The Malliavin calculus for processes with conditionally independent increments

Aleh L. Yablonski

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Department of Functional Analysis, Belarusian State University,
F.Skaryna av.,4, 220050, Minsk, BELARUS
Email: yablonski@bsu.by

Department of Mathematics , University of Oslo
Box 1053 Blindern , N-0316 Oslo, Norway ,
Email: alehy@math.uio.no

Abstract

The purpose of this paper is to construct the analog of Malliavin derivative D and Skorohod integral δ for some class of processes which include, in particular, processes with conditionally independent increments. We introduce the family of orthogonal polynomials. By using these polynomials it is proved the chaos decomposition theorem of $L^2(\Omega)$. The definition of Malliavin derivative and Skorohod integral for a certain class of stochastic processes is given and it is shown that they are equal respectively to the annihilation and the creation operators on the Fock space representation of $L^2(\Omega)$. The analogue of Clark-Haussmann-Ocone formula for processes with conditionally independent increments is also established.

Key words and phrases: processes with conditionally independent increments, Malliavin calculus, Skorohod integral, multiple integral, orthogonal polynomials, chaos expansion, Clark-Haussmann-Ocone formula.

1   Introduction

It was shown by Karatzas and Ocone [14] how the stochastic calculus of variations developed by Malliavin [20] can be used in mathematical finance. This discovery led to an increase in the interest in the Malliavin calculus.

In the Brownian setup the calculus of variations has a complete form (see the elegant presentation of Nualart [22]). It is based on the operators D and δ which are called Malliavin derivative and Skorohod integral, respectively. There are two equivalent approaches to definition of the operator D: as a variational derivative and through the chaos decomposition.

For discontinuous processes it is possible to develop the Malliavin-type calculus by using some "generalized" or "weak" derivatives (see, e.g., [1, 4, 15] and references therein).

Nevertheless, it was shown in [24] that in the Poisson case small perturbations of the trajectories lead to a certain difference operator. This idea was extended for Lévy processes in [26, 28, 31, 33].

Alternatively, the operator D can be defined by its action on the chaos representation of L^2-functionals. The case of normal martingales with chaotic representation property was considered in [19]. But, in general, a Lévy process has no chaotic representation property in the sense that Brownian motion and Poisson process do. There are two different chaotic expansions introduced in [11] and [23]. By using these expansions two types of Malliavin operators for some classes of Lévy processes have been studied in the papers [2, 7, 8, 16, 18, 25, 31]. The relationship between them has been shown in [2, 31]. The connection of such derivative to the difference operator from [26, 28] was studied in [18, 31, 33]. For random Lévy measures the Skorohod integral and Malliavin derivative were considered in [5, 6].

The purpose of the paper is to construct the Malliavin calculus for some class of processes which includes, in particular, the processes with conditionally independent increments. It is also proved the chaos representation theorem for such type of processes.

In general the processes with conditionally independent increments can be described in terms of their triplets of characteristics (B, μ, ν), where B represents the "drift" part, μ is connected with continuous martingale part (Gaussian part for Lévy processes) and ν is a compensator of measure associated to the jumps of the original process. Since B is a bounded variation process then for our purposes we can set $B = 0$ without loss of generality. Therefore we start with two random measures μ and ν on certain measurable spaces which describe, respectively, the continuous and discontinuous parts of the process. In Section 2 we define the appropriate Hilbert space H connected to these measures and stochastic process indexed by elements of H. This construction allows us to consider aa rather general class of processes. The system of generalized orthogonal polynomials defined in [33] is used in the proof of the chaos decomposition of L^2 functionals.

Section 3 deals with multiple integrals with respect to the L^2-valued measure generated by the considering process. Their connection to generalized orthogonal polynomials and chaos expansion is also proved.

In Sections 4 and 5 we define the operator D and its adjoint operator δ. Then we show that they are generalizations of the Malliavin derivative and Skorohod integral. It is also proved that their action on the Fock space representation of L^2-functionals coincides with annihilation and creation operators. In the end of the last section we prove the analogue of Clark-Haussmann-Ocone formula for processes with conditionally independent increments.

2 The chaos decomposition

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space. Suppose that μ and ν are random measures defined on the measurable spaces $(T, \mathcal{A})$ and $(T \times X_0, \mathcal{B})$ respectively, such that the following conditions are satisfied:

1. $\mu(A, \cdot)$ and $\nu(B, \cdot)$ are $\mathcal{F}$-measurable for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

2. $\mu(\cdot, \omega)$ and $\nu(\cdot, \omega)$ are σ-finite measures for all $\omega \in \Omega$.

Consider $\Delta \notin X_0$ and denote $X = X_0 \cup \{\Delta\}$, $\mathcal{G} = \sigma(\mathcal{A} \times \{\Delta\}, \mathcal{B})$. Define a new measure $\pi(dtdx) = \mu(dt)\delta_\Delta(dx) + \nu(dtdx \cap (T \times X_0))$ on the σ-algebra $\mathcal{G}$. Here $\delta_\Delta(dx)$ is the measure

which gives mass one to the point Δ. Let $\mathcal{H}$ be a σ-algebra generated by measure π and the collection $\mathcal{N}$ of P-null events of $\mathcal{F}$, i.e.,

$$\mathcal{H} = \sigma\{\pi(A) : A \in \mathcal{G}\} \vee \mathcal{N}.$$

Define a new measure M_π on σ-algebra $\mathcal{G} \otimes \mathcal{F}$ in the following way. For any $A \in \mathcal{G}$ and $B \in \mathcal{F}$ we set $M_\pi(A \times B) = \mathsf{E}[\mathbf{1}_B \pi(A)]$. Then extension of it on the σ-algebra $\mathcal{G} \otimes \mathcal{F}$ can be done as usual (see e.g., [27, Ch. 4]). Suppose that measure M_π is σ-finite. Then there exists the sequence of the sets $U_n \in \mathcal{G} \otimes \mathcal{F}$ such that $\bigcup_{n=1}^\infty U_n = T \times X \times \Omega$ and $M_\pi(U_n) < \infty$. We can choose $U_n = A_n \times B_n$, where $A_n \in \mathcal{G}$ and $\mathcal{B}_n \in \mathcal{F}$. Indeed for any $\epsilon > 0$ and each U_n we can find $A_n^k \in \mathcal{G}$ and $B_n^k \in \mathcal{F}$ such that $U_n \subset \bigcup_{k=1}^\infty A_n^k \times \mathcal{B}_n^k$ and $M_\pi(U_n) \leq \sum_{k=1}^\infty M_\pi(A_n^k \times B_n^k) \leq M_\pi(U_n) + \epsilon$. Hence $M_\pi(A_n^k \times B_n^k) < \infty$ and $T \times X \times \Omega = \bigcup_{k,n=1}^\infty A_n^k \times B_n^k$. Renumeration of the sets $A_n^k \times B_n^k$ yields the desired result.

If we consider the restriction of the measure M_π on the σ-algebra $\mathcal{G} \otimes \mathcal{H}$ than it will be σ-finite too. Indeed, let $U_n = A_n \times B_n$, $A_n \in \mathcal{G}$ and $\mathcal{B}_n \in \mathcal{F}$ be as above then $\bigcup_{n=1}^\infty A_n = T \times X$ and $\bigcup_{n=1}^\infty B_n = \Omega$. Denote by C_n^m the following sets: $C_n^m = \{\omega \in \Omega : \pi(A_n) \leq m\}$, $m = 1, 2, \ldots$. Then $C_n^m \in \mathcal{H}$ and $\bigcup_{m=1}^\infty C_n^m = \{\pi(A_n) < \infty\} \supset (B_n \setminus N_n)$, where $\mathsf{P}(N_n) = 0$. Hence $\bigcup_{n,m=1}^\infty A_n \times (C_n^m \cup N_n) = T \times X \times \Omega$ and $M_\pi(A_n \times (C_n^m \cup N_n)) \leq m < \infty$. In fact we have a stronger property: $\pi(A_n, \omega) < \infty$ for all $\omega \in \bigcup_{m=1}^\infty C_n^m$. This property implies the sigma-finiteness of the π in the sense [27, Ch. 4, Def. 21].

Consider the Hilbert space $H = L^2(T \times X \times \Omega, \mathcal{G} \otimes \mathcal{H}, M_\pi)$ and assume that it is separable. Denote by $\pi(f)$ the integral of f with respect to measure π:

$$\pi(f) = \int_{T \times X} f(t, x) \pi(dt dx).$$

It was shown in [27, Ch. 4] that if $\mathsf{E}[\pi(|f|)] < \infty$ then $\pi(f)$ is $\mathcal{H}$-measurable or $\mathcal{F}$-measurable whenever f is $\mathcal{G} \otimes \mathcal{H}$-measurable or $\mathcal{G} \otimes \mathcal{F}$-measurable respectively. The scalar product and the norm in H will be denoted by $\langle \cdot ; \cdot \rangle_H$ and $||\cdot||_H$ respectively, i.e. for any $f, g \in H$

$$\langle f; g \rangle_H = \mathsf{E}(\pi(hg)) = \mathsf{E} \int_{T \times X} h(t, x) g(t, x) \pi(dt dx), \ ||f||_H^2 = \mathsf{E}(\pi(h^2)).$$

Definition 2.1 *We say that a stochastic process $L = \{L(h), h \in H\}$ is a conditional additive process on H if the following conditions are satisfied.*

1. *For all $h, g \in H$ and $\alpha, \beta \in L^\infty(\Omega, \mathcal{H}, \mathsf{P})$ we have P-a.s.*

$$L(\alpha h + \beta g) = \alpha L(h) + \beta L(g),$$

2. *For all $z \in \mathbb{R}$ and $h \in H$*

$$\mathsf{E}[e^{izL(h)}|\mathcal{H}] = \exp\left(-\frac{1}{2}z^2 \int_T h^2(t, \Delta)\mu(dt) + \int_{T \times X_0} \left(e^{izh(t,x)} - 1 - izh(t,x)\right)\nu(dt dx)\right). \tag{2.1}$$

Remark 2.2 *1. In this definition we suppose that the process $L(h)$ can be defined on the original probability space $(\Omega, \mathcal{F}, \mathsf{P})$. If it is not the case then it is possible to define μ, ν, and $L(h)$ verifying the above conditions on some extension $(\Omega', \mathcal{F}', \mathsf{P}')$ of the original probability space. So we can always assume that original probability space $(\Omega, \mathcal{F}, \mathsf{P})$ is rich enough for defining all necessary objects.*

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2. The definition 2.1 shows that the random variable $L(h)$ has a conditionally infinitely divisible distribution.

3. If measure ν is zero and measure μ is deterministic then L is an isonormal Gaussian process (see, e.g., [22, Def. 1.1.1, p. 4]).

4. If measures μ and ν are deterministic then L is an isonormal Lévy process (see, e.g., [33]).

Example 2.3 *Let L_t, $t \geq 0$ be a càdlàg real-valued process with $\mathcal{H}$-conditionally independent increments on complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$, where $\mathcal{H} \subset \mathcal{F}$. Suppose that L_t is a quasi-left-continuous semimartingale with respect to filtration $\mathcal{F}_t$, $t \geq 0$ generated by the natural filtration and σ-algebra $\mathcal{H}$, i.e. $\mathcal{F}_t = \bigcap_{s>t}(\mathcal{F}_t^0 \vee \mathcal{H})$, where $\mathcal{F}_t^0 = \sigma\{L_s : s \leq t\}$. In this case there exists a version of the characteristics (B, μ, ν) of L_t (see, e.g., [12, 17]) such that:*

1. *B_t, $t \geq 0$ is a continuous process of locally bounded variation with $B_0 = 0$;*

2. *μ_t, $t \geq 0$ is a continuous nondecreasing process with $\mu_0 = 0$;*

3. *$\nu(dtdx, \omega)$ is a predictable random measure defined on the Borel σ-algebra of $\mathbb{R}_+ \times \mathbb{R}_0$, where $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$ such that $\nu(\{t\} \times \mathbb{R}_0) = 0$, $\int_0^t \int_{\mathbb{R}_0} (|x|^2 \wedge 1)\nu(dsdx) < \infty$ for all $t \geq 0$ P a.s.*

Moreover, B, μ and ν are $\mathcal{H}$-measurable and we have for all $z \in \mathbb{R}$ and $s \leq t$

$$\mathsf{E}[\exp(iz(L_t - L_s))|\mathcal{H}]$$

$$= \exp\left[iz(B_t - B_s) - \frac{1}{2}z^2(\mu_t - \mu_s) + \int_s^t \int_{\mathbb{R}_0} (e^{izx} - 1 - izx\mathbf{1}_{|x|\leq 1})\nu(dtdx)\right].$$

Here $\mu_t = \langle L^c; L^c \rangle_t$ is a quadratic variation of the continuous parts of L, ν is a compensator of the random measure $N(dtdx)$ associated to the jumps of L. Hence the following canonical representation holds:

$$L_t = L_0 + L_t^c + \int_0^t \int_{|x|\leq 1} x(N(dsdx) - \nu(dsdx)) + \int_0^t \int_{|x|>1} xN(dsdx) + B_t. \qquad (2.2)$$

We can define L^2-valued measure $L(dtdx)$ with conditionally independent values on the disjoint sets by

$$L(A) = \int_{A(0)} dL_t^c + \iint_{A \setminus A(0)} (N(dtdx) - \nu(dtdx)),$$

where $A \in \mathcal{G}$, $M_\pi(A) < \infty$, $A(0) = A \cap (T \times \{0\})$.

Suppose that $T = \mathbb{R}_+$, $X_0 = \mathbb{R}_0$, $\Delta = 0$. Let $\mathcal{A}$ and $\mathcal{B}$ be the Borel σ-algebras of T and $T \times X_0$ respectively. Let $\mu(dt)$ be the measure on $\mathcal{A}$ generated by process μ_t. Suppose that σ-algebra $\mathcal{H}$ is generated by the measures μ and ν. Construct measure π and Hilbert space $H = L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mathcal{G} \otimes \mathcal{H}, M_\pi)$ as above. Then it is easy to show that for any $h \in H$ the random variable

$$L(h) = \int_0^\infty h(s,0)dL_s^c + \int_0^\infty \int_{\mathbb{R}_0} h(s,x)(N(dsdx) - \nu(dsdx)) = \iint_{\mathbb{R}_+ \times \mathbb{R}} h(t,x)L(dtds)$$

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is well defined and $L(h)$ is a conditional additive process on H.

On the other hand if we have a conditional additive process $L(h)$ on H then the L^2-valued measure $L(dtdx)$ which is given by $L(A) = L(\mathbf{1}_A)$ for all $A \in \mathcal{G}$ with $M_\pi(A) < \infty$ has conditionally independent values on the disjoint sets. In order to express the process L_t in terms of process $L(h)$ we can not write $L_t = L(h_t)$, where $h_t(s,x) = \mathbf{1}_{[0;t]}(s)\mathbf{1}_{\{0\}}(x) + x\mathbf{1}_{[0;t]}(s)$ because, in general, $h_t \notin H$. Therefore we define for any $n \geq 1$ the random variable $\tau_n = \inf\{t > 0 : \mu_t \leq n\}$. Obviously τ_n is an increasing sequence and $h_{t \wedge \tau_n}\mathbf{1}_{\{x=0\}} \in H$ for all $t \geq 0$ and $n \geq 1$. Moreover, the process $L(h_{t \wedge \tau_n}\mathbf{1}_{\{x=0\}})$, $t \geq 0$ has a version with continuous sample paths and $L(h_{t \wedge \tau_n}\mathbf{1}_{\{x=0\}}) = L(h_{t \wedge \tau_m}\mathbf{1}_{\{x=0\}})$ if $t \leq \tau_n$ and $n < m$. Therefore $L^c(t) = \lim_{n \to \infty} L(h_{t \wedge \tau_n}\mathbf{1}_{\{x=0\}})$ well defined continuous process. Furthermore for any set $A \in \mathcal{B}$ such that $M_\pi(A) = \mathsf{E}[\nu(A)] < \infty$ the random variable $N(A) = L(\mathbf{1}_A) + \nu(A)$ is an integer valued random measure with compensator measure ν. And we can define

$$\bar{L}_t = L^c(t) + \int_0^t \int_{|x| \leq 1} x(N(dsdx) - \nu(dsdx)) + \int_0^t \int_{|x| > 1} xN(dsdx). \qquad (2.3)$$

Comparing equalities (2.2) and (2.3) we deduce that the only characteristics which cannot be determined from $L(h)$ is the drift process B and initial value $L(0)$.

