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A DISCRETE DE RHAM COMPLEX WITH ENHANCED SMOOTHNESS

XUE–CHENG TAI AND RAGNAR WINTHER

Abstract. Discrete de Rham complexes are fundamental tools in the construction of stable elements for mixed finite element methods. The purpose of this paper is to discuss a new discrete de Rham complex in three space dimensions, where the finite element spaces have some extra smoothness compared to the standard requirements. The motivation for this construction is to produce discretization which have uniform stability properties for certain families of singular perturbation problems. In particular, we will show how the spaces constructed here lead to discretizations of Stokes type systems which have uniform convergence properties as the Stokes flow approaches a Darcy flow.

1. Introduction

In [8] a robust finite element discretization of Darcy–Stokes flow in two space dimensions was proposed. More precisely, given a domain $\Omega \subset \mathbb{R}^2$ the following singular perturbation problem was studied:

$$
\begin{aligned}
(\boldsymbol{I} - \varepsilon^2 \boldsymbol{\Delta})\boldsymbol{u} - \mathbf{grad}\, p &= \boldsymbol{f} \quad \text{in } \Omega, \\
\operatorname{div} \boldsymbol{u} &= g \quad \text{in } \Omega, \\
\boldsymbol{u} &= 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{1.1}
$$

where $\varepsilon \in (0, 1]$ is the perturbation parameter. The unknowns are the vector field $\boldsymbol{u}$ and the scalar field p, which in flow problems correspond to velocity and pressure, respectively. We note that when ε is not too small this problem is simply a standard linear Stokes problem, but with an additional non–harmful lower order term. However, if ε approaches zero the model problem formally tends to a mixed formulation of the Poisson equation with homogeneous Neumann boundary conditions, i.e. a Darcy flow. Hence, the model covers the transition from fluid flow to porous medium flow. In this respect, the singular perturbation system (1.1) is a prototype for problems arising in multiscale modelling.

The main motivation for the finite element method constructed in [8] was to construct a discretization which has convergence properties that are uniform with respect to the perturbation parameter ε. Hence,

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for ε bounded away from zero the method should behave as a finite element method for the linear Stokes problem, while as ε tends to zero the method should approach a mixed method for Poisson equation. The approach taken in [8] was to construct a pair of finite element spaces $(\boldsymbol{V}_h, Q_h)$, for approximating the solution $(\boldsymbol{u}, p)$, such that the Brezzi stability conditions, cf. [3], are satisfied with stability constants independent of ε. The purpose of present paper is to design a corresponding finite element method in three space dimensions.

The construction and analysis presented in [8] is closely related to discrete de Rham complexes. In two space dimensions the de Rham complex, with minimal smoothness measured in L^2, can be stated as

$$\mathbb{R} \xrightarrow{\subset} H^1 \xrightarrow{\textbf{curl}} \boldsymbol{H}(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \longrightarrow 0, \tag{1.2}$$

where **curl** in the two dimensional case denotes the operator which maps a scalar field ϕ to the vector field $(-\partial_{x_2}\phi, \partial_{x_1}\phi)$. The precise definitions of the involved spaces will be given in the next section. Note that the function spaces (1.2) have exactly the property that it consists of all L^2 fields such that the image of the differential operator mapping to the right also is in L^2. The statement that this is a complex simply means that the composition of two consecutive maps is zero. If the domain Ω is simply connected the sequence (1.2) is exact in the sense that the range of each map is exactly the null space of the succeeding map.

The Sobolev spaces H^1, $\boldsymbol{H}(\mathrm{div})$, and L^2 occurring in (1.2) are fundamental function spaces used for weak formulations of a large collection of differential systems. Furthermore, corresponding finite element spaces, and, in particular, various discrete de Rham complexes are important tools in designing stable finite element discretizations of these systems.

A discrete de Rham complex in two dimensions can be written on the form

$$\mathbb{R} \xrightarrow{\subset} W_h \xrightarrow{\textbf{curl}} \boldsymbol{V}_h \xrightarrow{\mathrm{div}} Q_h \longrightarrow 0, \tag{1.3}$$

where $W_h \subset H^1$, $\boldsymbol{V}_h \subset \boldsymbol{H}(\mathrm{div})$, and $Q_h \subset L^2$ are finite element spaces with respect to a given triangulation $\mathcal{T}_h$ of Ω. The most well known examples involves the Raviart–Thomas spaces [13] or the Brezzi–Douglas–Marini spaces [4] as the choice of $\boldsymbol{V}_h$, while W_h and Q_h consist of standard piecewise polynomial scalar fields which are globally continuous or discontinuous, respectively.

In [8] we constructed an discrete sequence of the form (1.3), but with the additional property that the finite element spaces are nonconforming approximations of spaces with extra smoothness. More precisely, $\boldsymbol{V}_h \subset \boldsymbol{H}(\mathrm{div})$, i.e. the elements of $\boldsymbol{V}_h$ has continuous normal components over all edges of the mesh. In addition, at each edge the tangential component of the vector fields in $\boldsymbol{V}_h$ have continuous mean value. Correspondingly, $W_h \subset H^1$ is a nonconforming approximation of H^2. Hence, the spaces constructed in [8] is a discrete analog of the complex

$$\mathbb{R} \xrightarrow{\subset} H^2 \xrightarrow{\mathbf{curl}} \boldsymbol{H}^1 \xrightarrow{\mathrm{div}} L^2 \longrightarrow 0, \tag{1.4}$$

which is an exact sequence if the domain Ω is simply connected.

In three space dimensions the Sobolev space version of the de Rham complex can be written in the form

$$\mathbb{R} \xrightarrow{\subset} H^1 \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl}) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \longrightarrow 0, \tag{1.5}$$

which is an exact sequence if the domain Ω is contractable. Here $\boldsymbol{H}(\mathbf{curl})$ consists of all vector fields $\boldsymbol{u} \in \boldsymbol{L}^2$ with $\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{L}^2$. A corresponding discrete de Rham complex of the form

$$\mathbb{R} \xrightarrow{\subset} S_h \xrightarrow{\mathbf{grad}} \boldsymbol{W}_h \xrightarrow{\mathbf{curl}} \boldsymbol{V}_h \xrightarrow{\mathrm{div}} Q_h \longrightarrow 0 \tag{1.6}$$

where $S_h \subset H^1$, $\boldsymbol{W}_h \subset \boldsymbol{H}(\mathbf{curl})$, $\boldsymbol{V}_h \subset \boldsymbol{H}(\mathrm{div})$, and $Q_h \subset L^2$ is referred to as a conforming approximation of the complex (1.5). Well known examples of such finite element spaces are the Nedelec families described in [10] and [11], cf. also [1].

A three dimensional example of a complex with extra smoothness, corresponding to (1.4), is given by

$$\mathbb{R} \xrightarrow{\subset} H^2 \xrightarrow{\mathbf{grad}} \boldsymbol{H}^1(\mathbf{curl}) \xrightarrow{\mathbf{curl}} \boldsymbol{H}^1 \xrightarrow{\mathrm{div}} L^2 \longrightarrow 0. \tag{1.7}$$

Here $\boldsymbol{H}^1(\mathbf{curl})$ consists of all vector fields $\boldsymbol{u} \in \boldsymbol{H}^1$ with $\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{H}^1$. The sequence (1.7) is obviously a complex. Furthermore, if Ω is convex polyhedron then the sequence is exact, cf. [7, Chapter I.3.5].

The main purpose of the present paper is to construction an analog to the one given in [8] for three space dimensions. Given a tetrahedral mesh $\mathcal{T}_h$ we construct a conforming approximation of the complex (1.5) of the form (1.6), which, at the same time, is a nonconforming approximation of (1.7) in the sense that the discrete spaces of (1.6) are nonconforming approximations of $H^2, \boldsymbol{H}^1(\mathbf{curl})$, and $\boldsymbol{H}^1$, respectively. We will show that the constructed spaces $\boldsymbol{V}_h$ and Q_h lead to a robust discretization of the Darcy–Stokes system (1.1) in the sense that the method is uniformly stable both with respect to the perturbation parameter ε and the discretization parameter h.

In a similar manner the finite element spaces S_h and $\boldsymbol{W}_h$, constructed below, can potentially be used to design uniform discretizations of other singular perturbation problems. For example, the space S_h is a three dimensional analog of the finite element space used in [12] to discretize fourth order problems which are perturbations of a second order problem. However, we will not perform such discussions here.

