CONSTRUCTION OF POLYNOMIAL PRESERVING COCHAIN EXTENSIONS BY BLENDING

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Abstract. A classical technique to construct polynomial preserving extensions of scalar functions defined on the boundary of an n simplex to the interior is to use so-called rational blending functions. The purpose of this paper is to generalize the construction by blending to the de Rham complex. More precisely, we define polynomial preserving extensions which map traces of k-forms defined on the boundary of the simplex to k-forms defined in the interior. Furthermore, the extensions are cochain maps, i.e., they commute with the exterior derivative.

1. Introduction

In applications such as the finite element approximation of partial differential equations and in computer aided geometrical design problems, there often arises a need for a method for extending a piecewise smooth function given on the boundary of a domain to the entire domain, in particular when the domain is a hypercube or a simplex. There is a considerable literature on this subject, dating back to early work of Coons [11] in the context of computer aided design, while more mathematical oriented studies of such problems were initiated in [7, 10]. The constructions were often referred to as *transfinite interpolation* or *blending function methods*, since the extension is obtained by combining, or blending, the transfinite boundary data using rational or polynomial basis functions. In [21], the two dimensional scheme described in [7] was generalized to the case of tetrahedra, and with a brief discussion of its generalization to n simplices. Alternative approaches, using polynomial rather than rational blending functions, were studied in [8, 19, 20, 23]. A summary of much of this early work can be found in [6].

More recently, the study of extension operators that preserve a polynomial structure of the boundary data have played a key role in the analysis of finite element methods of high polynomial order, cf. [1, 5, 9, 22]. In particular, the importance of such extensions that commute with the exterior derivative, i.e., *cochain extensions*, was illustrated by the analysis given in [13]. The results of these papers further

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motivated the theory developed in the series of three papers, [14, 15, 16], where polynomial preserving cochain extensions are constructed for the de Rham complex in three dimensions. An important additional property of these extensions is that they require only weak regularity of the boundary data to be well defined.

The purpose of this paper is to extend the method of blending to define polynomial preserving cochain extensions for differential forms of arbitrary order on n simplices. More specifically, for $\mathcal{S}_n = [x_0, x_1, \ldots, x_n] \subset \mathbb{R}^n$, an n-dimensional simplex, we define extensions E_n^k which map single-valued piecewise smooth k-forms defined on the boundary, $\partial\mathcal{S}_n$, to smooth forms on $\mathcal{S}_n$, such that they preserve polynomial structures and commute with the exterior derivative. The extension operators E_n^k are defined without relying on any explicit decomposition of the given differential forms defined on $\partial\mathcal{S}_n$. The focus of this paper is entirely on the algebraic properties of the operators E_n^k. Stability in normed spaces, such as for example L^2, is not considered. For scalar-valued functions, or zero-forms, the extensions presented here correspond to operators defined in [21], while the general construction for higher order forms appears to be new.

An outline of the paper is as follows. In Section 2, we introduce some basic notation and recall the construction of extensions by blending in the case of zero-forms, i.e., for scalar-valued functions. In particular, we verify that these extensions are polynomial preserving. We also present a summary of the main results of the paper and an application of the construction. In Section 3, we discuss the extension in the case of one-forms. The explicit construction in this basic case provides a motivation for the general construction for k-forms to follow. The discussion in Section 3 also motivates the construction of a family of order-reduction operators, presented in Section 4, which will play a key role in designing the coefficient operators, $A_{I,J}^k$, that will be used to define the extensions E_n^k. A precise definition of the coefficient operators $A_{I,J}^k$ is given in Section 5, and three key properties of these operators are established. In Section 6, we then show that the operators E_n^k are polynomial preserving cochain extensions.

2. PRELIMINARIES

2.1. **Notation.** We will use $[\cdot, \ldots, \cdot]$ to denote the simplex obtained by convex combination of the arguments. The simplex $\mathcal{S}_n = [x_0, x_1, \ldots, x_n] \subset \mathbb{R}^n$, $n \geq 1$, will be considered to be a fixed n-simplex throughout this paper. The barycentric coordinate associated to the vertex x_i will be denoted by $\lambda_i = \lambda_i(x)$, i.e., λ_i is a linear function on $\mathcal{S}_n$ satisfying $\lambda_i(x_j) = \delta_{i,j}$ and such that

$$x = \sum_{i=0}^{n} \lambda_i(x) x_i, \quad x \in \mathcal{S}_n.$$

Furthermore, the boundary of $\mathcal{S}_n$, $\partial\mathcal{S}_n$, consists of all $x \in \mathcal{S}_n$ such that $\lambda_i(x) = 0$ for at least one index $i \in \{0, \ldots, n\}$. We will use $\Lambda^k(\partial\mathcal{S}_n)$ to denote the space of single-valued piecewise smooth k-forms defined on $\partial\mathcal{S}_n$. More precisely, each element $u \in \Lambda^k(\partial\mathcal{S}_n)$ is smooth on each $(n-1)$-dimensional subsimplex of $\partial\mathcal{S}_n$, and with single-valued traces at the interfaces. Correspondingly, $\Lambda^k(\mathcal{S}_n)$ will denote the space of smooth k-forms defined on $\mathcal{S}_n$ and $\mathring{\Lambda}^k(\mathcal{S}_n)$ will denote the space of forms

in $\Lambda^k(\mathcal{S}_n)$ with vanishing trace on $\partial\mathcal{S}_n$. Our main goal is to construct extension operators $E_n^k : \Lambda^k(\partial\mathcal{S}_n) \to \Lambda^k(\mathcal{S}_n)$, $k = 0, 1, \ldots, n-1$, with desired properties.

We will use $d = d^k : \Lambda^k(\mathcal{S}_n) \to \Lambda^{k+1}(\mathcal{S}_n)$ to denote the exterior derivative defined by

$$du_x(v_1, \ldots, v_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} \partial_{v_j} u_x(v_1, \ldots, \hat{v}_j, \ldots, v_{k+1}),$$

where the hat symbol, e.g., $\hat{v}_j$, is used to indicate a suppressed argument and the vectors v_j are elements of $\mathbb{R}^n$. We also use $u^1 \wedge u^2$ to denote the wedge product mapping a j-form u^1 and k-form u^2 into a $(j+k)$-form. A smooth map $F : \mathcal{S}_n \to \partial\mathcal{S}_n$ provides a pullback of a differential form from $\partial\mathcal{S}_n$ to $\mathcal{S}_n$, i.e., a map from $\Lambda^k(\partial\mathcal{S}_n) \to \Lambda^k(\mathcal{S}_n)$ given by

$$(F^* u)_x(v_1, \ldots, v_k) = u_{F(x)}(DF_x(v_1), \ldots, DF_x(v_k)).$$

The pullback respects exterior products and differentiation, i.e.,

$$F^*(u^1 \wedge u^2) = F^* u^1 \wedge F^* u^2, \qquad F^*(du) = d(F^* u).$$

The pullback of the inclusion map of $\partial\mathcal{S}_n$ into $\mathcal{S}_n$ is the trace map, $\mathrm{tr}_{\partial S_n}$, and we note that the spaces $\Lambda^k(\partial\mathcal{S}_n)$ are precisely defined so that $\Lambda^k(\partial\mathcal{S}_n) = \mathrm{tr}_{\partial S_n} \Lambda^k(\mathcal{S}_n)$.

For each integer $r > 0$, the polynomial subspaces $\mathcal{P}_r\Lambda^k(\mathcal{S}_n)$ consist of all elements $u \in \Lambda^k(\mathcal{S}_n)$ such that for fixed vectors $v_1, \ldots, v_k$, the function $u_x(v_1, \ldots, v_k)$, as a function of x, is an element of $\mathcal{P}_r(\mathcal{S}_n)$, i.e., the space of polynomials of degree less than or equal to r defined on $\mathcal{S}_n$. The trimmed space $\mathcal{P}_r^-\Lambda^k(\mathcal{S}_n)$ is the subspace consisting of all $u \in \mathcal{P}_r\Lambda^k(\mathcal{S}_n)$ such that $u\lrcorner(x-a) \in \mathcal{P}_r\Lambda^{k-1}(\mathcal{S}_n)$ for any fixed $a \in \mathbb{R}^n$. Here, the symbol $\lrcorner$ is used to denote contraction, i.e.,

$$(u\lrcorner(x-a))_x(v_1, \ldots, v_{k-1}) = u_x(x-a, v_1, \ldots, v_{k-1}).$$

The corresponding spaces on the boundary are defined by

$$\mathcal{P}_r\Lambda^k(\partial\mathcal{S}_n) = \mathrm{tr}_{\partial S_n} \mathcal{P}_r\Lambda^k(\mathcal{S}_n) \quad \text{and} \quad \mathcal{P}_r^-\Lambda^k(\partial\mathcal{S}_n) = \mathrm{tr}_{\partial S_n} \mathcal{P}_r^-\Lambda^k(\mathcal{S}_n).$$

We refer to [2, 3, 4] for more details on the polynomial and piecewise polynomial spaces of differential forms.

