{\infty} \sqrt{\lambda_j} \int_0^t \int_0^\tau \langle \sigma(\cdot,\theta,u),\ u(\cdot,\tau) e_j\rangle_2\, \beta_j^H(d\theta)\, d\tau \right| \\ \leq \quad C\widehat{G}_\infty \left[\int_0^t \sup_{0\leq\gamma\leq\tau} \|u(\cdot,\gamma)\|_2^2\, d\tau\ \left(1 \ + \ \|L_{\partial_2\rho}\| \ + \ \|L_\rho\|\right) + \ \|\partial_2\rho_0\|^2 \ + \ \|\rho_0\|^2 \right].$$

Proof: Chosing $v = u$ in Lemma 5.3, performing a partial integration, and using Cauchy's inequality gives the bound,

$$\begin{aligned}
C\widehat{G}_\infty & \int_0^t \int_\theta^t \|u(\cdot,\tau)\|_2\, d\tau \Bigg[\|\partial_2\rho_0(\cdot,\cdot,\theta)\|_{L^2(I\times U)} \ + \ \int_\theta^t \frac{\|\rho_0(\cdot,\eta,\theta)\|_2}{(\eta-\theta)^\alpha}\, d\eta \\
& + \|u(\cdot,\theta)\|_2 \left(\|L_{\partial_2\rho}(\cdot,\cdot,\theta)\|_{L^2(I;L^\infty(U))} \ + \ \int_\theta^t \frac{\|L_\rho(\cdot,\eta,\theta)\|_\infty}{(\eta-\theta)^\alpha}\, d\eta \right) \Bigg]\, d\theta \\
\leq \quad & C\widehat{G}_\infty \int_0^t \Bigg[\|u(\cdot,\theta)\|_2 \int_0^\theta \left(\|\partial_2\rho_0(\cdot,\cdot,\tau)\|_{L^2(I\times U)} \ + \ \int_\tau^t \frac{\|\rho_0(\cdot,\eta,\tau)\|_2}{(\eta-\tau)^\alpha}\, d\eta \right) d\tau \\
& + \int_\theta^t \sup_{0\leq\gamma\leq\tau} \|u(\cdot,\gamma)\|_2^2\, d\tau \left(\|L_{\partial_2\rho}(\cdot,\cdot,\theta)\|_{L^2(I;L^\infty(U))} \ + \ \int_\theta^t \frac{\|L_\rho(\cdot,\eta,\theta)\|_\infty}{(\eta-\theta)^\alpha}\, d\eta \right) \Bigg]\, d\theta \\
\leq \quad & C\widehat{G}_\infty \Bigg[\int_0^t \|u(\cdot,\theta)\|_2^2\, d\theta \ + \ \|\partial_2\rho_0\|^2 \ + \ \|\rho_0\|^2 \\
& + \int_0^t \sup_{0\leq\gamma\leq\tau} \|u(\cdot,\gamma)\|_2^2\, d\tau\ \left(\|L_{\partial_2\rho}\| \ + \ \|L_\rho\| \right) \Bigg] \\
\leq \quad & C\widehat{G}_\infty \left[\int_0^t \sup_{0\leq\gamma\leq\tau} \|u(\cdot,\gamma)\|_2^2\, d\tau\ \left(1 \ + \ \|L_{\partial_2\rho}\| \ + \ \|L_\rho\|\right) + \ \|\partial_2\rho_0\|^2 \ + \ \|\rho_0\|^2 \right].
\end{aligned}$$

This proves the Corollary. □

6 The finite-dimensional solution

By the former Lemmas we can now prove a simple result which will be the basis of all further investigations

Corollary 6.1. *Let u satisfy the regularity requrements* (1) - (2) *of Definition 5.1. Then u is a variational solution to (5.6) if and only if*

$$
\begin{aligned}
\langle u(\cdot,t),\, w_n\rangle_2 \;&=\; t\,\langle h,\, w_n\rangle_2 \,+\, \langle g,\, w_n\rangle_2 \,+\, \int_0^t \left\langle \int_0^\tau b\cdot Du(\cdot,\theta)\,d\theta,\, w_n \right\rangle_2 d\tau \\
&\quad - \int_0^t \int_U \left\langle \int_0^\tau Du(x,\theta)\,d\theta,\, A(x)Dw_n(x) \right\rangle_{\mathbb{R}^{\mathrm{d}}} dx\,d\tau \\
&\quad + \int_0^t \int_0^\tau \langle f(\cdot,\theta,u(\cdot,\theta)),\, w_n\rangle_2\, d\theta\, d\tau \\
&\quad + \sum_{j=1}^{\infty} \sqrt{\lambda_j} \int_0^t \int_0^\tau \left\langle \sigma(\cdot,\theta,u),\, w_n e_j\right\rangle_2 \beta_j^H(d\theta)\, d\tau \qquad (6.1)
\end{aligned}
$$

holds a.s. for every $n \in \mathbb{Z}_+$ and every $t \in I$.

Proof: Any variational solution is clearly a solution to (6.1) so we need only show the if part. Let $v \in H_0^1(U)$ have the orthogonal decomposition

$$
v(x) \;=\; \sum_{j=1}^{\infty} v_n w_n(x). \qquad (6.2)
$$

By using the properties (1)-(2) it is then trivial, except perhaps for the stochastic integral term, to note that the finite sums of (6.2) together with (6.1) will give us a sequence of equations with each term converging a.s. in $L^\infty(I;L^2(U))$ to the corresponding one in (5.6). To verify this for the stochastic integral, let $v_N(x) = \sum_1^N v_n w_n(x)$ and replace v with $v - v_N$ in Corollary 5.4. By the general assumptions, convergence follows. □

6.1 Galerkin approximation

Let V_N be the linear span of $w_1,\ldots,w_N$. To state the results in this section we need the following space:

Definition 6.2. *Let $L^\infty(I;\mathbb{R})$ be endowed with the equivalent norms*

$$
|f|_\beta \;=\; \sup_{t\in I} e^{-\beta t}|f(t)|
$$

and denote by $L^{\infty,2}(I;V_N)$ the space of functions $f : I \mapsto V_N$ equipped with the equivalent norms

$$
\|f\|_\beta \;=\; \sup_{t\in I} e^{-\beta t}\|f(\cdot,t)\|_2.
$$

Note that, since V_N is finite dimensional the norms on $L^2(V_N)$ and $H_0^1(V_N)$ are equivalent. In particular, if $u(x) = \sum_{n=1}^N c_n w_n(x)$,

$$
\|Du\|_2^2 \;=\; \sum_{n=1}^N |c_n|^2 \|Dw_n\|_2^2 \;\le\; C_N^2 \sum_{n=1}^N |c_n|^2 \;=\; C_N^2\|u\|_2. \qquad (6.3)
$$

Let φ_N denote the orthonormal projection of $\varphi \in L^2(U)$ onto V_N.

Definition 6.3. *A random field u_N is an N'th order Galerkin approximation to (6.1) if*

(1) $u_N \in L^\infty(I;L^2(U))$ *a.s.*

(2) $\int_0^\cdot Du_N(\cdot,\tau)\,d\tau \in L^\infty(I;(L^2(U)))^{\otimes d})$ *a.s.*

(3) The following equation holds a.s. for every $n \in \{1,\dots,N\}$ and every $t \in I$:

$$\begin{aligned}\langle u_N(\cdot,t),\, w_n\rangle_2 &= t\langle h_N,\, w_n\rangle_2 + \langle g_N,\, w_n\rangle_2 \\ &\quad + \int_0^t \left\langle \int_0^\tau b\cdot Du_N(\cdot,\theta)\,d\theta,\, w_n\right\rangle_2 d\tau \\ &\quad - \int_0^t \int_U \left\langle \int_0^\tau Du_N(x,\theta)\,d\theta,\, A(x)Dw_n(x)\right\rangle_{\mathbb{R}^d} dx\,d\tau \\ &\quad + \int_0^t \int_0^\tau \langle f(\cdot,\theta,u_N(\cdot,\theta)),\, w_n\rangle_2\,d\theta\,d\tau \\ &\quad + \sum_{j=1}^\infty \sqrt{\lambda_j}\int_0^t \int_0^\tau \langle \sigma(\cdot,\theta,u_N),\, w_n e_j\rangle_2\,\beta_j^H(d\theta)\,d\tau. \end{aligned} \tag{6.4}$$

Introduce the V_N-valued mapping

$$\Phi_N(u)(x,t) = \sum_{n=1}^N \langle \Phi_N(u)(\cdot,t), w_n\rangle_2 w_n(x)$$

by specifying the fourier coefficients a.s. as the right hand side of (6.4) with u_N replaced by u:

$$\begin{aligned}&\langle \Phi_N(u)(\cdot,t),\, w_n\rangle_2 \\ &= t\langle h_N,\, w_n\rangle_2 + \langle g_N,\, w_n\rangle_2 + \int_0^t \left\langle \int_0^\tau b\cdot Du(\cdot,\theta)\,d\theta,\, w_n\right\rangle_2 d\tau \\ &\quad - \int_0^t \int_U \left\langle \int_0^\tau Du(x,\theta)\,d\theta,\, A(x)Dw_n(x)\right\rangle_{\mathbb{R}^d} dx\,d\tau \\ &\quad + \int_0^t \int_0^\tau \langle f(\cdot,\theta,u(\cdot,\theta)),\, w_n\rangle_2\,d\theta\,d\tau \\ &\quad + \sum_{j=1}^\infty \sqrt{\lambda_j}\int_0^t \int_0^\tau \langle \sigma(\cdot,\theta,u),\, w_n e_j\rangle_2\,\beta_j^H(d\theta)\,d\tau \\ &= t\langle h_N,\, w_n\rangle_2 + \langle g_N,\, w_n\rangle_2 + T_n(u)(t) - E_n(u)(t) + F_n(u)(t) + S_n(u)(t)\end{aligned}$$

for every $n \in \{1,\dots,N\}$ and every $t \in I$. To solve the N'th order Galerkin approximation problem we will show existence of a fixpoint

$$\Phi_N(u_N) = u_N$$

in $L^{\infty,2}(I;V_N)$. In line of doing this we need some results concerning Lipschitz continuity with respect to u of the terms T_n, E_N, F_n and S_n. Let us start with the simplest ones; the linear terms. Introduce the notation $\Delta T_n(t) = T_n(u)(t) - T_n(u^*)(t)$ and similarly for $E_n(t),\ F_n(t)$ and $S_n(t)$.

