AN EXTENSION OF BOHR'S THEOREM

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Abstract. The following extension of Bohr's theorem is established: If a somewhere convergent Dirichlet series f has an analytic continuation to the half-plane $\mathbb{C}_\theta = \{s = \sigma + it \,:\, \sigma > \theta\}$ that maps $\mathbb{C}_\theta$ to $\mathbb{C} \setminus \{\alpha, \beta\}$ for complex numbers $\alpha \neq \beta$, then f converges uniformly in $\mathbb{C}_{\theta+\varepsilon}$ for any $\varepsilon > 0$. The extension is optimal in the sense that the assertion no longer holds should $\mathbb{C} \setminus \{\alpha, \beta\}$ be replaced with $\mathbb{C} \setminus \{\alpha\}$.

1. Introduction

Let $\mathfrak{D}$ denote the class of Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{1}$$

that converge in at least one point $s = \sigma + it$ in the complex plane. Associated to each Dirichlet series f in $\mathfrak{D}$ is a number $\sigma_{\mathrm{c}}(f)$, called the *abscissa of convergence*, with the property that f converges if $\sigma > \sigma_{\mathrm{c}}(f)$ and f does not converge if $\sigma < \sigma_{\mathrm{c}}(f)$. This note concerns an extension of Bohr's classical theorem on uniform convergence of Dirichlet series [3]. We therefore define the *abscissa of uniform convergence* $\sigma_{\mathrm{u}}(f)$ as the infimum of the real numbers θ such that f converges uniformly in the half-plane $\mathbb{C}_\theta$. Here and in what follows, we set

$$\mathbb{C}_\theta = \{s = \sigma + it \,:\, \sigma > \theta\}.$$

Our starting point reads as follows.

Bohr's theorem. *Let f be in $\mathfrak{D}$. If there is a real number θ and a bounded set Ω such that f has an analytic continuation to $\mathbb{C}_\theta$ that maps $\mathbb{C}_\theta$ to Ω, then $\sigma_{\mathrm{u}}(f) \leq \theta$.*

Queffélec and Seip [10] (see also [9, Theorem 8.4.1]) showed that the assumption that Ω is a bounded set may be replaced with the weaker assumption that Ω is a half-plane. This extension of Bohr's theorem was applied to obtain the canonical formulation of the Gordon–Hedenmalm characterization of composition operators [6], which has proven to be essential for further developments (see e.g. [5, Section 6]).

The purpose of the present note is to delineate precisely the limits to how far Bohr's theorem may be extended in terms of the mapping properties of f in the half-plane $\mathbb{C}_\theta$. We will achieve this by establishing the following results.

Theorem 1. *Let f be in $\mathfrak{D}$. If there is a real number θ and complex numbers $\alpha \neq \beta$ such that f has an analytic continuation to $\mathbb{C}_\theta$ that maps $\mathbb{C}_\theta$ to $\mathbb{C} \setminus \{\alpha, \beta\}$, then $\sigma_{\mathrm{u}}(f) \leq \theta$.*

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Theorem 2. *There is a Dirichlet series f with $\sigma_{\mathrm{c}}(f) \leq 1/2$, $\sigma_{\mathrm{u}}(f) = 1$, and*

$$f(\mathbb{C}_\theta) = \mathbb{C} \setminus \{0\}$$

for any $1/2 \leq \theta \leq 1$.

It must be stressed that both results are fairly direct consequences of well-known techniques and results. The proof of Theorem 1 uses Schottsky's theorem similarly to how it is used by Titchmarsh in the introduction to [12, Chapter XI], while Theorem 2 is deduced from results of Bohr [2] and Helson [8] on the Riemann zeta function.

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2. Proof of Theorem 1 and Theorem 2

We begin with some preparation for the proof of Theorem 1. Let $\mathbb{D}(c,r)$ denote the open disc with center c and radius $r > 0$. If f is analytic and different from 0 and 1 in $\mathbb{D}(c,r)$, then the effective version of Schottsky's theorem due to Ahlfors [1] states that

$$|f(s)| \leq \exp\left(\frac{r+|s-c|}{r-|s-c|}\big(7+\max(0,\log|f(c)|)\big)\right) \tag{2}$$

for all s in $\mathbb{D}(c,r)$. (We do not actually require the effective version of Schottsky's theorem, but we find it more convenient to work with explicit expressions.)