Lemma 2.4 *Let $\mathcal{N}$ be a collection of P-null events of $\mathcal{F}$. Then*

$$\mathcal{H} \subset \mathcal{F}_L = \sigma\{L(h) : h \in H\} \vee \mathcal{N}.$$

PROOF. Let $U \in \mathcal{H}$ be an arbitrary set. We have to show that $\xi = \mathbf{1}_U$ is $\mathcal{F}_L$ measurable. Since the measure M_π is σ-finite then there exists a sequence of pairwise-disjoint sets $A_n \times B_n$, $A_n \in \mathcal{G}$ and $B_n \in \mathcal{H}$ such that $\bigcup_{n=1}^\infty A_n \times B_n = T \times X \times \Omega$ and $M_\pi(A_n \times B_n) < \infty$. Therefore $\mathbf{1}_{A_n \times B_n} \in H$. Denote by A_n^0 and A_n^Δ the intersections of A_n with $T \times X_0$ and $T \times \{\Delta\}$ respectively. Then $\bigcup_{n=1}^\infty [(A_n^0 \times B_n) \cup (A_n^\Delta \times B_n)] = T \times X \times \Omega$. Definition 2.1 of L implies $\xi L(h) = L(\xi h)$ for any $h \in H$ and $\xi L(h)^2 = L(\xi h)^2$ since ξ is an indicator function. Then

$$\xi \sum_{k=1}^n (L(\mathbf{1}_{A_k^0 \times \mathcal{B}_k})^2 + L(\mathbf{1}_{A_k^\Delta \times \mathcal{B}_k})^2) = \sum_{k=1}^n (L(\xi \mathbf{1}_{A_k^0 \times \mathcal{B}_k})^2 + L(\xi \mathbf{1}_{A_k^\Delta \times \mathcal{B}_k})^2)$$

and

$$\xi(1 - \mathbf{1}_{V_n}) = \frac{\sum_{k=1}^n (L(\xi \mathbf{1}_{A_k^0 \times \mathcal{B}_k})^2 + L(\xi \mathbf{1}_{A_k^\Delta \times \mathcal{B}_k})^2)}{\sum_{k=1}^n (L(\mathbf{1}_{A_k^0 \times \mathcal{B}_k})^2 + L(\mathbf{1}_{A_k^\Delta \times \mathcal{B}_k})^2)}, \qquad (2.4)$$

where $V_n = \bigcap_{k=1}^n (\{L(\mathbf{1}_{A_k^0 \times \mathcal{B}_k}) = 0\} \cap \{L(\mathbf{1}_{A_k^\Delta \times \mathcal{B}_k}) = 0\})$. Therefore $V_n \in \mathcal{F}_L$ and $\xi(1 - \mathbf{1}_{V_n})$ is $\mathcal{F}_L$-measurable for all $n \geq 1$.

Let us find the limit of $\mathbf{1}_{V_n}$ as $n \to \infty$. From Definition 2.1 we have that $L(\mathbf{1}_A)$ and $L(\mathbf{1}_B)$ are conditionally independent if $\mathbf{1}_A \mathbf{1}_B = 0$. Moreover $L(\mathbf{1}_{A_k^\Delta \times \mathcal{B}_k})$ has a conditionally Gaussian distribution with mean 0 and variation $\pi(A_k^\Delta \times \mathcal{B}_k)$, and $L(\mathbf{1}_{A_k^0 \times \mathcal{B}_k}) + \pi(A_k^0 \times \mathcal{B}_k)$ has a conditionally Poisson distribution with parameter $\pi(A_k^0 \times \mathcal{B}_k)$. Hence

$$\mathsf{P}[L(\mathbf{1}_{A_k^\Delta \times \mathcal{B}_k}) = 0 | \mathcal{H}] = \mathbf{1}_{\{\pi(A_k^\Delta \times \mathcal{B}_k) = 0\}}$$

and

$$\mathsf{P}[L(\mathbf{1}_{A_k^0 \times \mathcal{B}_k}) = 0 | \mathcal{H}] = \phi(\pi(A_k^0 \times \mathcal{B}_k)),$$

where $\phi(x) = \mathbf{1}_{\{0\}}(x) + \sum_{m=1}^{\infty} \mathbf{1}_{\{m\}}(x)e^{-m}m^m/m!$. Therefore we get

$$\mathsf{P}[V_n|\mathcal{H}] = \prod_{k=1}^{n} \mathsf{P}[L(\mathbf{1}_{A_k^{\triangle} \times \mathcal{B}_k}) = 0|\mathcal{H}] \prod_{k=1}^{n} \mathsf{P}[L(\mathbf{1}_{A_k^0 \times \mathcal{B}_k}) = 0|\mathcal{H}]$$

$$= \prod_{k=1}^{n} \mathbf{1}_{\{\pi(A_k^{\triangle} \times \mathcal{B}_k)=0\}} \prod_{k=1}^{n} \phi(\pi(A_k^0 \times \mathcal{B}_k)). \tag{2.5}$$

By using Stirling's formula we have

$$m! = \sqrt{2\pi m}\, m^m e^{-m} e^{\frac{\theta_m}{12m}},$$

where $\theta_m \in (0;1)$, $m = 1, 2, \ldots$. Then we can deduce that $\prod_{k=1}^{n} e^{-m_k}m_k^{m_k}/m_k! \leq 1/(2\pi)^{n/2}$. Hence

$$\prod_{k=1}^{n} \phi(m_k) = \prod_{k=1}^{n} \mathbf{1}_{\{0\}}(m_k) + \prod_{k=1}^{n} \mathbf{1}_{\mathbb{N}}(m_k)e^{-m_k}m_k^{m_k}/m_k! \to \prod_{k=1}^{\infty} \mathbf{1}_{\{0\}}(m_k),$$

as $n \to \infty$. This expression and formula (2.5) imply that

$$\mathsf{P}[V_n|\mathcal{H}] \to \prod_{k=1}^{n} \mathbf{1}_{\{\pi(A_k^{\triangle} \times \mathcal{B}_k)=0\}} \prod_{k=1}^{n} \mathbf{1}_{\{\pi(A_k^0 \times \mathcal{B}_k)=0\}} = \mathbf{1}_{\{\pi(T \times X \times \Omega)=0\}},$$

P a.s. as $n \to \infty$. Thus we get $\mathsf{E}((\mathbf{1}_{V_n} - \mathbf{1}_{\{\pi(T \times X \times \Omega)=0\}})^2) \to 0$ as $n \to \infty$. Since $\mathbf{1}_{V_n}$ is a monotone then $\mathbf{1}_{V_n} \to \mathbf{1}_{\{\pi(T \times X \times \Omega)=0\}}$ a.s. and therefore $\mathbf{1}_{\{\pi(T \times X \times \Omega)=0\}}$ is $\mathcal{F}_L$-measurable. Letting $n \to \infty$ in (2.4) we obtain that $\xi(1 - \mathbf{1}_{\{\pi(T \times X \times \Omega)=0\}})$ is $\mathcal{F}_L$-measurable. It is easy to show that any $\mathcal{H}$ measurable random variable is equal to some constant $c \in \mathbb{R}$ on the set $\{\pi(T \times X \times \Omega) = 0\}$ a.s. Therefore $\xi\mathbf{1}_{\{\pi(T \times X \times \Omega)=0\}} = c\mathbf{1}_{\{\pi(T \times X \times \Omega)=0\}}$ is $\mathcal{F}_L$-measurable which implies the $\mathcal{F}_L$-measure-ability of ξ. $\square$

In what follows we will always assume that $\mathcal{F}$ is a completion of $\mathcal{F}_L = \sigma\{L(h), h \in H\}$. Denote by K the following subset of H:

$$K = \{h \in H : h\mathbf{1}_{X_0} \in L^{\infty}(T \times X \times \Omega, \mathcal{G} \otimes \mathcal{H}, M_{\pi}), \ \pi(h^2) \in L^{\infty}(\Omega, \mathcal{H}, \mathsf{P})\} \tag{2.6}$$

The elements of K satisfy the following properties.

Lemma 2.5 *Suppose that $h \in K$ then*

1. *$|h| \in K$ and $zh \in K$ for all $z \in \mathbb{R}$.*

2. *$\mathsf{E}[\exp(\pi(h^2)/2)] < \infty$, $\mathsf{E}[\pi(|h|^k\mathbf{1}_{X_0})] < \infty$ and $\mathsf{E}[\pi(h^2)^k] < \infty$ for all $k \geq 2$.*

3. *$\mathsf{E}\left[\exp\left(\frac{1}{2}\int_T h^2(t,\Delta)\mu(dt) + \int_{T \times X_0}\left(e^{h(t,x)} - 1 - h(t,x)\right)\nu(dtdx)\right)\right] < \infty$.*

PROOF. The first two properties are evident. The proof of the last statement is based on the inequality $e^x - 1 - x \leq e^x x^2/2$, which can be proved by using Taylor's formula. We omit the details. $\square$

Lemma 2.6 *The set K is dense in H.*

PROOF.   Choose a $h \in H$. Then $\pi(h^2) < \infty$ a.s. Consider two sequences of sets $B_m = \{\pi(h^2) \leq m\} \in \mathcal{H}$ and $C_k = \{(t, x, \omega) : |h(t, x, \omega)| \leq k\} \in \mathcal{G} \otimes \mathcal{H}$. Denote $h_{k,m} = h\mathbf{1}_{C_k}\mathbf{1}_{B_m}$ It is evident that $h_{k,m} \in K$ for all integers k, $m \geq 1$. By dominated convergence theorem we have $||h_{k,m} - h\mathbf{1}_{B_m}||_H \to 0$ as $k \to \infty$ and $||h\mathbf{1}_{B_m} - h||_H = \mathsf{E}[\pi(h^2)\mathbf{1}_{\Omega \backslash B_m}] \to 0$ as $m \to \infty$ which completes the proof of the lemma.   □

The following lemma describes some properties of $L(h)$.

Lemma 2.7   *1. If $h \in H$ then $L(h) \in L^2(\Omega, \mathcal{F}, \mathsf{P})$. Furthermore $\mathsf{E}[L(h)|\mathcal{H}] = 0$ and $\mathsf{E}[L(h)L(g)|\mathcal{H}] = \pi(hg)$.*

2. Let $h_n \in H$ be a sequence such that $||h_n - h||_H \to 0$ as $n \to \infty$ for some $h \in H$. Then $\mathsf{E}[(L(h_n) - L(h))^2|\mathcal{H}] \to 0$ as $n \to \infty$ in $L^1(\Omega, \mathcal{F}, \mathsf{P})$.

3. If $h \in K$ then $L(h) \in L^p(\Omega, \mathcal{F}, \mathsf{P})$ for all $p \geq 1$, $\mathsf{E}[\exp(c|L(h)|)] < \infty$ for all $c \in \mathbb{R}$, and

$$\mathsf{E}[\exp(L(h))|\mathcal{H}] = \exp\left(\frac{1}{2}\int_T h^2(t, \Delta)\mu(dt) + \int_{T \times X_0} \left(e^{h(t,x)} - 1 - h(t, x)\right)\nu(dtdx)\right).$$

4. Let $h_1, \ldots h_n \in K$ be such that $h_i h_j = 0$ M_π-a.s. if $i \neq j$. Then for any integers $p_1, \ldots, p_n \geq 1$ we have

$$\mathsf{E}[L(h_1)^{p_1} \cdots L(h_n)^{p_n}|\mathcal{H}] = \mathsf{E}[L(h_1)^{p_1}|\mathcal{H}] \cdots \mathsf{E}[L(h_n)^{p_n}|\mathcal{H}].$$

PROOF.
1. Choose a $B \in \mathcal{H}$, $\mathsf{P}(B) > 0$. Denote by $f_B(z)$ the characteristic function of the random variable $L(h)$ with respect to restriction of probability on the set B, i.e., $f_B(z) = \mathsf{E}[\mathbf{1}_B e^{izL(h)}]$. Equality (2.1) implies that $f_B(z)$ can be written in the following form:

$$f_B(z) = \mathsf{E}\left[\mathbf{1}_B \exp\left(-\frac{z^2}{2}\int_T h^2(t, \Delta)\mu(dt) + \int_{T \times X_0} \left(e^{izh(t,x)} - 1 - izh(t, x)\right)\nu(dtdx)\right)\right].$$

By using this equality it is easy to show that $f_B(z)$ two times differentiable function. Hence $L(h) \in L^2(\Omega, \mathcal{F}, \mathsf{P})$. Taking the first and second derivatives at $z = 0$ in both sides of the equality above yields $\mathsf{E}[L(h)\mathbf{1}_B] = 0$ and $\mathsf{E}[L(h)^2\mathbf{1}_B] = \mathsf{E}[\pi(h^2)\mathbf{1}_B]$. Then $\mathsf{E}[L(h)|\mathcal{H}] = 0$, $\mathsf{E}[L(h)^2|\mathcal{H}] = \pi(h^2)$ and $\mathsf{E}[L(h)L(g)|\mathcal{H}] = \mathsf{E}[L(h + g)^2 - L(h - g)^2|\mathcal{H}]/4 = \pi(hg)$ which completes the proof of the first statement of the lemma.
2. Suppose that $h_n \to h$ as $n \to \infty$ in H. It means that $\mathsf{E}(\pi((h_n - h)^2)) \to 0$ as $n \to \infty$. Then from the first part of the lemma we have $\mathsf{E}[(L(h_n) - L(h))^2|\mathcal{H}] = \pi((h_n - h)^2)$ which implies the proof of second statement of the lemma.
3. Denote by $u(z)$ the following expression:

$$u(z) = \mathsf{E}\left[\exp\left(\frac{z^2}{2}\int_T h^2(t, \Delta)\mu(dt) + \int_{T \times X_0} \left(e^{zh(t,x)} - 1 - zh(t, x)\right)\nu(dtdx)\right)\right].$$

Since $h \in K$ then Lemma 2.5 implies that $u(z)$ is finite for all $z \in \mathbb{R}$. The right hand side in the equality above is meaningful even z is complex. Indeed if $z = a + ib$ then

$\mathrm{Re}\,(e^{zh} - 1 - zh) = e^{ah}\cos(bh) - 1 - ah = (e^{ah} - 1 - ah) + e^{ah}(\cos(bh) - 1) \leq (e^{ah} - 1 - ah)$.
Hence from Lemma 2.5 we have

$$\left| \mathsf{E}\left[\exp\left(\frac{z^2}{2} \int_T h^2(t,\Delta)\mu(dt) + \int_{T \times X_0} \left(e^{zh(t,x)} - 1 - zh(t,x) \right) \nu(dtdx) \right) \right] \right|$$

$$\leq \mathsf{E}\left[\exp\left(\frac{a^2}{2} \int_T h^2(t,\Delta)\mu(dt) + \int_{T \times X_0} \left(e^{ah(t,x)} - 1 - ah(t,x) \right) \nu(dtdx) \right) \right] < \infty.$$

The function $u(z)$ is analytic function for all $z \in \mathbb{C}$. If $z = it$, $t \in \mathbb{R}$ then $u(z) = f(t) = \mathsf{E}[e^{itL(h)}]$ coincides with characteristic function of $L(h)$. Hence characteristic function $f(t)$ infinitely differentiable for all $t \in \mathbb{R}$ and $L(h)$ has finite moments of all orders, i.e. $L(h) \in L^p(\Omega, \mathcal{F}, \mathsf{P})$ for all $p \geq 1$. Moreover

$$u(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \mathsf{E}[L(h)^k],$$

where the radius of convergence of the series being infinite. It follows that

$$\mathsf{E}[\exp(c|L(h)|)] = \sum_{k=0}^{\infty} \frac{c^k}{k!} \mathsf{E}[|L(h)|^k] < \infty$$

for all $c \in \mathbb{R}$. Hence $f(z) = \mathsf{E}[\exp(zL(h))]$ is analytic for all complex z. The uniqueness theorem yields $u(z) = f(z)$ which implies the third statement of the lemma.

4. From the previous parts of the lemma we have $L(h_k) \in L^p(\Omega, \mathcal{F}, \mathsf{P})$ for all $p \geq 1$ and conditional characteristic function of random variables $L(h_k)$ is infinitely differentiable if $h_1, \ldots h_n \in K$. Since $h_i h_j = 0$ M_π-a.s. if $i \neq j$ then we have the following equality:

$$\mathsf{E}\left[\exp\left(i \sum_{k=1}^{n} z_k L(h_k) \right) \middle| \mathcal{H} \right]$$

$$= \exp\left(\frac{1}{2} \sum_{k=1}^{n} z_k^2 \int_T h_k^2(t,\Delta)\mu(dt) + \sum_{k=1}^{n} \int_{T \times X_0} \left(e^{iz_k h_k(t,x)} - 1 - iz_k h_k(t,x) \right) \nu(dtdx) \right)$$

$$= \mathsf{E}\left[\exp\left(iz_1 L(h_1) \right) \middle| \mathcal{H} \right] \cdots \mathsf{E}\left[\exp\left(iz_n L(h_n) \right) \middle| \mathcal{H} \right].$$

Taking the $(p_1 + p_2 + \cdots + p_n)$th partial derivative $\frac{\partial^{p_1 + p_2 + \cdots + p_n}}{\partial z_1^{p_1} \cdots \partial z_n^{p_n}}$ at $z_1 = z_2 = \cdots = 0$ in both sides of the above equality yields

$$\mathsf{E}[L(h_1)^{p_1} \cdots L(h_n)^{p_n} | \mathcal{H}] = \mathsf{E}[L(h_1)^{p_1} | \mathcal{H}] \cdots \mathsf{E}[L(h_n)^{p_n} | \mathcal{H}].$$

$\square$

Now we will introduce the generalized orthogonal polynomials P_n (see, e.g. [33]). Denote by $\overline{x} = (x_1, x_2, \ldots, x_n, \ldots)$ a sequence of real numbers.