Lemma 6.4. *Let $u \in L^{\infty,2}(I;V_N)$. Then $T_n(u), E_n(u), F_n(u) \in L^\infty(I;\mathbb{R})$. In particular,*

$$|T_n(u)_\beta \leq C\beta^{-1}\|u\|_\beta, \tag{6.5}$$

and similarly for E_n and F_n. Moreover, for every $\beta \in [1,\infty)$, the mappings

$$T_n, E_n, F_n : L^{\infty,2}(I;V_N) \mapsto L^\infty(I;\mathbb{R})$$

are Lipschitz continuous, i.e., if $\beta \geq 1$ then there is some $C = C(N)$ such that

$$|T_n(u) - T_n(u^*)|_\beta \quad \leq \quad C\beta^{-1}\|u - u^*\|_\beta \tag{6.6}$$

and similarly for $E_n(t)$, $F_n(t)$ and $S_n(t)$.

Proof: Starting with the transport term T_n we have, by Minkowski's and Hölder's inequalities together with (6.3) and (T),

$$\begin{aligned}
&e^{-\beta t}|T_n(u)(t) - T_n(u^*)(t)| \\
&\leq \quad \|b\|_\infty \, \|w_n\|_2 \, e^{-\beta t} \int_0^t \left\| \int_0^\tau (Du(\cdot,\theta) - Du^*(\cdot,\theta))\, d\theta \right\|_2 d\tau \\
&\leq \quad C_N\|b\| \int_0^t \int_0^\tau e^{-\beta\theta}\|u(\cdot,\theta) - u^*(\cdot,\theta)\|_2 e^{-\beta(t-\theta)}\, d\theta\, d\tau \\
&\leq \quad C_N\|b\| \, \|u - u^*\|_\beta \int_0^t \int_0^\tau e^{-\beta(t-\theta)}\, d\theta\, d\tau \\
&\leq \quad C\beta^{-2}\|u - u^*\|_\beta.
\end{aligned}$$

Taking the supremum over I now gives the result for T_n. Proceeding with the diffusion term E_n we get, by (5.4),

$$\begin{aligned}
&\left| \left\langle \int_0^\tau Du(x,\theta)\, d\theta,\ A(x)Dw_n(x) \right\rangle_{\mathbb{R}^d} - \left\langle \int_0^\tau Du^*(x,\theta)\, d\theta,\ A(x)Dw_n(x) \right\rangle_{\mathbb{R}^d} \right| \\
&\leq \quad \int_0^\tau |Du(x,\theta) - Du^*(x,\theta)|_{\mathbb{R}^d}\, d\theta A_0 |Dw_n(x)|_{\mathbb{R}^d}.
\end{aligned}$$

Integrating with respect to x and using Minkowski's inequality for integrals, Hölder's inequality and (6.3), results in

$$\begin{aligned}
&\int_U \left| \left\langle \int_0^\tau [Du(x,\theta) - Du^*(x,\theta)]\, d\theta,\ A(x)Dw_n(x) \right\rangle_{\mathbb{R}^d} \right| dx \\
&\leq \quad A_0 \int_0^\tau \|Du(\cdot,\theta) - Du^*(\cdot,\theta)\|_2\, d\theta \|Dw_n\|_2 \\
&\leq \quad A_0 C_N^2 \int_0^\tau \|u(\cdot,\theta) - u^*(\cdot,\theta)\|_2\, d\theta
\end{aligned} \tag{6.7}$$

Using this estimate in the β-norm gives

$$\begin{aligned}
&e^{-\beta t}|E_n(u)(t) - E_n(u^*)(t)| \\
&\leq \quad A_0 C_N^2 \int_0^t \int_0^\tau e^{-\beta\theta}\|u(\cdot,\theta) - u^*(\cdot,\theta)\|_2 e^{-\beta(t-\theta)}\, d\theta\, d\tau \\
&\leq \quad A_0 C_N^2 \|u - u^*\|_\beta \int_0^t \int_0^\tau e^{-\beta(t-\theta)}\, d\theta\, d\tau \\
&\leq \quad A_0 C\|u - u^*\|_\beta \beta^{-2}.
\end{aligned} \tag{6.8}$$

Coming to the drift term F_n, note that, by Hölder's inequality and (D),

$$|\langle f(\cdot,\theta,u),\ w_n\rangle_2 - \langle f(\cdot,\theta,u^*),\ w_n\rangle| \quad \leq \quad \|w_n\|_2 \, \|L_f(\cdot,\theta)\|_\infty \, \|u(\cdot,\theta) - u^*(\cdot,\theta)\|_2$$

Hence, again by Hölder's inequality,

$$\begin{aligned}
&e^{-\beta t}|F_n(u)(t)-F_n(u^*)(t)| \\
&\leq \int_0^t\int_0^\tau e^{-\beta(t-\theta)}\|L_f(\cdot,\theta)\|_\infty\, e^{-\beta\theta}\|u(\cdot,\theta)-u^*(\cdot,\theta)\|_2\,d\theta\,d\tau \\
&\leq \|u-u^*\|_\beta\int_0^t\int_0^\tau e^{-\beta(t-\theta)}\|L_f(\cdot,\theta)\|_\infty\,d\theta\,d\tau \\
&\leq \|u-u^*\|_\beta\int_0^t e^{-\beta(t-\tau)}\int_0^\tau \|L_f(\cdot,\theta)\|_\infty\,d\theta\,d\tau \\
&\leq C\|L_f\|_{L^1(I;L^\infty(U))}\|u-u^*\|_\beta\beta^{-1}.
\end{aligned}$$

By chosing $u^*=0$ we obtain the special case

$$|T_n(u)|_\beta \quad\leq\quad C\beta^{-2}\|u\|_\beta$$

by linearity and similarly for E_n which proves (6.5) for these terms. Because of the nonlinearity, that argument does not work for F_n. Instead we estimate the β-norm of $F_n(u)(t)$ at $u=0$ as follows:

$$\begin{aligned}
|e^{-\beta t}F_n(0)(t)| &\leq e^{-\beta t}\int_0^t\int_0^\tau |\langle f_0(\cdot,\theta),\, w_n\rangle_2|\,d\theta\,d\tau \\
&\leq C\int_0^t e^{-\beta(t-\tau)}\int_0^\tau \|f_0(\cdot,\theta)\|_2\,d\theta\,d\tau \\
&\leq C\|f_0\|\beta^{-1}
\end{aligned}$$

By the triangle inequality we now obtain

$$|F_n(u)|_\beta \;\leq\; |F_n(u)-F_n(0)|_\beta \;+\; |F_n(0)|_\beta \quad\leq\quad C\beta^{-1}(1\,+\,\|u\|_\beta) \quad<\quad \infty. \qquad \square$$

We will now prove an analogue of (6.5) for S_n together with an estimate.

Lemma 6.5. *Let $u\in L^{\infty,2}(I;V_N)$. Then $S_n(u)\in L^\infty(I;\mathbb{R})$ and the following estimate holds for all $\beta\geq 1$*

$$|S_n(u)|_\beta \quad\leq\quad C\widehat{G}_\infty\,\beta^{-1}\left[\|u\|_\beta\,\left(\|L_{\partial_2\rho}\|\,+\,\|L_\rho\|_\infty\right)+\|\partial_2\rho_0\|\,+\,\|\rho_0\|\right].$$

Proof: By (3.1), Lemma 5.3 with $v=w_n$, Hölder's inequality and a partial integration in η,

$$\begin{aligned}
&e^{-\beta t}|S_n(u)(t)| \\
&\leq C\widehat{G}_\infty\int_0^t e^{-\beta(t-\theta)}(t-\theta)\|w_n\|_2 \\
&\qquad\times\Bigg[e^{-\beta\theta}\|u(\cdot,\theta)\|_2\,\left(\|L_{\partial_2\rho}(\cdot,\cdot,\theta)\|_{L^2(I;L^\infty(U))}\,+\,\int_\theta^t\frac{\|L_\rho(\cdot,\eta,\theta)\|_\infty}{(\eta-\theta)^\alpha}\,d\eta\right) \\
&\qquad\quad+\,\|\partial_2\rho_0(\cdot,\cdot,\theta)\|_{L^2(I\times U)}\,+\,\int_\theta^t\frac{\|\rho_0(\cdot,\eta,\theta)\|_2}{(\eta-\theta)^\alpha}\,d\eta\Bigg]\,d\theta \\
&\leq C\widehat{G}_\infty\,\beta^{-1}\int_0^t\Bigg[\|u\|_\beta\,\left(\|L_{\partial_2\rho}(\cdot,\cdot,\theta)\|_{L^2(I;L^2(U))}\,+\,\int_\theta^t\frac{\|L_\rho(\cdot,\eta,\theta)\|_\infty}{(\eta-\theta)^\alpha}\,d\eta\right) \\
&\qquad\quad+\,\|\partial_2\rho_0(\cdot,\cdot,\theta)\|_{L^2(I\times U)}\,+\,\int_\theta^t\frac{\|\rho_0(\cdot,\eta,\theta)\|_2}{(\eta-\theta)^\alpha}\,d\eta\Bigg]\,d\theta \\
&\leq C\widehat{G}_\infty\,\beta^{-1}\Bigg[\|u\|_\beta\,\left(\|L_{\partial_2\rho}\|\,+\,\|L_\rho\|\right)+\,\|\partial_2\rho_0\|\,+\,\|\rho_0\|\Bigg]. \qquad\square
\end{aligned}$$

In order to prove existence of a fixpoint to Φ_N, the following invariance result is useful.

