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ANALYSIS OF A CLASS OF DEGENERATE REACTION-DIFFUSION SYSTEMS AND THE BIDOMAIN MODEL OF CARDIAC TISSUE

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Abstract. We prove well-posedness (existence and uniqueness) results for a class of degenerate reaction-diffusion systems. A prototype system belonging to this class is provided by the bidomain model, which is frequently used to study and simulate electrophysiological waves in cardiac tissue. The existence result, which constitutes the main thrust of this paper, is proved by means of a nondegenerate approximation system, the Faedo-Galerkin method, and the compactness method.

Contents

1. Introduction

Our point of departure is a widely accepted model, the so-called *bidomain* model, for describing the cardiac electric activity in a physical domain $\Omega \subset \mathbb{R}^3$ (the cardiac muscle) over a time span $(0, T)$, $T > 0$. In this model the cardiac muscle is viewed as two superimposed (anisotropic) continuous media, referred to as the intracellular

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(i) and extracellular (e), which occupy the same volume and are seperated from each other by the cell membrane.

To state the model, we let $u_i = u_i(t,x)$ and $u_e = u_e(t,x)$ represent the spatial cellular at time $t \in (0,T)$ and location $x \in \Omega$ of the *intracellular* and *extracellular* electric potentials, respectively. The difference $v = v(t,x) = u_i - u_e$ is known as the *transmembrane* potential. The anisotropic properties of the two media are modeled by conductivity tensors $\mathbf{M}_i(t,x)$ and $\mathbf{M}_e(t,x)$. The surface capacitance of the membrane is represented by a constant $c_m > 0$. The transmembrane ionic current is represented by a nonlinear (cubic polynomial) function $h(t,x,v)$ depending on time t, location x, and the value of the potential v. The stimulation currents applied to the intra- and extracellular space are represented by a function $I_{\text{app}} = I_{\text{app}}(t,x)$.

A prototype system that governs the cardiac electric activity is the following degenerate reaction-diffusion system (known as the *bidomain equations*)

$$\begin{aligned} &c_m \partial_t v - \operatorname{div}\left(\mathbf{M}_i(t,x)\nabla u_i\right) + h(t,x,v) = I_{\text{app}}, \qquad (t,x) \in Q_T, \\ &c_m \partial_t v + \operatorname{div}\left(\mathbf{M}_e(t,x)\nabla u_e\right) + h(t,x,v) = I_{\text{app}}, \qquad (t,x) \in Q_T, \end{aligned} \tag{1.1}$$

where Q_T denotes the time-space cylinder $(0,T) \times \Omega$. We complete the bidomain system (1.1) with Dirichlet boundary conditions for the intra- and extracellular electric potentials:

$$u_j = 0 \quad \text{on } \partial\Omega \times (0,T), \qquad j = i,e, \tag{1.2}$$

and with initial data for the transmembrane potential:

$$v(0,x) = v_0(x), \quad x \in \Omega. \tag{1.3}$$

For the boundary we could have dealt with Neumann type conditions as well, which seem to be used frequently in the applicative literature, i.e.,

$$(\mathbf{M}_j(t,x)\nabla u_j) \cdot \eta = 0 \quad \text{on } \partial\Omega \times (0,T), \qquad j = i,e,$$

where η denotes the outer unit normal to the boundary $\partial\Omega$ of Ω

For the sake of completeness we have included a brief derivation of the bidomain model in Section 2, but we refer to the papers [7, 8, 9, 10, 14, 18, 30] and the books [16, 25, 29] for detailed accounts on the bidomain model.

If $\mathbf{M}_i \equiv \lambda \mathbf{M}_e$ for some constant $\lambda \in \mathbb{R}$, then the system (1.1) is equivalent to a scalar parabolic equation for the transmembrane potential v. This nondegenerate case, which assumes an equal anisotropic ratio for the intra- and extracelluar media, is known as the monodomain model. Being a scalar equation, the monodomain model is well understood from a mathematical point of view, see for example [26].

On the other hand, the bidomain system (1.1) was studied only recently from a well-posedness (existence and uniqueness of solutions) point view [10]. Indeed, standard elliptic/parabolic theory does not apply directly to the bidomain equations due to their degenerate structure, which is a consequence of the unequal anisotropic ratio of the intra- and extracellular media. In fact, a distinguishing feature of the bidomaim model lies in the structure of the coupling between the intra- and extracellular media, which takes into account the anisotropic conductivity of both media. When the degree of anisotropy is different in the two media, we end up with a system (1.1) that is of *degenerate parabolic* type.

In this paper we shall not exclusively investigate the bidomain system (1.1) but also a class of systems that are characterized by a combination of general nonlinear diffusivities and the degenerate structure seen in the bidomain equations. These reaction-diffusion systems read

$$
\begin{aligned}
&c_m \partial_t v - \operatorname{div} M_i(t,x,\nabla u_i) + h(t,x,v) = I_{\text{app}}, \qquad (t,x) \in Q_T,\\
&c_m \partial_t v + \operatorname{div} M_e(t,x,\nabla u_e) + h(t,x,v) = I_{\text{app}}, \qquad (t,x) \in Q_T,
\end{aligned}
\tag{1.4}
$$

where the nonlinear vector fields $M_j(t,x,\xi) : Q_T \times \mathbb{R}^3 \to \mathbb{R}^3$, $j = i,e$, are assumed to be Leray-Lions operators, p-coercive, and behave like $|\xi|^{p-1}$ for large values of $\xi \in \mathbb{R}^3$ for some $p > 2$, see Subsection 3.2 precise conditions. We complete the nonlinear system (1.4) with Dirichlet boundary conditions (1.2) for the intra- and extracellular potentials and initial data (1.3) for the transmembrane potential.

Formally, by taking $M_j(t,x,\xi) = \mathbf{M}_j\xi$, $j = i,e$, in (1.4) we obtain the bidomain equations (1.1). An example of a nonlinear diffusion part in (1.4) is provided by

$$
M_j(t,x,\xi) = |\xi|^{p-2}\,\mathbf{M}_j(t,x)\xi, \quad p > 2, \quad j = i,e. \tag{1.5}
$$

Although (1.4) can be viewed as a generalization of the bidomain equations in view of its more general diffusion part, it turns out that to prove existence of solutions we need to impose different conditions on the nonlinearity h than those employed in the bidomain system (1.1) (precise conditions will be given later). These conditions resemble those commonly used in the study of scalar semilinear elliptic or parabolic equations. From a technical point of view, this means that the proofs for the bidomain system (1.1) and the nonlinear system (1.4) will be different on various occations (typically more involved in the latter case).

The bidomain system contains the term h describing the flow of ions accross the cell membrane. This is the simplest possible model, and in this model it is customary to assume that the current is a cubic polynomial of the transmembrane potential. In a more realistic setup the reaction-diffusion system (1.1) is coupled with a system of ODEs for the ionic gating variables and for the ions concentration. However, since the main interest in this paper lies with the degenerate structure of the system (1.1), we neglect the ODE coupling and assume that the relevant effects are already taken care of by the nonlinear function h.

When it comes to well-posedness analyis for the bidomain model we know of only one paper, namely [10] (it treats both microscopic and macroscopic models). In that paper the authors propose a variational formulation of the model and show after an abstract change of variable that it has a structure that fits into the framework of evolution variational inequalities in Hilbert spaces. This allows them to obtain a series of results about existence, uniqueness, and regularity of solutions.

Somewhat related, based on the theory in [10] the author of [27] proves error estimates for a Galerkin method for the bidomain model. Let us also mention the paper [1] in which the authors use tools from Γ-convergence theory to study the asymptotic behaviour of anisotropic energies arising in the bidomain model.

Let us now put our own contributions into a perspective. With reference to the bidomain equations (1.1) and the work [10], we give a different and constructive proof for the existence of weak solutions. Our proof is based on introducing nondegenerate approximation systems to which we can apply the Faedo-Galerkin scheme. To prove convergence to weak solutions of the approximate solutions we utilize monotonicity and compactness methods. Additionally, we analyze for the first time the fully nonlinear and degenerate reaction-diffusion system (1.4).

As already alluded to, we prove existence of weak solutions for the bidomain system (1.1) and the nonlinear system (1.4) using specific nondegenerate approximation systems. The approximation systems read

$$\begin{aligned} &c_m\partial_t v + \varepsilon\partial_t u_i - \operatorname{div} M_i(t,x,\nabla u_i) + h(t,x,v) = I_{\text{app}}, \qquad (t,x)\in Q_T,\\ &c_m\partial_t v - \varepsilon\partial_t u_e + \operatorname{div} M_e(t,x,\nabla u_e) + h(t,x,v) = I_{\text{app}}, \qquad (t,x)\in Q_T, \end{aligned} \tag{1.6}$$

where $\varepsilon > 0$ is a small number. Notationally, we have let (1.6) cover both the bidomain case $p = 2$ and the nonlinear case $p > 2$. We supplement (1.6) with Dirichlet boundary conditions (1.2) and initial data

$$u_j(0,x) = u_{j,0}(x), \quad x \in \Omega, \quad j = i, e. \tag{1.7}$$

Since we use the non-degenerate problem (1.6) to produce approximate solutions, it becomes necessary to decompose the initial condition v_0 in (1.3) as $v_0 = u_{i,0} - u_{e,0}$ for some functions $u_{i,0}, u_{e,0}$, see Sections 6 and 7 for details. We prove existence of solutions to (1.6) (for each fixed $\varepsilon > 0$) by applying the Faedo-Galerkin method, deriving a priori estimates, and then passing to the limit in the approximate solutions using monotonicity and compactness arguments. Having proved existence for the nondegenerate systems, the goal is to send the regularization parameter ε to zero in sequences of such solutions to fabricate weak solutions of the original systems (1.1), (1.4). Again convergence is achieved by priori estimates and monotonicity/compactness arguments. On the technical side, we point out that in the nonlinear case ($p > 2$) we must prove strong convergence of the gradients of the approximate solutions to ensure that the limit functions in fact solve the orginal system (1.4), whereas in the "linear" bidomain model (1.1) we can achieve this with just weakly converging gradients.

Finally, let us mention that it is possible to analyze systems like the bidomain model by means of different methods than the ones utillized in [10] or in this paper, see for example [6, 12] and also the discussion in [10].

The plan of the paper is as follows: In Section 2 we recall briefly the derivation of the bidomain model. In Section 3 we introduce some notations/functional spaces and recall a few basic mathematical facts needed later on for the analysis. Section 4 is devoted to stating the definitions of weak solutions as well as the main results. In Section 5 we prove existence of solutions for the nondegenerate systems. The main results stated in Section 4 are proved in Section 6 for the bidomain system (1.1) and in Section 7 for the nonlinear system (1.4). We conclude the paper in Section 8 by proving uniqueness of weak solutions.

2. The bidomain model

We devote this section to a brief derivation of the bidomain model of cardiac tissue. As principal references on this model we use [14, 16, 25, 29].

The cardiac tissue (represented by the domain $\Omega \subset \mathbb{R}^3$) is conceived as the coupling of two anisotropic continuous superimposed media, one intracellular and the other extracellular, which are separated by the cell membrane. The electrical potentials in these media are denoted by u_i, the *intracellular potential*, and u_e, the *extracellular potential*. Inside each medium the current flows J_j are assumed to obey (the local form of) Ohm's law:

$$J_j = -\mathbf{M}_j \nabla u_j, \quad j = i, e, \tag{2.1}$$

where the matrices $\mathbf{M}_j = \mathbf{M}_j(x)$, $j = i, e$, represent the conductivities in the intra- and extracellular media. These media have preferred directions of conductivity, which is because the cardiac cells are long and thin with a specific direction of alignment. The conductivity matrices are of the form

$$\mathbf{M}_j = \sigma_t^j I + \left(\sigma_l^j - \sigma_t^j\right) a(x)a(x)^\top, \qquad j = i, e, \tag{2.2}$$

where I denotes the identity matrix, σ_l^j and σ_t^j, $j = i, e$, are the conductivity coefficients respectively along and across the cardiac fibers for the intracellular ($j = i$), extracellular ($j = e$) media, which are assumed to be the positive constants, while $a = a(x)$ is the unit vector tangent to the fibers at a point x. The conductivity is assumed to be greater along than across the fibers, that is, $\sigma_l^j > \sigma_t^j$, $j = i, e$.

The matrices $\mathbf{M}_j$, $j = i, e$, are symmetric and positive definite, and possess two different positive eigenvalues $\sigma_{l,t}^j$. The multiplicity of σ_l^j is 1, while it is 2 for $\sigma_t^{i,e}$. The conductivity of the composite medium is characterized by $\mathbf{M} := \mathbf{M}_i + \mathbf{M}_e$.

By the law of current conservation we have

$$\nabla \cdot J_i + \nabla \cdot J_e = 0. \tag{2.3}$$

The divergence currents in (2.3) go between the intra- and extracellular media, and are thus crossing the membrane. Hence they must be related to the transmembrane current per unit volume, which we denote by I_m, and to the applied stimulation current I_{app}. The transmembrane current I_m is most easily expressed in terms of current per unit area of membrane surface. The transmembrane current per unit volume is then obtained by multiplying I_m with a scaling factor χ, which is the membrane surface area per unit volume tissue. Since the currents fields can be considered quasi-static, we thus obtain from (2.3)

$$\nabla \cdot J_i = -\chi I_m + I_{\text{app}}, \qquad \nabla \cdot J_e = \chi I_m - I_{\text{app}}. \tag{2.4}$$

As a primary unknown we introduce the transmembrane potential v, which is defined as the difference between the intra- and extracellular potentials: $v = u_i - u_e$. Now the next step is express the membrane current I_m in terms of the unknown v. To this end, we need a model describing the electrical properties of the cell membrane. The model that we adopt here resides in representing the membrane by a capacitor and passive resistor in parallel. We recall that a capacitor is defined by

$$q = c_m v, \tag{2.5}$$

where q and c_m denote respectively the amount of charge and the capacitance. The capacitive current, denoted by I_c, is the amount of charge that flows per unit time, so by taking derivatives in (2.5) we bring about

$$I_c = \partial_t q = c_m \partial_t v. \tag{2.6}$$

The transmembrane current I_m is the sum of the capacitive current and the transmembrane ionic current, i.e., $I_m = I_c + I_{ion}$, where the ionic current I_{ion} is assumed (for simplicity) to depend only the transmembrane potential v. Exploiting (2.6) we can express the membrane current I_m as

$$I_m = c_m \partial_t v + I_{ion}(v). \tag{2.7}$$

We mention that in [10] (see also [27]) the authors employed the FitzHugh-Nagumo model for the ionic current. The FitzHugh-Nagumo membrane kinetics was introduced first as a simplified version of the membrane model of Hodgkin and Huxley describing the transmission of nervous electric impulses. The ionic current in this model is represented as (see for example [21])

$$I_{ion} = I_{ion}(v, w) = F(v) + \delta w, \tag{2.8}$$

where and $F : \mathbb{R} \to \mathbb{R}$ is a cubic polynomial, $\delta > 0$ is a constant, and w is the recovery variable. The recovery variable satisfies a single ODE that depends on v.

In this work we assume there is no recovery variable w and the scaling factor χ is set to 1, so that the ionic current can be represented as

$$I_{ion} = I_{ion}(v) = h(v), \tag{2.9}$$

for some given function h that depends only on the transmembrane potential v. The cell model (I_{ion}) that we employ herein is simple. Many more advanced models exist, see, e.g., [2, 15, 20, 22, 31]. We refer also to [25] for an overview of many relevant cell models, which consist of systems of ODEs that are coupled to the partial differential equations for the electrical current flow.

Finally, combining (2.9), (2.7), and (2.4) we obtain the bidomain system (1.1).