Define a function $F(z, \overline{x})$ by

$$F(z, \overline{x}) = \exp\left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{z^k}{k} x_k \right). \tag{2.7}$$

If $R(\overline{x}) = (\limsup |x_k|^{1/k})^{-1} > 0$ then the series in (2.7) converges for all $|z| < R(\overline{x})$. So the function $F(z, \overline{x})$ is analytic for $|z| < R(\overline{x})$.

Consider an expansion in powers of z of the function $F(z, \overline{x})$

$$F(z, \overline{x}) = \sum_{n=0}^{\infty} z^n P_n(\overline{x}).$$

Using this development, one can easily show the following equalities:

$$(n+1)P_{n+1}(\overline{x}) = \sum_{k=0}^{n} (-1)^k x_{k+1} P_{n-k}(\overline{x}), \quad n \geq 0, \qquad (2.8)$$

$$\frac{\partial}{\partial x_l} P_n(\overline{x}) = \begin{cases} 0, & \text{if } l > n, \\ (-1)^{l+1} \frac{1}{l} P_{n-l}(\overline{x}), & \text{if } l \leq n. \end{cases} \qquad (2.9)$$

Indeed, (2.8) and (2.9) follow from $\frac{\partial F}{\partial z} = \sum_{k=0}^{\infty}(-1)^k z^k x_{k+1} F$, respectively, and $\frac{\partial F}{\partial x_l} = (-1)^{l+1} \frac{F}{l} z^l$. From (2.9) it follows that P_n depends only on finite number of variables, namely $x_1, x_2, \ldots, x_n$. Since $P_0 \equiv 1$, then (2.8) implies that $P_n(x_1, x_2, \ldots, x_n)$ is a polynomial with the highest order term $\frac{x_1^n}{n!}$. The first polynomials are $P_1(x_1) = x_1$ and $P_2(x_1, x_2) = \frac{1}{2}(x_1^2 - x_2)$.

Using the equality $F(z, \overline{x} + \overline{y}) = F(z, \overline{x}) F(z, \overline{y})$, where $\overline{y} = (y_1, y_2, \ldots, y_n, \ldots)$ and $\overline{x} + \overline{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n, \ldots)$ it is easy to show that

$$P_n(\overline{x} + \overline{y}) = \sum_{k=0}^{n} P_k(\overline{x}) P_{n-k}(\overline{y}). \qquad (2.10)$$

If $\overline{u}(y) = (y, y^2, y^3, \ldots, y^n, \ldots)$ then $F(z, \overline{u}(y)) = 1 + zy$ for $|zy| < 1$. Hence $P_1(\overline{u}(y)) = y$ and $P_n(\overline{u}(y)) = 0$ for all $n \geq 2$. Furthermore, equation (2.10) implies that

$$P_n(\overline{x} + \overline{u}(y)) - P_n(\overline{x}) = y P_{n-1}(\overline{x}). \qquad (2.11)$$

It is possible to find the explicit formula for polynomials P_n. Indeed, P_n can be written in the following form:

$$P_n(x_1, x_2, \ldots x_n) = \sum_{i_1 + i_2 + \cdots + i_n \leq n} a_{i_1, i_2, \ldots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}.$$

It is easy to see that

$$\frac{\partial^{i_1 + i_2 + \cdots + i_n} P_n}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}} = i_1! i_2! \cdots i_n! a_{i_1, i_2, \ldots i_n}.$$

It follows from the equality (2.9) that

$$\frac{\partial^{i_1 + i_2 + \cdots + i_n} P_n}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}} = \begin{cases} 0, & \text{if } i_1 + 2i_2 + 3i_3 \cdots + n i_n \neq n, \\ (-1)^{n + i_1 + i_2 + \cdots + i_n} 2^{-i_2} 3^{-i_3} \cdots n^{-i_n}, & \text{if } i_1 + 2i_2 + 3i_3 \cdots + n i_n = n. \end{cases}$$

Hence

$$P_n(x_1, x_2, \ldots x_n) = \sum_{i_1 + 2i_2 + 3i_3 \cdots + n i_n = n} (-1)^{n + i_1 + i_2 + \cdots + i_n} \frac{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}}{i_1! i_2! \cdots i_n! 2^{i_2} 3^{i_3} \cdots n^{i_n}}. \qquad (2.12)$$

For $h \in K$ let $\overline{x}(h) = (x_1(h), x_2(h), \ldots x_n(h), \ldots)$ denote the sequence of the random variables, such that $x_1(h) = L(h)$, $x_2(h) = L(h^2 \mathbf{1}_{X_0}) + \int_{T \times X} h^k(t, x) \pi(dtdx) = L(h^2 \mathbf{1}_{X_0}) + \pi(h^2)$, $x_k(h) = L(h^k \mathbf{1}_{X_0}) + \int_{T \times X_0} h^k(t, x) \nu(dtdx) = L(h^k \mathbf{1}_{X_0}) + \pi(h^k \mathbf{1}_{X_0})$, $k = 3, 4, \ldots$.

The relationship between generalized orthogonal polynomials and conditional additive processes on H is given by the following result.

Lemma 2.8 *Let h and $g \in K$. Then for all $n, m \geq 0$ we have $P_n(\overline{x}(h))$ and $P_m(\overline{x}(g)) \in L^2(\Omega)$, and*

$$\mathsf{E}[P_n(\overline{x}(h))P_m(\overline{x}(g))|\mathcal{H}] = \begin{cases} 0, & \text{if } n \neq m, \\ \frac{1}{n!}\left(\mathsf{E}[L(h)L(g)|\mathcal{H}]\right)^n, & \text{if } n = m. \end{cases}$$

PROOF.   Since $h, g \in K$ and P_n, P_m are the polynomials, then by Lemma 2.7 we have $P_n(\overline{x}(h))$ and $P_m(\overline{x}(g)) \in L^2(\Omega, \mathcal{F}, \mathsf{P})$.

Denote by $\phi(z, \overline{x})$ the power of the exponent in the formula (2.7):

$$\phi(z, \overline{x}) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{z^k}{k}x_k.$$

Since

$$\frac{1}{R} = \limsup_{k\to\infty} ||x_k(h)||_{L^2(\Omega)}^{1/k} = \lim_{k\to\infty}\left[\mathsf{E}(L(h^k\mathbf{1}_{X_0})^2) + \mathsf{E}(\pi(h^k\mathbf{1}_{X_0})^2)\right]^{1/2k}$$

$$\leq \lim_{k\to\infty}\left(||h\mathbf{1}_{X_0}||_{L^\infty}^{2k-2}\mathsf{E}(\pi(h^2\mathbf{1}_{X_0})) + ||h\mathbf{1}_{X_0}||_{L^\infty}^{2k-4}\mathsf{E}(\pi(h^2\mathbf{1}_{X_0})^2)\right)^{1/2k} \leq ||h\mathbf{1}_{X_0}||_{L^\infty}.$$

Then the series

$$\sum_{k=1}^{\infty}\frac{|z|^k}{k}||x_k(h)||_{L^2(\Omega)}$$

converges if $|z| < 1/||h\mathbf{1}_{X_0}||_{L^\infty} \leq R$, which implies that $\phi(z, \overline{x}(h)) \in L^2(\Omega)$ for all $|z| < 1/||h\mathbf{1}_{X_0}||_{L^\infty}$.

Let's note that for all $|z| < 1/||h\mathbf{1}_{X_0}||_{L^\infty}$ we have $\ln(1 + zh\mathbf{1}_{X_0}) \in H$. Indeed, by using Taylor's formula, we get

$$(\ln(1 + zh\mathbf{1}_{X_0}))^2 \leq \frac{z^2h^2\mathbf{1}_{X_0}}{(1 - |z|\,||h\mathbf{1}_{X_0}||_{L^\infty})^2}.$$

In the same way one can obtain the following inequality

$$|\ln(1 + zh\mathbf{1}_{X_0}) - zh\mathbf{1}_{X_0}| \leq \frac{z^2h^2\mathbf{1}_{X_0}}{2(1 - |z|\,||h\mathbf{1}_{X_0}||_{L^\infty})^2},$$

which implies that $\ln(1 + zh(t,x)\mathbf{1}_{X_0}) - zh(t,x)\mathbf{1}_{X_0}$ is integrable with respect to measure M_π for all $|z| < 1/||h\mathbf{1}_{X_0}||_{L^\infty}$.

So by using the linearity and the continuity of the mapping $h \to L(h)$ we have for all $|z| < 1/||h\mathbf{1}_{X_0}||_{L^\infty}$

$$\phi(z, \overline{x}(h)) = \sum_{k=2}^{\infty}(-1)^{k+1}\frac{z^k}{k}\left(L(h^k\mathbf{1}_{X_0}) + \pi(h^k\mathbf{1}_{X_0})\right) + zL(h) - \frac{z^2}{2}\pi(h\mathbf{1}_\Delta)$$

$$= L\left(\ln(1 + zh\mathbf{1}_{X_0}) + zh\mathbf{1}_\Delta\right) + \int_{T\times X_0}(\ln(1 + zh(t,x)) - zh(t,x))\nu(dtdx) - \frac{z^2}{2}\pi(h\mathbf{1}_\Delta). \quad (2.13)$$

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We claim that $w = \ln(1 + zh\mathbf{1}_{X_0}) + zh\mathbf{1}_\Delta \in K$ for all $|z| < 1/\|h\mathbf{1}_{X_0}\|_{L^\infty}$. Indeed

$$|w\mathbf{1}_{X_0}| = |\ln(1 + zh\mathbf{1}_{X_0})| \le \frac{|z|\|h\mathbf{1}_{X_0}\|_{L^\infty}}{|1 - |z|\|h\mathbf{1}_{X_0}\|_{L^\infty}|}$$

and since $h \in K$ then we have fo some constant $C_1 > 0$

$$\pi(w^2) = z^2\pi(h^2\mathbf{1}_\Delta) + \pi(\ln^2(1 + zh\mathbf{1}_{X_0})) \le z^2\pi(h^2\mathbf{1}_\Delta) + \frac{z^2\pi(h^2\mathbf{1}_{X_0})}{(1 - |z|\|h\mathbf{1}_{X_0}\|_{L^\infty})^2} \le C_1.$$

Hence from inequality $\ln(1 + x) \le x$, equality (2.13) and Lemma 2.7 we have for all $|z| < 1/\|h\mathbf{1}_{X_0}\|_{L^\infty}$

$$\mathsf{E}(F(z, \overline{x}(h))^2) = \mathsf{E}[\exp(2\phi(z, \overline{x}(h)))] \le \mathsf{E}[\exp(2L(\ln(1 + zh\mathbf{1}_{X_0}) + zh\mathbf{1}_\Delta))] < \infty.$$

So $F(z, \overline{x}(h)) \in L^2(\Omega)$ if $|z| < 1/\|h\mathbf{1}_{X_0}\|_{L^\infty}$.

Consequently for $|z| < 1/\|h\mathbf{1}_{X_0}\|_{L^\infty}$ and $|y| < 1/\|g\mathbf{1}_{X_0}\|_{L^\infty}$ we get from (2.13)

$$\mathsf{E}[F(z, \overline{x}(h))F(y, \overline{x}(g))|\mathcal{H}] = \mathsf{E}[\exp(\phi(z, \overline{x}(h)) + \phi(y, \overline{x}(g)))|\mathcal{H}]$$

$$= \mathsf{E}[\exp(L(\ln[(1 + zh\mathbf{1}_{X_0})(1 + yg\mathbf{1}_{X_0})])$$

$$+ \int_{T \times X_0} (\ln[(1 + zh(t,x))(1 + yg(t,x))] - zh(t,x) - yg(t,x))\nu(dtdx)$$

$$+ L(zh\mathbf{1}_\Delta + yg\mathbf{1}_\Delta) - \frac{1}{2}\int_T (z^2h^2(t,\Delta) + y^2g^2(t,\Delta))\mu(dt))|\mathcal{H}]$$

$$= \exp(\int_{T \times X_0} (e^{\ln[(1 + zh(t,x))(1 + yg(t,x))]} - 1 - \ln[(1 + zh(t,x))(1 + yg(t,x))])\nu(dtdx)$$

$$+ \int_{T \times X_0} (\ln[(1 + zh(t,x))(1 + yg(t,x))] - zh(t,x) - yg(t,x))\nu(dtdx)$$

$$+ \frac{1}{2}\int_T ((zh(t,\Delta) + yg(t,\Delta))^2 - z^2h^2(t,\Delta) - y^2g^2(t,\Delta))\mu(dt))$$

$$= \exp(zy\int_{T \times X} h(t,x)g(t,x)\pi(dtdx)) = \exp(zy\mathsf{E}[L(h)L(g)|\mathcal{H}]),$$

where we have used Lemma 2.7 to calculate the conditional expectation.

Taking the $(n + m)$-th partial derivative $\frac{\partial^{n+m}}{\partial z^n \partial y^m}$ at $z = y = 0$ in both sides of the above equality yields

$$\mathsf{E}[n!m!P_n(\overline{x}(h))P_m(\overline{x}(g))|\mathcal{H}] = \begin{cases} 0, & \text{if } n \ne m, \\ n!\,(\mathsf{E}[L(h)L(g)|\mathcal{H}])^n, & \text{if } n = m. \end{cases}$$

$\square$

Lemma 2.9 *The random variables $\{e^{L(h)},\ h \in K\}$ form a total subset of $L^2(\Omega, \mathcal{F}, P)$.*

PROOF. It follows from Lemma 2.7 that $e^{L(h)} \in L^2(\Omega)$ if $h \in K$.

Let $\xi \in L^2(\Omega)$ be such that $\mathsf{E}(\xi e^{L(h)}) = 0$ for all $h \in K$. The linearity of the mapping $h \to L(h)$ implies

$$\mathsf{E}\left(\xi \exp \sum_{k=1}^{n} z_k L(h_k)\right) = 0 \tag{2.14}$$

for any $z_1, \ldots, z_n \in \mathbb{R}$, $h_1, \ldots, h_n \in K$, $n \geq 1$. Suppose that $n \geq 1$ and $h_1, \ldots, h_n \in K$ are fixed. Then (2.14) says that Laplace transform of the signed measure

$$\tau(B) = \mathsf{E}(\xi \mathbf{1}_B(L(h_1), \ldots, L(h_n))),$$

where B is a Borel subset of $\mathbb{R}^n$, is identically zero on $\mathbb{R}^n$. Consequently, this measure is zero, which implies $\mathsf{E}(\xi \mathbf{1}_G) = 0$ for any $G \in \mathcal{F}$. So $\xi = 0$, completing the proof of the lemma. $\square$

For each $n \geq 0$ we will denote by $\mathcal{P}_n$ the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $\{\xi P_n(\overline{x}(h)) : h \in K, \ \xi \in L^\infty(\Omega, \mathcal{H}, \mathsf{P})\}$. $\mathcal{P}_0$ will be the set $L^2(\Omega, \mathcal{H}, P)$ of $\mathcal{H}$-measurable square integrable random variables. For $n = 1$, $\mathcal{P}_1$ coincides with the set of random variables $\{L(h) : h \in H\}$. From Lemma 2.8 we obtain that $\mathcal{P}_n$ and $\mathcal{P}_m$ are orthogonal whenever $n \neq m$. We will call the space $\mathcal{P}_n$ chaos of order n.