If $J = \{j_0, j_1, \ldots, j_m\}$ is an ordered subset of $\{0, 1, \ldots, n\}$, but not necessarily increasingly ordered, we will refer to J as an index set. We will use Γ to denote the set of all increasingly ordered subsets of $\{0, 1, \ldots, n\}$. In other words, if $J \in \Gamma$, then J is an index set of the form

$$J = \{j_0, j_1, \ldots, j_m\}, \quad \text{where } 0 \le j_0 < j_1 \ldots < j_m \le n.$$

The number of elements in J will be denoted $|J|$, and Γ_m is the subset of Γ consisting of all $J \in \Gamma$ with $|J| = m+1$. Furthermore, for any $I \in \Gamma$, we let

$$\Gamma_m(I) = \{J \in \Gamma_m \,:\, J \subset I\}.$$

For an index set J, the complement of J relative to the set $\{0, 1, \ldots, n\}$ will be denoted J^c and $[x_J]$ will denote the simplex generated by the vertices $\{x_j\}_{j \in J}$, with orientation induced by the order of J. Furthermore, $\lambda_J = \lambda_J(x)$ will denote

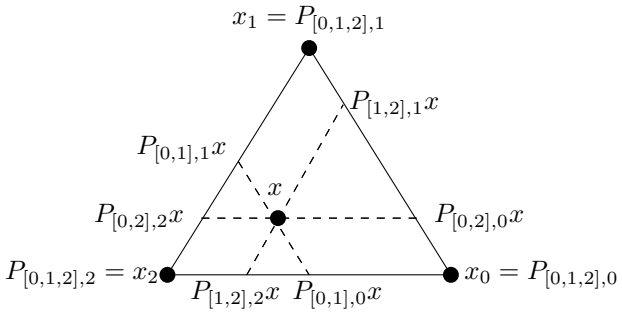

FIGURE 2.1. Projections $P_{I,j}x$ for $n = 2$.

the corresponding sum of barycentric coordinates, i.e., $\lambda_J = \sum_{j\in J} \lambda_j$, and we will use ϕ_J to denote the associated Whitney form, given by

$$\phi_J = \sum_{i=0}^{m} (-1)^i \lambda_{j_i} d\lambda_{j_0} \wedge \ldots \wedge \widehat{d\lambda_{j_i}} \wedge \ldots \wedge d\lambda_{j_m}, \quad \text{if } J = \{j_0, j_1, \ldots, j_m\}.$$

We note that if $J = \{j\} \in \Gamma_0$, then $\phi_J = \lambda_J = \lambda_j$. Furthermore, the set $\{\phi_J\}_{J\in\Gamma_k}$ is a basis for $\mathcal{P}_1^-\Lambda^k(\mathcal{S}_n)$. To simplify the presentation below, we introduce the difference operator δ. In particular, we let

$$(\delta\phi)_J = \sum_{i=0}^{m} (-1)^i \phi_{J(\hat{i})}, \qquad J = \{j_0, j_1, \ldots, j_m\},$$

where $J(\hat{i})$ refers to the set J, but with the index j_i removed.

2.2. **Scalar-valued functions.** We next give a quick review of the extensions studied in [21] for scalar-valued functions, or zero-forms, but in a slightly different notation. For each index set $I \in \Gamma_m$, $m \geq 1$, and $j \in \{0, \ldots, n\}$, we define the map $P_{I,j} : \mathcal{S}_n \to \partial\mathcal{S}_n$ by

$$P_{I,j}x = x + \sum_{i\in I} \lambda_i(x)(x_j - x_i).$$

In fact, $P_{I,j}$ is a projection onto the set $\{x \in \mathcal{S}_n : \lambda_i(x) = 0,\, i \in I,\, i \neq j\}$. Note that if $j \in I$, then $P_{I,j} = P_{I',j}$, where I' represents the index set obtained from I by removing j. In Figure 2.1, we show all the possible points $P_{I,j}x$, for $j \in I$, in the case when $n = 2$.

To define the extension $E_n^0 : \Lambda^0(\partial\mathcal{S}_n) \to \Lambda^0(\mathcal{S}_n)$, we will utilize the corresponding pullbacks $P_{I,j}^*$, defined by $(P_{I,j}^* u)_x = u(P_{I,j}x)$. We define

$$E_n^0 u = \frac{1}{n} \sum_{\substack{I\in\Gamma_m \\ 1\leq m\leq n}} (-1)^{m+1} \sum_{j\in I} \frac{\lambda_j}{\lambda_I} P_{I,j}^* u.$$

To see that E_n^0 is an extension, we need to check that $\operatorname{tr}_{\partial\mathcal{S}_n} E_n^0 u = u$. Without loss of generality, consider a point $x \in \partial\mathcal{S}_n$ such that $\lambda_0(x) = 0$. Then we have that

$$\frac{1}{n}\sum_{\substack{I\in\Gamma_1\\ 0\in I}}\sum_{j\in I}(\frac{\lambda_j}{\lambda_I}P_{I,j}^* u)_x = u_x.$$

In addition, if $I \in \Gamma$ such that $0 \notin I$, and $I' = \{0, I\}$ then

$$\Big(\frac{\lambda_j}{\lambda_I}P_{I,j}^* u\Big)_x = \Big(\frac{\lambda_j}{\lambda_{I'}}P_{I',j}^* u\Big)_x.$$

Therefore, the terms corresponding to I and I' cancel at the point x and we can conclude that $(E_n^0 u)_x = u_x$. This shows that E_n^0 *is an extension.*

Because of the rational factors in the definition of E_n^0, it is not entirely obvious that the operator preserves the polynomial structure of u. However, since $P_{I,j}$ is a linear map, the pullbacks $P_{I,j}^*$ have the property that if $u \in \mathcal{P}_r\Lambda^0(\partial\mathcal{S}_n)$, then $P_{I,j}^* u \in \mathcal{P}_r\Lambda^0(\mathcal{S}_n)$. To show that the same property holds for the extension E_n^0, we fix an $I \in \Gamma$ and consider the sum $\sum_{j\in I}(\lambda_j/\lambda_I)P_{I,j}^* u$. If I is the maximal set $I = \{0, \dots, n\}$, then $\lambda_I(x) \equiv 1$ on $\mathcal{S}_n$, so the sum is simply $\sum_{j=0}^n \lambda_j(x)u_{x_j}$, which is the linear function that interpolates u at the vertices. For any other I, there is at least a vertex x_p such that $p \notin I$, and

$$\sum_{j\in I}(\lambda_j/\lambda_I)P_{I,j}^* u = \sum_{j\in I}(\lambda_j/\lambda_I)(P_{I,j}^* u \pm P_{I,p}^* u) = P_{I,p}^* u + \sum_{j\in I}(\lambda_j/\lambda_I)(P_{I,j}^* u - P_{I,p}^* u).$$

The first term on the right hand side is polynomial preserving. Furthermore, since $P_{I,j}x - P_{I,p}x = \lambda_I(x)(x_j - x_p)$, we have

$$\begin{aligned}(P_{I,j}^* u)_x - (P_{I,p}^* u)_x &= \int_0^1 \frac{d}{d\tau} u_{[(1-\tau)P_{I,p}x + \tau P_{I,j}x]}\, d\tau \\ &= \lambda_I(x)\int_0^1 (du)_{[(1-\tau)P_{I,p}x+\tau P_{I,j}x]\lrcorner}(x_j - x_p)\, d\tau,\end{aligned} \tag{2.1}$$

which has λ_I as a factor. Furthermore, the curve $(1-\tau)P_{I,p}x + \tau P_{I,j}x$ for $\tau \in [0,1]$ belongs to the set $\{x \in \mathcal{S}_n : \lambda_i(x) = 0,\, i \in I(\hat{j})\} \subset \partial\mathcal{S}_n$. Therefore, we can conclude that $E_n^0(\mathcal{P}_r\Lambda^0(\partial\mathcal{S}_n)) \subset \mathcal{P}_r\Lambda^0(\mathcal{S}_n)$.

2.3. **The main result.** The discussion above shows that the operator $E_n^0 : \Lambda^0(\partial\mathcal{S}_n) \to \Lambda^0(\mathcal{S}_n)$ is a polynomial preserving extension operator. The main result of this paper is to construct corresponding operators $E_n^k : \Lambda^k(\partial\mathcal{S}_n) \to \Lambda^k(\mathcal{S}_n)$, $1 \le k < n$, satisfying $\operatorname{tr}_{\partial\mathcal{S}_n} E_n^k u = u$ and which are cochain extensions in the sense that the diagram

$$\begin{array}{ccccccc}
\Lambda^0(\partial\mathcal{S}_n) & \xrightarrow{d} & \Lambda^1(\partial\mathcal{S}_n) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Lambda^{n-1}(\partial\mathcal{S}_n) \\
\downarrow{\scriptstyle E_n^0} & & \downarrow{\scriptstyle E_n^1} & & & & \downarrow{\scriptstyle E_n^{n-1}} \\
\Lambda^0(\mathcal{S}_n) & \xrightarrow{d} & \Lambda^1(\mathcal{S}_n) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Lambda^{n-1}(\mathcal{S}_n),
\end{array}$$

commutes. In other words, $dE_n^{k-1} = E_n^k d$ for $1 \le k < n$, and we will also show that $dE_n^{n-1} = 0$. Furthermore, the extensions E_n^k are polynomial preserving in the

sense that

$$E_n^k(\mathcal{P}_r\Lambda^k(\partial\mathcal{S}_n)) \subset \mathcal{P}_r\Lambda^k(\mathcal{S}_n) \quad \text{and } E_n^k(\mathcal{P}_r^-\Lambda^k(\partial\mathcal{S}_n)) \subset \mathcal{P}_r^-\Lambda^k(\mathcal{S}_n),$$

for $0 \leq k < n$ and $r \geq 1$.