Remark 2.1. *We refer to Subsection 3.2 for precise conditions on the function h in* (2.9)*. Here it suffices to say that a representative example of h is the cubic polynomial*

$$h(v) = \chi\, G\, v \left(1 - \frac{v}{v_{th}}\right)\left(1 - \frac{v}{v_p}\right),$$

where we assign the following meanings to the constants χ, G, v_{th}, v_p: χ is the ratio of the membrane area per unit tissue, G is the maximum membrane conductance per unit area, and v_{th}, v_p are respectively the threshold and plateau values of v.

Remark 2.2. *The conductivity tensors $\mathbf{M}_j$, $j = i, e$, do not typically depend on time t in the bidomain application, but we have included this dependency in* (1.1) *for the sake of generality. The same applies to the (t, x) dependency in h, see* (1.1)*.*

Remark 2.3. *Although we do not claim any relevance of the nonlinear system* (1.4) *when it comes to representing the electrical properties of cardiac tissue, it can be illuminating to observe that* (1.4) *can be derived as above by assuming simply that the flows J_j are nonlinear functions of the potentials u_j (instead of* (2.1)*):*

$$J_j = J_j(t, x, \nabla u_j),$$

which would correspond to a nonlinear Ohm's law.

3. Preliminaries

3.1. **Mathematical preliminaries.** The purpose of this subsection is to introduce some notations as well as recalling a few well-known and basic mathematical results. As general books of reference, see [13, 24].

Let Ω be a bounded open subset of $\mathbb{R}^3$ with a smooth (say C^2) boundary $\partial\Omega$. For $1 \le q < \infty$, we denote by $W^{1,q}(\Omega)$ the Sobolev space of functions $u : \Omega \to \mathbb{R}$ for which $u \in L^q(\Omega)$ and $\nabla u \in L^q(\Omega; \mathbb{R}^3)$. We let $W_0^{1,q}(\Omega)$ denote the functions in $W^{1,q}(\Omega)$ that vanish on the boundary. For $q = 2$ we write $H_0^1(\Omega)$ instead of $W_0^{1,2}(\Omega)$. If $1 \le q < \infty$ and X is a Banach space, then $L^q(0, T; X)$ denotes the space of measurable function $u : (0, T) \to X$ for which $t \mapsto \|u(t)\|_X \in L^q(0, T)$.

Moreover, $C([0,T];X)$ denotes the space of continuous functions $u:[0,T]\to X$ for which $\|u\|_{C([0,T];X)} := \max_{t\in(0,T)} \|u(t)\|_X$ is finite.

For $1 \le q < \infty$, we denote by q' the conjugate exponent of q: $q' = \frac{q}{q-1}$. We will use Young's inequality (with ε) frequently:

$$ab \le \varepsilon a^q + C(\varepsilon) b^{q'}, \qquad C(\varepsilon) = \frac{1}{q'(\varepsilon q)^{q'/q}}, \qquad a,b,\varepsilon > 0.$$

For $u \in W_0^{1,q}(\Omega)$ with $q \in [1,\infty)$, the Poincaré inequality reads

$$\|u\|_{L^q(\Omega)} \le \begin{cases} C\|\nabla u\|_{L^q(\Omega)}, & 1 < q < \infty, \\ C\|\nabla u\|_{L^3(\Omega)}, & q = 1, \end{cases}$$

for some constant C (independent of the particular u).

Let H be a Hilbert space equipped with a scalar product $(\cdot,\cdot)_H$. Let X be a Banach space such that $X \hookrightarrow H \backsimeq H^1 \hookrightarrow X'$ and X is dense in H (X' denotes the dual of X, etc.). Suppose $u \in L^p(0,T;X)$ is such that $\partial_t u$ belongs to $L^{p'}(0,T;X')$ for some $p \in (1,\infty)$. Then $u \in C([0,T];H)$. Moreover, for every pair (u,v) of such functions we have the integration-by-parts formula

$$\begin{aligned} &(u(t),v(t))_H - (u(s),v(s))_H \\ &\qquad = \int_s^t \langle \partial_t u(\tau), v(\tau)\rangle_{X',X}\, d\tau + \int_s^t \langle \partial_t v(\tau), u(\tau)\rangle_{X',X}\, d\tau, \end{aligned}$$

for all $s,t \in [0,T]$. Specifically when $u = v$ there holds

$$\|u(t)\|_H^2 - \|u(s)\|_H^2 = 2\int_s^t \langle \partial_t u(\tau), u(\tau)\rangle_{X',X}\, d\tau.$$

We will make use of the last two results with $X = L^p(\Omega)$ ($p \ge 2$) and $H = L^2(\Omega)$.

Next we recall the Aubin-Lions compactness result (see, e.g., [19]). Let X be a Banach space, and let X_0, X_1 be separable and reflexive Banach spaces. Suppose $X_0 \hookrightarrow X \hookrightarrow X_1$, with a compact imbedding of X_0 into X. Let $\{u_n\}_{n\ge1}$ be a sequence that is bounded in $L^\alpha(0,T;X_0)$ and for which $\{\partial_t u_n\}_{n\ge1}$ is bounded in $L^\beta(0,T;X_1)$, with $1 < \alpha,\beta < \infty$. Then $\{u_n\}_{n\ge1}$ is precompact in $L^\alpha(0,T;X)$.

Let us also recall the following well-known compactness result (see, e.g., [28]): Let $X \hookrightarrow Y \hookrightarrow Z$ be Banach spaces, with a compact imbedding of X into Y. Let $\{u_n\}_{n\ge1}$ be a sequence that is bounded in $L^\infty(0,T;X)$ and equicontinuous as Z-valued distributions. Then the sequence $\{u_n\}_{n\ge1}$ is precompact in $C([0,T];Y)$.

3.2. **Assumptions.** In this subsection we intend to provide precise conditions on the "data" of our problems. First of all, we assume that the physical domain Ω is a bounded open subset of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Recall that the bidomain system (1.1) results if specify $M_j(t,x,\xi) = \mathbf{M}_j(t,x)\xi$ in the nonlinear system (1.4). Therefore the conditions stated next for the vector fields $M_j(t,x,\xi)$ cover also the bidomain system. On the other hand, the conditions that we impose on the function h for the nonlinear system do not reduce (by setting $p = 2$) to those that we impose on h for the bidomain system, so we operate with two sets of conditions for h.

3.2.1. *Conditions on the diffusive vector fields $M_j(t,x,\xi)$.* Let $2 \le p < +\infty$. We assume $M_j = M_j(t,x,\xi) : Q_T \times \mathbb{R}^3 \to \mathbb{R}^3$, $j = i, e$, are functions that are measurable in $(t,x) \in Q_T$ for each $\xi \in \mathbb{R}^3$ and continuous in $\xi \in \mathbb{R}^3$ for a.e. $(t,x) \in Q_T$, i.e., M_i, M_e are vector-valued Carathéodory functions.

Our basic requirements are

$$|M_j(t,x,\xi)| \le C_M \left(|\xi|^{p-1} + f_1(t,x)\right), \quad j = i, e, \tag{3.1}$$

$$(M_j(t,x,\xi) - M_j(t,x,\xi')) \cdot (\xi - \xi') \ge C_M |\xi - \xi'|^p, \quad j = i, e, \tag{3.2}$$

$$M_j(t,x,\xi) \cdot \xi \ge C_M |\xi|^p, \quad j = i, e, \tag{3.3}$$

for a.e. $(t,x) \in Q_T$, $\forall \xi, \xi' \in \mathbb{R}^3$, and with C_M being a positive constant and f_1 belonging to $L^{p'}(Q_T)$. Moreover, we assume there exist Carathéodory functions $\mathcal{M}_j(t,x,\xi) : Q_T \times \mathbb{R}^3 \to \mathbb{R}$, $j = i, e$, such that for a.e. $(t,x) \in Q_T$ and $\forall \xi \in \mathbb{R}^3$

$$\frac{\partial}{\partial \xi_l} \mathcal{M}_j(t,x,\xi) = M_{j,l}(t,x,\xi), \quad l = 1, 2, 3, \tag{3.4}$$

$$|\partial_t \mathcal{M}_j(t,x,\xi)| \le K_1 \mathcal{M}_j(t,x,\xi) + f_2, \tag{3.5}$$

for some constant K_1 and function $f_2 \in L^1(Q_T)$.

Remark 3.1. *Typical examples of vector fields M_j that satisfy conditions* (3.1)-(3.3) *are the p-Laplace type operators in* (1.5)*. Concerning* (1.5)*, the vector fields $\mathcal{M}_j(t,x,\xi)$ satisfying* (3.4) *are given by $\frac{1}{p} |\xi|^p \mathbf{M}_j(t,x)$, and they satisfy* (3.5) *trivially if the matrices $\mathbf{M}_j$ are independent of time t (the representative case).*

Remark 3.2. *Referring to the bidomain model and the above discussion we perceive that conditions* (3.1)-(3.3) *are satisfied with $M_j = \mathbf{M}_j(t,x)\xi$, $p = 2$ provided*

$$\mathbf{M}_j \in L^\infty(Q_T; \mathbb{R}^{N \times N}), \quad j = i, e,$$

$$\mathbf{M}_j(t,x)\xi \cdot \xi \ge C'_M |\xi|^2, \quad \textit{for a.e. } (t,x) \in Q_T \textit{ and } \forall \xi \in \mathbb{R}^3, \quad j = i, e.$$

3.2.2. *Conditions on the "ionic current" $h(t,x,v)$.* We assume $h : Q_T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. For $p \ge 2$, we assume there exist constants $C_h, K_2 > 0$ such that

$$h(t,x,0) = 0, \qquad \frac{h(t,x,v_1) - h(t,x,v_2)}{v_1 - v_2} \ge -C_h, \quad \forall v_1 \ne v_2, \tag{3.6}$$

$$|\partial_t H(t,x,v)| \le K_2 H(t,x,v) + f_3, \qquad H(t,x,v) = \int_0^v h(t,x,\rho)\, d\rho, \tag{3.7}$$

for a.e. $(t,x) \in Q_T$ and for some function $f_3 \in L^1(Q_T)$.

For $p = 2$, we assume additionally

$$0 < \liminf_{|v| \to \infty} \frac{h(\cdot,\cdot,v)}{v^3} \le \limsup_{|v| \to \infty} \frac{h(\cdot,\cdot,v)}{v^3} < \infty, \tag{3.8}$$

while for $p > 2$ we assume

$$0 < \liminf_{|v| \to \infty} \frac{h(\cdot,\cdot,v)}{|v|^{p-2} v} \le \limsup_{|v| \to \infty} \frac{h(\cdot,\cdot,v)}{|v|^{p-2} v} < \infty. \tag{3.9}$$

Remark 3.3. *One should be aware that condition* (3.7) *is trivially satisfied when h is independent of time t, which is the representative case for the bidomain model.*

Remark 3.4. *A consequence of* (3.6) *and* (3.8) *is that for a.e.* $(t,x) \in Q_T$ *and* $\forall v \in \mathbb{R}$ *there holds*

$$|h(t,x,v)| \le C\left(|v|^3 + 1\right) \qquad (p=2), \tag{3.10}$$

while a consequence of (3.6) *and* (3.9) *is that for a.e.* $(t,x) \in Q_T$ *and* $\forall v \in \mathbb{R}$

$$C'\,|v|^{p-1} \le |h(t,x,v)| \le C''\left(|v|^{p-1} + 1\right) \qquad (p>2), \tag{3.11}$$

for some constants $C, C', C'' > 0$.

Remark 3.5. *A fact that will be used several times in this paper is*

$$\left(h(t,x,v_1) - h(t,x,v_2)\right)(v_1 - v_2) + C_h\,(v_1 - v_2)^2 \ge 0, \tag{3.12}$$

$\forall v_1, v_2 \in \mathbb{R}$ *and for a.e.* $(t,x) \in Q_T$. *This inequality is an outcome of* (3.6).

Remark 3.6. *A typical example of a nonlinearity* h *in* (1.4) *that satisfies* (3.6) *(trivially) and* (3.9) *for* $p > 2$ *is* $h(t,x,v) = |v|^{p-2}\,v$.

Remark 3.7. *In the fully nonlinear case* ($p > 2$)*, condition* (3.9) *is used to prove strong* L^p *convergence of the gradients of the approximate solutions, which is needed in the existence proof, see in particular Section 7.*

3.3. **A basis for the Faedo-Galerkin method.** Later on we use the Faedo-Galerkin method to prove existence of solutions. For that purpose we need a basis. The material presented in this subsection is standard, and we have included it just for the sake of completeness.

Let $q > 0$ be such that $q < p^* = \frac{3p}{3-p}$ and $s \in \mathbb{N}$ satisfy $s > \frac{5}{2}$.Then

$$W_0^{s,2}(\Omega) \subset W_0^{1,p}(\Omega) \subset L^q(\Omega) \subset (W_0^{s,2}(\Omega))',$$

with continuous and dense inclusions. We denote by $W_0^{s,2}(\Omega)$ the higher order Sobolev space $\left\{u, D^\alpha u \in L^2(\Omega), |\alpha| \le s, u = 0 \text{ on } \partial\Omega\right\}$. In particular, the inclusion $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ is compact. The Aubin-Lions compactness criterion says that

$$\text{the inclusion } W \subset L^p(0,T;L^q(\Omega)) \text{ is compact,}$$

where $W = \left\{u \in L^p(0,T;W_0^{1,p}(\Omega)) : \partial_t u \in L^{p'}\left(0,T;(W_0^{s,2}(\Omega))'\right)\right\}$.

Consider the following spectral problem: Find $w \in W_0^{s,2}(\Omega)$ and a number λ such that

$$\begin{cases} (w,\phi)_{W_0^{s,2}(\Omega)} = \lambda(w,\phi)_{L^2(\Omega)}, & \forall \phi \in W_0^{s,2}(\Omega), \\ w = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.13}$$

where $(\cdot,\cdot)_{W_0^{s,2}(\Omega)}$ and $(\cdot,\cdot)_{L^2(\Omega)}$ denote the inner products of $W_0^{s,2}(\Omega)$ and $L^2(\Omega)$ respectively. By the Riesz representation theorem there is a unique Θe such that

$$\Phi(e) := (e,\phi)_{L^2(\Omega)} = (\Theta e,\phi)_{W_0^{s,2}(\Omega)}, \qquad \forall \phi \in W_0^{s,2}(\Omega).$$

Clearly, the operator $L^2(\Omega) \ni e \mapsto \Theta e \in L^2(\Omega)$ is linear, symmetric, bounded, and compact. Moreover, Θ is positive since

$$(e,\Theta e)_{L^2(\Omega)} = (\Theta e,\Theta e)_{W_0^{s,2}(\Omega)} \ge 0,$$

Hence, problem (3.13) posseses a sequence of positive eigenvalues $\{\lambda_l\}_{l=1}^\infty$ and the corresponding eigenfunctions $\{e_l\}_{l=1}^\infty$ form a sequence that is orthogonal in $W_0^{s,2}(\Omega)$ and orthonormal in $L^2(\Omega)$, see, e.g., [24, p.267].

4. Statement of main results

In this section we define what we mean by weak solutions of the bidomain system (1.1) and the nonlinear system (1.4), starting with the former model. We also supply our main existence results.

