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An $\mathcal{O}$-acyclic variety of even index

With an appendix by Olivier Wittenberg

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Abstract

We give the first examples of $\mathcal{O}$-acyclic smooth projective geometrically connected varieties over the function field of a complex curve, whose index is not equal to one. More precisely, we construct a family of Enriques surfaces over $\mathbb{P}^1$ such that any multi-section has even degree over the base $\mathbb{P}^1$ and show moreover that we can find such a family defined over $\mathbb{Q}$. This answers affirmatively a question of Colliot-Thélène and Voisin. Furthermore, our construction provides counterexamples to: the failure of the Hasse principle accounted for by the reciprocity obstruction; the integral Hodge conjecture; and universality of Abel–Jacobi maps.

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1 Introduction

In a letter to Grothendieck [10, p. 152], Serre asked whether a smooth projective geometrically connected variety Y over the function field of a complex curve should always have a rational point if it is $\mathcal{O}$-acyclic, that is, $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$. This indeed holds for rationally connected varieties, as proved by Graber–Harris–Starr [15], generalizing a classical theorem of Tsen. However, Graber–Harris–Mazur–Starr

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[14] gave a counterexample for the general case; in fact, they showed that there exist Enriques surfaces with no rational points over the function field of a complex curve. Later, more explicit constructions of such Enriques surfaces were given by Lafon [19] and Starr [26]. It is remarkable that the example of Lafon is defined over $\mathbb{Q}(t)$ and has no rational point over the local field $\mathbb{C}((t))$.

In light of these examples, one might still hope that a weaker statement could be true. We recall that the *index* of a proper variety Y over a field F is defined to be

$$I(Y) = \gcd \left\{ \deg_F(\alpha) \mid \alpha \in CH_0(Y) \right\}.$$

One can then ask:

Question 1.1 *Does an $\mathcal{O}$-acyclic smooth projective geometrically connected variety Y over the function field of a complex curve always have $I(Y) = 1$?*

In other words, we ask whether Serre's question has a positive answer if we replace a rational point on Y with a 0-cycle of degree 1.

It is important to note that there is no local obstruction here: the Riemann–Roch theorem implies that Y as in Question 1.1 always has indices one everywhere locally, or equivalently, that Y gives a one-parameter family $X \to C$ of $\mathcal{O}$-acyclic varieties with no multiple fiber (see also [9, Proposition 7.3] and [12, Theorem 1]).

Nevertheless, it was expected by several mathematicians that Question 1.1 would have a negative answer (see [26] for expectations of Esnault on the indices of the examples of Graber–Harris–Mazur–Starr and Lafon). In particular, Colliot-Thélène and Voisin asked [9, Question 7.9] whether one can construct an $\mathcal{O}$-acyclic surface of index not equal to one. The aim of this paper is to give the first counterexamples to Question 1.1 and thereby to answer affirmatively the question raised by Colliot-Thélène and Voisin. Our main result is the following:

Theorem 1.2 (=Theorem 3.1, 4.1) *Let $X \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ be the rank one degeneracy locus of a map of vector bundles*

$$\mathcal{O}^{\oplus 3} \to \mathcal{O}(2, 2, 0) \oplus \mathcal{O}(2, 0, 2).$$

If X is very general, then the first projection gives a family $X \to \mathbb{P}^1$ of Enriques surfaces such that any multi-section has even degree over the base $\mathbb{P}^1$. That is, the index $I(X_\eta)$ is even, where X_η is the generic fiber. Moreover, we can find threefolds with these properties defined over $\mathbb{Q}$.

Remark 1.3 Our construction can be generalized to give a counterexample to Question 1.1 when $\dim Y = 2n$ for any positive integer n (besides ones obtained from Theorem 1.2 by taking the product with a projective space); see Theorem 3.3.

Remark 1.4 It would be natural to ask an analogue of Question 1.1 over the function field of a curve over the algebraic closure of a finite field. We will prove some conditional positive results in Proposition 4.4 and Corollary 4.6.

Our construction has consequences for certain questions in number theory. We say that the Hasse principle holds for 0-cycles of degree 1 on a smooth projective geometrically connected variety Y over the function field $F = \mathbb{C}(C)$ of a complex curve C if there is a 0-cycle of degree 1 on Y whenever there is such a cycle on Y_{F_p} for any point $p \in C$, where $F_p \cong \mathbb{C}((t))$ is the completion of F at p. The reciprocity obstruction to the Hasse principle for 0-cycles of degree 1 on a variety over the function field of a complex curve, which is an analogue of the Brauer–Manin obstruction for rational points on a variety over a number field, was defined and pointed out to the authors by Colliot-Thélène (see also [7, Sect. 5]).

As a consequence of our construction, we prove that the failure of the Hasse principle for 0-cycles of degree 1 on an Enriques surface over $\mathbb{C}(\mathbb{P}^1)$ cannot always be accounted for by the reciprocity obstruction.

Theorem 1.5 (=Theorem 5.2) *Let X_η be the generic fiber of a very general family $X \to \mathbb{P}^1$ of Enriques surfaces as in Theorem 1.2. Then the Hasse principle fails for 0-cycles of degree 1 on X_η, while there is no reciprocity obstruction for X_η.*

Question 1.1 is also related to the integral Hodge conjecture. We recall that the integral Hodge conjecture in degree $2i$ on a smooth complex projective variety X is the statement that degree $2i$ integral Hodge classes on X are algebraic, i.e., the image $H^{2i}_{\mathrm{alg}}(X, \mathbb{Z}) \subseteq H^{2i}(X, \mathbb{Z})$ of the cycle class map $\mathrm{cl}^i : CH^i(X) \to H^{2i}(X, \mathbb{Z})$ generates the entire group $Hdg^{2i}(X, \mathbb{Z}) = H^{i,i}(X) \cap H^{2i}(X, \mathbb{Z})$ of integral Hodge classes. While the statement holds for $i = 0, 1, \dim X$, it is known to fail in general for $2 \le i \le \dim X - 1$. The first counterexample was constructed by Atiyah–Hirzebruch [1] and many others have been found since then [2, 3, 9, 20, 23, 25, 29].

As pointed out by Colliot-Thélène and Voisin [9, Theorem 7.6], a counterexample to Question 1.1 gives a one-parameter family $X \to C$ of $\mathcal{O}$-acyclic varieties for which the integral Hodge conjecture fails in degree $2d - 2$, where $d = \dim X$. This means that the defect of the integral Hodge conjecture in degree $2d - 2$, defined as

$$Z^{2d-2}(X) = Hdg^{2d-2}(X, \mathbb{Z})/H^{2d-2}_{\mathrm{alg}}(X, \mathbb{Z}),$$

is non-zero.

It follows that the integral Hodge conjecture fails in degree 4 for the threefold X in Theorem 1.2, and that the defect $Z^4(X)$ is non-zero. In the last part of the paper, we determine completely the 2-torsion subgroup $Z^4(X)[2]$. In addition, this allows us to compute explicitly the degree 3 unramified cohomology group $H^3_{\mathrm{nr}}(X, \mathbb{Z}/2)$, a stable birational invariant of smooth complex projective varieties defined in the framework of the Bloch–Ogus theory [5]. A key input is a theorem of Colliot-Thélène and Voisin [9, Theorem 3.9] together with the fact that we have $CH_0(X) = \mathbb{Z}$ (this can be deduced from a result of Bloch–Kas–Lieberman [4]).

Theorem 1.6 (=Theorem 6.1, Corollary 6.2) *Let X be the total space of a very general family of Enriques surfaces as in Theorem 1.2. Then we have*

$$H^3_{\mathrm{nr}}(X, \mathbb{Z}/2) = Z^4(X)[2] = (\mathbb{Z}/2)^{46}.$$

Remark 1.7 Note that there is a 2-torsion element in the Néron–Severi group of the geometric generic fiber of the family $X \to \mathbb{P}^1$. In contrast, Colliot-Thélène and Voisin proved that if $X \to C$ is a family of $\mathcal{O}$-acyclic surfaces such that the geometric generic fiber has torsion free Néron–Severi group, then the degree 3 unramified cohomology group with torsion coefficients is conjecturally of rank at most one [9, Theorem 7.7, 8.21,Remark 8.22].

We note that Theorem 1.2 also has an application to universality of the Abel-Jacobi maps. A classical question of Murre asks whether the Abel-Jacobi map is universal among all regular homomorphisms (see [20, Sect, 4] and [27, Sect, 1] for more precise statements). Recently, a negative answer to the question was given by a fourfold constructed by the authors [20]. In fact, the threefold X of Theorem 1.2 can be used to construct another such fourfold. We refer the reader to the papers [20] and [27] for the details of the argument.

This paper is organized as follows. In Sect. 2, we introduce certain families of Enriques surfaces parametrized by $\mathbb{P}^1$ and study their basic properties. In Sect. 3, we prove the main theorem over $\mathbb{C}$ using an explicit geometric construction. The proof involves a combination of monodromy and specialization arguments, and a key congruence obtained previously by the authors in [20]. In Sect. 4, we refine this construction to get counterexamples defined over $\mathbb{Q}$. In Sect. 5, we discuss the failure of the Hasse principle and the reciprocity obstruction on our examples. In Sect. 6, we compute the defect of the integral Hodge conjecture in degree 4 on the total space of the family of Enriques surfaces of the main theorem, and in addition, its degree 3 unramified cohomology group with $\mathbb{Z}/2$ coefficients. Finally, in the Appendix, Olivier Wittenberg proves that the vanishing of the reciprocity obstruction obtained in Theorem 5.2 is in fact a completely general phenomenon.

Notation We work over the complex numbers in Sects. 2, 3, 5, and 6. In Sect. 4, we work over $\overline{\mathbb{Q}}$. We use Grothendieck's notation for projective bundles: for a vector bundle $\mathcal{E}$, $\mathbb{P}(\mathcal{E})$ parameterizes one-dimensional quotients of $\mathcal{E}$. We write $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ for the relative hyperplane bundle.

We will let $\mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^s}(a, b)$ and $\mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^s \times \mathbb{P}^t}(a, b, c)$ denote line bundles on products of projective spaces (i.e., these are $\mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^r}(a) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^s}(b)$ and $\mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^r}(a) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^s}(b) \otimes \mathrm{pr}_3^* \mathcal{O}_{\mathbb{P}^t}(c)$ respectively). Similarly, we will write $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}(\mathcal{E})}(a, b)$ for the line bundle $\mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(b)$ on $\mathbb{P}^1 \times \mathbb{P}(\mathcal{E})$. To simplify notation we will usually drop the subscripts when the context is clear.

2 Families of Enriques surfaces parametrized by $\mathbb{P}^1$

We will fix the following notation:

- $\mathbb{P}_A = \mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^2}(\mathcal{O}(2, 0) \oplus \mathcal{O}(0, 2))$, $E_1 = \mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^2}(\mathcal{O}(2, 0))$, $E_2 = \mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^2}(\mathcal{O}(0, 2))$
- $\mathbb{P}_B = \mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^2}(\mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1))$, $F_1 = \mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^2}(\mathcal{O}(1, 0))$, $F_2 = \mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^2}(\mathcal{O}(0, 1))$
- $\mathbb{P}_C = \mathbb{P}(H^0(\mathbb{P}_B, \mathcal{O}(1)))$, $P_1 = \mathbb{P}(H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(1, 0)))$, $P_2 = \mathbb{P}(H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(0, 1)))$.