Lemma 6.6. *Let $u \in L^{\infty,2}(I;V_N)$. Then*

$$\Phi_N(u) \in L^{\infty,2}(I;V_N) \text{ a.s.}$$

and there exists a large enough random variable β_0 taking values in $(1,\infty)$ such that the closed (random) ball

$$B_N \quad = \quad \left\{u \in L^{\infty,2}(I;V_N) \,:\, \|u\|_{\beta_0} \leq 1 + 2CN\|g_N\|_2\right\}$$

is invariant a.s. with respect to Φ_N, i.e., $\Phi_N(B_N) \subset B_N$ a.s.

Proof: We have

$$\|\Phi_N(u)(t)\|_2 \quad \leq \quad \sum_{n=1}^{N} |\langle \Phi_N(u)(\cdot,t),\, w_n\rangle_2|$$

By a trivial maximization procedure the linear term has the β-norm

$$|t\,\langle h_N,\, w_n\rangle_2\,|_\beta \quad = \quad C|\,\langle h_N,\, w_n\rangle_2\,|\beta^{-1} \quad \leq \quad C\|h_N\|_2\beta^{-1}.$$

Using this estimate together with Lemmas 6.4 and 6.5 we obtain, since $\beta \geq 1$,

$$\begin{aligned}
&\|\Phi_N(u)\|_\beta \\
&\leq \; CN\left(\|h_N\|_2|\beta^{-1} + \|g_N\|_2 + |T_n(u)|_\beta + |E_n(u)|_\beta + |F_n(u)|_\beta + |S_n(u)|_\beta\right) \\
&\leq \; CN\Big(\|h_N\|_2\beta^{-1} + \|g_N\|_2 + \|u\|_\beta\beta^{-1} + \; (1+\|u\|_\beta)\beta^{-1} + (1+\|u\|_\beta)\beta^{-1}\widehat{G}_\infty\Big) \\
&\leq \; CN\Big(\|g_N\|_2 + \beta^{-1}(1+\|u\|_\beta)(1+\widehat{G}_\infty)(1+\|h_N\|_2)\Big).
\end{aligned}$$

Hence, a.s., $\Phi_N(u) \in L^{\infty,2}(I;V_N)$. Chosing the random variable β_0 to take values in the interval

$$\Big(\max(1,[(1+\widehat{G}_\infty)(1+\|h_N\|_2)2CN]),\infty\Big)$$

ensures $CN\beta_0^{-1}(1+\widehat{G}_\infty)(1+\|h_N\|_2) \leq \frac{1}{2}$ and we obtain

$$\|\Phi_N(u)\|_{\beta_0} \quad \leq \quad CN\|g_N\|_2 + \frac{1}{2}(1+\|u\|_{\beta_0}).$$

If $u \in B_N$, then $\Phi_N(u) \in B_N$ since

$$\|\Phi_N(u)\|_{\beta_0} \leq \; CN\|g_N\|_2 + \frac{1}{2}\,(1+1+2CN\|g_N\|_2) \; = \; 1+2CN\|g_N\|_2. \qquad \square$$

We will on several occassions need various continuity properties of the stochastic integral. The following Lemma provides the basis of these.

Lemma 6.7. *If $u,u^*,v \in L^\infty(I;L^2(U))$ then*

$$\begin{aligned}
&\Bigg|\sum_{j=1}^{\infty}\sqrt{\lambda_j}\int_0^t\int_0^\tau \left\langle \sigma(\cdot,\theta,u),\, v(\cdot,\tau)e_j\right\rangle_2 \beta_j^H(d\theta)\,d\tau \\
&\qquad - \sum_{j=1}^{\infty}\sqrt{\lambda_j}\int_0^t\int_0^\tau \left\langle \sigma(\cdot,\theta,u^*),\, v(\cdot,\tau)e_j\right\rangle_2 \beta_j^H(d\theta)\,d\tau\Bigg| \\
&\leq \; C\widehat{G}_\infty\int_0^t\Bigg\langle |u(\cdot,\theta)-u^*(\cdot,\theta)|,\, \int_\theta^t |v(\cdot,\tau)|\,d\tau\int_\theta^T \frac{\|L_\rho(\cdot,\eta,\theta)\|_\infty}{(\eta-\theta)^\alpha}\,d\eta \\
&\qquad + \int_\theta^t |v(\cdot,\tau)|\,d\tau\,\|L_{\partial_2\rho}(\cdot,\cdot,\theta)\|_{L^2(I;L^\infty(U))}\Bigg\rangle_2 d\theta
\end{aligned} \tag{6.9}$$

Proof: Let us use the notation $\Delta(\theta,\tau)) = \big\langle \sigma(\cdot,\theta,u) \,-\, \sigma(\cdot,\theta,u^*),\, v(\cdot,\tau)e_j \big\rangle_2$. Then, by (3.1), the expression to the left in (6.9) is bounded by

$$\begin{aligned}
&\sum_{j=1}^{\infty}\sqrt{\lambda_j}\int_0^t \left|\int_0^\tau \Delta(\theta,\tau)\, d\beta_j(\theta)\right| d\tau \\
&\quad= \sum_{j=1}^{\infty}\sqrt{\lambda_j}G_j\int_0^t\int_0^\tau \frac{|\Delta(\theta,\tau)|}{\theta^\alpha}\, d\theta\, d\tau \\
&\qquad+ \sum_{j=1}^{\infty}\sqrt{\lambda_j}G_j\int_0^t\int_0^\tau\int_0^\theta \frac{|\Delta(\theta,\tau)-\Delta(y,\tau)|}{(\theta-y)^{1+\alpha}}\, dy\, d\theta\, d\tau. \qquad (6.10)
\end{aligned}$$

Estimating $\Delta(\theta,\tau)$, using (5.3), $(\delta 1)$ and Fubini's theorem, we get

$$\begin{aligned}
|\Delta(\theta,\tau)| &\le \|e_j\|_\infty \left\langle \left|\int_0^\theta [\rho(\cdot,\theta,\xi,u(\cdot,\xi)) - \rho(\cdot,\theta,\xi,u^*(\cdot,\xi))]\, d\xi\right|,\, |v(\cdot,\tau)| \right\rangle_2 \\
&\le \|e_j\|_\infty \left\langle \int_0^\theta L_\rho(\cdot,\theta,\xi)|u(\cdot,\xi)-u^*(\cdot,\xi)|\, d\xi,\, |v(\cdot,\tau)| \right\rangle_2 \\
&= \|e_j\|_\infty \int_0^\theta \big\langle |u(\cdot,\xi)-u^*(\cdot,\xi)|,\, \|L_\rho(\cdot,\theta,\xi)\|_\infty\, |v(\cdot,\tau)| \big\rangle\, d\xi.
\end{aligned}$$

By changing the order of integration the first sum in (6.10) is bounded by

$$\begin{aligned}
&\widehat{G}_\infty \int_0^t\int_0^\tau \frac{1}{\theta^\alpha}\int_0^\theta \big\langle |u(\cdot,\xi)-u^*(\cdot,\xi)|,\, \|L_\rho(\cdot,\theta,\xi)\|_\infty\, |v(\cdot,\tau)| \big\rangle_2\, d\xi\, d\theta\, d\tau \\
&\quad= \widehat{G}_\infty \int_0^t \left\langle |u(\cdot,\xi)-u^*(\cdot,\xi)|,\, \int_\xi^t \frac{\|L_\rho(\cdot,\theta,\xi)\|_\infty}{\theta^\alpha}\int_\theta^t |v(\cdot,\tau)\|_\infty\, d\tau\, d\theta \right\rangle_2 d\xi \\
&\quad\le \widehat{G}_\infty \int_0^t \left\langle |u(\cdot,\xi)-u^*(\cdot,\xi)|,\, \int_\xi^T \frac{\|L_\rho(\cdot,\theta,\xi)\|_\infty}{(\theta-\xi)^\alpha}\, d\theta \int_\xi^t |v(\cdot,\tau)|\, d\tau \right\rangle_2 d\xi
\end{aligned}$$

Let $y \le \theta$ and manipulate the difference in the second sum as

$$\begin{aligned}
&\Delta(\theta,\tau) - \Delta(y,\tau) \\
&\quad= \Big\langle \int_0^\theta [\rho(\cdot,\theta,\xi,u(\cdot,\xi)) - \rho(\cdot,\theta,\xi,u^*(\cdot,\xi))]\, d\xi \\
&\qquad\qquad - \int_0^y [\rho(\cdot,y,\xi,u(\cdot,\xi)) - \rho(\cdot,y,\xi,u^*(\cdot,\xi))]\, d\xi,\, v(\cdot,\tau)e_j \Big\rangle_2 \\
&\quad= \Big\langle \int_y^\theta [\rho(\cdot,\theta,\xi,u(\cdot,\xi)) - \rho(\cdot,\theta,\xi,u^*(\cdot,\xi))]\, d\xi \\
&\qquad\qquad + \int_0^y \big[\rho(\cdot,\theta,\xi,u(\cdot,\xi)) - \rho(\cdot,\theta,\xi,u^*(\cdot,\xi)) \\
&\qquad\qquad - \rho(\cdot,y,\xi,u(\cdot,\xi)) + \rho(\cdot,y,\xi,u^*(\cdot,\xi))\big]\, d\xi,\, v(\cdot,\tau)e_j \Big\rangle_2 \\
&\quad= \Big\langle \int_y^\theta [\rho(\cdot,\theta,\xi,u(\cdot,\xi)) - \rho(\cdot,\theta,\xi,u^*(\cdot,\xi))]\, d\xi \\
&\qquad + \int_0^y\int_y^\theta [\partial_2\rho(\cdot,\lambda,\xi,u(\cdot,\xi)) - \partial_2\rho(\cdot,\lambda,\xi,u^*(\cdot,\xi))]\, d\lambda\, d\xi,\, v(\cdot,\tau)e_j \Big\rangle_2 .
\end{aligned}$$