In §2 we introduce the notation to be used in this paper, and we define the finite element spaces $S_h, \boldsymbol{W}_h, \boldsymbol{V}_h$, and Q_h. The properties of these discrete spaces are discussed in §3, and then in §4 we proceed to show that the pair of spaces $(\boldsymbol{V}_h, Q_h)$ leads to a uniformly stable discretization of the Darcy–Stokes system (1.1).

2. Notation and preliminaries

We will use $H^m = H^m(\Omega)$ to denote the L^2–based Sobolev spaces of order m on the polygonal domain $\Omega \subset \mathbb{R}^3$, and the corresponding norm by $\|\cdot\|_m$. The subspace H_0^m is the closure in H^m of $C_0^\infty(\Omega)$, while L_0^2 consists of all elements of L^2 with mean value zero. The notation $(\cdot,\cdot)$ will be used to denote the standard L^2 inner product over the domain Ω. In general we will use boldface symbols for vector fields and function spaces of vector fields. In particular $\boldsymbol{H}(\mathbf{curl}) = \boldsymbol{H}(\mathbf{curl};\Omega)$ is the spaces of all $\boldsymbol{L}^2$ vector fields $\boldsymbol{v}$ with $\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^2$, and with $\boldsymbol{H}(\mathrm{div}) = \boldsymbol{H}(\mathrm{div};\Omega)$ defined in a similar manner. The gradient of a vector field $\boldsymbol{v}$ is denoted $\boldsymbol{D}\boldsymbol{v}$, i.e. $\boldsymbol{D}\boldsymbol{v}$ is the 3×3 matrix with elements

$$(\boldsymbol{D}\boldsymbol{v})_{i,j} = \partial \boldsymbol{v}_i/\partial x_j \quad 1 \le i,j \le 3.$$

For a subset $T \subset \mathbb{R}^n$, the notation $\mathbb{P}_k = \mathbb{P}_k(T)$ is used for the space of polynomials of degree k defined on T, and $\mathbb{P}_k^n$ denotes the corresponding space of polynomial vector fields. If $T \subset \mathbb{R}^3$ is a tetrahedron then $\Delta_2(T)$ denotes the set of the four 2–dimensional faces, $\Delta_1(T)$ is the set of the six 1–dimensional edges, and $\Delta_0(T)$ the set of the four vertices.

In order to define the finite element spaces $S_h, \boldsymbol{W}_h, \boldsymbol{V}_h$, and Q_h we will first define the restriction of these spaces to one tetrahedron. Througout this paper $\{\mathcal{T}_h\}$ is a family of shape regular tetrahedral meshes, where h is the maximal diameter. For $T \in \mathcal{T}_h$ let $b = b_T \in \mathbb{P}_4$ be the quartic bubble function with respect to T, i.e. $b = \lambda_1\lambda_2\lambda_3\lambda_4$, where λ_i are the barycentric coordinates with respect to the vertices of T. The restriction of the space S_h to T will be denoted $S(T)$ and is given by

$$S(T) = \{s = s_2 + b\, s_1 : s_i \in \mathbb{P}_i, \quad i = 1,2\}.$$

Hence, the space $S(T)$ is a linear space of dimension 14. The corresponding spaces $\boldsymbol{W}(T)$ is a space of dimension 36 given by

$$\boldsymbol{W}(T) = \boldsymbol{N}_1 + \mathbf{grad}(b\mathbb{P}_1) + b\mathbb{P}_1^3.$$

Here $\boldsymbol{N}_1 = \boldsymbol{N}_1(T)$ is the polynomial space which corresponds to the restriction of the second lowest order $\boldsymbol{H}(\mathbf{curl})$ space of Nedelec's first family to one tetrahedron, cf. [10]. Hence,

$$\boldsymbol{N}_1 = \{\boldsymbol{w} \in \mathbb{P}_2^3 : \boldsymbol{w}\cdot\boldsymbol{x} \in \mathbb{P}_2\}.$$

This space has dimension 20, and a $\boldsymbol{w} \in \boldsymbol{N}_1$ is uniquely determined by the two lowest order moments of the tangential components on each edge, and the lowest order moment of the two tangential components on each face. We refer to [10] for more details. The restriction of the space $\boldsymbol{V}_h$ to T, $\boldsymbol{V}(T)$, is given as

$$\boldsymbol{V}(T) = \mathbb{P}_1^3 + \mathbf{curl}(b\mathbb{P}_1^3), \tag{2.1}$$

which is a space of dimension 24. Finally, $Q(T)$ is simply taken to be $\mathbb{P}_0$. It is straightforward to check that $\mathbf{grad}\, S(T) \subset \boldsymbol{W}(T)$, $\mathbf{curl}\, \boldsymbol{W}(T) \subset \boldsymbol{V}(T)$, and $\operatorname{div} \boldsymbol{V}(T) \subset Q(T)$. Hence, the polynomial sequence

$$\mathbb{R} \xrightarrow{\subset} S(T) \xrightarrow{\mathbf{grad}} \boldsymbol{W}(T) \xrightarrow{\mathbf{curl}} \boldsymbol{V}(T) \xrightarrow{\operatorname{div}} Q(T) \longrightarrow 0 \tag{2.2}$$

is a complex. In fact, it can be easily checked that (2.2) is exact.

The finite element spaces S_h, $\boldsymbol{W}_h$, $\boldsymbol{V}_h$, and Q_h will be defined from the corresponding spaces of restrictions to a given tetrahedron, introduced above, by specifying degrees of freedom for these local spaces. As for the degree of freedom for the one dimensional space $Q(T) = \mathbb{P}_0$ we use the mean value of the function over T. Hence, the corresponding global space Q_h is a subspace L^2.

Any function $s \in S(T)$ is determined by the values of s at each vertex and

$$\int_e s\, dx_e \quad e \in \Delta_1(T), \qquad \int_f \frac{\partial s}{\partial \boldsymbol{n}}\, d\boldsymbol{x}_f \quad f \in \Delta_2(T). \tag{2.3}$$

Here and below dx_e and $d\boldsymbol{x}_f$ indicate integration with respect to arc length or surface area, and $\boldsymbol{n}$ is a unit normal vector on f.

It is straightforward to check that these degrees of freedom uniquely determines an element of $s \in S(T)$. If the the degrees of freedom associated $\Delta_0(T)$ and $\Delta_1(T)$ are all zero then $s = bs_1$, where $s_1 \in \mathbb{P}_1$. Furthermore, on a face $f \in \Delta_2(T)$

$$\frac{\partial s}{\partial \boldsymbol{n}} = c_f b_f s_1,$$

where $c_f \neq 0$ is a constant, and b_f is the cubic bubble function associated the face f. However, b_f is nonzero in the interior of f. Hence, if the zero order moment of $\partial s/\partial \boldsymbol{n}$ is zero on each face $f \in \Delta_2(T)$ there must exist an interior root of s_1 on each face f, and therefore $s = s_1 = 0$.

The local space $S(T)$ and the degrees of freedom determined by (2.3) defines the corresponding global space S_h. It is clear that the elements of S_h are continuous, i.e. $S_h \subset H^1$. Furthermore, the normal derivatives are weakly continuous over inter–element faces f, in the sense that

$$\int_f [\frac{\partial s}{\partial \boldsymbol{n}}]\, d\boldsymbol{x}_f = 0,$$

where $[\cdot]$ denotes the jump across the face f. Hence, the space S_h is a nonconforming approximation of H^2.

Finally, we have to design proper degrees of freedom for the spaces of vector fields, $\boldsymbol{W}(T)$ and $\boldsymbol{V}(T)$. Recall that rigid motions in two and three dimensional spaces are vector fields $\boldsymbol{r} \in \mathbb{R}^n, n = 2, 3$ of the form

$$\boldsymbol{r}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b}\boldsymbol{x}, \tag{2.4}$$

where $\boldsymbol{a} \in \mathbb{R}^n$ and $\boldsymbol{b}$ is a skew symmetric $n \times n$ real matrix (n=2,3). The space of rigid motions will be denoted $\boldsymbol{RM}$. Furthermore, if $f \subset \mathbb{R}^3$ is a two dimensional affine space , i.e. a plane in $\mathbb{R}^3$, then $\boldsymbol{RM}(f)$ denotes the rigid motions on f, i.e. $\boldsymbol{RM}(f)$ is the space which only contains tangential to f and all vector fields from $\boldsymbol{RM}(f)$ are of the form (2.4). Hence, $\boldsymbol{RM}(f)$ is a linear space of dimension 3. In fact, all the vectors $\boldsymbol{r}(\boldsymbol{x})$ from $\boldsymbol{RM}(f)$ for $\boldsymbol{x} \in \mathbb{R}^3$ is of the form

$$\boldsymbol{r}(\boldsymbol{x}) = \boldsymbol{r}_0 + \beta(\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{n}, \ \forall \beta \in \mathbb{R},$$

where $\boldsymbol{x}_0 \in f$ is a fixed point and $\boldsymbol{r}_0$ is a fixed tangent vector.