As is explained in [20], these spaces are related by the following geometric construction: $\mathbb{P}_C$ is a 5-dimensional projective space, and P_1 and P_2 define disjoint planes in it via the isomorphism

$$H^0(\mathbb{P}_B, \mathcal{O}(1)) = H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(1, 0)) \oplus H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(0, 1)).$$
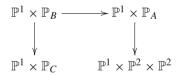
The projective bundle $\mathbb{P}_B$ is then identified with the blow-up of $\mathbb{P}_C$ along the union of P_1 and P_2, and F_1 and F_2 are the corresponding exceptional divisors. Furthermore, there is an involution ι on $\mathbb{P}_C$ induced by the involution on $H^0(\mathbb{P}_B, \mathcal{O}(1))$ with the (± 1)-eigenspaces $H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(1, 0))$ and $H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(0, 1))$, respectively. The involution ι lifts to an involution on the blow-up $\mathbb{P}_B$, and we have $\mathbb{P}_A = \mathbb{P}_B/\iota$. Thus there is a double cover $\mathbb{P}_B \to \mathbb{P}_A$ over $\mathbb{P}^2 \times \mathbb{P}^2$, which is ramified along $F_1 \cup F_2$, and the divisors F_i are mapped isomorphically onto E_i for $i = 1, 2$.

The varieties $\mathbb{P}_A, \mathbb{P}_B, \mathbb{P}_C$ were used in [20] to give projective models of Enriques surfaces. In this paper, we will use them to study the threefolds X in Theorem 1.2; these are Enriques surface fibrations over $\mathbb{P}^1$. We now explain the main construction.

Let $X \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ be the rank one degeneracy locus of a general map of vector bundles

$$\mathcal{O}^{\oplus 3} \to \mathcal{O}(2, 2, 0) \oplus \mathcal{O}(2, 0, 2). \tag{1}$$

Then X is a smooth threefold and the first projection $X \to \mathbb{P}^1$ defines a family of Enriques surfaces (see [20, Lemma 2.1]). There is a natural diagram

$$
\begin{array}{ccc}
\mathbb{P}^1 \times \mathbb{P}_B & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}_A \\
\downarrow & & \downarrow \\
\mathbb{P}^1 \times \mathbb{P}_C & & \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2
\end{array}
$$

in which $\mathbb{P}^1 \times \mathbb{P}_A \to \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ is the natural projection; $\mathbb{P}^1 \times \mathbb{P}_B \to \mathbb{P}^1 \times \mathbb{P}_A$ is the quotient map by the involution ι (which acts trivially on $\mathbb{P}^1$); and $\mathbb{P}^1 \times \mathbb{P}_B \to \mathbb{P}^1 \times \mathbb{P}_C$ is the blow-up along the union of $\mathbb{P}^1 \times P_1$ and $\mathbb{P}^1 \times P_2$.

The above diagram restricts to a diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & X' \\
\downarrow & & \downarrow \simeq \\
Y_{\min} & & X
\end{array}
$$

where $X' \subset \mathbb{P}^1 \times \mathbb{P}_A$, $Y \subset \mathbb{P}^1 \times \mathbb{P}_B$, and $Y_{\min} \subset \mathbb{P}^1 \times \mathbb{P}_C$ are respectively defined by a section of $\mathcal{O}(2, 1)^{\oplus 3}$ on $\mathbb{P}^1 \times \mathbb{P}_A$ and ι-invariant sections of $\mathcal{O}(2, 2)^{\oplus 3}$ on $\mathbb{P}^1 \times \mathbb{P}_B$ and $\mathbb{P}^1 \times \mathbb{P}_C$ induced by the map of vector bundles defining X.

Note that each of the intersections $Y_{\min} \cap (\mathbb{P}^1 \times P_i)$ is a complete intersection of three divisors of type $(2, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^2$; thus they consist of 24 points $y_{i,1}, \ldots, y_{i,24}$.

Moreover, the map $Y \to Y_{\min}$ is the blow-up of $Y_{\min}$ along the 48 points $y_{i,j}$, with the corresponding exceptional divisors $F_{i,j}$ being the components of $Y \cap (\mathbb{P}^1 \times F_i)$. The double cover $Y \to X'$ is ramified exactly along the union of the $F_{i,j}$, and each $F_{i,j}$ is mapped isomorphically onto $E_{i,j}$ (the components of $X' \cap (\mathbb{P}^1 \times E_i)$). If X is general, the map $\mathbb{P}^1 \times \mathbb{P}_A \to \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ restricts to an isomorphism $X' \to X$.

Remark 2.1 The minimal model $X_{\min}$ of X can be obtained by contracting the projective planes $E_{i,j}$ to points; $X_{\min}$ is singular exactly at the images of $E_{i,j}$, and at each of the singular points the tangent cone is the affine cone over a Veronese surface.
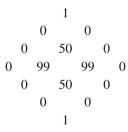
Lemma 2.2 *The threefold X has the following properties:*

(1) *The degree homomorphism* $\deg \colon CH_0(X) \to \mathbb{Z}$ *is an isomorphism.*
(2) *The canonical divisor of X is of the form*

$$K_X = 4F + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{24} E_{i,j},$$

  where F is the class of a fiber of the projection $X \to \mathbb{P}^1$. Thus X has Kodaira dimension $\kappa(X) = 1$.
(3) *The topological Euler characteristic equals $\chi_{\mathrm{top}}(X) = -96$ and Hodge diamond is given by*

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 50 & & 0 & \\
0 & & 99 & & 99 & & 0 \\
 & 0 & & 50 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

(4) *X is simply connected and the cohomology groups $H^i(X, \mathbb{Z})$ are torsion-free for all i.*

Proof The arguments are entirely analogous to those in [20, Section 2], where the case of the rank one degeneracy locus of a general map of vector bundles

$$\mathcal{O}^{\oplus 3} \to \mathcal{O}(1, 2, 0) \oplus \mathcal{O}(1, 0, 2)$$

is considered. The properties (1) to (4) correspond to the statements of [20, Lemma 2.4, 2.5, 2.6, 2.7] respectively. We note that the property (1) can be deduced from a result of Bloch–Kas–Lieberman [4] on the Chow group of 0-cycles on an Enriques surface. $\square$

We will also need the following:

Lemma 2.3 *Let X be the threefold defined by* (1). *Then we have*

$$H^2(X, \mathbb{Z}) = \frac{\mathbb{Z}[F] \oplus \mathbb{Z}[H_1] \oplus \mathbb{Z}[H_2] \oplus \bigoplus_{i=1}^2 \bigoplus_{j=1}^{24} \mathbb{Z}[E_{i,j}]}{\langle -2[H_1] + \sum_{j=1}^{24} [E_{1,j}] = -2[H_2] + \sum_{j=1}^{24} [E_{2,j}] \rangle},$$

where F is the class of a fiber of the first projection $X \to \mathbb{P}^1$ and H_1 (resp. H_2) is the pullback of the class of a line in $\mathbb{P}^2$ via the composition $X \to \mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{pr_1} \mathbb{P}^2$ (resp. $X \to \mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{pr_2} \mathbb{P}^2$).

Proof Let $X^\circ = X \backslash \bigcup_{i=1}^2 \bigcup_{j=1}^{24} E_{i,j}$. The long exact sequence for cohomology with supports yields

$$0 \to \bigoplus_{i=1}^2 \bigoplus_{j=1}^{24} \mathbb{Z}[E_{i,j}] \to H^2(X, \mathbb{Z}) \to H^2(X^\circ, \mathbb{Z}) \to 0. \tag{2}$$

Let $Y^\circ = Y \backslash \bigcup_{i=1}^2 \bigcup_{j=1}^{24} F_{i,j}$. Since X° is the quotient of Y° by the group $\langle \iota \rangle = \mathbb{Z}/2$, which acts freely, we can apply the Cartan–Leray spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}/2, H^q(Y^\circ, \mathbb{Z})) \Rightarrow H^{p+q}(X^\circ, \mathbb{Z}).$$

We have $H^1(Y^\circ, \mathbb{Z}) = H^1(Y_{\min}, \mathbb{Z}) = 0$ by the Lefschetz hyperplane section theorem, so we have a short exact sequence

$$0 \to \mathbb{Z}/2 \to H^2(X^\circ, \mathbb{Z}) \to H^2(Y^\circ, \mathbb{Z})^\iota \to 0.$$

The long exact sequence for cohomology with supports yields

$$0 \to \bigoplus_{i=1}^2 \bigoplus_{j=1}^{24} \mathbb{Z}[F_{i,j}] \to H^2(Y, \mathbb{Z}) \to H^2(Y^\circ, \mathbb{Z}) \to 0.$$

Applying the Lefschetz hyperplane section theorem to $Y_{\min}$, it is straightforward to compute

$$H^2(Y, \mathbb{Z}) = \frac{\mathbb{Z}[F] \oplus \mathbb{Z}[H_1] \oplus \mathbb{Z}[H_2] \oplus \bigoplus_{i=1}^2 \bigoplus_{j=1}^{24} \mathbb{Z}[F_{i,j}]}{\langle -[H_1] + \sum_{j=1}^{24} [F_{1,j}] = -[H_2] + \sum_{j=1}^{24} [F_{2,j}] \rangle}.$$

Thus we obtain

$$H^2(Y^\circ, \mathbb{Z}) = \frac{\mathbb{Z}[F] \oplus \mathbb{Z}[H_1] \oplus \mathbb{Z}[H_2]}{\langle [H_1] = [H_2] \rangle}$$

and $H^2(Y^\circ, \mathbb{Z})$ is ι-invariant. This, combined with the equality

$$-2[H_1] + \sum_{j=1}^{24}[E_{1,j}] = -2[H_2] + \sum_{j=1}^{24}[E_{2,j}]$$

in $H^2(X, \mathbb{Z})$, implies that

$$H^2(X^\circ, \mathbb{Z}) = \frac{\mathbb{Z}[F] \oplus \mathbb{Z}[H_1] \oplus \mathbb{Z}[H_2]}{\langle 2[H_1] = 2[H_2] \rangle},$$

and the claim follows immediately from (2). □

3 Proof of the main theorem

We will now prove Theorem 1.2 over the complex numbers.

Theorem 3.1 *Let* $X \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ *be the rank one degeneracy locus of a very general map of vector bundles*

$$\mathcal{O}^{\oplus 3} \to \mathcal{O}(2, 2, 0) \oplus \mathcal{O}(2, 0, 2).$$

Then the first projection gives a family $X \to \mathbb{P}^1$ *of Enriques surfaces such that any multi-section has even degree over the base* $\mathbb{P}^1$. *That is, the index* $I(X_\eta)$ *is even, where* X_η *is the generic fiber.*

Proof The first goal will be to prove that for any 1-cycle α on X and for any 12-tuple of integers $1 \le j_1 < \cdots < j_{12} \le 24$, there is a congruence

$$\deg(\alpha/\mathbb{P}^1) \equiv \sum_{k=1}^{12} \alpha \cdot E_{1,j_k} \quad \mod 2. \tag{3}$$

These congruences will imply the theorem. Indeed, from (3) we obtain

$$\alpha \cdot E_{1,1} \equiv \cdots \equiv \alpha \cdot E_{1,24} \quad \mod 2, \tag{4}$$

which in turn implies that $\deg(\alpha/\mathbb{P}^1)$ is even.

To prove the congruence (3), we combine monodromy and specialization arguments. First, we prove that a certain monodromy group acts on the set of 24 planes $E_{1,1}, \ldots, E_{1,24}$ by permutations, and every permutation of the $E_{1,j}$ is realized by this action. This will allow us to reduce to proving (3) for a fixed 12-tuple $1 \le j_1 < \cdots < j_{12} \le 24$.

Consider the universal family

$$\mathcal{X} \to G = \mathrm{Gr}(3, H^0(\mathbb{P}^1 \times \mathbb{P}_A, \mathcal{O}(2, 1)))$$

of complete intersections in $\mathbb{P}^1 \times \mathbb{P}_A$ of three divisors of type $(2, 1)$. Let $\mathcal{E}_1$ denote the pullback of the Cartier divisor E_1 via the projection map $\mathcal{X} \to \mathbb{P}_A$. The corresponding family $\mathcal{E}_1 \to G$ is the union of the planes $E_{1,1}, \ldots, E_{1,24}$ in the fibers of $\mathcal{X} \to G$. Let $\widetilde{G} \to G$ be the Stein factorization of $\mathcal{E}_1 \to G$, which is a finite morphism of degree 24, and let $U \subset G$ be the largest open set over which $\widetilde{G} \to G$ is étale. We will now prove the following:

Lemma 3.2 *The monodromy representation*

$$\rho \colon \pi_1(U) \to S_{24},$$

uniquely determined up to the choice of a base point, is surjective.

