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HOPF ALGEBRA STRUCTURE ON TOPOLOGICAL HOCHSCHILD HOMOLOGY

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Abstract. The topological Hochschild homology $THH(R)$ of a commutative S-algebra (E_∞ ring spectrum) R naturally has the structure of a Hopf algebra over R, in the homotopy category. We show that under a flatness assumption this makes the Bökstedt spectral sequence converging to the mod p homology of $THH(R)$ into a Hopf algebra spectral sequence. We then apply this additional structure to study some interesting examples, including the commutative S-algebras ku, ko, tmf, ju and j, and to calculate the homotopy groups of $THH(ku)$ and $THH(ko)$ after smashing with suitable finite complexes. This is part of a program to make systematic computations of the algebraic K-theory of S-algebras, using topological cyclic homology.

1. Introduction

The topological Hochschild homology $THH(R)$ of an S-algebra R (or an A_∞ ring spectrum, or a functor with smash product) was constructed in the mid-1980's by Bökstedt [Bö1], as the natural promotion of the classical Hochschild homology of an algebra in the category of vector spaces (equipped with tensor product) to one in the category of spectra (equipped with smash product). It is the initial ingredient in the construction by Bökstedt, Hsiang and Madsen [BHM93] of the topological cyclic homology $TC(R;p)$ of the S-algebra R, which in many cases closely approximates the algebraic K-theory $K(R)$ of R [Mc97], [Du97], [HM97].

When R is the valuation ring in a local number field, systematic computations of the topological cyclic homology of R were made in [HM03], thereby verifying the Lichtenbaum–Quillen conjectures for the algebraic K-theory of these fields. Particular computations for other S-algebras, related to topological K-theory, have revealed a more general pattern of how algebraic K-theory creates a “red-shift” in chromatic filtration [AR02], and satisfies a Galois descent property [Au]. These results indicate that the algebraic K-theory of a commutative S-algebra R may be governed by some form of “S-algebraic geometry” associated to R, where the chromatic filtration is related to a Zariski topology and Galois covers are related to an étale topology.

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Further systematic computations of the topological Hochschild homology, topological cyclic homology and algebraic K-theory of commutative S-algebras can therefore be expected to shed light on these new geometries, and on the algebraic K-theory functor. The present paper advances the algebraic-topological foundations for making such systematic computations, especially by taking into account the Hopf algebra structure present in the topological Hochschild homology of commutative S-algebras. This program is continued in [BR] and [L-N], which analyze the homological homotopy fixed point spectral sequence approximating the cyclic fixed points of topological Hochschild homology, and the action by Steenrod operations on the homology of the latter, respectively.

When R is a commutative S-algebra (or an E_∞ ring spectrum, or a commutative FSP), there is an equivalence $THH(R) \simeq R \otimes S^1$ due to McClure, Schwänzl and Vogt [MSV97], where S^1 is the topological circle and $(-) \otimes S^1$ refers to the topologically tensored structure in the category of commutative S-algebras. The pinch map $S^1 \to S^1 \vee S^1$ and reflection $S^1 \to S^1$ then induce maps $\psi \colon THH(R) \to THH(R) \wedge_R THH(R)$ and $\chi \colon THH(R) \to THH(R)$, which make $THH(R)$ into a Hopf algebra over R in the homotopy category [EKMM97, IX.3.4]. See also theorem 3.9.

We wish to apply this added structure for computations. Such computations are usually made by starting with the simplicial model for $THH(R)$, with $[q] \mapsto R^{\wedge(q+1)} = R \otimes S^1_q$, where now $[q] \mapsto S^1_q$ is the simplicial circle. The resulting skeleton filtration on $THH(R)$ gives rise to the Bökstedt spectral sequence in homology

$$E^2_{**}(R) = HH_*(H_*(R;\mathbb{F}_p)) \Longrightarrow H_*(THH(R);\mathbb{F}_p)\,,$$

as we explain in section 4. Compare [HM97] and [MS93]. But the coproduct and conjugation maps are not induced by simplicial maps in this model, so some adjustment is needed in order for these structures to carry over to the spectral sequence. This we arrange in section 3, by using a doubly subdivided simplicial circle to provide an alternative simplicial model for $THH(R)$, for which the Hopf algebra structure maps can be simplicially defined. The verification of the Hopf algebra relations then also involves a triply subdivided simplicial circle.

In section 4 we transport the Hopf algebra structure on $THH(R)$ to the Bökstedt spectral sequence, showing in theorem 4.4 that if its initial term $E^2_{**}(R)$ is flat as a module over $H_*(R;\mathbb{F}_p)$, then this term is an A_*-comodule $H_*(R;\mathbb{F}_p)$-Hopf algebra and the d^2-differentials respect this structure. Furthermore, if every E^r-term is flat over $H_*(R;\mathbb{F}_p)$, then the Bökstedt spectral sequence is one of A_*-comodule $H_*(R;\mathbb{F}_p)$-Hopf algebras.

Thereafter we turn to the desired computational applications. The mod p homology of the topological Hochschild homology of the Eilenberg–Mac Lane S-algebras $H\mathbb{F}_p$ and $H\mathbb{Z}$ was already computed by Bökstedt [Bö2]. The first non-algebraic example, namely the topological Hochschild homology of the Adams summand $\ell = BP\langle 1\rangle$ of p-local connective topological K-theory, was computed for p odd by McClure and Staffeldt in [MS93].

We show in section 5 how to extend these computations to include the case of $BP\langle 1\rangle = ku_{(2)}$ at $p = 2$, and to more general Johnson–Wilson S-algebras $BP\langle n\rangle$

when p and n are such that these are commutative. In particular, we provide a proof of Bökstedt's formula saying that the suspension map $\sigma\colon \Sigma R \to THH(R)$ takes the Dyer–Lashof operations Q^k on the homology of the commutative S-algebra R compatibly to the corresponding operations on the homology of the commutative S-algebra $THH(R)$. See proposition 5.9.

In section 6 we do the same for the higher real commutative S-algebras ko and tmf at $p = 2$. As sample results we have corollary 5.13(a) and theorem 6.2(a):

$$H_*(THH(ku); \mathbb{F}_2) \cong H_*(ku; \mathbb{F}_2) \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3)$$

and

$$H_*(THH(ko); \mathbb{F}_2) \cong H_*(ko; \mathbb{F}_2) \otimes E(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3)\,.$$

Here $E(-)$ and $P(-)$ denote the exterior and polynomial algebras on the indicated generators, respectively, and the classes $\bar{\xi}_k$ are the conjugates of Milnor's generators for the dual Steenrod algebra A_*.

In the more demanding section 7 we proceed to the p-local real and complex image of J spectra j and ju, which are connective, commutative S-algebras. At odd primes the two are homotopy equivalent. We identify the mod p homology algebra of ju at $p = 2$ and of $j = ju$ at odd primes in proposition 7.12(a) and (b), and make essential use of our results about Hopf algebra structures to show that the corresponding Bökstedt spectral sequences for THH collapse at the E^2- and E^p-terms, respectively, in proposition 7.13(a) and (b). Finally, in theorem 7.15 we resolve the algebra extension questions to obtain $H_*(THH(ju); \mathbb{F}_p)$ as an algebra, both for $p = 2$ and for p odd. This proof involves a delicate comparison with the case of $THH(ku)$ for $p = 2$, and with $THH(\ell)$ for p odd. Again as a sample result, we have theorem 7.15(a):

$$H_*(THH(ju); \mathbb{F}_2) \cong H_*(ju; \mathbb{F}_2) \otimes E(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3) \otimes \Gamma(\sigma b)\,.$$

Here $\Gamma(\sigma b) = E(\gamma_{2^k}(\sigma b) \mid k \geq 0)$ is the divided power algebra on a class σb in degree 4.

The algebra structure of $H_*(j; \mathbb{F}_2)$ is described as a split square-zero extension of $(A/\!/A_2)_*$ in proposition 7.12(c):

$$0 \to A_* \,\square_{A_{2*}}\, \Sigma^7 K_* \to H_*(j; \mathbb{F}_2) \to (A/\!/A_2)_* \to 0\,.$$

Here $A_2 = \langle Sq^1, Sq^2, Sq^4 \rangle \subset A$, and $K_* \subset A_{2*}$ is dual to a cyclic A_2-module K of rank 17 over $\mathbb{F}_2$. The A-module structure of $H^*(j; \mathbb{F}_2)$ was given in [Da75], but this identification of the algebra structure seems to be new. The E^2-term of the Bökstedt spectral sequence for j is described in proposition 7.13(c), but it is not flat over $H_*(j; \mathbb{F}_2)$, so the coproduct on $H_*(THH(j); \mathbb{F}_2)$ is not conveniently described by this spectral sequence. We have therefore not managed to evaluate the homology of $THH(j)$ at $p = 2$ by these methods.

Next we consider the passage from the homology of $THH(R)$ to its homotopy, with suitably chosen finite coefficients. This has been a necessary technical switch in past computations of topological cyclic homology $TC(R; p)$, since TC is defined as the homotopy inverse limit of a diagram of fixed-point spectra derived from THH, and the interaction between inverse limits and homology can be difficult to control. The homotopy groups of an inverse limit are much better behaved. Nonetheless, it may be that future computations of the topological cyclic homology of S-algebras will follow a purely homological approach, see [BR] and [L-N].

In section 8 we follow the strategy of [MS93] to compute the homotopy groups of $THH(ku) \wedge M$, where $M = C_2$ is the mod 2 Moore spectrum, and of $THH(ko) \wedge Y$, where $Y = C_2 \wedge C_\eta$ is the 4-cell spectrum employed by Mahowald [M82]. The results appear in theorems 8.13 and 8.14, respectively. In each case the method is to use the Adams spectral sequence to pass from homology to homotopy, and to use a computation with a Morava $K(1)$-based Bökstedt spectral sequence to obtain enough information about the v_1-periodic towers in the abutment to completely determine the differential structure of the Adams spectral sequence.

The present paper is based on the first author's Master's thesis [An02] at the University of Oslo, from June 2002.

2. Hochschild and topological Hochschild homology

Let k be a graded field, i.e., a graded commutative ring such that every graded k-module is free, and Λ a graded k-algebra. We recall the definition of the Hochschild homology of Λ, e.g. from [Ma75, X.4]. The Hochschild complex $C_*(\Lambda) = C^k_*(\Lambda)$ is the chain complex of graded k-modules with $C_q(\Lambda) = \Lambda^{\otimes(q+1)}$ in degree q (all tensor products are over k) and boundary homomorphisms $\partial\colon C_q(\Lambda) \to C_{q-1}(\Lambda)$ given by

$$\partial(\lambda_0 \otimes \cdots \otimes \lambda_q) = \sum_{i=0}^{q-1} (-1)^i \lambda_0 \otimes \cdots \otimes \lambda_i\lambda_{i+1} \otimes \cdots \otimes \lambda_q + (-1)^{q+\epsilon} \lambda_q\lambda_0 \otimes \cdots \otimes \lambda_{q-1}$$

where $\epsilon = |\lambda_q|(|\lambda_0| + \cdots + |\lambda_{q-1}|)$. The Hochschild homology $HH_*(\Lambda) = HH^k_*(\Lambda)$ is defined to be the homology of this chain complex. It is bigraded, first by the Hochschild degree q and second by the internal grading from Λ. When Λ is commutative the shuffle product of chains defines a product

$$\phi\colon HH_*(\Lambda) \otimes_\Lambda HH_*(\Lambda) \to HH_*(\Lambda)$$

that makes $HH_*(\Lambda)$ a commutative Λ-algebra, with unit corresponding to the inclusion of 0-chains $\Lambda \to HH_*(\Lambda)$.

When Λ is commutative there is also a chain level coproduct $\psi\colon C_*(\Lambda) \to C_*(\Lambda) \otimes_\Lambda C_*(\Lambda)$ given in degree q by

$$\psi(\lambda_0 \otimes \lambda_1 \otimes \cdots \otimes \lambda_q) = \sum_{i=0}^{q} (\lambda_0 \otimes \lambda_1 \otimes \cdots \otimes \lambda_i) \otimes_\Lambda (1 \otimes \lambda_{i+1} \otimes \cdots \otimes \lambda_q) . \tag{2.1}$$

It is essential to tensor over Λ in the target of this chain map. When $HH_*(\Lambda)$ is flat as a Λ-module, the chain level coproduct ψ induces a coproduct

$$\psi\colon HH_*(\Lambda) \to HH_*(\Lambda) \otimes_\Lambda HH_*(\Lambda)$$

on Hochschild homology. Here the right hand side is identified with the homology of $C_*(\Lambda)\otimes_\Lambda C_*(\Lambda)$ by the Künneth theorem. We shall now compare this chain level definition of the coproduct on $HH_*(\Lambda)$ with an equivalent definition given in more simplicial terms.

Let $B_*(\Lambda) = B_*(\Lambda, \Lambda, \Lambda)$ be the two-sided bar construction [ML75, X.2] for the k-algebra Λ. It has $B_q(\Lambda) = \Lambda \otimes \Lambda^{\otimes q} \otimes \Lambda$ in degree q, and is a free resolution of Λ in the category of Λ-bimodules. We use the bar notation $\lambda_0[\lambda_1|\dots|\lambda_q]\lambda_{q+1}$ for a typical generator of $B_q(\Lambda)$. The Hochschild complex is obtained from the two-sided bar construction by tensoring it with Λ viewed as a Λ-bimodule: $C_*(\Lambda) = \Lambda \otimes_{\Lambda-\Lambda} B_*(\Lambda)$.

When Λ is commutative, $B_*(\Lambda)$ is the chain complex $\mathrm{Ch}(\Lambda \otimes \Delta^1)$ associated to the simplicial Λ-bimodule $[q] \mapsto \Lambda \otimes \Delta^1_q$. Here Δ^1 is the simplicial 1-simplex, and $\Lambda \otimes \Delta^1_q$ denotes the tensor product of one copy of Λ for each element of Δ^1_q. See section 3 below for more on this notation. The Λ-bimodule structure on $B_*(\Lambda)$ is derived from the inclusion of the two boundary points $\partial\Delta^1 \to \Delta^1$, and $C_*(\Lambda)$ equals the chain complex $\mathrm{Ch}(\Lambda \otimes S^1)$ associated to the simplicial Λ-module $[q] \mapsto \Lambda \otimes S^1_q$, where $S^1 = \Delta^1/\partial\Delta^1$ is the simplicial circle.