Theorem 2.10 *The space $L^2(\Omega, \mathcal{F}, P)$ can be decomposed into the infinite orthogonal sum of the subspaces $\mathcal{P}_n$:*

$$L^2(\Omega, \mathcal{F}, P) = \bigoplus_{n=0}^{\infty} \mathcal{P}_n.$$

PROOF. Let $\xi \in L^2(\Omega, \mathcal{F}, P)$ such that ξ is orthogonal to all $\mathcal{P}_n$, $n \geq 0$. We have to show that $\xi = 0$. For all $h \in K$ and $\eta \in L^\infty(\Omega, \mathcal{H}, P)$ we get $\mathsf{E}(\xi \eta P_n(\overline{x}(h))) = 0$. Hence $\mathsf{E}[\xi P_n(\overline{x}(h))|\mathcal{H}] = 0$. Since from the proof of Lemma 2.8 we have that $F(z, \overline{x}(h)) \in L^2(\Omega)$ for all $z < 1/\|h\mathbf{1}_{X_0}\|_{L^\infty}$, then $\mathsf{E}[\xi F(z, \overline{x}(h))|\mathcal{H}] = 0$ for $z < 1/\|h\mathbf{1}_{X_0}\|_{L^\infty}$. Using equality (2.13) we obtain

$$0 = \mathsf{E}[\xi F(z, \overline{x}(h))|\mathcal{H}] = \mathsf{E}[\xi e^{\phi(z,\overline{x}(h))}|\mathcal{H}] = \mathsf{E}[\xi \exp(L(\ln(1 + zh\mathbf{1}_{X_0}))$$

$$+ \int_{T \times X_0} (\ln(1 + zh(t,x)) - zh(t,x))\nu(dtdx) + L(zh\mathbf{1}_\Delta) - \frac{1}{2}\int_T z^2 h^2(t,\Delta)\mu(dt))|\mathcal{H}].$$

Thus for any $z < 1/\|h\mathbf{1}_{X_0}\|_{L^\infty}$

$$\mathsf{E}[\xi \exp(L(\ln(1 + zh\mathbf{1}_{X_0})) + L(zh\mathbf{1}_\Delta))] = 0. \tag{2.15}$$

We claim that if $h \in K$ such that $h \geq \varepsilon - 1$ M_π-a.e. for some $1 \geq \varepsilon > 0$ then equality (2.15) holds for $z = 1$. Indeed the right-hand side of the expression (2.15) meaningful in this case for all $z \in [0; 1]$. The extension to the complex numbers $\operatorname{Re} z \in [0; 1]$ is evident. Denote this function by $\Phi(z)$. Since $|h\mathbf{1}_{X_0}/(1 + zh\mathbf{1}_{X_0})| \leq \|h\mathbf{1}_{X_0}\|_{L^\infty}/\varepsilon$ and $\pi(h^2\mathbf{1}_{X_0}/(1 + zh\mathbf{1}_{X_0})^2) \leq \pi(h^2)/\varepsilon^2$ then $h\mathbf{1}_{X_0}/(1 + zh\mathbf{1}_{X_0}) \in K$ and the straightforward calculation shows that $\Phi(z)$ is differentiable and

$$\Phi'(z) = \mathsf{E}[\xi L(h\mathbf{1}_{X_0}/(1 + zh\mathbf{1}_{X_0}) + h\mathbf{1}_\Delta) \exp(L(\ln(1 + zh\mathbf{1}_{X_0})) + L(zh\mathbf{1}_\Delta))].$$

Hence $\Phi(z)$ is an analytical function for $\mathrm{Re}\, z \in [0; 1]$. Consequently the uniqueness theorem yields the desired statement.

For any $g \in K$ we have $(e^g - 1) \in K$ and $(e^g - 1)\mathbf{1}_{X_0} > -1 + \varepsilon$ M_π-a.e. for some $1 \geq \varepsilon > 0$. Putting in (2.15) $h = (e^g - 1)\mathbf{1}_{X_0} + g\mathbf{1}_\Delta$ and $z = 1$ we deduce that $\mathsf{E}(\xi e^{L(g)}) = 0$ for all $g \in K$. By Lemma 2.9 we get $\xi = 0$, which completes the proof of the theorem. $\qquad\square$

3 Multiple integrals

The purpose of the section is to define multiple stochastic integrals with respect to L and to show that the nth chaos $\mathcal{P}_n$ is generated by these multiple stochastic integrals. The construction of multiple stochastic integrals for processes with independent increments provided by Itô in [11]. For its generalization to other classes of processes the reader referred to [9, 13, 21, 32].

Henceforth we will assume that random measure $\pi(dtdx, \omega)$ has no atoms for all $\omega \in \Omega$. It means that neither measure μ nor measure ν has no atoms for all $\omega \in \Omega$. Since a separable Hilbert space H has the form $H = L^2(T \times X \times \Omega, \mathcal{G} \otimes \mathcal{H}, M_\pi)$, where $M_\pi(dtdxd\omega) = \pi(dtdx, \omega)P(d\omega)$ is a σ-finite measure, then the process L is characterized by the family of random variables $\{L(A), A \in \mathcal{G} \otimes \mathcal{H}, M_\pi(A) < \infty\}$, where $L(A) = L(\mathbf{1}_A)$. We can consider $L(A)$ as a $L^2(\Omega, \mathcal{F}, P)$-valued measure on the parametric space $(T \times X \times \Omega, \mathcal{G} \otimes \mathcal{H})$, which takes conditionally independent values on any family of disjoint subsets of $T \times X \times \Omega$.

Fix $m \geq 1$. Denote by M_π^m the following measure

$$M_\pi^m(dt_1 dx_1 \cdots dt_m dx_m d\omega) = \pi(dt_1 dx_1, \omega) \cdots \pi(dt_m dx_m, \omega)\mathsf{P}(d\omega),$$

defined on the σ-algebra $\mathcal{G}^{\otimes m} \otimes \mathcal{F}$. In this section will consider only the restriction of this measure on the σ-algebra $\mathcal{G}^{\otimes m} \otimes \mathcal{H}$. Since the measure M_π is σ-finite then M_π^m will be σ-finite. Indeed if the sequence of pairwise-disjoint sets $A_n \times B_n$, $A_n \in \mathcal{G}$ and $B_n \in \mathcal{H}$ such that $\bigcup_{n=1}^\infty A_n \times B_n = T \times X \times \Omega$ and $M_\pi(A_n \times B_n) < \infty$, then setting $B_n^k = \{k - 1 \leq \pi(A_n \times B_n) < k\}$ we have $\bigcup_{k=1}^\infty A_n \times B_n^k = A_n \times B_n$. Denote $B_{n_1 n_2 \ldots n_m}^{k_1 k_2 \ldots k_m} = \bigcap_{j=1}^m B_{n_j}^{k_j}$ then $M_\pi^m(A_{n_1} \times A_{n_2} \times \cdots A_{n_m} \times B_{n_1 n_2 \ldots n_m}^{k_1 k_2 \ldots k_m}) \leq k_1 k_2 \cdots k_m < \infty$ and $(T \times X)^m \times \Omega = \bigcup_{n_1, n_2, \ldots, n_m=1}^\infty \bigcup_{k_1, k_2, \ldots, k_m=1}^\infty A_{n_1} \times A_{n_2} \times \cdots A_{n_m} \times B_{n_1 n_2 \ldots n_m}^{k_1 k_2 \ldots k_m}$.

For any $\omega \in \Omega$ we can define measure $\pi^{\otimes m}(dt_1 dx_1 \cdots dt_m dx_m, \omega)$ on the σ-algebra $\mathcal{G}^{\otimes m}$ as a m-th power of the measure π. Since measure π is σ-finite and without atoms then measure $\pi^{\otimes m}$ is σ-finite and without atoms. Moreover, $\pi^{\otimes m}(\Delta_m, \omega) = 0$ for all $\omega \in \Omega$, where $\Delta_m = \{(t_1, \ldots, t_m) : \exists t_i = t_j, i \neq j\}$ is a 'diagonal' set. Indeed, for fixed $\omega \in \Omega$ σ-finiteness of π implies that $T \times X = \bigcup_{i=1}^\infty T_i$, where $T_1, T_2, \ldots$ are pairwise-disjoint sets in $\mathcal{G}$ and $\pi(T_i) < \infty$. Then $(T \times X)^m = \bigcup_{i_1, \ldots, i_m=1}^\infty T_{i_1} \times \cdots \times T_{i_m}$ and $\pi^{\otimes m}(T_{i_1} \times \cdots \times T_{i_m}) < \infty$. Define $C_{i_1, \ldots, i_m} = (T_{i_1} \times \cdots \times T_{i_m})\bigcap \Delta_m$. Then $\Delta_m = \bigcup_{i_1, \ldots, i_m=1}^\infty C_{i_1, \ldots, i_m}$. It is easy to see that $C_{i_1, \ldots, i_m} = \emptyset$ if all the indices $i_1, \ldots, i_m$ are different. Hence it is enough to prove that $\pi^{\otimes m}(C_{i_1, \ldots, i_m}) = 0$ if some of indices $i_1, \ldots, i_m$ are equal. Suppose that $i_1 = i_2$. Using the nonexistence of atoms for the measure π for any $n \in \mathbb{N}$ we can determine a system of pairwise-disjoints sets $\{B_1, \ldots, B_n\} \subset \mathcal{G}$, such that $\bigcup_{i=1}^n B_i = A_{i_1}$ and $\pi(B_i) = \pi(A_{i_1})/n$ for every $i = 1, \ldots, n$. Then $C_{i_1, \ldots, i_m} \subset \bigcup_{i=1}^n B_i \times B_i \times A_{i_3} \times \cdots \times A_{i_m}$. Hence $\pi^{\otimes m}(C_{i_1, \ldots, i_m}) \leq \sum_{i=1}^n \pi(B_i)^2 \pi(A_{i_3}) \cdots \pi(A_{i_m}) = \pi(A_{i_1})^2 \pi(A_{i_2})\pi(A_{i_3}) \cdots \pi(A_{i_m})/n$. Letting n tend to ∞ we obtain the desired result.

Precisely speaking the set Δ_m may not be an element of the σ-algebra $\mathcal{G}^{\otimes m}$ but the calculations above show that Δ_m belongs to completion $\overline{\mathcal{G}^{\otimes m}}^\omega$ of $\mathcal{G}^{\otimes m}$ with respect to the measure $\pi^{\otimes m}(\cdot, \omega)$ for all $\omega \in \Omega$. Since $\pi^{\otimes m}(\cdot, \omega)$ can be extended to the σ-algebra $\overline{\mathcal{G}^{\otimes m}} = \bigcap_{\omega \in \Omega} \overline{\mathcal{G}^{\otimes m}}^\omega$ and $\Delta_m \in \overline{\mathcal{G}^{\otimes m}}$ then measure M_π^m can be extended on the σ-algebra $\overline{\mathcal{G}^{\otimes m}} \otimes \mathcal{H}$ and $M_\pi^m(\Delta_m \times \Omega) = 0$. This fact is very important for definition of the multiple stochastic integral.

Set $\mathcal{G}_0 = \{ A \in \mathcal{G} \otimes \mathcal{H} : \mathbf{1}_A \in K \}$. We will define the multiple stochastic integral $I_m(f)$ of a function $f \in L^2((T \times X)^m \times \Omega, \mathcal{G}^{\otimes m} \otimes \mathcal{H}, M_\pi^m)$. Denote by $\mathcal{E}_m$ the set of elementary functions of the form

$$f(t_1, x_1, \ldots, t_m, x_m, \omega) = \sum_{i_1, \ldots, i_m = 1}^n a_{i_1, \ldots, i_m}(\omega) \mathbf{1}_{A_{i_1}}(t_1, x_1, \omega) \cdots \mathbf{1}_{A_{i_m}}(t_m, x_m, \omega), \qquad (3.1)$$

where $A_1, \ldots, A_n$ are pairwise-disjoint sets in $\mathcal{G}_0$, and the coefficients $a_{i_1, \ldots, i_m} \in L^\infty(\Omega, \mathcal{H}, \mathsf{P})$ are zero if any two of indices $i_1, \ldots, i_m$ are equal.

For a function of the form (3.1) we define the multiple integral of the m-th order

$$I_m(f) = \sum_{i_1, \ldots, i_m = 1}^n a_{i_1, \ldots, i_m} L(A_{i_1}) \cdots L(A_{i_m}).$$

The definition does not depend on particular representation of f, and the following properties hold:

(i) $I_m(\alpha f + \beta g) = \alpha I_m(f) + \beta I_m(g)$ for all α and $\beta \in L^\infty(\Omega, \mathcal{H}, \mathsf{P})$, f and g in $\mathcal{E}_m$.

(ii) $I_m(f) = I_m(\widetilde{f})$, where $\widetilde{f}$ denotes the symmetrization of f with respect to pairs of nonrandom variables, which is defined by

$$\widetilde{f}(t_1, x_1, \ldots, t_m, x_m, \omega) = \frac{1}{m!} \sum_\sigma f(t_{\sigma(1)}, x_{\sigma(1)}, \ldots, t_{\sigma(m)}, x_{\sigma(m)}, \omega),$$

σ running over all permutations of $\{1, \ldots, m\}$.

(iii)
$$\mathsf{E}[I_m(f) I_p(g) | \mathcal{H}] = \begin{cases} 0, & \text{if } p \neq m, \\ m! \pi(\widetilde{f}\widetilde{g}), & \text{if } p = m. \end{cases}$$

The properties can be proved using Lemma 2.7 and exactly the same arguments as those used, for example, in [22, p. 8-9].

In order to extend the multiple stochastic integral to the space $L^2(M_\pi^m)$ we have to prove the following lemma.

Lemma 3.1 *The space $\mathcal{E}_m$ is dense in $L^2((T \times X)^m \times \Omega, \mathcal{G}^{\otimes m} \otimes \mathcal{H}, M_\pi^m)$.*

PROOF.   In order to show that $\mathcal{E}_m$ is dense in $L^2(M_\pi^m)$ it is suffices to show that the indicator function of any set $A \times B = A_1 \times A_2 \times \cdots \times A_m \times B$, where $A_1, \ldots, A_m \in \mathcal{G}$, $B \in \mathcal{H}$ and $M_\pi^m(A \times B) < \infty$ can be approximated by elementary functions in $\mathcal{E}_m$.

Denote by B_k' the following set $B_k' = \{\omega \in \Omega : \pi(A_1, \omega) \pi(A_2, \omega) \cdots \pi(A_m, \omega) \leq k\}$. Since $M_\pi^m(A \times B) < \infty$ then $\mathbf{1}_{A_1 \times \cdots \times A_m \times B_k} \to \mathbf{1}_{A \times B}$ as $k \to \infty$. Hence it is possible to assume

that $\pi(A_1,\omega)\pi(A_2,\omega)\cdots\pi(A_m,\omega)\mathbf{1}_B \le C$ a.s. for some positive constant C. This implies that $A_i \times B \in \mathcal{G}_0$, $i = 1,\ldots,m$. Furthermore, it is possible to suppose that any sets A_i and A_j either equal or disjoint. Indeed, there exists a finite system of pairwise disjoint sets $\{A_1',\ldots,A_n'\} \subset \mathcal{G}$ such that each A_i can be expressed as a disjoint union of some of A_j'. Then the indicator function of the set $A \times B$ can be represent as a finite sum of the indicator functions of the sets $A_{i_1}' \times A_{i_2}' \times \cdots \times A_{i_m}' \times B$. If all indices $i_1,\ldots,i_m$ are different then it is an element of $\mathcal{E}_m$. For other indices some of the sets A_{i_k}' are equal.

Since $M_\pi^m(\Delta_m \times \Omega) = 0$ then $M_\pi^m((A \times B) \bigcap (\Delta_m \times \Omega)) = 0$ and for any $\epsilon > 0$ there exists a system of sets $U_k^\epsilon = A_1^k \times \cdots A_m^k \times B_k$, $k = 1,2,\ldots$ such that $\bigcup_{k=1}^\infty U_k^\epsilon \supset ((A \times B) \bigcap (\Delta_m \times \Omega))$, $\sum_{k=1}^\infty M_\pi^m(U_k^\epsilon) < \epsilon$, $\bigcup_{k=1}^\infty A_i^k = A_i$, $i = 1,\ldots,m$ and $\bigcup_{k=1}^\infty B_k = B$. Therefore $A_i^k \times B_k \in \mathcal{G}_0$, $i = 1,\ldots,m$, $k = 1,2,\ldots$ and $\bigcup_{k=1}^\infty U_k^\epsilon \subset A \times B$.

For any $n \ge 1$ we can find a system of pairwise disjoint sets $\{C_1^n, C_2^n, \ldots, C_{p_n}^n\} \subset \mathcal{G}$, such that each A_i^k, $i = 1,\ldots,m$, $k = 1,\ldots,n$ can be expressed as a disjoint union of some of C_j^n. Notice that $C_j^n \times B_k \in \mathcal{G}_0$ for all $j = 1,\ldots,p_n$, $k = 1,\ldots,n$. We have

$$\mathbf{1}_{\bigcup_{k=1}^n U_k^\epsilon} = \sum_{k=1}^n \sum_{j_1,\ldots,j_m=1}^{p_n} \epsilon_{j_1,\ldots,j_m,k}^n \mathbf{1}_{C_{j_1}^n \times C_{j_2}^n \times \cdots C_{j_m}^n \times B_k},$$

where $\epsilon_{j_1,\ldots,j_m,k}^n$ is 0 or 1. Let J_n be the set of m-tuples $(j_1,\ldots,j_m)$, $j_i \in \{1,2,\ldots,p_n\}$, $i = 1,\ldots,m$, where all the indices are different. We set

$$\mathbf{1}_{V_n} = \sum_{k=1}^n \sum_{(j_1,\ldots,j_m) \in J_n} (1 - \epsilon_{j_1,\ldots,j_m,k}^n) \mathbf{1}_{C_{j_1}^n \times C_{j_2}^n \times \cdots C_{j_m}^n \times B_k}.$$

Then $\mathbf{1}_{V_n}$ belongs to $\mathcal{E}_m$ and

$$V_n = \left(\left(\bigcup_{k=1}^n A_1^k \right) \times \left(\bigcup_{k=1}^n A_2^k \right) \times \cdots \times \left(\bigcup_{k=1}^n A_m^k \right) \times \left(\bigcup_{k=1}^n B_k \right) \right) \setminus \left(\bigcup_{k=1}^n U_k^\epsilon \right)$$

for all $n \in \mathbb{N}$. Since $\bigcup_{k=1}^n U_k^\epsilon \subset \left(\bigcup_{k=1}^n A_1^k \right) \times \cdots \times \left(\bigcup_{k=1}^n A_m^k \right) \times \left(\bigcup_{k=1}^n B_k \right)$ then

$$\mathbf{1}_{V_n} \to \mathbf{1}_{A \times B} - \mathbf{1}_{\bigcup_{k=1}^\infty U_k^\epsilon}$$

M_π^m-a.e. and in $L^2(M_\pi^m)$ as $n \to \infty$. Finally the fact $\mathbf{1}_{A \times B} - \mathbf{1}_{\bigcup_{k=1}^\infty U_k^\epsilon} \to \mathbf{1}_{A \times B}$ in $L^2(M_\pi^m)$ as $\epsilon \to 0$ implies the proof of the lemma. $\qquad\square$

Letting $f = g$ in property (iii) obtains

$$\mathsf{E}(I_n(f)^2) = m! ||\tilde{f}||_{L^2(M_\pi^m)}^2 \le m! \, ||f||_{L^2(M_\pi^m)}^2.$$

Therefore, the operator I_m can be extended to a linear and continuous operator from $L^2(M_\pi^m)$ to $L^2(\Omega, \mathcal{F}, \mathsf{P})$, which satisfies properties (i), (ii) and (iii).