Definition 4.1 (Bidomain model). *A weak solution of* (1.1)*,* (1.2)*,* (1.3) *is a triple of functions* $u_i, u_e, v \in L^2(0,T;H_0^1(\Omega))$ *with* $v = u_i - u_e$ *such that* $\partial_t v$ *belongs to* $L^2\left(0,T,\left(H_0^1(\Omega)\right)'\right)$, $v(0) = v_0$ *a.e. in* Ω*, and*

$$\begin{aligned}\int_0^T c_m \langle \partial_t v, \varphi_i \rangle \, dt + \iint_{Q_T} \mathbf{M}_i(t,x)\nabla u_i \cdot \nabla \varphi_i \, dx \, dt \\ + \iint_{Q_T} h(t,x,v)\varphi_i \, dx \, dt = \iint_{Q_T} I_{\text{app}} \varphi_i \, dx \, dt,\end{aligned} \tag{4.1}$$

$$\begin{aligned}\int_0^T c_m \langle \partial_t v, \varphi_e \rangle \, dt - \iint_{Q_T} \mathbf{M}_e(t,x)\nabla u_e \cdot \nabla \varphi_e \, dx \, dt \\ + \iint_{Q_T} h(t,x,v)\varphi_e \, dx \, dt = \iint_{Q_T} I_{\text{app}} \varphi_e \, dx \, dt,\end{aligned} \tag{4.2}$$

for all $\varphi_j \in L^2(0,T;H_0^1(\Omega))$*,* $j = i, e$*. Here,* $\langle \cdot, \cdot \rangle$ *denotes the duality pairing between* $H_0^1(\Omega)$ *and* $(H_0^1(\Omega))'$*.*

Remark 4.1. *In view of* (3.8) *and Sobolev's imbedding theorem (the latter tells us that* $H_0^1(\Omega) \subset L^6(\Omega)$*), we conclude* $h(t,x,v) \in L^2(Q_T)$ *and thus* $\iint_{Q_T} h(t,x,v)\varphi_j \, dx \, dt$*,* $j = i, e$*, are well-defined integrals. Moreover, consult Subsection 3.1, it follows from Definition 4.1 that* $v \in C([0,T];L^2(\Omega))$*, and thus the initial condition* (1.3) *is valid.*

Theorem 4.1 (Bidomain model, $p = 2$). *Assume conditions* (3.1)*-*(3.8) *hold. If* $v_0 \in L^2(\Omega)$ *and* $I_{\text{app}} \in L^2(Q_T)$*, then the bidomain problem* (1.1)*,* (1.2)*,* (1.3) *possesses a unique weak solution. If* $v_0 = u_{i,0} - u_{e,0}$ *with* $u_{i,0}, u_{i,0} \in H_0^1(\Omega)$ *and* $I_{\text{app}} \in L^2(Q_T)$*, then this weak solution obeys* $\partial_t v \in L^2(Q_T)$*.*

Definition 4.2 (Nonlinear model, $p > 2$). *A weak solution of* (1.4)*,* (1.2)*,* (1.3) *is a triple of functions* $u_i, u_e, v \in L^p(0,T;W_0^{1,p}(\Omega))$ *with* $v = u_i - u_e$ *such that* $\partial_t v \in L^{p'}\left(0,T;(W_0^{1,p}(\Omega))'\right)$, $v(0) = v_0$ *a.e. in* Ω*, and*

$$\begin{aligned}\int_0^T c_m \langle \partial_t v, \varphi_i \rangle \, dt + \iint_{Q_T} M_i(t,x,\nabla u_i) \cdot \nabla \varphi_i \, dx \, dt \\ + \iint_{Q_T} h(t,x,v)\varphi_i \, dx \, dt = \iint_{Q_T} I_{\text{app}} \varphi_i \, dx \, dt,\end{aligned} \tag{4.3}$$

$$\begin{aligned}\int_0^T c_m \langle \partial_t v, \varphi_e \rangle \, dt - \iint_{Q_T} M_e(t,x,\nabla u_e) \cdot \nabla \varphi_e \, dx \, dt \\ + \iint_{Q_T} h(t,x,v)\varphi_e \, dx \, dt = \iint_{Q_T} I_{\text{app}} \varphi_e \, dx \, dt,\end{aligned} \tag{4.4}$$

for all $\varphi_j \in L^p(0,T;W_0^{1,p}(\Omega))$*,* $j = i, e$*. Here,* $\langle \cdot, \cdot \rangle$ *denotes the duality pairing between* $W_0^{1,p}(\Omega)$ *and* $(W_0^{1,p}(\Omega))'$*.*

Remark 4.2. *Due to condition* (3.9) *the function* $h(t,x,v)$ *belongs to* $L^{p'}(Q_T)$, *and thus the integrals* $\iint_{Q_T} h(t,x,v)\varphi_j \,dx\,dt$, $j=i,e$, *are well-defined. Moreover, by Definition 4.2, there holds* $v \in C([0,T];L^2(\Omega))$. *Consequently,* (1.3) *has a meaning.*

Theorem 4.2 (Nonlinear model, $p>2$). *Assume conditions* (3.1)-(3.7) *and* (3.9) *hold. If* $v_0 \in L^2(\Omega)$ *and* $I_{\text{app}} \in L^2(Q_T)$, *then the nonlinear problem* (1.4), (1.2), (1.3) *possseses a unique weak solution. If* $v_0 = u_{i,0} - u_{e,0}$ *with* $u_{i,0}, u_{e,0} \in W_0^{1,p}(\Omega)$ *and* $I_{\text{app}} \in L^2(Q_T)$, *then this weak solution obeys* $\partial_t v \in L^2(Q_T)$.

Now we are ready to embark on the proofs of Theorem 4.1 and 4.2.

5. Existence of solutions for the approximate problems

This section is devoted to proving existence of solutions to the approximate problems (1.6), (1.2), (1.7) introduced and discussed in the introduction. The existence proof is based on the Faedo-Galerkin method, a priori estimates, and the compactness method.

Definition 5.1 (Approximate problems). *A solution of problem* (1.6), (1.2), (1.7) *is a triple of functions* $u_i, u_e, v \in L^p(0,T;W_0^{1,p}(\Omega))$ *with* $v = u_i - u_e$ *such that* $\partial_t u_j \in L^2(Q_T)$, $u_j(0) = u_{j,0}$ *a.e. in* Ω, *for* $j=i,e$, *and*

$$\begin{aligned}\iint_{Q_T} c_m \partial_t v \varphi_i \,dx\,dt + \iint_{Q_T} \varepsilon \partial_t u_i \varphi_i \,dx\,dt + \iint_{Q_T} M_i(t,x,\nabla u_i)\cdot\nabla\varphi_i \,dx\,dt \\ + \iint_{Q_T} h(t,x,v)\varphi_i \,dx\,dt = \iint_{Q_T} I_{\text{app}}\varphi_i \,dx\,dt,\end{aligned} \tag{5.1}$$

$$\begin{aligned}\iint_{Q_T} c_m \partial_t v \varphi_e \,dx\,dt - \iint_{Q_T} \varepsilon \partial_t u_e \varphi_e \,dx\,dt - \iint_{Q_T} M_e(t,x,\nabla u_e)\cdot\nabla\varphi_e \,dx\,dt \\ + \iint_{Q_T} h(t,x,v)\varphi_e \,dx\,dt = \iint_{Q_T} I_{\text{app}}\varphi_e \,dx\,dt,\end{aligned} \tag{5.2}$$

for all $\varphi_j \in L^p(0,T;W_0^{1,p}(\Omega))$, $j=i,e$.

Remark 5.1. *"Cosmetically speaking", we have chosen to let Definition 5.1 cover both the bidomain case* $p=2$ *and the nonlinear case* $p>2$. *Although in this section we keep the same notation for the two cases, we will at various places in the presentation that follows employ different proofs.*

Supplied with the basis $\{e_l\}_{l=1}^{+\infty}$ introduced in Subsection 3.3, we look for finite dimensional approximate solutions to the regularized problem (1.6), (1.2), (1.7) as sequences $\{u_{i,n}\}_{n>1}$, $\{u_{e,n}\}_{n>1}$, $\{v_n\}_{n>1}$ defined for $t \geq 0$ and $x \in \overline{\Omega}$ by

$$u_{i,n}(t,x) = \sum_{l=1}^{n} c_{i,n,l}(t) e_l(x), \qquad u_{e,n}(t,x) = \sum_{l=1}^{n} c_{e,n,l}(t) e_l(x), \tag{5.3}$$

and

$$v_n(t,x) = \sum_{l=1}^{n} d_{n,l}(t) e_l(x), \qquad d_{n,l}(t) := c_{i,n,l}(t) - c_{e,n,l}(t). \tag{5.4}$$

The goal is to determine the coefficients $\{c_{i,n,l}(t)\}_{l=1}^n$, $\{c_{e,n,l}(t)\}_{l=1}^n$, $\{d_{n,l}(t)\}_{l=1}^n$ such that for $k = 1, \dots, n$

$$
\begin{aligned}
&(c_m \partial_t v_n, e_k)_{L^2(\Omega)} + (\varepsilon \partial_t u_{i,n}, e_k)_{L^2(\Omega)} \\
&\qquad + \int_\Omega M_i(t,x,\nabla u_{i,n}) \cdot \nabla e_k \, dx + \int_\Omega h(t,x,v) e_k \, dx = \int_\Omega I_{\mathrm{app},n} e_k \, dx, \\
&(c_m \partial_t v_n, e_k)_{L^2(\Omega)} - (\varepsilon \partial_t u_{e,n}, e_k)_{L^2(\Omega)} \\
&\qquad - \int_\Omega M_e(t,x,\nabla u_{e,n}) \cdot \nabla e_k \, dx + \int_\Omega h(t,x,v_n) e_k \, dx = \int_\Omega I_{\mathrm{app},n} e_k \, dx,
\end{aligned}
\tag{5.5}
$$

and, with reference to the initial conditions (1.7),

$$
\begin{aligned}
u_{i,n}(0,x) &= u_{0,i,n}(x) := \sum_{l=1}^n c_{i,n,l}(0) e_l(x), \qquad c_{i,n,l}(0) := (u_{i,0}, e_l)_{L^2(\Omega)}, \\
u_{e,n}(0,x) &= u_{0,e,n}(x) := \sum_{l=1}^n c_{e,n,l}(0) e_l(x), \qquad c_{e,n,l}(0) := (u_{e,0}, e_l)_{L^2(\Omega)}, \\
v_n(0,x) &= v_{0,n}(x) := \sum_{l=1}^n d_{n,l}(0) e_l(x), \qquad d_{n,l}(0) := c_{i,n,l}(0) - c_{e,n,l}(0),
\end{aligned}
\tag{5.6}
$$

ln (5.5), we have used a finite dimensional approximation of I_{app}:

$$
I_{\mathrm{app},n}(t,x) = \sum_{l=1}^n (I_{\mathrm{app}}, e_l)_{L^2(\Omega)} (t) e_l(x).
$$

By our choice of basis, $u_{i,n}$ and $u_{e,n}$ satisfy the Dirichlet boundary condition (1.2). With $I_{\mathrm{app}} \in L^2(Q_T)$ and $u_{0,j} \in W_0^{1,p}(\Omega)$, it is clear that, as $n \to \infty$, $I_{\mathrm{app},n} \to I_{\mathrm{app}}$ in $L^2(Q_T)$ and $u_{0,j,n} \to u_{0,j}$ in $W_0^{1,p}(\Omega)$, for $j = i, e$.

Using the orthonormality of the basis, we can write (5.5) more explicitly as a system of ordinary differential equations:

$$
\begin{aligned}
c_m d'_{n,k}(t) + \varepsilon c'_{i,n,k}(t) &+ \int_\Omega M_i(t,x,\nabla u_{i,n}) \cdot \nabla e_k \, dx \\
&+ \int_\Omega h(t,x,v_n) e_k \, dx = \int_\Omega I_{\mathrm{app},n} e_k \, dx, \\
c_m d'_{n,k}(t) - \varepsilon c'_{e,n,k}(t) &- \int_\Omega M_e(t,x,\nabla u_{e,n}) \cdot \nabla e_k \, dx \\
&+ \int_\Omega h(t,x,v_n) e_k \, dx = \int_\Omega I_{\mathrm{app},n} e_k \, dx.
\end{aligned}
\tag{5.7}
$$

Adding together the two equations in (5.7) yields for $k = 1, \dots, n$

$$
\begin{aligned}
(2c_m + \varepsilon)\, d'_{n,k}(t) &= \int_\Omega \left(M_e(t,x,\nabla u_{e,n}) - M_i(t,x,\nabla u_{i,n}) \right) \cdot \nabla e_k \, dx \\
&\qquad - 2\int_\Omega h(t,x,v_n) e_k \, dx + 2\int_\Omega I_{\mathrm{app},n} e_k \, dx \\
&=: F^k \left(t, \{d_{n,l}\}_{l=1}^n, \{c_{i,n,l}\}_{l=1}^n, \{c_{e,n,l}\}_{l=1}^n \right).
\end{aligned}
\tag{5.8}
$$

Plugging the equation (5.8) for $d'_{n,k}(t)$ back into (5.7), we find for $k = 1, \dots, n$

$$
\begin{aligned}
\varepsilon c'_{i,n,k}(t) &= -\frac{c_m}{2c_m+\varepsilon} F^k\left(t, \{d_{n,l}\}_{l=1}^n, \{c_{i,n,l}\}_{l=1}^n, \{c_{e,n,l}\}_{l=1}^n\right) \\
&\quad - \int_\Omega M_i(t,x,\nabla u_{i,n}) \cdot \nabla e_k \, dx - \int_\Omega h(t,x,v_n) e_k \, dx + \int_\Omega I_{\text{app},n} e_k \, dx \\
&=: F_i^k\left(t, \{d_{n,l}\}_{l=1}^n, \{c_{i,n,l}\}_{l=1}^n, \{c_{e,n,l}\}_{l=1}^n\right)
\end{aligned}
\tag{5.9}
$$

and

$$
\begin{aligned}
\varepsilon c'_{e,n,k}(t) &= \frac{c_m}{2c_m+\varepsilon} F^k\left(t, \{d_{n,l}\}_{l=1}^n, \{c_{i,n,l}\}_{l=1}^n, \{c_{e,n,l}\}_{l=1}^n\right) \\
&\quad - \int_\Omega M_e(t,x,\nabla u_{e,n}) \cdot \nabla e_l \, dx + \int_\Omega h(t,x,v_n) e_k \, dx - \int_\Omega I_{\text{app},n} e_k \, dx \\
&=: F_e^k\left(t, \{d_{n,l}\}_{l=1}^n, \{c_{i,n,l}\}_{l=1}^n, \{c_{e,n,l}\}_{l=1}^n\right).
\end{aligned}
\tag{5.10}
$$

The next step is to prove existence of a local solution to the ODE system (5.8), (5.9), (5.10), (5.6). To this end, let $\rho \in (0,T)$ and set $U = [0,\rho]$. We choose $r > 0$ so large that the ball $B_r \subset \mathbb{R}^{3n}$ contains the three vectors $\{d_{n,l}(0)\}_{l=1}^n$, $\{c_{i,n,l}(0)\}_{l=1}^n$, $\{c_{e,n,l}(0)\}_{l=1}^n$, and then we set $V := \overline{B_r}$. We also set $F = \{F^k\}_{k=1}^n$, $F_i = \{F_i^k\}_{k=1}^n$, and $F_e = \{F_e^k\}_{k=1}^n$. Thanks to our assumptions (3.1)-(3.9) the functions $F, F_j : U \times V \to \mathbb{R}^n$, $j = i, e$, are Carathéodory functions. Moreover, the components of F F_j can be estimated on $U \times V$ as follows:

$$
\begin{aligned}
&\left|F^k\left(t, \{d_{n,l}\}_{l=1}^n, \{c_{i,n,l}\}_{l=1}^n, \{c_{e,n,l}\}_{l=1}^n\right)\right| \\
&\le 2\left\|I_{\text{app},n}\right\|_{L^2(\Omega)} \left\|e_k\right\|_{L^2(\Omega)} \\
&\quad + \sum_{j=i,e} \left(\int_\Omega \left|M_j\left(t,x,\sum_{l=1}^n c_{j,n,l}\nabla e_l\right)\right|^{p'} dx\right)^{1/p'} \left(\int_\Omega |\nabla e_k|^p\right)^{1/p} \\
&\quad + 2\left(\int_\Omega \left|h\left(t,x,\sum_{l=1}^n d_{n,l} e_l\right)\right|^{p'}\right)^{1/p'} \left(\int_\Omega |e_k|^p\right)^{1/p}
\end{aligned}
\tag{5.11}
$$

and for $j = i, e$

(5.12)
$$
\begin{aligned}
&\left|F_j^k\left(t, \{d_{n,l}\}_{l=1}^n, \{c_{i,n,l}\}_{l=1}^n, \{c_{e,n,l}\}_{l=1}^n\right)\right| \\
&\le \frac{c_m}{2c_m+\varepsilon}\Bigg[2\left\|I_{\text{app},n}\right\|_{L^2(\Omega)}\left\|e_k\right\|_{L^2(\Omega)} \\
&\qquad + \sum_{j=i,e}\left(\int_\Omega\left|M_j\left(t,x,\sum_{l=1}^n c_{j,n,l}\nabla e_l\right)\right|^{p'} dx\right)^{1/p'}\left(\int_\Omega |\nabla e_k|^p\right)^{1/p} \\
&\qquad + 2\left(\int_\Omega\left|h\left(t,x,\sum_{l=1}^n d_{n,l}e_l\right)\right|^{p'}\right)^{1/p'}\left(\int_\Omega |e_k|^p\right)^{1/p}\Bigg] \\
&\quad + \left(\int_\Omega\left|M_j\left(t,x,\sum_{l=1}^n c_{j,n,l}\nabla e_l\right)\right|^{p'} dx\right)^{1/p'}\left(\int_\Omega |\nabla e_k|^p\right)^{1/p} \\
&\quad + \left(\int_\Omega\left|h\left(t,x,\sum_{l=1}^n d_{n,l}e_l\right)\right|^{p'}\right)^{1/p'}\left(\int_\Omega |e_k|^p\right)^{1/p} + \left\|I_{\text{app},n}\right\|_{L^2(\Omega)}\left\|e_k\right\|_{L^2(\Omega)}.
\end{aligned}
$$