Proof Recall from Sect. 2 that the planes $E_{1,1}, \ldots, E_{1,24}$ are parameterized by the 24 intersection points of three divisors of type $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. To prove the lemma, we restrict over a certain line l on G defined as follows. Let $\widetilde{l} \subset \mathbb{P}^1 \times \mathbb{P}^2$ be a general complete intersection of two divisors of type $(2, 2)$. Taking a general pencil in $|\mathcal{O}_{\widetilde{l}}(2, 2)|$, we obtain a Lefschetz pencil $\widetilde{l} \to \mathbb{P}^1$ by [11, Theorem XVII. 2.5]. This defines a line

$$l \subset \mathrm{Gr}(3, H^0(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(2, 2, 0))) \subset G$$

such that $\widetilde{l} = l \times_G \widetilde{G}$, where the inclusion between the Grassmannians is via the identification

$$H^0(\mathbb{P}^1 \times \mathbb{P}_A, \mathcal{O}(2, 1)) = H^0(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(2, 2, 0))$$
$$\oplus H^0(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(2, 0, 2)).$$

We let $l^\circ = l \cap U$; this is the maximal open set where $\widetilde{l} \to l$ is étale.

We claim that the induced monodromy representation

$$\rho_{l^\circ} \colon \pi_1(l^\circ) \to S_{24}$$

is surjective. Indeed, $\pi_1(l^\circ)$ is generated by loops around branch points $b \in B$ of $\widetilde{l} \to l$, and the image of each such loop is a transposition in S_{24}. The image of ρ_{l° is moreover transitive since $\widetilde{l}$ is irreducible. Any transitive subgroup of S_{24} which is generated by transpositions must be S_{24} itself, so it follows that $\rho \colon \pi_1(U) \to S_{24}$ is surjective. $\qquad\square$

By the above lemma, we reduce to proving the congruence (3) for a single 12-tuple $1 \leq j_1 < \cdots < j_{12} \leq 24$. Indeed, if $g \in \pi_1(U)$ is a lift of a permutation $\sigma \in S_{24}$, then it will imply for any 1-cycle α on X, there is a congruence

$$\deg(\alpha/\mathbb{P}^1) = \deg(g^*(\alpha)/\mathbb{P}^1) \equiv \sum_{k=1}^{12} g^*(\alpha) \cdot E_{j_k} \equiv \sum_{k=1}^{12} \alpha \cdot E_{\sigma^{-1}(j_k)} \mod 2.$$

Here we have used $g^*(\alpha) \cdot E_{\sigma(j)} = \alpha \cdot E_j$ for each j, and the fact that $g^*(\alpha)$ is again an algebraic cycle because X is very general. We also have $\deg(g^*(\alpha)/\mathbb{P}^1) = \deg(\alpha/\mathbb{P}^1)$, because the degree is obtained by intersecting with the class of a fiber over $\mathbb{P}^1$, which is invariant under monodromy. Letting σ run over all permutations, we see that the congruence will hold for all 12-tuples.

To finish the proof of the congruence (3), we use a specialization argument. We consider X as the complete intersection of three divisors D_1, D_2, D_3 in $|\mathcal{O}(2, 1)|$ on $\mathbb{P}^1 \times \mathbb{P}_A$. If we degenerate each D_i to a union $D_i' + D_i''$, where $D_i' \in |\mathcal{O}(1, 1)|$ and $D_i'' \in |\mathcal{O}(1, 0)|$ are very general divisors, we obtain a family of threefolds $\mathcal{X}_T \to T$, with special fiber equal to

$$X_0 \cup R_1 \cup R_2 \cup R_3,$$

where X_0 is a very general intersection of three divisors in $|\mathcal{O}(1, 1)|$, and R_1, R_2, R_3 are intersections of two divisors of type $\mathcal{O}(1, 1)$ and one of type $\mathcal{O}(1, 0)$. In particular, the R_i are pairwise disjoint and can be regarded as complete intersections of two relative hyperplane sections in $\mathbb{P}_A$. By the geometric construction in Sect. 2, we may regard X_0 as the rank one degeneracy locus in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ of a very general map of vector bundles

$$\mathcal{O}^{\oplus 3} \to \mathcal{O}(1, 2, 0) \oplus \mathcal{O}(1, 0, 2).$$

By construction, X_0 is also the only dominant component with respect to the projection $X_0 \cup R_1 \cup R_2 \cup R_3 \to \mathbb{P}^1$. Furthermore, again by genericity, we may assume that $X_0 \cup R_1 \cup R_2 \cup R_3$ is a simple normal crossing variety and the intersection $(X_0 \cup R_1 \cup R_2 \cup R_3) \cap (\mathbb{P}^1 \times E_1)$ is transversal.

This degeneration allows us to specialize cycles on X to cycles on $X_0 \cup R_1 \cup R_2 \cup R_2$. On the level of divisors, the union of 24 components $E_{1,1}, \ldots, E_{1,24}$ on X specializes to the union of 12 components $E_{1,1}^{(0)}, \ldots, E_{1,12}^{(0)}$ on X_0 and 4 components $E_{1,1}^{(l)}, \ldots, E_{1,4}^{(l)}$ on R_l for $l = 1, 2, 3$ given by the intersections with $\mathbb{P}^1 \times E_1$. Thus the chosen specialization gives a 12-tuple $1 \le j_1 < \cdots < j_{12} \le 24$ such that $E_{1,j_1}, \ldots, E_{1,j_{12}}$ specialize to $E_{1,1}^{(0)}, \ldots, E_{1,12}^{(0)}$.

By [13, Sect. 20.3] there is moreover a specialization map of Chow groups

$$CH_1(X) \to CH_1(X_0 \cup R_1 \cup R_2 \cup R_3)$$

which is compatible with intersections with Cartier divisors. If α_0 is the specialization of a 1-cycle α on X, we may write $\alpha_0 = \alpha_0^{(0)} + \alpha_0^{(R)}$, where

$\alpha_0^{(0)}$ is a 1-cycle on X_0 and

$\alpha_0^{(R)}$ is supported in $R_1 \cup R_2 \cup R_3$.

Now we recall a key congruence obtained in the course of the proof of [20, Theorem 3.1]: we have, for any 1-cycle $\alpha_0^{(0)}$ on X_0, a congruence

$$\deg(\alpha_0^{(0)}/\mathbb{P}^1) \equiv \sum_{j=1}^{12} \alpha_0^{(0)} \cdot E_{1,j}^{(0)} \quad \mathrm{mod}\ 2. \tag{5}$$

Note that $\deg(\alpha/\mathbb{P}^1) = \deg(\alpha_0^{(0)}/\mathbb{P}^1)$ and $\alpha \cdot E_{1,j_k} = \alpha_0 \cdot E_{1,k}^{(0)} = \alpha_0^{(0)} \cdot E_{1,k}^{(0)}$ since $E_{1,k}^{(0)}$ is disjoint from R_1, R_2 and R_3. Thus from the congruence (5), we deduce the congruence (3) for $1 \leq j_1 < \cdots < j_{12} \leq 24$. This completes the proof. $\square$

Theorem 3.1 can be generalized to higher dimensions:

Theorem 3.3 *For a positive integer n, we let $X \subset \mathbb{P}^1 \times \mathbb{P}^{2n} \times \mathbb{P}^{2n}$ be the rank one degeneracy locus of a very general map of vector bundles*

$$\mathcal{O}^{\oplus(2n+1)} \to \mathcal{O}(2,2,0) \oplus \mathcal{O}(2,0,2).$$

Then the first projection gives a family $X \to \mathbb{P}^1$ of $\mathcal{O}$-acyclic 2n-folds such that any multi-section has even degree over the base $\mathbb{P}^1$. That is, the index $I(X_\eta)$ is even, where X_η is the generic fiber.

Proof The geometry of the family of $\mathcal{O}$-acyclic $2n$-folds is similar to that of Lemma 2.2. An alternative projective model of X is given by a complete intersection in

$$\mathbb{P}^1 \times \mathbb{P}_{\mathbb{P}^{2n} \times \mathbb{P}^{2n}}(\mathcal{O}(2,0) \oplus \mathcal{O}(0,2))$$

of $(2n+1)$ divisors of type $(2,1)$, and the intersection

$$X \cap (\mathbb{P}^1 \times \mathbb{P}_{\mathbb{P}^{2n} \times \mathbb{P}^{2n}}(\mathcal{O}(2,0)))$$

consists of $(2n+1)2^{2n+1}$ components $E_{1,1}, \ldots, E_{1,(2n+1)2^{2n+1}}$. The theorem follows from a congruence

$$\deg(\alpha/\mathbb{P}^1) \equiv \sum_{k=1}^{(2n+1)2^{2n}} \alpha \cdot E_{1,j_k} \quad \mathrm{mod}\ 2$$

for any 1-cycle α on X and for any $(2n+1)2^{2n}$-tuple $1 \leq j_1 < \cdots < j_{(2n+1)2^{2n}} \leq (2n+1)2^{2n+1}$. We leave the details of the proof to the reader. $\square$

4 Degenerations and examples over $\mathbb{Q}$

We now explain how to give examples as in Theorem 1.2 defined over the rational numbers. The construction is similar to the one used in the previous section, but the

degeneration argument now uses Enriques fibrations defined in terms 2×3-minors, rather than complete intersections of three divisors.

We will work over $\overline{\mathbb{Q}}$ and set

$$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 = \operatorname{Proj} \overline{\mathbb{Q}}[s, t] \times \operatorname{Proj} \overline{\mathbb{Q}}[x_0, x_1, x_2] \times \operatorname{Proj} \overline{\mathbb{Q}}[y_0, y_1, y_2].$$

The goal is to prove the following:

Theorem 4.1 *Let p_i, q_i, r_i, s_i ($i = 0, 1, 2$) be general homogeneous polynomials of tridegree $(1, 2, 0), (0, 0, 2), (2, 2, 0), (2, 0, 2)$ in variables s, t, x_i, y_i defined over $\mathbb{Q}$. Then there exists a prime number p such that, if $X \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ is the rank one degeneracy locus of a map of vector bundles*

$$\mathcal{O}^{\oplus 3} \to \mathcal{O}(2, 2, 0) \oplus \mathcal{O}(2, 0, 2)$$

given by the matrix

$$M = \begin{pmatrix} sp_0 + pr_0 & (s-t)p_1 + pr_1 & (s+t)p_2 + pr_2 \\ stq_0 + ps_0 & t(s-t)q_1 + ps_1 & t(s+t)q_2 + ps_2 \end{pmatrix}, \tag{6}$$

then the first projection gives a family $X \to \mathbb{P}^1$ of Enriques surfaces such that any multi-section has even degree over the base $\mathbb{P}^1$. That is, the index $I(X_\eta)$ is even, where X_η is the generic fiber.

Note that for general p_i, q_i, r_i, s_i defined over $\mathbb{Q}$ and large p, the threefold X is smooth and irreducible.