If $f \in L^2(M_\pi^p)$ is symmetric function and $g \in K$ the contraction of one index of f and g is denoted by $f \otimes_1 g$ and is defined by

$$(f \otimes_1 g)(t_1, x_1, \ldots, t_{p-1}, x_{p-1}, \omega)$$

$$= \int_{T \times X} f(t_1, x_1, \ldots, t_{p-1}, x_{p-1}, s, z, \omega) g(s, z, \omega) \pi(dsdz, \omega).$$

The tensor product $f \otimes g$ will be understood as tensor product with respect to nonrandom variables, i.e. $(f \otimes g)(t_1, x_1, \ldots, t_{p+1}, x_{p+1}, \omega) = f(t_1, x_1, \ldots, t_p, x_p, \omega)g(t_{p+1}, x_{p+1}, \omega)$. Notice that $f \otimes_1 g \in L^2(M_\pi^{p-1})$ and $f \otimes g \in L^2(M_\pi^{p+1})$ if $g \in K$.

The tensor product $f \otimes g$ and the contractions $f \otimes_1 g$ are not necessarily symmetric. We will denote their symmetrization by $f \widetilde{\otimes} g$ and $f \widetilde{\otimes}_1 g$ respectively.

The following, so called product formula, will be useful in the sequel. It was initially derived by Itô [10] for Gaussian case and by Kabanov [13] for Poisson case, then extended by Russo and Vallois [29] to products of two multiple stochastic integrals with respect to a normal martingale.

Proposition 3.2 Let $f \in L^2(M_\pi^p)$ be a symmetric function and let $g \in K$. Then

$$I_p(f)I_1(g) = I_{p+1}(f \otimes g) + pI_{p-1}(f \otimes_1 g) + pI_p(fg\mathbf{1}_{X_0}). \tag{3.2}$$

PROOF.   Since $g \in K$ then $fg\mathbf{1}_{X_0} \in L^2(M_\pi^p)$ and the right-hand side is correctly defined.

By the density of elementary functions in $L^2(M_\pi^p)$ and by properties (i) and (ii) we can assume that f is the symmetrization of the function $\mathbf{1}_{A_1}(t_1, x_1, \omega)\mathbf{1}_{A_2}(t_2, x_2, \omega) \cdots \mathbf{1}_{A_p}(t_p, x_p, \omega)$, where the A_i are pairwise-disjoint sets of $\mathcal{G}_0$, and $g = \mathbf{1}_{A_1}$ or $\mathbf{1}_{A_0}$, where $A_0 \in \mathcal{G}_0$ is disjoint with $A_1, \ldots, A_p$. The case $g = \mathbf{1}_{A_0}$ is immediate because $f \otimes_1 g = fg = 0$ and $f \otimes g \in \mathcal{E}_{p+1}$. So, we assume $g = \mathbf{1}_{A_1}$. Since $A_i \in \mathcal{G}_0$ then $\pi(A_1)\pi(A_2) \cdots \pi(A_p) \leq C$ for some real constant $C > 0$. Given $\epsilon > 0$, according to Lemma 3.1 we can find the function $v'_\epsilon \in \mathcal{E}_2$ such that $||v'_\epsilon - \mathbf{1}_{A_1} \otimes \mathbf{1}_{A_1}||_{L^2(M_\pi^2)} \leq \epsilon$. It is possible to write v'_ϵ in the following form

$$v'_\epsilon(t_1, x_1, t_2, x_2, \omega) = \sum_{i,j=1}^{k_\epsilon} a_{ij}^\epsilon(\omega)\mathbf{1}_{C_i}(t_1, x_1, \omega)\mathbf{1}_{C_j}(t_2, x_2, \omega), \tag{3.3}$$

where $C_1, \ldots, C_{k_\epsilon}$ are pairwise disjoint subsets of A_1 in $\mathcal{G}_0$, $a_{ij}^\epsilon \in L^\infty(\Omega, \mathcal{H}, \mathsf{P})$ and $a_{ii}^\epsilon = 0$, $i, j = 1, \ldots, k_\epsilon$. Set $A_1^0 = A_1 \cap (T \times X_0 \times \Omega)$ and

$$v_\epsilon(t_1, x_1, \ldots, t_{p+1}, x_{p+1}, \omega) = v'_\epsilon(t_1, x_1, t_2, x_2, \omega)\mathbf{1}_{A_2}(t_3, x_3, \omega) \cdots \mathbf{1}_{A_p}(t_{p+1}, x_{p+1}, \omega).$$

Then v_ϵ is an elementary function and we have

$$I_p(f)I_1(g) = L(A_1)^2 L(A_2) \cdots L(A_p)$$

$$= I_{p+1}(v_\epsilon) + \pi(A_1)L(A_2) \cdots L(A_p) + L(A_1^0)L(A_2) \cdots L(A_p)$$

$$+ [(L(A_1)^2 - \pi(A_1) - L(A_1^0))L(A_2) \cdots L(A_p) - I_{p+1}(v_\epsilon)]$$

$$= I_{p+1}(v_\epsilon) + pI_{p-1}(f \otimes_1 g) + pI_p(fg\mathbf{1}_{X_0}) + R_\epsilon. \tag{3.4}$$

Indeed

$$f \otimes_1 g = \frac{1}{p}\pi(A_1)\,\mathrm{symm}(\mathbf{1}_{A_2} \otimes \cdots \otimes \mathbf{1}_{A_p}),$$

and

$$fg\mathbf{1}_{X_0} = \frac{1}{p}\mathbf{1}_{A_1^0} \otimes \mathrm{symm}(\mathbf{1}_{A_2} \otimes \cdots \otimes \mathbf{1}_{A_p}),$$

where $\mathrm{symm}(\cdot)$ denotes the symmetrization with respect to the pairs of nonrandom variables of the function in parentheses. We have

$$\left|\left|\tilde{v}_\epsilon - f\widetilde{\otimes}g\right|\right|_{L^2(M_\pi^{p+1})}^2 = \left|\left|\tilde{v}_\epsilon - \mathrm{symm}(\mathbf{1}_{A_1} \otimes \mathbf{1}_{A_1} \otimes \mathbf{1}_{A_2} \otimes \cdots \otimes \mathbf{1}_{A_p})\right|\right|_{L^2(M_\pi^{p+1})}^2$$

$$\leq \left|\left|v_\epsilon - \mathbf{1}_{A_1} \otimes \mathbf{1}_{A_1} \otimes \mathbf{1}_{A_2} \otimes \cdots \otimes \mathbf{1}_{A_p}\right|\right|^2_{L^2(M_\pi^{p+1})}$$

$$= \mathsf{E}\left[\pi(A_2)\cdots\pi(A_p) \int_{(T\times X)^2} (v'_\epsilon(t_1,x_1,t_2,x_2) - \mathbf{1}_{A_1}(t_1,x_1)\mathbf{1}_{A_1}(t_2,x_2))^2 \, \pi(dt_1 dx_1)\pi(dt_2 dx_2)\right]$$

$$\leq C \left|\left|v'_\epsilon - \mathbf{1}_{A_1} \otimes \mathbf{1}_{A_1}\right|\right|^2_{L^2(M_\pi^2)} \leq C\epsilon^2, \tag{3.5}$$

and

$$\mathsf{E}(R_\epsilon^2) = \mathsf{E}\Big([L(A_1)^2 - \pi(A_1) - L(A_1^0) - I_2(v'_\epsilon)]^2 L(A_2)^2 L(A_3)^2 \cdots L(A_p)^2\Big).$$

Lemma 2.7 and properties of multiple integral imply

$$\mathsf{E}(R_\epsilon^2) = \mathsf{E}\Big(\big[L(A_1)^4 + \pi(A_1)^2 + \pi(A_1^0) + 2\pi^{\otimes 2}(\widetilde{v'}_\epsilon^2)$$

$$-2\pi(A_1)^2 - 2L(A_1)^2 L(A_1^0) - 2L(A_1)^2 I_2(v'_\epsilon)\big]\pi(A_2)\cdots\pi(A_p)\Big). \tag{3.6}$$

Taking fourth derivative at $z = 0$ of conditional characteristic function of L given by formula (2.1) yields

$$\mathsf{E}[L(A_1)^4|\mathcal{H}] = 3\pi(A_1)^2 + \pi(A_1^0). \tag{3.7}$$

By using the same arguments since $A_1 = A_1^0 \cup (A_1 \setminus A_1^0)$ we get

$$\mathsf{E}[L(A_1)^2 L(A_1^0)|\mathcal{H}] = \mathsf{E}[L(A_1^0)^3 + 2L(A_1 \setminus A_1^0)L(A_1^0)^2 + L(A_1 \setminus A_1^0)^2 L(A_1^0)|\mathcal{H}] = \pi(A_1^0). \tag{3.8}$$

It follows from equality (3.3) and Lemma 2.7 that

$$\mathsf{E}[L(A_1)^2 I_2(v'_\epsilon)|\mathcal{H}] = \sum_{i,j=1}^{k_\epsilon} a_{ij}^\epsilon \mathsf{E}[L(A_1)^2 L(C_i)L(C_j)|\mathcal{H}]$$

$$= \sum_{i,j=1}^{k_\epsilon} a_{ij}^\epsilon \mathsf{E}[2L(C_i)^2 L(C_j)^2|\mathcal{H}] = 2\sum_{i,j=1}^{k_\epsilon} a_{ij}^\epsilon \pi(C_i)\pi(C_j) = 2\int_{(T\times X)^2} \widetilde{v'}_\epsilon d\pi^{\otimes 2} = 2\pi^{\otimes 2}(\widetilde{v'}_\epsilon). \tag{3.9}$$

Substituting expressions (3.7), (3.8) and (3.9) into (3.6) we have

$$\mathsf{E}(R_\epsilon^2) = \mathsf{E}[(2\pi(A_1)^2 + 2\pi^{\otimes 2}(\widetilde{v'}_\epsilon^2) - 4\pi^{\otimes 2}(\widetilde{v'}_\epsilon))\pi(A_2)\cdots\pi(A_p)]$$

$$= 2\mathsf{E}[\pi^{\otimes 2}((\widetilde{v'}_\epsilon - \mathbf{1}_{A_1} \otimes \mathbf{1}_{A_1})^2)\pi(A_2)\cdots\pi(A_p)] \leq 2C \left|\left|v'_\epsilon - \mathbf{1}_{A_1} \otimes \mathbf{1}_{A_1}\right|\right|^2_{L^2(M_\pi^2)} \leq 2C\epsilon^2. \tag{3.10}$$

From formulae (3.4), (3.5) and (3.10) we obtain the desired result. □

The next result gives the relationship between generalized orthogonal polynomials and multiple stochastic integrals.

Theorem 3.3 *Let P_n be the nth generalized orthogonal polynomial, and $\bar{x}(h) = (x_k(h))_{k=1}^\infty$, where $x_1(h) = L(h)$, $x_2(h) = L(h^2 \mathbf{1}_{X_0}) + ||h||_H^2$, $x_k(h) = L(h^k \mathbf{1}_{X_0}) + \int_{T\times X_0} h^k(t,x)\nu(dtdx)$, $k = 3, 4, \ldots$ and $h \in K$. Then it holds that*

$$n! P_n(\bar{x}(h)) = I_n(h^{\otimes n}), \tag{3.11}$$

where $h^{\otimes n}(t_1, x_1, \ldots, t_n, x_n, \omega) = h(t_1, x_1, \omega) \cdots h(t_n, x_n, \omega)$.

PROOF. We will prove the theorem by induction on n. For $n = 1$ it is immediate. Assume it holds for $1, 2, \ldots, n$. Using the product formula (3.2) and recursive relation for generalized orthogonal polynomials (2.8), we have

$$I_{n+1}(h^{\otimes(n+1)}) = I_n(h^{\otimes n})I_1(h) - nI_{n-1}\left(h^{\otimes(n-1)}\pi(h^2)\right) - nI_n(h^{\otimes(n-1)} \otimes (h^2\mathbf{1}_{X_0}))$$

$$= n!P_n(\overline{x}(h))L(h) - n!\pi(h^2)P_{n-1}(\overline{x}(h)) - nI_{n-1}(h^{\otimes(n-1)})I_1(h^2\mathbf{1}_{X_0})$$
$$+n(n-1)I_{n-2}(h^{\otimes(n-2)})\pi(h^3\mathbf{1}_{X_0}) + n(n-1)I_{n-1}(h^{\otimes(n-2)} \otimes (h^3\mathbf{1}_{X_0}))$$

$$= n!\sum_{k=0}^{1}(-1)^{k+1}x_{k+1}(h)P_{n-k}(\overline{x}(h)) + n!P_{n-2}(\overline{x}(h))\pi(h^3\mathbf{1}_{X_0}) + n(n-1)I_{n-1}(h^{\otimes(n-2)} \otimes (h^3\mathbf{1}_{X_0}))$$

$$= \cdots = n!\sum_{k=0}^{n-1}(-1)^{k+1}x_{k+1}(h)P_{n-k}(\overline{x}(h)) + n!(-1)^n P_0(\overline{x}(h))\pi(h^{n+1}\mathbf{1}_{X_0}) + n!(-1)^n I_1(h^{n+1})$$

$$= n!\sum_{k=0}^{n}(-1)^{k+1}x_{k+1}(h)P_{n-k}(\overline{x}(h)) = (n+1)!P_{n+1}(\overline{x}(h)),$$

which completes the proof of the theorem. □

From this theorem and Theorem 2.10 we deduce the following result.

Corollary 3.4 *Any square integrable random variable $\xi \in L^2(\Omega, \mathcal{F}, \mathsf{P})$ can be expanded into a series of multiple stochastic integrals:*

$$\xi = \sum_{k=0}^{\infty} I_k(f_k).$$

Here $f_0 = \mathsf{E}[\xi|\mathcal{H}]$, and I_0 is the identity mapping on the $L^2(\Omega, \mathcal{H}, \mathsf{P})$. Furthermore, this representation is unique provided the functions $f_k \in L^2(M_\pi^k)$ are symmetric with respect to the pairs of nonrandom variables.

PROOF. The proof uses the same arguments as those used, for example, in [22, Th. 1.1.2], so we omit it. □

The following technical lemma will be needed in the sequel.