We will show below that a vector field $\boldsymbol{v} \in \boldsymbol{V}(T)$ is uniquely determined by 24 degrees of freedom. For all $f \in \Delta_2(T)$ we specify the moments

$$\int_f (\boldsymbol{v} \cdot \boldsymbol{n}) p \, d\boldsymbol{x}_f \quad p \in \mathbb{P}_1(f), \quad \int_f \boldsymbol{v}_t \cdot \boldsymbol{r} \, d\boldsymbol{x}_f \quad \boldsymbol{r} \in \boldsymbol{RM}(f). \tag{2.5}$$

In the above, $\mathbb{P}_1(f)$ is the space of linear functions on f. Here and also later, we use $\boldsymbol{v}_t$ to denote the tangential component of $\boldsymbol{v}$ on f, i.e.,

$$\boldsymbol{v}_t = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n}) \boldsymbol{n}. \tag{2.6}$$

Note that if we introduce a basis for the spaces $\mathbb{P}_1(f)$ and $\boldsymbol{RM}(f)$ then these moment conditions lead to 24 degrees of freedom for the elements $\boldsymbol{v} \in \boldsymbol{V}(T)$.

The proof that these degrees of freedom are unisolvent for $\boldsymbol{V}(T)$ will be given in the next section. Once $V(T)$ has been defined, the finite element space V_h is defined by these degree of freedoms over the finite elements of $\mathcal{T}_h$. It can easily be seen that the elements of the corresponding global space $\boldsymbol{V}_h$ have continuous normal derivatives. Therefore, $\boldsymbol{V}_h \subset \boldsymbol{H}(\text{div})$. Furthermore, the tangential components are weakly continuous, so $\boldsymbol{V}_h$ is a nonconforming approximation of $\boldsymbol{H}^1$.

Remark. Note that if $\boldsymbol{v} \in \boldsymbol{V}_h$ then the jumps of $\boldsymbol{v}$ on the inter-element faces are orthogonal to rigid motions. ¿From the observation done in [9], based on the general nonconforming theory of [2], it therefore follows that the elements of the nonconforming $\boldsymbol{H}^1$ space $\boldsymbol{V}_h$ will indeed satisfy Korn's inequality. □

The degrees of freedom for $\boldsymbol{W}(T)$ are determined from the moments

$$\int_e (\boldsymbol{w} \cdot \boldsymbol{t}) p \, dx_e \quad e \in \Delta_1(T), \quad p \in \mathbb{P}_1(e), \tag{2.7}$$

where $\boldsymbol{t}$ is a tangent vector on e, and for all $f \in \Delta_2(T)$

$$\int_f \boldsymbol{w} \, d\boldsymbol{x}_f, \quad \text{and} \int_f (\mathbf{curl}\, \boldsymbol{w})_t \cdot \boldsymbol{r} \, d\boldsymbol{x}_f \quad \boldsymbol{r} \in \boldsymbol{RM}(f). \tag{2.8}$$

It is a consequence of the discussion in §3 below that these degrees of freedom determine an element of $\boldsymbol{W}(T)$ uniquely. It can also be seen that the elements of the corresponding global space, $\boldsymbol{W}_h$, have continuous tangential components, and therefore $\boldsymbol{W}_h \subset \boldsymbol{H}(\mathbf{curl})$. Furthermore, the normal components of $\boldsymbol{w}$, and the components of $\mathbf{curl}\,\boldsymbol{w}$ are weakly continuous, and hence the space $\boldsymbol{W}_h$ is a nonconforming approximation of $\boldsymbol{H}^1(\mathbf{curl})$.

3. The discrete de Rham complex

The purpose of this section is to complete the discussion of the finite element spaces S_h, $\boldsymbol{W}_h$, $\boldsymbol{V}_h$, and Q_h. In particular, we will show that the corresponding complex (1.6) is exact. In order to show that an element $\boldsymbol{v} \in \boldsymbol{V}(T)$ and $\boldsymbol{w} \in \boldsymbol{W}(T)$ are uniquely determined by the degrees of freedom specified by (2.5), or (2.7) and (2.8), respectively, we will need some preliminary results in two space dimensions.

3.1. **Some preliminary results in two space dimensions.** Throughout this subsection $f \subset \mathbb{R}^2$ will be a general triangle, and $\hat{f}$ the reference triangle given by

$$\hat{f} = \{\boldsymbol{x} \in \mathbb{R}^2 \,:\, x_1, x_2 \geq 0,\, x_1 + x_2 \leq 1\}.$$

We let λ_i for $i = 1, 2, 3$ be the barycentric cooordinates on f, and $b = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$ the cubic bubble function on f. The integral of b over f is denoted $|b|_f$. For example, if $f = \hat{f}$ then $b(\boldsymbol{x}) = x_1 x_2 (1 - x_1 - x_2)$ and $|b|_f = 1/120$.

Furthermore, for each triangle f there is a $1-1$ linear transformation Φ of the form

$$\Phi(\hat{\boldsymbol{x}}) = B\hat{\boldsymbol{x}} + \boldsymbol{x}_0$$

mapping $\hat{f}$ onto f. If $\lambda_i(\boldsymbol{x})$ are the barycentric cooordinates on f and $\hat{\lambda}_i(\hat{\boldsymbol{x}})$ the corresponding functions on $\hat{f}$, then

$$\lambda_i(\boldsymbol{x}) = \hat{\lambda}_i(\Phi^{-1}(\boldsymbol{x})).$$

The corresponding Piola transform, $\mathcal{P}$, maps 2–vectorfield defined on $\hat{f}$ to corresponding vectorfields on f. If $\hat{\boldsymbol{z}}$ is a vectorfield on $\hat{f}$ then

$$\boldsymbol{z}(\boldsymbol{x}) = \mathcal{P}\hat{\boldsymbol{z}}(x) = J^{-1}B\hat{\boldsymbol{z}}(\Phi^{-1}(\boldsymbol{x})).$$

Here J is the determinant of B. The Piola transform maps constant vectors to constant vectors. In addition, we have $\mathcal{P}\hat{\boldsymbol{z}}(\boldsymbol{x}) = J^{-1}(\boldsymbol{x} - \boldsymbol{x}_0)$ if $\hat{\boldsymbol{z}}(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{x}}$.

The following identities, which can be established by straightforward calculations, will be useful below.

Lemma 3.1. *If* $f = \hat{f}$ *then*

$$\int_f x_1 b\, d\boldsymbol{x} = \int_f x_2 b\, d\boldsymbol{x}_f = 1/360 = |b|_f/3,$$

$$\int_f x_1^2 b\, d\boldsymbol{x} = \int_f x_2^2 b\, d\boldsymbol{x}_f = |b|_f/7,$$

$$\int_f x_1 x_2 b\, d\boldsymbol{x} = 2|b|_f/21.$$

For a general triangle f, we define the barycentre $\boldsymbol{x}^b \in f$ by

$$\lambda_i(\boldsymbol{x}^b) = 1/3 \quad i = 1, 2, 3.$$

It is a direct consequence of the lemma above that the integration rule

$$\int_f bp\, d\boldsymbol{x} = |b|_f p(\boldsymbol{x}^b) \tag{3.1}$$

is exact for $p \in \mathbb{P}_1$ and $f = \hat{f}$. By a change of variables this formula then holds for any triangle f.

Assume that $\boldsymbol{v} \in \mathbb{P}_1^2(f)$ is of the form

$$\boldsymbol{v} = \sum_{i=1}^{3} c_i(\lambda_i - \frac{1}{3})\,\mathbf{grad}\,\lambda_i. \tag{3.2}$$

By (3.1), it follows that

$$\int_f b(\boldsymbol{v} \cdot \boldsymbol{z})\, d\boldsymbol{x} = 0$$

for all constant vector fields $\boldsymbol{z}$. In addition we have the following characterization.