In view of (3.1)-(3.9), we can uniformly (on $U \times V$) bound (5.11) and (5.12):

$$
\left|F^k\left(t, \{d_{n,l}\}_{l=1}^n, \{c_{i,n,l}\}_{l=1}^n, \{c_{e,n,l}\}_{l=1}^n\right)\right| \le C(r,n)M(t), \tag{5.13}
$$

$$
\left|F_j^k\left(t, \{d_{n,l}\}_{l=1}^n, \{c_{i,n,l}\}_{l=1}^n, \{c_{e,n,l}\}_{l=1}^n\right)\right| \le C_j(r,n)M_j(t), \quad j = i, e, \tag{5.14}
$$

where $C(r,n)$, $C_j(r,n)$ are constants that depend on r, n and $M(t), M_j(t)$ are $L^1(U)$ functions that are independent of k, n, r.

Hence, according to standard ODE theory, there exist absolutely continuous functions $\{d_{n,l}\}_{l=1}^n, \{c_{i,n,l}\}_{l=1}^n, \{c_{e,n,l}\}_{l=1}^n$ satisfying (5.8), (5.9), (5.10), (5.6) for a.e. $t \in [0, \rho')$ for some $\rho' > 0$. Moreover, the following equations hold on $[0, \rho')$:

(5.15)
$$
\begin{aligned}
d_{n,l}(t) &= d_{n,l}(0) \\
&\quad + \frac{1}{2c_m+\varepsilon}\int_0^t F^l\left(\tau, \{d_{n,k}(\tau)\}_{k=1}^n, \{c_{i,n,k}(\tau)\}_{k=1}^n, \{c_{e,n,k}(\tau)\}_{k=1}^n\right) d\tau
\end{aligned}
$$

and for $j = i, e$

$$
\begin{aligned}
c_{j,n,l}(t) &= c_{j,n,l}(0) \\
&\quad + \frac{1}{\varepsilon}\int_0^t F_j^l\left(\tau, \{d_{n,k}(\tau)\}_{k=1}^n, \{c_{i,n,k}(\tau)\}_{k=1}^n, \{c_{e,n,k}(\tau)\}_{k=1}^n\right) d\tau.
\end{aligned} \tag{5.16}
$$

To summarize our findings so far, on $[0, \rho')$ the functions $u_{i,n}, u_{e,n}, v_n$ defined by (5.3) and (5.4) are well-defined and constitute our approximate solutions to the regularized system (1.6) with data (1.2), (1.7).

To prove global existence of the Faedo-Galerkin solutions we derive n-independent a priori estimates bounding $v_n, u_{i,n}, u_{e,n}$ in various Banach spaces.

Given some (absolutely continuous) coefficients $b_{j,n,l}(t)$, $j = i, e$, we form the functions $\varphi_{i,n}(t,x) := \sum_{l=1}^n b_{i,n,l}(t)e_l(x)$ and $\varphi_{e,n}(t,x) := \sum_{l=1}^n b_{e,n,l}(t)e_l(x)$. It follows from (5.7) that the Faedo-Galerkin solutions satisfy the following weak formulations for each fixed t, which will be the starting point for deriving a series of a priori esitmates:

$$\begin{aligned}&\int_\Omega c_m \partial_t v_n \varphi_{i,n}\,dx + \int_\Omega \varepsilon \partial_t u_{i,n}\varphi_{i,n}\,dx\\ &\qquad + \int_\Omega M_i(t,x,\nabla u_{i,n})\cdot\nabla\varphi_{i,n}\,dx + \int_\Omega h(t,x,v_n)\varphi_{i,n}\,dx\\ &\qquad\quad = \int_\Omega I_{\mathrm{app},n}\varphi_{i,n}\,dx,\end{aligned} \tag{5.17}$$

$$\begin{aligned}&\int_\Omega c_m \partial_t v_n \varphi_{e,n}\,dx - \int_\Omega \varepsilon \partial_t u_{e,n}\varphi_{e,n}\,dx\\ &\qquad - \int_\Omega M_e(t,x,\nabla u_{e,n})\cdot\nabla\varphi_{e,n}\,dx + \int_\Omega h(t,x,v_n)\varphi_{e,n}\,dx\\ &\qquad\quad = \int_\Omega I_{\mathrm{app},n}\varphi_{e,n}\,dx.\end{aligned} \tag{5.18}$$

Remark 5.2. *From* (5.17) *until* (5.37)*, we will intentionally commit a notational crime by reserving the letter T for an arbitrary time in the existence interval* $[0,\rho')$ *for the Faedo-Galerkin solutions (and not the final time used elsewhere).*

Lemma 5.1. *Assume conditions* (3.1)-(3.9) *hold and $p \geq 2$. If $u_{i,0}, u_{e,0} \in L^2(\Omega)$ and $I_{\mathrm{app}} \in L^2(Q_T)$, then there exist constants c_1, c_2, c_3 not depending on n such that*

$$\|v_n\|_{L^\infty(0,T;L^2(\Omega))} + \sum_{j=i,e} \left\|\sqrt{\varepsilon} u_{j,n}\right\|_{L^\infty(0,T;L^2(\Omega))} \leq c_1, \tag{5.19}$$

$$\sum_{j=i,e} \|\nabla u_{j,n}\|_{L^p(Q_T)} \leq c_2, \tag{5.20}$$

$$\sum_{j=i,e} \|u_{j,n}\|_{L^p(Q_T)} \leq c_3. \tag{5.21}$$

If, in addition, $u_{i,0}, u_{e,0} \in W_0^{1,p}(\Omega)$, then there exists a constant $c_4 > 0$ not depending on n such that

$$\|\partial_t v_n\|_{L^2(Q_T)} + \sum_{j=i,e} \left\|\sqrt{\varepsilon}\partial_t u_{j,n}\right\|_{L^2(Q_T)} \leq c_4. \tag{5.22}$$

Proof. Substituting $\varphi_{i,n} = u_{i,n}$ and $\varphi_{e,n} = -u_{e,n}$ in (5.17) and (5.18), respectively, and then summing the resullting equations, we procure the equation

$$\begin{aligned}&\frac{c_m}{2}\frac{d}{dt}\int_\Omega |v_n|^2\,dx + \frac{\varepsilon}{2}\sum_{j=i,e}\frac{d}{dt}\int_\Omega |u_{j,n}|^2\,dx\\ &\quad + \sum_{j=i,e}\int_\Omega M_j(t,x,\nabla u_{j,n})\cdot\nabla u_{j,n}\,dx + \int_\Omega h(t,x,v_n)v_n\,dx\\ &\quad = \int_\Omega I_{\mathrm{app},n} v_n\,dx.\end{aligned} \tag{5.23}$$

By Young's inequality, there exist constants $C_1, C_2 > 0$ independent of n such that

$$\iint_{Q_T} I_{\mathrm{app},n} v_n \, dx\, dt \le C_1 + C_2 \iint_{Q_T} |v_n|^2 \, dx\, dt. \tag{5.24}$$

Integrating (5.23) over $(0,T)$ and then exploiting (5.24) and also (3.3), (3.6), we obtain

$$\begin{aligned}
&\frac{c_m}{2}\int_\Omega |v_n(T,x)|^2 \, dx + \frac{\varepsilon}{2}\sum_{j=i,e}\int_\Omega |u_j(T,x)|^2 \, dx \\
&\quad + C_M \sum_{j=i,e}\iint_{Q_T} |\nabla u_{j,n}|^p \, dx\, dt + \iint_{Q_T}\Big(h(t,x,v_n)v_n + C_h |v_n|^2\Big)\, dx\, dt \\
&\le C_1 + (C_2 + C_h)\iint_{Q_T} |v_n|^2 \, dx\, dt \\
&\quad + \frac{c_m}{2}\int_\Omega |v_0(x)|^2 \, dx + \frac{\varepsilon}{2}\sum_{j=i,e}\int_\Omega |u_{j,0}(x)|^2 \, dx \\
&\le \tilde{C}_1 + (C_2 + C_h)\iint_{Q_T} |v_n|^2 \, dx\, dt.
\end{aligned} \tag{5.25}$$

In view of (3.12) and Gronwall's inequality, it follows from (5.25) that

$$\int_\Omega |v_n(T,x)|^2 \, dx + \varepsilon \sum_{j=i,e}\int_\Omega |u_j(T,x)|^2 \, dx \le C_3, \tag{5.26}$$

for some constant $C_3 > 0$ independent of n, which proves (5.19).

From (5.25) and (5.26) we also conclude that

$$\begin{aligned}
&C_M \sum_{j=i,e}\iint_{Q_T} |\nabla u_{j,n}|^p \, dx\, dt \le \tilde{C}_1 + (C_h + C_2)TC_3, \\
&0 \le \iint_{Q_T}\Big(h(t,x,v_n)v_n + C_h |v_n|^2\Big)\, dx\, dt \le \tilde{C}_1 + (C_h + C_2)TC_3,
\end{aligned} \tag{5.27}$$

where the first estimate proves assertion (5.20).

By (3.12),

$$-h(t,x,v_n)v_n \le h(t,x,v_n)v_n + 2C_h |v_n|^2 \quad \text{a.e. in } Q_T.$$

On the other hand, it is trivial that the same bound holds for $h(t,x,v_n)v_n$. Hence

$$|h(t,x,v_n)v_n| \le h(t,x,v_n)v_n + 2C_h |v_n|^2 \quad \text{a.e. in } Q_T. \tag{5.28}$$

Thanks to (5.28) and (5.26), (5.27) there exist a constant $C_4 > 0$ independent of n such that

$$\iint_{Q_T} |h(t,x,v_n)v_n| \, dx\, dt \le C_4. \tag{5.29}$$

A consequence of (5.29) and (3.11) is an L^p bound on v_n in the fully nonlinear case:

$$\iint_{Q_T} |v_n|^p \, dx\, dt \le C_5, \qquad p > 2, \tag{5.30}$$

for some constant $C_5 > 0$ being independent on n.

The Poincaré inequality implies the existence of a constant $C_6 > 0$ independent of n such that for each fixed t

$$\|u_{j,n}(t,\cdot)\|_{L^p(\Omega)} \leq C_6 \|\nabla u_{j,n}(t,\cdot)\|_{L^p(\Omega)}, \qquad j = i, e,$$

and therefore, by (5.27),

$$\int_0^T \|u_{j,n}(t,\cdot)\|_{L^p(\Omega)}^p \, dt \leq C_7, \qquad j = i, e. \tag{5.31}$$

This concludes the proof of (5.21).

Now we turn to the proof of (5.22), and start by reminding the reader of the functions $\mathcal{M}_j$ and H defined respectively in (3.4) and (3.7). We substitute $\varphi_{i,n}(t,\cdot) = \partial_t u_{i,n}(t,\cdot)$ in (5.17) and $\varphi_{e,n}(t,\cdot) = -\partial_t u_{e,n}(t,\cdot)$ in (5.18), and sum the resulting equations to bring about an equation that is integrated over $(0,T)$. The final outcome reads

$$\begin{aligned}
&\iint_{Q_T} |\partial_t v_n|^2 \, dx\, dt + \varepsilon \sum_{j=i,e} \iint_{Q_T} |\partial_t u_{j,n}|^2 \, dx\, dt \\
&\qquad + \iint_{Q_T} \sum_{j=i,e} M_j(t,x,\nabla u_{j,n}) \cdot \nabla(\partial_t u_{j,n}) \, dx\, dt + \int_\Omega h(t,x,v_n)\partial_t v_n \, dx\, dt \\
&= \iint_{Q_T} |\partial_t v_n|^2 \, dx\, dt + \varepsilon \sum_{j=i,e} \iint_{Q_T} |\partial_t u_{j,n}|^2 \, dx\, dt \\
&\qquad + \int_0^T \partial_t \int_\Omega \left(\sum_{j=i,e} \mathcal{M}_j(t,x,\nabla u_{j,n}) + H(t,x,v_n) \right) dx\, dt \\
&\qquad - \iint_{Q_T} \left(\sum_{j=i,e} \partial_t \mathcal{M}_j(t,x,\nabla u_{j,n}) + \partial_t H(t,x,v_n) \right) dx\, dt \\
&= \iint_{Q_T} I_{\mathrm{app},n} \partial_t v_n \, dx\, dt \leq \frac{1}{2} \iint_{Q_T} |\partial_t v_n|^2 \, dx\, dt + C_8,
\end{aligned} \tag{5.32}$$

where we have used Young's inequality and the uniform L^2 boundedness of $I_{\mathrm{app},n}$ to derive the last inequality.

Taking into account (3.5) and (3.7) in (5.32), we conclude that there exist two constants $C_9, C_{10} > 0$ independent of n such that

$$\begin{aligned}
&\frac{1}{2} \iint_{Q_T} |\partial_t v_n|^2 \, dx\, dt + \varepsilon \sum_{j=i,e} \iint_{Q_T} |\partial_t u_{j,n}|^2 \, dx\, dt \\
&\qquad + \int_\Omega \left(\sum_{j=i,e} \mathcal{M}_j(T,x,\nabla u_{j,n}(T,x)) + H(T,x,v_n(T,x)) \right) dx \\
&\leq C_9 \iint_{Q_T} \left(\sum_{j=i,e} \mathcal{M}_j(t,x,\nabla u_{j,n}) + H(t,x,v_n) \right) dx\, dt \\
&\qquad + \int_\Omega \left(\sum_{j=i,e} \mathcal{M}_j(0,x,\nabla u_{j,n}(0,x)) + H(0,x,v_n(0,x)) \right) dx + C_{10}.
\end{aligned} \tag{5.33}$$

By definitions of $\mathcal{M}_j$ and H, (3.1), $u_{i,0}, u_{e,0} \in W_0^{1,p}(\Omega)$, and (3.8) for $p = 2$ and (3.9) for $p > 2$, we deduce

$$\int_\Omega \left| \sum_{j=i,e} \mathcal{M}_j(0, x, \nabla u_{j,n}(0, x)) + H(0, x, v_n(0, x)) \right| dx \le C'_{10},$$

for some constant $C'_{10} > 0$ independent of n.