Continuing by taking the absolute value, using Fubibi's theorem, $(\delta 1)$ and $(\delta 2)$, gives the bound

$$\begin{aligned}
&|\Delta(\theta,\tau)-\Delta(y,\tau)| \\
&\le \int_y^\theta \left\langle L_\rho(\cdot,\theta,\xi)|u(\cdot,\xi)-u^*(\cdot,\xi)|,\ |v(\cdot,\tau)|\ |e_j|\right\rangle_2 d\xi \\
&\quad +\int_0^y\int_y^\theta \left\langle L_{\partial_2\rho}(\cdot,\lambda,\xi)|u(\cdot,\xi)-u^*(\cdot,\xi)|,\ |v(\cdot,\tau)|\ |e_j|\right\rangle_2 d\lambda\, d\xi \\
&\le \|e_j\|_\infty \int_y^\theta \left\langle |u(\cdot,\xi)-u^*(\cdot,\xi)|,\ \|L_\rho(\cdot,\theta,\xi)\|_\infty\, |v(\cdot,\tau)|\right\rangle_2 d\xi \\
&\quad + \|e_j\|_\infty \int_0^y \left\langle |u(\cdot,\xi)-u^*(\cdot,\xi)|,\ \int_y^\theta \|L_{\partial_2\rho}(\cdot,\lambda,\xi)\|_\infty d\lambda |v(\cdot,\tau)|\right\rangle d\xi
\end{aligned}$$

We plug this into (6.10) one term at the time. Starting with the first one and changing the order of integration gives

$$\widehat{G}_\infty \int_0^t\int_0^\tau\int_0^\theta \frac{dy\, d\theta\, d\tau}{(\theta-y)^{1+\alpha}}\int_y^\theta \left\langle |u(\cdot,\xi)-u^*(\cdot,\xi)|,\ \|L_\rho(\cdot,\theta,\xi)\|_\infty d\lambda |v(\cdot,\tau)|\right\rangle_2 d\xi \le \widehat{G}_\infty \int_0^t \langle |u(\cdot,\xi)-u^*(\cdot,\xi)|,\ I\rangle_2 \tag{6.11}$$

where

$$\begin{aligned}
I &= \int_0^\xi\int_\xi^t\int_\xi^\tau \frac{\|L_\rho(\cdot,\theta,\xi)\|_\infty\, |v(\cdot,\tau)|}{(\theta-y)^{1+\alpha}}\, d\theta\, d\tau\, dy \\
&\le C\int_\xi^t |v(\cdot,\tau)|\int_\xi^\tau \frac{\|L_\rho(\cdot,\theta,\xi)\|_\infty}{(\theta-\xi)^\alpha}\, d\theta\, d\tau \\
&\le C\int_\xi^t |v(\cdot,\tau)|\, d\tau \int_\xi^T \frac{\|L_\rho(\cdot,\theta,\xi)\|_\infty}{(\theta-\xi)^\alpha}\, d\theta
\end{aligned} \tag{6.12}$$

Continuing with the second term we get

$$\widehat{G}_\infty \int_0^t\int_0^\tau\int_0^\theta \frac{dy\, d\theta\, d\tau}{(\theta-y)^{1+\alpha}}\int_0^y\int_y^\theta \|L_{\partial_2\rho}(\cdot,\lambda,\xi)\|_\infty d\lambda \left\langle |u(\cdot,\xi)-u^*(\cdot,\xi)|,\ |v(\cdot,\tau)|\right\rangle_2 d\xi \le \widehat{G}_\infty \int_0^t \langle |u(\cdot,\xi)-u^*(\cdot,\xi)|,\ I\rangle_2\, d\xi \tag{6.13}$$

where the maximal and Hölder's inequalities gives

$$\begin{aligned}
I &= \int_\xi^t\int_y^t\int_\theta^t \frac{|v(\cdot,\tau)|}{(\theta-y)^{1+\alpha}}\int_y^\theta \|L_{\partial_2\rho}(\cdot,\lambda,\xi)\|_\infty d\lambda\, d\tau\, d\theta\, dy \\
&\le \int_\xi^t |v(\cdot,\tau)|\, d\tau \int_\xi^t\int_\xi^\theta \frac{1}{(\theta-y)^{1+\alpha}}\int_y^\theta \|L_{\partial_2\rho}(\cdot,\lambda,\xi)\|_\infty d\lambda\, dy\, d\theta \\
&\le \int_\xi^t |v(\cdot,\tau)\, d\tau| \int_\xi^t\int_\xi^\theta \frac{1}{(\theta-y)^{\alpha}} H_2(\|L_{\partial_2\rho}(\cdot,\lambda,\xi)\|_\infty)(\theta)\, dy\, d\theta \\
&\le \int_\xi^t |v(\cdot,\tau)\, d\tau| \int_\xi^t (\theta-\xi)^{1-\alpha} H_2(\|L_{\partial_2\rho}(\cdot,\lambda,\xi)\|_\infty)(\theta)\, d\theta \\
&\le \int_\xi^t |v(\cdot,\tau)\, d\tau\, \|L_{\partial_2\rho}(\cdot,\cdot,\xi)\|_{L^2(I;L^\infty(U))}
\end{aligned} \tag{6.14}$$

Putting (6.12) back into (6.11) and (6.14) back into (6.13) and summing we find the estimate

$$C\widehat{G}_\infty \int_0^t \Big\langle |u(\cdot,\theta)-u^*(\cdot,\theta)|,\ \int_\theta^t |v(\cdot,\tau)|\,d\tau \int_\theta^T \frac{\|L_\rho(\cdot,\eta,\theta)\|_\infty}{(\eta-\theta)^\alpha}\,d\eta$$
$$+\int_\theta^t |v(\cdot,\tau)|\,d\tau\ \|L_{\partial_2\rho}(\cdot,\cdot,\theta)\|_{L^2(I;L^\infty(U))}\Big\rangle_2\,d\theta$$

which is the desired upper bound since it dominates the first sum in (6.10). □

Specilizing to S_n, i.e., changing v fir w_n, gives

Lemma 6.8. *If $u,u^*\in L^\infty(I;L^2(U))$ then*

$$\begin{aligned}&|S_n(u)(t)-S_n(u^*)(t)|\\ &\le\ C\widehat{G}_\infty\int_0^t (t-\theta)\Big\langle |u(\cdot,\theta)-u^*(\cdot,\theta)|,\ |w_n|\int_\theta^T \frac{\|L_\rho(\cdot,\eta,\theta)\|_\infty}{(\eta-\theta)^\alpha}\,d\eta\\ &\quad+|w_n|\ \|L_{\partial_2\rho}(\cdot,\cdot,\theta)\|_{L^2(I;L^\infty(U))}\Big\rangle_2\,d\theta.\end{aligned}$$

Lemma 6.8 allows us to prove an analogue of Lemma 6.4 for the term S_n.

Lemma 6.9. *For every $\beta\in[1,\infty)$, the mapping $S_n: L^{\infty,2}(I;V_N)\mapsto L^\infty(I;\mathbb{R})$ is Lipschitz continuous and we have the estimate*

$$|S_n(u)-S_n(u^*)|_\beta\ \le\ C\widehat{G}_\infty\|u-u^*\|_\beta\beta^{-1}.$$

Proof: Using Lemma 6.8, Hölder's inequality, and a trivial maximization procedure

$$\begin{aligned}&e^{-\beta t}|S_n(u)(t)-S_n(u^*)(t)|\\ &\le\ C\widehat{G}_\infty\int_0^t e^{-\beta(t-\theta)}(t-\theta)e^{-\beta\theta}\|u(\cdot,\theta)-u^*(\cdot,\theta)\|_2\ \|w_n\|_2\\ &\quad\times\Big[\int_\theta^T\frac{\|L_\rho(\cdot,\eta,\theta)\|_\infty}{(\eta-\theta)^\alpha}\,d\eta+\|L_{\partial_2\rho}(\cdot,\cdot,\theta)\|_{L^2(I;L^\infty(U))}\Big]\,d\theta\\ &\le\ C\widehat{G}_\infty\,\|u-u^*\|_\beta\beta^{-1}\int_0^t\Big[\int_\theta^T\frac{\|L_\rho(\cdot,\eta,\theta)\|_\infty}{(\eta-\theta)^\alpha}\,d\eta\ +\ \|L_{\partial_2\rho}(\cdot,\cdot,\theta)\|_{L^2(I;L^\infty(U))}\Big]\,d\theta\\ &=\ C\widehat{G}_\infty\,\|u-u^*\|_\beta\beta^{-1}\Big[\|L_\rho\|+\ \|L_{\partial_2\rho}\|\Big].\end{aligned}$$
□

The next Lemma is on a contraction property of Φ_N, crucial in the fixpoint argument which will provide the N'th order Galerkin approximation to (6.1).