Lemma 3.2. *If* $\boldsymbol{v} \in \mathbb{P}_1^2(f)$ *is of the form* (3.2) *and satisfies*

$$\int_f b(\boldsymbol{v} \cdot \boldsymbol{x})\, d\boldsymbol{x} = 0$$

then $c_1 + c_2 + c_3 = 0$.

Proof. Let $\hat{\boldsymbol{z}}$ and $\boldsymbol{\psi}$ be smooth vector fields on $\hat{f}$ and f, respectively. We have

$$\int_f b(\boldsymbol{\psi} \cdot \mathcal{P}\hat{\boldsymbol{z}})\, d\boldsymbol{x} = \int_{\hat{f}} \hat{b}(B^T\hat{\boldsymbol{\psi}} \cdot \hat{\boldsymbol{z}})\, d\hat{\boldsymbol{x}} \tag{3.3}$$

where $\hat{\boldsymbol{\psi}} = \boldsymbol{\psi} \circ \Phi$, $\hat{b} = b \circ \Phi$ and B^T is the transpose of the matrix B. Note, that if $\boldsymbol{\psi} = \mathbf{grad}\, q$ then $\mathbf{grad}_{\hat{\boldsymbol{x}}}\, \hat{q} = B^T\hat{\boldsymbol{\psi}}$. Therefore, if $\boldsymbol{v} \in \mathbb{P}_1^2(f)$ is of the form (3.2) then

$$B^T\hat{\boldsymbol{v}} = \sum_{i=1}^{3} c_i(\hat{\lambda}_i - \frac{1}{3})\,\mathbf{grad}_{\hat{\boldsymbol{x}}}\,\hat{\lambda}_i.$$

Hence, by the assumption and (3.3), the coefficients c_i satisfies

$$
\begin{aligned}
(\sum_{i=1}^{3} c_i \int_{\hat{f}} \hat{b}(\hat{\lambda}_i - \frac{1}{3}) \, \mathbf{grad}_{\hat{\boldsymbol{x}}} \hat{\lambda}_i) \cdot \hat{\boldsymbol{x}} \, d\hat{\boldsymbol{x}} \\
&= \int_{\hat{f}} \hat{b}(B^T \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{x}}) \, d\hat{\boldsymbol{x}} \\
&= J^{-1} \int_f b\boldsymbol{v} \cdot (\boldsymbol{x} - \boldsymbol{x}_0) \, d\boldsymbol{x} = 0,
\end{aligned}
$$

where we have used that $(\mathcal{P}\hat{\boldsymbol{x}})(\boldsymbol{x}) = J^{-1}(\boldsymbol{x} - \boldsymbol{x}_0)$. However, from the identities of Lemma 3.1 we easily compute

$$
\begin{aligned}
&(\sum_{i=1}^{3} c_i \int_{\hat{f}} \hat{b}(\hat{\lambda}_i - \frac{1}{3}) \, \mathbf{grad}_{\hat{\boldsymbol{x}}} \hat{\lambda}_i) \cdot \hat{\boldsymbol{x}} \, d\hat{\boldsymbol{x}} \\
&= \int_{\hat{f}} [c_1 x_1 (x_1 - \frac{1}{3}) + c_2 x_2 (x_2 - \frac{1}{3}) + c_3 (x_1 + x_2 - \frac{2}{3})(x_1 + x_2)] \, dx_1 dx_2 \\
&= \frac{2}{63} |b|_f (c_1 + c_2 + c_3),
\end{aligned}
$$

and therefore $c_1 + c_2 + c_3 = 0$. □

3.2. **Unisolvent degrees of freedom.** We now return to the discussion of polynomial spaces defined on a tetrahedron $T \subset \mathbb{R}^3$. We recall that $b = b_T = \prod_{j=1}^{4} \lambda_j$ is the quartic bubble function on T. Furthermore, on the face $f = f_i = \{\boldsymbol{x} : \lambda_i(\boldsymbol{x}) = 0\} \in \Delta_2(T)$ we associate the cubic bubble function $b_f = \prod_{j \neq i} \lambda_j$. We need to show that the functions in the spaces $\boldsymbol{V}(T)$ and $\boldsymbol{W}(T)$ are uniquely determined by the moment conditions given by (2.5) and (2.7)–(2.8), respectively. We first establish the following lemma.

Lemma 3.3. *Assume that* $\boldsymbol{v} \in \mathbb{P}_1^3(T)$ *satisfies*

$$
\int_f b_f (\boldsymbol{v} \times \boldsymbol{n}) \cdot \boldsymbol{r} d\boldsymbol{x}_f = 0 \quad \boldsymbol{r} \in \boldsymbol{RM}(f), \ f \in \Delta_2(T). \tag{3.4}
$$

Then $\boldsymbol{v} = 0$.

Proof. If $\boldsymbol{v} \in \mathbb{P}_1^3(T)$ satisfies (3.4) then

$$
\int_f b_f \boldsymbol{v}_t \cdot \boldsymbol{z} d\boldsymbol{x}_f = 0 \quad \boldsymbol{z} \in \mathbb{P}_0^2(f), \ f \in \Delta_2(f).
$$

This follows since $\boldsymbol{v} \times \boldsymbol{n} = R\boldsymbol{v}_t$, where the matrix R represents a rotation by 90 degrees in the tangent space of f, and since $\boldsymbol{RM}(f)$ contains all constant tangential vector fields. Therefore, using (3.1), we conclude that

$$
\boldsymbol{v}_t(x_f^b) = 0, \quad f \in \Delta_2(f), \tag{3.5}
$$

where x_f^b is the barycentre of a face f. The space of functions in $\mathbb{P}_1^3(T)$ satisfying (3.5) is a four dimensional subspace which is given as the span of the functions $(\lambda_i - \frac{1}{3})\,\mathbf{grad}\,\lambda_i \quad i = 1,2,3,4$. Hence, $\boldsymbol{v}$ is of the form

$$\boldsymbol{v} = \sum_{i=1}^{4} c_i(\lambda_i - \frac{1}{3})\,\mathbf{grad}\,\lambda_i \tag{3.6}$$

for some constants c_1, c_2, c_3, c_4. Restricting $\boldsymbol{v}$ to the face f_1, given by $\lambda_1 = 0$, the tangential component $\boldsymbol{v}_t$ has the form

$$\boldsymbol{v}_t = \sum_{i=2}^{4} c_i(\lambda_i - \frac{1}{3})\,\mathbf{grad}_t\,\lambda_i,$$

where $\mathbf{grad}_t\,\lambda = (\mathbf{grad}\,\lambda)_t$ is the tangential component of $\mathbf{grad}\,\lambda$. Note also that (3.4) implies that for any fixed $\boldsymbol{x}_0 \in f_1$,

$$\int_{f_1} b_1 \boldsymbol{v}_t \cdot (\boldsymbol{x} - \boldsymbol{x}_0)\, d\boldsymbol{x}_f = \int_{f_1} b_1(\boldsymbol{v} \times \boldsymbol{n}) \cdot \big((\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{n}\big)\, d\boldsymbol{x}_f = 0.$$

As a consequence of Lemma 3.2 we conclude that $c_2 + c_3 + c_4 = 0$. By considering all the four faces we conclude that

$$\sum_{i \neq j} c_i = 0 \quad j = 1,2,3,4,$$

and this implies that $c_1 = c_2 = c_3 = c_4 = 0$. □

Next we will show that the elements of $\boldsymbol{V}(T)$ are uniquely determined by the 24 degrees of freedom given by (2.5).

Lemma 3.4. *Assume that* $\boldsymbol{v} \in \boldsymbol{V}(T) = \mathbb{P}_1^3 + \mathbf{curl}(b\mathbb{P}_1^3)$ *and that all the degrees of freedom represented by* (2.5) *are zero. Then* $v = 0$.