There is a canonical chain level coproduct $\psi\colon B_*(\Lambda) \to B_*(\Lambda) \otimes_\Lambda B_*(\Lambda)$ of Λ-bimodules, given in degree q by

$$\psi(\lambda_0[\lambda_1|\dots|\lambda_q]\lambda_{q+1}) = \sum_{i=0}^{q} \lambda_0[\lambda_1|\dots|\lambda_i]1 \otimes_\Lambda 1[\lambda_{i+1}|\dots|\lambda_q]\lambda_{q+1}\,.$$

When Λ is commutative the chain level coproduct ψ on $C_*(\Lambda)$ is derived from this, as the obvious composite map

$$C_*(\Lambda) = \Lambda \otimes_{\Lambda-\Lambda} B_*(\Lambda) \xrightarrow{1\otimes\psi} \Lambda \otimes_{\Lambda-\Lambda} (B_*(\Lambda) \otimes_\Lambda B_*(\Lambda)) \xrightarrow{\psi'} C_*(\Lambda) \otimes_\Lambda C_*(\Lambda)\,.$$

Second, there is a shuffle equivalence $sh\colon B_*(\Lambda) \otimes_\Lambda B_*(\Lambda) \to dB_*(\Lambda)$ of Λ-bimodules, by the Eilenberg–Zilber theorem [Ma75, VIII.8.8] applied to two copies of the simplicial Λ-bimodule $\Lambda \otimes \Delta^1$. Here

$$dB_*(\Lambda) = \mathrm{Ch}(\Lambda \otimes d\Delta^1) = \mathrm{Ch}((\Lambda \otimes \Delta^1) \otimes_\Lambda (\Lambda \otimes \Delta^1))$$

is the chain complex associated to the simplicial tensor product of two copies of $\Lambda \otimes \Delta^1$, considered as simplicial Λ-modules by way of the right and left actions, respectively. This simplicial tensor product equals $\Lambda \otimes d\Delta^1$, where the "double 1-simplex" $d\Delta^1 = \Delta^1 \cup_{\Delta^0} \Delta^1$ is the union of two 1-simplices that are compatibly oriented. (So $d\Delta^1$ is the 2-fold edgewise subdivision of Δ^1 [BHM93, §1].) More explicitly,

$$sh(x \otimes_\Lambda y) = \sum_{(\mu,\nu)} \mathrm{sgn}(\mu,\nu)(s_\nu(x) \otimes_\Lambda s_\mu(y))\,,$$

where $x \in B_i(\Lambda)$, $y \in B_{q-i}(\Lambda)$, the sum is taken over all $(i, q-i)$-shuffles (μ,ν), $\mathrm{sgn}(\mu,\nu)$ is the sign of the associated permutation, and $s_\nu(x)$ and $s_\mu(y)$ are the appropriate iterated degeneracy operations on x and y, respectively.

Third, there is a chain equivalence $\pi\colon dB_*(\Lambda) \to B_*(\Lambda)$ of Λ-bimodules, induced by the simplicial map $\pi\colon d\Delta^1 = \Delta^1 \cup_{\Delta^0} \Delta^1 \to \Delta^1$ that collapses the second Δ^1 in $d\Delta^1$ to a point. It is given by

$$\pi(x \otimes_\Lambda y) = x \cdot \epsilon(y)$$

for $x, y \in B_q(\Lambda)$, where $\epsilon(\lambda_0[\lambda_1|\dots|\lambda_q]\lambda_{q+1}) = \lambda_0\lambda_1\dots\lambda_q\lambda_{q+1}$ is the augmentation.

Lemma 2.2. *Let Λ be a commutative k-algebra. The maps $sh \circ \psi\colon B_*(\Lambda) \to dB_*(\Lambda)$ and $\pi\colon dB_*(\Lambda) \to B_*(\Lambda)$ of Λ-bimodule complexes are mutual chain inverses. Hence the induced composite*

$$dC_*(\Lambda) = \Lambda \otimes_{\Lambda-\Lambda} dB_*(\Lambda) \xrightarrow{1\otimes\pi} C_*(\Lambda) \xrightarrow{1\otimes(sh\circ\psi)} dC_*(\Lambda)$$

is chain homotopic to the identity.

Proof. All three chain complexes $B_*(\Lambda)$, $B_*(\Lambda) \otimes_\Lambda B_*(\Lambda)$ and $dB_*(\Lambda)$ are free Λ-bimodule resolutions of Λ, and the maps ψ, sh and π are Λ-bimodule chain maps, so it suffices to verify that the composite $\pi \circ sh \circ \psi\colon B_*(\Lambda) \to B_*(\Lambda)$ covers the identity on Λ. In degree zero, $\psi(\lambda_0[]\lambda_1) = \lambda_0[]1 \otimes_\Lambda 1[]\lambda_1$, $sh(\lambda_0[]1 \otimes_\Lambda 1[]\lambda_1) = \lambda_0[]1 \otimes_\Lambda 1[]\lambda_1$ and $\pi(\lambda_0[]1 \otimes_\Lambda 1[]\lambda_1) = \lambda_0[]\lambda_1$, as required. If desired, the explicit formulas can be composed also in higher degrees, to show that $(\pi \circ sh \circ \psi)(x) \equiv x$ modulo simplicially degenerate terms, for all $x \in B_q(\Lambda)$. In either case, we see that $\pi \circ sh \circ \psi$ is chain homotopic to the identity. The remaining conclusions follow by uniqueness of inverses. □

Note that $dC_*(\Lambda)$ defined above is the chain complex associated to the simplicial Λ-module $\Lambda \otimes d'S^1$, where the "double circle" $d'S^1 = d\Delta^1/\partial d\Delta^1$ is the quotient of the double 1-simplex $d\Delta^1 = \Delta^1 \cup_{\Delta^0} \Delta^1$ by its two end-points $\partial d\Delta^1$. The chain equivalence $1 \otimes \pi\colon dC_*(\Lambda) = \mathrm{Ch}(\Lambda \otimes d'S^1) \to \mathrm{Ch}(\Lambda \otimes S^1) = C_*(\Lambda)$ obtained by tensoring down the Λ-bimodule equivalence induced from the collapse map $\pi\colon d\Delta^1 \to \Delta^1$, is then more directly obtained from the collapse map $\pi\colon d'S^1 \to S^1$ that collapses the second of the two 1-simplices in $d'S^1$ to a point.

There is also a simplicial pinch map $\psi\colon d'S^1 \to S^1 \vee S^1$ to the one-point union (wedge sum) of two circles, that collapses the 0-skeleton of $d'S^1$ to a point. It induces a map

$$\psi'\colon dC_*(\Lambda) = \mathrm{Ch}(\Lambda \otimes d'S^1) \to \mathrm{Ch}(\Lambda \otimes (S^1 \vee S^1))\,.$$

The target is the chain complex associated to the simplicial tensor product of two copies of $\Lambda \otimes S^1$, considered as a simplicial Λ-module. It is therefore also the target of another shuffle equivalence $sh\colon C_*(\Lambda) \otimes_\Lambda C_*(\Lambda) \to \mathrm{Ch}(\Lambda \otimes (S^1 \vee S^1))$.

Proposition 2.3. *Let Λ be a commutative k-algebra. The composite map*

$$dC_*(\Lambda) \underset{\simeq}{\xrightarrow{1\otimes\pi}} C_*(\Lambda) \xrightarrow{\psi} C_*(\Lambda) \otimes_\Lambda C_*(\Lambda) \underset{\simeq}{\xrightarrow{sh}} \mathrm{Ch}(\Lambda \otimes (S^1 \vee S^1))$$

of the chain level coproduct ψ in $C_*(\Lambda)$ (see formula (2.1)), with the chain equivalence $1 \otimes \pi$ induced by the simplicial collapse map $\pi \colon d'S^1 \to S^1$ and the shuffle equivalence sh, is chain homotopic to the map

$$dC_*(\Lambda) = \mathrm{Ch}(\Lambda \otimes d'S^1) \xrightarrow{\psi'} \mathrm{Ch}(\Lambda \otimes (S^1 \vee S^1))$$

induced by the simplicial pinch map $\psi \colon d'S^1 \to S^1 \vee S^1$.

Hence, when $HH_*(\Lambda)$ is flat over Λ, the coproduct ψ on Hochschild homology agrees, via the identifications induced by $1 \otimes \pi$ and sh, with the map ψ' induced by the simplicial pinch map.

Proof. Consider the following diagram.

$$\begin{tikzcd}
C_*(\Lambda) \arrow[r, "1\otimes\psi"] & \Lambda \otimes_{\Lambda-\Lambda} (B_*(\Lambda) \otimes_\Lambda B_*(\Lambda)) \arrow[r, "\psi'"] \arrow[d, "1\otimes sh"] & C_*(\Lambda) \otimes_\Lambda C_*(\Lambda) \arrow[d, "sh"] \\
 & dC_*(\Lambda) \arrow[ul, "1\otimes\pi"] \arrow[r, "\psi'"] & \mathrm{Ch}(\Lambda \otimes (S^1 \vee S^1))
\end{tikzcd}$$

The composite along the upper row is the coproduct ψ, the composite around the triangle is chain homotopic to the identity, by lemma 2.2, and the square commutes by naturality of the shuffle map with respect to the pinch map ψ'. A diagram chase provides the claimed chain homotopy. □

We shall make use of the following standard calculations of Hochschild homology. The formulas for the coproduct ψ follow directly from the chain level formula (2.1) above. Let $P(x) = k[x]$ and $E(x) = k[x]/(x^2)$ be the polynomial and exterior algebras over k in one variable x, and let $\Gamma(x) = k\{\gamma_i(x) \mid i \geq 0\}$ be the *divided power algebra* with multiplication

$$\gamma_i(x) \cdot \gamma_j(x) = (i,j)\ \gamma_{i+j}(x)\,,$$

where $(i,j) = (i+j)!/i!j!$ is the binomial coefficient.

Proposition 2.4. *For $x \in \Lambda$ let $\sigma x \in HH_1(\Lambda)$ be the homology class of the cycle $1 \otimes x \in C_1(\Lambda)$ in the Hochschild complex. For $\Lambda = P(x)$ there is a $P(x)$-algebra isomorphism*

$$HH_*(P(x)) = P(x) \otimes E(\sigma x)\,.$$

The class σx is coalgebra primitive, i.e., $\psi(\sigma x) = \sigma x \otimes 1 + 1 \otimes \sigma x$. For $\Lambda = E(x)$ there is an $E(x)$-algebra isomorphism

$$HH_*(E(x)) = E(x) \otimes \Gamma(\sigma x)\,.$$

The ith divided power $\gamma_i(\sigma x)$ is the homology class of the cycle $1 \otimes x \otimes \cdots \otimes x \in C_i(\Lambda)$. The coproduct is given by

$$\psi(\gamma_k(\sigma x)) = \sum_{i+j=k} \gamma_i(\sigma x) \otimes \gamma_j(\sigma x)\,.$$

There is a Künneth formula

$$HH_*(\Lambda_1 \otimes \Lambda_2) \cong HH_*(\Lambda_1) \otimes HH_*(\Lambda_2)\,.$$

Let $P_h(x) = k[x]/(x^h)$ be the truncated polynomial algebra of height h. We write $P(x_i \mid i \geq 0) = P(x_0, x_1, \dots) = P(x_0) \otimes P(x_1) \otimes \dots$, and so on. When k is of prime characteristic p it is a standard calculation with binomial coefficients that

$$\Gamma(x) = P_p(\gamma_{p^i}(x) \mid i \geq 0) \tag{2.5}$$

as a k-algebra.

We have already noted that the Hochschild complex $C_*(\Lambda)$ is the chain complex associated to a simplicial graded k-module $[q] \mapsto C_q(\Lambda) = \Lambda^{\otimes(q+1)}$, with face maps d_i corresponding to the individual terms in the alternating sum defining the Hochschild boundary ∂. In fact, this is a *cyclic* graded k-module, in the sense of Connes, with cyclic structure maps t_q that cyclically permute the $(q+1)$ tensor factors in $C_q(\Lambda)$, up to sign. It follows that the geometric realization $HH(\Lambda) = |[q] \mapsto C_q(\Lambda)|$ admits a natural S^1-action $\alpha\colon HH(\Lambda) \wedge S^1_+ \to HH(\Lambda)$, and that the Hochschild homology groups are the homotopy groups of this space: $HH_*(\Lambda) = \pi_* HH(\Lambda)$.

The basic idea in the definition of topological Hochschild homology is to replace the ground ring k by the sphere spectrum S, and the symmetric monoidal category of graded k-modules under the tensor product $\otimes = \otimes_k$ by the symmetric monoidal category of spectra, interpreted as S-modules, under the smash product $\wedge = \wedge_S$. A monoid in the first category is a graded k-algebra Λ, which is then replaced by a monoid in the second category, i.e., an S-algebra R. To make sense of this we will work in the framework of [EKMM97], but we could also use [HSS00] or any other reasonable setting that gives a symmetric monoidal category of spectra.

The original definition of topological Hochschild homology was given by Bökstedt in the mid 1980's [Bö1], inspired by work and conjectures of Goodwillie and Waldhausen. The following definition is not the one originally used by Bökstedt, since he did not have the symmetric monoidal smash product from [EKMM97] or [HSS00] available, but it agrees with the heuristic definition that his more complicated definition managed to make sense of, with the more elementary technology that he had at hand.

Definition 2.6. Let R be an S-algebra, with multiplication $\mu\colon R \wedge R \to R$ and unit $\eta\colon S \to R$. The topological Hochschild homology of R is (the geometric realization of) a simplicial S-module $THH(R)$ with

$$THH_q(R) = R^{\wedge(q+1)}$$

in simplicial degree q. The simplicial structure is like that on the simplicial k-module underlying the Hochschild complex. More precisely, the i-th face map $d_i\colon R^{\wedge(q+1)} \to R^{\wedge q}$ equals $id_R^i \wedge \mu \wedge id_R^{q-i-1}$ for $0 \leq i < q$, while $d_q = (\mu \wedge id_R^{q-1})t_q$, where t_q cyclically permutes the $(q+1)$ smash factors R by moving the last factor to the front. The j-th degeneracy map $s_j\colon R^{\wedge(q+1)} \to R^{\wedge(q+2)}$ equals $id_R^{j+1} \wedge \eta \wedge id_R^{q-j}$.

Furthermore, the cyclic operators $t_q\colon THH_q(R) \to THH_q(R)$ make $THH(R)$ a cyclic S-module, so that its geometric realization has a natural S^1-action

$$\alpha\colon THH(R) \wedge S^1_+ \to THH(R)\,.$$

The topological Hochschild homology groups of R are defined to be the homotopy groups $\pi_* THH(R)$.

We note that when R is a commutative S-algebra there is a product that makes $THH(R)$ a commutative R-algebra, with unit corresponding to the inclusion of 0-simplices $R \to THH(R)$. When R is cofibrant as an S-module, we can also describe the homotopy type of $THH(R)$ as the smash product $R \wedge_{R \wedge R^{op}} R$. See [EKMM97, IX] for further discussion on these definitions of $THH(R)$.