All the operators E_n^k, $0 \leq k < n$, that we construct will be rational functions of the form

$$E_n^k u = \frac{1}{n} \sum_{\substack{I\in\Gamma_m \\ 1\leq m\leq n}} (-1)^{m+1} \sum_{\substack{J\in\Gamma_s(I) \\ 0\leq s\leq k}} \frac{\phi_J}{\lambda_I^{s+1}} \wedge A_{I,J}^k u, \tag{2.2}$$

where the coefficient operators $A_{I,J}^k$ map $\Lambda^k(\partial\mathcal{S}_n)$ to $\Lambda^{k-s}(\mathcal{S}_n)$ for $J \in \Gamma_s(I)$, and such that $A_{I,J}^k = P_{I,j}^*$ for $J = \{j\} \in \Gamma_0$. However, in contrast to what was done in the series of papers [14, 15, 16], we will not perform a careful discussion of bounds for the corresponding operator norms. It is clear that the extensions constructed by blending will not be well defined on spaces with weak regularity, such as $L^2(\partial\mathcal{S}_n)$. On the other hand, a lesson to be learned from the series mentioned above, see also [17, 18], is that some additional averaging technique has to be used to be able to construct such weak regularity extensions.

2.4. **Polynomial complexes.** As an example of a direct application of the extensions E_n^k constructed below, we will briefly consider polynomial complexes of differential forms on $\mathbb{R}^n$, cf. [3, Section 3.5] and [12, Section 4.2]. For any $r > 0$, the full polynomial complex

$$\mathbb{R} \hookrightarrow \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n \to 0,$$

and the trimmed polynomial complex

$$\mathbb{R} \hookrightarrow \mathcal{P}_r^-\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n \to 0,$$

are both exact. In fact, by combining the full polynomial spaces and the trimmed polynomial spaces, we can obtain 2^{n-1} different exact polynomial complexes of the form

$$\mathbb{R} \hookrightarrow \mathcal{P}\Lambda^0 \xrightarrow{d} \mathcal{P}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}\Lambda^n \to 0, \tag{2.3}$$

where $\mathcal{P}\Lambda^0 = \mathcal{P}_r\Lambda^0 = \mathcal{P}_r^-\Lambda^0$ and $\mathcal{P}_{r-k}\Lambda^k \subset \mathcal{P}\Lambda^k \subset \mathcal{P}_r^-\Lambda^k$. In particular, each space $\mathcal{P}\Lambda^k$ is either a trimmed or a full polynomial space. Furthermore, the exactness of these complexes can be established by a standard Poincaré-type operator of the form

$$(Q^k u)_x = \int_0^1 \tau^{k-1} u_{\lfloor (1-\tau)a+\tau x \rfloor}(x-a)\, d\tau,$$

where $a \in \mathbb{R}^n$ is fixed. More precisely, if the spaces $\mathcal{P}\Lambda^{k-1}$ and $\mathcal{P}\Lambda^k$ are related as in (2.3), then the operator Q^k maps $\mathcal{P}\Lambda^k$ to $\mathcal{P}\Lambda^{k-1}$, and if $u \in \mathcal{P}\Lambda^k$ satisfies $du = 0$, then $u = dQ^k u$. In fact, any element u of the polynomial spaces $\mathcal{P}\Lambda^k$ admits the representation

$$u = dQ^k u + Q^{k+1} du.$$

It also well-known that the corresponding polynomial complexes with boundary conditions are exact. More precisely, we consider complexes of the form

$$\mathring{\mathcal{P}}\Lambda^0(\mathcal{S}_n) \xrightarrow{d} \mathring{\mathcal{P}}\Lambda^1(\mathcal{S}_n) \xrightarrow{d} \cdots \xrightarrow{d} \mathring{\mathcal{P}}\Lambda^n(\mathcal{S}_n) \to \mathbb{R},$$

where the spaces $\mathring{\mathcal{P}}\Lambda^k(\mathcal{S}_n)$ are the restriction of the spaces $\mathcal{P}\Lambda^k$ of (2.3) to $\mathcal{S}_n$, and with vanishing traces on ∂S_n. The exactness can in this case be established by counting degrees of freedom, using the results of [3, Section 4] or [4, Section 5]. On the other hand, in this setting it is not straightforward to construct a simple operator $\mathring{Q}^k$ which maps $\mathring{\mathcal{P}}\Lambda^k(\mathcal{S}_n)$ to $\mathring{\mathcal{P}}\Lambda^{k-1}(\mathcal{S}_n)$, and which has the property that $u = d\mathring{Q}^k u$ if $du = 0$. For example, if we consider an operator of the form Q^k above, then it seems impossible to choose the point $a \in \mathcal{S}_n$ such that the vanishing trace condition is preserved. However, by using the extension operator E_n^k constructed in this paper, we can define $\mathring{Q}^k$ as

$$\mathring{Q}^k u = (I - E_n^{k-1} \circ \mathrm{tr}_{\partial S}) Q^k u,$$

where the operator Q^k is a Poincaré-type operator of the form above, with $a \in \mathcal{S}_n$. It follows from the properties of E_n^{k-1} and Q^k that the operator $\mathring{Q}^k$ maps $\mathring{\mathcal{P}}_r\Lambda^k(\mathcal{S}_n)$ to $\mathring{\mathcal{P}}^-_{r+1}\Lambda^{k-1}(\mathcal{S}_n)$ and $\mathring{\mathcal{P}}^-_r\Lambda^k(\mathcal{S}_n)$ to $\mathring{\mathcal{P}}_r\Lambda^{k-1}(\mathcal{S}_n)$. Furthermore, we have

$$d\mathring{Q}^k u + \mathring{Q}^{k+1} du = u - E_n^k \circ \mathrm{tr}_{\partial \mathcal{S}_n} u = u, \quad u \in \mathring{\Lambda}^k(\mathcal{S}_n).$$

3. The case of one-forms

To motivate the general construction for k-forms, we first consider the construction for one-forms. The operator E_n^1 must satisfy the commuting property, $E_n^1 du = dE_n^0 u$. The right hand side of this identity is known, and given by

$$dE_n^0 u = \frac{1}{n} \sum_{\substack{I \in \Gamma_m \\ 1 \le m \le n}} (-1)^{m+1} \sum_{j \in I} \Big[d\Big(\frac{\lambda_j}{\lambda_I}\Big) P_{I,j}^* u + \frac{\lambda_j}{\lambda_I} P_{I,j}^* du \Big],$$

where we have used the fact that $P_{I,j}^*$ commutes with the exterior derivative. The goal is to write the complete right hand side in terms of du. For a fixed $j \in I$, we have

$$d\Big(\frac{\lambda_j}{\lambda_I}\Big) = \frac{1}{\lambda_I^2}[\lambda_I d\lambda_j - \lambda_j d\lambda_I] = \frac{1}{\lambda_I^2} \sum_{i \in I, i \neq j} [\lambda_i d\lambda_j - \lambda_j d\lambda_i] = \sum_{i \in I, i \neq j} \frac{\phi_{i,j}}{\lambda_I^2}.$$

Since $\phi_{j,i} = -\phi_{i,j}$, we then obtain

$$\sum_{j \in I} d\Big(\frac{\lambda_j}{\lambda_I}\Big) P_{I,j}^* u = \sum_{J \in \Gamma_1(I)} \frac{\phi_J}{\lambda_I^2} (\delta P_I^* u)_J,$$

where $(\delta P_I^* u)_J = P_{I,j}^* u - P_{I,i}^* u$ if $J = \{i, j\}$. Furthermore, as in (2.1), we obtain

$$((\delta P_I^* u)_J)_x = \lambda_I(x) \int_0^1 (du)_{[(1-\tau)P_{I,i}x + \tau P_{I,j}x]} \lrcorner (x_j - x_i)\, d\tau, \quad J = \{i, j\}.$$

Since the above formula depends on du, rather than u, this leads to a possible definition of $E_n^1 u$ such that the desired commuting relation holds. More precisely, we can define

$$E_n^1 u = \frac{1}{n} \sum_{\substack{I \in \Gamma_m \\ 1 \le m \le n}} (-1)^{m+1} \Big[\sum_{j \in I} \frac{\lambda_j}{\lambda_I} P_{I,j}^* u + \sum_{J \in \Gamma_1(I)} \frac{\phi_J}{\lambda_I^2} R_{I,J}^1 u \Big],$$

where the operator $R^1_{I,J}$ is defined by

$$(R^1_{I,J}u)_x = \lambda_I(x)\int_0^1 u_{[(1-\tau)P_{I,i}x+\tau P_{I,j}x]}(x_j - x_i)\,d\tau$$

for any $I \in \Gamma$ and J an index set of length 2. Alternatively, we have

$$(R^1_{I,J}u)_x = \int_{P_{I,i}x}^{P_{I,j}x} u, \quad J = \{i,j\},$$

where we have used differential form notation for writing the integral of a one-form u over the one-dimensional space $[P_{I,i}x, P_{I,j}x]$. The problem with the definition of E^1_n above is that the line $[P_{I,i}x, P_{I,j}x]$ will in general not belong to the boundary $\partial\mathcal{S}_n$, cf. Figure 2.1. We will therefore replace the operator $R^1_{I,J}$ by an alternative operator, $A^1_{I,J}$, given by

$$A^1_{I,J}u = \frac{1}{n-1}\sum_{p\notin J}[R^1_{I,\{p,j\}}u - R^1_{I,\{p,i\}}u], \quad J = \{i,j\}. \tag{3.1}$$

This operator will still satisfy the key relation $A^1_{I,J}du = (\delta P^*_I u)_J$, and the line from $[P_{I,j}x, P_{I,p}x]$ will belong to the set $\lambda_i = 0$ if $J = \{i,j\}$ and $p \notin J$. As a consequence, the operator $A^1_{I,J}$ maps one-forms defined on $\partial\mathcal{S}_n$ to scalar functions defined on $\mathcal{S}_n$, and the operator E^1_n, defined by

$$E^1_n u = \frac{1}{n}\sum_{\substack{I\in\Gamma_m\\ 1\le m\le n}} (-1)^{m+1}\Big[\sum_{j\in I}\frac{\lambda_j}{\lambda_I}P^*_{I,j}u + \sum_{J\in\Gamma_1(I)}\frac{\phi_J}{\lambda_I^2}A^1_{I,J}u\Big], \tag{3.2}$$

will satisfy the commuting relation $dE^0_n u = E^1_n du$.