Proof. Let $\boldsymbol{v} = \boldsymbol{p} + \mathbf{curl}\, b\boldsymbol{q}$, where $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}_1^3$. On each face $f \in \Delta_2(T)$ the normal component of $\mathbf{curl}\, b\boldsymbol{q}$ is zero since it only depends on tangential derivatives of $b\boldsymbol{q}$. Therefore, we have $\boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{p} \cdot \boldsymbol{n}$. Hence, if the 12 constraints on the normal component of $\boldsymbol{v}$ are all zero, we can conclude that $\boldsymbol{p} \cdot \boldsymbol{n} = 0$ on each face. Since three faces are meeting at a vertex, we conclude that $\boldsymbol{p} = 0$ on each vertex. However, this means that $\boldsymbol{p} = 0$, or $\boldsymbol{v} = \mathbf{curl}(b\boldsymbol{q})$. As a consequence, on each face

$$\boldsymbol{v}_t = (\mathbf{curl}\, b\boldsymbol{q})_t = -\frac{\partial b}{\partial \boldsymbol{n}}(\boldsymbol{q} \times \boldsymbol{n}).$$

However, $\partial b / \partial \boldsymbol{n}$ is proportional to b_f. Therefore, if the conditions on $\boldsymbol{v}_t$ in (2.5) all vanish, then

$$\int_f b_f(\boldsymbol{q} \times \boldsymbol{n}) \cdot \boldsymbol{r}\, d\boldsymbol{x}_f = 0 \quad \boldsymbol{r} \in \boldsymbol{RM}(f)$$

for all $f \in \Delta_2(T)$, and by Lemma 3.3 this implies that $\boldsymbol{q} = 0$. □

A similar argument can be given to show that the elements of $\boldsymbol{W}(T)$ are uniquely determined by the degrees of freedom given by (2.7)–(2.8). Recall that a vector field $\boldsymbol{w}$ is in $\boldsymbol{W}(T)$ if it is on the form

$$\boldsymbol{w} = \boldsymbol{w}^0 + \mathbf{grad}(bp) + b\boldsymbol{q}$$

where $p \in \mathbb{P}_1$, $\boldsymbol{q} \in \mathbb{P}_1^3$ and $\boldsymbol{w}^0 \in \boldsymbol{N}_1$. From the definition of $\boldsymbol{N}_1$, we see that $\boldsymbol{w}^0 \in \mathbb{P}_2^3$ and satisfies $\boldsymbol{w}^0 \cdot \boldsymbol{x} = 0$. If all the degrees of freedom given by (2.7)–(2.8) are zero then we quickly derive that $\boldsymbol{w}^0 = 0$ from the standard 20 degrees of freedom of $\boldsymbol{N}_1$ (two lowest order moments of the tangential component on each edge and the loweat order moment of the tangential components on each face). Furthermore, since $\boldsymbol{w} = \mathbf{grad}(bp) + b\boldsymbol{q}$ we obtain $\boldsymbol{w} \cdot \boldsymbol{n}$ is proportional to $b_f p$ on each face. Hence, we conclude that

$$\int_f b_f p \, d\boldsymbol{x}_f = 0 \quad f \in \Delta_2(T)$$

and therefore, by (3.1), $p = 0$ at the barycentre of each face. But then $p = 0$. Finally, if $\boldsymbol{w} = b\boldsymbol{q}$ then the tangential component $(\mathbf{curl}\,\boldsymbol{w})_t$ is proportional to $b_f(\boldsymbol{q} \times \boldsymbol{n})$ on each face, and therefore Lemma 3.3 again implies that $\boldsymbol{q} = 0$.

3.3. **The discrete complex.** We have seen that the polynomial spaces $S(T)$, $\boldsymbol{W}(T)$, $\boldsymbol{V}(T)$, and $Q(T)$, defined on a single tetrahedron T all have a set of unisovent degrees of freedom specified in §2. Given a tetrahedral mesh $\mathcal{T}_h$ the spaces S_h, $\boldsymbol{W}_h$, $\boldsymbol{V}_h$, and Q_h are all defined as the functions which belong to the corresponding polynomial space on each tetrahedron T, and where the continuity conditions are implicitly defined by the degrees of freedom on vertices, edges, and faces.

It is straightforward to check that in the sequence

$$\mathbb{R} \xrightarrow{\subset} S_h \xrightarrow{\mathbf{grad}} \boldsymbol{W}_h \xrightarrow{\mathbf{curl}} \boldsymbol{V}_h \xrightarrow{\mathrm{div}} Q_h \longrightarrow 0 \tag{3.7}$$

each space is mapped into the succeeding space by the given operator, and hence the sequence is a complex. Furthermore, if Ω is contractible the sequence is exact. In fact, this is an easy consequence of the similar property for more standard discrete spaces. To see this, let S_h^0 be the standard continuous piecewise linear space with respect to the triangulation $\mathcal{T}_h$, $\boldsymbol{W}_h^0$ the second lowest order Nedelec space corresponding to piecewise polynomials in $\boldsymbol{N}_1$, and $\boldsymbol{V}_h^0$ the space of piecewise linear vector fields with $\boldsymbol{H}(\mathrm{div})$ continuity, i.e. $\boldsymbol{V}_h^0$ is the the lowest order space in Nedelec's second family. For these spaces it is well known that the sequence

$$\mathbb{R} \xrightarrow{\subset} S_h^0 \xrightarrow{\mathbf{grad}} \boldsymbol{W}_h^0 \xrightarrow{\mathbf{curl}} \boldsymbol{V}_h^0 \xrightarrow{\mathrm{div}} Q_h \longrightarrow 0 \tag{3.8}$$

is exact, cf. [1], [11].

By definition the restriction $\boldsymbol{v}_T$ of an element $\boldsymbol{v} \in \boldsymbol{V}_h$ to a tetrahedron $T \in \mathcal{T}_h$ is of the form

$$\boldsymbol{v}_T = \boldsymbol{v}_T^0 + \mathbf{curl}(b_T \boldsymbol{q}_T) \quad \text{with } \boldsymbol{v}_T^0, \boldsymbol{q}_T \in \mathbb{P}_1^3. \tag{3.9}$$

However, on each face $f \in \Delta_2(T)$ the normal component of $\mathbf{curl}(b_T \boldsymbol{q}_T)$ is zero. Therefore, the continuity requirements on $\boldsymbol{v}$ imply that the piecewise polynomial $\boldsymbol{v}^0$ has continuous normal components, and hence $\boldsymbol{v}^0$ is an element of the space $\boldsymbol{V}_h^0$. Furthermore, the weak continuity of the tangential components of $\boldsymbol{v}$ implies that for each face f of $\mathcal{T}_h$

$$\int_f [(\boldsymbol{v}^0 + \mathbf{curl}\, b\boldsymbol{q})_t] \cdot \boldsymbol{r}\, d\boldsymbol{x}_f = 0 \quad \boldsymbol{r} \in \boldsymbol{RM}(f), \tag{3.10}$$

where $[\cdot]$ denote the jump across f.

Assume now that $\operatorname{div} \boldsymbol{v} = 0$. In order to show that the sequence (3.7) is exact we need to show that there is a $\boldsymbol{w} \in \boldsymbol{W}_h$ such that $\mathbf{curl}\, \boldsymbol{w} = \boldsymbol{v}$. However, if $\operatorname{div} \boldsymbol{v} = 0$ and $\boldsymbol{v}$ is of the form (3.9) then $\operatorname{div} \boldsymbol{v}^0 = 0$, and by the exactness of the sequence (3.8) we can conclude that there is a $\boldsymbol{w}^0 \in \boldsymbol{W}_h^0$ such that

$$\boldsymbol{v}_T = \mathbf{curl}(\boldsymbol{w}_T^0 + b_T \boldsymbol{q}_T)$$

on each triangle T. Furthermore, from (3.10) we obtain

$$\int_f [(\mathbf{curl}\, \boldsymbol{w}^0 + b\boldsymbol{q})_t] \cdot \boldsymbol{r}\, d\boldsymbol{x}_f = 0 \quad \boldsymbol{r} \in \boldsymbol{RM}(f).$$

Hence, if $\boldsymbol{w} = \boldsymbol{w}^0 + b\boldsymbol{q}$ we conclude from (2.7)–(2.8) that $\boldsymbol{w} \in \boldsymbol{W}_h$.