3. Hopf algebra structure on THH

Already Bökstedt noted that the simplicial structure on $THH(R)$, as defined above, is derived from the simplicial structure on the standard simplicial circle $S^1 = \Delta^1/\partial\Delta^1$. This can be made most precise in the case when R is a commutative S-algebra, in which case there is a formula $THH(R) \cong R \otimes S^1$ in terms of the simplicial tensor structure on the category of commutative S-algebras. A corresponding formula $THH(R) \simeq R \otimes |S^1|$ in terms of the topological tensor structure was discussed by McClure, Schwänzl and Vogt in [MSV97]. We shall keep to the simplicial context, since we will make use of the resulting skeletal filtrations to form spectral sequences.

We now make this "tensored structure" explicit. Let R be a commutative S-algebra and X a finite set, and let

$$R \otimes X = \bigwedge_{x \in X} R$$

be the smash product of one copy of R for each element of X. It is again a commutative S-algebra. Now let $f\colon X \to Y$ be a function between finite sets, and let $R \otimes f\colon R \otimes X \to R \otimes Y$ be the smash product over all $y \in Y$ of the maps

$$R \otimes f^{-1}(y) = \bigwedge_{x \in f^{-1}(y)} R \to R = R \otimes \{y\}$$

that are given by the iterated multiplication from the $\#f^{-1}(y)$ copies of R on the left to the single copy of R on the right. If $f^{-1}(y)$ is empty, this is by definition the unit map $\eta\colon S \to R$. Since R is commutative, there is no ambiguity in how these iterated multiplications are to be formed.

Note that the construction $R \otimes X$ is functorial in both R and X. Given an injection $X \to Y$ and any function $X \to Z$, there is a natural isomorphism

$$R \otimes (Y \cup_X Z) \cong (R \otimes Y) \wedge_{(R \otimes X)} (R \otimes Z)\,.$$

There is also a natural map

$$R \wedge X_+ = \bigvee_{x \in X} R \to \bigwedge_{x \in X} R = R \otimes X$$

whose restriction to the wedge summand indexed by $x \in X$ is $R \otimes i_x$, where $i_x \colon \{x\} \to X$ is the inclusion.

By naturality, these constructions all extend degreewise to simplicial finite sets $X \colon [q] \mapsto X_q$, simplicial maps $f \colon X \to Y$, etc. In particular, we can define the simplicial commutative S-algebra

$$R \otimes X = \Big([q] \mapsto R \otimes X_q\Big)$$

with structure maps $R \otimes d_i$, $R \otimes s_j$, etc. There is then a useful natural map

$$\omega \colon R \wedge X_+ \to R \otimes X\,. \tag{3.1}$$

Consider the special case $X = S^1 = \Delta^1/\partial\Delta^1$. Here Δ^1_q has $(q+2)$ elements $\{x_0, \dots, x_{q+1}\}$ where $x_t \colon [q] \to [1]$ has $\# x_t^{-1}(0) = t$. The quotient S^1_q has $(q+1)$ elements, obtained by identifying $x_0 \sim x_{q+1}$. Then $d_i(x_t) = x_t$ for $t \le i$ and $d_i(x_t) = x_{t-1}$ for $t > i$, while $s_j(x_t) = x_t$ for $t \le j$ and $s_j(x_t) = x_{t+1}$ for $t > j$. So a direct check shows that there is a natural isomorphism

$$THH(R) \cong R \otimes S^1 \tag{3.2}$$

of (simplicial) commutative S-algebras. In degree q it is the obvious identification $R^{\wedge(q+1)} \cong R \otimes S^1_q$.

There are natural maps $\eta \colon * \to S^1$, $\epsilon \colon S^1 \to *$ and $\phi \colon S^1 \vee S^1 \to S^1$ that map to the base point of S^1, retract to $*$, and fold two copies of S^1 to one, respectively. By naturality, these induce the following maps of commutative S-algebras:

$$\begin{aligned} &\eta \colon R \to THH(R) \\ &\epsilon \colon THH(R) \to R \\ &\phi \colon THH(R) \wedge_R THH(R) \to THH(R)\,. \end{aligned} \tag{3.3}$$

In the last case, the product map involves the identification $THH(R) \wedge_R THH(R) \cong R \otimes (S^1 \cup_* S^1)$, where $S^1 \cup_* S^1 = S^1 \vee S^1$. Taken together, these maps naturally make $THH(R)$ an augmented commutative R-algebra.

There is also a natural map

$$\omega \colon R \wedge S^1_+ \to THH(R) \tag{3.4}$$

derived from (3.1), which captures part of the circle action upon $THH(R)$. More precisely, the map ω admits the following factorization:

$$\omega = \alpha \circ (\eta \wedge id) \colon R \wedge S^1_+ \to THH(R) \wedge S^1_+ \to THH(R)\,. \tag{3.5}$$

This is clear by inspection of the definition of the circle action α on the 0-simplices of $THH(R)$.

We would like to have a coproduct on $THH(R)$, coming from a pinch map $S^1 \to S^1 \vee S^1$, but there is no such simplicial map with our basic model for S^1. To fix this we again consider a "double model" for S^1, denoted dS^1, with

$$dS^1 = (\Delta^1 \sqcup \Delta^1) \cup_{(\partial\Delta^1 \sqcup \partial\Delta^1)} \partial\Delta^1\,.$$

Here $\partial\Delta^1 \sqcup \partial\Delta^1 \to \partial\Delta^1$ is the identity map on each summand, so the two non-degenerate 1-simplices of dS^1 have opposing orientations in the geometrically realized circle. It is the quotient of the barycentric subdivision of Δ^1 by its boundary.

Then we have a simplicial pinch map $\psi\colon dS^1 \to S^1 \vee S^1$ that collapses $\partial\Delta^1 \subset dS^1$ to $*$, as well as a simplicial flip map $\chi\colon dS^1 \to dS^1$ that interchanges the two copies of Δ^1.

Remark 3.6. The simplicial set dS^1 introduced here differs from the double circle $d'S^1$ considered in section 2, in that the orientation of the second 1-simplex has been reversed. The switch is necessary here to make the flip map χ simplicial. In principle, we could have used the same dS^1 in section 2 as here, but this would have entailed the cost of discussing the anti-simplicial involution $\lambda_0[\lambda_1|\dots|\lambda_q]\lambda_{q+1} \mapsto \pm\lambda_{q+1}[\lambda_q|\dots|\lambda_1]\lambda_0$ of $B_*(\Lambda)$, and complicating the formula (2.1) for the chain level coproduct ψ. We choose instead to suppress this point.

We define a corresponding "double model" for $THH(R)$, denoted $dTHH(R)$, by

$$dTHH(R) = R \otimes dS^1 \,.$$

The pinch and flip maps now induce the following natural maps of commutative S-algebras:

$$\begin{aligned} \psi'&\colon dTHH(R) \to THH(R) \wedge_R THH(R) \\ \chi'&\colon dTHH(R) \to dTHH(R)\,. \end{aligned} \tag{3.7}$$

Lemma 3.8. *Let R be cofibrant as an S-module. Then the collapse map $\pi\colon dS^1 \to S^1$ that takes the second Δ^1 to $*$ induces a weak equivalence*

$$\pi\colon dTHH(R) \xrightarrow{\simeq} THH(R)\,.$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
B(R) \arrow[d, equal] & R \wedge R \arrow[l] \arrow[r] \arrow[d, equal] & B(R) \arrow[d] \\
B(R) & R \wedge R \arrow[l] \arrow[r] & R
\end{tikzcd}$$

of commutative S-algebras. Here $B(R) = B(R, R, R) = R \otimes \Delta^1$ is the two-sided bar construction, its augmentation $B(R) \to R$ is a weak equivalence, and the inclusion $R \wedge R \to B(R)$ is a cofibration of S-modules. From [EKMM97, III.3.8] we know that the categorical pushout (balanced smash product) in this case preserves weak equivalences. Pushout along the upper row gives $dTHH(R)$ and pushout along the lower row gives $THH(R)$, so the induced map $\pi\colon dTHH(R) \to THH(R)$ is indeed a weak equivalence. $\square$

Theorem 3.9. *Let R be a commutative S-algebra. Its topological Hochschild homology $THH(R)$ is naturally an augmented commutative R-algebra, with unit, counit and product maps η, ϵ and ϕ defined as in (3.3) above. In the stable homotopy category, these maps, the coproduct map*

$$\psi = \psi' \circ \pi^{-1} \colon THH(R) \to THH(R) \wedge_R THH(R)$$

and the conjugation map

$$\chi = \pi \circ \chi' \circ \pi^{-1} \colon THH(R) \to THH(R)$$

naturally make $THH(R)$ an R-Hopf algebra.

Proof. To check that $THH(R)$ is indeed a Hopf algebra over R in the stable homotopy category, we must verify that a number of diagrams commute. We will do one case that illustrates the technique, and leave the rest to the reader.

Let $T = THH(R)$. In order to show that the diagram

$$\begin{tikzcd}
T \arrow[r, "\psi"] \arrow[d, "\psi"'] \arrow[drr, "\epsilon"] & T \wedge_R T \arrow[drr, "id\wedge\chi"] & & & \\
T \wedge_R T \arrow[drr, "\chi\wedge id"'] & & R \arrow[drr, "\eta"] & & T \wedge_R T \arrow[d, "\phi"] \\
& & T \wedge_R T \arrow[rr, "\phi"'] & & T
\end{tikzcd} \tag{3.10}$$

commutes in the stable homotopy category it suffices to check that the diagram of simplicial sets

$$\begin{tikzcd}
tS^1 \arrow[r, "\psi_1"] \arrow[d, "\psi_2"'] \arrow[drr, "\epsilon"] & S^1 \vee dS^1 \arrow[drr, "id\vee\chi'"] & & & \\
dS^1 \vee S^1 \arrow[drr, "\chi'\vee id"'] & & * \arrow[drr, "\eta"] & & S^1 \vee dS^1 \arrow[d, "\phi(id\vee\pi)"] \\
& & dS^1 \vee S^1 \arrow[rr, "\phi(\pi\vee id)"'] & & S^1
\end{tikzcd}$$

homotopy commutes (simplicially).

Here $tS^1 = \partial\Delta^2$ is a "triple model" for S^1, with three non-degenerate 1-simplices. The pinch map ψ_1 identifies the vertices 0 and 1 in $\partial\Delta^2$ and takes the face δ_0 to the first Δ^1 in dS^1. Then the composite $\phi(id \vee \pi)(id \vee \chi')\psi_1$ factors as

$$\partial\Delta^2 \subset \Delta^2 \xrightarrow{s_1} \Delta^1 \to S^1 \,,$$

and Δ^1 is simplicially contractible. Similarly, ψ_2 identifies the vertices 1 and 2 and takes δ_2 to the first Δ^1 in dS^1. Then $\phi(\pi \vee id)(\chi' \vee id)\psi_2$ factors as

$$\partial\Delta^2 \subset \Delta^2 \xrightarrow{s_0} \Delta^1 \to S^1 \,,$$

and again this map is simplicially contractible.

Finally we use a weak equivalence $tTHH(R) = R \otimes tS^1 \to THH(R)$, as in lemma 3.8, to deduce that the diagram (3.10) indeed homotopy commutes. □

Remark 3.11. As noted above, $THH(R)$ is commutative as an R-algebra. The pinch map $\psi\colon dS^1 \to S^1 \vee S^1$ is not homotopy cocommutative, so we do not expect that the coproduct $\psi\colon THH(R) \to THH(R) \wedge_R THH(R)$ will be cocommutative in any great generality.

The inclusion of the base point $\eta\colon * \to S^1$ induces a cofiber sequence of R-modules

$$R = R \wedge *_+ \xrightarrow{1\wedge\eta_+} R \wedge S^1_+ \xrightarrow{j} R \wedge S^1 = \Sigma R$$

which is canonically split by the retraction map

$$R \wedge S^1_+ \xrightarrow{1\wedge\epsilon_+} R \wedge *_+ = R\,.$$

Hence in the stable homotopy category there is a canonical section $\kappa\colon \Sigma R \to R \wedge S^1_+$ to the map labeled j above. We let

$$\sigma = \omega \circ \kappa\colon \Sigma R \to R \wedge S^1_+ \to THH(R) \tag{3.12}$$

be the composite map. It induces an operator $\sigma\colon H_*(\Sigma R; \mathbb{F}_p) \to H_*(THH(R); \mathbb{F}_p)$, which we in proposition 4.8 shall see is compatible with that of proposition 2.4.

4. The Bökstedt spectral sequence

Let R be an S-algebra. To calculate the mod p homology $H_*(THH(R); \mathbb{F}_p)$ of its topological Hochschild homology, Bökstedt constructed a strongly convergent spectral sequence

$$E^2_{s,*}(R) = HH_s(H_*(R; \mathbb{F}_p)) \Longrightarrow H_{s+*}(THH(R); \mathbb{F}_p)\,, \tag{4.1}$$

using the skeleton filtration on $THH(R)$. In fact,

$$E^1_{s,*}(R) = H_*(R^{\wedge(s+1)}; \mathbb{F}_p) \cong H_*(R; \mathbb{F}_p)^{\otimes(s+1)} = C_s(H_*(R; \mathbb{F}_p))$$

equals the Hochschild s-chains of the algebra $\Lambda = H_*(R; \mathbb{F}_p)$ over $k = \mathbb{F}_p$, and the d^1-differential can as usual be identified with the Hochschild boundary operator ∂. (To be quite precise, the E^1-term is really the associated normalized complex $\Lambda \otimes \bar{\Lambda}^{\otimes s}$, with $\bar{\Lambda} = \Lambda/k$, but this change does not affect the E^2-term.)

This is naturally a spectral sequence of A_*-comodules, where $A_* = H_*(H\mathbb{F}_p; \mathbb{F}_p)$ is the dual of the mod p Steenrod algebra, since it is obtained by applying mod p homology to a filtered spectrum. If R is a commutative S-algebra, the spectral sequence admits more structure.

Proposition 4.2. *Let R be a commutative S-algebra. Then:*

(a) $H_*(THH(R);\mathbb{F}_p)$ is an augmented commutative A_*-comodule $H_*(R;\mathbb{F}_p)$-algebra.

(b) The Bökstedt spectral sequence $E^r_{**}(R)$ is an augmented commutative A_*-comodule $H_*(R;\mathbb{F}_p)$-algebra spectral sequence, converging to $H_*(THH(R);\mathbb{F}_p)$.