In order to prove Theorem 4.1, it will be convenient to introduce the following 1-dimensional family of degeneracy loci of vector bundles

$$\mathcal{O}^{\oplus 3} \to \mathcal{O}(2, 2, 0) \oplus \mathcal{O}(2, 0, 2)$$

on $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$. We set $B = \operatorname{Proj} \overline{\mathbb{Q}}[\lambda, \mu]$ and define the total space $\mathcal{X}$ as the subvariety of $B \times \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by the maximal minors of the matrix

$$M_{(\lambda, \mu)} = \begin{pmatrix} \lambda sp_0 + \mu r_0 & \lambda(s-t)p_1 + \mu r_1 & \lambda(s+t)p_2 + \mu r_2 \\ \lambda stq_0 + \mu s_0 & \lambda t(s-t)q_1 + \mu s_1 & \lambda t(s+t)q_2 + \mu s_2 \end{pmatrix}, \tag{7}$$

where the p_i, q_i, r_i, s_i have tridegrees $(1, 2, 0), (0, 0, 2), (2,2,0)$ and $(2, 0, 2)$ respectively.

Let $\mathcal{X} \to B$ denote the natural projection map onto the first factor. By construction, the generic fiber $\mathcal{X}_{\eta_B}$ is a smooth threefold with an Enriques surface fibration $\mathcal{X}_{\eta_B} \to \mathbb{P}^1_{\eta_B}$. The morphism $\mathcal{X} \to B$ is flat outside of the fiber $(\lambda, \mu) = (1, 0)$; we will compute the flat closure of $\mathcal{X}_{\eta_B}$ in $B \times \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ below. In any case, in order to prove Theorem 4.1, we will mainly be interested in the fiber over $(\lambda, \mu) = (1, p)$.

For now, let $\mathcal{E}_1 \subset \mathcal{X}$ denote the codimension 1 subscheme defined by the top row of $M_{(\lambda, \mu)}$, i.e.,

$$\lambda s p_0 + \mu r_0 = \lambda(s - t)p_1 + \mu r_1 = \lambda(s + t)p_2 + \mu r_2 = 0. \qquad (8)$$

By Bertini, $\mathcal{E}_1$ is smooth and irreducible for general p_i, r_i. Let $\mathcal{E}_1 \to \widetilde{B} \to B$ denote the Stein factorization of $\mathcal{E}_1 \to B$. The morphism $\widetilde{B} \to B$ is finite of degree 24; over a general point $b \in B$ the fiber corresponds to the 24 distinct planes $E_{1,1}, \ldots, E_{1,24}$ in $\mathcal{X}_b$. We let $B^\circ \subset B$ denote the maximal open set over which $\widetilde{B} \to B$ is étale. There is an associated monodromy representation

$$\rho : \pi_1^{\text{ét}}(B^\circ) \to S_{24}.$$

Lemma 4.2 *For general p_i, r_i as above, the map ρ is surjective.*

Proof Note that $\widetilde{B}$ is defined by (8) inside $B \times \mathbb{P}^1 \times \mathbb{P}^2$. It is straightforward to check that the cover $\widetilde{B} \to B$ is Lefschetz for general p_i, r_i. Now the assertion follows from an argument similar to that in the proof of Theorem 3.1. We note that $\pi_1^{\text{ét}}(B^\circ)$ is generated by loops around the branch points of $\widetilde{B} \to B$ [16, XIII, Corollaire 2.12] and the image of each loop is a transposition in S_{24}. $\qquad\square$

The parameter space for the families of threefolds given by (7) is a certain rational variety, hence has a Zariski dense set of $\mathbb{Q}$-rational points. As a consequence, we can choose p_i, q_i, r_i, s_i defined over $\mathbb{Q}$ such that ρ is surjective. We will therefore in the following choose p_i, q_i, r_i, s_i satisfying the above conditions: thus for the family $\mathcal{X} \to B$ over $\mathbb{Q}$, the generic fiber is smooth and irreducible; $\mathcal{E}_1$ is smooth and irreducible; and the monodromy map

$$\rho : \pi_1^{\text{ét}}(B^\circ) \to S_{24}$$

is surjective.

Lemma 4.3 *There are infinitely many prime numbers p such that if $x \in B$ is given by $(\lambda, \mu) = (1, p)$, then the induced map*

$$\rho_x : \pi_1^{\text{ét}}(x, \overline{x}) = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to S_{24}$$

is surjective.

Proof The setting resembles that of [28, Sect. 1] (but is more classical). By Hilbert's irreducibility theorem, the set $\{x \in B^\circ(\mathbb{Q}) \mid \rho_x \text{ is surjective}\}$ is the complement of a thin set in $B(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q})$. Moreover, the complement of a thin set in $\mathbb{P}^1(\mathbb{Q})$ contains infinitely many points with $(\lambda, \mu) = (1, p)$ for some prime number p (see [24, Sect. 9.6, Theorem]), which gives us the desired conclusion. $\qquad\square$

To conclude the proof of Theorem 4.1, we again use a specialization argument as in Theorem 3.1. We begin by computing the flat limit of the family $\mathcal{X} \to B$ over $(\lambda, \mu) = (1, 0)$.

Note that $\mathcal{X}$ contains $\{\mu = t = 0\}$ as a component. Removing this component reveals that the flat closure of $\mathcal{X}_{\eta_B}$ in $B \times \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ is defined by the 3×3-minors of the matrix

$$
\begin{pmatrix}
\lambda s p_0 + \mu r_0 & \lambda(s-t)p_1 + \mu r_1 & \lambda(s+t)p_2 + \mu r_2 & 0 \\
\lambda s q_0 & \lambda(s-t)q_1 & \lambda(s+t)q_2 & \mu \\
s_0 & s_1 & s_2 & -t
\end{pmatrix}.
$$

The corresponding family $\overline{\mathcal{X}} \to B$ is flat and has special fiber $\overline{\mathcal{X}}_0$ over $(\lambda, \mu) = (1, 0)$ given by a union $\widetilde{X}_0 \cup R_0 \cup R_1 \cup R_2 \cup R_3$, where $\widetilde{X}_0$ is given by the minors of the matrix

$$
N = \begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix},
$$

R_0 is given by

$$
t = \det \begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \\ s_0 & s_1 & s_2 \end{pmatrix} = 0,
$$

and R_1, R_2, R_3 are respectively given by

$$
s = p_2 q_1 - p_1 q_2 = 0, \ s - t = p_2 q_0 - p_0 q_2 = 0, \ s + t = p_1 q_0 - p_0 q_1 = 0.
$$

Note that the R_i are pairwise disjoint, and $\widetilde{X}_0$ is regular if the p_i, q_i are general.

Similarly, the subfamily $\mathcal{E}_1 \to B$, given by (7), has a special fiber over $(\lambda, \mu) = (1, 0)$ which consists of the union of 12 components $E_{1,1}^{(0)}, \ldots, E_{1,12}^{(0)}$ supported on X_0 given by
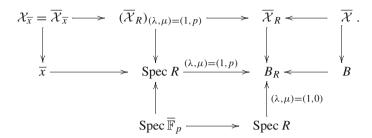
$$
p_0 = p_1 = p_2 = 0
$$

and the unions of 4 components $E_{1,1}^{(l)}, \ldots, E_{1,4}^{(l)}$ supported on R_l for $l = 1, 2, 3$ respectively given by

$$
s = p_1 = p_2 = 0, \ s - t = p_0 = p_2 = 0, \ s + t = p_0 = p_1 = 0.
$$

It is important to note that $E_{1,1}^{(0)}, \ldots, E_{1,12}^{(0)}$ are Cartier divisors on $\overline{\mathcal{X}}_0$ since they are supported on $\overline{\mathcal{X}}_0 \backslash (R_0 \cup R_1 \cup R_2 \cup R_3)$ which is regular.

Let p and $x \in B$ be as in Lemma 4.3. For any valuation ring $R \subset \overline{\mathbb{Q}}$ whose maximal ideal contains p, we have the following diagram of restrictions:

$$
\begin{array}{ccccccc}
\mathcal{X}_{\overline{x}} = \overline{\mathcal{X}}_{\overline{x}} & \longrightarrow & (\overline{\mathcal{X}}_R)_{(\lambda,\mu)=(1,p)} & \longrightarrow & \overline{\mathcal{X}}_R & \longleftarrow & \overline{\mathcal{X}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{x} & \longrightarrow & \operatorname{Spec} R & \xrightarrow{(\lambda,\mu)=(1,p)} & B_R & \longleftarrow & B \\
& & \uparrow & & \uparrow\scriptstyle{(\lambda,\mu)=(1,0)} & & \\
& & \operatorname{Spec} \overline{\mathbb{F}}_p & \longrightarrow & \operatorname{Spec} R & &
\end{array}
$$

Proof of Theorem 4.1 Let p_i, q_i, r_i, s_i be general and defined over $\mathbb{Q}$. Let p be a sufficiently large prime number which satisfies Lemma 4.3 and let $X = \mathcal{X}_{\overline{x}}$. We prove that any multi-section of $X \to \mathbb{P}^1$ has even degree over the base $\mathbb{P}^1$. As in the proof of Theorem 3.1, it is enough to prove, for any 1-cycle α on X and for any 12-tuple $1 \le j_1 < \cdots < j_{12} \le 24$, a congruence

$$
\deg(\alpha/\mathbb{P}^1) \equiv \sum_{k=1}^{12} \alpha \cdot E_{1,j_k} \quad \mod 2.
$$

By Lemma 4.3, it suffices to verify this congruence for some 12-tuple $1 \le j_1 < \cdots < j_{12} \le 24$. To establish this, we use the above family over $\operatorname{Spec} R$, which allows us to specialize cycles from X to cycles on $((\overline{\mathcal{X}}_R)_{(\lambda,\mu)=(1,p)})_{\overline{\mathbb{F}}_p}$.

For a sufficiently large valuation ring $R \subset \overline{\mathbb{Q}}$ whose maximal ideal contains p, the specialization $((\mathcal{E}_1)_R)_{(\lambda,\mu)=(1,p)} \subset (\overline{\mathcal{X}}_R)_{(\lambda,\mu)=(1,p)}$ is a disjoint union of 24 components $\mathcal{E}_{1,1}, \ldots, \mathcal{E}_{1,24}$, each of which is isomorphic to $\mathbb{P}^2_R$. Let $\mathcal{E}_{1,j_1}, \ldots, \mathcal{E}_{1,j_{12}}$ be the components which restrict to $E_{1,1}^{(0)}, \cdots, E_{1,12}^{(0)}$ on the special fiber $((\overline{\mathcal{X}}_R)_{(\lambda,\mu)=(1,p)})_{\overline{\mathbb{F}}_p}$. Then $\underline{\mathcal{E}}_{1,j_1}, \ldots, \underline{\mathcal{E}}_{1,j_{12}} \subset (\overline{\mathcal{X}}_R)_{(\lambda,\mu)=(1,p)}$ are Cartier divisors since they are supported on $(\overline{\mathcal{X}}_R)_{(\lambda,\mu)=(1,p)} \backslash (R_0 \cup R_1 \cup R_2 \cup R_3)$ which is regular.

Now by the specialization homomorphism for Chow groups [13, Ex. 20.3.5], the desired congruence follows from a congruence in the proof of [20, Theorem 3.1]: we have, for any 1-cycle $\widetilde{\alpha}_0$ on $\widetilde{X}_0$, a congruence

$$
\deg(\widetilde{\alpha}_0/\mathbb{P}^1) \equiv \sum_{j=1}^{12} \widetilde{\alpha}_0 \cdot E_{1,j}^{(0)} \quad \mod 2.
$$

The proof is complete. $\qquad\square$

The above proof uses a specialization argument which does not extend in general to other fields. One natural question is whether one can find such examples defined over the algebraic closure of a finite field. In contrast to the examples above, we prove some positive results in this situation, conditional on the Tate conjecture. We recall that the

Tate conjecture in degree $2i$ on a smooth projective variety V over a finite field k of characteristic p asserts that the image of the cycle class map

$$\mathrm{cl}^i \otimes \mathbb{Q}_l : CH^i(V_{\bar{k}}) \otimes \mathbb{Q}_l \to H^{2i}_{\text{ét}}(V_{\bar{k}}, \mathbb{Q}_l(i))$$

generates the subspace of classes in $H^{2i}_{\text{ét}}(V_{\bar{k}}, \mathbb{Q}_l(i))$ fixed by some open subgroup of $\mathrm{Gal}(\bar{k}/k)$ for any prime number $l \neq p$. The integral Tate conjecture is an integral analogue of the Tate conjecture (with $\mathbb{Z}_l$ instead of $\mathbb{Q}_l$).