Lemma 6.10. *There exists a random variable $\beta_1\in[1,\infty)$ such that the map Φ_N is a contraction on $\Phi_N(B_N)$ with respect to the norm $\|\cdot\|_{\beta_1}$: if $u,u^*\in B_N$ then*

$$\|\Phi_N(u)-\Phi_N(u^*)\|_{\beta_1}\ \le\ \frac{1}{2}\|u-u^*\|_{\beta_1} \tag{6.15}$$

Proof: Let $u,u^*\in L^{\infty,2}(I;V_N)$. Then, by the Lipschitz continuity of the terms T_n, E_n, F_n and S_n (Lemmas 6.4 and 6.9) we find that

$$\begin{aligned}\|\Phi_N(u)-\Phi_N(u^*)\|_\beta\ &\le\ \sum_{n=1}^N\|\langle\Phi_N(u)(\cdot,\cdot),\ w_n\rangle_2\,w_n-\langle\Phi_N(u^*)(\cdot,\cdot),\ w_n\rangle_2\,w_n\|_\beta\\ &\le\ \sum_{n=1}^N\big(|\Delta T_n|_\beta+|\Delta E_n|_\beta+|\Delta F_n|_\beta+|\Delta S_n|_\beta\big)\\ &\le\ CN(1+\widehat{G}_\infty)\|u-u^*\|_\beta\beta^{-1}\end{aligned} \tag{6.16}$$

Let $u, u^* \in B_N$. Then, chosing the random variable β_1 to take any value in the interval

$$\left(\max\left(1,\ 2CN(1+\widehat{G}_\infty)\right),\ \infty\right)$$

ensures the conclusion (6.15). □

Proposition 6.11. *The map Φ_N has a fix point $u_N \in L^{\infty,2}(I;V_N)$ for every positive integer N. Moreover, $u_N \in B_N$.*

Proof: The argument is identical to the existence part of Proposition 2 in [Nualart b] and is included for easy reference only. Chose a random variable $\beta_2 \geq \max(\beta_0, \beta_1)$ for which the conclusions of both Lemma 6.6 and 6.10 hold. Fix any $u_{0,N} \in B_N$ and define recursively $u_{m+1,N} = \Phi_N(u_{m,N})$. Then the invariance property of Lemma 6.6 implies that $u_{m,N} \in \Phi_N(B_N)$ for every $m \in \mathbb{N}$. By the contraction property of Lemma 6.10 we may conclude that, for $m \leq n$,

$$\begin{aligned} \|u_{n,N} - u_{m,N}\|_{\beta_2} &\leq \left(\frac{1}{2}\right)^m \|u_{n-m,N} - u_{0,N}\|_{\beta_2} \\ &\leq \left(\frac{1}{2}\right)^m 2 \sup_{u \in B_N} \|u\|_{\beta_2} \\ &\leq \left(\frac{1}{2}\right)^{m-1} (1 + 2CN\|g\|_2) \ \to\ 0, \text{ as } m, n \to \infty. \end{aligned}$$

Hence the sequence $\{u_{m,N}\}_{m=0}^{\infty}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{\beta_2}$. Defining the limit as $u_N = \lim_{m\to\infty} u_{m,N}$ we have, by the invariance property of Lemma 6.6 once more, $u_{m,N} \in B_N$ and hence $u_N \in B_N$ since B_N is closed. The contractivity property of Lemma 6.10 now allows us to deduce that

$$\|\Phi_N(u_N) - \Phi_N(u_{m,N})\|_{\beta_2} \quad \leq \quad \frac{1}{2}\|u_N - u_{m,N}\|_{\beta_2} \ \to\ 0 \text{ as } m \to \infty.$$

Since the norms are equivalent for all $\beta \geq 1$ the conclusion holds for every particular choice of β, say, $\beta = 1$. This implies the following set of equalitites in $L^{\infty,2}(I;V_N)$ and finishes the proof

$$\Phi_N(u_N) \;=\; \lim_{m\to\infty} \Phi_N(u_{m,N}) \;=\; \lim_{m\to\infty} u_{m+1,N} \;=\; u_N.$$ □

This far we have shown that the Galerkin approximation has a unique solution. Note how all arguments are done pathwisely, for a fixed, but arbitrary path ω.

7 Existence of solutions

The next step is to let $N \to \infty$ in the Galerkin sequence u_N and we will discover that $\{u_N\}_{N=1}^{\infty}$ is suitably bounded and has a subsequence that converges a.s. to a solution of equation (6.1). We start with a simple Lemma.

Lemma 7.1. *Let $c \in \mathbb{R}$ and $v \in L^1(I)$. If*

$$\eta(t) \;=\; c + \int_0^t v(\tau)\,d\tau, \qquad t \in I, \tag{7.1}$$

then we have

$$\eta^2(t) \;=\; c^2 + 2\int_0^t \eta(\tau)v(\tau)\,d\tau, \qquad t \in I. \tag{7.2}$$

Proof: We may w.l.o.g. assume $c = 0$ and $t > 0$. By partial integration

$$\begin{aligned}
\int_0^t \eta(\tau)v(\tau)\,d\tau &= \int_0^t \left(\int_0^\tau v(\theta)\,d\theta\right) v(\tau)\,d\tau \\
&= \left(\int_0^t v(\tau)\,d\tau\right)^2 - \int_0^t \left(\int_0^\tau v(\theta)\,d\theta\right) v(\tau)\,d\tau \\
&= \left(\int_0^t v(\tau)\,d\tau\right)^2 - \int_0^t \eta(\tau)v(\tau)\,d\tau,
\end{aligned}$$

which gives (7.2). □

Remark. Let

$$\begin{aligned}
c &= \langle g_N,\, w_n\rangle \\
v(\tau) &= \langle h_N,\, w_n\rangle + \left\langle \int_0^\tau b\cdot Du_N(\cdot,\theta)\,d\theta,\, w_n \right\rangle_2 \\
&\quad + \int_0^\tau \langle f(\cdot,\theta,u_N(\cdot,\theta)),\, w_n\rangle_2\,d\theta - \left\langle \int_0^\tau Du_N(\cdot,\theta)\,d\theta,\, ADw_n \right\rangle_{(L^2(U))^{\otimes d}} \\
&\quad + \sum_{j=1}^\infty \sqrt{\lambda_j} \int_0^\tau \left\langle \sigma(\cdot,\theta,u_N),\, w_n e_j \right\rangle_2 \beta_j(d\theta).
\end{aligned}$$

Hence, with this choice of c and v, (7.1) is the same equation as (6.4) with $\eta(t) = \langle u_N(\cdot,t),\, w_n\rangle_2$. It follows from the proof of Lemma 5.2 and Lemma 5.4 that this choice of parameters satisfy the conditions of Lemma 7.1.

Proposition 7.2. *$\{u_N\}_{N=1}^\infty$ and $\{\int_0^{\cdot} Du_N(\cdot,\theta)\,d\theta\}_{N=1}^\infty$ are a.s. bounded sequences in $L^\infty(I;L^2(U))$ and $L^\infty(I;(L^2(U))^{\otimes d})$ respectively.*

Proof: Square (6.4) using Lemma 7.1 to get

$$\begin{aligned}
&\langle u_N(\cdot,t),\, w_n\rangle_2^2 \\
&\quad = \langle g_N,\, w_n\rangle_2^2 + 2\langle h_N,\, w_n\rangle_2 t \int_0^t \langle u_N(\cdot,\tau),\, w_n\rangle_2\,d\tau \\
&\qquad + 2\int_0^t \left\langle \int_0^\tau b\cdot Du_N(\cdot,\theta)\,d\theta,\, \langle u_N(\cdot,\tau),\, w_n\rangle_2 w_n \right\rangle_2 d\tau \\
&\qquad - 2\int_0^t \left\langle A\int_0^\tau Du_N(\cdot,\theta)\,d\theta,\, \langle u_N(\cdot,\tau),\, w_n\rangle_2 Dw_n \right\rangle_2 d\tau \\
&\qquad + 2\int_0^t \int_0^\tau \langle f(\cdot,\theta,u_N(\cdot,\theta)),\, \langle u_N(\cdot,\tau),\, w_n\rangle_2 w_n\rangle_2\,d\theta\,d\tau \\
&\qquad + 2\sum_{j=1}^\infty \sqrt{\lambda_j} \int_0^t \int_0^\tau \left\langle \sigma(\cdot,\theta,u_N),\, e_j\,\langle u_N(\cdot,\tau),\, w_n\rangle_2 w_n \right\rangle d\beta_j(\theta)\,d\tau.
\end{aligned}$$

Summing over $n \in \{1,\dots,N\}$ gives

$$\begin{aligned}
&\|u_N(\cdot,t)\|_2^2 + 2\int_0^t \left\langle A\int_0^\tau Du_N(\cdot,\theta)\,d\theta,\ Du_N(\cdot,\tau)\right\rangle_2 d\tau \\
&= \|g_N\|_2^2 + 2t\int_0^t \langle h_N,\ u_N(\cdot,\tau)\rangle_2\,d\tau + 2\int_0^t \left\langle \int_0^\tau b\cdot Du_N(\cdot,\theta)\,d\theta,\ u_N(\cdot,\tau)\right\rangle_2 d\tau \\
&\quad + 2\int_0^t\int_0^\tau \langle f(\cdot,\theta,u_N(\cdot,\theta)),\ u_N(\cdot,\tau)\rangle\,d\theta\,d\tau \\
&\quad + 2\sum_{j=1}^\infty \sqrt{\lambda_j}\int_0^t\int_0^\tau \left\langle \sigma(\cdot,\theta,u_N),\ u_N(\cdot,\tau)e_j\right\rangle d\beta_j(\theta)\,d\tau. \qquad (7.3)
\end{aligned}$$