To see that the operator E^1_n, defined by (3.2), is an extension, we have to show that $\mathrm{tr}_{\partial\mathcal{S}_n} E^1_n u = u$. This follows by essentially the same argument as for zero-forms, (cf. also the proof of Theorem 6.1 below). As in the case of zero-forms, we will have

$$\mathrm{tr}_{\lambda_0=0}\,\frac{1}{n}\sum_{\substack{I\in\Gamma_m\\ 1\le m\le n}} (-1)^{m+1}\sum_{j\in I}\frac{\lambda_j}{\lambda_I}P^*_{I,j}u = \mathrm{tr}_{\lambda_0=0}\,u.$$

To complete the argument, it will therefore be enough to show that

$$\mathrm{tr}_{\lambda_0=0}\Big[\sum_{\substack{I\in\Gamma_m\\ 1\le m\le n}} (-1)^{m+1}\sum_{J\in\Gamma_1(I)}\frac{\phi_J}{\lambda_I^2}A^1_{I,J}u\Big] = 0. \tag{3.3}$$

If $0 \in J$, then $\mathrm{tr}_{\lambda_0=0}\,\phi_J = 0$. On the other hand, if $0 \notin J$, then we can conclude from (3.1) that for any $I \in \Gamma$ such that $0 \notin I$ and $J \in \Gamma_1(I)$,

$$\mathrm{tr}_{\lambda_0=0}(A^1_{I,J}u - A^1_{I',J}u) = 0, \quad I' = \{0, I\}.$$

Therefore, the terms corresponding to I and I' on the left hand side of (3.3) cancel. We can therefore conclude that the identity (3.3) holds, and as a consequence, E^1_n is an extension.

The fact that the operator E^1_n maps single-valued, piecewise smooth one-forms defined on $\partial\mathcal{S}_n$, i.e., elements of $\Lambda^1(\partial\mathcal{S}_n)$, to smooth one-forms defined on $\mathcal{S}_n$ and also preserves the polynomial structure is not at all obvious, since the operator $R^k_{I,J}$

and hence the operator $A^k_{I,J}$ has one factor of λ_I, but not two. However, we will show in Section 6, using an alternative representation of the operator E^1_n, that E^1_n does, in fact, have these properties.

4. The operators $R^k_{I,J}$ and their properties

We have seen above that to define the extensions E^1_n for one-forms, such that $dE^0_n = E^1_n d$, we had to rely on the operators $R^1_{I,J}$ for index sets J with $|J| = 2$. Furthermore, to make sure that the operator E^1_n was well-defined for functions defined on the boundary $\partial\mathcal{S}_n$, we needed to use special combinations of these operators, referred to as $A^1_{I,J}$, cf. (3.1). To develop a formula for E^k_n of the form (2.2), in the general case, we will first consider how to generalize the operators $R^k_{I,J}$, introduced above, to the case when $k > 1$ and J is a more general index set. If $|J| = s+1$, then the operators $R^k_{I,J}$ are order reduction operators in the sense that they map $\Lambda^k(\mathcal{S}_n)$ to $\Lambda^{k-s}(\mathcal{S}_n)$. The special linear combinations that we need for the definition of the extensions E^k_n are then given in the next section, cf. formula (5.1) below.

We recall that when J is an index set, the set $[x_J]$ is the simplex generated by the vertices $\{x_j\}_{j\in J}$. For any $x \in \mathcal{S}_n$ and index set $I \in \Gamma$, the corresponding simplex $[P_{I,J}x]$ is the convex combinations of the points $\{P_{I,j}x\}_{j\in J}$. In general, the simplex $[P_{I,J}x]$ will not be a subset of $\partial\mathcal{S}_n$, cf. Figure 2.1. However, if $I \cap J^c$ is nonempty, where we recall that J^c is the complement of J, we will indeed have $[P_{I,J}x] \subset \partial\mathcal{S}_n$, since any λ_i, with $i \in I \cap J^c$, will be identically zero on $[P_{I,J}x]$.

Key tools for the construction of the operators $R^k_{I,J}$ are the maps $F_I : \mathcal{S}_n \times \partial\mathcal{S}_n \to \mathcal{S}_n$ given by

$$F_I(x, y) = x + \sum_{i\in I} \lambda_i(x)(y - x_i),$$

where $I \in \Gamma$. We observe that $F_I(x, x_j) = P_{I,j}x$, cf. Figure 2.1, and that $F_I(\cdot, y)$ is linear in y. Therefore, if $y \in \partial\mathcal{S}_n$ is a convex combination of some of the vertices $\{x_j\}$, then $F_I(x, y)$ is the corresponding convex combination of $\{P_{I,j}x\}$. Furthermore, if we restrict the domain of the map F_I to $\mathcal{S}_n \times [x_J]$, where $I \cap J^c$ is nonempty, then the range is a subset of $\partial\mathcal{S}_n$. The pullback F^*_I, is a map

$$F^*_I : \Lambda^k(\mathcal{S}_n) \to \Lambda^k(\mathcal{S}_n \times \partial\mathcal{S}_n).$$

In the discussion below, we will be interested in operators of the form $\mathrm{tr}_{\mathcal{S}_n\times[x_J]} \circ F^*_I$, mapping $\Lambda^k(\mathcal{S}_n)$ to $\Lambda^k(\mathcal{S}_n \times [x_J])$. In particular, it follows from the discussion above that if $I \cap J^c$ is nonempty, then this operator maps $\Lambda^k(\partial\mathcal{S}_n)$ to $\Lambda^k(\mathcal{S}_n \times [x_J])$.

A space of k-forms on a product space can be expressed using the tensor product $\otimes$ as

$$\Lambda^k(\mathcal{S}_n \times \partial\mathcal{S}_n) = \sum_{s=0}^{k} \Lambda^{k-s}(\mathcal{S}_n) \otimes \Lambda^s(\partial\mathcal{S}_n).$$

In other words, elements $U \in \Lambda^{k-s}(\mathcal{S}_n) \otimes \Lambda^s(\partial\mathcal{S}_n)$ can be written as a sum of terms of the form

$$a(x, y)dx^{k-s} \otimes dy^s,$$

where dx^{k-s} and dy^s run over bases in $\mathrm{Alt}^{k-s}(\mathcal{S}_n)$ and $\mathrm{Alt}^s(\partial\mathcal{S}_n)$, respectively, and where a is a scalar function on $\mathcal{S}_n \times \partial\mathcal{S}_n$. Here Alt^k is the corresponding space of algebraic k-forms. Furthermore, for each s, $0 \le s \le k$, there is a canonical map $\Pi_s : \Lambda^k(\mathcal{S}_n \times \partial\mathcal{S}_n) \to \Lambda^{k-s}(\mathcal{S}_n) \otimes \Lambda^s(\partial\mathcal{S}_n)$ such that

$$U = \sum_{s=0}^{k} \Pi_s U, \quad U \in \Lambda^k(\mathcal{S}_n \times \partial\mathcal{S}_n).$$

The functions $\Pi_s F_I^* u \in \Lambda^{k-s}(\mathcal{S}_n) \otimes \Lambda^s(\partial\mathcal{S}_n)$ can be identified as

$$\begin{aligned}(\Pi_s F_I^* u)_{x,y}&(v_1, \dots, v_{k-s}, t_1, \dots, t_s) \\ &= u_{F_I(x,y)}(D_x F_I v_1, , \dots, D_x F_I v_{k-s}, D_y F_I t_1, \dots, D_y F_I t_s),\end{aligned}$$

where the tangent vectors $v_i \in T(\mathcal{S}_n)$ and $t_i \in T_y(\partial\mathcal{S}_n)$. For the special function F_I in our case, we have $D_y F_I = \lambda_I(x) I$, while

$$D_x F_I = D_x x + \sum_{i \in I} (y - x_i) d_x \lambda_i = \sum_{\ell \in I^c} (x_\ell - y) d_x \lambda_\ell,$$

where $D_x x$ is the identity matrix.

The basic commuting property for pull-backs, namely $dF^* = F^* d$, can be expressed in the present setting as

$$\Pi_s F_I^* du = \Pi_s dF_I^* u = d_{\mathcal{S}} \Pi_s F_{I,J}^* u - (-1)^{k-s} d_{\partial\mathcal{S}} \Pi_{s-1} F_I^* u, \quad u \in \Lambda^k(\mathcal{S}_n), \tag{4.1}$$

where $0 \le s \le k+1$, and where $d_{\mathcal{S}}$ and $d_{\partial\mathcal{S}}$ denote the exterior derivative with respect to the spaces $\mathcal{S}_n$ and $\partial\mathcal{S}_n$, respectively.