Proposition 4.4 *Let X be a smooth projective variety over $\overline{\mathbb{F}}_p$ with fibration $X \to C$ over a smooth projective curve C. Assume that*

(1) *the generic fiber X_η is smooth with $\chi(\mathcal{O}_{X_\eta}) = 1$;*
(2) *$b_2 = \rho$ on X, where b_2 is the second Betti number and ρ is the Neron–Severi rank;*
(3) *the Tate conjecture holds in degree 2 on surfaces over finite fields of characteristic p.*

Then the fibration $X \to C$ admits multi-sections whose degrees over the base C add up to a power of p. That is, the index $I(X_\eta)$ is a power of p, where X_η is the generic fiber.

Remark 4.5 A similar assertion was proved by Colliot-Thélène and Szamuely [8, Theorem 6.1], where, among other things, the torsion-freeness of the Picard group $\mathrm{Pic}(X_{\bar{\eta}})$ of the geometric generic fiber $X_{\bar{\eta}}$ is assumed.

Proof of Proposition 4.4 Let $X \to C$ be a fibration as in the statement and $d = \dim X$. Under the assumption (1), the Riemann-Roch formula together with the Poincaré duality shows that the push-forward homomorphism

$$H^{2d-2}(X, \mathbb{Z}_l(d-1)) \to H^0(C, \mathbb{Z}_l) = \mathbb{Z}_l$$

is surjective for any prime $l \neq p$ (the arguments are analogous to the proofs of Proposition A.6 and Corollary A.7 due to Wittenberg).

On the other hand, if $b_2 = \rho$ on X, the cokernel of the cycle class map

$$\mathrm{cl}^{2d-2} \otimes \mathbb{Z}_l : CH^{2d-2}(X) \otimes \mathbb{Z}_l \to H^{2d-2}(X, \mathbb{Z}_l(d-1))$$

is finite by the hard Lefschetz theorem due to Deligne. If we further assume that the Tate conjecture holds in degree 2 on surfaces over finite fields of characteristic p, then the integral Tate conjecture holds in degree $2d-2$ on X (viewed as the base extension of a smooth projective variety over a finite field of characteristic p), according to a theorem of Schoen [22, Theorem 0.5]. This implies that the cokernel of $\mathrm{cl}^{2d-2} \otimes \mathbb{Z}_l$ is torsion-free, hence $\mathrm{cl}^{2d-2} \otimes \mathbb{Z}_l$ is surjective. Combined with the argument in the previous paragraph, the statement now follows. □

Corollary 4.6 *Let $X \to C$ be a one-parameter family of $\mathcal{O}$-acyclic varieties over $\overline{\mathbb{Q}}$. Assume that the Tate conjecture holds in degree 2 on surfaces over finite fields. Then the reduction $X_p \to C_p$ over $\overline{\mathbb{F}}_p$ admits multi-sections with coprime degrees over the*

base C_p for any large prime number p. That is, $I((X_p)_\eta) = 1$, where $(X_p)_\eta$ is the generic fiber.

Proof We note that the $\mathcal{O}$-acyclicity of fibers of the family $X \to C$ implies $b_2 = \rho$ on X by [9, Proposition 7.3], thus we also have $b_2 = \rho$ on the good reductions of X by specialization. Now the statement is immediate from Proposition 4.4 by observing that there exist 1-cycles on X_p obtained by spreading out 1-cycles on X over valuations rings inside $\overline{\mathbb{Q}}$, whose degrees over the base C_p do not depend on p. □

5 Failure of the Hasse principle and the reciprocity obstruction

The reciprocity obstruction to the Hasse principle for 0-cycles of degree 1 on a smooth projective geometrically connected variety Z over the function field $F = \mathbb{C}(C)$ of a complex curve C was defined and pointed out to the authors by Colliot-Thélène (see also [7, Sect. 5]). We explain the construction in the following. We will assume that $H^1_{\text{ét}}(Z_{\overline{F}}, \mathbb{Z}/2) = \mathbb{Z}/2$ for simplicity.

The Leray spectral sequence for the étale sheaf $\mathbb{Z}/2$ and the morphism $Z \to \operatorname{Spec} F$ yields a short exact sequence

$$0 \to H^1_{\text{ét}}(F, \mathbb{Z}/2) \to H^1_{\text{ét}}(Z, \mathbb{Z}/2) \to H^1_{\text{ét}}(Z_{\overline{F}}, \mathbb{Z}/2) \to 0.$$

Note that the Galois group $\operatorname{Gal}(\overline{F}/F)$ acts trivially on $H^1_{\text{ét}}(Z_{\overline{F}}, \mathbb{Z}/2) = \mathbb{Z}/2$. We then choose a lift $\xi \in H^1_{\text{ét}}(Z, \mathbb{Z}/2)$ of the non-trivial class in $H^1_{\text{ét}}(Z_{\overline{F}}, \mathbb{Z}/2) = \mathbb{Z}/2$.

The evaluation pairing

$$Z(F) \times H^1_{\text{ét}}(Z, \mathbb{Z}/2) \to H^1_{\text{ét}}(F, \mathbb{Z}/2)$$

extends to an evaluation pairing on the Chow group of 0-cycles

$$CH_0(Z) \times H^1_{\text{ét}}(Z, \mathbb{Z}/2) \to H^1_{\text{ét}}(F, \mathbb{Z}/2).$$

Thus we get the evaluation map of ξ

$$CH_0(Z) \to H^1_{\text{ét}}(F, \mathbb{Z}/2).$$

Similarly, we get the local evaluation map of ξ

$$CH_0(Z_{F_p}) \to H^1_{\text{ét}}(F_p, \mathbb{Z}/2) = \mathbb{Z}/2$$

for any $p \in C$, where $F_p \cong \mathbb{C}((t))$ is the completion of F at p. The local evaluation maps are identically zero for all but finitely many $p \in C$ by an argument of good reduction.

The diagonal embedding

$$F \hookrightarrow \prod_{p \in C} F_p$$

yields a complex

$$H^1_{\text{ét}}(F, \mathbb{Z}/2) \to \bigoplus_{p \in C} H^1_{\text{ét}}(F_p, \mathbb{Z}/2) \to \mathbb{Z}/2$$

that is

$$F^*/F^{*2} \to \bigoplus_{p \in C} \mathbb{Z}/2 \to \mathbb{Z}/2,$$

where the first map is induced by the divisor map and the second map is the summation map. Then it follows that the image of the diagonal map

$$CH_0(Z) \to \prod_{p \in C} CH_0(Z_{F_p})$$

is contained in the kernel of the sum of the local evaluations

$$\theta : \prod_{p \in C} CH_0(Z_{F_p}) \to \mathbb{Z}/2.$$

Proposition 5.1 *(Reciprocity obstruction) If for each family* $\{\alpha_p\}_{p \in C}$ *of 0-cycles of degree 1, we have* $\theta(\{\alpha_p\}) = 1 \in \mathbb{Z}/2$*, then there is no 0-cycle of degree 1 on Z.*

As a consequence of our construction in Sect. 3, we prove that the failure of the Hasse principle for 0-cycles of degree 1 on an Enriques surface over $\mathbb{C}(\mathbb{P}^1)$ cannot always be accounted for by the reciprocity obstruction.

Theorem 5.2 *Let* X_η *be the generic fiber of the family* $X \to \mathbb{P}^1$ *of Enriques surfaces of Theorem* 3.1*. Then the Hasse principle fails for 0-cycles of degree 1 on* X_η*, while the assumption of Proposition* 5.1 *is not satisfied.*

Remark 5.3 In fact, a direct computation shows that X_η has rational points everywhere locally. Hence the Hasse principle already fails for rational points on X_η. The proof in the following also shows that there is no reciprocity obstruction to the Hasse principle for rational points on X_η. Therefore it follows that the reciprocity obstruction to the Hasse principle for rational points on an Enriques surface over $\mathbb{C}(\mathbb{P}^1)$ is not the only obstruction.

Proof of Theorem 5.2 Let $F = \mathbb{C}(\mathbb{P}^1)$. Theorem 3.1 shows that there is no 0-cycle of degree 1 on X_F. On the other hand, it is automatic from the $\mathcal{O}$-acyclicity of Enriques surfaces and the Riemann-Roch theorem that the family $X \to \mathbb{P}^1$ has no multiple fiber (in fact this is easy to see directly from the defining equations). It then follows from Hensel's lemma that there is a 0-cycle of degree 1 on X_{F_p} for any $p \in \mathbb{P}^1$. Therefore the Hasse principle fails for 0-cycles of degree 1 on X_F.

By choosing a lift $\xi \in H^1_{\text{ét}}(X_F, \mathbb{Z}/2)$ of the non-zero class in $H^1_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}/2) = \mathbb{Z}/2$, we obtain the map θ in Proposition 5.1. To see that X_F does not satisfy the

assumption of Proposition 5.1, it is enough to verify the following: for each i and j, if $p_{i,j} \in \mathbb{P}^1$ is the image of $E_{i,j}$, then the local evaluation map

$$CH_0(X_{F_{p_{i,j}}}) \to H^1_{\text{ét}}(F_{p_{i,j}}, \mathbb{Z}/2) = \mathbb{Z}/2$$

restricts to a surjection on 0-cycles of degree 1; this will then provide a family $\{\alpha_p\}_{p \in C}$ of 0-cycles of degree 1 such that $\theta(\{\alpha_p\}) = 0 \in \mathbb{Z}/2$.

Recall that by construction in Sect. 2, X admits a natural double cover $Y \to X$ over $\mathbb{P}^1$ and the cover is ramified along $\bigcup F_{i,j}$ and branched over $\bigcup E_{i,j}$. Then one can in fact assume that ξ is given by the étale double cover $Y^\circ \to X^\circ$, where $X^\circ = X \setminus \bigcup E_{i,j}$ and $Y^\circ = Y \setminus \bigcup F_{i,j}$, since evaluation maps only differ by classes in $H^1_{\text{ét}}(F, \mathbb{Z}/2)$.