We can use a similar argument to show that all curl–free elements of $\boldsymbol{W}_h$ are gradients of functions in S_h. First note that any $\boldsymbol{w} \in \boldsymbol{W}_h$ is on the form

$$\boldsymbol{w}_T = \boldsymbol{w}_T^0 + \mathbf{grad}(b_T p_T) + b_T \boldsymbol{q}_T \tag{3.11}$$

on each tetrahedron T, where $\boldsymbol{w}_T^0$ is in the class $\boldsymbol{N}_1$ and p and $\boldsymbol{q}$ are linears. Furthermore, $\boldsymbol{w}^0 \in \boldsymbol{W}_h^0$ since the two other terms on the right hand side of (3.11) vanish for the standard degrees of freedom of $\boldsymbol{W}_h^0$. If $\mathbf{curl}\, \boldsymbol{w} = 0$ then clearly

$$\mathbf{curl}\, \boldsymbol{w}^0 = 0 \quad \text{and} \quad \mathbf{curl}\, b\boldsymbol{q} = 0.$$

However, if $\mathbf{curl}\, b\boldsymbol{q} = 0$ then, in particular, the tangential component $(\mathbf{curl}\, b\boldsymbol{q})_t = 0$ on all faces, and hence, by (2.7)–(2.8) the element $b\boldsymbol{q}$ of $\boldsymbol{W}_h$ is zero. Furthermore, since $\mathbf{curl}\, \boldsymbol{w}^0 = 0$ we can use (3.8) to obtain $\boldsymbol{w}^0 = \mathbf{grad}\, s^0$ for a suitable $s^0 \in S_{h,0}$. So $\boldsymbol{w} = \mathbf{grad}\, s$, where $s = s^0 + bp \in S_h$.

3.4. **Commuting diagrams.** The finite element spaces S_h, $\boldsymbol{W}_h$, $\boldsymbol{V}_h$, and Q_h introduced above are subspaces of H^1, $\boldsymbol{H}(\mathbf{curl})$, $\boldsymbol{H}(\mathrm{div})$, and L^2, respectively. In addition, due to additional weak continuity over interelement faces, the spaces S_h, $\boldsymbol{W}_h$, and $\boldsymbol{V}_h$ are nonconforming approximations of H^2, $\boldsymbol{H}^1(\mathbf{curl})$, and $\boldsymbol{H}^1$.

The degrees of freedom, or more pricesly the moment conditions, specified above define canonical interpolation operators

$$\Pi_h^S : H^2 \to S_h, \quad \boldsymbol{\Pi}_h^W : \boldsymbol{H}^1(\mathbf{curl}) \to \boldsymbol{W}_h, \quad \boldsymbol{\Pi}_h^V : \boldsymbol{H}^1 \to \boldsymbol{V}_h,$$

and $\Pi_h^Q : L^2 \to Q_h$. Furthermore, the following diagram commutes.

$$\begin{array}{ccccccccccc} \mathbb{R} & \longrightarrow & H^2 & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}^1(\mathbf{curl}) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}^1 & \xrightarrow{\mathrm{div}} & L^2 & \longrightarrow & 0 \\ & & \big\downarrow \Pi_h^S & & \big\downarrow \boldsymbol{\Pi}_h^W & & \big\downarrow \boldsymbol{\Pi}_h^V & & \big\downarrow \Pi_h^Q & & \\ \mathbb{R} & \longrightarrow & S_h & \xrightarrow{\mathbf{grad}} & \boldsymbol{W}_h & \xrightarrow{\mathbf{curl}} & \boldsymbol{V}_h & \xrightarrow{\mathrm{div}} & Q_h & \longrightarrow & 0 \end{array}$$

In other words the identities

$$\mathbf{grad}\,\Pi_h^S = \boldsymbol{\Pi}_h^W\,\mathbf{grad}, \quad \mathbf{curl}\,\boldsymbol{\Pi}_h^W = \boldsymbol{\Pi}_h^V\,\mathbf{curl}, \quad \mathrm{div}\,\boldsymbol{\Pi}_h^V = \Pi_h^Q\,\mathrm{div},$$

all holds. It is a straightforward and standard argument to verify these identities, and we therefore omit the details here.

In the analysis for the finite element solutions, we need the corresponding spaces to be with homogeneous boundary conditions. Hence, the complex (1.7) should be replaced by

(3.12)

$$0 \xrightarrow{\subset} H_0^2 \xrightarrow{\mathbf{grad}} \boldsymbol{H}_0^1(\mathbf{curl}) \xrightarrow{\mathbf{curl}} \boldsymbol{H}_0^1 \xrightarrow{\mathrm{div}} L_0^2 \longrightarrow 0,$$

where

$$\boldsymbol{H}_0^1(\mathbf{curl}) = \{\boldsymbol{w} \in \boldsymbol{H}_0^1 : (\mathbf{curl}\,\boldsymbol{w})_t = 0 \quad \text{on } \partial\Omega\}.$$

A corresponding discrete, nonconforming, approximation is obtained by restricting the spaces S_h, $\boldsymbol{W}_h$, and $\boldsymbol{V}_h$ to the subspaces with vanishing degrees of freedom on the boundary $\partial\Omega$. For example, the space $\boldsymbol{V}_h$ is replaced by $\boldsymbol{V}_{h,0}$ given as all $\boldsymbol{v} \in \boldsymbol{V}_h$ such that

$$\int_f (\boldsymbol{v} \cdot \boldsymbol{n}) p \, d\boldsymbol{x}_f = 0 \quad p \in \mathbb{P}_1(f), \qquad \int_f \boldsymbol{v}_t \cdot \boldsymbol{r} \, d\boldsymbol{x}_f = 0 \quad \boldsymbol{r} \in \boldsymbol{RM}(f),$$

for all faces f in $\partial\Omega$. Hence, $\boldsymbol{v} \cdot \boldsymbol{n}$ vanishes on the boundary for any $\boldsymbol{v} \in \boldsymbol{V}_{h,0}$, while the tangential component is zero in a weak sense. Hence, $\boldsymbol{V}_{h,0} \subset \boldsymbol{H}_0(\mathrm{div})$, where

$$\boldsymbol{H}_0(\mathrm{div}) = \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}) : \boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega\}.$$

Furthermore, $\boldsymbol{V}_{h,0}$ is a nonconforming approximation of $\boldsymbol{H}_0^1$. We also let $Q_{h,0} = Q_h \cap L_0^2$.

As above we obtain the commuting diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H_0^2 & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}_0^1(\mathbf{curl}) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}_0^1 & \xrightarrow{\mathrm{div}} & L_0^2 & \longrightarrow & 0 \\ & & \big\downarrow \Pi_h^S & & \big\downarrow \boldsymbol{\Pi}_h^W & & \big\downarrow \boldsymbol{\Pi}_h^V & & \big\downarrow \Pi_h^Q & & \\ 0 & \longrightarrow & S_{h,0} & \xrightarrow{\mathbf{grad}} & \boldsymbol{W}_{h,0} & \xrightarrow{\mathbf{curl}} & \boldsymbol{V}_{h,0} & \xrightarrow{\mathrm{div}} & Q_{h,0} & \longrightarrow & 0 \end{array}$$

where the upper and lower rows are complexes.

4. Uniform error estimates for the Darcy–Stokes system

In this section we shall discuss how the finite element space $\boldsymbol{V}_{h,0} \times Q_{h,0} \subset \boldsymbol{H}_0(\mathrm{div}) \times L_0^2$ can be used to construct a discretization of the singular perturbation problem (1.1) in $\mathbb{R}^3$ with uniform convergence properties with respect to the parameter ε. The results are similar to the corresponding results obtained in [8] for the two dimensional case. Therefore, the discussion here will be rather brief, and we will only focus the attention on the parts where the analysis from [8] needs to be essentially modified.

In order to avoid some technical difficulties, we will restrict the discussion to the case when the source term $g = 0$. The standard weak formulation of the system (1.1) is to find $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1 \times L_0^2$ such that

$$\begin{aligned} a_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) + (p, \mathrm{div}\, \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}) \qquad \boldsymbol{v} \in \boldsymbol{H}_0^1, \\ (\mathrm{div}\, \boldsymbol{u}, q) &= 0 \qquad\qquad\;\, q \in L_0^2. \end{aligned} \tag{4.1}$$

Here we assume that data $\boldsymbol{f} \in \boldsymbol{H}^{-1} \equiv (\boldsymbol{H}_0^1)^*$ and a_ε is the bilinear form

$$a_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v}) + \varepsilon^2 (\boldsymbol{D}\boldsymbol{u}, \boldsymbol{D}\boldsymbol{v}),$$

defined on $\boldsymbol{H}_0^1 \times \boldsymbol{H}_0^1$. The corresponding finite element $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_{h,0} \times Q_{h,0}$ is given by the following equations:

$$\begin{aligned} a_\varepsilon(\boldsymbol{u}_h, \boldsymbol{v}) + (p_h, \mathrm{div}\, \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}) \qquad \boldsymbol{v} \in \boldsymbol{V}_{h,0}, \\ (\mathrm{div}\, \boldsymbol{u}_h, q) &= 0 \qquad\qquad\;\, q \in Q_{h,0}. \end{aligned} \tag{4.2}$$