Let $I \in \Gamma$ and J an index set with $|J| = s+1$, $0 \le s \le k \le n$. We introduce a family of operators $R_{I,J}^k$, mapping $\Lambda^k(\mathcal{S}_n)$ to $\Lambda^{k-s}(\mathcal{S}_n)$, defined by

$$(R_{I,J}^k u)_x = \int_{[x_J]} (\Pi_s F_I^* u)_x. \tag{4.2}$$

For $s > k$, we define $R_{I,J}^k$ to be zero. If $v_1, \dots, v_{k-s}$ are vectors in $\mathbb{R}^{n+1}$ and $t_1, \dots, t_s$ is any orthonormal basis for the tangent space $T[x_J] = T[P_{I,J} x]$, then

$$\begin{aligned}(R_{I,J}^k u)_x(v_1, \dots, v_{k-s}) &= \lambda_I(x)^s \int_{[x_J]} u_{F_I(x,y)} \lrcorner D_x F_I v_1 \lrcorner \dots \lrcorner D_x F_I v_{k-s} \\ &= \lambda_I(x)^s \int_{[x_J]} (u_{F_I(x,y)} \lrcorner D_x F_I v_1 \lrcorner \dots \lrcorner D_x F_I v_{k-s})(t_1, \dots, t_s)\, dy.\end{aligned}$$

Note that since u is a k-form, $u_{F_I(x,\cdot)} \lrcorner D_x F_I v_1 \lrcorner \dots \lrcorner D_x F_I v_{k-s}$ is an s-form with respect to y, which we can then integrate over the s-dimensional space $[x_J]$. In the final formula, we see that the integral to be evaluated is an integral over $[x_J]$ for a fixed x and vectors $v_1, \dots, v_{k-s}$.

In the special case when $s = 0$, i.e., when the simplex $[x_J]$ degenerates to a vertex x_j, the operator $R_{I,J}^k$ is interpreted as $P_{I,j}^*$. When $s = 1$ and $J = \{i, j\}$, we can utilize the parameterization $y = (1-\tau)x_i + \tau x_j$, $\tau \in [0,1]$ of $[x_J]$ to verify that the definition of $R_{I,J}^1$ given in Section 3 corresponds exactly to the definition given by (4.2). If I and J are related such that $I \cap J^c$ is nonempty, then $R_{I,J}^k u$ will only depend on $\mathrm{tr}_{\partial\mathcal{S}_n} u$. Furthermore, if $i \in I \cap J^c$, then all the vectors

$D_xF_Iv_1,\dots,D_xF_Iv_{k-s}$ and $t_1,\dots,t_s$ will belong to the tangent space of the boundary simplex $\{x\in\mathcal{S}_n\,:\,\lambda_i(x)=0\}$, a space of dimension $<n$. Hence, for $k=n$, it follows that all operators of the form $R^n_{I,J}$ are identically zero in this case.

We will summarize the key properties of the operator $R^k_{I,J}$ in the three lemmas given below.

Lemma 4.1. *Let $I\in\Gamma$ and assume that $j\in\{0,\dots,n\}$ is such that $j\notin I$. Then for all index sets J,*

$$\operatorname{tr}_{\lambda_j=0} R^k_{I,J}u=\operatorname{tr}_{\lambda_j=0} R^k_{I',J}u,$$

where $I'\in\Gamma$ is equal to $\{j,I\}$ up to a reordering. Moreover, if $I\cap J^c$ is nonempty, then $R^k_{I,J}u$ only depends on $\operatorname{tr}_{\partial\mathcal{S}}u$, and the operators $R^n_{I,J}$ are all identically zero.

Proof. The properties obtained when $I\cap J^c$ is nonempty are already observed above. Furthermore, we have for all $y\in\partial\mathcal{S}_n$

$$F_I(x,y)\equiv F_{I'}(x,y),\quad \{x\in\mathcal{S}_n\,:\,\lambda_j(x)=0\}.$$

As a consequence,

$$\operatorname{tr}_{\lambda_j=0} F_I(\cdot,y)^*u=\operatorname{tr}_{\lambda_j=0} F_{I'}(\cdot,y)^*u,\quad y\in\partial\mathcal{S}_n.$$

The desired result follows directly from the definition of the operators $R^k_{I,J}$. □

It follows directly from the definition of the operators $R^k_{I,J}$ and the smoothness of the functions $F_I(\cdot,y)$ on $\mathcal{S}_n$ for any fixed $y\in\partial\mathcal{S}_n$, that the operator $\lambda_I^{-s}R^k_{I,J}$ maps $\Lambda^k(\mathcal{S}_n)$ to $\Lambda^{k-s}(\mathcal{S}_n)$. The corresponding result in the polynomial case is given below.

Lemma 4.2. *Let $I\in\Gamma$ and J an index set with $|J|=s+1$.*

i) *If $u\in\mathcal{P}_r\Lambda^k(\mathcal{S}_n)$, then $\lambda_I^{-s}R^k_{I,J}u\in\mathcal{P}_r\Lambda^{k-s}(\mathcal{S}_n)$;*
ii) *If $u\in\mathcal{P}^-_r\Lambda^k(\mathcal{S}_n)$ then $\lambda_I^{-s}R^k_{I,J}u\in\mathcal{P}^-_r\Lambda^{k-s}(\mathcal{S}_n)$.*

Furthermore, if $I\cap J^c$ is nonempty, then the assumptions in the two cases can be reduced to the trace conditions $u\in\mathcal{P}_r\Lambda^k(\partial\mathcal{S}_n)$ and $u\in\mathcal{P}^-_r\Lambda^k(\partial\mathcal{S}_n)$, respectively.

Proof. Recall that

$$\lambda_I(x)^{-s}(R^k_{I,J}u)_x(v_1,\dots,v_{k-s})\\ =\int_{[x_J]}(u_{F_I(x,y)}\lrcorner D_xF_Iv_1\lrcorner\dots\lrcorner D_xF_Iv_{k-s})(t_1,\dots,t_s)\,dy,$$

where $t_1,\dots,t_s$ is any orthonormal basis for the tangent space $T[x_J]$. Since F_I is linear in x, the integrand preserves the polynomial degree of u for each fixed $y\in[x_J]$. Since the same will be true for the integral with respect to y, the first part of the lemma follows.

To show the $\mathcal{P}^-_r$ spaces are also preserved, we look at $\lambda_I^{-s}R^k_{I,J}u\lrcorner(x-x_j)$, where $j\in J$. In fact, we choose $j\in I\cap J$ if this set is nonempty. It then follows that

$$DF_x(x-x_j)=\sum_{\ell\in I^c}(x_\ell-y)\lambda_\ell(x)=F_I(x,y)-y,$$

which gives

$$(R^k_{I,J}u\lrcorner(x-x_j))_x = \int_{[x_J]} (\Pi_s F_I^* u')_x, \tag{4.3}$$

where for each fixed y, we have $u'_x = u\lrcorner(x-y)$. In other words, $R^k_{I,J}u\lrcorner(x-x_j)$ satisfies the same definition as $R^k_{I,J}u$, but with u replaced by u'. However, if $u \in \mathcal{P}_r^-\Lambda^k(\mathcal{S}_n)$, then $u' \in \mathcal{P}_r\Lambda^k(\mathcal{S}_n)$ for each y, and hence the same argument as above shows that $\lambda_I^{-s} R^k_{I,J}u\lrcorner(x-x_j) \in \mathcal{P}_r\Lambda^k(\mathcal{S}_n)$. Alternatively, if $I \cap J$ is empty, we obtain

$$DF_x(x-x_j) = x - x_j + \sum_{i\in I}(y-x_i)\lambda_i(x) = F_I(x,y) - x_j.$$

Also in this case, we obtain an expression of the form (4.3), but now with $u' = u\lrcorner(x-x_j)$. As above, we again can conclude that $\lambda_I^{-s} R^k_{I,J}u\lrcorner(x-x_j) \in \mathcal{P}_r\Lambda^k(\mathcal{S}_n)$ if $u \in \mathcal{P}_r^-\Lambda^k(\mathcal{S}_n)$. Hence, the second part of lemma has been established. The final conclusion, related to the assumption $I \cap J^c$ nonempty, again follows from the fact that $R^k_{I,J}u$ only depends on $\mathrm{tr}_{\partial\mathcal{S}_n} u$ in this case. □

For $I \in \Gamma$ and index sets J with $|J| = s+1$, we define

$$(\delta R^k_I u)_J = \sum_{i=0}^{s} (-1)^i R^k_{I,J(\hat{i})}.$$

The following relation will be a key tool to show that the extensions E^k_n are cochain maps.