For each i and j, working locally around $p_{i,j}$, we consider the base change of the Enriques fibration $X \to \mathbb{P}^1$

$$X \times_{\mathbb{P}^1} \text{Spec}\,\widehat{\mathcal{O}}_{\mathbb{P}^1, p_{i,j}} \to \text{Spec}\,\widehat{\mathcal{O}}_{\mathbb{P}^1, p_{i,j}},$$

where $\widehat{\mathcal{O}}_{\mathbb{P}^1, p_{i,j}}$ is the completion of the local ring $\mathcal{O}_{\mathbb{P}^1, p_{i,j}}$. One can compute that the special fiber is reduced and consists of $E_{i,j}$ and a residual component $R_{i,j}$. Then, by Hensel's lemma, there is a section S_1 (resp. S_2) which intersects transversally with $E_{i,j}$ (resp. $R_{i,j}$) at one point. Now we consider the double cover

$$Y \times_{\mathbb{P}^1} \text{Spec}\,\widehat{\mathcal{O}}_{\mathbb{P}^1, p_{i,j}} \to X \times_{\mathbb{P}^1} \text{Spec}\,\widehat{\mathcal{O}}_{\mathbb{P}^1, p_{i,j}},$$

whose branched locus is $E_{i,j}$. Then it is straightforward to see that the inverse image of S_1 gives degree 2 integral multi-section, while that of S_2 splits into two disjoint sections. Therefore the $F_{p_{i,j}}$-rational points of $X_{F_{p_{i,j}}}$ corresponding to the sections S_1 and S_2 take values 1 and 0 in $\mathbb{Z}/2$ respectively under the local evaluation map. This concludes that X_F does not satisfy the assumption of Proposition 5.1, hence the proof of the theorem. □

6 Defect of the integral Hodge conjecture in degree 4 and degree 3 unramified cohomology with $\mathbb{Z}/2$ coefficients

Theorem 6.1 *Let X be the total space of the family of Enriques surfaces of Theorem 3.1. Then we have $Z^4(X)[2] = (\mathbb{Z}/2)^{46}$.*

Proof By Lemma 2.2, the Hodge structure of $H^4(X, \mathbb{Z})$ is trivial and $H^4(X, \mathbb{Z})$ is free of rank 50. By the Tor long exact sequence, we have

$$Z^4(X)[2] = \text{Ker}(H^4_{\text{alg}}(X, \mathbb{Z})/2 \to H^4(X, \mathbb{Z}/2)).$$

We define

$$H^4_{\text{alg}}(X, \mathbb{Z}/2) = \text{Im}\left(\text{cl}^2 \otimes \mathbb{Z}/2 \colon CH^2(X)/2 \to H^4(X, \mathbb{Z}/2)\right).$$

Since $H_{\text{alg}}^4(X, \mathbb{Z})/2 = (\mathbb{Z}/2)^{50}$, we are reduced to proving that $H_{\text{alg}}^4(X, \mathbb{Z}/2) = (\mathbb{Z}/2)^4$.

We first prove that

$$\text{Im}\left(CH^2(X)/2 \xrightarrow{\text{cl} \otimes \mathbb{Z}/2} H^4(X, \mathbb{Z}/2) \xrightarrow{(i_X)_*} H^{10}(\mathbb{P}^1 \times \mathbb{P}_A, \mathbb{Z}/2) \right) = (\mathbb{Z}/2)^2,$$

where $i_X \colon X \to \mathbb{P}^1 \times \mathbb{P}_A$ is the inclusion map. The rank of the image is ≤ 2 as a result of Theorem 3.1 and the congruence (4) in the proof. Thus it suffices to find two linearly independent classes in the image. It is easy to see that lines $l_1 \subset E_{1,1}$ and $l_2 \subset E_{2,1}$ give such classes.

We define

$$H_{\text{van}}^4(X, \mathbb{Z}/2) = \text{Ker}\left((i_X)_* \colon H^4(X, \mathbb{Z}/2) \to H^{10}(\mathbb{P}^1 \times \mathbb{P}_A, \mathbb{Z}/2) \right).$$

We have rank $H_{\text{van}}^4(X, \mathbb{Z}/2) = 46$. Indeed, it is enough to observe that the push-forward homomorphism

$$(i_X)_* \colon H^4(X, \mathbb{Z}) \to H^{10}(\mathbb{P}^1 \times \mathbb{P}_A, \mathbb{Z})$$

is surjective, which follows from the fact that the pullback homomorphism

$$(i_X)^* \colon H^2(\mathbb{P}^1 \times \mathbb{P}_A, \mathbb{Z}) \to H^2(X, \mathbb{Z})$$

is injective with torsion-free cokernel by Lemma 2.3. We prove that $H_{\text{van}}^4(X, \mathbb{Z}/2)$ is generated by classes $c_{i,j_1,j_2} \in H^4(X, \mathbb{Z})$ ($i = 1, 2$, $1 \leq j_1 < j_2 \leq 24$) with intersection properties

$$c_{i,j_1,j_2} \cdot E_{i',j'} = \delta_{i,i'} \cdot (\delta_{j_1,j'} - \delta_{j_2,j'}),$$
$$c_{i,j_1,j_2} \cdot F = c_{i,j_1,j_2} \cdot H_1 = c_{i,j_1,j_2} \cdot H_2 = 0.$$

It is enough to show that

$$H_{\text{van}}^4(X, \mathbb{Z}) = \text{Ker}\left((i_X)_* \colon H^4(X, \mathbb{Z}) \to H^{10}(\mathbb{P}^1 \times \mathbb{P}_A, \mathbb{Z}) \right),$$

which is of rank 46, is generated by the above classes. Let $i_Y \colon Y \to \mathbb{P}^1 \times \mathbb{P}_B$ be the inclusion map and let

$$H_{\text{van}}^4(Y, \mathbb{Z}) = \text{Ker}\left((i_Y)_* \colon H^4(Y, \mathbb{Z}) \to H^{10}(\mathbb{P}^1 \times \mathbb{P}_B, \mathbb{Z}) \right).$$

The group $H_{\text{van}}^4(Y, \mathbb{Z})$ has rank 46. Using Lemma 2.3, it is straightforward to see that

$$\text{Coker}\left(f_* \colon H_{\text{van}}^4(Y, \mathbb{Z}) \to H_{\text{van}}^4(X, \mathbb{Z}) \right) = (\mathbb{Z}/2)^{46},$$

where $f : Y \to X$ is the natural map, thus the push-forward homomorphism $f_* : H^4_{\mathrm{van}}(Y, \mathbb{Z}) \to H^4_{\mathrm{van}}(X, \mathbb{Z})$ can be identified with the multiplication homomorphism $\mathbb{Z}^{46} \xrightarrow{\times 2} \mathbb{Z}^{46}$. Now it is enough to observe that $H^4_{\mathrm{van}}(Y, \mathbb{Z})$ is generated by classes $d_{i,j_1,j_2} \in H^4(Y, \mathbb{Z})$ $(i = 1, 2, 1 \le j_1 < j_2 \le 24)$ with intersection properties

$$d_{i,j_1,j_2} \cdot F_{i',j'} = \delta_{i,i'} \cdot (\delta_{j_1,j'} - \delta_{j_2,j'}),$$
$$d_{i,j_1,j_2} \cdot F = d_{i,j_1,j_2} \cdot H_1 = d_{i,j_1,j_2} \cdot H_2 = 0,$$

which is immediate.

We prove that $H^4_{\mathrm{alg}}(X, \mathbb{Z}/2) \cap H^4_{\mathrm{van}}(X, \mathbb{Z}/2) = (\mathbb{Z}/2)^2$. We note that we have the congruence (4) in the proof of Theorem 3.1, and moreover, we may also assume a congruence

$$\alpha \cdot E_{2,1} \equiv \cdots \equiv \alpha \cdot E_{2,24} \mod 2 \tag{9}$$

for any 1-cycle α on X. Then the congruences (4) and (9) imply

$$\mathrm{rank} \left(H^4_{\mathrm{alg}}(X, \mathbb{Z}/2) \cap H^4_{\mathrm{van}}(X, \mathbb{Z}/2) \right) \le 2.$$

Now it is enough to find two linearly independent classes in $H^4_{\mathrm{alg}}(X, \mathbb{Z}/2) \cap H^4_{\mathrm{van}}(X, \mathbb{Z}/2)$. It is a simple matter to check that $C_1 = (H_1)^2$ and $C_2 = (H_2)^2$ indeed give such classes.

It follows that

$$H^4_{\mathrm{alg}}(X, \mathbb{Z}/2) = \mathbb{Z}/2[l_1] \oplus \mathbb{Z}/2[l_2] \oplus \mathbb{Z}/2[C_1] \oplus \mathbb{Z}/2[C_2] = (\mathbb{Z}/2)^4.$$

The proof is complete. $\qquad\square$

Let n be a positive integer. We recall that the degree 3 unramified cohomology group $H^3_{\mathrm{nr}}(X, \mathbb{Z}/n)$ for a smooth projective variety X is defined to be

$$H^3_{\mathrm{nr}}(X, \mathbb{Z}/n) = H^0(X_{\mathrm{Zar}}, \mathcal{H}^3(\mathbb{Z}/n)),$$

where $\mathcal{H}^3(\mathbb{Z}/n)$ is the Zariski sheaf associated to the presheaf $U \mapsto H^3(U, \mathbb{Z}/n)$ [5]. The group $H^3_{\mathrm{nr}}(X, \mathbb{Z}/n)$ is a stable birational invariant of smooth projective varieties [5, Theorem 4.2]. As an application of the Bloch–Kato conjecture settled by Voevodsky, it was proved by Colliot-Thélène and Voisin [9, Theorem 3.9] that we have

$$H^3_{\mathrm{nr}}(X, \mathbb{Z}/n) = Z^4(X)[n]$$

if $CH_0(X)$ is supported on a surface. This theorem, together with Lemma 2.2 (1) and Theorem 6.1, implies:

Corollary 6.2 *Let X be the total space of the family of Enriques surfaces of Theorem 3.1. Then we have $H^3_{\mathrm{nr}}(X, \mathbb{Z}/2) = (\mathbb{Z}/2)^{46}$.*

Declarations

Appendix A: Vanishing of the reciprocity obstruction by Olivier Wittenberg

In this appendix, we prove that the vanishing of the reciprocity obstruction to the existence of a 0-cycle of degree 1 is a general fact that holds for all $\mathcal{O}$-acyclic varieties over the function field F of a complex curve, and, in fact, for all smooth proper varieties Y over F such that $\chi(Y, \mathcal{O}_Y) = 1$. We actually prove the following slightly more general statement, in the spirit of [12].

Theorem A.1 *Let $F = \mathbb{C}(C)$ be the function field of a smooth proper irreducible complex curve C. Let Y be a smooth proper variety over F and E be a coherent sheaf on Y. Then there exists a collection $(\alpha_p)_{p \in C(\mathbb{C})} \in \prod_{p \in C(\mathbb{C})} \mathrm{CH}_0(Y_{F_p})$ of local 0-cycle classes of degree $\chi(Y, E)$ that belongs to the left kernel of the natural pairing*

$$\left(\prod_{p \in C(\mathbb{C})} \mathrm{CH}_0(Y_{F_p}) \right) \times H^1_{et}(Y, \mathbb{Q}/\mathbb{Z}(1)) \to \mathbb{Q}/\mathbb{Z}. \tag{10}$$

In other words, there is no reciprocity obstruction to the existence of a 0-cycle of degree $\chi(Y, E)$ on Y.

Theorem A.1 builds on a purely cohomological reinterpretation of the reciprocity obstruction (presented in Sects. A.1–A.4) and on a variant of an argument of Colliot-Thélène and Voisin itself based on the Riemann–Roch theorem (see Sect. A.5).

In the situation of Theorem A.1, local 0-cycles of degree $\chi(Y, E)$ had previously been shown to exist in [12, Theorem 1]. It may seem surprising that the existence of

a collection of local 0-cycles that globally survives the reciprocity obstruction comes "for free", without having to make any additional assumption on Y, especially in view of the negative answer to Question 1.1 now provided by Ottem and Suzuki.

A.1 Recollections on the reciprocity obstruction

Let us first recall how the pairing (10), introduced by Colliot-Thélène and Gille in [7, §5], is defined.

For $p \in C(\mathbb{C})$, the Galois cohomology group $H^1(F_p, \mathbb{Q}/\mathbb{Z}(1))$, where $\mathbb{Q}/\mathbb{Z}(1)$ denotes the torsion subgroup of $\mathbb{C}^*$, is canonically isomorphic to $\mathbb{Q}/\mathbb{Z}$. We denote this canonical isomorphism by $\mathrm{inv}_p : H^1(F_p, \mathbb{Q}/\mathbb{Z}(1)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$. Mapping a closed point $q \in Y_{F_p}$ and a class $\beta \in H^1_{\text{ét}}(Y_{F_p}, \mathbb{Q}/\mathbb{Z}(1))$ to $\mathrm{inv}_p \, \mathrm{Cores}_{F_p(q)/F_p} \, \beta(q) \in \mathbb{Q}/\mathbb{Z}$, where Cores denotes the corestriction map in Galois cohomology, uniquely extends to a bilinear pairing

$$\mathrm{CH}_0(Y_{F_p}) \times H^1_{\text{ét}}(Y_{F_p}, \mathbb{Q}/\mathbb{Z}(1)) \to \mathbb{Q}/\mathbb{Z}. \tag{11}$$

Denoting the latter by angle brackets, the pairing (10) is then defined as the sum

$$((\alpha_p)_{p \in C(\mathbb{C})}, \beta) \mapsto \sum_{p \in C(\mathbb{C})} \langle \alpha_p, \beta \rangle, \tag{12}$$

which can be checked to have only finitely many non-zero terms.