Lemma 3.5 *Let $f_k \in L^2((T \times X)^k \times \Omega, \mathcal{G}^{\otimes k} \otimes \mathcal{H}, M_\pi^k)$ and $g_m \in L^2((T \times X)^m \times \Omega, \mathcal{G}^{\otimes m} \otimes \mathcal{H}, M_\pi^m)$ be a symmetric with respect to pairs of nonrandom variables functions and $p \le k \wedge m$. Then there exist $\mathcal{G}^{\otimes p} \otimes \mathcal{F}$ measurable versions of the processes $I_{k-p}(f_k(\cdot, t_1, x_1, \ldots, t_p, x_p))$ and $I_{m-p}(g_m(\cdot, t_1, x_1, \ldots, t_p, x_p))$ which belong to $L^2((T \times X)^p \times \Omega, \mathcal{G}^{\otimes p} \otimes \mathcal{F}, M_\pi^p)$ and the following equality holds*

$$\mathsf{E}\left[\int_{(T \times X)^p} I_{k-p}(f_k(\cdot, t_1, x_1, \ldots, t_p, x_p))I_{m-p}(g_m(\cdot, t_1, x_1, \ldots, t_p, x_p))\pi(dt_1 dx_1)\cdots\pi(dt_p dx_p)\right]$$

$$= \begin{cases} 0, & \text{if } m \ne k, \\ (m-p)!\mathsf{E}\left[\int_{(T \times X)^m}(g_m f_m)(t_1, x_1, \ldots, t_m, x_m)\pi(dt_1 dx_1)\cdots\pi(dt_m dx_m)\right], & \text{if } m = k. \end{cases}$$

PROOF. It is easy to verify that statement of the lemma is valid for elementary functions from $\mathcal{E}_k$ and $\mathcal{E}_m$. The general case will follow by the limit argument. $\qquad\square$

Now let $T = \mathbb{R}_+$, $X_0 = \mathbb{R}_0$, $\Delta = 0$ and $L(dtdx)$ be as in Example 2.3. Denote by Σ_n the 'increasing simplex' of $(\mathbb{R}_+ \times \mathbb{R})^n$:

$$\Sigma_n = \{(t_1, x_1, \ldots, t_n, x_n) \in (\mathbb{R}_+ \times \mathbb{R})^n : 0 < t_1 < \cdots < t_n\},$$

and we extend a function f_n defined on $\Sigma_n \times \Omega$ by making f_n symmetric with respect to pairs of nonrandom variables. If the function f_n square integrable with respect to measure M_π^n then we have

$$I_n(f_n) = n! \int_{\Sigma_n} f_n(t_1, x_1, \ldots, t_n, x_n) L(dt_1 dx_1) \cdots L(dt_n dx_n).$$

Indeed, this equality is clear if f_n is an elementary function of the form (3.1), and in the the general case equality will follow by the density argument, taking in account that iterated integral verifies the same properties as the multiple integral. In particular Lemma 3.5 holds for iterated integral. Note that the domain Σ_n and its symmetrization do not cover $(\mathbb{R}_+ \times \mathbb{R})^n$: we are ignoring the 'diagonal set'. Since in the beginning of this section was proved that the 'diagonal set' has M_π^n measure zero and we consider the functions as an elements of L^2 which are the the equivalence classes, then we will always choose the representative that vanishes on the 'diagonal set'.

4 The derivative operator

In this section we introduce the operator D. Then we will show that it is equal to the Malliavin derivatives in the Gaussian case (see, e.g., [22]) and to the difference operator defined in [24, 26] in the Poisson case. We will also proof that the derivatives operators defined via the chaos decomposition in [2, 3, 18, 19, 25, 28, 31] for certain Lévy processes coincide with the operator D.

We denote by $C_b^\infty(\mathbb{R}^n)$ the set of all infinitely continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all of its partial derivatives are bounded.

Let $\mathcal{S}$ denote the class of smooth random variables such that a random variable $\xi \in \mathcal{S}$ has the form

$$\xi = f(L(h_1), \ldots, L(h_n)), \tag{4.1}$$

where f belongs to $C_b^\infty(\mathbb{R}^n)$, $h_1, \ldots, h_n$ are in K, and $n \geq 1$.

Lemma 4.1   *1. The set $\mathcal{S}$ is dense in $L^p(\Omega, \mathcal{F}, \mathsf{P})$, for any $p \geq 1$.*

   2. The set $\{\xi h : \xi \in \mathcal{S}, h \in K\}$ is dense in $L^2(T \times X \times \Omega, \mathcal{G} \otimes \mathcal{F}, M_\pi)$.

   3. The set $\{u e^{L(v)} : u, v \in K\}$ is a total set of $L^2(T \times X \times \Omega, \mathcal{G} \otimes \mathcal{F}, M_\pi)$.

PROOF. 1. Let $\{h_k\}_{k=1}^\infty \subset K$ be a dense subset of H. Define $\mathcal{F}_n = \sigma(L(h_1), \ldots, L(h_n))$. Then $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and $\mathcal{F}$ is the smallest σ-algebra containing all the $\mathcal{F}_n$'s. Choose a $g \in L^p(\Omega)$. Then

$$g = \mathsf{E}(g|\mathcal{F}) = \lim_{n \to \infty} \mathsf{E}(g|\mathcal{F}_n).$$

By the Doob-Dynkin Lemma we have that for each n, there exist a Borel measurable function $g_n : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathsf{E}(g|\mathcal{F}_n) = g_n(L(h_1), \ldots, L(h_n)).$$

Each such g_n can be approximated by functions $f_m^{(n)}$ where $f_m^{(n)} \in C_b^\infty(\mathbb{R}^n)$ such that $||f_m^{(n)}(L(h_1), \ldots, L(h_n)) - g_n(L(h_1), \ldots, L(h_n))||_{L^p(\Omega)}$ converges to zero as $m \to \infty$. Since $f_m^{(n)}(L(h_1), \ldots, L(h_n)) \in \mathcal{S}$ we have the first statement of the lemma.

2. It is enough to show that indicator function $\mathbf{1}_{A \times B}$, where $A \in \mathcal{G}$, $B \in \mathcal{F}$ and $M_\pi(A \times B) < \infty$ can be approximated by the processes of the form ξh, where $\xi \in \mathcal{S}$ and $h \in K$. It follows from the previous part of the lemma that there exists the sequence ξ_n in $\mathcal{S}$ such that $\xi_n \to \mathbf{1}_B$ as $n \to \infty$ in $L^2(\Omega)$. Set $C_m = \{\pi(A) \le m\}$. Then $C_m \in \mathcal{H}$ and $\bigcup_{m \ge 1} C_m = \{\pi(A) < \infty\} \supset B$ a.s. The processes $\mathbf{1}_{A \times C_m} \xi_n$ have required form and letting $m \to \infty$ and then $n \to \infty$ we obtain the desired result.

3. Lemma 2.9 implies that finite linear combinations of the random variables $e^{L(v)}$, $v \in K$ are dense in $L^2(\Omega)$. The same arguments as in previous part of the lemma yield the density of the set of the linear combinations of the processes $ue^{L(v)}$, $u, v \in K$, which completes the proof of the lemma. $\qquad\square$

Definition 4.2 *The stochastic derivative of a smooth random variable ξ of the form (4.1) is the stochastic process $D\xi = \{D_{t,x}\xi, \ (t,x) \in T \times X\}$ indexed by the parameter space $T \times X$ given by*

$$D_{t,x}\xi = \sum_{k=1}^n \frac{\partial f}{\partial y_k}(L(h_1), \ldots, L(h_n))h_k(t,x)\mathbf{1}_\Delta(x)$$

$$+ \Big(f(L(h_1) + h_1(t,x), \ldots, L(h_n) + h_n(t,x)) - f(L(h_1), \ldots, L(h_n))\Big)\mathbf{1}_{X_0}(x). \qquad (4.2)$$

Remark 4.3   *1. If the measure ν is zero and the measure μ is deterministic then $D\xi$ coincides with the Malliavin derivative (see, for example, [22, Def. 1.2.1, p. 24]).*

    2. If the measure μ is zero and the measure ν is deterministic then D coincides with the difference operator defined in [26].

    3. If $T = \mathbb{R}_+$, the measure μ is the Lebesgue measure and X is a metric space, and the measure ν is the product of the Lebesgue measure times the measure β satisfying $\int_M (|x|^2 \wedge 1)\beta(dx)$, then D is the operator ∇^- from [28].

    4. If measures μ and ν are both deterministic then D coincides with operator defined in [33], see also [31].

Lemma 4.4 *Suppose that ξ is smooth functional of the form (4.1) and $h \in H$. Then*

$$\mathsf{E}\left[\int_{T \times X} D_{t,x}\xi h(t,x)\pi(dtdx) \,\Big|\, \mathcal{H}\right] = \mathsf{E}[\xi L(h)|\mathcal{H}]. \qquad (4.3)$$

PROOF.   The proof will be done in three steps.

Step 1. Suppose first that

$$\xi = e^{iz_1 L(h_1)} \cdots e^{iz_n L(h_n)}.$$

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Then $\mathrm{Re}\,\xi \in \mathcal{S}$ and $\mathrm{Im}\,\xi \in \mathcal{S}$ and

$$\mathsf{E}[\xi L(h)|\mathcal{H}] = \frac{1}{i}\frac{d}{dz}\left(\mathsf{E}\left[\exp\left(i\sum_{k=1}^{n}z_k L(h_k) + izL(h)\right)\bigg|\mathcal{H}\right]\right)\bigg|_{z=0}$$

$$= \frac{1}{i}\frac{d}{dz}\exp\left(-\frac{1}{2}\int_T\left(\sum_{k=1}^{n}z_k h_k(t,\Delta) + zh(t,\Delta)\right)^2\mu(dt)\right.$$

$$+ \int_{T\times X_0}\left(\exp(i\sum_{k=1}^{n}z_k h_k(t,x) + izh(t,x)) - 1\right.$$

$$\left.-i\left(\sum_{k=1}^{n}z_k h_k(t,x) + zh(t,x)\right)\right)\nu(dtdx)\right)\bigg|_{z=0}$$

$$= \left(\int_{T\times X_0}h(t,x)\left(\exp(i\sum_{k=1}^{n}z_k h_k(t,x)) - 1\right)\nu(dtdx)\right.$$

$$\left.+i\int_T h(t,\Delta)\sum_{k=1}^{n}z_k h_k(t,\Delta)\mu(dt)\right)\exp\left(-\frac{1}{2}\int_T\left(\sum_{k=1}^{n}z_k h_k(t,\Delta)\right)^2\mu(dt)\right.$$

$$\left.+\int_{T\times X_0}\left(\exp(i\sum_{k=1}^{n}z_k h_k(t,x)) - 1 - i\sum_{k=1}^{n}z_k h_k(t,x)\right)\nu(dtdx)\right)$$

$$= \mathsf{E}[\xi|\mathcal{H}]\left(\int_{T\times X_0}h(t,x)\left(\exp(i\sum_{k=1}^{n}z_k h_k(t,x)) - 1\right)\nu(dtdx)\right.$$

$$\left.+i\int_T h(t,\Delta)\sum_{k=1}^{n}z_k h_k(t,\Delta)\mu(dt)\right).$$

On the other hand

$$\mathsf{E}\left[\int_{T\times X}D_{t,x}\xi h(t,x)\pi(dtdx)\bigg|\mathcal{H}\right]$$

$$= \mathsf{E}\left[\int_{T\times X_0}\left(\exp(i\sum_{k=1}^{n}z_k(L(h_k) + h_k(t,x))) - \exp(i\sum_{k=1}^{n}z_k L(h_k))\right)h(t,x)\nu(dtdx)\bigg|\mathcal{H}\right]$$

$$+\mathsf{E}\left[\int_T i\sum_{j=1}^{n}z_j\exp(i\sum_{k=1}^{n}z_k L(h_k))h_j(t,\Delta)\mu(dt)\bigg|\mathcal{H}\right]$$

$$= \mathsf{E}[\xi|\mathcal{H}]\left(\int_{T\times X_0}h(t,x)\left(\exp(i\sum_{k=1}^{n}z_k h_k(t,x)) - 1\right)\nu(dtdx)\right.$$

$$\left.+i\int_T h(t,\Delta)\sum_{k=1}^{n}z_k h_k(t,\Delta)\mu(dt)\right).$$

Hence we have (4.3). By linearity we deduce that (4.3) also holds for smooth variables of the form (4.1), where the function f is a trigonometric polynomial.

Step 2. Assume that ξ of the form (4.1) such that $f \in C_b^\infty(\mathbb{R}^n)$ is periodic on every variable function. Then there is a sequence of trigonometric polynomials g_m such that $g_m \to f$ and $\partial g_m / \partial x_k \to \partial f / \partial x_k$ for every $k = 1, \ldots n$ uniformly on $\mathbb{R}^n$ as $m \to \infty$. Denote $\eta_m = g_m(L(h_1), \ldots, L(h_n))$. Then $\eta_m \in \mathcal{S}$, and by Step 1 we get

$$\mathsf{E}[\eta_m L(h) | \mathcal{H}] = \mathsf{E}\left[\left. \int_{T \times X} D_{t,x} \eta_m h(t,x) \pi(dtdx) \right| \mathcal{H} \right]. \tag{4.4}$$

Since $\eta_m \to \xi$ in $L^2(\Omega)$ and $D\eta_m \to D\xi$ in $L^2(T \times X \times \Omega, \mathcal{G} \otimes \mathcal{F}, M_\pi)$ then letting $m \to \infty$ in (4.4) we obtain (4.3).

Step 3. Assume that ξ of the form (4.1). Consider the sequence $\{\chi_m, m = 1, 2, \ldots\}$ of functions, such that $\chi_m \in C^\infty(\mathbb{R}^n)$, $0 \leq \chi_m \leq 1$, $\chi_m(x) = 1$ if $|x| \leq m$, $\chi(x) = 0$, if $|x| > m + 1$ and $|\nabla \chi_m| \leq 2$. Define g_m as a periodic extension on all variables of the function $f\chi_m$. Then $\zeta_m = g_m(L(h_1), \ldots, L(h_n))$ is smooth variable such that $|\zeta_m| \leq \|f\|_{L^\infty}$ and $|D\zeta_m| \leq \|\nabla f\|_{L^\infty} \sum_{i=1}^n |h_i|$. Hence by the dominated convergence theorem $\zeta_m \to \xi$ in $L^2(\Omega)$ and $D\zeta_m \to D\xi$ in $L^2(T \times X \times \Omega, \mathcal{G} \otimes \mathcal{F}, M_\pi)$ as $m \to \infty$. Since by Step 2 formula (4.3) is true for ζ_m, then letting $m \to \infty$ completes the proof of the lemma. $\square$

Applying this lemma to the product of two smooth functionals we obtain the "integration by parts" formula.

Lemma 4.5 *Suppose ξ and η are the smooth functionals and $h \in H$, then*

$$\mathsf{E}[\xi\eta L(h) | \mathcal{H}] = \mathsf{E}\left[\left. \int_{T \times X} (\xi D_{t,x}\eta + \eta D_{t,x}\xi + \mathbf{1}_{X_0} D_{t,x}\xi D_{t,x}\eta) h(t,x) \pi(dtdx) \right| \mathcal{H} \right]. \tag{4.5}$$

As a consequence of the above lemma it can be shown in the same way as in [33] that the expression of the derivative $D\xi$ given in (4.2) does not depend on the particular representation of ξ in (4.1).

For $p \geq 1$ define a norm for $\mathcal{G} \otimes \mathcal{F}$ measurable function by the following expression

$$\|u\|_{2,p} = \left(\mathsf{E}\left[\left(\int_{T \times X} u(t,x)^2 \pi(dtdx) \right)^{p/2} \right] \right)^{1/p}.$$

Let $L^{2,p}(M_\pi) = L^{2,p}(T \times X \times \Omega, \mathcal{G} \otimes \mathcal{F}, M_\pi)$ be the set of all (equivalent classes of) functions $u(t,x,\omega)$ on $T \times X \times \Omega$ such that $\|u\|_{2,p} < \infty$.

Lemma 4.6 *The operator D is closable as an operator from $L^p(\Omega, \mathcal{F}, \mathsf{P})$ to $L^{2,p}(M_\pi)$, for any $p \geq 1$.*

PROOF. Let $\{\xi_n, n \geq 1\}$ be a sequence of smooth random variables such that $\mathsf{E}|\xi_n|^p \to 0$ and $D\xi_n$ converges to ζ in $L^{2,p}(M_\pi)$. Then from Lemma 4.5 it follows that for any $h \in K$ and $\eta \in \mathcal{S}$ we have

$$\mathsf{E}(\xi_n \eta L(h)) = \langle D\eta; \xi_n h \rangle_{L^2(M_\pi)} + \langle D\xi_n; \eta h \rangle_{L^2(M_\pi)} + \langle D\xi_n; \mathbf{1}_{X_0} h D\eta \rangle_{L^2(M_\pi)}.$$

Taking the limit as $n \to \infty$, since $\eta, D\eta$ are bounded, and $h \in K$ we obtain

$$\langle \zeta; \eta h \rangle_{L^2(M_\pi)} + \langle \zeta; \mathbf{1}_{X_0} h D\eta \rangle_{L^2(M_\pi)} = 0. \tag{4.6}$$

If $h(t,x) = 0$ for $x \neq \Delta$, then (4.6) implies, that

$$\langle \zeta; \eta h \rangle_{L^2(M_\pi)} = 0.$$

Thus from Lemma 4.1 we deduce $\zeta_{t,\Delta} = 0$ for M_π-almost all $(t, \Delta, \omega) \in T \times \{\Delta\} \times \Omega$. Substituting this expression into (4.6) we have for any $h \in H$

$$\langle \zeta; h D\eta \rangle_{L^2(M_\pi)} = 0. \tag{4.7}$$

Let $\phi_n \in C_b^\infty(\mathbb{R})$ such that $0 \leq \phi_n(x) \leq e^x$ and $\phi_n(x) \to e^x$ for all $x \in \mathbb{R}$. Putting in (4.7) $\eta = \phi_n(L(g))$ and $h(t,x) = u(t,x)e^{-g(t,x)}$, where $u, g \in K$ and then letting $n \to \infty$ we get

$$\langle \zeta; u e^{L(g)} \rangle_{L^2(M_\pi)} = 0.$$

It follows from Lemma 4.1 that $\zeta_{t,x} = 0$ for M_π-a.a. $(t, x, \omega) \in T \times X \times \Omega$ completing the proof of the lemma. $\qquad\square$

We will denote the closure of D again D and its domain in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$.