Proof of Theorem 1. We may assume without loss of generality that $\alpha = 0$ and $\beta = 1$. It is well-known (see e.g. [9, Chapter 4.2]) that $\sigma_{\mathrm{u}}(f) \leq \sigma_{\mathrm{c}}(f) + 1$, so every Dirichlet series in $\mathfrak{D}$ converges uniformly in some half-plane. For $\vartheta > \theta$, we set

$$M(f,\vartheta) = \sup_{t \in \mathbb{R}} |f(\vartheta + it)|.$$

It is plain that $M(f,\vartheta) < \infty$ if $\vartheta > \sigma_{\mathrm{u}}(f)$. We fix $\vartheta > \sigma_{\mathrm{u}}(f)$ and apply (2) with $c = \vartheta + it$, $r = \vartheta - \theta$, and $s = \sigma + it$, to infer that if $\theta < \sigma < \vartheta$, then

$$|f(s)| \leq \exp\left(\frac{2(\vartheta-\theta)}{\sigma-\theta}\big(7+\max(0,\log|M(f,\vartheta)|)\big)\right). \tag{3}$$

This demonstrates that f is bounded in $\mathbb{C}_{\theta+\varepsilon}$ for any $\varepsilon > 0$, and, consequently, that $\sigma_{\mathrm{u}}(f) \leq \theta$ by Bohr's theorem. $\square$

Ritt [11, Theorem II] established a version of Schottsky's theorem for convergent Dirichlet series. This result provides an upper bound similar to (3) that is valid in all of $\mathbb{C}_\theta$ and that only depends on θ and a_1, under the additional assumption that a_1 is not equal to 0 or 1. Here a_1 denotes the first coefficient in the series (1).

To prepare for the proof of Theorem 2, we consider the vertical translation

$$V_\tau f(s) = f(s + i\tau).$$

The *vertical limit functions* of a Dirichlet series f in $\mathfrak{D}$ are the functions which can be obtained as uniform limits of sequences of vertical translations $(V_{\tau_k} f)_{k\geq 1}$ in $\mathbb{C}_\theta$ for any fixed $\theta > \sigma_{\mathrm{u}}(f)$. Recall from [7, Section 2.3] that the vertical limit functions of the Dirichlet series (1) coincide with the Dirichlet series of the form

$$f_\chi(s) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s},$$

where χ is a completely multiplicative function from the natural numbers to the unit circle.

Certain properties of f are preserved under vertical limits. For instance, Bohr's theorem implies that if f is in $\mathfrak{D}$, then $\sigma_{\mathrm{u}}(f) = \sigma_{\mathrm{u}}(f_\chi)$ for any χ. A consequence of Rouché's theorem (see e.g. [4, Lemma 1]) is that $f_\chi(\mathbb{C}_\theta) = f(\mathbb{C}_\theta)$ for any χ and any $\theta \geq \sigma_{\mathrm{u}}(f)$. However, the abscissa of convergence for f and f_χ may in general be different (see [7, 8] or [9, Chapter 8.4]).

Proof of Theorem 2. We begin with the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

which satisfies $\sigma_{\mathrm{c}}(\zeta) = \sigma_{\mathrm{u}}(\zeta) = 1$. A result of Bohr [2] (see also [9, Chapter 4.5]) asserts that $\zeta(\mathbb{C}_1) = \mathbb{C} \setminus \{0\}$. By the discussion above, it follows that $\sigma_{\mathrm{u}}(\zeta_\chi) = 1$ and that $\zeta_\chi(\mathbb{C}_1) = \mathbb{C} \setminus \{0\}$ for any χ. Helson [8] established that there are χ such that the Dirichlet series ζ_χ converges and does not vanish in the half-plane $\mathbb{C}_{1/2}$. Choosing $f = \zeta_\chi$ for such a χ, we obtain the stated result. $\square$

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