Proposition 4.3. *Let $I \in \Gamma$ and J an index set with $|J| = s+1$, where $0 \le s \le k+1$. The operators $R^k_{I,J}$ satisfy the relations*

$$d(R^k_{I,J}u) = R^{k+1}_{I,J}du + (-1)^{k-s}(\delta R^k_I u)_J, \quad u \in \Lambda^k(\mathcal{S}).$$

Proof. By applying (4.1), we get

$$\begin{aligned} dR^k_{I,J}u &= \int_{[x_J]} d_{\mathcal{S}}\Pi_s F_I^* u = \int_{[x_J]} [\Pi_s F_I^* du + (-1)^{k-s} d_{[x_J]}\Pi_{s-1}F_I^* u] \\ &= R^{k+1}_{I,J}du + (-1)^{k-s}\int_{\partial[x_J]} \Pi_{s-1}F_I^* u, \end{aligned}$$

where we have used Stokes theorem for the last equality. The proof of the proposition is completed by observing that

$$\int_{\partial[x_J]} \Pi_{s-1}F_I^* u = (\delta R^k_I u)_J.$$

□

5. The operators $A^k_{I,J}$ and their properties

In this section, we define the coefficient operators $A^k_{I,J}$ for $I \in \Gamma_m$, $1 \le m \le n$, and $J \in \Gamma_s(I)$, $0 \le s \le k \le n-1$, as operators mapping $\Lambda^k(\partial\mathcal{S}_n)$ to $\Lambda^{k-s}(\mathcal{S}_n)$. As we saw already in Section 3, the simplices $[P_{I,J}x]$, defined as all convex combinations of the points $P_{I,j}x$, $j \in J$, will in general not be a subset of $\partial\mathcal{S}_n$, unless $I \cap J^c$ is nonempty. As a consequence, the operators $R^k_{I,J}$ are not well defined for functions in $\Lambda^k(\partial\mathcal{S}_n)$ unless this condition holds. We will therefore define the operators $A^k_{I,J}$ as linear combinations of these order reduction operators in such a way that they are well defined for functions in $\Lambda^k(\partial\mathcal{S}_n)$. We have already defined the operators $A^k_{I,J}$ for $J = \{j\} \in \Gamma_0(I)$ by $A^k_{I,J}u = R^k_{I,J}u = P^*_{I,j}u$, and in the case $J \in \Gamma_1(I)$ by (3.1). The general definition given below will generalize the definitions given in these special cases.

In the general case, we define the operators $A^k_{I,J}$ of the form

$$A^k_{I,J} = c^k_{s,n}[(n-s)R^k_{I,J} - \sum_{p\in J^c}(\delta R^k_I)_{\{p,J\}}], \tag{5.1}$$

where $c^k_{s,n}$ are constants to be specified below. Here

$$(\delta R^k_I)_{\{p,J\}} = R^k_{I,J} - \sum_{i=0}^{s}(-1)^i R^k_{I,\{p,J(\hat{i})\}},$$

and the index set $\{p, J(\hat{i})\}$ is given by

$$\{p, J(\hat{i})\} = \{p, j_0, \dots j_{i-1}, j_{i+1}, \dots, j_s\} \quad \text{if } J = \{j_0, j_1, \dots, j_s\}.$$

Alternatively, we can express the operator $A^k_{I,J}$ as

$$A^k_{I,J} = c^k_{s,n}\sum_{p\in J^c}\sum_{i=0}^{s}(-1)^i R^k_{I,\{p,J(\hat{i})\}}. \tag{5.2}$$

The constants $c^k_{n,s}$ are given by

$$c^k_{s,n} = \frac{(-1)^{1+ks}(s!)^2}{(n-1)\cdots(n-s)}, \qquad 1 \le s \le n-1, \quad \text{and } c^k_{0,n} = -1.$$

In fact, the key relations we will use below are that these constants satisfy

$$\frac{c^{k+1}_{s,n}}{c^k_{s,n}} = (-1)^s \quad \text{and} \quad \frac{c^k_{s,n}}{c^k_{s-1,n}} = \frac{s^2}{(n-s)}(-1)^k, \tag{5.3}$$

which can be easily checked. Note that when $s = k = 1$ and $n > 1$, we have $c^1_{1,n} = 1/(n-1)$, and as consequence, we see that (5.2) generalizes the definition given in (3.1). Furthermore, $A^k_{I,J} = 0$ for $J \in \Gamma_s(I)$, $s > k$.

We next establish three key properties of the operator $A^k_{I,J}$, using analogous properties established for the operator $R^k_{I,J}$.

Lemma 5.1. *The operators $A^k_{I,J}u$ only depend on the boundary traces of u and satisfy for any $J \in \Gamma(I)$ and $j \in I^c$,*

$$\operatorname{tr}_{\lambda_j=0} A^k_{I',J}u = \operatorname{tr}_{\lambda_j=0} A^k_{I,J}u, \tag{5.4}$$

where $I' \in \Gamma$ is equal to $\{j, I\}$ up to a possible reordering. Furthermore, the operators $A^n_{I,J}$ are all identically zero.

Proof. For each $i \in J \subset \Gamma_s(I)$, we have that $i \in I \cap \{p, J(\hat{i})\}^c$. As a consequence, $A^k_{I,J}u$ only depends on $\mathrm{tr}_{\partial\mathcal{S}_n} u$. The rest of the results follow directly from the corresponding properties of the operators $R^k_{I,J}$ given in Lemma 4.1. □

Lemma 5.2. *Let $I \in \Gamma$ and $J \in \Gamma_s(I)$. Then the operator $\lambda_I^{-s} A^k_{I,J}u$ maps the spaces $\Lambda^k(\partial\mathcal{S}_n) \to \Lambda^{k-s}(\mathcal{S}_n)$, $\mathcal{P}_r\Lambda^k(\partial\mathcal{S}_n) \to \mathcal{P}_r\Lambda^{k-s}(\mathcal{S}_n)$, and $\mathcal{P}^-_r\Lambda^k(\partial\mathcal{S}_n) \to \mathcal{P}^-_r\Lambda^{k-s}(\mathcal{S}_n)$.*

Proof. Since $A^k_{I,J}u$ is a linear combination of operators of the form $R_{I,\{p,J(\hat{i})\}}u$, for which $I \cap \{p, J(\hat{i})\}^c$ is nonempty, the lemma follows directly from corresponding properties of the operators $R^k_{I,J}$, cf. Lemma 4.2. □

Proposition 5.3. *Let $I \in \Gamma$ and $J \in \Gamma_s(I)$. The operators $A^k_{I,J}u$ satisfy the relations*

$$A^{k+1}_{I,J}du = (-1)^s d(A^k_{I,J}u) + s(\delta A^k_I u)_J, 0 \le s \le k+1,\ 0 \le k \le n-1. \tag{5.5}$$

Proof. For $s = 0$, the relation $dA^k_{I.J}u = A^k_{I,J}du$ follows from the corresponding property of the pullbacks $P^*_{I,j}$. Next we show that for for $s \ge 1$ we have

$$(\delta A^k_I u)_J = -sc^k_{s-1,n}(\delta R^k_I u)_J. \tag{5.6}$$

From (5.1), it follows that

$$(\delta A^k_I u)_J = c^k_{s-1,n}[(n-s+1)(\delta R^k_I u)_J - (\delta W)_J],$$

where $W_J = \sum_{p\in J^c}(\delta R^k_I u)_{\{p,J\}}$ for I and k fixed. However,

$$\begin{aligned}(\delta W)_J &= \sum_{a=0}^{s}(-1)^a[(\delta R^k_I u)_{\{a,J(\hat{a})\}} + \sum_{p\in J^c}(\delta R^k_I u)_{\{p,J(\hat{a})\}}]\\ &= (s+1)(\delta R^k_I u)_J + \sum_{p\in J^c}\sum_{a=0}^{s}(-1)^a R^k_{I,J(\hat{a})}u - \sum_{p\in J^c}((\delta\circ\delta)R^k_{I,p}u)_J\\ &= (n+1)(\delta R^k_I u)_J,\end{aligned}$$

where we have used the fact that for each fixed p,

$$((\delta\circ\delta)R^k_{I,p}u)_J = \sum_{a=0}^{s}(-1)^a\Big[\sum_{i=0}^{a-1}(-1)^i R_{I,\{p,J(\hat{a},\hat{i})\}} - \sum_{i=a+1}^{s}(-1)^i R_{I,\{p,J(\hat{a},\hat{i})\}}\Big] = 0.$$

This implies (5.6). For $s \ge 1$, it follows from Proposition 4.3, (5.3), (5.6), and the property $\delta\circ\delta = 0$ that

$$\begin{aligned}d(A^k_{I,J}u) &= c^k_{s,n}[(n-s)dR^k_{I,J}u - \sum_{p\in J^c}(\delta dR^k_I u)_{\{p,J\}}]\\ &= c^k_{s,n}[(n-s)R^{k+1}_{I,J}du - \sum_{p\in J^c}(\delta R^{k+1}_I du)_{\{p,J\}}] + c^k_{s,n}(n-s)(-1)^{k-s}(\delta R^k_I u)_J\end{aligned}$$

$$= \frac{c^k_{s,n}}{c^{k+1}_{s,n}} A^{k+1}_{I,J}\,du + \frac{c^k_{s,n}(n-s)}{s c^k_{s-1,n}}(-1)^{k-s+1}(\delta A^k_I u)_J.$$
$$= (-1)^s A^{k+1}_{I,J}\,du + s(-1)^{s+1}(\delta A^k_I u)_J.$$

This completes the proof. □

6. The operator E^k_n

In this final section, we prove that the operators E^k_n, defined by (2.2), are polynomial preserving cochain extensions.