Now we will state the chain rule.

Proposition 4.7 *Suppose $p \geq 1$ is fixed and $\xi = (\xi^1, \ldots, \xi^m)$ is a random vector whose components belong to the space $\mathbb{D}^{1,p}$. Let $\phi \in C^1(\mathbb{R}^m)$ be a function with bounded partial derivatives. Then $\phi(\xi) \in \mathbb{D}^{1,p}$ and*

$$D_{t,x}\phi(\xi) = \begin{cases} \sum_{k=1}^m \frac{\partial \phi}{\partial x_k}(\xi) D_{t,\Delta}\xi^k, & \text{if } x = \Delta, \\ \phi(\xi^1 + D_{t,x}\xi^1, \ldots, \xi^m + D_{t,x}\xi^m) - \phi(\xi^1, \ldots, \xi^m), & \text{if } x \neq \Delta. \end{cases} \tag{4.8}$$

PROOF.   The proof can be easily obtain by approximation ξ by smooth random variables and the function ϕ by smooth functions with compact support. $\qquad\square$

Applying the above proposition we obtain, that $L(h) \in \mathbb{D}^{1,2}$ for all $h \in H$ and $D_{t,x}L(h) = h(t,x)$.

Lemma 4.8 *It holds that $P_n(\overline{x}(h)) \in \mathbb{D}^{1,p}$ for all $p \geq 1$, $h \in K$, $n = 1, 2, \ldots$ and*

$$D_{t,x}P_n(\overline{x}(h)) = P_{n-1}(\overline{x}(h))h(t,x). \tag{4.9}$$

PROOF.   As in the proof of Proposition 4.7 one can obtain that $P_n(\overline{x}(h)) \in \mathbb{D}^{1,p}$ for all $p \geq 1$, $h \in K$, $n = 1, 2, \ldots$ and (4.8) holds. Then the definition of $\overline{x}(h)$ and equality (2.9) imply

$$D_{t,\Delta}P_n(\overline{x}(h)) = \frac{\partial P_n}{\partial x_1}(\overline{x}(h))h(t, \Delta) = P_{n-1}(\overline{x}(h))h(t, \Delta).$$

It follows from the relationships (4.8) and (2.11) that for $x \neq \Delta$ we have

$$D_{t,x}P_n(\overline{x}(h)) = P_n(\overline{x}(h) + \overline{u}(h(t,x))) - P_n(\overline{x}(h)) = h(t,x)P_{n-1}(\overline{x}(h)),$$

where $\overline{u}(y) = (y, y^2, \ldots, y^k, \ldots)$. The proof is complete. $\qquad\square$

The product rule can be proved in the same manner.

Proposition 4.9 *Let $\xi \in \mathbb{D}^{1,p}$, $p \geq 1$ and η is a smooth variable from $\mathcal{S}$. Then $\xi\eta \in \mathbb{D}^{1,p}$ and*

$$D(\xi\eta) = \xi D\eta + \eta D\xi + D\xi D\eta \mathbf{1}_{X_0}. \tag{4.10}$$

PROOF.   The equation (4.10) holds if ξ and η are smooth variables. Then, the general case follows by a limit argument, using the fact that D is closed. $\square$

The following proposition is more or less evident.

Proposition 4.10 *Let ξ be $\mathcal{H}$-measurable random variable such that $\xi \in L^p(\Omega, \mathcal{H}, \mathsf{P})$ for some $p \geq 1$. Then $\xi \in \mathbb{D}^{1,p}$ and $D\xi = 0$ M_π-a.e.*

PROOF.   By the density arguments we can assume that $\xi = \mathbf{1}_U$, where $U \in \mathcal{H}$. Then for any $h \in K$ as in the proof of Lemma 2.4 we have $\xi\mathbf{1}_{\{L(h)\neq 0\}} = L(\xi h)^2/L(h)^2\mathbf{1}_{\{L(h)\neq 0\}}$. Since $h \in K$ it easy to show that $\eta_\varepsilon(h) = L(\xi h)^2/(L(h)^2 + \varepsilon) = \xi L(h)^2/(L(h)^2 + \varepsilon) \to \xi\mathbf{1}_{\{L(h)\neq 0\}}$ as $\varepsilon \to 0$ in $L^p(\Omega)$. If we will show that $\eta_\varepsilon(h) \in \mathbb{D}^{1,p}$ and $D\eta_\varepsilon(h) \to 0$ in $L^{2,p}(M_\pi)$ for any $h \in K$ then $\xi\mathbf{1}_{\{L(h)\neq 0\}}$ will be in $\mathbb{D}^{1,p}$ and $D(\xi\mathbf{1}_{\{L(h)\neq 0\}}) = 0$ for any $h \in K$ implying as in the proof of Lemma 2.4 that $\xi \in \mathbb{D}^{1,p}$ and $D\xi = 0$.

Let us show that $\eta_\varepsilon(h) \in \mathbb{D}^{1,p}$ and $D\eta_\varepsilon(h) \to 0$. Set $f(x,y) = x^2/(y^2 + \varepsilon)$. Then $f(x,y)e^{-(x^2+y^2)/n} = f_n(x,y) \in C_b^\infty(\mathbb{R}^2)$ and by dominated convergence theorem we have $f_n(L(\xi h), L(h)) \to f(L(\xi h), L(h)) = \eta_\varepsilon(h)$ as $n \to \infty$ in $L^p(\Omega)$. In the same way we obtain that the derivative $Df_n(L(\xi h), L(h))$ converges in $L^{2,p}(M_\pi)$ which implies $\eta_\varepsilon(h) \in \mathbb{D}^{1,p}$ and the limit $D(\eta_\varepsilon(h))$ is given by

$$D(\eta_\varepsilon(h)) = \left(\frac{2L(\xi h)\xi h}{L(h)^2 + \varepsilon} - \frac{2L(h)L(\xi h)^2 h}{(L(h)^2 + \varepsilon)^2} \right)\mathbf{1}_\Delta + \left(\frac{(L(\xi h) + \xi h)^2}{(L(h) + h)^2 + \varepsilon} - \frac{L(\xi h)^2}{L(h)^2 + \varepsilon} \right)\mathbf{1}_{X_0}$$

$$= \left(\frac{2\xi^2 L(h)h}{(L(h)^2 + \varepsilon)^2}\mathbf{1}_\Delta + \xi^2 \frac{(L(h) + h)^2 - L(h)^2}{((L(h) + h)^2 + \varepsilon)(L(h)^2 + \varepsilon)}\mathbf{1}_{X_0} \right)\varepsilon.$$

Letting $\varepsilon \to 0$ in the equality above we obtain the desired result. $\square$

The following lemma shows the action of the operator D via the chaos decomposition.

Proposition 4.11 *Let $\xi \in L^2(\Omega)$ with a development*

$$\xi = \sum_{k=0}^{\infty} I_k(f_k), \tag{4.11}$$

where $f_k \in L^2(M_\pi^k)$ are symmetric with respect to pairs of nonrandom variables. Then $\xi \in \mathbb{D}^{1,2}$ if and only if

$$\sum_{k=1}^{\infty} kk! \, \|f_k\|_{L^2(M_\pi^k)}^2 < \infty \tag{4.12}$$

and in this case we have

$$D_{t,x}\xi = \sum_{k=1}^{\infty} kI_{k-1}(f_k(\cdot, t, x)). \tag{4.13}$$

Moreover

$$\mathsf{E}\left[\int_{T \times X} (D_{t,x}\xi)^2 \pi(dtdx) \,\Big|\, \mathcal{H} \right] = \sum_{k=1}^{\infty} kk! \int_{(T \times X)^k} f_k(t_1, x_1, \ldots, t_k, x_k)^2 \pi(dt_1dx_1) \cdots \pi(dt_kdx_k).$$

PROOF. The proof will be done in three steps.

Step 1. Suppose first that $k \geq 1$ and

$$\xi = P_k(\overline{x}(h)) = \frac{1}{k!} I_k(h^{\otimes k}) = I_k(f_k), \qquad (4.14)$$

with $h \in K$. Then by Lemma 4.8 $\xi \in \mathbb{D}^{1,2}$ and by equality (4.9) we get

$$D_{t,x} P_k(\overline{x}(h)) = P_{k-1}(\overline{x}(h)) h(t,x).$$

Hence

$$D_{t,x}\xi = k I_{k-1}(f_k(\cdot, t, x)). \qquad (4.15)$$

From Proposition 4.10 and formula (4.10) we deduce that equality (4.15) holds for any linear combination of random variables of the form $\eta P_k(\overline{x}(h))$, where η is $\mathcal{H}$-measurable bounded random variable. Since formula (4.15) implies that $\|D\xi\|^2_{L^2(M_\pi)} = k \mathsf{E} \xi^2$ then it follows that $\mathcal{P}_k$, $k \geq 1$ is included in $\mathbb{D}^{1,2}$.

If $k = 0$ then Proposition 4.10 implies that $\mathcal{P}_0 = L^2(\Omega, \mathcal{H}, \mathsf{P}) \subset \mathbb{D}^{1,2}$.

Step 2. Let $\xi \in L^2(\Omega)$ has an expansion (4.11). Suppose that (4.12) holds. Define

$$\xi_n = \sum_{k=0}^{n} I_k(f_k).$$

Then the sequence ξ_n converges to ξ in $L^2(\Omega)$, and by Step 1 we have $\xi_n \in \mathbb{D}^{1,2}$ and $D_{t,x}\xi = \sum_{k=1}^{n} k I_{k-1}(f_k(\cdot, t, x))$. It follows from Lemma 3.5 and equality (4.12) that $D_{t,x}\xi_n$ converges in $L^2(M_\pi)$ to the right-hand side of (4.13). Therefore $\xi \in \mathbb{D}^{1,2}$ and (4.13) holds.

Step 3. Suppose $\xi \in \mathbb{D}^{1,2}$. Note that formula (4.5) holds for $\xi \in \mathbb{D}^{1,2}$ and $\eta \in \mathbb{D}^{1,p}$ for some $p > 2$ if $h \in K$. Since by Proposition 4.8 $\eta = P_m(\overline{x}(g)) \in \mathbb{D}^{1,p}$ for all $p \geq 1$ and $g \in K$, then we have

$$\lim_{n \to \infty} \left(\langle D\xi_n; \eta h \rangle_{L^2(M_\pi)} + \langle D\xi_n; D\eta h \mathbf{1}_{X_0} \rangle_{L^2(M_\pi)} \right) = \lim_{n \to \infty} \left(\mathsf{E}(\xi_n \eta L(h)) - \langle D\eta; \xi_n h \rangle_{L^2(M_\pi)} \right)$$

$$= \mathsf{E}(\xi \eta L(h)) - \langle D\eta; \xi h \rangle_{L^2(M_\pi)} = \langle D\xi; \eta h \rangle_{L^2(M_\pi)} + \langle D\xi; D\eta h \mathbf{1}_{X_0} \rangle_{L^2(M_\pi)}.$$

It follows from equation (4.8) that $\eta + \mathbf{1}_{X_0} D\eta = P_m(\overline{x}(g)) + \mathbf{1}_{X_0} g P_{m-1}(\overline{x}(g))$. Then for all $m = 1, 2, \ldots$ we obtain

$$\lim_{n \to \infty} \left(\langle D\xi_n; P_m(\overline{x}(g)) h \rangle_{L^2(M_\pi)} + \langle D\xi_n; P_{m-1}(\overline{x}(g)) g h \mathbf{1}_{X_0} \rangle_{L^2(M_\pi)} \right)$$

$$= \langle D\xi; P_m(\overline{x}(g)) h \rangle_{L^2(M_\pi)} + \langle D\xi; P_{m-1}(\overline{x}(g)) g h \mathbf{1}_{X_0} \rangle_{L^2(M_\pi)}.$$

Since $P_0 = 1$ and $\lim_{n \to \infty} \langle D\xi_n; P_0(\overline{x}(g)) h \rangle_{L^2(M_\pi)} = \langle D\xi; P_0(\overline{x}(g)) h \rangle_{L^2(M_\pi)}$ for all $h \in L^2(M_\pi)$, then we deduce by induction that

$$\lim_{n \to \infty} \langle D\xi_n; P_m(\overline{x}(g)) h \rangle_{L^2(M_\pi)} = \langle D\xi; P_m(\overline{x}(g)) h \rangle_{L^2(M_\pi)}.$$

From Lemma 3.5 we deduce that for $n > m$ the expression $\langle D\xi_n; P_m(\overline{x}(g)) h \rangle_{L^2(M_\pi)}$ is equal to

$$\mathsf{E} \left(\int_{T \times X} (m+1) I_m \left(f_{m+1}(\cdot, t, x) \right) h(t,x) \pi(dt dx) P_m(\overline{x}(g)) \right).$$

Hence the projection of $\int_{T\times X} D_{t,x}\xi h(t,x)\pi(dtdx)$ on the m-th chaos is equal to

$$\int_{T\times X} (m+1)I_m\left(f_{m+1}(\cdot,t,x)\right)h(t,x)\pi(dtdx).$$

Thus for any $\zeta \in L^2(\Omega,\mathcal{F},\mathsf{P})$ we have

$$\langle D\xi; h\zeta\rangle_{L^2(M_\pi)} = \mathsf{E}\left(\int_{T\times X} D_{t,x}\xi h(t,x)\pi(dtdx)\zeta\right)$$

$$= \mathsf{E}\left(\sum_{m=0}^{\infty}\int_{T\times X}(m+1)I_m\left(f_{m+1}(\cdot,t,x)\right)h(t,x)\pi(dtdx)\zeta\right)$$

$$= \langle\sum_{m=0}^{\infty}(m+1)I_m\left(f_{m+1}(\cdot,t,x)\right);h\zeta\rangle_{L^2(M_\pi)}.$$

Since the set $\{h\zeta : h \in K, \zeta \in L^2(\Omega,\mathcal{F},\mathsf{P})\}$ is dense in $L^2(T\times X\times\Omega,\mathcal{G}\otimes\mathcal{F},M_\pi)$ then

$$D_{t,x}\xi = \sum_{m=0}^{\infty}(m+1)I_m\left(f_{m+1}(\cdot,t,x)\right),$$

which completes the proof of the proposition. $\qquad\square$

Remark 4.12 *This proposition implies that the operator D is an annihilation operator on the Fock space on Hilbert space H.*

The equations (4.13) can be considered as a definition of the operator D. This approach was developed for pure jump Lévy process, the particular case of Poisson processes, the case of general Lévy process with no drift and the case of certain class of martingales in [2, 3, 18, 19, 25, 28].

Let $A \in \mathcal{G}\otimes\mathcal{H}$. We will denote by $\mathcal{F}_A^0$ the σ-algebra generated by the random variables $\{L(B), B \subset A, B \in \mathcal{G}_0\}$. Set $\mathcal{F}_A = \mathcal{F}_A^0 \vee \mathcal{H}$. The following results are modification of Proposition 1.2.5 from [22, p. 32] and it shows how to compute the derivative of a conditional expectation with respect to a σ-algebra generated by stochastic process.

Lemma 4.13 *Suppose that $\xi \in L^2(\Omega,\mathcal{F},\mathsf{P})$ with the expansion (4.11). Let $A \in \mathcal{G}\otimes\mathcal{H}$. Then*

$$\mathsf{E}[\xi|\mathcal{F}_A] = \sum_{k=0}^{\infty}I_k(f_k\mathbf{1}_A^{\otimes k}). \tag{4.16}$$

PROOF.   By the density of elementary functions in $L^2(M_\pi^k)$ and by linearity we can assume that $\xi = I_k(f_k)$, where $f_k = \eta\mathbf{1}_{A_1}\otimes\cdots\otimes\mathbf{1}_{A_k}$ with pairwise-disjoint sets $A_1,\ldots,A_k \in \mathcal{G}_0$ and $\eta \in L^\infty(\Omega,\mathcal{H},\mathsf{P})$. Then we have

$$\mathsf{E}[\xi|\mathcal{F}_A] = \mathsf{E}[\eta L(A_1)\cdots L(A_k)|\mathcal{F}_A]$$

$$= \eta\mathsf{E}\left[\prod_{i=1}^{k}(L(A_i\cap A) + L(A_i\setminus A))\Bigg|\mathcal{F}_A\right] = \eta\mathsf{E}[L(A_1\cap A)\cdots L(A_m\cap A)|\mathcal{F}_A] =$$

$$= I_k(\eta\mathbf{1}_{A_1\cap A}\otimes\cdots\otimes\mathbf{1}_{A_k\cap A}) = I_k(f_k\mathbf{1}_A^{\otimes k}).$$

$\qquad\square$

Proposition 4.14 *Suppose that* $\xi \in \mathbb{D}^{1,2}$, *and* $A \in \mathcal{G} \otimes \mathcal{H}$. *Then* $\mathsf{E}(\xi|\mathcal{F}_A) \in \mathbb{D}^{1,2}$ *and we have*

$$D_{t,x}(\mathsf{E}(\xi|\mathcal{F}_A)) = \mathsf{E}(D_{t,x}\xi|\mathcal{F}_A)\mathbf{1}_A(t,x)$$

M_π*-a.e. in* $T \times X \times \Omega$.