By the monotonicity conditions (3.3) and (3.6),

$$\sum_{j=i,e} \int_\Omega \mathcal{M}_j(T, x, \nabla u_{j,n}(T, x))\, dx \ge 0 \tag{5.34}$$

and

$$\begin{aligned} &\int_\Omega H(T, x, v_n(T, x))\, dx + C_h \int_\Omega |v_n(T, x)|^2 \, dx \\ &\qquad \ge \int_\Omega \int_0^{v_n} \Big(h(T, x, \rho) + C_h \rho \Big)\, d\rho\, dx \ge 0. \end{aligned} \tag{5.35}$$

Using (5.34) and (5.35) in (5.33) we obtain

$$\begin{aligned} &\int_\Omega \left(\sum_{j=i,e} \mathcal{M}_j(T, x, \nabla u_{j,n}(T, x)) + H(T, x, v_n(T, x)) + C_h |v_n(T, x)|^2 \right) dx \\ &\le C_9 \iint_{Q_T} \left(\sum_{j=i,e} \mathcal{M}_j(t, x, \nabla u_{j,n}) + H(t, x, v_n) + C_h |v_n|^2 \right) dx\, dt \\ &\qquad + C_h \int_\Omega |v_n(T, x)|^2 \, dx + C''_{10}, \qquad C''_{10} = C_{10} + C'_{10}. \end{aligned} \tag{5.36}$$

Now (5.36), (5.19), and an application of Gronwall's lemma in (5.36) furnish

$$\sum_{j=i,e} \int_\Omega \mathcal{M}_j(T, x, \nabla u_{j,n}(T, x))\, dx + \int_\Omega H(T, x, v_n(T, x))\, dx \le C_{11}, \tag{5.37}$$

for some constant $C_{11} > 0$ independent of n.

Finally, combining (5.34), (5.35), (5.37) in (5.33) delivers (5.22). □

We want to show that the local solution constructed above can be extended to the whole time interval $[0, T)$ (independently of n). To this end, observe that for an arbitrary t in the existence interval $[0, \rho')$ there holds, thanks to (5.19),

$$\begin{aligned} &\left| \{d_{n,l}(t)\}_{l=1,\dots,n} \right|_{\mathbb{R}^n}^2 + \sum_{j=i,e} \left| \{c_{j,n,l}(t)\}_{l=1,\dots,n} \right|_{\mathbb{R}^n}^2 \\ &\qquad = \|v_n(t, \cdot)\|_{L^2(\Omega)} + \sum_{j=i,e} \|u_{j,n}(t, \cdot)\|_{L^2(\Omega)} \le C, \end{aligned} \tag{5.38}$$

where $C > 0$ is a constant independent of t and n. We continue by introducing

$$S := \{ t \in [0, T) : \text{there exist a solution of (5.5), (5.6) on } [0, t) \},$$

and observing that S is nonempty due to the above local existence result.

We claim that S is an open set. To see this, let $\bar{t} \in S$ and $0 < t_1 < t_2 < \bar{t}$. In view of (5.15), (5.13) and (5.16), (5.14) we then obtain for $l = 1, \dots, n$

$$|d_{n,l}(t_1) - d_{n,l}(t_2)| \le c(C, n, c_m, \varepsilon) \int_{t_1}^{t_2} |M(\tau)| \, d\tau \tag{5.39}$$

and

$$|c_{j,n,l}(t_1) - c_{j,n,l}(t_2)| \le c(C, n, c_m, \varepsilon) \int_{t_1}^{t_2} |M_j(\tau)| \, d\tau, \qquad j = i, e. \tag{5.40}$$

Since $M, M_j \in L^1$, $j = i, e$, we use (5.39) and (5.40) to conclude respectively that $t \mapsto d_{n,l}(t)$ and $t \mapsto c_{j,n,l}(t)$, $j = i, e$, are uniformly continuous. At time $\bar{t}$, we solve the ODE system (5.8), (5.9), (5.10) with initial data

$$\lim_{t \uparrow \bar{t}} \left(d_{n,l}(t), c_{i,n,l}(t), c_{e,n,l}(t)\right), \qquad l = 1, \dots, n,$$

which provides us with a solution on $[0, t + \varepsilon)$ for some $\varepsilon = \varepsilon(\bar{t}) > 0$, and thus S is open. It remains to prove that S is closed. We consider a sequence $\{t_\ell\}_{\ell > 1} \subset S$ such that $t_\ell \to \bar{t}$ as $\ell \to \infty$. Let $\left\{ (d^\ell_{n,l}(t), c^\ell_{i,n,l}(t), c^\ell_{e,n,l}(t)) \right\}_{l=1}^{n}$ denote the solution of (5.8), (5.9), (5.10), (5.6) on $[0, t_\ell)$, and define for $l = 1, \dots, n$

$$\tilde{d}^\ell_{n,l}(t) = \begin{cases} d^\ell_{n,l}(t), & \text{if } t \in [0, t_\ell), \\ d^\ell_{n,l}(t_\ell), & \text{if } t \in [t_\ell, \bar{t}\,), \end{cases}$$

and for $j = i, e$

$$\tilde{c}^\ell_{j,n,l}(t) = \begin{cases} c^\ell_{j,n,l}(t), & \text{if } t \in [0, t_\ell), \\ c^\ell_{j,n,l}(t_\ell), & \text{if } t \in [t_\ell, \bar{t}\,). \end{cases}$$

It follows from what we have said before that the sequences

$$\left\{ \tilde{d}^\ell_{n,l}(t) \right\}_{\ell > 1}, \qquad \left\{ \tilde{c}^\ell_{j,n,l}(t) \right\}_{\ell > 1}, \quad j = i, e, \qquad l = 1, \dots, n,$$

are equibounded and equicontinuous on $[0, \bar{t})$. Hence there exist subsequences that converge uniformly on $[0, \bar{t})$ to continuous functions $\tilde{d}_{n,k}(t)$ and $\tilde{c}_{j,n,l}(t)$, $j = i, e$. By (5.15), (5.16), and Lebesgue's dominated convergence theorem, it is easy to see that these functions must solve the ODE system (5.8), (5.9), (5.10), (5.6) on $[0, \bar{t})$. Hence $\bar{t} \in S$, and we infer that S is closed. Consequently, $S = [0, T)$.

Having proved that the Faedo-Galerkin solutions (5.3), (5.4) are well-defined, we are now ready to prove existence of solutions to our nondegenerate system (1.6).

Theorem 5.1 (Regularized system)**.** *Assume* (3.1)-(3.9) *hold and* $p \ge 2$*. If* $u_{j,0} \in W_0^{1,p}(\Omega)$*,* $j = i, e$*, and* $I_{\text{app}} \in L^2(Q_T)$*, then the regularized system* (1.6)-(1.2)-(1.7) *possesses a solution for each fixed* $\varepsilon > 0$*.*

The remaining part of this section is devoted to proving Theorem 5.1.

Lemma 5.1 shows that $\{v_n\}_{n>1}$, $\{u_{j,n}\}_{n>1}$, $j = i, e$, are bounded in $L^p(0, T; W_0^{1,p}(\Omega))$ and $\{\partial_t v_n\}_{n>1}$, $\{\partial_t u_{j,n}\}_{n>1}$, $j = i, e$, are bounded in $L^2(Q_T)$. Therefore, possibly at the cost of extracting subsequences, which we do not bother to relabel, we can assume there exist limit functions u_i, u_e, v with $v = u_i - u_e$ such that as $n \to \infty$

$$\begin{cases} u_{j,n} \to u_j \text{ a.e. in } Q_T, \text{ strongly in } L^2(Q_T), \\ \text{and weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \\ v_n \to v \text{ a.e. in } Q_T, \text{ strongly in } L^2(Q_T), \\ \text{and weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \\ M_j(t,x,\nabla u_{j,n}) \to \Sigma_j \text{ weakly in } L^{p'}(Q_T;\mathbb{R}^3), \\ h(t,x,v_n) \to h(t,x,v) \text{ a.e. in } Q_T \text{ and weakly in } L^{p'}(Q_T). \end{cases} \tag{5.41}$$

Lemma 5.2. *As $n \to \infty$, $h(t,x,v_n) \to h(t,x,v)$ strongly in $L^q(Q_T)$ $\forall q \in [1,p')$.*

Proof. Because of (5.21) and Remarks 4.1 and 4.2, the sequence $\{h(t,x,v_n)\}_{n>1}$ is bounded in $L^{p'}(Q_T)$. The lemma is then a consequence of (5.41) and Vitali's theorem. □

Keeping in mind (5.41) and Lemma 5.2 we infer, by integrating (5.17) and (5.18) over $(0,T)$ and then letting $n \to \infty$,

$$\begin{aligned} &\iint_{Q_T} c_m \partial_t v \varphi_i \,dx\,dt + \varepsilon \iint_{Q_T} \partial_t u_i \varphi_i \,dx\,dt \\ &\quad + \iint_{Q_T} \Sigma_i \cdot \nabla \varphi_i \,dx\,dt + \iint_{Q_T} h(t,x,v)\varphi_i \,dx\,dt \\ &\quad\quad = \iint_{Q_T} I_{\text{app}} \varphi_i \,dx\,dt, \end{aligned} \tag{5.42}$$

$$\begin{aligned} &\iint_{Q_T} c_m \partial_t v \varphi_e \,dx\,dt - \varepsilon \iint_{Q_T} \partial_t u_e \varphi_e \,dx\,dt \\ &\quad - \iint_{Q_T} \Sigma_e \cdot \nabla \varphi_e \,dx\,dt + \iint_{Q_T} h(t,x,v)\varphi_e \,dx\,dt \\ &\quad\quad = \iint_{Q_T} I_{\text{app}} \varphi_e \,dx\,dt, \end{aligned} \tag{5.43}$$

for any $\varphi_j \in L^p(0,T;W_0^{1,p}(\Omega))$, $j = i,e$. To conclude that the limit functions in (5.41) satisfy the weak form of (1.6), we need to identify $\Sigma_j(t,x)$ as $M_j(t,x,\nabla u_j)$, which boils down to proving strong convergence in L^p of the gradients $\nabla u_{j,n}$. We remark that in the case $p = 2$ (i.e., $M_j(t,x,\xi) = \mathbf{M}_j(t,x)\xi$) we do not need strong convergence of the gradients, so Lemma 5.3 below is needed only in the fully non-linear case ($p > 2$).

Lemma 5.3. *For $j = i,e$, $\nabla u_{j,n} \to \nabla u_j$ strongly in $L^p(Q_T)$ as $n \to \infty$ and $\Sigma_j(t,x) = M_j(t,x,\nabla u_j)$ for a.e. $(t,x) \in Q_T$ and in $L^{p'}(Q_T;\mathbb{R}^3)$.*

Proof. Fixing an integer $N \geq 1$, we consider functions $w_j = w_j(t,x)$ of the form

$$w_j(t,x) = \sum_{l=1}^{N} a_{j,l}(t) e_l(x), \qquad j = i,e, \tag{5.44}$$

where $\{a_{j,l}\}_{l=1}^N$ are given $C^1([0,T])$ functions and $\{e_l\}_{l=1}^\infty$ is the basis introduced in Subsection 3.3. We also set $w := w_i - w_e$. Assuming that $n \geq N$, we add together

(5.17) with $\varphi_i(t,\cdot) = (u_{i,n} - w_i)(t,\cdot)$ and (5.18) with $\varphi_e(t,\cdot) = -(u_{e,n} - w_e)(t,\cdot)$. Integrating the resulting equation over $(0,T)$ and then adding it to (3.6) we get

$$
\begin{aligned}
&\sum_{j=i,e} \iint_{Q_T} (M_j(t,x,\nabla u_{j,n}) - M_j(t,x,\nabla w_j)) \cdot (\nabla u_{j,n} - \nabla w_j)\, dx\, dt \\
&= -\iint_{Q_T} c_m \partial_t v_n (v_n - w)\, dx\, dt - \sum_{j=i,e} \iint_{Q_T} \varepsilon \partial_t u_{j,n}(u_{j,n} - w_j)\, dx\, dt \\
&\quad - \sum_{j=i,e} \iint_{Q_T} M_j(t,x,\nabla w_j) \cdot (\nabla u_{j,n} - \nabla w_j)\, dx\, dt \\
&\quad - \iint_{Q_T} \left[(h(v_n) - h(w))(v_n - w) + C_h \left| v_n - w \right|^2 \right] dx\, dt \\
&\quad - \iint_{Q_T} h(w)(v_n - w)\, dx\, dt + C_h \iint_{Q_T} \left| v_n - w \right|^2\, dx\, dt \\
&\quad + \iint_{Q_T} I_{\mathrm{app},n}(v_n - w)\, dx\, dt \\
&\leq -\iint_{Q_T} c_m \partial_t v_n (v_n - w)\, dx\, dt - \sum_{j=i,e} \iint_{Q_T} \varepsilon \partial_t u_{j,n}(u_{j,n} - w_j)\, dx\, dt \\
&\quad - \sum_{j=i,e} \iint_{Q_T} M_j(t,x,\nabla w_j) \cdot (\nabla u_{j,n} - \nabla w_j)\, dx\, dt \\
&\quad - \iint_{Q_T} h(w)(v_n - w)\, dx\, dt + C_h \iint_{Q_T} \left| v_n - w \right|^2\, dx\, dt \\
&\quad + \iint_{Q_T} I_{\mathrm{app},n}(v_n - w)\, dx\, dt =: E_1 + E_2 + E_3 + E_4 + E_5 + E_6.
\end{aligned}
\tag{5.45}
$$

By Lemma 5.1 and (5.41), we draw the conclusions that

$$
\begin{aligned}
\lim_{n\to\infty} E_1 &= -\iint_{Q_T} c_m \partial_t v (v - w)\, dx\, dt, \\
\lim_{n\to\infty} E_2 &= -\sum_{j=i,e} \iint_{Q_T} \varepsilon \partial_t u_j (u_j - w_j)\, dx\, dt.
\end{aligned}
$$

From (3.1), (3.8), (3.9), and (5.41), it follows that $M_j(t,x,\nabla w_j) \in L^{p'}(Q_T;\mathbb{R}^3)$, $j = i,e$, $h(w) \in L^{p'}(Q_T)$, and thus

$$
\begin{aligned}
\lim_{n\to\infty} E_3 &= -\sum_{j=i,e} \iint_{Q_T} M_j(t,x,\nabla w_j) \cdot (\nabla u_j - \nabla w_j)\, dx\, dt, \\
\lim_{n\to\infty} E_4 &= -\iint_{Q_T} h(w)(v - w)\, dx\, dt.
\end{aligned}
$$

The term E_5 is sorted out using the convergence $v_n \to v$ in $L^2(Q_T)$, cf. (5.41):

$$
\lim_{n\to\infty} E_5 = C_h \iint_{Q_T} \left| v - w \right|^2\, dx\, dt.
$$

Bringing to mind that $\{I_{\mathrm{app},n}\}_{n>1}$ is bounded in $L^2(Q_T)$ and exploiting again the convergence $v_n \to v$ in $L^2(Q_T)$, we deduce

$$\lim_{n\to\infty} E_6 = \iint_{Q_T} I_{\mathrm{app},n}(v-w)\,dx\,dt.$$

Now we can pass to the limit in (5.45) to obtain, keeping in mind (3.2),

$$\begin{aligned}
&C_M \lim_{n\to\infty} \iint_{Q_T} |\nabla u_{j,n} - \nabla w_j|^p \,dx\,dt \\
&\le \lim_{n\to\infty} \iint_{Q_T} \sum_{j=i,e} (M_j(t,x,\nabla u_{j,n}) - M_j(t,x,\nabla u_j)) \cdot (\nabla u_{j,n} - \nabla w_j)\,dx\,dt \\
&\le -\iint_{Q_T} c_m \partial_t v(v-w)\,dx\,dt - \sum_{j=i,e} \iint_{Q_T} \varepsilon \partial_t u_{j,n}(u_j - w_j)\,dx\,dt \\
&\quad - \sum_{j=i,e} \iint_{Q_T} M_j(t,x,\nabla w_j)\cdot(\nabla u_j - \nabla w_j)\,dx\,dt \\
&\quad - \iint_{Q_T} h(w)(v-w)\,dx\,dt + C_h \iint_{Q_T} |v-w|^2 \,dx\,dt \\
&\quad + \iint_{Q_T} I_{\mathrm{app},n}(v-w)\,dx\,dt.
\end{aligned} \tag{5.46}$$

Since functions of the form (5.44) are dense in $L^p(0,T;W_0^{1,p}(\Omega))$, inequality (5.46) holds in fact for all functions $w_j \in L^p(0,T;W_0^{1,p}(\Omega))$. Hence, choosing $w_j = u_j$, $j=i,e$, in (5.46) gives us

$$\begin{aligned}
&C_M \lim_{n\to\infty} \iint_{Q_T} |\nabla u_{j,n} - \nabla u_j|^p \,dx\,dt \\
&\le \lim_{n\to\infty} \iint_{Q_T} \sum_{j=i,e} (M_j(t,x,\nabla u_{j,n}) - M_j(t,x,\nabla u_j)) \cdot (\nabla u_{j,n} - \nabla u_j)\,dx\,dt \\
&\le 0,
\end{aligned} \tag{5.47}$$

which proves the first part of the lemma.