The second term on the left is bounded from below as

$$\begin{aligned}
&2\int_0^t \left\langle A\int_0^\tau Du_N(\cdot,\theta)\,d\theta,\ Du_N(\cdot,\tau)\right\rangle_2 d\tau \\
&= \int_0^t \frac{d}{d\tau}\left\|A^{1/2}\int_0^\tau Du_N(\cdot,\theta)\,d\theta\right\|_2^2 d\tau \\
&= \left\|A^{1/2}\int_0^t Du_N(\cdot,\theta)\,d\theta\right\|_2^2 \\
&= \left\langle A\int_0^t Du_N(\cdot,\theta)\,d\theta,\ \int_0^t Du_N(\cdot,\theta)\,d\theta\right\rangle_2 \\
&\ge a_0\left\|\int_0^t Du_N(\cdot,\theta)\,d\theta\right\|_2^2 \qquad (7.4)
\end{aligned}$$

by the ellipticity condition (Δ). The third term on the right is bounded by

$$\begin{aligned}
&\left|2\int_0^t \left\langle \int_0^\tau b\cdot Du_N(\cdot,\theta)\,d\theta,\ u_N(\cdot,\tau)\right\rangle_2 d\tau\right| \\
&\le 2\|b\|_\infty \int_0^t \left\|\int_0^\tau Du_N(\cdot,\theta)\,d\theta\right\|_2 \|u_N(\cdot,\tau)\|_2\,d\tau \\
&\le \|b\|_\infty \int_0^t \left(\left\|\int_0^\tau Du_N(\cdot,\theta)\,d\theta\right\|_2^2 + \|u_N(\cdot,\tau)\|_2^2\right) d\tau \qquad (7.5)
\end{aligned}$$

using Hölder's and Cauchy's inequalities. For the fourth term on the right we use (5.7) to get

$$\begin{aligned}
&\left|2\int_0^t\int_0^\tau \langle f(\cdot,\theta,u_N(\cdot,\theta)),\ u_N(\cdot,\tau)\rangle\,d\theta\,d\tau\right| \\
&\le 2\int_0^t\int_0^\tau \left(\|L_f(\cdot,\theta)\|_\infty\,\|u_N(\cdot,\theta)\|_2 + \|f_0(\cdot,\theta)\|_2\right)\|u_N(\cdot,\tau)\|_2\,d\theta\,d\tau \\
&\le \int_0^t \left(\|L_f\|_{L^1(I;L^\infty(U))}\sup_{0\le\gamma\le\tau}\|u_N(\cdot,\gamma)\|_2^2 + \|f_0\|_{L^1(I;L^2(U))}\|u_N(\cdot,\tau)\|_2\right) d\tau \\
&\le C\|f_0\|_{L^1(I;L^2(U))}^2 + \left(1+\|L_f\|_{L^1(I;L^\infty(U))}\right)\int_0^t \sup_{0\le\gamma\le\tau}\|u_N(\cdot,\gamma)\|_2^2\,d\tau. \qquad (7.6)
\end{aligned}$$

By Corollary 5.5 with $u = u_N$, the last term in (7.3) is bounded in the following way

$$C\widehat{G}_\infty\left[\int_0^t \sup_{0\le\gamma\le\tau}\|u_N(\cdot,\gamma)\|_2^2\,d\tau\ \left(1 + \|L_{\partial_2\rho}\| + \|L_\rho\|\right) + \|\partial_2\rho_0\|^2 + \|\rho_0\|^2\right]. \qquad (7.7)$$

Using the estimates (7.4), (7.5), (7.6) and (7.7), together with Hölder's inequality, in (7.3) we get

$$
\begin{aligned}
&\|u_N(\cdot,t)\|_2^2 + a_0 \left\| \int_0^t Du_N(\cdot,\theta)\,d\theta \right\|_2^2 \\
&\quad\le\ \|g\|_2^2 \,+\, 2\|h\|_2 t \int_0^t \|u_N(\cdot,\tau)\|_2\,d\tau \\
&\qquad + \|b\| \int_0^t \left(\left\| \int_0^\tau Du_N(\cdot,\theta)\,d\theta \right\|_2^2 \,+\, \|u_N(\cdot,\tau)\|_2^2 \right) d\tau \\
&\qquad + C\|f_0\|^2 + (1+\|L_f\|) \int_0^t \sup_{0\le\gamma\le\tau} \|u_N(\cdot,\gamma)\|_2^2\,d\tau \,+\, C\widehat{G}_\infty \left(\|\rho_0\|^2 \,+\, \|\partial_2\rho_0\|^2 \right) \\
&\qquad + C\widehat{G}_\infty \left(1 \,+\, \|L_\rho\| \,+\, \|L_{\partial_2\rho}\| \right) \int_0^t \sup_{0\le\gamma\le\tau} \|u_N(\cdot,\gamma)\|_2^2\,d\tau
\end{aligned}
$$

By Cauchy's inequality and rearranging terms on the right hand side results in

$$
\begin{aligned}
&\|g\|_2^2 \,+\, \|h\|_2^2 + C \left[\|f_0\| \,+\, \widehat{G}_\infty \left(\|\rho_0\|^2 \,+\, \|\partial_2\rho_0\|^2 \right) \right] \\
&\qquad + C \left[1 \,+\, \|b\| \,+\, \|L_f\| \,+\, \widehat{G}_\infty \left(1 \,+\, \|L_\rho\| \,+\, \|L_{\partial_2\rho}\| \right) \right] \\
&\qquad\quad \times \int_0^t \sup_{0\le\gamma\le\tau} \left(\left\| \int_0^\gamma Du_N(\cdot,\eta)\,d\eta \right\|_2^2 \,+\, \|u_N(\cdot,\gamma)\|_2^2 \right) d\tau \\
&\le\ \|g\|_2^2 \,+\, \|h\|_2^2 \,+\, CQ_0 \,+\, CQ \int_0^t M(\tau)\,d\tau,
\end{aligned}
\tag{7.8}
$$

where we have defined

$$
\begin{aligned}
Q_0 &= \|f_0\| \,+\, \widehat{G}_\infty \left(\|\rho_0\|^2 \,+\, \|\partial_2\rho_0\|^2 \right), \\
Q &= 1 \,+\, \|b\| \,+\, \|L_f\| \,+\, \widehat{G}_\infty \left(1 \,+\, \|L_\rho\| \,+\, \|L_{\partial_2\rho}\| \right), \text{ and} \\
M(\tau) &= \sup_{0\le\gamma\le\tau} \left(\left\| \int_0^\gamma Du_N(\cdot,\eta)\,d\eta \right\|_2^2 \,+\, \|u_N(\cdot,\gamma)\|_2^2 \right)
\end{aligned}
$$

Hence we have the inequality

$$
M(t) \ \le\ C \left(\|g\|_2^2 \,+\, \|h\|_2^2 \,+\, Q_0 \,+\, Q \int_0^t M(\tau)\,d\tau \right)
$$

since the right hand side of (7.8) is a nondecreasing function of t. By Gronwall's

$$
M(t) \ \le\ C(\|g\|_2^2 + \|h\|_2^2 + Q_0)\left(1 + Qe^{CQt}\right) \ \le\ C(\|g\|_2^2 + \|h\|_2^2 + Q_0)Qe^{CQt}.
$$

Putting $t = T$ finishes the Proposition. □

Proposition 7.3. *There is an element $\tilde{u} \in L^\infty(I; L^2(U))$ such that, in distribution sense,*

$$
\int_0^{\cdot} D\tilde{u}(\cdot,\tau)\,d\tau \in L^\infty(I; (L^2(U))^{\otimes d}).
$$

Moreover, there is a subsequence of $\{u_N\}_{N=1}^{\infty}$ *such that, a.s.,*

$$(1) \qquad \int_I \int_U u_N(x,t)\,\psi(x,t)\,dx\,dt \to \int_I \int_U \tilde{u}(x,t)\,\psi(x,t)\,dx\,dt$$

$$(2) \qquad \int_I \int_U \left(\int_0^t Du_N(x,\tau)\,d\tau\right)\Gamma(x,t)\,dx\,dt \to \int_I \int_U \left(\int_0^t D\tilde{u}(x,\tau)\,d\tau\right)\Gamma(x,t)\,dx\,dt$$

as $N \to \infty$, *for every* $\psi \in L^1(I;L^2(U))$ *and every* $\Gamma \in L^1(I;(L^2(U))^{\otimes d})$.