The "reciprocity law", in this context, is the equality

$$\sum_{p \in C(\mathbb{C})} \mathrm{inv}_p \, \gamma = 0, \tag{13}$$

valid for any global class $\gamma \in H^1(F, \mathbb{Q}/\mathbb{Z}(1))$, and which amounts to the assertion that any principal divisor on C has degree 0. Applied to $\gamma = \mathrm{Cores}_{F(q)/F} \, \beta(q)$ for a closed point $q \in Y$, it implies that the diagonal map $\mathrm{CH}_0(Y) \to \prod_{p \in C(\mathbb{C})} \mathrm{CH}_0(Y_{F_p})$ takes values in the left kernel of (10). Equivalently, an element of $\prod_{p \in C(\mathbb{C})} \mathrm{CH}_0(Y_{F_p})$ that does not belong to the left kernel of (10) cannot come from $\mathrm{CH}_0(Y)$; in this situation one says that there is a "reciprocity obstruction".

A.2 From Chow groups to cohomology

Let us fix a smooth and proper variety X over $\mathbb{C}$ and a morphism $f : X \to C$ with generic fibre Y. Let X_p denote the fibre of X above $p \in C$. For any scheme Z of finite type over $\mathbb{C}$, over F_p or over $\widehat{\mathcal{O}}_{C,p}$, and all integers q, j, we set $H^q_{\text{ét}}(Z, \hat{\mathbb{Z}}(j)) = \varprojlim_{n \geq 1} H^q_{\text{ét}}(Z, \mathbb{Z}/n\mathbb{Z}(j))$. Let $d = \dim(Y)$ and $\mathbb{Z}(d) = (\sqrt{-1})^d \mathbb{Z}$. Combining the inverse of the isomorphism $H^{2d}_{\text{ét}}(X \times_C \mathrm{Spec}(\widehat{\mathcal{O}}_{C,p}), \hat{\mathbb{Z}}(d)) \xrightarrow{\sim} H^{2d}_{\text{ét}}(X_p, \hat{\mathbb{Z}}(d))$ given by the proper base change theorem with the canonical identification between singular and étale cohomology $H^{2d}(X_p(\mathbb{C}), \mathbb{Z}(d)) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} = H^{2d}_{\text{ét}}(X_p, \hat{\mathbb{Z}}(d))$, we obtain a

canonical injection

$$H^{2d}(X_p(\mathbb{C}), \mathbb{Z}(d)) \hookrightarrow H^{2d}_{\text{ét}}(X \times_C \text{Spec}(\widehat{\mathcal{O}}_{C,p}), \hat{\mathbb{Z}}(d)). \tag{14}$$

We shall consider the pull-back map

$$H^{2d}_{\text{ét}}(X \times_C \text{Spec}(\widehat{\mathcal{O}}_{C,p}), \hat{\mathbb{Z}}(d)) \to H^{2d}_{\text{ét}}(Y_{F_p}, \hat{\mathbb{Z}}(d)) \tag{15}$$

and its composition

$$H^{2d}(X_p(\mathbb{C}), \mathbb{Z}(d)) \to H^{2d}_{\text{ét}}(Y_{F_p}, \hat{\mathbb{Z}}(d)) \tag{16}$$

with this injection.

Proposition A.2 *For any $p \in C(\mathbb{C})$, the image of the cycle class map to étale cohomology* $\text{cl} : \text{CH}_0(Y_{F_p}) \to H^{2d}_{\text{ét}}(Y_{F_p}, \hat{\mathbb{Z}}(d))$ *is equal to the image of* (16).

Proof Let $X_{p,1}, \ldots, X_{p,n}$ denote the irreducible components of X_p, endowed with the reduced subscheme structure. Let Z^h_1 be the group of horizontal 1-cycles on the scheme $X \times_C \text{Spec}(\widehat{\mathcal{O}}_{C,p})$, that is, the group of those 1-cycles whose support is flat over $\widehat{\mathcal{O}}_{C,p}$. Given $z \in Z^h_1$, let $(z \cdot X_{p,i})$ denote the intersection number of z with the Cartier divisor $X_{p,i} \subset X_p$. The map $Z^h_1 \to \mathbb{Z}^n$, $z \mapsto ((z \cdot X_{p,1}), \ldots, (z \cdot X_{p,n}))$ is surjective as a consequence of [6, 9.1/9], and fits into a commutative diagram

$$
\begin{array}{ccccc}
\mathbb{Z}^n & \twoheadleftarrow & Z^h_1 & \twoheadrightarrow & \text{CH}_0(Y_{F_p}) \\
\| & & \downarrow & & \downarrow {\scriptstyle \text{cl}} \\
H^{2d}(X_p(\mathbb{C}), \mathbb{Z}(d)) & \hookrightarrow & H^{2d}_{\text{ét}}(X \times_C \text{Spec}(\widehat{\mathcal{O}}_{C,p}), \hat{\mathbb{Z}}(d)) & \longrightarrow & H^{2d}_{\text{ét}}(Y_{F_p}, \hat{\mathbb{Z}}(d)),
\end{array}
\tag{17}
$$

whose middle vertical arrow is the cycle class map (see [21, (1.12)] for its definition), whose lower horizontal arrows are the injection (14) and the pull-back map (15), and whose leftmost vertical map comes from the canonical isomorphisms $H^{2d}(X_{p,i}(\mathbb{C}), \mathbb{Z}(d)) = \mathbb{Z}$ for $i \in \{1, \ldots, n\}$ and from the decomposition $H^{2d}(X_p(\mathbb{C}), \mathbb{Z}(d)) = \bigoplus^n_{i=1} H^{2d}(X_{p,i}(\mathbb{C}), \mathbb{Z}(d))$. The desired statement now follows from the diagram. $\square$

A.3 Extension to a pairing between cohomology classes

Let us denote by $\prod'_{p \in C(\mathbb{C})} H^{2d}_{\text{ét}}(Y_{F_p}, \hat{\mathbb{Z}}(d))$ the subgroup of $\prod_{p \in C(\mathbb{C})} H^{2d}_{\text{ét}}(Y_{F_p}, \hat{\mathbb{Z}}(d))$ consisting of those families $(\alpha_p)_{p \in C(\mathbb{C})}$ such that for all but finitely many $p \in C(\mathbb{C})$, the class α_p belongs to the image of the pull-back map (15). Letting f also stand for the morphisms $Y \to \text{Spec}(F)$ and $Y_{F_p} \to \text{Spec}(F_p)$ obtained from $f : X \to C$ by

base change, we consider the pairing

$$\left(\prod_{p\in C(\mathbb{C})}' H^{2d}_{\text{ét}}(Y_{F_p}, \hat{\mathbb{Z}}(d)) \right) \times H^1_{\text{ét}}(Y, \mathbb{Q}/\mathbb{Z}(1)) \to \mathbb{Q}/\mathbb{Z} \tag{18}$$

defined by $((\alpha_p)_{p\in C(\mathbb{C})}, \beta) \mapsto \sum_{p\in C(\mathbb{C})} \text{inv}_p\, f_*(\alpha_p \smile \beta)$, where

$$f_* : H^{2d+1}_{\text{ét}}(Y_{F_p}, \mathbb{Q}/\mathbb{Z}(d+1)) \to H^1(F_p, \mathbb{Q}/\mathbb{Z}(1)) \tag{19}$$

is induced by the trace morphism associated with f (see [17, XVIII, (2.13.2)]). This pairing is well-defined as $\alpha_p \smile \beta$ vanishes for any p such that both α_p and the image of β in $H^1_{\text{ét}}(Y_{F_p}, \mathbb{Q}/\mathbb{Z}(1))$ come from the cohomology of $X \times_C \text{Spec}(\hat{\mathcal{O}}_{C,p})$, by the proper base change theorem.

Proposition A.3 *The pairing* $\left(\prod_{p\in C(\mathbb{C})} \text{CH}_0(Y_{F_p}) \right) \times H^1_{\text{ét}}(Y, \mathbb{Q}/\mathbb{Z}(1)) \to \mathbb{Q}/\mathbb{Z}$ *induced by* (18) *via the maps* $\text{cl} : \text{CH}_0(Y_{F_p}) \to H^{2d}_{\text{ét}}(Y_{F_p}, \hat{\mathbb{Z}}(d))$ *(whose product takes values in* $\prod_{p\in C(\mathbb{C})}' H^{2d}_{\text{ét}}(Y_{F_p}, \hat{\mathbb{Z}}(d))$ *by Proposition A.2) is equal to the pairing* (10).

Proof Letting $i : \text{Spec}(F_p(q)) \to Y_{F_p}$ denote the inclusion of a closed point q of Y_{F_p} and 1 the unit of $H^0(F_p(q), \hat{\mathbb{Z}})$, we have $\text{cl}(q) = i_*1$. For $\beta \in H^1(Y, \mathbb{Q}/\mathbb{Z}(1))$, the equality $\text{Cores}_{F_p(q)/F_p}\, \beta(q) = (f \circ i)_*\beta(q) = f_*i_*i^*\beta$ and the projection formula $i_*i^*\beta = i_*1 \smile \beta$ therefore yield the desired compatibility. $\qquad\square$

A.4 Cohomological reinterpretation

We are now in a position to reformulate the conclusion of Theorem A.1 in purely cohomological terms.

Proposition A.4 *For $m \in \mathbb{Z}$, the following conditions are equivalent:*

(1) *There exists a collection* $(\alpha_p)_{p\in C(\mathbb{C})} \in \prod_{p\in C(\mathbb{C})} \text{CH}_0(Y_{F_p})$ *belonging to the left kernel of the pairing* (10), *with* $\deg(\alpha_p) = m$ *for all* $p \in C(\mathbb{C})$.
(2) *There exists* $\alpha \in H^{2d}(X(\mathbb{C}), \mathbb{Z}(d))$ *with* $f_*\alpha = m$ *in* $H^0(C(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}$.

Proof Let us consider the following variants of (2):

(2′) There exists $\alpha \in H^{2d}_{\text{ét}}(X, \hat{\mathbb{Z}}(d))$ with $f_*\alpha = m$ in $H^0_{\text{ét}}(C, \hat{\mathbb{Z}}) = \hat{\mathbb{Z}}$.
(2″) Same as (2′), except that we impose, in addition, that the image of α in $H^{2d}_{\text{ét}}(Y_{F_p}, \hat{\mathbb{Z}}(d))$ lies in the image of (16) for all $p \in C(\mathbb{C})$.