In view of (5.47), along subsequences the following convergences hold:

$$\nabla u_{j,n} \to \nabla u_j \quad \text{a.e. in } Q_T, \quad j=i,e.$$

Hence, $\Sigma_j(t,x) = M_j(t,x,\nabla u_j)$ a.e. in Q_T and also in $L^{p'}(Q_T)$. This concludes the proof of the lemma. □

Finally, we prove that the limits u_i, u_e in (5.41) obey the initial data (1.7).

Lemma 5.4. *For $j=i,e$, there holds $u_j(0,x) = u_{j,0}(x)$ for a.e. $x \in \Omega$.*

Proof. The proof adapts a standard argument given in [13]. Pick a test function φ_e of the form (5.44) with $\varphi_e(T,\cdot)=0$. We use $\varphi_e(t,\cdot)$ in (5.18) and then integrate with respect to $t \in (0,T)$. In the resulting equation we send $n\to\infty$, followed by an integration by parts in the obtained limit equation, thereby obtaining

$$
\begin{aligned}
&-\iint_{Q_T} c_m v\partial_t\varphi_e\,dx\,dt + \iint_{Q_T} \varepsilon u_e\partial_t\varphi_e\,dx\,dt \\
&\quad -\iint_{Q_T} M_e(t,x,\nabla u_e)\cdot\nabla\varphi_e\,dx\,dt + \iint_{Q_T} h(t,x,v)\varphi_e\,dx\,dt \\
&= \iint_{Q_T} I_{\mathrm{app}}\varphi_e\,dx\,dt + \int_\Omega c_m v(0,x)\varphi_e(0,x)\,dx - \int_\Omega \varepsilon u_e(0,x)\varphi_e(0,x)\,dx.
\end{aligned}
\tag{5.48}
$$

On the other hand, integration by parts in (5.18) yields

$$
\begin{aligned}
&-\iint_{Q_T} c_m v_n\partial_t\varphi_e\,dx\,dt + \iint_{Q_T} \varepsilon u_{e,n}\partial_t\varphi_e\,dx\,dt \\
&\quad -\iint_{Q_T} M_e(t,x,\nabla u_{e,n})\cdot\nabla\varphi_e\,dx\,dt + \iint_{Q_T} h(t,x,v_n)\varphi_e\,dx\,dt \\
&= \iint_{Q_T} I_{\mathrm{app},n}\varphi_e\,dx\,dt + \int_\Omega c_m v_n(0,x)\varphi_e(0,x)\,dx - \int_\Omega \varepsilon u_{e,n}(0,x)\varphi_e(0,x)\,dx,
\end{aligned}
\tag{5.49}
$$

for all φ_e of the form (5.44) with $\varphi_e(T,\cdot)=0$.

Since by construction $u_{j,n}(0,\cdot)\to u_{j,0}(\cdot)$ in $W_0^{1,p}(\Omega)$ for $j=i,e$, and in view of the convergences established for the approximate solutions, sending $n\to\infty$ in (5.49) delivers

$$
\begin{aligned}
&-\iint_{Q_T} c_m v_n\partial_t\varphi_e\,dx\,dt + \iint_{Q_T} \varepsilon u_{e,n}\partial_t\varphi_e\,dx\,dt \\
&\quad -\iint_{Q_T} M_e(t,x,\nabla u_{e,n})\cdot\nabla\varphi_e\,dx\,dt + \iint_{Q_T} h(t,x,v_n)\varphi_e\,dx\,dt \\
&= \iint_{Q_T} I_{\mathrm{app},n}\varphi_e\,dx\,dt + \int_\Omega c_m v_0(x)\varphi_e(0,x)\,dx - \int_\Omega \varepsilon u_{e,0}(x)\varphi_e(0,x)\,dx,
\end{aligned}
\tag{5.50}
$$

for all φ_e of the form (5.44) with $\varphi_e(T,\cdot)=0$.

Comparing (5.48) and (5.50), using also that functions of the form (5.44) are dense in $L^p(0,T;W_0^{1,p}(\Omega))$, yields $u_e(0,x)=u_{e,0}(x)$ for a.e. $x\in\Omega$. Reasoning along the same lines for u_i yields $u_i(0,x)=u_{i,0}(x)$ for a.e. $x\in\Omega$. □

6. Existence of solutions for the bidomain model

6.1. **Proof of Theorem 4.1.**

6.1.1. *The case $v_0=u_{i,0}-u_{e,0}$ with $u_{i,0},u_{e,0}\in H_0^1(\Omega)$.* From the previous section we know there exist sequences $\{u_{i,\varepsilon}\}_{\varepsilon>0}$, $\{u_{e,\varepsilon}\}_{\varepsilon>0}$, and $\{v_\varepsilon=u_{i,\varepsilon}-u_{e,\varepsilon}\}_{\varepsilon>0}$ of solutions to (1.6), (1.2), (1.7), cf. Definition 5.1 (with $p=2$). Furthermore, we have immediately at our disposal a series of a priori estimates, which we collect in a lemma.

Lemma 6.1. *Assume conditions* (3.1)-(3.8) *hold.*

If $u_{i,0},u_{e,0}\in L^2(\Omega)$ and $I_{\mathrm{app}}\in L^2(Q_T)$, then there exist constants c_1,c_2,c_3 not depending on ε such that

$$
\|v_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \sum_{j=i,e}\left\|\sqrt{\varepsilon}u_{j,\varepsilon}\right\|_{L^\infty(0,T;L^2(\Omega))} \le c_1,
$$

$$\|\nabla u_{j,\varepsilon}\|_{L^2(Q_T)} \le c_2, \qquad \sum_{j=i,e} \|u_{j,\varepsilon}\|_{L^2(Q_T)} \le c_3, \qquad j = i, e.$$

If, in addition, $u_{i,0}, u_{e,0} \in H_0^1(\Omega)$*, then there exists a constant* $c_4 > 0$ *independent of* ε *such that*

$$\|\partial_t v_\varepsilon\|_{L^2(Q_T)} + \sum_{j=i,e} \left\|\sqrt{\varepsilon}\partial_t u_{j,\varepsilon}\right\|_{L^2(Q_T)} \le c_4. \tag{6.1}$$

Proof. By the (weak) lower semicontinuity properties of norms, the estimates in Lemma 5.1 hold with $v_n, u_{i,n}, u_{e,n}$ replaced by $v_\varepsilon, u_{i,\varepsilon}, u_{e,\varepsilon}$, respectively. Moreover, the constants c_1, c_2, c_3, c_4 are independent of ε (consult the proof of Lemma 5.1). □

In view of Lemma 6.1, we can assume there exist limit functions u_i, u_e, v with $v = u_i - u_e$ such that as $\varepsilon \to 0$ the following convergences hold (modulo extraction of subsequences, which we do not bother to relabel):

$$\begin{cases} v_\varepsilon \to v \text{ a.e. in } Q_T, \text{ strongly in } L^2(Q_T), \text{ and weakly in } L^2(0,T;H_0^1(\Omega)), \\ u_{i,\varepsilon} \to u_i \text{ weakly in } L^2(0,T;H_0^1(\Omega)),\ u_{e,\varepsilon} \to u_e \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \\ h(t,x,v_\varepsilon) \to h(t,x,v) \text{ a.e. in } Q_T \text{ and weakly in } L^2(Q_T), \end{cases}$$

and, according to (6.1), $v \in C^{1/2}([0,T];L^2(\Omega))$. Additionally, $\partial_t v_\varepsilon \to \partial_t v$ and $\varepsilon\partial_t u_{j,\varepsilon} \to 0$, $j = i, e$, weakly in $L^2(Q_T)$. Arguing as in the proof of Lemma 5.2, we conclude also that $h(t,x,v_\varepsilon) \to h(t,x,v)$ strongly in $L^q(Q_T)$ $\forall q \in [1,2)$. Thanks to all these convergences and repeating the argument from the previous section to prove that the initial condition (1.3) is satisfied, it is easy to see that the limit triple $(u_i, u_e, v = u_i - u_e)$ is a weak solution of the bidomain model (1.1), (1.2), (1.3), cf. Definition 4.1, thereby proving Theorem 4.1 in the case $v_0 = u_{i,0} - u_{e,0}$ with $u_{i,0}, u_{e,0} \in H_0^1(\Omega)$.

6.1.2. *The case* $v_0 \in L^2(\Omega)$. To deal with this case, we approximate the initial data v_0 by a sequence $\{v_{0,\rho}\}_{\rho>0}$ of functions satisfying

$$v_{0,\rho} \in H_0^1(\Omega), \quad \|v_{0,\rho}\|_{L^2(\Omega)} \le \|v_0\|_{L^2(\Omega)}, \quad v_{0,\rho} \to v_0 \text{ in } L^2(\Omega) \text{ as } \rho \to 0.$$

For $\rho > 0$, we then introduce an artificial decomposition $v_{0,\rho} = u_{i,0,\rho} - u_{e,0,\rho}$ with $u_{i,0,\rho}, u_{e,0,\rho} \in H_0^1(\Omega)$. From the previous subsection, there exist sequences $\{u_{i,\rho}\}_{\rho>0}$, $\{u_{e,\rho}\}_{\rho>0}$, $\{v_\rho = u_{i,\rho} - u_{e,\rho}\}_{\rho>0}$ for which $u_{i,\rho}, u_{e,\rho} \in L^2(0,T;H_0^1(\Omega))$, $\partial_t v_\rho \in L^2(Q_T)$, and

$$\begin{aligned} &\iint_{Q_T} c_m \partial_t v_\rho \varphi_i \,dx\,dt + \iint_{Q_T} M_i(t,x)\nabla u_{i,\rho} \cdot \nabla\varphi_i \,dx\,dt \\ &\quad + \iint_{Q_T} h(t,x,v_\rho)\varphi_i \,dx\,dt = \iint_{Q_T} I_{\mathrm{app}}\varphi_i \,dx\,dt \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} &\iint_{Q_T} c_m \partial_t v_\rho \varphi_e \,dx\,dt - \iint_{Q_T} M_e(t,x)\nabla u_{e,\rho} \cdot \nabla\varphi_e \,dx\,dt \\ &\quad + \iint_{Q_T} h(t,x,v_\rho)\varphi_e \,dx\,dt = \iint_{Q_T} I_{\mathrm{app}}\varphi_e \,dx\,dt, \end{aligned} \tag{6.3}$$

for any $\varphi_j \in L^2(0,T;H_0^1(\Omega))$.

To pass to the limit $\rho \to 0$ in (6.2) and (6.3) we need a priori estimates. The ones from Lemma 5.1 that survive the test of being ρ-independent are

$$\|v_\rho\|_{L^\infty(0,T;L^2(\Omega))} \le c, \quad \|\nabla u_{j,\rho}\|_{L^2(Q_T)} \le c, \quad \|u_{j,\rho}\|_{L^2(Q_T)} \le c, \quad j = i, e. \tag{6.4}$$

We conclude from (6.4) that the sequences $\{u_{i,\rho}\}_{\rho>0}$, $\{u_{e,\rho}\}_{\rho>0}$, $\{v_\rho\}_{\rho>0}$ are bounded in $L^2(0,T;H_0^1(\Omega))$. In view of the equations satisfied by v_ρ this implies that $\{\partial_t v_\rho\}_{\rho>0}$ is bounded in $L^2\left(0,T;(H_0^1(\Omega))'\right)$, but there are no bounds on $\{\partial_t u_{i,\rho}\}_{\rho>0}$, $\{\partial_t u_{e,\rho}\}_{\rho>0}$! Therefore, possibly at the cost of extracting subsequences (which are not relabeled), we can assume that there exist limits $u_i, u_e, v \in L^2(0,T;H_0^1(\Omega))$ with $v = u_i - u_e$ and $\partial_t v \in L^2\left(0,T;(H_0^1(\Omega))'\right)$ such that as $\rho \to 0$

$$\begin{cases} v_\rho \to v \text{ a.e. in } Q_T, \text{ strongly in } L^2(Q_T), \text{ and weakly in } L^2(0,T;H_0^1(\Omega)),\\ u_{i,\rho} \to u_i \text{ weakly in } L^2(0,T;H_0^1(\Omega)),\ u_{e,\rho} \to u_e \text{ weakly in } L^2(0,T;H_0^1(\Omega)),\\ h(t,x,v_\rho) \to h(t,x,v) \text{ a.e. in } Q_T \text{ and weakly in } L^2(Q_T), \end{cases}$$

and $v \in C([0,T];L^2(\Omega))$. In addition, $\partial_t v_\rho \to \partial_t v$ weakly in $L^2\left(0,T;(H_0^1(\Omega))'\right)$. Arguing as in the proof of Lemma 5.2, we obtain $h(t,x,v_\rho) \to h(t,x,v)$ strongly in $L^q(Q_T)$ $\forall q \in [1,2)$. Equipped with these convergences it is not difficult to pass to the limit as $\rho \to 0$ in (6.2), (6.3) to conclude that the limit triple $(u_i, u_e, v = u_i - u_e)$ is a weak solution to the bidomain model (1.1), (1.2), (1.3). This proves Theorem 4.1 in the case $v_0 \in L^2(\Omega)$.

7. Existence of solutions for the nonlinear model

7.1. Proof of Theorem 4.2.

7.1.1. *The case $v_0 = u_{i,0} - u_{e,0}$ with $u_{i,0}, u_{e,0} \in W_0^{1,p}(\Omega)$.* In view of the results in Section 5, there exist sequences $\{u_{i,\varepsilon}\}_{\varepsilon>0}$, $\{u_{e,\varepsilon}\}_{\varepsilon>0}$, and $\{v_\varepsilon = u_{i,\varepsilon} - u_{e,\varepsilon}\}_{\varepsilon>0}$ of solutions to (1.6), (1.2), (1.7), cf. Definition 5.1, and the following weak formulations hold for each $\varepsilon > 0$:

$$\begin{aligned} &\iint_{Q_T} c_m \partial_t v_\varepsilon \varphi_i \,dx\,dt + \varepsilon \iint_{Q_T} \partial_t u_{i,\varepsilon} \varphi_i \,dx\,dt \\ &\quad + \iint_{Q_T} M_i(t,x,\nabla u_{i,\varepsilon}) \cdot \nabla \varphi_i \,dx\,dt \\ &\qquad + \iint_{Q_T} h(t,x,v_\varepsilon)\varphi_i \,dx\,dt = \iint_{Q_T} I_{\text{app}} \varphi_i \,dx\,dt, \end{aligned} \tag{7.1}$$

$$\begin{aligned} &\iint_{Q_T} c_m \partial_t v_\varepsilon \varphi_e \,dx\,dt - \varepsilon \iint_{Q_T} \partial_t u_{e,\varepsilon} \varphi_e \,dx\,dt \\ &\quad - \iint_{Q_T} M_e(t,x,\nabla u_{e,\varepsilon}) \cdot \nabla \varphi_e \,dx\,dt \\ &\qquad + \iint_{Q_T} h(t,x,v_\varepsilon)\varphi_e \,dx\,dt = \iint_{Q_T} I_{\text{app}} \varphi_e \,dx\,dt, \end{aligned} \tag{7.2}$$

for any $\varphi_j \in L^p(0,T;W_0^{1,p}(\Omega))$, $j = i, e$.