Theorem 6.1. *The operators E^k_n, defined by* (2.2)*, are extension operators.*

Proof. We first note since the coefficients $A^k_{I,J}u$ only depend on the boundary traces of u, the same is true of $E^k_n u$. To establish the extension property, we will, without loss of generality, show that $\operatorname{tr}_{\lambda_0=0} E^k_n u = \operatorname{tr}_{\lambda_0=0} u$. In fact, the primal operator $E^k_{n,0}$, given by

$$E^k_{n,0}u = \frac{1}{n}\sum_{\substack{I\in\Gamma_m\\ 1\le m\le n}} (-1)^{m+1}\sum_{j\in I}\frac{\lambda_j}{\lambda_I}P^*_{I,j}u,$$

is already an extension. To see this, we can argue as we have done above for scalar-valued functions and one-forms. The maps $P_{I,j}$, where $I\in\Gamma_1$ with $0, j\in I$ and $j\neq 0$ are projections onto $\{x : \lambda_0(x) = 0\}$. As a consequence,

$$\operatorname{tr}_{\lambda_0=0}\Big[\frac{1}{n}\sum_{\substack{I\in\Gamma_1\\ 0\in I}}\sum_{j\in I}\frac{\lambda_j}{\lambda_I}P^*_{I,i}\Big]u = \operatorname{tr}_{\lambda_0=0} u,$$

while the rest of the terms of $E^k_{n,0}u$ give no additional contribution to the trace due to the cancellation of terms corresponding to I and $I' = \{0, I\}$, where $0\notin I$. So to complete the proof, we need to show that

$$\operatorname{tr}_{\lambda_0=0}\Big[\frac{1}{n}\sum_{\substack{I\in\Gamma_m\\ 1\le m\le n}}(-1)^{m+1}\sum_{\substack{J\in\Gamma_s(I)\\ 1\le s\le k}}\frac{\phi_J}{\lambda_I^{s+1}}\wedge A^k_{I,J}u\Big] = 0.$$

However, for terms corresponding to pairs (I, J), with $0\in J$, we have $\operatorname{tr}_{\lambda_0=0}\phi_J = 0$, while if $0\notin J$, the trace of the terms corresponding to the pairs (I, J) and $(\{0, I\}, J)$ will cancel due to the trace property (5.4) of the coefficients. □

Theorem 6.2. *The extensions E^k_n are cochain maps, i.e., they satisfy $dE^k_n = E^{k+1}_n d$ for $0\le k\le n-2$. In addition, $dE^{n-1}_n u = 0$.*

Proof. We first observe that for $I\in\Gamma$ and $J\in\Gamma_s(I)$, we have

$$d\Big(\frac{\phi_J}{\lambda_I^{s+1}}\Big) = (s+1)\sum_{i\in I\setminus J}\frac{\phi_{\{i,J\}}}{\lambda_I^{s+2}}. \tag{6.1}$$

In particular, the right hand side is zero if $J = I$. By the Leibniz rule, we have

$$dE_n^k u = \frac{1}{n} \sum_{\substack{I\in\Gamma_m\\ 1\le m\le n}} (-1)^{m+1} \sum_{\substack{J\in\Gamma_s(I)\\ 0\le s\le k}} d\Big[\frac{\phi_J}{\lambda_I^{s+1}} \wedge A_{I,J}^k u\Big]$$

$$= \frac{1}{n} \sum_{\substack{I\in\Gamma_m\\ 1\le m\le n}} (-1)^{m+1} \sum_{s=0}^{k} \sum_{J\in\Gamma_s(I)} \Big[d\Big(\frac{\phi_J}{\lambda_I^{s+1}}\Big) \wedge (A_{I,J}^k u) + (-1)^s \frac{\phi_J}{\lambda_I^{s+1}} d(A_{I,J}^k u)\Big].$$

However, by using (6.1), we obtain for each fixed $I \in \Gamma$,

$$\begin{aligned}\sum_{s=0}^{k} \sum_{J\in\Gamma_s(I)} d\Big(\frac{\phi_J}{\lambda_I^{s+1}}\Big) \wedge (A_{I,J}^k u) &= \sum_{s=0}^{k}(s+1) \sum_{J\subset\Gamma_s(I)} \sum_{i\in I\setminus J} \frac{\phi_{i,J}}{\lambda_I^{s+2}} \wedge (A_{I,J}^k u)\\ &= \sum_{s=0}^{k}(s+1) \sum_{J\in\Gamma_{s+1}(I)} \frac{\phi_J}{\lambda_I^{s+2}} \wedge \sum_{i=0}^{s+1}(-1)^i A_{I,J(\hat{i})}^k u\\ &= \sum_{s=0}^{k+1} s \sum_{J\in\Gamma_s(I)} \frac{\phi_J}{\lambda_I^{s+1}} \wedge (\delta A_I^k u)_J.\end{aligned}$$

Combining these results, noting that $A_{I,J}^k u = 0$ for $J \in \Gamma_{k+1}(I)$, and using (5.5), we get that

$$dE_n^k u = E_n^{k+1} du, \quad 0 \le k \le n-2, \quad \text{and } dE_n^{n-1} = 0.$$

This completes the proof.

□

The final result we need to prove is that the extensions E_n^k preserve smoothness and polynomial properties. The operator E_n^k can be expressed as

$$E_n^k u = \frac{1}{n} \sum_{\substack{I\in\Gamma_m\\ 1\le m\le n}} (-1)^{m+1} E_n^k(I) u,$$

where each operator $E_n^k(I)$ is given by

$$E_n^k(I) = \sum_{\substack{J\in\Gamma_s(I)\\ 0\le s\le k}} \frac{\phi_J}{\lambda_I^{s+1}} \wedge A_{I,J}^k u.$$

We will show below that each operator $E_n^k(I)$ preserves smoothness and polynomial properties. It is worth noting that it follows from Lemma 5.2 that the operators $E_n^k(I)$ map $\Lambda^k(\partial\mathcal{S}_n)$ to $\lambda_I^{-1}\Lambda^k(\mathcal{S}_n)$, and also

$$\mathcal{P}_r\Lambda^k(\partial\mathcal{S}_n) \to \lambda_I^{-1}\mathcal{P}_{r+1}\Lambda^k(\mathcal{S}_n), \quad \text{and } \mathcal{P}_r^-\Lambda^k(\partial\mathcal{S}_n) \to \lambda_I^{-1}\mathcal{P}_{r+1}^-\Lambda^k(\mathcal{S}_n). \tag{6.2}$$

In fact, to obtain the result for the trimmed spaces, we also need to use the wedge product property for these spaces, for example expressed by formula (3.16) of [3]. To obtain preservation of smoothness and polynomial properties, we need to remove the singular factor λ_I^{-1}. The analysis below will lead to the following fundamental result.

Theorem 6.3. *The extension operator E_n^k maps the spaces $\Lambda^k(\partial\mathcal{S}_n) \to \Lambda^k(\mathcal{S}_n)$, $\mathcal{P}_r\Lambda^k(\partial\mathcal{S}_n) \to \mathcal{P}_r\Lambda^k(\mathcal{S}_n)$, and $\mathcal{P}_r^-\Lambda^k(\partial\mathcal{S}_n) \to \mathcal{P}_r^-\Lambda^k(\mathcal{S}_n)$.*

The proof of this result will utilize the following alternative representation of the operators $E_n^k(I)$.

Lemma 6.4. *The operators $E_n^k(I)$ admit the representation*

$$E_n^k(I)u = \frac{1}{m+1}\Big[\sum_{j\in I} P_{I,j}^* u + dQ_n^k u + Q_n^{k+1} du\Big], \tag{6.3}$$

where the operator $Q_n^k = Q_n^k(I)$ is given by

$$Q_n^k u = \sum_{\substack{J\in\Gamma_s(I)\\ 1\le s\le k}} \frac{1}{s}\frac{(\delta\phi)_J}{\lambda_I^s} \wedge A_{I,J}^k u.$$

Since the derivation of the alternative representation of the operators E_n^k is slightly technical, we will first use the representation (6.3) to prove Theorem 6.3.

Proof. (of Theorem 6.3) It is enough to show the desired mapping properties for each operator $E_n^k(I)$. From the result of Lemma 5.2, it easily follows that the operator $Q_n^k(I)$ maps $\Lambda^k(\partial\mathcal{S}_n)$ to $\Lambda^{k-1}(\mathcal{S}_n)$ and $\mathcal{P}_r\Lambda^k(\partial\mathcal{S}_n)$ to $\mathcal{P}_{r+1}\Lambda^{k-1}(\mathcal{S}_n)$. By combining this with the fact that the operators $P_{I,j}$ are linear, the desired result in the smooth case and the full polynomial case follows. Furthermore, since $\mathcal{P}_r^-\Lambda^k(\partial\mathcal{S}_n)$ is a subspace of $\mathcal{P}_r\Lambda^k(\partial\mathcal{S}_n)$, it follows from (6.2) that $E_n^k(I)$ maps $\mathcal{P}_r^-\Lambda^k(\partial\mathcal{S}_n)$ into

$$\mathcal{P}_r\Lambda^k(\mathcal{S}_n) \cap \lambda_I^{-1}\mathcal{P}_{r+1}^-\Lambda^k(\mathcal{S}_n). \tag{6.4}$$

However, this space is identical to $\mathcal{P}_r^-\Lambda^k(\mathcal{S}_n)$. To see this, let $u \in \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{S}_n)$ such that $\lambda_I^{-1}u \in \mathcal{P}_r\Lambda^k(\mathcal{S}_n)$. It follows from the definition of the trimmed spaces that for any $x_j \in \Delta_0(\mathcal{S}_n)$

$$u\lrcorner(x - x_j) \in \mathcal{P}_{r+1}\Lambda^{k-1}(\mathcal{S}_n).$$

But since $\lambda_I^{-1}u$ is also in $\mathcal{P}_r\Lambda^k(\mathcal{S}_n)$ we have

$$\lambda_I^{-1}(u\lrcorner(x - x_j)) = (\lambda_I^{-1}u)\lrcorner(x - x_j) \in \mathcal{P}_{r+1}\Lambda^{k-1}(\mathcal{S}_n).$$

In other words, the polynomial form $u\lrcorner(x - x_j)$ has λ_I as a linear factor, and as a consequence, $\lambda_I^{-1}(u\lrcorner(x - x_j))$ is also a polynomial form, which then must be in $\mathcal{P}_r\Lambda^{k-1}(\mathcal{S}_n)$. This implies that $\lambda_I^{-1}u \in \mathcal{P}_r^-\Lambda^k(\mathcal{S}_n)$. This shows that the space given by (6.4) is included in $\mathcal{P}_r^-\Lambda^k(\mathcal{S}_n)$, and the opposite inclusion is straightforward to check. □

It remains to establish the alternative representation (6.3) for the operators $E_n^k(I)$. Throughout the discussion below, the index set $I \in \Gamma_m$, $1 \le m \le n$, will be considered fixed. We first study the primal operator $E_{n,0}^k = E_{n,0}^k(I)$ given by

$$E_{n,0}^k u = \sum_{J\in\Gamma_0(I)} \frac{\phi_J}{\lambda_I} \wedge A_{I,J}^k u = \sum_{i\in I} \frac{\lambda_i}{\lambda_I} \wedge P_{I,i}^* u.$$

In particular, for $k = 0$ we have $E_{n,0}^0 = E_n^0(I)$.