Proof. Assume that R is a commutative S-algebra. Then $THH(R)$ is an augmented commutative R-algebra by theorem 3.9, and the relevant structure maps (3.3) are all maps of simplicial spectra. Hence they respect the skeleton filtration on $THH(R)$, and we have in particular a composite map of spectral sequences

$$E^r_{**}(R) \otimes_\Lambda E^r_{**}(R) \to {}'E^r_{**} \xrightarrow{\phi} E^r_{**}(R)$$

with ${}'E^r_{**}$ the spectral sequence associated to the skeleton filtration on $THH(R) \wedge_R THH(R)$. The left hand map is induced by the usual homology cross product from the E^1-term and onwards. This defines the algebra structure on $E^r_{**}(R)$, and the remainder is straightforward. □

Corollary 4.3. *If $H_*(R;\mathbb{F}_p)$ is a polynomial algebra over $\mathbb{F}_p$, then $E^r_{**}(R)$ collapses at the E^2-term, so $E^2_{**}(R) = E^\infty_{**}(R)$. Furthermore, there are no nontrivial $H_*(R;\mathbb{F}_p)$-module extensions.*

Proof. If $H_*(R;\mathbb{F}_p) = P(x_i)$ is a polynomial algebra, where the index i ranges through some set I, then $E^2_{**}(R) = HH_*(P(x_i)) = P(x_i) \otimes E(\sigma x_i)$ by proposition 2.4 (and passage to colimits). All the $H_*(R;\mathbb{F}_p)$-algebra generators are in filtration $n = 1$, so all differentials on these classes are zero, since the Bökstedt spectral sequence is a right half plane homological spectral sequence. Hence the E^∞-term is a free $H_*(R;\mathbb{F}_p)$-module, and so also $H_*(THH(R);\mathbb{F}_p) \cong H_*(R;\mathbb{F}_p) \otimes E(\sigma x_i)$ is a free $H_*(R;\mathbb{F}_p)$-module. □

There may, as we shall see in section 5, be multiplicative extensions between $E^\infty_{**}(R)$ and $H_*(THH(R);\mathbb{F}_p)$, as well as A_*-comodule extensions.

A flatness hypothesis is required for the spectral sequence to carry the coproduct and full Hopf algebra structure. Our sections 5, 6 and 7 will show many examples of Bökstedt spectral sequences with this structure.

Theorem 4.4. *Let R be a commutative S-algebra and write $\Lambda = H_*(R;\mathbb{F}_p)$.*

(a) If $H_*(THH(R);\mathbb{F}_p)$ is flat over Λ, then there is a coproduct

$$\psi\colon H_*(THH(R);\mathbb{F}_p) \to H_*(THH(R);\mathbb{F}_p) \otimes_\Lambda H_*(THH(R);\mathbb{F}_p)$$

and $H_*(THH(R);\mathbb{F}_p)$ is an A_*-comodule Λ-Hopf algebra.

(b) If each term $E^r_{**}(R)$ for $r \geq 2$ is flat over Λ, then there is a coproduct

$$\psi\colon E^r_{**}(R) \to E^r_{**}(R) \otimes_\Lambda E^r_{**}(R)$$

and $E^r_{**}(R)$ is an A_*-comodule Λ-Hopf algebra spectral sequence. In particular, the differentials d^r respect the coproduct ψ.

Proof. Write $T = THH(R)$. There is a Künneth spectral sequence with

$$E^2_{**} = \mathrm{Tor}^{\Lambda}_{**}(H_*(T;\mathbb{F}_p), H_*(T;\mathbb{F}_p)) \Longrightarrow H_*(T \wedge_R T;\mathbb{F}_p)\,.$$

When $H_*(T;\mathbb{F}_p)$ is flat over Λ the higher Tor-groups vanish, the spectral sequence collapses, and the map

$$\psi\colon H_*(T;\mathbb{F}_p) \to H_*(T \wedge_R T;\mathbb{F}_p) \cong H_*(T;\mathbb{F}_p) \otimes_\Lambda H_*(T;\mathbb{F}_p)$$

induces the coproduct in part (a).

For part (b) let dE^r_{**} be the spectral sequence associated to the skeleton filtration on $dTHH(R) = R \otimes dS^1$, and let $'E^r_{**}$ be the spectral sequence associated to $T \wedge_R T = R \otimes (S^1 \vee S^1)$, as in the proof of proposition 4.2. Then there are natural maps of spectral sequences

$$E^r_{**}(R) \xleftarrow{\pi} dE^r_{**} \xrightarrow{\psi'} {}'E^r_{**} \xleftarrow{sh} E^r_{**}(R) \otimes_\Lambda E^r_{**}(R)\,. \tag{4.5}$$

Here $\pi\colon dE^r_{**} \to E^r_{**}(R)$ is an isomorphism for $r \geq 2$, by the algebraic analogue of lemma 3.8. The map $\psi'\colon dE^r_{**} \to {}'E^r_{**}$ is induced by the simplicial pinch map ψ' from (3.7). As regards the final (shuffle) map sh, we have

$${}'E^1_{s,*} \cong \Lambda^{\otimes(s+1)} \otimes_\Lambda \Lambda^{\otimes(s+1)} \cong E^1_{s,*}(R) \otimes_\Lambda E^1_{s,*}(R)$$

by the collapsing Künneth spectral sequence for $H_*(THH_s(R) \wedge_R THH_s(R);\mathbb{F}_p)$, and the map sh for $r = 1$ is the shuffle equivalence from the bigraded tensor product $[E^1_{**}(R) \otimes_\Lambda E^1_{**}(R)]_{s,*}$. By assumption $E^2_{**}(R) = HH_*(\Lambda)$ is flat over Λ, so by the algebraic Künneth spectral sequence and the Eilenberg–Zilber theorem

$$sh\colon E^2_{**}(R) \otimes_\Lambda E^2_{**}(R) \cong H_*(E^1_{**}(R) \otimes_\Lambda E^1_{**}(R)) \xrightarrow{\cong} H_*({}'E^1_{**}) = {}'E^2_{**}$$

is an isomorphism. Inductively, suppose that sh is an isomorphism for a fixed $r \geq 2$, and that $E^{r+1}_{**}(R)$ is flat over Λ. Then by the algebraic Künneth spectral sequence again

$$sh\colon E^{r+1}_{**}(R) \otimes_\Lambda E^{r+1}_{**}(R) \cong H_*(E^r_{**}(R) \otimes_\Lambda E^r_{**}(R)) \xrightarrow{\cong} H_*({}'E^r_{**}) = {}'E^{r+1}_{**}$$

is also an isomorphism, as desired. The coproduct ψ on $E^r_{**}(R)$ is then the composite map $(sh)^{-1} \circ \psi' \circ \pi^{-1}$, for $r \geq 2$. The conjugation χ on $E^r_{**}(R)$ is more simply defined, as the composite map $\pi \circ \chi' \circ \pi^{-1}$. □

Remark 4.6. The same proof shows that if only an initial sequence of terms

$$E^2_{**}(R), \dots, E^{r_0}_{**}(R)$$

of the Bökstedt spectral sequence are flat over Λ, then these are all A_*-comodule Λ-Hopf algebras and the differentials $d^2, \dots, d^{r_0}$ respect that structure.

By proposition 2.3, the coproduct (and Hopf algebra structure) on the E^2-term $E^2_{**}(R) = HH_*(H_*(R;\mathbb{F}_p))$ that is derived from the R-Hopf algebra structure on

$THH(R)$ agrees with the algebraically defined structure on the Hochschild homology $HH_*(\Lambda)$ of the commutative algebra $\Lambda = H_*(R; \mathbb{F}_p)$.

There are natural examples that show that the flatness hypothesis is not always realistic. For example, Ausoni [Au] has studied the case of $R = ku$ at an odd prime p, where $H_*(ku; \mathbb{F}_p) = H_*(\ell; \mathbb{F}_p) \otimes P_{p-1}(x)$ for a class x in degree 2, and already the E^2-term $E^2_{**}(ku) = HH_*(H_*(ku; \mathbb{F}_p))$ is not flat over $H_*(ku; \mathbb{F}_p)$.

We shall also see in proposition 7.13(c) that for $R = j$, the connective, real image of J-spectrum at $p = 2$, the E^2-term $E^2_{**}(j) = HH_*(H_*(j; \mathbb{F}_2))$ is not flat over $H_*(j; \mathbb{F}_2)$.

We say that the graded k-algebra Λ is *connected* when it is trivial in negative degrees and the unit map $\eta\colon k \to \Lambda$ is an isomorphism in degree 0.

Proposition 4.7. *Let R be a commutative S-algebra with $\Lambda = H_*(R; \mathbb{F}_p)$ connected and such that $HH_*(\Lambda)$ is flat over Λ. Then the E^2-term of the first quadrant Bökstedt spectral sequence*

$$E^2_{**}(R) = HH_*(\Lambda)$$

is an A_*-comodule Λ-Hopf algebra, and a shortest non-zero differential $d^r_{s,t}$ in lowest total degree $s + t$, if there exists any, must map from an algebra indecomposable to a coalgebra primitive and A_*-comodule primitive, in $HH_*(\Lambda)$.

Proof. If $d^2, \dots, d^{r-1}$ are all zero, then $E^2_{**}(R) = E^r_{**}(R)$ is still an A_*-comodule Λ-Hopf algebra. If $d^r(xy) \neq 0$, where xy is decomposable (a product), then the Leibniz formula

$$d^r(xy) = d^r(x)y \pm x d^r(y)$$

implies that $d^r(x) \neq 0$ or $d^r(y) \neq 0$, so xy cannot be in the lowest possible total degree for the source of a differential. Dually, if $d^r(z)$ is not (coalgebra) primitive, with $\psi(z) = z \otimes 1 + 1 \otimes z + \sum_i z'_i \otimes z''_i$, then the co-Leibniz formula

$$\psi \circ d^r = (d^r \otimes 1 \pm 1 \otimes d^r)\psi$$

(tensor products over Λ) implies that some term $d^r(z'_i) \neq 0$ or $d^r(z''_i) \neq 0$, so z cannot be in the lowest possible total degree. Finally, if $d^r(z)$ is not A_*-comodule primitive, with $\nu(z) = 1 \otimes z + \sum_i a_i \otimes z_i$, then the co-linearity condition

$$\nu \circ d^r = (1 \otimes d^r)\nu$$

implies that some term $d^r(z_i) \neq 0$, so z cannot be in the lowest possible total degree. (The last two arguments are perhaps easier to visualize in the $\mathbb{F}_p$-vector space dual spectral sequence.) □

Proposition 4.8. *Let R be a commutative S-algebra. For each element $x \in H_t(R; \mathbb{F}_p)$ the image $\sigma x \in H_{t+1}(THH(R); \mathbb{F}_p)$ is a coalgebra primitive*

$$\psi(\sigma x) = \sigma x \otimes 1 + 1 \otimes \sigma x$$

that is represented in $E^r_{**}(R)$ by the class $\sigma x = [1 \otimes x] \in E^2_{1,t}(R) = HH_1(\Lambda)_t$.

Proof. Note that the coproduct on $THH(R)$ is compatible under $\omega\colon R \wedge S^1_+ \to THH(R)$ with the pinch map $R \wedge dS^1_+ \to R \wedge (S^1 \vee S^1)_+$, and thus under $\sigma\colon \Sigma R \to THH(R)$ with the pinch map $\Sigma R \to \Sigma R \vee \Sigma R$. The claims then follow by inspection of the definitions in section 3. □

5. Differentials and algebra extensions

We now apply the Bökstedt spectral sequence (4.1) to compute the mod p homology of $THH(R)$ for the S-algebras $R = BP\langle n\rangle$ for $-1 \le n \le \infty$, in the cases when these are commutative. In each case we can replace R by its p-localization $R_{(p)}$ or p-completion R_p without changing the mod p homology of $THH(R)$, so we will sometimes do so without further comment. The cases $R = H\mathbb{F}_p$ and $R = H\mathbb{Z}$ were first treated by Bökstedt [Bö2], and the case $R = \ell_p \subset ku_p$ (the Adams summand of p-complete connective K-theory) is due to McClure and Staffeldt [MS93, 4.2].

5.1. The dual Steenrod algebra. Let $A = H^*(H\mathbb{F}_p;\mathbb{F}_p)$ be the Steenrod algebra, with generators Sq^i for $p = 2$ and β and P^i for p odd. We recall the structure of its dual $A_* = H_*(H\mathbb{F}_p;\mathbb{F}_p)$ from [Mi60, Thm. 2]. When $p = 2$ we have

$$A_* = P(\xi_k \mid k \ge 1) = P(\bar\xi_k \mid k \ge 1)$$

where ξ_k has degree $2^k - 1$ and $\bar\xi_k = \chi(\xi_k)$ is the conjugate class. Most of the time it will be more convenient for us to use the conjugate classes. The coproduct is given by

$$\psi(\bar\xi_k) = \sum_{i+j=k} \bar\xi_i \otimes \bar\xi_j^{2^i} ,$$

where as usual we read $\bar\xi_0$ to mean 1. When p is odd we have

$$A_* = P(\bar\xi_k \mid k \ge 1) \otimes E(\bar\tau_k \mid k \ge 0)$$

with $\bar\xi_k = \chi(\xi_k)$ in degree $2(p^k - 1)$ and $\bar\tau_k = \chi(\tau_k)$ in degree $2p^k - 1$. The coproduct is given by

$$\psi(\bar\xi_k) = \sum_{i+j=k} \bar\xi_i \otimes \bar\xi_j^{p^i} \qquad \text{and} \qquad \psi(\bar\tau_k) = 1 \otimes \bar\tau_k + \sum_{i+j=k} \bar\tau_i \otimes \bar\xi_j^{p^i} .$$

The mod p homology Bockstein satisfies $\beta(\bar\tau_k) = \bar\xi_k$.

Any commutative S-algebra R has a canonical structure as an E_∞ ring spectrum [EKMM97, II.3.4]. In particular, its mod p homology $H_*(R;\mathbb{F}_p)$ admits natural Dyer–Lashof operations

$$Q^k \colon H_*(R;\mathbb{F}_2) \to H_{*+k}(R;\mathbb{F}_2)$$

for $p = 2$ and

$$Q^k \colon H_*(R;\mathbb{F}_p) \to H_{*+2k(p-1)}(R;\mathbb{F}_p)$$

for p odd. Their formal properties are summarized in [BMMS86, III.1.1], and include Cartan formulas, Adem relations and Nishida relations. For $p = 2$, $Q^k(x) = 0$ when $k < |x|$ and $Q^k(x) = x^2$ when $k = |x|$. For p odd, $Q^k(x) = 0$ when $k < 2|x|$ and $Q^k(x) = x^p$ when $k = 2|x|$. In the special case of $R = H\mathbb{F}_p$, the Dyer–Lashof operations in $A_* = H_*(H\mathbb{F}_p;\mathbb{F}_p)$ satisfy

$$Q^{p^k}(\bar\xi_k) = \bar\xi_{k+1}$$

for all primes p, and

$$Q^{p^k}(\bar\tau_k) = \bar\tau_{k+1}$$

for p odd. These formulas were first obtained by Leif Kristensen (unpublished), and appeared in print in [BMMS86, III.2.2 and III.2.3].