Similar to Lemma 6.1 for the bidomain model, we have the following a priori estimates for the nonlinear model:

Lemma 7.1. *Assume conditions* (3.1)-(3.7) *and* (3.9) *hold.*

If $u_{i,0}, u_{e,0} \in L^2(\Omega)$ *and* $I_{\mathrm{app}} \in L^2(Q_T)$, *then there exist constants* c_1, c_2, c_3 *not depending on* ε *such that*

$$\|v_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \sum_{j=i,e} \left\|\sqrt{\varepsilon} u_{j,\varepsilon}\right\|_{L^\infty(0,T;L^2(\Omega))} \le c_1,$$

$$\|\nabla u_{j,\varepsilon}\|_{L^p(Q_T)} \le c_2, \qquad \|u_{j,\varepsilon}\|_{L^p(Q_T)} \le c_3, \qquad j = i, e.$$

If, in addition, $u_{i,0}, u_{e,0} \in W_0^{1,p}(\Omega)$, *then there exists a constant* $c_4 > 0$ *independent of* ε *such that*

$$\|\partial_t v_\varepsilon\|_{L^2(Q_T)} + \sum_{j=i,e} \left\|\sqrt{\varepsilon}\partial_t u_{j,\varepsilon}\right\|_{L^2(Q_T)} \le c_4. \tag{7.3}$$

In view of Lemma 7.1, we can assume there exist limit functions u_i, u_e, v with $v = u_i - u_e$ and Σ_i, Σ_e such that as $\varepsilon \to 0$ the following convergences are true (again modulo extraction of subsequences, which we do not relabel):

$$\begin{cases} v_\varepsilon \to v \text{ a.e. in } Q_T, \text{ strongly in } L^p(Q_T), \\ \text{and weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \\ u_{j,\varepsilon} \to u_j \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)),\ j = i, e, \\ M_j(t,x,\nabla u_{j,\varepsilon}) \to \Sigma_j \text{ weakly in } L^{p'}(Q_T;\mathbb{R}^3),\ j = i, e, \\ h(t,x,v_\varepsilon) \to h(t,x,v) \text{ a.e. in } Q_T \text{ and weakly in } L^{p'}(Q_T), \end{cases} \tag{7.4}$$

and, according to (7.3), $v \in C^{1/2}([0,T];L^2(\Omega))$. Besides, $\partial_t v_\varepsilon \to \partial_t v$, $\varepsilon\partial_t u_{j,\varepsilon} \to 0$, $j = i, e$, weakly in $L^2(Q_T)$. Arguing as in the proof of Lemma 5.2, we conclude additionally that $h(t,x,v_\varepsilon) \to h(t,x,v)$ strongly in $L^q(Q_T)$ $\forall q \in [1, p')$.

Different from the bidomain case, to continue we need to establish L^p convergence of the gradients, so that we can identify Σ_j as $M_j(t,x,\nabla u_j)$.

Lemma 7.2. *For* $j = i, e$,

(7.5)
$$\limsup_{\varepsilon\to 0} \sum_{j=i,e} \int_0^T \int_\Omega M_j(t,x,\nabla u_{j,\varepsilon}) \cdot \nabla u_{j,\varepsilon}\,dx\,dt \le \sum_{j=i,e} \int_0^T \int_\Omega \Sigma_j(t.x) \cdot \nabla u_j\,dx\,dt.$$

Proof. Choose $\varphi_i = u_{i,\varepsilon} - u_i$ in (7.1) and $\varphi_e = -(u_{e,\varepsilon} - u_e)$ in (7.2). Adding the resulting equations delivers

$$J_\varepsilon^0 + J_\varepsilon^1 + J_\varepsilon^2 = J_\varepsilon^3, \tag{7.6}$$

where

$$J_\varepsilon^0 = \int_0^T \int_\Omega \Big(\partial_t v_\varepsilon (v_\varepsilon - v) + \sum_{j=i,e} \varepsilon\partial_t u_{j,\varepsilon}(u_{j,\varepsilon} - u_j)\Big)\,dx\,dt,$$

$$J_\varepsilon^1 = \sum_{j=i,e} \int_0^T \int_\Omega M_j(t,x,\nabla u_{j,\varepsilon}) \cdot \nabla(u_{j,\varepsilon} - u_j)\,dx\,dt,$$

$$J_\varepsilon^2 = \int_0^T \int_\Omega h(t,x,v_\varepsilon)(v_\varepsilon - v)\,dx\,dt, \quad J_\varepsilon^3 = \int_0^T \int_\Omega I_{\mathrm{app}}(v_\varepsilon - v)\,dx\,dt.$$

The goal is to take the limit $\varepsilon \to 0$ in (7.6).

First, we claim that

$$\lim_{\varepsilon\to 0} J^0_\varepsilon = 0. \tag{7.7}$$

To see this, observe that

$$\begin{aligned} \left|J^0_\varepsilon\right| &\le \|\partial_t v_\varepsilon\|_{L^2(Q_T)} \|v_\varepsilon - v\|_{L^2(Q_T)} \\ &\quad + \sum_{j=i,e} \sqrt{\varepsilon} \left\|\sqrt{\varepsilon}\partial_t u_{j,\varepsilon}\right\|_{L^2(Q_T)} \|u_{j,\varepsilon} - u_j\|_{L^2(Q_T)}. \end{aligned} \tag{7.8}$$

On account of (7.4), in particular the convergence $v_\varepsilon \to v$ in $L^2(Q_T)$ and the L^2 boundness of $\partial_t v_\varepsilon$, $\sqrt{\varepsilon}\partial_t u_{j,\varepsilon}$, $j = i, e$, sending $\varepsilon \to 0$ in (7.8) yields (7.7).

By the weak convergence of $h(t, x, v_\varepsilon)$ to $h(t, x, v)$ in $L^{p'}(Q_T)$ and the strong convergence of v_ε to v in $L^p(Q_T)$, cf. (7.4),

$$\lim_{\varepsilon\to 0} J^2_\varepsilon = 0.$$

Clearly, again by (7.4),

$$\lim_{\varepsilon\to 0} J^3_\varepsilon = 0.$$

Summarizing our findings, taking the lim sup in (7.6) as $\varepsilon \to 0$ yields

$$\limsup_{\varepsilon\to 0} \sum_{j=i,e} \int_0^T \int_\Omega M_j(t, x, \nabla u_{j,\varepsilon}) \cdot \nabla(u_{j,\varepsilon} - u_j)\, dx\, dt \le 0. \tag{7.9}$$

We deduce from (7.9) and (7.4) that

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \sum_{j=i,e} \int_0^T \int_\Omega M_j(t, x, \nabla u_{j,\varepsilon}) \cdot \nabla u_{j,\varepsilon}\, dx\, dt \\ &\le \limsup_{\varepsilon\to 0} \sum_{j=i,e} \int_0^T \int_\Omega M_j(t, x, \nabla u_{j,\varepsilon}) \cdot \nabla u_j\, dx\, dt = \int_0^T \int_\Omega \Sigma_j \cdot \nabla u_j\, dx\, dt, \end{aligned}$$

which proves the lemma. □

A consequence of the previous lemma is strong convergence of the gradients.

Lemma 7.3. *For $j = i, e$, $\nabla u_{j,\varepsilon} \to \nabla u_j$ strongly in $L^p(Q_T)$ as $\varepsilon \to 0$ and $\Sigma_j(t, x) = M_j(t, x, \nabla u_j)$ for a.e. $(t, x) \in Q_T$ and in $L^{p'}(Q_T; \mathbb{R}^3)$.*

Proof. Since $\nabla u_j \in L^p(Q_T; \mathbb{R}^3)$ and, by (3.1), $M_j(t, x, \nabla u_j)$ is bounded in $L^{p'}(Q_T; \mathbb{R}^3)$, it follows from (7.4)

(7.10)

$$\lim_{\varepsilon\to 0} \sum_{j=i,e} \int_0^T \int_\Omega M_j(t, x, \nabla u_{j,\varepsilon}) \cdot \nabla u_j\, dx\, dt = \sum_{j=i,e} \int_0^T \int_\Omega \Sigma_j(t, x) \cdot \nabla u_j\, dx\, dt,$$

$$\lim_{\varepsilon\to 0} \sum_{j=i,e} \int_0^T \int_\Omega M_j(t, x, \nabla u_j) \cdot (\nabla u_{j,\varepsilon} - \nabla u_j)\, dx\, dt = 0.$$

We use (7.5) and (7.10) to infer

(7.11)

$$\limsup_{\varepsilon\to 0} \sum_{j=i,e} \int_0^T \int_\Omega (M_j(t, x, \nabla u_{j,\varepsilon}) - M_j(t, x, \nabla u_j)) \cdot (\nabla u_{j,\varepsilon} - \nabla u_j)\, dx\, dt \le 0.$$

Combining (3.2) with (7.11) yields

(7.12)
$$\begin{aligned}
&C_M \lim_{\varepsilon\to 0} \sum_{j=i,e} \int_0^T \int_\Omega |\nabla u_{j,\varepsilon} - \nabla u_j|^p \, dx\, dt \\
&\le \lim_{\varepsilon\to 0} \sum_{j=i,e} \int_0^T \int_\Omega (M_j(t,x,\nabla u_{j,\varepsilon}) - M_j(t,x,\nabla u_j)) \cdot (\nabla u_{j,\varepsilon} - \nabla u_j)\, dx\, dt = 0,
\end{aligned}$$

and thus the lemma is proved. □

Putting to use the convergences in (7.4) and Lemma 7.3 and the argument from Section 5 to prove that the initial condition (1.3) is satisfied, we can send $\varepsilon \to 0$ in (7.1) and (7.2) to obtain that the limit triple $(u_i, u_e, v = u_i - u_e)$ is a weak solution to the nonlinear model (1.4), (1.2), (1.3), cf. Definition 4.1, thereby proving Theorem 4.2 in the case $v_0 = u_{i,0} - u_{e,0}$ with $u_{i,0}, u_{e,0} \in W_0^{1,p}(\Omega)$.

7.1.2. *The case* $v_0 \in L^2(\Omega)$. To deal with this case, we approximate the initial data v_0 by a sequence $\{v_{0,\rho}\}_{\rho>0}$ of functions satisfying

$$v_{0,\rho} \in W_0^{1,p}(\Omega), \quad \|v_{0,\rho}\|_{L^2(\Omega)} \le \|v_0\|_{L^2(\Omega)}, \quad v_{0,\rho} \to v_0 \text{ in } L^2(\Omega) \text{ as } \rho \to 0,$$

where we call to mind that $W_0^{1,p}(\Omega) \subset H_0^1(\Omega)$ since $p \ge 2$. Alike the bidomain case, we introduce an artificial decomposition $v_{0,\rho} = u_{i,0,\rho} - u_{e,0,\rho}$ with $u_{i,0,\rho}, u_{e,0,\rho} \in W_0^{1,p}(\Omega)$. From the previous subsection, we can produce sequences $\{u_{i,\rho}\}_{\rho>0}$, $\{u_{e,\rho}\}_{\rho>0}$, and $\{v_\rho = u_{i,\rho} - u_{e,\rho}\}_{\rho>0}$ such that $u_{i,\rho}, u_{e,\rho} \in L^p(0,T; W_0^{1,p}(\Omega))$, $\partial_t v_\rho \in L^2(Q_T)$, and

$$\begin{aligned}
&\iint_{Q_T} c_m \partial_t v_\rho \varphi_i \, dx\, dt + \iint_{Q_T} M_i(t,x,\nabla u_{i,\rho}) \cdot \nabla \varphi_i \, dx\, dt \\
&\quad + \iint_{Q_T} h(t,x,v_\rho)\varphi_i \, dx\, dt = \iint_{Q_T} I_{\mathrm{app}} \varphi_i \, dx\, dt,
\end{aligned} \tag{7.13}$$

$$\begin{aligned}
&\iint_{Q_T} c_m \partial_t v_\rho \varphi_e \, dx\, dt - \iint_{Q_T} M_e(t,x,\nabla u_{e,\rho}) \cdot \nabla \varphi_e \, dx\, dt \\
&\quad + \iint_{Q_T} h(t,x,v_\rho)\varphi_e \, dx\, dt = \iint_{Q_T} I_{\mathrm{app}} \varphi_e \, dx\, dt,
\end{aligned} \tag{7.14}$$

for any $\varphi_j \in L^p(0,T; W_0^{1,p}(\Omega))$, $j = i,e$.

To pass to the limit $\rho \to 0$ in (7.13) and (7.14) we need a priori estimates. Among the ones in Lemma 7.1, the following estimates are independent of ρ:

$$\|v_\rho\|_{L^\infty(0,T;L^2(\Omega))} \le c, \quad \|\nabla u_{j,\rho}\|_{L^p(Q_T)} \le c, \quad \|u_{j,\rho}\|_{L^p(Q_T)} \le c, \quad j = i,e. \tag{7.15}$$

We conclude from (7.15) that the sequences $\{u_{i,\rho}\}_{\rho>0}$, $\{u_{e,\rho}\}_{\rho>0}$, and $\{v_\rho\}_{\rho>0}$ are bounded in $L^p(0,T; W_0^{1,p}(\Omega))$. In view of the equations satisfied by v_ρ, $\{\partial_t v_\rho\}_{\rho>0}$ is bounded in $L^{p'}\left(0,T; (W_0^{1,p}(\Omega))'\right)$. Therefore, possibly at the cost of extracting subsequences, which are not relabeled, we can assume there exist limit functions $u_i, u_e, v \in L^p(0,T; W_0^{1,p}(\Omega))$ with $v = u_i - u_e$ and $\partial_t v \in L^{p'}\left(0,T; (W_0^{1,p}(\Omega))'\right)$, such that as $\rho \to 0$

$$
(7.16)\qquad
\begin{cases}
v_\rho \to v \text{ a.e. in } Q_T, \text{ strongly in } L^p(Q_T),\\
\text{and weakly in } L^p(0,T;W_0^{1,p}(\Omega)),\\
\partial_t v_\rho \to \partial_t v_\rho \text{ weakly in } L^{p'}\Big(0,T;(W_0^{1,p}(\Omega))'\Big),\\
u_{j,\rho} \to u_i \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)),\ j=i,e,\\
M_j(t,x,\nabla u_{j,\rho}) \to \Sigma_j \text{ weakly in } L^{p'}(Q_T;\mathbb{R}^3),\ j=i,e,\\
h(t,x,v_\rho) \to h(t,x,v) \text{ a.e. in } Q_T \text{ and weakly in } L^{p'}(Q_T),
\end{cases}
$$

and $v \in C([0,T];L^p(\Omega))$. We argue again as in the proof of Lemma 5.2 to obtain $h(t,x,v_\rho) \to h(t,x,v)$ strongly in $L^q(Q_T)$ $\forall q \in [1,p')$. Equipped with all these convergences it is not difficult to send $\rho \to 0$ in (7.13), (7.14) to conclude that that the limit triple $(u_i, u_e, v = u_i - u_e)$ is a weak solution to the nonlinear model (1.4), (1.2), (1.3), provided we can make the identification $\Sigma_j = M_j(t,x,\nabla u_j)$, in which case the proof of Theorem 4.1 is completed. The remaining part of this section is devoted to this identification task.