Lemma 6.5. *For $I \in \Gamma_m$, the operator $E^k_{n,0} = E^k_{n,0}(I)$ has a representation of the form*

$$E^k_{n,0}u = \frac{1}{m+1}\Big[\sum_{j\in I} P^*_{I,j}u + \lambda_I^{-1}\sum_{J\in\Gamma_1(I)}(\delta\phi)_J \wedge (\delta A^k_I u)_J\Big]. \tag{6.5}$$

Proof.

$$\begin{aligned}\lambda_I^{-1}\sum_{i\in I}\lambda_i P^*_{I,i}u &= \frac{1}{m+1}\sum_{j\in I}[P^*_{I,j}u + \lambda_I^{-1}\sum_{i\in I}\lambda_i(P^*_{I,i}u - P^*_{I,j}u)]\\ &= \frac{1}{m+1}\Big[\sum_{j\in I}P^*_{I,j}u + \lambda_I^{-1}\sum_{i\in I}\sum_{\substack{j\in I\\ j<i}}(\lambda_i-\lambda_j)(P^*_{I,i}u - P^*_{I,j}u)\Big]\\ &= \frac{1}{m+1}\Big[\sum_{j\in I}P^*_{I,j}u + \lambda_I^{-1}\sum_{J\in\Gamma_1(I)}(\delta\phi)_J\wedge(\delta A^k_I u)_J\Big].\end{aligned}$$

The representation for $E^k_{n,0}$ follows immediately. □

To establish an analogous result more generally, we will also need the following preliminary result.

Lemma 6.6. *For $I \in \Gamma_m$ and $0 \le s \le m$, the following identity holds.*

$$\begin{aligned}\sum_{J\in\Gamma_s(I)} d\Big(\frac{(\delta\phi)_J}{\lambda_I^s}\Big)\wedge A^k_{I,J}u + \frac{s}{\lambda_I^{s+1}}\sum_{J\in\Gamma_{s+1}(I)}(\delta\phi)_J\wedge(\delta A^k_I u)_J\\ = \frac{(m+1)s}{\lambda_I^{s+1}}\sum_{J\in\Gamma_s(I)}\phi_J\wedge A^k_{I,J}u.\end{aligned}$$

Proof. If $J\in\Gamma_{s-1}(I)$, we get from (6.1) that

$$d\Big(\frac{\phi_J}{\lambda_I^s}\Big) = s\sum_{p\in I\setminus J}\frac{\phi_{p,J}}{\lambda_I^{s+1}}.$$

As a further consequence, we obtain that if $J\in\Gamma_s(I)$, then

$$\begin{aligned}d\Big(\frac{(\delta\phi)_J}{\lambda_I^s}\Big) &= \frac{s}{\lambda_I^{s+1}}\sum_{i=0}^{s}(-1)^i\sum_{p\in I\setminus J(\hat{i})}\phi_{p,J(\hat{i})}\\ &= \frac{s(s+1)}{\lambda_I^{s+1}}\phi_J + \frac{s}{\lambda_I^{s+1}}\sum_{i=0}^{s}(-1)^i\sum_{p\in I\setminus J}\phi_{p,J(\hat{i})}.\end{aligned}$$

If we wedge the second term with $A^k_{I,J}u$ and sum over $J\in\Gamma_s(I)$, we obtain

$$\begin{aligned}\sum_{J\in\Gamma_s(I)}\sum_{i=0}^{s}(-1)^i\sum_{p\in I\setminus J}\phi_{p,J(\hat{i})}\wedge A^k_{I,J}u &= -\sum_{J\in\Gamma_{s+1}(I)}\sum_{\substack{a,i=0\\ a\neq i}}^{s+1}(-1)^{a+i}\phi_{J(\hat{i})}\wedge A_{I,J(\hat{a})}u\\ &= -\sum_{J\in\Gamma_{s+1}(I)}(\delta\phi)_J\wedge(\delta A^k_I u)_J + (m-s)\sum_{J\in\Gamma_s(I)}\phi_J\wedge A^k_{I,J}u.\end{aligned}$$

The desired result follows by collecting terms. □

To establish Lemma 6.4, we will make use of the following operators,

$$E^k_{n,\ell} = E^k_{n,\ell}(I) = \sum_{\substack{J\in\Gamma_s(I)\\ 0\le s\le \ell}} \frac{\phi_J}{\lambda_I^{s+1}} \wedge A^k_{I,J}u, \quad 0\le \ell\le k.$$

In particular, we note that $E^k_{n,k}(I) = E^k_n(I)$.

Proof. (of Lemma 6.4) We will prove by induction that the operator $E^k_{n,\ell}$ admits a representation of the form

$$E^k_{n,\ell}u = \frac{1}{m+1}\Big[\sum_{j\in I} P^*_{I,j}u + \sum_{J\in\Gamma_{\ell+1}(I)} \frac{(\delta\phi)_J}{\lambda_I^{\ell+1}} \wedge (\delta A^k_I u)_J + dQ^k_{n,\ell}u + Q^{k+1}_{n,\ell}du\Big], \tag{6.6}$$

where,

$$Q^k_{n,\ell} = \sum_{\substack{J\in\Gamma_s(I)\\ 1\le s\le \ell}} \frac{1}{s}\frac{(\delta\phi)_J}{\lambda_I^s} \wedge A^k_{I,J}u,$$

such that $Q^k_{n,k} = Q^k_n$ and $Q^k_{n,0} = 0$. Note that when $\ell = 0$, (6.6) is exactly the identity (6.5), while for $l = k$, $A^{k+1}_{I,J}du = (k+1)(\delta A^k u)_{I,J}$. This gives

$$Q^{k+1}_{n,k}du + \sum_{J\in\Gamma_{k+1}} \frac{(\delta\phi)_J}{\lambda_I^{k+1}} \wedge (\delta A^k_I u)_J = Q^{k+1}_n du.$$

Therefore, the representation formula (6.3) follows from (6.6) for $l = k$.

It remains to carry out the induction argument to establish (6.6). If we assume that (6.6) holds for $\ell - 1$, then

$$\begin{aligned}(m+1)E^k_{n,\ell}u - \sum_{j\in I} P^*_{I,j}u &= dQ^k_{n,\ell-1}u + Q^{k+1}_{n,\ell-1}du\\ &+ \sum_{J\in\Gamma_\ell(I)} \Big[(m+1)\frac{\phi_J}{\lambda_I^{\ell+1}} \wedge A^k_{I,J}u + \frac{(\delta\phi)_J}{\lambda_I^\ell} \wedge (\delta A^k_I u)_J\Big]\\ &= dQ^k_{n,\ell-1}u + Q^{k+1}_{n,\ell}du\\ &+ \sum_{J\in\Gamma_\ell(I)} \Big[(m+1)\frac{\phi_J}{\lambda_I^{\ell+1}} \wedge A^k_{I,J}u + (-1)^{(\ell-1)}\frac{1}{\ell}\frac{(\delta\phi)_J}{\lambda_I^\ell} \wedge dA^k_{I,J}u\Big],\end{aligned}$$

where we have used the relation (5.5). Now from Lemma 6.6, we get that

$$\begin{aligned}d(Q^k_{n,\ell} - Q^k_{n,\ell-1}) &= \sum_{J\in\Gamma_\ell(I)} \frac{1}{\ell} d\Big[\frac{(\delta\phi)_J}{\lambda_I^\ell} \wedge A^k_{I,J}u\Big]\\ &= \sum_{J\in\Gamma_\ell(I)} \frac{1}{\ell}\Big[(-1)^{\ell-1}\frac{(\delta\phi)_J}{\lambda_I^\ell} \wedge dA^k_{I,J}u + (m+1)\ell\frac{\phi_J}{\lambda_I^{\ell+1}} \wedge A^k_{I,J}u\Big]\\ &\quad - \sum_{J\in\Gamma_{\ell+1}(I)} \frac{(\delta\phi)_J}{\lambda_I^{\ell+1}} \wedge (\delta A^k_I u)_J.\end{aligned}$$

Combining these results, we get

$$(m+1)E^k_{n,\ell}u - \sum_{j\in I} P^*_{I,j}u = dQ^k_{n,\ell}u + Q^{k+1}_{n,\ell}du + \sum_{J\in\Gamma_{\ell+1}(I)} \frac{(\delta\phi)_J}{\lambda_I^{\ell+1}} \wedge (\delta A^k_I u)_J,$$

which completes the induction argument, and the proof of the lemma. □

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