5.2. The Johnson–Wilson spectra $BP\langle n\rangle$. For any prime p let $R = BP\langle n\rangle$ for $-1 \le n \le \infty$ be the spectrum introduced by Johnson and Wilson in [JW73], with mod p cohomology $H^*(R; \mathbb{F}_p) \cong A//E_n = A \otimes_{E_n} \mathbb{F}_p$. Here $E_n \subset A$ is the subalgebra generated by the Milnor primitives $Q_0, \dots, Q_n$, which are inductively defined by $Q_0 = Sq^1$ and $Q_k = [Sq^{2^k}, Q_{k-1}]$ for $p = 2$, and by $Q_0 = \beta$ and $Q_{k+1} = [P^{p^k}, Q_k]$ for p odd, see [Mi60, §1].

This spectrum has homotopy groups

$$\pi_* BP\langle n\rangle = \mathbb{Z}_{(p)}[v_1, \dots, v_n] = \pi_* BP/(v_k \mid k > n)\,,$$

where $\pi_* BP = \mathbb{Z}_{(p)}[v_k \mid k \ge 1]$ with $v_0 = p$. The class v_k is detected in the Adams spectral sequence $E_2^{**} = \mathrm{Ext}_{A_*}^{**}(\mathbb{F}_p, H_*(BP; \mathbb{F}_p))$ for $\pi_* BP$ by the normalized cobar cocycle $\sum_{i+j=k+1} [\bar{\xi}_i] \bar{\xi}_j^{2^i}$ for $p = 2$ (the term for $i = 0$ is zero) and $-\sum_{i+j=k} [\bar{\tau}_i] \bar{\xi}_j^{p^i}$ for p odd [Ra04, p. 63]. Under the change-of-rings isomorphism to $E_2^{**} \cong \mathrm{Ext}_{E_*}^{**}(\mathbb{F}_p, \mathbb{F}_p)$, where $E_* = E(\bar{\xi}_k \mid k \ge 1)$ for $p = 2$ and $E(\bar{\tau}_k \mid k \ge 0)$ for p odd, these cobar cocycles correspond to $[\bar{\xi}_{k+1}]$ and $-[\bar{\tau}_k]$, respectively. Modulo decomposables, we have $\bar{\xi}_{k+1} \equiv \xi_{k+1}$ for $p = 2$ and $-\bar{\tau}_k \equiv \tau_k$ for p odd.

The special cases $BP\langle -1\rangle = H\mathbb{F}_p$, $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$, $BP\langle 1\rangle = \ell \subset ku_{(p)}$ (the Adams summand in p-local connective topological K-theory) and $BP\langle \infty\rangle = BP \subset MU_{(p)}$ (the p-local Brown–Peterson spectrum) have been previously studied. For $p = 2$ we emphasize that $\ell = ku_{(2)}$.

In each case $-1 \le n \le \infty$, the spectrum $BP\langle n\rangle$ admits the structure of an S-algebra, see e.g. [BJ02, 3.5]. It is well known that $H\mathbb{F}_p$ and $H\mathbb{Z}_{(p)}$ admit unique structures as commutative S-algebras, and that the p-complete Adams summand $\ell_p \subset ku_p$ admits at least one such structure [MS93, §9]. It remains a well-known open problem whether $BP\langle n\rangle$ is a commutative S-algebra for $2 \le n \le \infty$. For our purposes it will suffice if the p-completion $BP\langle n\rangle_p$ admits such a structure.

Proposition 5.3. *For $-1 \le n \le \infty$ the unique map of S-algebras $R = BP\langle n\rangle \to H\mathbb{F}_p$ identifies $\Lambda = H_*(R; \mathbb{F}_p)$ with the following sub Hopf algebra of A_*:*

$$H_*(BP\langle n\rangle; \mathbb{F}_2) = P(\bar{\xi}_1^2, \dots, \bar{\xi}_{n+1}^2, \bar{\xi}_k \mid k \ge n+2)$$

when $p = 2$, and

$$H_*(BP\langle n\rangle; \mathbb{F}_p) = P(\bar{\xi}_k \mid k \ge 1) \otimes E(\bar{\tau}_k \mid k \ge n+1)$$

when p is odd.

Proof. See [Wi75, 1.7] and dualize. □

In particular, $H_*(H\mathbb{Z}; \mathbb{F}_2) = P(\bar{\xi}_1^2, \bar{\xi}_k \mid k \ge 2)$, $H_*(ku; \mathbb{F}_2) = P(\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_k \mid k \ge 3)$, $H_*(BP; \mathbb{F}_2) = P(\bar{\xi}_k^2 \mid k \ge 1)$ for $p = 2$, and $H_*(H\mathbb{Z}; \mathbb{F}_p) = P(\bar{\xi}_k \mid k \ge 1) \otimes E(\bar{\tau}_k \mid k \ge 1)$, $H_*(\ell; \mathbb{F}_p) = P(\bar{\xi}_k \mid k \ge 1) \otimes E(\bar{\tau}_k \mid k \ge 2)$, $H_*(BP; \mathbb{F}_p) = P(\bar{\xi}_k \mid k \ge 1)$ for p odd.

The E^2-term of the Bökstedt spectral sequence for $BP\langle n\rangle$ can now be computed from proposition 2.4. It is

$$E^2_{**}(BP\langle n\rangle) = H_*(BP\langle n\rangle; \mathbb{F}_2) \otimes E(\sigma\bar{\xi}_1^2, \dots, \sigma\bar{\xi}_{n+1}^2, \sigma\bar{\xi}_k \mid k \ge n+2)$$

when $p = 2$, and

$$E^2_{**}(BP\langle n\rangle) = H_*(BP\langle n\rangle; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_k \mid k \geq 1) \otimes \Gamma(\sigma\bar{\tau}_k \mid k \geq n+1)$$

when p is odd.

In all the cases for $p = 2$, as well as the case $R = BP$ for p odd, the E^2-term is generated as an algebra by classes in filtration 1. Hence if furthermore n is such that $R = BP\langle n\rangle_p$ is a commutative S-algebra (which includes the cases $n \leq 1$) the Bökstedt spectral sequence collapses at the E^2-term by corollary 4.3, so $E^2 = E^\infty$. The added hypothesis that $BP\langle n\rangle$ is commutative is unrealistic for $p = 2$ and $n \geq 2$, by [St99, 6.5], but there is a replacement spectrum $BP\langle n\rangle'$ that may serve as a substitute, see [St99, 2.10].

5.4. Odd-primary differentials. In the remaining cases, with p odd and $n < \infty$, there are non-trivial d^{p-1}-differentials in the Bökstedt spectral sequence for $THH(BP\langle n\rangle)$. These are all determined by naturality with respect to the map of S-algebras $BP\langle n\rangle \to H\mathbb{F}_p$ and the case $R = H\mathbb{F}_p$, since the map of E^2-terms

$$E^2_{**}(BP\langle n\rangle) \to E^2_{**}(H\mathbb{F}_p) = A_* \otimes E(\sigma\bar{\xi}_k \mid k \geq 1) \otimes \Gamma(\sigma\bar{\tau}_k \mid k \geq 0)$$

is injective. Now $H\mathbb{F}_p$ is a connective, commutative S-algebra, and the E^2-term is free, hence flat, over $A_* = H_*(H\mathbb{F}_p; \mathbb{F}_p)$, so we are in the situation of proposition 4.7. The algebra generators (indecomposables) of the E^2-term are in filtrations p^i for $i \geq 0$, by formula (2.5), and the coalgebra primitives are all in filtration 1, by proposition 2.4, so the shortest non-zero differential in lowest total degree, if any, must go from some filtration p^i to filtration 1.

Indeed, Bökstedt [Bö2, 1.3] found that in the case $R = H\mathbb{F}_p$ there are non-trivial d^{p-1}-differentials in his spectral sequence computing $H_*(THH(R); \mathbb{F}_p)$. In our notation, they are given by the formula

$$d^{p-1}(\gamma_j(\sigma\bar{\tau}_k)) = \sigma\bar{\xi}_{k+1} \cdot \gamma_{j-p}(\sigma\bar{\tau}_k) \tag{5.5}$$

for $j \geq p$, up to units in $\mathbb{F}_p$. This way of writing Bökstedt's formula first appears in [MS93, p. 21]. A proof of a more general result that implies (5.5) was published by Hunter [Hu96, Thm. 1], as recalled below. A short, direct proof of Bökstedt's formula, in some particular cases, was later found by Ausoni [Au].

Proposition 5.6. *Let R be a commutative S-algebra, and $x \in H_{2i-1}(R; \mathbb{F}_p)$ with $i \geq 1$. Then in the Bökstedt spectral sequence $E^r_{**}(R)$ the differentials d^r vanish for $2 \leq r \leq p-2$, and there is a differential*

$$d^{p-1}(\gamma_p(\sigma x)) = \sigma(\beta Q^i(x))$$

up to a unit in $\mathbb{F}_p$.

In the case $R = H\mathbb{F}_p$ with $x = \bar{\tau}_k$, we get $i = p^k$ and $\beta Q^i(x) = \bar{\xi}_{k+1}$. This implies Bökstedt's formula (5.5) for $j = p$. The general case follows from this by induction on $j \geq p$ and the coalgebra structure on the Bökstedt spectral sequence. (In more detail, a comparison of $\psi d^{p-1}(\gamma_j(\sigma\bar{\tau}_k)) = (d^{p-1} \otimes 1 + 1 \otimes d^{p-1})(\psi\gamma_j(\sigma\bar{\tau}_k))$ and

$\psi(\sigma\bar{\xi}_{k+1} \cdot \gamma_{j-p}(\sigma\bar{\tau}_k))$ shows that the difference $d^{p-1}(\gamma_j(\sigma\bar{\tau}_k)) - \sigma\bar{\xi}_{k+1} \cdot \gamma_{j-p}(\sigma\bar{\tau}_k)$ must be a coalgebra primitive, and there are none such other than zero in its bidegree when $j > p$.)

By naturality, the formula (5.5) for d^{p-1} holds also in the Bökstedt spectral sequence for $BP\langle n\rangle$ at odd primes p (whether this S-algebra is commutative or not). Consider its $E^2 = E^{p-1}$-term as the tensor product of the complexes $E(\sigma\bar{\xi}_{k+1}) \otimes \Gamma(\sigma\bar{\tau}_k)$ for $k \geq n+1$ and the remaining terms $H_*(BP\langle n\rangle; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \dots, \sigma\bar{\xi}_{n+1})$. Applying the Künneth formula, we compute its homology to be

$$E^p_{**}(BP\langle n\rangle) = H_*(BP\langle n\rangle; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \dots, \sigma\bar{\xi}_{n+1}) \otimes P_p(\sigma\bar{\tau}_k \mid k \geq n+1)\,.$$

At this stage the map $E^p_{**}(BP\langle n\rangle) \to E^p_{**}(H\mathbb{F}_p)$ is no longer injective, so we seem obliged to assume that $BP\langle n\rangle_p$ is commutative. Then the E^p-term above is generated as an algebra over $H_*(BP\langle n\rangle; \mathbb{F}_p)$ by classes in filtration 1, and the Bökstedt spectral sequence therefore collapses at $E^p = E^\infty$.

Proposition 5.7. *Let p be any prime and $-1 \leq n \leq \infty$ be such that $R = BP\langle n\rangle_p$ admits the structure of a commutative S-algebra (e.g., $n \leq 1$). Then the Bökstedt spectral sequence for R collapses at the $E^p = E^\infty$-term, which equals*

$$E^\infty_{**}(BP\langle n\rangle) = H_*(BP\langle n\rangle; \mathbb{F}_2) \otimes E(\sigma\bar{\xi}_1^2, \dots, \sigma\bar{\xi}_{n+1}^2, \sigma\bar{\xi}_k \mid k \geq n+2)$$

when $p = 2$, and

$$E^\infty_{**}(BP\langle n\rangle) = H_*(BP\langle n\rangle; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_k \mid 1 \leq k \leq n+1) \otimes P_p(\sigma\bar{\tau}_k \mid k \geq n+1)$$

when p is odd.

5.8. Algebra extensions. The E^∞-terms above compute $H_*(THH(BP\langle n\rangle; \mathbb{F}_p))$ as a (free) $\Lambda = H_*(BP\langle n\rangle; \mathbb{F}_p)$-module. To determine the rest of the Λ-Hopf algebra structure, we need to resolve the possible algebra extensions. For this we use the Dyer–Lashof operations that stem from the (assumed) commutative S-algebra structure on $THH(BP\langle n\rangle)$.

The map $\sigma\colon \Sigma R \to THH(R)$ relates the Dyer–Lashof operations on R to those of $THH(R)$, by the following formula of Bökstedt [Bö2, 2.9]:

$$Q^k(\sigma x) = \sigma Q^k(x)\,.$$

See also [MS93, p. 22]. There appears to be no published proof of this key relation, so we offer the following, slightly more general result. Let $\alpha\colon THH(R) \wedge S^1_+ \to THH(R)$ be the S^1-action map, inducing the homomorphism

$$\begin{aligned}\alpha\colon H_*(THH(R); \mathbb{F}_p) &\otimes H_*(S^1_+; \mathbb{F}_p) \\ &\cong H_*(THH(R) \wedge S^1_+; \mathbb{F}_p) \to H_*(THH(R); \mathbb{F}_p)\end{aligned}$$

in homology. Let $s_1 \in H_1(S^1_+; \mathbb{F}_p)$ be the canonical generator.

Proposition 5.9. *Let R be a commutative S-algebra. Then we have*

$$Q^k(\alpha(x \otimes s_1)) = \alpha(Q^k(x) \otimes s_1)$$

for all integers k and classes $x \in H_*(THH(R);\mathbb{F}_p)$. In particular, we have

$$Q^k(\sigma x) = \sigma Q^k(x)$$

for all integers k and classes $x \in H_*(R;\mathbb{F}_p)$.

Proof. The circle acts on $THH(R)$ by commutative S-algebra maps, so the right adjoint map

$$\widetilde{\alpha}\colon THH(R) \to F(S^1_+, THH(R))$$

is a map of commutative S-algebras, where the product structure on the right is given by pointwise multiplication, using the diagonal map $S^1_+ \to S^1_+ \wedge S^1_+$.

Let $DS^1_+ = F(S^1_+, S)$ be the functional dual of S^1_+, also with the pointwise multiplication. It has mod p homology $H_*(DS^1_+;\mathbb{F}_p) \cong H^{-*}(S^1_+;\mathbb{F}_p) = E(\iota_1)$ for a canonical class $\iota_1 \in H^1(S^1_+;\mathbb{F}_p)$ dual to the class $s_1 \in H_1(S^1_+;\mathbb{F}_p)$. The Dyer–Lashof operations Q^k on $H_*(DS^1_+;\mathbb{F}_p)$ correspond [BMMS86, III.1.2 and VIII.3] to the Steenrod operations P^{-k} on $H^{-*}(S^1_+;\mathbb{F}_p)$, hence are trivial for $k \neq 0$. (And similarly, with different notation, when $p = 2$.)