As $\boldsymbol{V}_{h,0}$ is nonconforming, the bilinear form $a_\varepsilon(\cdot, \cdot)$ is understood to be the sum of the corresponding integrals over each tetrahedron of $\mathcal{T}_h$. Recall that for a smooth vector field $\boldsymbol{v}$

$$\boldsymbol{\Delta v} = \mathbf{grad}\, \mathrm{div}\, \boldsymbol{v} - \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{v},$$

and, as a consequence,

$$(\boldsymbol{D}\boldsymbol{u}, \boldsymbol{D}\boldsymbol{v}) = (\mathrm{div}\, \boldsymbol{u}, \mathrm{div}\, \boldsymbol{v}) + (\mathbf{curl}\, \boldsymbol{u}, \mathbf{curl}\, \boldsymbol{v}), \boldsymbol{u} \in \boldsymbol{H}_0^1, \boldsymbol{v} \in \boldsymbol{H}^1.$$

If $\boldsymbol{u}$ is the solution of (4.1), we define the consistency error by

$$E_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) = a_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) + (p, \mathrm{div}\, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v}), \quad \boldsymbol{v} \in \boldsymbol{V}_{h,0}. \tag{4.3}$$

¿From Green's formula we obtain that

$$E_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) = \varepsilon^2 \sum_{f \in \Delta_2^h} \int_f (\mathbf{curl}\, \boldsymbol{u}) \;\; [\boldsymbol{v} \times \boldsymbol{n}]\, dx_f, \tag{4.4}$$

Here, Δ_2^h denotes all the faces for the tetrahedral mesh $\mathcal{T}_h$.

The uniform analysis of the discretization of the system (1.1) will be based on the ε–dependent function space $(\boldsymbol{H}_0(\mathrm{div}) \cap \varepsilon \cdot \boldsymbol{H}_0^1) \times L_0^2$. The corresponding norm is given by

$$\|\!|\boldsymbol{v}|\!\|_\varepsilon^2 = \|\boldsymbol{v}\|_0^2 + \|\, \mathrm{div}\, \boldsymbol{v}\|_0^2 + \varepsilon^2 \|\boldsymbol{D}\boldsymbol{v}\|_0^2.$$

For convenience we also introduce $\| \cdot \|_a$ as the norm associated the bilinear form a_ε. For elements of $\boldsymbol{V}_{h,0}$ these norms should be interpreted as the corresponding broken norms.

Using the commuting diagram property $\operatorname{div} \boldsymbol{\Pi}_h^V = \Pi_h^Q \operatorname{div}$ and the $\boldsymbol{H}^1$ boundedness of $\boldsymbol{\Pi}_h^V$ we obtain that there exists a constant $\alpha_1 > 0$, independent of h and ε, such that

$$(4.5) \qquad \sup_{\boldsymbol{v} \in \boldsymbol{V}_{h,0}} \frac{(q, \operatorname{div} \boldsymbol{v})}{\|\!|\boldsymbol{v}|\!\|_\varepsilon} \geq \sup_{\boldsymbol{v} \in \boldsymbol{V}_{h,0}} \frac{(q, \operatorname{div} \boldsymbol{v})}{\|\!|\boldsymbol{v}|\!\|_1} \geq \alpha_1 \|q\|_0 \quad \text{for all } q \in Q_{h,0}.$$

Hence, the proper uniform inf–sup condition is satisfied.

Remark. Recall from [8] that most standard Stokes elements will not lead to a uniformly stable discretization in the present case. This is due to the fact the bilinear form a_ε is not uniformly coercive with respect to the energy norm $\|\!| \cdot |\!\|_\varepsilon$ on the space of weakly divergence free vector fields, i.e. the second Brezzi condition is violated. However, in the present case, where the divergence operator maps $\boldsymbol{V}_{h,0}$ onto $Q_{h,0}$, this conditions is obvious. In particula, we have that $\operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{u}_h = 0$. □

Taking $\boldsymbol{v} = \boldsymbol{\Pi}_h^V \boldsymbol{u} - \boldsymbol{u}_h$ in the first equation of (4.2) and (4.3), we obtain

$$(4.6) \qquad \|\boldsymbol{u} - \boldsymbol{u}_h\|_a \leq 2\left(\|\boldsymbol{u} - \boldsymbol{\Pi}_h^V \boldsymbol{u}\|_a + \sup_{\boldsymbol{v} \in \boldsymbol{V}_{h,0}} \frac{|E_{\varepsilon,h}(\boldsymbol{u}, \boldsymbol{v})|}{\|\boldsymbol{v}\|_a} \right).$$

Since the polynomial space $\boldsymbol{V}(T)$ contains all linears, and the family $\{\mathcal{T}_h\}$ is shape regular, we obtain from a scaling argument that

$$(4.7) \qquad \|\boldsymbol{\Pi}_h^V \boldsymbol{v} - \boldsymbol{v}\|_a \leq c(h^2 + \varepsilon h)\|\boldsymbol{v}\|_2, \quad \boldsymbol{v} \in \boldsymbol{H}^2 \cap \boldsymbol{H}_0^1,$$

where the constant c is independent of $\boldsymbol{v}$, ε and h. Under the assumption that the solution $\boldsymbol{u}$ of (4.1) is in $\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1$ we can use a trace theorem and a scaling argument (cf. [8, Lemma 5.1]), to establish that

$$(4.8) \qquad \sup_{\boldsymbol{v} \in \boldsymbol{V}_{h,0}} \frac{|E_\varepsilon(\boldsymbol{u}, \boldsymbol{v})|}{\|\boldsymbol{v}\|_a} \leq c\,\varepsilon \begin{cases} h\|\operatorname{\mathbf{curl}} \boldsymbol{u}\|_1 \\ h^{1/2}\|\operatorname{\mathbf{curl}} \boldsymbol{u}\|_1^{1/2}\|\operatorname{\mathbf{curl}} \boldsymbol{u}\|_0^{1/2}. \end{cases}$$

By combining (4.5)–(4.8) we obtain the following error estimates (cf. [8, Theorem 5.1])

$$(4.9) \qquad \|\boldsymbol{u} - \boldsymbol{u}_h\|_0 + \varepsilon\|\operatorname{\mathbf{curl}}(\boldsymbol{u} - \boldsymbol{u}_h)\|_0 \leq c(h^2 + \varepsilon h)\|\boldsymbol{u}\|_2,$$

$$(4.10) \qquad \|p - p_h\|_0 \leq c\,h(\|p\|_1 + (\varepsilon + h)\|\boldsymbol{u}\|_2).$$

These estimates are uniform in the sense that the constant c is independent of $\boldsymbol{u}$, ε and h. However, in general the term $\|\boldsymbol{u}\|_2$ is not bounded uniformly in ε. A real uniform estimate, corresponding to a result obtained in [8] in the two dimensional case, is of the form

$$(4.11) \qquad \|\boldsymbol{u} - \boldsymbol{u}_h\|_0 + \varepsilon\|\ \boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\|_0 \leq c\,h^{1/2}\|\boldsymbol{f}\|_1.$$

As illustrated by the development in [8] the key ingredient in deriving such an estimate is proper uniform bounds on the solution $\boldsymbol{u}$, cf.

Lemma 4.1 below. However, the argument given in [8] for this result cannot easily be extended to three dimensions. For completeness, we therefore give an alternative proof here, valid in both two and three dimensions.

4.1. **Uniform regularity.** Throughout this section we assume that the domain Ω is a convex polyhedron. By an energy argument it is straightforward to show that the weak solution $(\boldsymbol{u}, p)$ of (1.1) satifies the uniform bound

$$\|\!|\boldsymbol{u}|\!\|_\varepsilon + \|p\|_0 \le c\,\|\boldsymbol{f}\|_0. \tag{4.12}$$

Hence, for a fixed $\boldsymbol{f} \in \boldsymbol{L}^2$, the quantity $\|\boldsymbol{D}\boldsymbol{u}\|_0$ is at most proportional to ε^{-1} as ε tends to zero. However, if $\boldsymbol{f}$ is more regular an improved estimate can be obtained. To see this we let $(\boldsymbol{u}^0, p^0) \in \boldsymbol{H}_0(\mathrm{div}) \times L_0^2$ be the weak solution of the corresponding reduced problem

$$\begin{aligned} \boldsymbol{u}^0 - \mathbf{grad}\, p^0 &= \boldsymbol{f} \quad \text{in } \Omega,\\ \mathrm{div}\, \boldsymbol{u}^0 &= 0 \quad \text{in } \Omega,\\ \boldsymbol{u}^0 \cdot \boldsymbol{n} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.13}$$

Below we shall consider this problem with $\boldsymbol{f} \in \boldsymbol{H}^1$. Hence, it is an immediate consequence of elliptic regularity that $(\boldsymbol{u}^0, p^0)$ is a classical solution in $(\boldsymbol{H}^1 \cap \boldsymbol{H}_0(\mathrm{div})) \times H^2$, with corresponding norms depending continuously on $\|\boldsymbol{f}\|_1$.