Proof: Since, by Lemma 7.2, $\{u_N\}$ is, a.s., a bounded sequence in $L^\infty(I;L^2(U))$, which is the dual of $L^1(I;L^2(U))$, Alaoglu's Theorem implies there is a subsequence, also denoted by $\{u_N\}$, and an element $\tilde{u} \in L^\infty(I;L^2(U))$ such that $u_N \to \tilde{u}$, a.s., in the weak* topology of $L^\infty(I;L^2(U))$. This means that, with probability one,

$$\langle u_N,\ \psi\rangle \to \langle \tilde{u}, \psi\rangle, \qquad \forall \psi \in L^1(I;L^2(U)), \tag{7.9}$$

where $\langle f_1,\ f_2\rangle$ is short for the integral of the product $f_1 f_2$ over $U \times I$. For later use we note that (7.9) is equivalent to

$$\langle |u_N - \tilde{u}|,\ \psi\rangle \ \to\ 0 \qquad \forall \psi \in L^1(I;L^2(U))$$

since $|u_N - \tilde{u}| = (u_N - \tilde{u})\mathrm{sgn}\,(u_N - \tilde{u})$ and $\psi \in L^1(I;L^2(U))$ if and only if $\psi\mathrm{sgn}\,(u_N - \tilde{u}) \in L^1(I;L^2(U))$. Similarly, by passing to still another subsequence we have, for some $v \in L^\infty(I;(L^2(U))^{\otimes d})$,

$$\left\langle \int_0^{\cdot} Du_N(\cdot,\tau)\,d\tau,\ \Gamma\right\rangle \to \langle v,\ \Gamma\rangle, \qquad \forall \Gamma \in L^1(I;(L^2(U))^{\otimes d}).$$

We will now identify v. Let $\varphi : U \times I \mapsto \mathbb{R}^{\mathrm{d}}$ be smooth. By a partial integration on I and since u_N is 0 on the boundary ∂U, the Gauss' divergence theorem gives

$$\begin{aligned}
&\int_U \int_I \left\langle \int_0^t Du_N(x,\tau)\,d\tau,\ \varphi(x,t)\right\rangle_{\mathbb{R}^{\mathrm{d}}} dt\,dx \\
&= \int_U \Bigg(\left\langle \int_I Du_N(x,\tau)\,d\tau,\ \int_I \varphi(x,\tau)\,d\tau \right\rangle_{\mathbb{R}^{\mathrm{d}}} \\
&\qquad - \int_I \left\langle Du_N(x,t),\ \int_0^t \varphi(x,\tau)\,d\tau \right\rangle_{\mathbb{R}^{\mathrm{d}}} dt \Bigg)\,dx \\
&= \int_I \int_U \left\langle Du_N(x,t),\ \int_I \varphi(x,\tau)\,d\tau - \int_0^t \varphi(x,\tau)\,d\tau \right\rangle_{\mathbb{R}^{\mathrm{d}}} dx\,dt \\
&= \int_I \Bigg(\int_{\partial U} u_N(x,t) \left\langle \int_I \varphi(x,\tau)\,d\tau - \int_0^t \varphi(x,\tau)\,d\tau,\ N(x) \right\rangle_{\mathbb{R}^{\mathrm{d}}} dS(x) \\
&\qquad - \int_U u_N(x,t)\ \mathrm{div}_x \left[\int_I \varphi(x,\tau)\,d\tau - \int_0^t \varphi(x,\tau)\,d\tau \right] dx \Bigg)\,dt \\
&= \int_I \int_U u_N(x,t)\ \mathrm{div}_x \left[\int_0^t \varphi(x,\tau)\,d\tau - \int_I \varphi(x,\tau)\,d\tau \right] dx\,dt
\end{aligned}$$

where $N(x)$ is the exterior unit normal at $x \in \partial U$. Hence, in the limit we get

$$\langle v,\ \varphi\rangle = \left\langle \tilde{u},\ \mathrm{div}_x \left[\int_0^{\cdot} \varphi(\cdot,\tau)\,d\tau - \int_I \varphi(\cdot,\tau)\,d\tau \right] \right\rangle$$

by (7.9). On the other hand, by treating $\tilde{u}$ as an element of the space of compactly supported distributions $\mathscr{E}'(\mathbb{R}^{d+1})$ and noting that

$$\mathrm{div}_x \left[\int_0^{\cdot} \varphi(\cdot,\tau)\, d\tau - \int_I \varphi(\cdot,\tau)\, d\tau \right] \in C^\infty(\mathbb{R}^{d+1}) \;=\; \mathscr{E}(\mathbb{R}^{d+1})$$

we can do the following calculations which are true almost by definition.

$$\begin{aligned}
&\left\langle \tilde{u},\ \mathrm{div}_x \left[\int_0^{\cdot} \varphi(\cdot,\tau)\, d\tau - \int_I \varphi(\cdot,\tau)\, d\tau \right] \right\rangle \\
&= \quad -\left\langle D\tilde{u},\ \int_0^{\cdot} \varphi(\cdot,\tau)\, d\tau - \int_I \varphi(\cdot,\tau)\, d\tau \right\rangle \\
&= \quad -\left\langle \left(\int_0^t D\tilde{u}(\cdot,\tau)\, d\tau \right)',\ \int_0^{\cdot} \varphi(\cdot,\tau)\, d\tau - \int_I \varphi(\cdot,\tau)\, d\tau \right\rangle \\
&= \quad \left\langle \int_0^{\cdot} D\tilde{u}(\cdot,\tau)\, d\tau,\ \varphi \right\rangle .
\end{aligned}$$

This proves that, in distribution sense, i.e., multiplied by smooth functions φ and put under the integral sign,

$$\int_0^t D\tilde{u}(x,\tau)\, d\tau \quad = \quad v(x,t) \in L^\infty(I; L^2(U)).$$

By a standard limiting procedure (2) then holds for every $\Gamma \in L^1(I; L^2(U))$. $\square$

It is natural to hope $\tilde{u}$ qualifies as a solution to (6.1). We will send N to ∞ in each term of (6.1) separately and discover this very fact.

Theorem 7.4. *There exists a solution to (6.1).*

Proof: With the help of Proposition 7.3 we will now show that each term in (6.4) converges a.s. on $\mathbb{R}$ to the corresponding term in (6.1) from which the theorem follows. It is immediate that the terms involving initial data converge to the same ones with u_N replaced by $\tilde{u}$. For the next term we have

$$\begin{aligned}
\int_0^t \left\langle b \cdot \int_0^\tau Du_N(\cdot,\theta)\, d\theta,\ w_n \right\rangle_2 d\tau \quad &= \quad \int_I \left\langle \int_0^\tau Du_N(\cdot,\theta)\, d\theta,\ 1_{[0,t]}(\tau) w_n b \right\rangle_2 d\tau \\
&\to \quad \int_I \left\langle \int_0^\tau D\tilde{u}(\cdot,\theta)\, d\theta,\ 1_{[0,t]}(\tau) w_n b \right\rangle_2 d\tau \\
&= \quad \int_0^t \left\langle b \cdot \int_0^\tau D\tilde{u}(\cdot,\theta)\, d\theta, w_n \right\rangle_2 d\tau,
\end{aligned}$$

where the limit holds true by Proposition 7.3 since, the function $\Gamma = 1_{[0,t]} w_n b$ belongs to $\in L^1(I; L^2(U))^{\otimes d}$. Coming next to the diffusion term we have similarly

$$\begin{aligned}
&\int_0^t \int_U \left\langle \int_0^\tau Du_N(x,\theta)\, d\theta,\ A(x) Dw_n(x) \right\rangle_{\mathbb{R}^d} dx\, d\tau \\
&= \quad \int_I \int_U \left\langle \int_0^\tau Du_N(x,\theta)\, d\theta,\ 1_{[0,t]}(\tau) A(x) Dw_n(x) \right\rangle_{\mathbb{R}^d} dx\, d\tau \\
&\to \quad \int_I \int_U \left\langle \int_0^\tau D\tilde{u}(x,\theta)\, d\theta,\ 1_{[0,t]}(\tau) A(x) Dw_n(x) \right\rangle_{\mathbb{R}^d} dx\, d\tau \\
&= \quad \int_0^t \int_U \left\langle \int_0^\tau D\tilde{u}(x,\theta)\, d\theta,\ A(x) Dw_n(x) \right\rangle_{\mathbb{R}^d} dx\, d\tau,
\end{aligned}$$

because $\Gamma = 1_{[0,t]} A D w_n \in L^1(I; L^2(U))^{\otimes d}$. As for the drift term we have, by (D),

$$\begin{aligned}
&\left| \int_0^t \int_0^\tau \langle f(\cdot,\theta,u_N(\cdot,\theta)) - f(\cdot,\theta,\tilde{u}(\cdot,\theta)),\ w_n \rangle_2 \, d\theta \, d\tau \right| \\
&= \quad \left| \int_I \langle f(\cdot,\tau,u_N(\cdot,\tau)) - f(\cdot,\tau,\tilde{u}(\cdot,\tau)),\ w_n \rangle_2 \, (t-\tau) 1_{[0,t]}(\tau) \, d\tau \right| \\
&\leq \quad T \int_I \big\langle L_f(\cdot,\tau) |u_N(\cdot,\tau) - \tilde{u}(\cdot,\tau)|,\ |w_n| \big\rangle_2 \, d\tau \\
&= \quad T \int_I \big\langle |u_N(\cdot,\tau) - \tilde{u}(\cdot,\tau)|,\ L_f(\cdot,\tau)|w_n| \big\rangle_2 \, d\tau \\
&\to \quad 0,
\end{aligned}$$

since $\psi = L_f |w_n| \in L^1(I; L^2(U))$. Finally, we discuss the noise term. With $u = u_N$ and $u^* = \tilde{u}$ in Lemma 6.8 we have the estimate

$$\begin{aligned}
&\left| \sum_{j=1}^\infty \sqrt{\lambda_j} \int_0^t \int_0^\tau \big\langle \sigma(\cdot,\theta,u_N) - \sigma(\cdot,\theta,\tilde{u}),\ w_n e_j \big\rangle_2 \, \beta_j^H(d\theta) \, d\tau \right| \\
&\leq \quad C \widehat{G}_\infty \int_0^t (t-\theta) \Big\langle |u_N(\cdot,\theta) - \tilde{u}(\cdot,\theta)|,\ |w_n| \int_\theta^T \frac{\|L_\rho(\cdot,\eta,\theta)\|_\infty}{(\eta-\theta)^\alpha} \, d\eta \\
&\qquad + |w_n| \, \|L_{\partial_2 \rho}(\cdot,\cdot,\theta)\|_{L^2(I;L^\infty(U))} \Big\rangle_2 \, d\theta. \qquad (7.10)
\end{aligned}$$

Since both terms to the right in the scalar product belong to $L^1(I; L^2(U))$ it also tends to zero. $\square$

8 Uniqueness and stability

We will now prove a general inequality from which both uniqueness as well as continuity with respect to initial data and will follow.