There is a canonical map of commutative S-algebras

$$\nu\colon THH(R) \wedge DS^1_+ \to F(S^1_+, THH(R))\,,$$

given by the composition of functions

$$F(S, THH(R)) \wedge F(S^1_+, S) \to F(S^1_+, THH(R))\,.$$

Compare [LMS86, III.1]. The map ν is an equivalence since S^1_+ is a finite CW complex (i.e., by Spanier–Whitehead duality). Hence there are homomorphisms

$$H_*(THH(R);\mathbb{F}_p) \xrightarrow{\widetilde{\alpha}} H_*(F(S^1_+, THH(R));\mathbb{F}_p) \xleftarrow[\cong]{\nu} H_*(THH(R);\mathbb{F}_p) \otimes H_*(DS^1_+;\mathbb{F}_p)$$

that take $x \in H_*(THH(R);\mathbb{F}_p)$ to

$$\nu^{-1}\widetilde{\alpha}(x) = x \otimes 1 + \alpha(x \otimes s_1) \otimes \iota_1\,.$$

Since $\widetilde{\alpha}$ and ν are maps of commutative S-algebras, we have

$$Q^k(\nu^{-1}\widetilde{\alpha}(x)) = \nu^{-1}\widetilde{\alpha}(Q^k(x))\,.$$

The Cartan formula for Dyer–Lashof operations [BMMS86, III.1.1(6)] then gives us

$$Q^k(x \otimes 1 + \alpha(x \otimes s_1) \otimes \iota_1) = Q^k(x) \otimes 1 + Q^k(\alpha(x \otimes s_1)) \otimes \iota_1$$

since $Q^i(1) = 0$ and $Q^i(\iota_1) = 0$ for $i \neq 0$. Hence we have

$$Q^k(x) \otimes 1 + Q^k(\alpha(x \otimes s_1)) \otimes \iota_1 = Q^k(x) \otimes 1 + \alpha(Q^k(x) \otimes s_1) \otimes \iota_1$$

and can read off $Q^k(\alpha(x \otimes s_1)) = \alpha(Q^k(x) \otimes s_1)$, as desired.

Specializing to classes $x \in H_*(R;\mathbb{F}_p)$, which map under $\eta\colon R \to THH(R)$ to $\eta(x) \in H_*(THH(R);\mathbb{F}_p)$, we have $\sigma x = \alpha(\eta(x) \otimes s_1)$, and thus obtain Bökstedt's formula $Q^k(\sigma x) = \sigma Q^k(x)$. □

The same ideas can be used to prove that $\sigma\colon H_*(R;\mathbb{F}_p) \to H_{*+1}(THH(R);\mathbb{F}_p)$ is a graded derivation.

Proposition 5.10. *Let R be a commutative S-algebra. Then we have*

$$\alpha(xy \otimes s_1) = x \cdot \alpha(y \otimes s_1) + (-1)^{|y|}\alpha(x \otimes s_1) \cdot y$$

for $x, y \in H_*(THH(R); \mathbb{F}_p)$. In particular, we have the Leibniz rule

$$\sigma(x \cdot y) = x \cdot \sigma(y) + (-1)^{|y|}\sigma(x) \cdot y$$

for $x, y \in H_*(R; \mathbb{F}_p)$.

Proof. We keep the notation of the proof of proposition 5.9. Since $\tilde{\alpha}$ and ν are maps of (commutative) S-algebras, $\nu^{-1}\tilde{\alpha}\colon H_*(THH(R); \mathbb{F}_p) \to H_*(THH(R); \mathbb{F}_p) \otimes E(\iota_1)$ is an algebra homomorphism. Thus

$$xy \otimes 1 + \alpha(xy \otimes s_1) \otimes \iota_1 = (x \otimes 1 + \alpha(x \otimes s_1) \otimes \iota_1) \cdot (y \otimes 1 + \alpha(y \otimes s_1) \otimes \iota_1)$$

for $x, y \in H_*(THH(R); \mathbb{F}_p)$. Multiplying out and comparing ι_1-coefficients gives the claimed formulas. $\square$

We also wish to describe the A_*-comodule structure on $H_*(THH(R); \mathbb{F}_p)$. In the cases $R = BP\langle n\rangle$ the following observations will suffice. The A_*-comodule coaction map

$$\nu\colon H_*(R; \mathbb{F}_p) \to A_* \otimes H_*(R; \mathbb{F}_p)$$

is in each case given by restricting the coproduct

$$\psi\colon A_* \to A_* \otimes A_*$$

given in subsection 5.1, to the subalgebra $H_*(R; \mathbb{F}_p) \subset A_*$, since the latter inclusion is induced by a spectrum map $R \to H\mathbb{F}_p$. The operator

$$\sigma\colon H_*(\Sigma R; \mathbb{F}_p) \to H_*(THH(R); \mathbb{F}_p)$$

is induced by a spectrum map, hence is an A_*-comodule homomorphism. Hence the coaction map

$$\nu\colon H_*(THH(R); \mathbb{F}_p) \to A_* \otimes H_*(THH(R); \mathbb{F}_p)$$

satisfies

$$\nu \circ \sigma = (1 \otimes \sigma)\nu\,. \tag{5.11}$$

A class x is called an *A_*-comodule primitive* if $\nu(x) = 1 \otimes x$.

Theorem 5.12. *Let p be any prime and $-1 \leq n \leq \infty$ be such that $BP\langle n\rangle_p$ admits the structure of a commutative S-algebra (e.g., $n \leq 1$). Then for $n < \infty$ we have*

$$H_*(THH(BP\langle n\rangle); \mathbb{F}_2) = H_*(BP\langle n\rangle; \mathbb{F}_2) \otimes E(\sigma\bar{\xi}_1^2, \dots, \sigma\bar{\xi}_{n+1}^2) \otimes P(\sigma\bar{\xi}_{n+2})$$

when $p = 2$, and

$$H_*(THH(BP\langle n\rangle);\mathbb{F}_p) = H_*(BP\langle n\rangle;\mathbb{F}_p) \otimes E(\sigma\bar\xi_1, \dots, \sigma\bar\xi_{n+1}) \otimes P(\sigma\bar\tau_{n+1})$$

when p is odd, as primitively generated $H_*(BP\langle n\rangle;\mathbb{F}_p)$-Hopf algebras.

For $n = \infty$ we have

$$H_*(THH(BP);\mathbb{F}_2) = H_*(BP;\mathbb{F}_2) \otimes E(\sigma\bar\xi_k^2 \mid k \geq 1)$$

when $p = 2$, and

$$H_*(THH(BP);\mathbb{F}_p) = H_*(BP;\mathbb{F}_p) \otimes E(\sigma\bar\xi_k \mid k \geq 1)$$

when p is odd, as primitively generated $H_*(BP;\mathbb{F}_p)$-Hopf algebras.

The A_*-comodule coaction on $H_*(THH(BP\langle n\rangle);\mathbb{F}_p)$ is given on $H_*(BP\langle n\rangle;\mathbb{F}_p)$ by restricting the coproduct on A_*. For $p = 2$ the algebra generators $\sigma\bar\xi_k^2$ for $1 \leq k \leq n+1$ are A_*-comodule primitives, while

$$\nu(\sigma\bar\xi_{n+2}) = 1 \otimes \sigma\bar\xi_{n+2} + \bar\xi_1 \otimes \sigma\bar\xi_{n+1}^2 .$$

For p odd the algebra generators $\sigma\bar\xi_k$ for $1 \leq k \leq n+1$ are A_*-comodule primitives, while

$$\nu(\sigma\bar\tau_{n+1}) = 1 \otimes \sigma\bar\tau_{n+1} + \bar\tau_0 \otimes \sigma\bar\xi_{n+1} .$$

(As usual, $\bar\xi_0$ is read as 1 in such formulas. Thus for $n = -1$, $\sigma\bar\xi_1$ and $\sigma\bar\tau_0$ are also primitive.)

Proof. We first resolve the algebra extensions. For $p = 2$ we find that the squares in $H_*(THH(R);\mathbb{F}_2)$ of the algebra generators in $E^\infty_{**}(R)$ are

$$(\sigma\bar\xi_k^2)^2 = Q^{2^{k+1}-1}(\sigma\bar\xi_k^2) = \sigma Q^{2^{k+1}-1}(\bar\xi_k^2) = 0$$

for $k = 1, \dots, n+1$ and

$$(\sigma\bar\xi_k)^2 = Q^{2^k}(\sigma\bar\xi_k) = \sigma Q^{2^k}(\bar\xi_k) = \sigma\bar\xi_{k+1}$$

for $k \geq n+2$. In the first case we have used the easy consequence $Q^k(y^2) = 0$ of the Cartan formula, for $p = 2$ and k odd.

For p odd the classes $\sigma\bar\xi_k$ remain exterior, since they are of odd degree in the graded commutative algebra $H_*(THH(R);\mathbb{F}_p)$. The p-th powers of the truncated polynomial generators in $E^\infty_{**}(R)$ are

$$(\sigma\bar\tau_k)^p = Q^{p^k}(\sigma\bar\tau_k) = \sigma Q^{p^k}(\bar\tau_k) = \sigma\bar\tau_{k+1}$$

for $k \geq n+1$. Hence these assemble to a polynomial algebra on the single generator $\sigma\bar\tau_{n+1}$.

All the algebra generators are of the form σx, in Hochschild filtration 1, and are primitive by proposition 4.8.

To compute the A_*-comodule structure we use that σ is a graded derivation, see proposition 5.10, so $\sigma(y^p) = 0$ and $\sigma(1) = 0$. Then for $p = 2$ and $1 \le k \le n+1$ we have

$$\nu(\sigma\bar{\xi}_k^2) = \sum_{i+j=k} \bar{\xi}_i^2 \otimes \sigma\bar{\xi}_j^{2^{i+1}} = 1 \otimes \sigma\bar{\xi}_k^2$$

while

$$\nu(\sigma\bar{\xi}_{n+2}) = \sum_{i+j=n+2} \bar{\xi}_i \otimes \sigma\bar{\xi}_j^{2^i} = 1 \otimes \sigma\bar{\xi}_{n+2} + \bar{\xi}_1 \otimes \sigma\bar{\xi}_{n+1}^2 \,.$$

For p odd and $1 \le k \le n+1$ we get

$$\nu(\sigma\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \sigma\bar{\xi}_j^{p^i} = 1 \otimes \sigma\bar{\xi}_k$$

while

$$\nu(\sigma\bar{\tau}_{n+1}) = 1 \otimes \sigma\bar{\tau}_{n+1} + \sum_{i+j=n+1} \bar{\tau}_i \otimes \sigma\bar{\xi}_j^{p^i} = 1 \otimes \sigma\bar{\tau}_{n+1} + \bar{\tau}_0 \otimes \sigma\bar{\xi}_{n+1} \,.$$

□

For later reference we extract the following special cases, which correspond to $n = 1$. Recall that $H_*(ku; \mathbb{F}_2) = (A/\!/E_1)_* \subset A_*$ and $H_*(\ell; \mathbb{F}_p) = (A/\!/E_1)_* \subset A_*$.

Corollary 5.13. *(a) There is an isomorphism*

$$H_*(THH(ku); \mathbb{F}_2) \cong H_*(ku; \mathbb{F}_2) \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3)$$

of primitively generated $H_*(ku; \mathbb{F}_2)$-Hopf algebras.

The A_*-comodule coaction $\nu\colon H_*(THH(ku); \mathbb{F}_2) \to A_* \otimes H_*(THH(ku); \mathbb{F}_2)$ is given on $H_*(ku; \mathbb{F}_2)$ by restricting the coproduct $\psi\colon A_* \to A_* \otimes A_*$, and on the algebra generators by $\nu(\sigma\bar{\xi}_1^2) = 1 \otimes \sigma\bar{\xi}_1^2$, $\nu(\sigma\bar{\xi}_2^2) = 1 \otimes \sigma\bar{\xi}_2^2$ and

$$\nu(\sigma\bar{\xi}_3) = 1 \otimes \sigma\bar{\xi}_3 + \bar{\xi}_1 \otimes \sigma\bar{\xi}_2^2 \,.$$

(b) There is an isomorphism

$$H_*(THH(\ell); \mathbb{F}_p) \cong H_*(\ell; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2) \otimes P(\sigma\bar{\tau}_2)$$

of primitively generated $H_*(\ell; \mathbb{F}_p)$-Hopf algebras.

The A_*-comodule coaction $\nu\colon H_*(THH(\ell); \mathbb{F}_p) \to A_* \otimes H_*(THH(\ell); \mathbb{F}_p)$ is given on $H_*(\ell; \mathbb{F}_p)$ by restricting the coproduct $\psi\colon A_* \to A_* \otimes A_*$, and on the algebra generators by $\nu(\sigma\bar{\xi}_1) = 1 \otimes \sigma\bar{\xi}_1$, $\nu(\sigma\bar{\xi}_2) = 1 \otimes \sigma\bar{\xi}_2^2$ and

$$\nu(\sigma\bar{\tau}_2) = 1 \otimes \sigma\bar{\tau}_2 + \bar{\tau}_0 \otimes \sigma\bar{\xi}_2 \,.$$

6. The higher real cases

For $p = 2$ there are a few more known examples of commutative S-algebras such that $H^*(R;\mathbb{F}_p)$ is a cyclic A-module. Let ko be the connective real K-theory spectrum, with $H^*(ko;\mathbb{F}_2) \cong A//A_1 = A \otimes_{A_1} \mathbb{F}_2$, and let tmf be the Hopkins–Mahowald topological modular forms spectrum, with $H^*(tmf;\mathbb{F}_2) \cong A//A_2 = A \otimes_{A_2} \mathbb{F}_2$. See e.g. [Re01, 21.5]. Here $A_n \subset A$ is the subalgebra generated by $Sq^1, \dots, Sq^{2^n}$, so A_1 has rank 8 and A_2 has rank 64. It is well known that ko is a commutative S-algebra, and in the case of tmf this is a consequence of the Hopkins–Miller theory, being presented by Goerss and Hopkins.