PROOF.   By Lemma 4.13 and Proposition 4.11 we obtain

$$\mathsf{E}(D_{t,x}\xi|\mathcal{F}_A)\mathbf{1}_A(t,x) = \sum_{k=1}^{\infty} kI_{k-1}(f_k(\cdot,t,x)\mathbf{1}_A^{\otimes(k-1)})\mathbf{1}_A(t,x) = D_{t,x}(\mathsf{E}(\xi|\mathcal{F}_A)).$$

$\square$

Remark 4.15 *In particular, if* ξ *is* $\mathcal{F}_A$*-measurable and belongs to* $\mathbb{D}^{1,2}$, *then* $D_{t,x}\xi = 0$ M_π*-a.e. in* A^c.

5   The Skorohod integral

In this section we consider the adjoint of the operator D, and we will show that it coincides with the Skorohod integral [30] in the Gaussian case and with the extended stochastic integral introduced by Kabanov [13] in the pure jump Lévy case. See also [2, 3, 18, 28]. So it can be considered as a generalization of the stochastic integral. We will call it Skorohod integral and will establish the expression of it in terms of the chaos expansion as well as prove some of its properties.

We recall that the derivative operator D is a closed and unbounded operator defined on the dense subset $\mathbb{D}^{1,2}$ of $L^2(\Omega)$ with values in $L^2(T \times X \times \Omega, \mathcal{G} \otimes \mathcal{F}, M_\pi)$.

Definition 5.1 *We denote by* δ *the adjoint of the operator* D *and will call it Skorohod integral.*

The operator δ is closed unbounded operator on $L^2(T \times X \times \Omega, \mathcal{G} \otimes \mathcal{F}, M_\pi)$ with values in $L^2(\Omega)$ defined on Dom δ, where Dom δ is the set of processes $u \in L^2(M_\pi)$ such that

$$\left| \mathsf{E} \int_{T \times X} D_{t,x}\xi u(t,x)\pi(dtdx) \right| \leq c \left\| \xi \right\|_{L^2(\Omega)}$$

for all $\xi \in \mathbb{D}^{1,2}$, where c is some constant depending on u.

If $u \in$ Dom δ, then $\delta(u)$ is the element of $L^2(\Omega)$ such that

$$\mathsf{E}(\xi\delta(u)) = \mathsf{E} \int_{T \times X} D_{t,x}\xi u(t,x)\pi(dtdx) \tag{5.1}$$

for any $\xi \in \mathbb{D}^{1,2}$.

The following proposition shows the behavior of δ in terms of the chaos expansion.

Proposition 5.2 *Let* $u \in L^2(T \times X \times \Omega, \mathcal{G} \otimes \mathcal{F}, M_\pi)$ *with the expansion*

$$u(t,x) = \sum_{k=0}^{\infty} I_k(f_k(\cdot, t, x)). \tag{5.2}$$

Then $u \in \mathrm{Dom}\ \delta$ *if and only if the series*

$$\delta(u) = \sum_{k=0}^{\infty} I_{k+1}(\tilde{f}_k) \tag{5.3}$$

converges in $L^2(\Omega)$.

Recall that $\tilde{f}_k$ is a symmetrization of f_k in all its pairs of nonrandom variables is given by

$$\tilde{f}_k(t_1, x_1, \ldots, t_k, x_k, t, x, \omega) = \frac{1}{k+1}(f_k(t_1, x_1, \ldots, t_k, x_k, t, x, \omega)$$

$$+ \sum_{i=1}^{k} f_k(t_1, x_1, \ldots, t_{i-1}, x_{i-1}, t, x, t_{i+1}, x_{i+1}, \ldots, t_i, x_i, \omega)).$$

PROOF.   The proof is the same as in the Gaussian case (see, e.g., [22, Prop. 1.3.1, p. 36]).

□

Remark 5.3 *It follows from Proposition 5.2 that the operator* δ *coincides with Skorohod integral in the Gaussian case and with extended stochastic integral introduced by Kabanov for pure jump Lévy processes (see, e.g., [30, 13, 22, 2, 3, 18, 28]).*

It follows from proposition above that Dom δ is the subspace of $L^2(M_\pi)$ formed by the processes that satisfy the following condition:

$$\sum_{k=1}^{\infty}(k+1)! ||\tilde{f}_k||^2_{L^2(M_\pi^{k+1})} < \infty. \tag{5.4}$$

If $u \in \mathrm{Dom}\ \delta$, then the sum of the series (5.4) is equal to $\mathsf{E}\delta(u)^2$.

Note that the Skorohod integral is a linear operator and has zero mean, e.g., $\mathsf{E}(\delta(u)) = 0$ if $u \in \mathrm{Dom}\ \delta$. The following statements prove some properties of δ.

Proposition 5.4 *Let* $u, v \in \mathrm{Dom}\ \delta$ *be arbitrary stochastic process. Then for all* α *and* β *in* $L^\infty(\Omega, \mathcal{H}, \mathsf{P})$ *we have* $\alpha u + \beta v \in \mathrm{Dom}\ \delta$ *and*

$$\delta(\alpha u + \beta v) = \alpha \delta(u) + \beta \delta(v).$$

Moreover $\mathsf{E}[\delta(u)|\mathcal{H}] = 0$.

PROOF.   The proof follows from the properties (i) and (iii) of the multiple integral.   □

Proposition 5.5 *Suppose that u is a Skorohod integrable process. Let $\xi \in \mathbb{D}^{1,2}$ such that $\mathsf{E}(\int_{T \times X}(\xi^2 + (D_{t,x}\xi)^2 \mathbf{1}_{X_0})u(t,x)^2 \pi(dtdx)) < \infty$. Then it holds that*

$$\delta((\xi + \mathbf{1}_{X_0}D\xi)u) = \xi\delta(u) - \int_{T \times X}(D_{t,x}\xi)u(t,x)\pi(dtdx), \tag{5.5}$$

provided that one of the two sides of the equality (5.5) exists.

PROOF.   Let $\eta \in \mathcal{S}$ be a smooth random variables. Then by the product rule (4.10) and by the duality relation (5.1), we get

$$\mathsf{E}(\int_{T \times X}(D_{t,x}\eta)(\xi + \mathbf{1}_{X_0}(x)D_{t,x}\xi)u(t,x)\pi(dtdx)) = \int_{T \times X} \mathsf{E}(u(t,x)(D_{t,x}(\xi\eta) - \eta D_{t,x}\xi))\pi(dtdx)$$

$$= \mathsf{E}\left(\eta(\xi\delta(u) - \int_{T \times X}(D_{t,x}\xi)u(t,x)\pi(dtdx))\right),$$

and the result follows. $\square$

As in the Gaussian case or in the case of processes with independent increments in order to prove some other properties of Skorohod integral we will define a class of processes contained in Dom δ (see [22], [33]).

Definition 5.6 *Let $\mathbb{L}^{1,2}$ denote the class of processes $u \in L^2(T \times X \times \Omega, \mathcal{G} \otimes \mathcal{F}, M_\pi)$ such that $u(t,x) \in \mathbb{D}^{1,2}$ for all $(t,x) \notin R$, where $R \subset T \times X$ and $M_\pi(R \times \Omega) = 0$, and there exists a measurable version of the multiparametrical process $D_{t,x}u(s,y)$ satisfying $\mathsf{E}\int_{T \times X}\int_{T \times X}(D_{t,x}u(s,y))^2\pi(dtdx)\pi(dsdy) < \infty$.*

If the process u has the expansion (5.2), then $u \in \mathbb{L}^{1,2}$ if and only if the series

$$\int_{T \times X}\int_{T \times X}\mathsf{E}\left(\sum_{k=1}^\infty kI_{k-1}(f_k(\cdot,t,x,s,y))\right)^2 \pi(dtdx)\pi(dsdy) = \sum_{k=1}^\infty kk!\,||f_k||^2_{L^2(M_\pi^{k+1})}$$

converges.

Since $||\tilde{f}_k||_{L^2(M_\pi^{k+1})} \le ||f_k||_{L^2(M_\pi^{k+1})}$ then from (5.4) we deduce that $\mathbb{L}^{1,2} \subset$ Dom δ.

The proofs of the following propositions use the chaos expansion therefore they can be done as in the Gaussian case (see, for instance [22, pp. 38 - 40]).

Proposition 5.7 *Suppose that $u \in \mathbb{L}^{1,2}$ and for all $(t,x) \notin R$, where $R \subset T \times X$ and $M_\pi(R \times \Omega) = 0$ the two-parameter process $\{D_{t,x}u(s,y), (s,y) \in T \times X\}$ is Skorohod integrable, and there exists a version of the process $\{\delta(D_{t,x}u(\cdot,\cdot)), (t,x) \in T \times X\}$ which belongs to $L^2(M_\pi)$. Then $\delta(u) \in \mathbb{D}^{1,2}$ and we have*

$$D_{t,x}\delta(u) = u(t,x) + \delta(D_{t,x}u(\cdot,\cdot)). \tag{5.6}$$

Proposition 5.8 *Suppose that $u \in \mathbb{L}^{1,2}$ and $v \in \mathbb{L}^{1,2}$. Then we have*

$$\mathsf{E}[\delta(u)\delta(v)|\mathcal{H}] = \mathsf{E}\left[\int_{T \times X}u(t,x)v(t,x)\pi(dtdx)\Big|\mathcal{H}\right]$$

$$+\mathsf{E}\left[\int_{T \times X}\int_{T \times X}D_{s,y}u(t,x)D_{t,x}v(s,y)\pi(dtdx)\pi(dsdy)\Big|\mathcal{H}\right]. \tag{5.7}$$

Now we will show that the operator δ is an extension of the Itô integral. Let L_t, $t \in [0; 1]$ be a processes with $\mathcal{H}$-conditionally independent increments. Assume that the canonical triplet of its characteristics (B, μ, ν) such that $B = 0$. Then as in Example 2.3 we have random measure $N(dtdx)$ associated to jumps of L with compensator measure ν, the measure μ connected with continuous part of L and conditional additive process $L(h)$ on H. We denote by L_p^2 the subset of $L^2(M_\pi)$ formed by $\mathcal{F}_t$-predictable processes.

Te following technical will be needed.

Lemma 5.9 *Let $A \in \mathcal{G} \otimes \mathcal{H}$ be a set with finite M_π measure, and let ξ be a square integrable random variable that is measurable with respect to the σ-algebra $\mathcal{F}_{A^c}$. Then the process $\xi \mathbf{1}_A$ is Skorohod integrable and*

$$\delta(\xi \mathbf{1}_A) = \xi L(A).$$

PROOF. Suppose first that $\xi \in \mathbb{D}^{1,2}$ and $\mathbf{1}_A \in K$. By using Proposition 5.5 and Remark 4.15 we have

$$\delta(\xi \mathbf{1}_A) = \delta((\xi + \mathbf{1}_{X_0} D\xi) \mathbf{1}_A) = \xi \delta(\mathbf{1}_A) - \int_{T \times X} (D_{t,x}\xi) \mathbf{1}_A(t, x) \pi(dtdx) = \xi L(A).$$

The general case follows by a limit argument, using the facts that $\mathbb{D}^{1,2}$ and K are dense and δ is closed. □

Proposition 5.10 *$L_p^2 \subset \mathrm{Dom}\ \delta$, and the restriction of the operator δ to the space L_p^2 coincides with the usual stochastic integral, that is*

$$\delta(u) = \int_0^1 u(t, 0) dL^c(t) + \int_0^1 \int_{\mathbb{R}_0} u(t, x)(N(dtdx) - \nu(dtdx)).$$

PROOF. Suppose that u is an elementary adapted processes of the form

$$u_{t,x} = \sum_{i=1}^n \xi_i \mathbf{1}_{A_i} \mathbf{1}_{(t_i; t_{i+1}]}(t) \mathbf{1}_{B_i}(x),$$

where $0 \leq t_1 < \cdots < t_{n+1} \leq 1$, $\mathcal{B}_i$ is a borel set of $\mathbb{R}$, $A_i \in \mathcal{H}$ such that $\mathbf{1}_{(t_i; t_{i+1}] \times B_i \times A_i}(t, x) \in K$ and ξ_i is square integrable and $\mathcal{F}_{t_i}$ measurable random variable. Then from the Lemma 5.9 we obtain $u \in \mathrm{Dom}\ \delta$ and

$$\delta(u) = \sum_{i=1}^\infty \xi_i \left((L_{t_{i+1}} - L_{t_i}) \mathbf{1}_{B_i}(0) + \int_{t_i}^{t_{i+1}} \int_{B_i} (N(dtdx) - \nu(dtdx)) \right).$$

The general case follows by monotone class argument since δ is closed. □

The predictable projection of a stochastic process indexed by $t \geq 0$ and $x \in \mathbb{R}$ can be defined similarly as in a one parametrical case (see, i.g, [12, 17, 31]). Let $Y = \{Y(t, x), t \geq 0, x \in \mathbb{R}\}$ be an measurable integrable process. There exists a predictable process $Z = \{Z(t, x), t \geq 0, x \in \mathbb{R}\}$ such that for every predictable stopping time τ

$$Z(\tau, x) \mathbf{1}_{\{\tau < \infty\}} = \mathsf{E}[Y(\tau, x) \mathbf{1}_{\{\tau < \infty\}} | \mathcal{F}_{\tau-}].$$

The following result is so-called Clark-Haussmann-Ocone formula.

Proposition 5.11 *Let $\xi \in \mathbb{D}^{1,2}$, and suppose that a process with conditionally independent increments L_t, $t \in [0;1]$ has the form*

$$L_t = L_t^c + \int_0^t \int_{|x| \leq 1} x(N(dsdx) - \nu(dsdx)) + \int_0^t \int_{|x| > 1} xN(dsdx).$$

Then

$$\xi = \mathsf{E}[\xi|\mathcal{H}] + \int_0^1 {}^p(D_{t,0}\xi)dL_t^c + \int_0^1 \int_{\mathbb{R}_0} {}^p(D_{t,x}\xi)(N(dtdx) - \nu(dtdx)). \tag{5.8}$$

PROOF.   Let $\xi \in \mathbb{D}^{1,2}$ have an expansion $\xi = \sum_{n=0}^\infty I_n(f_n)$. Using (4.13) and (4.16) we have

$$\mathsf{E}[D_{t,x}\xi|\mathcal{F}_t] = \sum_{n=1}^\infty n\mathsf{E}[I_{n-1}(f_n(\cdot,t,x))|\mathcal{F}_t] = \sum_{n=1}^\infty nI_{n-1}(f_n(\cdot,t,x)\mathbf{1}_{[0;t]}^{\otimes(n-1)}).$$

It follows from the arguments in the beginning of the Section 3, that $f_n(\cdot,t,x)\mathbf{1}_{[0;t]}^{\otimes(n-1)} = f_n(\cdot,t,x)\mathbf{1}_{[0;t)}^{\otimes(n-1)}$ as elements of $L^2(M_\pi^{n-1})$. Thus

$$\mathsf{E}[D_{t,x}\xi|\mathcal{F}_t] = \sum_{n=1}^\infty nI_{n-1}(f_n(\cdot,t,x)\mathbf{1}_{[0;t]}^{\otimes(n-1)}) = \sum_{n=1}^\infty nI_{n-1}(f_n(\cdot,t,x)\mathbf{1}_{[0;t)}^{\otimes(n-1)}).$$

Since $I_{n-1}(f_n(\cdot,t,x)\mathbf{1}_{[0;t)}^{\otimes(n-1)})$ is predictable then it is easy to show that ${}^p\big(I_{n-1}(f_n(\cdot,t,x))\big) = I_{n-1}(f_n(\cdot,t,x)\mathbf{1}_{[0;t)}^{\otimes(n-1)})$. Hence

$${}^p(D_{t,x}\xi) = \sum_{n=1}^\infty n\, {}^p(I_{n-1}(f_n(\cdot,t,x))) = \sum_{n=1}^\infty nI_{n-1}(f_n(\cdot,t,x)\mathbf{1}_{[0;t)}^{\otimes(n-1)}).$$

Therefore ${}^p(D_{t,x}\xi) = \mathsf{E}[D_{t,x}\xi|\mathcal{F}_t]$ as an $L^2(M_\pi)$ processes.

Set $\phi(t,x) = \mathsf{E}[D_{t,x}|\mathcal{F}_t]$ and $\psi(t,x) = {}^p(D_{t,x}\xi)$. Then from equality (5.3) we deduce

$$\delta(\psi) = \delta(\phi) = \sum_{n=1}^\infty I_n(f_n) = \xi - \mathsf{E}[\xi|\mathcal{H}],$$

which shows the desired result because Proposition 5.10 implies that

$$\delta(\psi) = \int_0^1 {}^p(D_{t,0}\xi)dL_t^c + \int_0^1 \int_{\mathbb{R}_0} {}^p(D_{t,x}\xi)(N(dtdx) - \nu(dtdx)).$$

$\square$

Acknowledgments

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