Lemma 4.1. *Assume that Ω is convex and that $\boldsymbol{f} \in \boldsymbol{H}^1$. There exists a constant $c > 0$, independent of ε and $\boldsymbol{f}$, such that*

$$\varepsilon^{1/2}\|\boldsymbol{u}\|_1 + \varepsilon^{3/2}\|\boldsymbol{u}\|_2 \le c\,\|\boldsymbol{f}\|_1, \tag{4.14}$$

$$\|\boldsymbol{u} - \boldsymbol{u}^0\|_0 + \|p - p^0\|_1 \le c\,\varepsilon^{1/2}\|\boldsymbol{f}\|_1. \tag{4.15}$$

Proof. When Ω is convex the solution of the standard Stokes problem

$$\begin{aligned} -\Delta\bar{\boldsymbol{u}} - \mathbf{grad}\,\bar{p} &= \boldsymbol{f} \quad \text{in } \Omega,\\ \mathrm{div}\,\bar{\boldsymbol{u}} &= 0 \quad \text{in } \Omega,\\ \bar{\boldsymbol{u}} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{4.16}$$

satisfies the regularity estimate, cf. [6],

$$\|\bar{\boldsymbol{u}}\|_2 + \|\bar{p}\|_1 \le c\,\|\boldsymbol{f}\|_0. \tag{4.17}$$

By considering the pair $(\boldsymbol{u}, \varepsilon^{-2}(p - p^0))$ as a weak solution of the system

$$\begin{aligned} -\Delta\boldsymbol{u} - \mathbf{grad}(\varepsilon^{-2}(p - p^0)) &= -\varepsilon^{-2}(\boldsymbol{u} - \boldsymbol{u}^0)\\ \mathrm{div}\,\boldsymbol{u} &= 0, \end{aligned} \tag{4.18}$$

we obtain from (4.17) that $\boldsymbol{u} \in \boldsymbol{H}_0^1 \cap \boldsymbol{H}^2$, $p \in L_0^2 \cap H^1$ and

$$\varepsilon^2\|\boldsymbol{u}\|_2 + \|p - p^0\|_1 \le c\,\|\boldsymbol{u} - \boldsymbol{u}^0\|_0, \tag{4.19}$$

with constant c independent of ε. Due to the enhanced regularity of the solution $\boldsymbol{u}$ we obtain from (4.18) that

$$a_\varepsilon(\boldsymbol{u}, \boldsymbol{v}) - (\boldsymbol{u}^0, \boldsymbol{v}) + (p - p^0, \mathrm{div}\,\boldsymbol{v}) = \langle \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}, \boldsymbol{v}\rangle, \quad \boldsymbol{v} \in \boldsymbol{H}^1 \cap \boldsymbol{H}_0(\mathrm{div}),$$

where, $\langle\cdot,\cdot\rangle$ is the L^2 inner product on $\partial\Omega$. Setting $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}^0$ this gives

$$\|\boldsymbol{u}-\boldsymbol{u}^0\|_0^2 + \varepsilon^2\|\boldsymbol{D}\boldsymbol{u}\|_0^2 = \varepsilon^2(\boldsymbol{D}\boldsymbol{u}, \boldsymbol{D}\boldsymbol{u}^0) + \varepsilon^2\langle\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}, \boldsymbol{u}^0\rangle. \tag{4.20}$$

By using a standard trace inequality we further obtain

$$\begin{aligned}
\varepsilon^2\langle\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}, \boldsymbol{u}^0\rangle &\le c\,\varepsilon^2\|\boldsymbol{u}\|_1^{1/2}\|\boldsymbol{u}\|_2^{1/2}\|\boldsymbol{u}^0\|_1 \\
&\le c_\delta\varepsilon\|\boldsymbol{u}^0\|_1^2 + \delta\varepsilon^3\|\boldsymbol{u}\|_1\|\boldsymbol{u}\|_2 \\
&\le c_\delta\varepsilon\|\boldsymbol{u}^0\|_1^2 + \frac{\delta\varepsilon^2}{2}\|\boldsymbol{u}\|_1^2 + \frac{\delta\varepsilon^4}{2}\|\boldsymbol{u}\|_2^2 \qquad (4.21)\\
&\le c_\delta\varepsilon\|\boldsymbol{f}\|_1^2 + C\,\delta(\varepsilon^2\|\boldsymbol{D}\boldsymbol{u}\|_0^2 + \|\boldsymbol{u}-\boldsymbol{u}^0\|_0^2),
\end{aligned}$$

where we have used (4.19) and the H^1–regularity of $\boldsymbol{u}^0$ in the last step. Here both the constants C and c_δ are independent of ε, but c_δ depends continuously on δ. For the first term on the right hand side of (4.20) we have

$$\varepsilon^2(\boldsymbol{D}\boldsymbol{u}, \boldsymbol{D}\boldsymbol{u}^0) \le \frac{\varepsilon^2}{4}\|\boldsymbol{D}\boldsymbol{u}\|_0^2 + \varepsilon^2\|\boldsymbol{D}\boldsymbol{u}^0\|_0^2 \le \frac{\varepsilon^2}{4}\|\boldsymbol{D}\boldsymbol{u}\|_0^2 + c\,\varepsilon^2\|\boldsymbol{f}\|_1^2,$$

where the constant c is independent of ε. However, together with (4.20) and (4.21), and by choosing δ sufficiently small, this implies

$$\|\boldsymbol{u}-\boldsymbol{u}^0\|_0^2 + \varepsilon^2\|\boldsymbol{D}\boldsymbol{u}\|_0^2 \le c\,\varepsilon\|\boldsymbol{f}\|_1^2 \tag{4.22}$$

with c independent of ε. Together with (4.19) this implies the desired estimates (4.14) and (4.15). □

¿From Lemma 4.1 and (4.8), we see that

$$\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h,0}} \frac{|E_\varepsilon(\boldsymbol{u},\boldsymbol{v}_h)|}{\|\boldsymbol{v}_h\|_a} \le c\,h^{\frac12}\varepsilon\|\,\mathbf{curl}\,\boldsymbol{u}\|_1^{\frac12}\|\,\mathbf{curl}\,\boldsymbol{u}\|_0^{\frac12} \le c\,h^{\frac12}\|\boldsymbol{f}\|_1.$$

Furthermore, since the interpolation operator $\boldsymbol{\Pi}_h^V$ is defined from traces on the two dimensional faces in Δ_2^h, the interpolation estimate

$$\|\boldsymbol{\Pi}_h^V\boldsymbol{v}-\boldsymbol{v}\|_0 \le c\,h^{1/2}\|\boldsymbol{v}\|_0^{1/2}\|\boldsymbol{v}\|_1^{1/2},$$

follows from a standard trace inequality and scaling. From this estimate, and by arguing exactly as in the proof of Theorem 6.1 of [8], we derive

$$\|\boldsymbol{u}-\boldsymbol{\Pi}_h^V\boldsymbol{u}\|_0 + \varepsilon\|\boldsymbol{u}-\boldsymbol{\Pi}_h^V\boldsymbol{u}\|_1 \le c\,h^{\frac12}\|\boldsymbol{f}\|_1.$$

Combining the two estimates above with the inf-sup condition (4.5) and error bound (4.6), we obtain the desired uniform estimate (4.11).

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Department of Mathematics, University of Bergen, Johannes Brunsgt. 12, 5007 Bergen, Norway

E-mail address: Xue-Cheng.Tai@mi.uib.no

Centre of Mathematics for Applications and Department of Informatics, University of Oslo, P.O. Box 1053 Blindern, 0316 Oslo, Norway

E-mail address: ragnar.winther@cma.uio.no