Proposition 6.1. *There are maps of S-algebras $tmf \to ko \to H\mathbb{F}_2$ that induce the following identifications:*

$$H_*(ko;\mathbb{F}_2) = P(\bar\xi_1^4, \bar\xi_2^2, \bar\xi_k \mid k \geq 3)$$

and

$$H_*(tmf;\mathbb{F}_2) = P(\bar\xi_1^8, \bar\xi_2^4, \bar\xi_3^2, \bar\xi_k \mid k \geq 4)\,.$$

Proof. This is immediate by dualization from $H^*(ko;\mathbb{F}_2) \cong A//A_1$, cf. [St63], and $H^*(tmf;\mathbb{F}_2) \cong A//A_2$. $\square$

We now follow the outline of section 5. By proposition 2.4 the E^2-terms of the respective Bökstedt spectral sequences are

$$E^2_{**}(ko) = H_*(ko;\mathbb{F}_2) \otimes E(\sigma\bar\xi_1^4, \sigma\bar\xi_2^2, \sigma\bar\xi_k \mid k \geq 3)$$

and

$$E^2_{**}(tmf) = H_*(tmf;\mathbb{F}_2) \otimes E(\sigma\bar\xi_1^8, \sigma\bar\xi_2^4, \sigma\bar\xi_3^2, \sigma\bar\xi_k \mid k \geq 4)\,.$$

By corollary 4.3 both spectral sequences collapse at the E^2-term, so $E^2_{**}(R) = E^\infty_{**}(R)$. To resolve the algebra extensions we use the Dyer–Lashof operations and proposition 5.9. The squares in $H_*(THH(ko);\mathbb{F}_2)$ of the algebra generators in $E^\infty_{**}(ko)$ are

$$(\sigma\bar\xi_1^4)^2 = Q^5(\sigma\bar\xi_1^4) = \sigma Q^5(\bar\xi_1^4) = 0$$
$$(\sigma\bar\xi_2^2)^2 = Q^7(\sigma\bar\xi_2^2) = \sigma Q^7(\bar\xi_2^2) = 0$$

by the formula $Q^k(y^2) = 0$ for $p = 2$ and k odd, and

$$(\sigma\bar\xi_k)^2 = Q^{2^k}(\sigma\bar\xi_k) = \sigma Q^{2^k}(\bar\xi_k) = \sigma\bar\xi_{k+1}$$

for all $k \geq 3$. Similar calculations show that $(\sigma\bar\xi_1^8)^2 = 0$, $(\sigma\bar\xi_2^4)^2 = 0$ and $(\sigma\bar\xi_3^2)^2 = 0$ in $H_*(THH(tmf);\mathbb{F}_2)$, while $(\sigma\bar\xi_k)^2 = \sigma\bar\xi_{k+1}$ for all $k \geq 4$. The A_*-comodule coaction map ν on the resulting algebra generators is obtained from the coproduct on A_* and formula (5.11), as in the proof of theorem 5.12. The result is as follows:

Theorem 6.2. *(a) There is an isomorphism*

$$H_*(THH(ko);\mathbb{F}_2) \cong H_*(ko;\mathbb{F}_2) \otimes E(\sigma\bar\xi_1^4, \sigma\bar\xi_2^2) \otimes P(\sigma\bar\xi_3)$$

of primitively generated $H_*(ko;\mathbb{F}_2)$-Hopf algebras. The A_*-comodule structure is given on $H_*(ko;\mathbb{F}_2)$ by restricting the coproduct on A_*, and on the algebra generators by $\nu(\sigma\bar\xi_1^4) = 1 \otimes \sigma\bar\xi_1^4$ and

$$\begin{aligned}\nu(\sigma\bar\xi_2^2) &= 1 \otimes \sigma\bar\xi_2^2 + \bar\xi_1^2 \otimes \sigma\bar\xi_1^4 \\ \nu(\sigma\bar\xi_3) &= 1 \otimes \sigma\bar\xi_3 + \bar\xi_1 \otimes \sigma\bar\xi_2^2 + \bar\xi_2 \otimes \sigma\bar\xi_1^4 .\end{aligned}$$

(b) There is an isomorphism

$$H_*(THH(tmf);\mathbb{F}_2) \cong H_*(tmf;\mathbb{F}_2) \otimes E(\sigma\bar\xi_1^8, \sigma\bar\xi_2^4, \sigma\bar\xi_3^2) \otimes P(\sigma\bar\xi_4)$$

of primitively generated $H_*(tmf;\mathbb{F}_2)$-Hopf algebras. The A_*-comodule structure is given on $H_*(tmf;\mathbb{F}_2)$ by restricting the coproduct on A_*, and on the algebra generators by $\nu(\sigma\bar\xi_1^8) = 1 \otimes \sigma\bar\xi_1^8$ and

$$\begin{aligned}\nu(\sigma\bar\xi_2^4) &= 1 \otimes \sigma\bar\xi_2^4 + \bar\xi_1^4 \otimes \sigma\bar\xi_1^8 \\ \nu(\sigma\bar\xi_3^2) &= 1 \otimes \sigma\bar\xi_3^2 + \bar\xi_1^2 \otimes \sigma\bar\xi_2^4 + \bar\xi_2^2 \otimes \sigma\bar\xi_1^8 \\ \nu(\sigma\bar\xi_4) &= 1 \otimes \sigma\bar\xi_4 + \bar\xi_1 \otimes \sigma\bar\xi_3^2 + \bar\xi_2 \otimes \sigma\bar\xi_2^4 + \bar\xi_3 \otimes \sigma\bar\xi_1^8 .\end{aligned}$$

Proof. Assemble the computations above. □

7. THE REAL AND COMPLEX IMAGE OF J

We now turn to the various image of J spectra that are commutative S-algebras. Their mod p cohomology is no longer cyclic as a module over the Steenrod algebra, but of rank 2, so extra work is needed to describe their homology as an A_*-comodule algebra.

7.1. The image of J spectra. For any prime p let the p-local, connective *complex image of J spectrum* be $ju = K(\mathbb{F}_r)_{(p)}$, where $r = 3$ for $p = 2$ and r is a prime power that topologically generates the p-adic units for p odd. Being the localized algebraic K-theory of a field, ju is a commutative S-algebra [Ma77, VIII.3.1]. For $p = 2$ there is a cofiber sequence of spectra

$$ju \xrightarrow{\kappa} ku_{(2)} \xrightarrow{\psi^3 - 1} bu_{(2)} , \tag{7.2}$$

where $bu \simeq \Sigma^2 ku$ is the 1-connected cover of ku and ψ^3 is the Adams operation. For odd p the cofiber sequence appears as

$$ju \xrightarrow{\kappa} \ell \xrightarrow{\psi^r - 1} \Sigma^q \ell \tag{7.3}$$

with $q = 2p - 2$, where $\ell \subset ku_{(p)}$ is the p-local, connective Adams summand and ψ^r is the r-th Adams operation [Ma77, V.5.16].

Let the 2-local, connective *real image of J spectrum* be $j = K\mathcal{N}(\mathbb{F}_3)_{(2)}$, as in [Ma77, VIII.3.1]. Being the localized algebraic K-theory of a symmetric bimonoidal category, j is a commutative S-algebra. There is a cofiber sequence of spectra

$$j \xrightarrow{\kappa} ko_{(2)} \xrightarrow{\psi^3-1} bspin_{(2)}\,, \tag{7.4}$$

where $bspin \simeq \Sigma^4 ksp$ is the 3-connected cover of ko [Ma77, V.5.16].

The fiber map $\kappa\colon ju \to \ell$ is a map of commutative S-algebras, at least after p-adic completion, because there is a (discrete) model $K(k')_p$ for ℓ_p with k' a suitable subfield of the algebraic closure of $\mathbb{F}_r$, and applying the functor $K(-)_p$ to the field inclusions $\mathbb{F}_r \subset k' \subset \overline{\mathbb{F}}_r$ produces the commutative S-algebra maps

$$ju_p \xrightarrow{\kappa} \ell_p \subset ku_p\,.$$

See [Ma77, VIII.3.2] and [MS93, §9]. Similarly, $\kappa\colon j \to ko$ becomes a map of commutative S-algebras after 2-adic completion, since there is a (discrete) model $K\mathcal{O}(\overline{\mathbb{F}}_3)_2$ for ko_2, and κ can be identified with the natural map $K\mathcal{N}(\mathbb{F}_3)_2 \to K\mathcal{O}(\overline{\mathbb{F}}_3)_2$. See [Ma77, VIII.2.6 and 3.2].

7.5. Cohomology modules. Recall that

$$H^*(\ell;\mathbb{F}_p) = A/\!/E_1 \cong \mathbb{F}_p\{1, P^1, \dots, P^p, \dots\}\,,$$

where $E_1 = E(Q_0, Q_1) \subset A$ is the exterior algebra generated by $Q_0 = \beta$ and $Q_1 = [P^1, \beta]$, and that

$$H^*(ko;\mathbb{F}_2) = A/\!/A_1 \cong \mathbb{F}_2\{1, Sq^4, Sq^2Sq^4 \equiv Sq^6, Sq^1Sq^2Sq^4 \equiv Sq^7, \dots\}\,,$$

where $A_1 = \langle Sq^1, Sq^2\rangle \subset A$ is the subalgebra generated by Sq^1 and Sq^2. There are also A-module isomorphisms

$$\begin{aligned} H^*(bo;\mathbb{F}_2) &\cong \Sigma A/ASq^2 \\ H^*(bso;\mathbb{F}_2) &\cong \Sigma^2 A/ASq^3 \\ H^*(bspin;\mathbb{F}_2) &\cong \Sigma^4 A/A\{Sq^1, Sq^2Sq^3\}\,. \end{aligned}$$

See [AP76, 2.5, 2.4 and p. 501]. Here $Sq^2Sq^3 = Sq^5 + Sq^4Sq^1$ in admissible form, but the shorter expression is perhaps more memorable. For p odd we let $A_n \subset A$ be the subalgebra generated by $\beta, P^1, \dots, P^{p^{n-1}}$. In particular, $A_1 = \langle \beta, P^1\rangle$ contains E_1, and

$$A/\!/A_1 \cong \mathbb{F}_p\{1, P^p, P^1P^p \equiv -P^{p+1}, \beta P^1 P^p \equiv Q_2, \dots\}\,.$$

Lemma 7.6. *(a) For $p = 2$ the map $\psi^3 - 1\colon ku_{(2)} \to \Sigma^2 ku_{(2)}$ induces right multiplication by Sq^2 on mod 2 cohomology:*

$$(\psi^3 - 1)^* = Sq^2\colon \Sigma^2 A/\!/E_1 \to A/\!/E_1\,.$$

(b) For p odd the map $\psi^r - 1\colon \ell \to \Sigma^q \ell$ induces right multiplication by P^1 on mod p cohomology:

$$(\psi^r - 1)^* = P^1\colon \Sigma^q A/\!/E_1 \to A/\!/E_1\,.$$

(c) For $p = 2$ the map $\psi^3 - 1\colon ko_{(2)} \to bspin_{(2)}$ induces right multiplication by Sq^4 on mod 2 cohomology:

$$(\psi^3 - 1)^* = Sq^4\colon \Sigma^4 A/A\{Sq^1, Sq^2Sq^3\} \to A/\!/A_1\,.$$

Case (c) is due to Mahowald and Milgram [MaMi76, 3.4].

Proof. The S-algebra unit map $e\colon S_{(p)} \to ju$ is well-known to be 2-connected for $p = 2$ and $(pq-2)$-connected for p odd, since this is the degree of the first element β_1 in the p-primary cokernel of J [Ra04, 1.1.14]. Hence $e^*: H^*(ju;\mathbb{F}_p) \to H^*(S;\mathbb{F}_p) = \mathbb{F}_p$ is 2-coconnected for $p = 2$ and $(pq - 2)$-coconnected for p odd, meaning that the homomorphism is injective in the stated degree, and an isomorphism in lower degrees. In particular, $H^2(ju;\mathbb{F}_2) = 0$ for $p = 2$ and $H^q(ju;\mathbb{F}_p) = 0$ for p odd.

So in the long exact cohomology sequence associated to the cofiber sequence (7.2)

$$\Sigma^2 A/\!/E_1 \xrightarrow{(\psi^3-1)^*} A/\!/E_1 \xrightarrow{\kappa^*} H^*(ju;\mathbb{F}_2) \tag{7.7}$$

the non-zero class of Sq^2 in $A/\!/E_1$ maps to zero under κ^*, hence is in the image of $(\psi^3 - 1)^*$. The latter is a (left) A-module homomorphism, and can only take $\Sigma^2(1)$ to Sq^2, hence is given by (right) multiplication by Sq^2. This proves (a). For part (b) we use the same argument for the exact sequence

$$\Sigma^q A/\!/E_1 \xrightarrow{(\psi^r-1)^*} A/\!/E_1 \xrightarrow{\kappa^*} H^*(ju;\mathbb{F}_p) \tag{7.8}$$

associated to (7.3), in cohomological degree q. The non-zero class of P^1 in $A/\!/E_1$ maps to zero under κ^*, hence must equal the image of $\Sigma^q(1)$ under $(\psi^r - 1)^*$. This proves (b).

The unit map $e\colon S_{(2)} \to j$ is likewise well-known to be 6-connected, since this is the degree of the first element ν^2 in the 2-primary cokernel of J, so $e^*: H^*(j;\mathbb{F}_2) \to H^*(S;\mathbb{F}_2) = \mathbb{F}_2$ is 6-coconnected. In particular, $H^4(j;\mathbb{F}_2) = 0$. So in the long exact cohomology sequence associated to the cofiber sequence (7.4)

$$\Sigma^4 A/A\{Sq^1, Sq^2Sq^3\} \xrightarrow{(\psi^3-1)^*} A/\!/A_1 \xrightarrow{\kappa^*} H^*(j;\mathbb{F}_2) \tag{7.9}$$

the non-zero class of Sq^4 in $A/\!/A_1$ maps to zero under κ^*, hence is in the image of $(\psi^3 - 1)^*$. The only class that can hit it is $\Sigma^4(1)$, which proves (c). □

Lemma 7.10. *(a) For $p = 2$ there is a uniquely split extension of A-modules*

$$0 \to A/\!/A_1 \to H^*(ju;\mathbb{F}_2) \to \Sigma^3 A/\!/A_1 \to 0\,.$$

Hence there is a canonical A-module isomorphism $H^*(ju;\mathbb{F}_2) \cong A/\!/A_1\{1, x\}$, with x a class in degree 3.

(b) For p odd there is a non-split extension of A-modules

$$0 \to A//A_1 \to H^*(ju;\mathbb{F}_p) \to \Sigma^{pq-1}A//A_1 \to 0\,.$$

As an A-module, $H^*(ju;\mathbb{F}_p)$ is generated by two classes 1 and x in degrees 0 and $(pq-1)$, respectively, with $\beta(x) = P^p(1)$.

(c) There is a (unique) non-split extension of A-modules

$$0 \to A//A_2 \to H^*(j;\mathbb{F}_2) \to A \otimes_{A_2} \Sigma^7 K \to 0\,.$$

The cyclic A_2-module $K = A_2/A_2\{Sq^1, Sq^7, Sq^4Sq^6+Sq^6Sq^4\}$ has rank 17 over $\mathbb{F}_2$. As an A-module, $H^*(j;\mathbb{F}_2)$ is generated by two classes 1 and x in degrees 0 and 7, respectively, with $Sq^1(x) = Sq^8(1)$.

For case (b), see also [Ro03, 5.1(b)]. Case (c) is due to Davis [Da75, Thm. 1], who also shows that $H^*(j;\mathbb{F}_2)$ is a free $A//A_3$-module.