It is clear that (2) $\Rightarrow$ (2″) $\Rightarrow$ (2′). On the other hand, we have (2′) $\Rightarrow$ (2) as the maps $f_* : H^{2d}_{\text{ét}}(X, \hat{\mathbb{Z}}(d)) \to H^0_{\text{ét}}(C, \hat{\mathbb{Z}})$ and $f_* : H^{2d}(X(\mathbb{C}), \mathbb{Z}(d)) \to H^0(C(\mathbb{C}), \mathbb{Z})$ share the same cokernel, by the comparison between singular and étale cohomology. Hence (2), (2′) and (2″) are equivalent. Now (2″) is in turn equivalent to the existence of $\alpha \in \varinjlim H^{2d}_{\text{ét}}(X_U, \hat{\mathbb{Z}}(d))$, where U ranges over the dense open subsets of C and X_U denotes the inverse image of U in X, whose image α_p in $H^{2d}_{\text{ét}}(Y_{F_p}, \hat{\mathbb{Z}}(d))$, for all

$p \in C(\mathbb{C})$, belongs to the image of (16) and satisfies $f_* \alpha_p = m \in \hat{\mathbb{Z}}$. Finally, this condition is equivalent to (1) in view of the next lemma, of Proposition A.2 and of Proposition A.3. □

Lemma A.5 *The left kernel of the pairing* (18) *coincides with the image of the diagonal map* $\varinjlim H^{2d}_{\text{ét}}(X_U, \hat{\mathbb{Z}}(d)) \rightarrow \prod'_{p \in C(\mathbb{C})} H^{2d}_{\text{ét}}(Y_{F_p}, \hat{\mathbb{Z}}(d))$, *where* U *ranges over the dense open subsets of* C.

Proof For any bounded complex $\mathcal{C}$ of constructible étale sheaves of abelian groups on C and for any small enough dense open subset U of C, one obtains, by proceeding exactly as in the proof of [18, Proposition 2.6], an exact sequence

$$H^0_{\text{ét}}(U, \mathcal{C}) \rightarrow \prod_{p \in C(\mathbb{C}) \setminus U(\mathbb{C})} H^0_{\text{ét}}(F_p, \mathcal{C}) \times \prod_{p \in U(\mathbb{C})} H^0_{\text{ét}}(\widehat{O}_{C,p}, \mathcal{C}) \rightarrow H^1_{\text{ét}}(F, \mathcal{C}')^D, \tag{20}$$

where $\mathcal{C}' = R\mathcal{H}om(\mathcal{C}, \mathbb{Q}/\mathbb{Z}(1))$, where D denotes the Pontrjagin dual, and where we still denote by $\mathcal{C}$ (resp. $\mathcal{C}'$) the pull-back of $\mathcal{C}$ (resp. $\mathcal{C}'$) to any of U, $\text{Spec}(F)$, $\text{Spec}(\widehat{O}_{C,p})$, $\text{Spec}(F_p)$. The lemma now follows by considering the exact sequences associated in this way with $\mathcal{C} = Rf_*\mathbb{Z}/n\mathbb{Z}(d)[2d]$ for $n \geq 1$ and applying $\varinjlim_U \varprojlim_n$ to these sequences, in view of the canonical isomorphism $\mathcal{C}' = Rf_*\mathbb{Z}/n\mathbb{Z}(1)$ given by Poincaré duality; as the groups $H^0_{\text{ét}}(U, \mathcal{C}) = H^{2d}_{\text{ét}}(X_U, \mathbb{Z}/n\mathbb{Z}(d))$ are all finite, the resulting sequence is still exact (Mittag–Leffler criterion). □

A.5 Applying the Riemann–Roch theorem

The next statement and its proof are a variation on a result of Colliot-Thélène and Voisin [9, Proposition 7.3 (ii)], in the style of [12]. When $E = \mathcal{O}_Y$, its formulation is parallel to [12, Proposition 2.4].

Proposition A.6 *Let E be a coherent sheaf on Y and $n \geq 1$ be an integer. If the class of the fibres of $f : X \rightarrow C$ in $\text{NS}(X)/(\text{torsion})$ is divisible by n, then $\chi(Y, E)$ is divisible by n.*

Proof As E is the restriction of a coherent sheaf on X and as any coherent sheaf on X admits a finite resolution by locally free sheaves, we may assume that E is the restriction of a locally free sheaf V on X, which we henceforth fix.

Let F be a fibre of f. By assumption, there exist divisors H and M on X such that the equality

$$n[H] = [M] + [F] \tag{21}$$

holds in $\text{NS}(X)$ and such that $[M]$ belongs to the torsion subgroup of $\text{NS}(X)$. By the last condition, the divisor M is numerically trivial. Hence so is the cycle class $H^2 \in \text{CH}^2(X)$, in view of (21) and of the equality $F^2 = 0 \in \text{CH}^2(X)$.

The Hirzebruch–Riemann–Roch theorem applied to the locally free sheaves V and $V \otimes_{\mathcal{O}_X} \mathcal{O}_X(H)$ on X therefore gives us the equality

$$\chi(X, V \otimes_{\mathcal{O}_X} \mathcal{O}_X(H)) - \chi(X, V) = \deg(\operatorname{ch}(V) \cdot (H + H^2/2 + \cdots) \cdot \operatorname{Td}(T_X))$$
$$= \deg(H \cdot Z) = \frac{1}{n}\deg(F \cdot Z),$$

where $Z \in \operatorname{CH}_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ denotes the 1-dimensional component of $\operatorname{ch}(V) \cdot \operatorname{Td}(T_X)$. By the Hirzebruch–Riemann–Roch theorem applied to the locally free sheaf E on Y, we also have the equality $\chi(Y, E) = \deg(\operatorname{ch}(E) \cdot \operatorname{Td}(T_Y))$, which can be rewritten as $\chi(Y, E) = \deg(F \cdot Z)$ since $\operatorname{Td}(T_X)|_Y = \operatorname{Td}(T_Y)$ and $\operatorname{ch}(V)|_Y = \operatorname{ch}(E)$. Hence

$$\chi(X, V \otimes_{\mathcal{O}_X} \mathcal{O}_X(H)) - \chi(X, V) = \frac{1}{n}\chi(Y, E) \tag{22}$$

and we conclude that n divides $\chi(Y, E)$ since the left-hand side is an integer. $\qquad\square$

Corollary A.7 *For any coherent sheaf E on Y, there exists $\alpha \in H^{2d}(X(\mathbb{C}), \mathbb{Z}(d))$ such that $f_* \alpha = \chi(Y, E)$ in $H^0(C(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}$.*

Proof According to Proposition A.6, the integer $\chi(Y, E)$ annihilates the kernel of $f^* : \operatorname{NS}(C) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \to \operatorname{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$. As $\operatorname{NS}(C) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = H^2(C(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(1))$ and $\operatorname{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \subset H^2(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(1))$, the latter kernel coincides with the kernel of $f^* : H^2(C(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(1)) \to H^2(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(1))$. Thus, Poincaré duality implies that $\chi(Y, E)$ also annihilates the cokernel of $f_* : H^{2d}(X(\mathbb{C}), \mathbb{Z}(d)) \to H^0(C(\mathbb{C}), \mathbb{Z})$. $\square$

Combining Proposition A.4 with Corollary A.7 now yields Theorem A.1.

References

1. Atiyah, M.F., Hirzebruch, F.: Analytic cycles on complex manifolds. Topology **1**, 25–45 (1962)
2. Ballico, E., Catanese, F., Ciliberto, C. (eds.): Classification of Irregular Varieties. Lecture Notes in Mathematics, vol. 1515. Springer, Berlin (1992)
3. Benoist, O., Ottem, J.C.: Failure of the integral Hodge conjecture for threefolds of Kodaira dimension zero. Comment. Math. Helv. **95**(1), 27–35 (2020)
4. Bloch, S., Kas, A., Lieberman, D.: Zero cycles on surfaces with $p_g = 0$. Compos. Math. **33**(2), 135–145 (1976)
5. Bloch, S., Ogus, A.: Gersten's conjecture and the homology of schemes. Ann. Sci. École Norm. Sup. (4) **7**, 181–201 (1974)
6. Bosch, S.; Lütkebohmert, W.; Raynaud, M.: Néron Models, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 21. Springer, Berlin, x+325 pp (1990)
7. Colliot-Thélène, J. -L.,Gille, P.: Remarques sur l'approximation faible sur un corps de fonctions d'une variable. In: Arithmetic of Higher-Dimensional Algebraic Varieties (Palo Alto, CA, 2002), pp. 121–134, Progr. Math., vol. 226. Birkhäuser Boston, Boston
8. Colliot-Thélène, J.-L., Szamuely, T.: Autour de la conjecture de Tate à coefficients $\mathbf{Z}_\ell$ *pour les variétés sur les corps finis*. In: The Geometry of Algebraic Cycles, pp. 83–98, Clay Math. Proc., vol. 9. Amer. Math. Soc., Providence
9. Colliot-Thélène, J.-L., Voisin, C.: Cohomologie non ramifiée et conjecture de Hodge entière. Duke Math. J. **161**(5), 735–801 (2012)

10. Colmez, P., Serre, J.-P.: Correspondance Grothendieck-Serre, Documents Mathématiques (Paris), vol. 2. Société Mathématique de France, Paris (2001)
11. Deligne, P., Katz, N.: Groupes de monodromie en géométrie algébrique. II, Lecture Notes in Mathematics, vol. 340. Springer, Berlin (1973)
12. Esnault, H., Levine, M., Wittenberg, O.: Index of varieties over Henselian fields and Euler characteristic of coherent sheaves. J. Algebr. Geom. **24**(4), 693–718 (2015)
13. Fulton, W.: Intersection Theory, 2nd edn, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 2. Springer, Berlin (1998)
14. Graber, T., Harris, J., Mazur, B., Starr, J.: Rational connectivity and sections of families over curves. Ann. Sci. École Norm. Sup. (4) **38**(5), 671–692 (2005)
15. Graber, T., Harris, J., Starr, J.: Families of rationally connected varieties. J. Am. Math. Soc. **16**(1), 57–67 (2003)
16. Grothendieck, A.: Revêtements étales et groupe fondamental. Fasc. I: Exposés 1 à 5, Séminaire de Géométrie Algébrique, 1960/61. Troisième édition, corrigée, Institut des Hautes Études Scientifiques, Paris (1963)
17. Grothendieck, A.: Théorie des topos et cohomologie étale des schémas. Tome 3, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4). Dirigé par M. Artin, A. Grothendieck et J.-L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat. Lecture Notes in Mathematics, vol. 305. Springer, Berlin, vi+640 pp (1973)
18. Izquierdo, D.: Théorèmes de dualité pour les corps de fonctions sur des corps locaux supérieurs. Math. Z. **284**(1–2), 615–642 (2016)
19. Lafon, G.: Une surface d'Enriques sans point sur $\mathbb{C}((t))$. C. R. Math. Acad. Sci. Paris **338**(1), 51–54 (2004)
20. Ottem, J.C., Suzuki, F.: A pencil of Enriques surfaces with non-algebraic integral Hodge classes. Math. Ann. **377**(1–2), 183–197 (2020)
21. Saito, S., Sato, K.: A finiteness theorem for zero-cycles over p-adic fields, with an appendix by U. Jannsen. Ann. Math. (2) **172**(3), 1593–1639 (2010)
22. Schoen, C.: An integral analog of the Tate conjecture for one-dimensional cycles on varieties over finite fields. Math. Ann. **311**(3), 493–500 (1998)
23. Schreieder, S.: Stably irrational hypersurfaces of small slopes. J. Am. Math. Soc. **32**(4), 1171–1199 (2019)
24. Serre, J.-P.: Lectures on the Mordell–Weil Theorem, Translated from the French and edited by Martin Brown from Notes by Michel Waldschmidt, 3rd edn. Aspects of Mathematics. Friedr. Vieweg & Sohn, Braunschweig (1997)
25. Soulé, C., Voisin, C.: Torsion cohomology classes and algebraic cycles on complex projective manifolds. Adv. Math. **198**(1), 107–127 (2005)
26. Starr, J.M.: A pencil of Enriques surfaces of index one with no section. Algebra Number Theory **3**(6), 637–652 (2009)
27. Suzuki, F.: A remark on a 3- fold constructed by Colliot-Thélène and Voisin. Math. Res. Lett. **27**(1), 301–317 (2020)
28. Terasoma, T.: Complete intersections with middle Picard number 1 defined over Q. Math. Z. **189**(2), 289–296 (1985)
29. Totaro, B.: On the integral Hodge and Tate conjectures over a number field. Forum Math. Sigma **1**, e4 (2013)