A chief difference between the present case and Subsection 7.1 is that now v_0 is not regular enough to ensure the boundedness of $\partial_t v_\rho$ in $L^2(Q_T)$, which was used in the proof of Lemma 7.2. To handle this difficulty we apply a time-regularization procedure, introduced first by Landes [17] and thereafter employed by many authors to solve nonlinear parabolic equations with L^1 or measure data (see [11, 4, 23, 3]).

Lemma 7.4. *For $j = i,e$*

$$
(7.17)\qquad
\begin{aligned}
\limsup_{\rho\to 0} \sum_{j=i,e} \int_0^T \int_0^t \int_\Omega M_j(t,x,\nabla u_{j,\rho}) \cdot \nabla u_{j,\rho}\, dx\, ds\, dt\\
\le \sum_{j=i,e} \int_0^T \int_0^t \int_\Omega \Sigma_j \cdot \nabla u_j \, dx\, ds\, dt
\end{aligned}
$$

Proof. First, we introduce the time regularization of v, where $v = u_i - u_e$ and u_i, u_e are the limit functions in (7.16). We denote this regularized function by $(v)_\mu$, where μ is a regularization parameter tending to infinity. We define $(v)_\mu$ as the unique solution in $L^p(0,T;W_0^{1,p}(\Omega))$ of the equation

$$
(7.18)\qquad \partial_t (v)_\mu + \mu((v)_\mu - v) = 0 \quad \text{in } \mathcal{D}'(Q_T),
$$

which is supplemented with the initial condition

$$
(7.19)\qquad (v)_\mu|_{t=0} = v_0^\mu \quad \text{in } \Omega,
$$

where $\{v_0^\mu\}_{\mu>1}$ is a sequence of functions such that

$$
(7.20)\qquad
\begin{aligned}
&v_0^\mu \in W_0^{1,p}(\Omega), \quad v_0^\mu \to v_0 \text{ strongly in } L^2(\Omega) \text{ as } \mu \to \infty, \text{ and}\\
&\frac{1}{\mu}\|v_0^\mu\|_{W_0^{1,p}(\Omega)} \to 0 \text{ as } \mu \to \infty.
\end{aligned}
$$

Following [17] we can derive easily the properties

$$
(7.21)\qquad
\begin{aligned}
&\partial_t (v)_\mu \in L^p(0,T;W_0^{1,p}(\Omega)) \text{ and}\\
&(v)_\mu \to v \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ as } \mu \to \infty.
\end{aligned}
$$

We claim that

$$\liminf_{\mu\to\infty}\lim_{\rho\to 0} J^0_{\rho,\mu}\geq 0, \qquad J^0_{\rho,\mu}=\int_0^T\int_0^t\int_\Omega \partial_t v_\rho(v_\rho-(v)_\mu)\,dx\,ds\,dt. \tag{7.22}$$

To see this, we exploit the regularity $\partial_t(v)_\mu\in L^p(0,T;W_0^{1,p}(\Omega))$ and calculate

$$\begin{aligned}
&\int_0^T\int_0^t\int_\Omega \partial_t v_\rho(v_\rho-(v)_\mu)\,dx\,dt\,ds\\
&=\int_0^T\int_0^t\int_\Omega \partial_t(v_\rho-(v)_\mu)(v_\rho-(v)_\mu)\,dx\,dt\,ds\\
&\qquad+\int_0^T\int_0^t\int_\Omega \partial_t(v)_\mu(v_\rho-(v)_\mu)\,dx\,dt\,ds,\\
&=\frac{1}{2}\int_0^T\int_\Omega \left|v_\rho-(v)_\mu\right|^2\,dx\,dt-\frac{T}{2}\int_\Omega \left|v_\rho-(v)_\mu\right|^2(t=0)\,dx\\
&\qquad+\int_0^T\int_0^t\int_\Omega \partial_t(v)_\mu(v_\rho-(v)_\mu)\,dx\,ds\,dt.
\end{aligned} \tag{7.23}$$

Using (7.16) and (7.21), by sending $\rho\to 0$ in (7.23) we come up with

$$\begin{aligned}
&\lim_{\rho\to 0}\int_0^T\int_0^t\int_\Omega \partial_t v_\rho(v_\rho-(v)_\mu)\,dx\,dt\,ds\\
&=\frac{1}{2}\int_0^T\int_\Omega \left|v-(v)_\mu\right|^2\,dx\,dt-\frac{T}{2}\int_\Omega \left|v_0-v_0^\mu\right|^2\,dx\\
&\qquad+\int_0^T\int_0^t\int_\Omega \partial_t(v)_\mu(v-(v)_\mu)\,dx\,ds\,dt.
\end{aligned} \tag{7.24}$$

Availing ourselves of (7.20), (7.21), and (7.18), we obtain from (7.24)

$$\liminf_{\mu\to\infty}\lim_{\rho\to 0}\int_0^T\int_0^t\int_\Omega \partial_t v_\rho(v_\rho-(v)_\mu)\,dx\,dt\,ds\geq 0, \tag{7.25}$$

which proves our claim (7.22).

Next, we choose $\varphi_i=u_{i,\rho}-(u_i)_\mu$ and $\varphi_e=-(u_{e,\rho}-(u_e)_\mu)$ in (7.13) and (7.14), respectively, and add the resulting equations to obtain

$$J^0_{\rho,\mu}+J^1_{\rho,\mu}+J^2_{\rho,\mu}=J^3_{\rho,\mu}, \tag{7.26}$$

where $J^0_{\rho,\mu}$, defined in (7.22), is nonnegative by (7.22) and

$$\begin{aligned}
J^1_{\rho,\mu}&=\sum_{j=i,e}\int_0^T\int_0^t\int_\Omega M_j(t,x,\nabla u_{j,\rho})\cdot\nabla(u_{j,\rho}-(u_j)_\mu)\,dx\,ds\,dt,\\
J^2_{\rho,\mu}&=\int_0^T\int_0^t\int_\Omega h(t,x,v_\rho)(v_\rho-(v)_\mu)\,dx\,ds\,dt,\\
J^3_{\rho,\mu}&=\int_0^T\int_0^t\int_\Omega I_{\mathrm{app}}(v_\rho-(v)_\mu)\,dx\,ds\,dt.
\end{aligned}$$

Our goal is to send first $\rho\to 0$ and second $\mu\to\infty$ in (7.26).

By the weak convergence of $h(t,x,v_\rho)$ to $h(t,x,v)$ in $L^{p'}(Q_T)$ and the strong convergence of v_ρ to v in $L^p(Q_T)$, cf. (7.16),

$$\lim_{\rho\to 0} J^2_{\rho,\mu} = \int_0^T \int_0^t \int_\Omega h(t,x,v)(v-(v)_\mu)\,dx\,ds\,dt, \tag{7.27}$$

and using (7.21) in (7.27) we obtain

$$\lim_{\mu\to\infty}\lim_{\rho\to 0} J^2_{\rho,\mu} = 0.$$

By (7.16),

$$\lim_{\rho\to 0} J^3_{\rho,\mu} = \int_0^T \int_0^t \int_\Omega I_{\mathrm{app}}(v-(v)_\mu)\,dx\,ds\,dt, \tag{7.28}$$

and using (7.21) and sending $\mu\to\infty$ in (7.28) we obtain

$$\lim_{\mu\to\infty}\lim_{\rho\to 0} J^3_{\rho,\mu} = 0.$$

Summarizing, sending first $\rho\to 0$ and second $\mu\to\infty$ in (7.26) produces

$$\limsup_{\mu\to\infty}\limsup_{\rho\to 0} J^1_{\rho,\mu} \le 0. \tag{7.29}$$

We deduce from (7.29) and (7.16)

$$\begin{aligned}
&\limsup_{\rho\to 0}\sum_{j=i,e}\int_0^T\int_0^t\int_\Omega M_j(t,x,\nabla u_{j,\rho})\cdot\nabla u_{j,\rho}\,dx\,ds\,dt\\
&\le \limsup_{\mu\to\infty}\limsup_{\rho\to 0}\sum_{j=i,e}\int_0^T\int_0^t\int_\Omega M_j(t,x,\nabla u_{j,\rho})\cdot\nabla (u_j)_\mu\,dx\,ds\,dt\\
&= \int_0^T\int_0^t\int_\Omega \Sigma_j\cdot\nabla u_j\,dx\,ds\,dt,
\end{aligned}$$

and by means of that proving the lemma. □

A consequence of the previous lemma is strong convergence of the gradients.

Lemma 7.5. *For $j=i,e$, $\nabla u_{j,\rho}\to\nabla u_j$ strongly in $L^p(Q_T)$ as $\rho\to 0$ and $\Sigma_j(t,x)=M_j(t,x,\nabla u_j)$ for a.e. $(t,x)\in Q_T$ and in $L^{p'}(Q_T)$.*

Proof. Since $\nabla u_j\in L^p(Q_T;\mathbb{R}^3)$ and $M_j(t,x,\nabla u_j)\in L^{p'}(Q_T;\mathbb{R}^3)$, it follows from (7.16) that

$$\begin{aligned}
&\lim_{\rho\to 0}\sum_{j=i,e}\int_0^T\int_0^t\int_\Omega M_j(t,x,\nabla u_{j,\rho})\cdot\nabla u_j\,dx\,ds\,dt\\
&\qquad=\sum_{j=i,e}\int_0^T\int_0^t\int_\Omega \Sigma_j(t,x)\cdot\nabla u_j\,dx\,ds\,dt,\\
&\lim_{\rho\to 0}\sum_{j=i,e}\int_0^T\int_0^t\int_\Omega M_j(t,x,\nabla u_j)\cdot(\nabla u_{j,\rho}-\nabla u_j)\,dx\,ds\,dt = 0.
\end{aligned} \tag{7.30}$$

Combining (7.17) and (7.30) gives

$$\lim_{\rho\to 0}\sum_{j=i,e}\int_0^T\int_0^t\int_\Omega (M_j(t,x,\nabla u_{j,\rho})-M_j(t,x,\nabla u_j)) \cdot (\nabla u_{j,\rho}-\nabla u_j)\,dx\,ds\,dt\le 0,$$

which, together with the monotonicity property (3.2), proves the lemma (consult the proof of Lemma 7.3 for more details). □

8. Uniqueness of weak solutions

The purpose of this final section is to prove uniqueness of weak solutions to our degenerate systems, thereby completing the well-posedness analysis.

Theorem 8.1. *Assume conditions* (3.1)-(3.9) *hold and* $p\ge 2$. *Let* $(u_{i,1},u_{e,1},v_1)$ *and* $(u_{i,2},u_{e,2},v_2)$ *be two weak solutions to the bidomain model* (1.1), (1.2), (1.3) *or the nonlinear model* (1.4), (1.2), (1.3), *with "data"* $v_0=v_{1,0}, I_{\mathrm{app}}=I_{\mathrm{app},1}$ *and* $v_0=v_{2,0}, I_{\mathrm{app}}=I_{\mathrm{app},2}$, *respectively. Then for any* $t\in[0,T]$

$$\begin{aligned}&\int_\Omega |v_1(t,x)-v_2(t,x)|^2\,dx\\ &\le \exp\left(\frac{2C_h+1}{c_m}t\right)\left[\int_\Omega |v_{1,0}(x)-v_{2,0}(x)|^2\,dx\right.\\ &\qquad\left.+\int_0^t\int_\Omega |I_{\mathrm{app},1}(s,x)-I_{\mathrm{app},2}(s,x)|^2\,dx\,ds\right].\end{aligned}\tag{8.1}$$

In particular, there exists at most one weak solution to the bidomain model (1.1), (1.2), (1.3) *and the nonlinear model* (1.4), (1.2), (1.3).

Proof. According to Definitions 4.1 and 4.2, the following equations hold for all test functions $\varphi_j\in L^p(0,T;W_0^{1,p}(\Omega)))$, $j=i,e$:

$$\begin{aligned}&\int_0^t c_m\left\langle \partial_t(v_1-v_2),\varphi_i\right\rangle\,ds\\ &\quad+\int_0^t\int_\Omega (M_i(s,x,\nabla u_{i,1})-M_i(s,x,\nabla u_{i,2}))\cdot\nabla\varphi_i\,dx\,ds\\ &\quad+\int_0^t\int_\Omega (h(s,x,v_1)-h(s,x,v_2))\varphi_i\,dx\,ds=\int_0^t\int_\Omega (I_{\mathrm{app},1}-I_{\mathrm{app},2})\varphi_i\,dx\,ds\end{aligned}\tag{8.2}$$

and

$$\begin{aligned}&\int_0^t c_m\left\langle \partial_t(v_1-v_2),\varphi_e\right\rangle\,ds\\ &-\int_0^t\int_\Omega (M_e(s,x,\nabla u_{e,1})-M_e(s,x,\nabla u_{e,2}))\cdot\nabla\varphi_e\,dx\,ds\\ &+\int_0^t\int_\Omega (h(s,x,v_1)-h(s,x,v_2))\varphi_e\,dx\,ds=\int_0^t\int_\Omega (I_{\mathrm{app},k}-I_{\mathrm{app},2})\varphi_e\,dx.\end{aligned}\tag{8.3}$$

We utilize $\varphi_i = u_{i,1} - u_{i,2}$ in (8.2), $\varphi_e = -(u_{e,1} - u_{e,2})$ in (8.3), and add the resulting equations to obtain

(8.4)
$$\begin{aligned}
&\int_0^t c_m \langle \partial_t(v_1 - v_2), (v_1 - v_2) \rangle \, ds \\
&\quad + \sum_{j=i,e} \int_0^t \int_\Omega (M_j(s,x,\nabla u_{j,1}) - M_j(s,x,\nabla u_{j,2})) \cdot (\nabla u_{j,1} - \nabla u_{j,2}) \, dx \, ds \\
&\quad + \int_0^t \int_\Omega (h(s,x,v_1) - h(s,x,v_2))(v_1 - v_2) \, dx \, ds + C_h \int_0^t \int_\Omega |v_1 - v_2|^2 \, dx \, ds \\
&\qquad = C_h \int_0^t \int_\Omega |v_1 - v_2|^2 \, dx \, ds + \int_0^t \int_\Omega (I_{\text{app},1} - I_{\text{app},2})(v_1 - v_2) \, dx \, ds.
\end{aligned}$$

By Young's inequality,

$$\begin{aligned}
&\int_0^t \int_\Omega (I_{\text{app},1} - I_{\text{app},2})(v_1 - v_2) \, dx \, ds \\
&\le \frac{1}{2} \int_0^t \int_\Omega |I_{\text{app},1} - I_{\text{app},2}|^2 \, dx \, ds + \frac{1}{2} \int_0^t \int_\Omega |v_1 - v_2|^2 \, dx \, ds.
\end{aligned} \tag{8.5}$$

By (3.2), (3.6), (8.4), (8.5), and the classical "weak chain rule" (see, e.g., [5]),

$$\begin{aligned}
&\frac{c_m}{2} \int_\Omega |v_1(t,x) - v_2(t,x)|^2 \, dx \\
&\le \frac{c_m}{2} \int_\Omega |v_{1,0} - v_{2,0}|^2 \, dx + \frac{1}{2} \int_0^t \int_\Omega |I_{\text{app},1} - I_{\text{app},2}|^2 \, dx \, ds \\
&\quad + \left(C_h + \frac{1}{2} \right) \int_0^t \int_\Omega |v_1 - v_2|^2 \, dx \, ds.
\end{aligned}$$

An application of Gronwall's inequality now yields

$$\begin{aligned}
&\int_\Omega |v_1(t,x) - v_2(t,x)|^2 \, dx \\
&\le \exp\left(\frac{2C_h + 1}{c_m} t \right) \int_\Omega |v_{1,0}(x) - v_{2,0}(x)|^2 \, dx \\
&\quad + \int_0^t \exp\left(\frac{2C_h + 1}{c_m} (t - s) \right) \int_\Omega |I_{\text{app},1}(s,x) - I_{\text{app},2}(s,x)|^2 \, dx \, ds.
\end{aligned} \tag{8.6}$$

which proves (8.1). □

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