Theorem 8.1. *Let u and u^* be solutions corresponding to initial data (g,h) and (g^*,h^*) respectively. Then*

$$|\langle u - u^*,\ \psi \rangle| \quad \leq \quad C \sqrt{\|g-g^*\|_2^2 + \|h-h^*\|_2^2} \, \|\psi\|_{L^1(I;L^2(U))}, \qquad a.s.$$

for every $\psi \in L^1(I; L^2(U))$.

Proof: Consider the difference of the equations (6.4) for two different sequences u_n and u_N^*. As in the proof of Proposition 7.2 we now square, using 7.1, and sum over

$n \in \{1, \ldots, N\}$ to get

$$\begin{aligned}
&\|u_N(\cdot,t) - u_N^*(\cdot,t)\|_2^2 \\
&\quad = \quad \|g_N - g_N^*\|_2^2 \\
&\qquad\quad +2t\int_0^t \langle h_N - h_N^*,\ u_N(\cdot,\tau) - u_N^*(\cdot,\tau)\rangle_2\, d\tau \\
&\qquad\quad +2\int_0^t \left\langle \int_0^\tau b\cdot(Du_N(\cdot,\theta) - Du_N^*(\cdot,\theta))\, d\theta,\ u_N(\cdot,\tau) - u_N^*(\cdot,\tau) \right\rangle_2 d\tau \\
&\qquad\quad -2\int_0^t \int_U \Big\langle \int_0^\tau (Du_N(x,\theta) - Du_N^*(x,\theta))\, d\theta, \\
&\qquad\qquad\qquad\qquad A(x)D\,(u_N(x,\tau) - u_N^*(x,\tau)) \Big\rangle_{\mathbb{R}^d} dx\, d\tau \\
&\qquad\quad +2\int_0^t \int_0^\tau \langle f(\cdot,\theta,u_N(\cdot,\theta)) - f(\cdot,\theta,u_N^*(\cdot,\theta)),\ u_N(\cdot,\tau) - u_N^*(\cdot,\tau)\rangle_2\, d\theta\, d\tau \\
&\qquad\quad +2\sum_{j=1}^\infty \sqrt{\lambda_j}\int_0^t \int_0^\tau \Big\langle \sigma(\cdot,\theta,u_N) - \sigma(\cdot,\theta,u_N^*), \\
&\qquad\qquad\qquad\qquad (u_N(\cdot,\tau) - u_N^*(\cdot,\tau))\, e_j \Big\rangle_2 \beta_j^H(d\theta)\, d\tau. \qquad (8.1)
\end{aligned}$$

By Hölder's and Cauchy's inequalities the second term on the right is bounded by

$$\|h_N - h_N^*\|_2^2 + C\int_0^t \|u_N(\cdot,\tau) - u_N^*(\cdot,\tau)\|_2^2\, d\tau \qquad (8.2)$$

and by the same inequalities the third term on the right is bounded by

$$\|b\|_\infty \int_0^t \left(\left\| \int_0^\tau (Du_N(\cdot,\theta) - Du_N^*(\cdot,\theta))\, d\theta \right\|_2^2 + \|u_N(\cdot,\tau) - u_N^*(\cdot,\tau)\|_2^2 \right) d\tau. \qquad (8.3)$$

The next term can be handled exactly as in (7.4) and we have

$$\begin{aligned}
&2\int_0^t \left\langle A\int_0^\tau (Du_N(\cdot,\theta) - Du_N^*(\cdot,\theta))\, d\theta,\ (Du_N(\cdot,\tau) - Du_N^*(x,\theta)) \right\rangle_2 d\tau \\
&\quad \geq \quad a_0 \left\| \int_0^t (Du_N(\cdot,\theta) - Du_N^*(\cdot,\theta))\, d\theta \right\|_2^2 \qquad (8.4)
\end{aligned}$$

The drift term is similarly estimated to

$$2\|L_f\|_{L^1(I;L^\infty(U))} \int_0^t \sup_{0\le\gamma\le\tau} \|u_N(\cdot,\gamma) - u_N^*(\cdot,\gamma)\|_2^2\, d\tau. \qquad (8.5)$$

For the last term we use Lemma 6.7 with $u = u_N$, $u^* = u_N^*$ and $v(x,\tau) = u_N(x,\tau) -$

$u_N^*(x,\tau)$, Hölder's and Minkowski's inequalities to get the bound

$$\begin{aligned}
& C\widehat{G}_\infty \int_0^t \Big\langle |u_N(\cdot,\theta)-u_N^*(\cdot,\theta)|,\ \int_\theta^t |u_N(\cdot,\tau)-u_N^*(\cdot,\tau)|\,d\tau \int_\theta^T \frac{\|L_\rho(\cdot,\eta,\theta)\|_\infty}{(\eta-\theta)^\alpha}\,d\eta \\
& \qquad + \int_\theta^t |u_N(\cdot,\tau)-u_N^*(\cdot,\tau)|\,d\tau\, \|L_{\partial_2\rho}(\cdot,\cdot,\theta)\|_{L^2(I;L^\infty(U))} \Big\rangle_2 d\theta \\
\leq\ & C\widehat{G}_\infty \int_0^t \|u_N(\cdot,\theta)-u_N^*(\cdot,\theta)\|_2 \int_\theta^t \|u_N(\cdot,\tau)-u_N^*(\cdot,\tau)\|_2\,d\tau \\
& \qquad \times\left(\int_\theta^T \frac{\|L_\rho(\cdot,\eta,\theta)\|_\infty}{(\eta-\theta)^\alpha}\,d\eta\ +\ \|L_{\partial_2\rho}(\cdot,\cdot,\theta)\|_{L^2(I;L^\infty(U))}\right) d\theta \\
\leq\ & C\widehat{G}_\infty \left(\|L_\rho\|\ +\ \|L_{\partial_2\rho}\|\right) \int_0^t \sup_{0\leq\gamma\leq\tau} \|u_N(\cdot,\gamma)-u_N^*(\cdot,\gamma)\|_2^2\,d\tau, \qquad (8.6)
\end{aligned}$$

Putting the bounds (8.2) - (8.6) into (8.1) we obtain

$$\begin{aligned}
& \|u_N(\cdot,t)-u_N^*(\cdot,t)\|_2^2\ +\ a_0 \left\| \int_0^t (Du_N(\cdot,\theta)-Du_N^*(\cdot,\theta))\,d\theta \right\|_2^2 \\
\leq\ & \|g_N-g_N^*\|_2^2 + \|h_N-h_N^*\|_2^2 + C\left(1+\|b\|+\|L_f\|\ +\widehat{G}_\infty\left(\|L_\rho\|+\|L_{\partial_2\rho}\|\right)\right) \\
& \qquad \times \int_0^t \sup_{0\leq\gamma\leq\tau} \|u_N(\cdot,\gamma)-u_N^*(\cdot,\gamma)\|_2^2\,d\tau \\
& \quad + \|b\| \int_0^t \sup_{0\leq\gamma\leq\tau} \left\| \int_0^\tau (Du_N(\cdot,\theta)-Du_N^*(\cdot,\theta))\,d\theta \right\|_2^2 d\tau. \qquad (8.7)
\end{aligned}$$

Let

$$\begin{aligned}
M(\tau) &= \sup_{0\leq\gamma\leq\tau} \left(\|u_N(\cdot,\gamma)-u_N^*(\cdot,\gamma)\|_2^2 + \left\| \int_0^\gamma (Du_N(\cdot,\theta)-Du_N^*(\cdot,\theta))\,d\theta \right\|^2 \right) \\
Q &= 1+\|b\|+\|L_f\|+\widehat{G}_\infty\left(\|L_\rho\|+\|L_{\partial_2\rho}\|\right)
\end{aligned}$$

and notice that (8.7) can be written

$$M(t) \leq \|g_N-g_N^*\|_2^2 + \|h_N-h_N^*\|_2^2 + CQ\int_0^t M(\tau)\,d\tau.$$

Then Gronwall's inequality gives

$$M(t) \leq \left(\|g_N-g_N^*\|_2^2 + \|h_N-h_N^*\|_2^2\right) e^{CQt}. \qquad (8.8)$$

Now, let $\psi \in L^1(I;L^2(U))$. Then we have, by Hölder's inequality and (8.8),

$$\begin{aligned}
|\langle u_N-u_N^*,\ \psi\rangle| &\leq \int_I \|u_N(\cdot,\tau)-u_N^*(\cdot,\tau)\|_2\ \|\psi(\cdot,\tau)\|_2\,d\tau \\
&\leq \sqrt{\|g_N-g_N^*\|_2^2 + \|h_N-h_N^*\|_2^2} \int_I e^{CQ\tau}\,\|\psi(\cdot,\tau)\|_2\,d\tau \\
&\leq e^{CQ}\sqrt{\|g_N-g_N^*\|_2^2 + \|h_N-h_N^*\|_2^2}\,\|\psi\|_{L^1(I;L^2(U))}.
\end{aligned}$$

Hence, in the limit $N \to \infty$ we get, by previous results,

$$|\langle u-u^*,\ \psi\rangle| \leq e^{CQ}\sqrt{\|g-g^*\|_2^2 + \|h-h^*\|_2^2}\,\|\psi\|_{L^1(I;L^2(U))}. \qquad \square$$

The following are immediate consequences.

Corollary 8.2. *The solution to (6.1) is unique.*

Corollary 8.3. *The solution to (6.1) depends continuously on the initial data.*

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