Proof. (a) In the long exact sequence (7.7) the A-module homomorphism $(\psi^3-1)^*$ is induced up from the A_1-module homomorphism

$$Sq^2 \colon \Sigma^2 A_1//E_1 \to A_1//E_1 = \mathbb{F}_2\{1, Sq^2\}$$

with kernel $\Sigma^2\mathbb{F}_2\{Sq^2\}$ and cokernel $\mathbb{F}_2\{1\}$. Since A is flat (in fact free) over A_1, it follows that $\ker(\psi^3-1)^* \cong \Sigma^4 A//A_1$ and $\operatorname{cok}(\psi^3-1)^* \cong A//A_1$. Hence there is an extension of A-modules

$$0 \to A//A_1 \to H^*(ju;\mathbb{F}_2) \to \Sigma^3 A//A_1 \to 0\,,$$

as asserted. The group of such extensions is trivial, by the change of rings isomorphism

$$\operatorname{Ext}^1_A(\Sigma^3 A//A_1, A//A_1) \cong \operatorname{Ext}^1_{A_1}(\Sigma^3\mathbb{F}_2, A//A_1)\,.$$

For in any A_1-module extension

$$0 \to A//A_1 \to E \to \Sigma^3\mathbb{F}_2 \to 0$$

let $x \in E$ be the unique class in degree 3 that maps to $\Sigma^3(1)$. Then $Sq^2x = 0$ since $A//A_1 = \mathbb{F}_2\{1, Sq^4, Sq^6, Sq^7, \dots\}$ is trivial in degree 5. Furthermore $Sq^1x = Sq^4(1)$ would contradict the Adem relation $Sq^2Sq^2 = Sq^3Sq^1$, since $Sq^3Sq^4 \equiv Sq^7$ in $A//A_1$. So $Sq^1x = 0$ and the extension E is trivial.

Two choices of splitting maps for the trivial extension describing $H^*(ju;\mathbb{F}_2)$ differ by an A-module homomorphism $\Sigma^3 A//A_1 \to A//A_1$, which must be zero since $A//A_1$ is trivial in degree 3. Therefore the splitting is unique, as claimed.

(b) Similarly, in (7.8) the A-module homomorphism $(\psi^r-1)^*$ is induced up from the A_1-module homomorphism

$$P^1 \colon \Sigma^q A_1//E_1 \to A_1//E_1 = \mathbb{F}_p\{1, P^1, \dots, P^{p-1}\}$$

with kernel $\Sigma^q \mathbb{F}_p\{P^{p-1}\}$ and cokernel $\mathbb{F}_p\{1\}$. As above it follows that $\ker(\psi^r - 1)^* \cong \Sigma^{pq} A/\!/A_1$ and $\mathrm{cok}(\psi^r - 1)^* \cong A/\!/A_1$. Hence there is an extension of A-modules

$$0 \to A/\!/A_1 \to H^*(ju; \mathbb{F}_p) \to \Sigma^{pq-1} A/\!/A_1 \to 0\,,$$

as asserted. This time the group of extensions is non-trivial; in fact it is isomorphic to $\mathbb{Z}/p$ and generated by the extension above. To see this, we again use the change of rings isomorphism

$$\mathrm{Ext}^1_A(\Sigma^{pq-1} A/\!/A_1, A/\!/A_1) \cong \mathrm{Ext}^1_{A_1}(\Sigma^{pq-1}\mathbb{F}_p, A/\!/A_1)$$

and consider A_1-module extensions

$$0 \to A/\!/A_1 \to E \to \Sigma^{pq-1}\mathbb{F}_p \to 0\,.$$

Let $x \in E$ be the unique class that maps to $\Sigma^{pq-1}(1)$. Then $P^1 x = 0$ since $A/\!/A_1 = \mathbb{F}_p\{1, P^p, P^1P^p, \dots\}$ is trivial in degree $(p+1)q-1$. But βx is a multiple of P^p, and this multiple classifies the extension.

To see that βx is non-zero in the case of $H^*(ju; \mathbb{F}_p)$, recall again that the first class β_1 in the cokernel of J is in degree $(pq-2)$ and has order p. Let $c \to S_{(p)} \to ju$ be the usual cofiber sequence. Then by the Hurewicz and universal coefficient theorems, the lowest class x in $H^*(\Sigma c; \mathbb{F}_p)$ sits in degree $(pq-1)$ and supports a non-trivial mod p Bockstein $\beta x \neq 0$. Furthermore, $H^*(\Sigma c; \mathbb{F}_p) \cong H^*(ju; \mathbb{F}_p)$ in positive degrees $* > 0$, so also in $H^*(ju; \mathbb{F}_p)$ we have $\beta x \neq 0$.

(c) In (7.9), the A-module homomorphism $(\psi^3 - 1)^*$ is induced up from the A_2-module homomorphism

$$Sq^4 \colon \Sigma^4 A_2/A_2\{Sq^1, Sq^2Sq^3\} \to A_2/\!/A_1\,.$$

A direct calculation shows that $A_2/A_2\{Sq^1, Sq^2Sq^3\}$ has rank 24 and $A_2/\!/A_1$ has rank 8, as $\mathbb{F}_2$-vector spaces. The homomorphism Sq^4 has cokernel $A_2/\!/A_2 = \mathbb{F}_2\{1\}$ of rank 1, so its kernel $\Sigma^8 K$ has rank 17. Here

$$\begin{aligned}\Sigma^4 K = \mathbb{F}_2\{&Sq^4,\ Sq^6,\ Sq^7,\ Sq^6Sq^2,\ Sq^9,\ Sq^{10} + Sq^8Sq^2,\ Sq^7Sq^3,\\ &Sq^{11} + Sq^9Sq^2,\ Sq^{10}Sq^2,\ Sq^{13} + Sq^{10}Sq^3,\ Sq^{11}Sq^2,\ Sq^{11}Sq^3,\\ &Sq^{13}Sq^2 + Sq^{12}Sq^3,\ Sq^{13}Sq^3,\ Sq^{17} + Sq^{15}Sq^2,\ Sq^{17}Sq^2 + Sq^{16}Sq^3,\ Sq^{17}Sq^3\}\end{aligned}$$

as a submodule of $A_2/A_2\{Sq^1, Sq^2Sq^3\}$. By another direct calculation, $\Sigma^4 K$ is in fact the cyclic A_2-submodule generated by $\Sigma^4 Sq^4$. The annihilator ideal turns out to be generated by Sq^1, Sq^7 and $Sq^4Sq^6 + Sq^6Sq^4 = Sq^{10} + Sq^8Sq^2 + Sq^7Sq^3$ (in admissible form), so

$$K \cong A_2/A_2\{Sq^1, Sq^7, Sq^4Sq^6 + Sq^6Sq^4\}\,.$$

Hence there is an extension of A-modules

$$0 \to A/\!/A_2 \to H^*(j; \mathbb{F}_2) \to A \otimes_{A_2} \Sigma^7 K \to 0$$

with $A \otimes_{A_2} \Sigma^7 K \cong \Sigma^7 A/A\{Sq^1, Sq^7, Sq^4Sq^6 + Sq^6Sq^4\}$. The group of such A-module extensions is

$$\mathrm{Ext}^1_A(A \otimes_{A_2} \Sigma^7 K, A//A_2) \cong \mathbb{Z}/2\,,$$

and the extension is determined by the action of Sq^1 on the generator x in degree 7 that maps to $\Sigma^7(1)$. ($Sq^7 x = 0$ by the Adem relation $Sq^1 Sq^7 = 0$ and the fact that Sq^1 acts injectively from degree 14 of $A//A_2$. $(Sq^4Sq^6 + Sq^6Sq^4)x = 0$ since $A//A_2$ is trivial in degree 17.)

To see that $Sq^1(x) = Sq^8(1) \neq 0$ in $H^*(j; \mathbb{Z}/2)$, we once again use the cofiber sequence $c \to S_{(2)} \to j$ and the fact that c is 5-connected with $\pi_6(c) = \mathbb{Z}/2\{\nu^2\}$. Hence the lowest class x in $H^*(\Sigma c; \mathbb{F}_2)$ sits in degree 7 and supports a non-trivial $Sq^1 x \neq 0$. Again, $H^*(\Sigma c; \mathbb{F}_2) \cong H^*(j; \mathbb{F}_2)$ in positive degrees, so also in $H^*(j; \mathbb{F}_2)$ we have $Sq^1 x \neq 0$. The only possible value is Sq^8 from $A//A_2$. □

We display the A_2-module $\Sigma^4 K \subset \Sigma^4 A_2/A_2\{Sq^1, Sq^2Sq^3\}$ below. Here (i) or (i,j) denotes an admissible class with lexicographically leading term Sq^i or Sq^iSq^j, respectively. The arrows indicate the Sq^1- and Sq^2-operations. The Sq^4-operations can be deduced from the relations $Sq^4(6) = (10)$ and $Sq^4(13) = (17)$.

$$\begin{tikzcd}
(4) \arrow[rr, bend left] & & (6) \arrow[r] & (7) \arrow[rr, bend left] & (6,2) \arrow[r] \arrow[rr, bend right] & (9) & (7,3)
\end{tikzcd}$$

$$\begin{tikzcd}
 & (11) \arrow[rr] & & (13) \arrow[dr] & & & \\
(10) \arrow[ur] \arrow[rr] & & (10,2) \arrow[rr] \arrow[dr] & & (11,3) \arrow[rr] & & (13,3) \\
 & & & (11,2) \arrow[rr] & & (13,2) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
(17) \arrow[r, bend left] & (17,2) \arrow[r] & (17,3)
\end{tikzcd}$$

7.11. Homology algebras. Let us write $(A//A_1)_* \subset A_*$ for the A_*-comodule subalgebra dual to the quotient A-module coalgebra $A//A_1$ of A. For $p = 2$ we recall from proposition 6.1 that:

$$(A//A_1)_* = P(\bar\xi_1^4, \bar\xi_2^2, \bar\xi_k \mid k \geq 3) \cong H_*(ko; \mathbb{F}_2)\,.$$

For p odd

$$(A//A_1)_* = P(\bar\xi_1^p, \bar\xi_k \mid k \geq 2) \otimes E(\bar\tau_k \mid k \geq 2)\,,$$

but there is no spectrum with mod p homology realizing $(A//A_1)_*$. Both for $p = 2$ and for p odd there is an extension of A_*-comodules

$$0 \to \Sigma^{pq-1}(A//A_1)_* \to H_*(ju; \mathbb{F}_p) \xrightarrow{\kappa} (A//A_1)_* \to 0$$

where κ is an A_*-comodule algebra homomorphism.

Likewise, we write $(A//A_2)_* \subset A_*$ for the A_*-comodule subalgebra dual to the quotient A-module coalgebra $A//A_2$ of A. For $p = 2$ we recall from proposition 6.1 that

$$(A//A_2)_* = P(\bar{\xi}_1^8, \bar{\xi}_2^4, \bar{\xi}_3^2, \bar{\xi}_k \mid k \geq 4) \cong H_*(tmf; \mathbb{F}_2) \,.$$

There is an extension of A_*-comodules

$$0 \to A_* \,\Box_{A_{2*}}\, \Sigma^7 K_* \to H_*(j; \mathbb{F}_2) \xrightarrow{\kappa} (A//A_2)_* \to 0$$

where κ is an A_*-comodule algebra homomorphism. Here $\Box$ denotes the cotensor product [MiMo65, 2.2], and $K_* \subset A_{2*}$ is the A_{2*}-comodule dual to the cyclic A_2-module $K = A_2/A_2\{Sq^1, Sq^7, Sq^4Sq^6 + Sq^6Sq^4\}$.

Proposition 7.12. *(a) For $p = 2$, let $b \in H_3(ju; \mathbb{F}_2)$ be the image of $\Sigma^3(1)$ in $\Sigma^3(A//A_1)_*$. Then there is an A_*-comodule algebra isomorphism*

$$H_*(ju; \mathbb{F}_2) \cong (A//A_1)_* \otimes E(b) \,,$$

where $(A//A_1)_*$ has the subalgebra structure from A_*, and b is A_*-comodule primitive.

The Dyer–Lashof operations in $H_*(ju; \mathbb{F}_2)$ satisfy $Q^4(b) = 0$, $Q^5(\bar{\xi}_1^4) = 0$, $Q^7(\bar{\xi}_2^2) = 0$ and $Q^{2^k}(\bar{\xi}_k) = \bar{\xi}_{k+1}$ for all $k \geq 3$, so $H_*(ju; \mathbb{F}_2)$ is generated by $\bar{\xi}_1^4$, $\bar{\xi}_2^2$, $\bar{\xi}_3$ and b as an algebra over the Dyer–Lashof algebra.

(b) For p odd, let $b \in H_{pq-1}(ju; \mathbb{F}_p)$ be the image of $\Sigma^{pq-1}(1)$ in $\Sigma^{pq-1}(A//A_1)_*$. There is an algebra isomorphism

$$(A//A_1)_* \otimes E(b) \cong H_*(ju; \mathbb{F}_p)$$

that takes the algebra generators $\bar{\xi}_1^p$, $\bar{\xi}_k$, $\bar{\tau}_k$ (for $k \geq 2$) and b to classes $\tilde{\xi}_1^p$, $\tilde{\xi}_k$, $\tilde{\tau}_k$ and b in $H_*(ju; \mathbb{F}_p)$, respectively. These map under κ to $\bar{\xi}_1^p$, $\bar{\xi}_k$, $\bar{\tau}_k$ and 0, respectively.

The Dyer–Lashof operations on $H_*(ju; \mathbb{F}_p)$ satisfy $Q^{pq/2}(b) = 0$ and $Q^{p^k}(\tilde{\tau}_k) = \tilde{\tau}_{k+1}$ for all $k \geq 2$, and $\beta(\tilde{\tau}_k) = \tilde{\xi}_k$ for all $k \geq 2$. Thus $H_*(ju; \mathbb{F}_p)$ is generated as an algebra over the Dyer–Lashof algebra by $\tilde{\xi}_1^p$, $\tilde{\xi}_2$, $\tilde{\tau}_2$ and b.

The A_*-comodule structure is determined by

$$\begin{aligned}
\nu(b) &= 1 \otimes b \\
\nu(\tilde{\xi}_1^p) &= 1 \otimes \tilde{\xi}_1^p - \tau_0 \otimes b + \bar{\xi}_1^p \otimes 1 \\
\nu(\tilde{\xi}_2) &= 1 \otimes \tilde{\xi}_2 + \bar{\xi}_1 \otimes \tilde{\xi}_1^p + \tau_1 \otimes b + \bar{\xi}_2 \otimes 1 \\
\nu(\tilde{\tau}_2) &= 1 \otimes \tilde{\tau}_2 + \bar{\tau}_0 \otimes \tilde{\xi}_2 + \bar{\tau}_1 \otimes \tilde{\xi}_1^p - \tau_0\tau_1 \otimes b + \bar{\tau}_2 \otimes 1 \,.
\end{aligned}$$

(The class $b \in H_{pq-1}(ju; \mathbb{F}_p)$ maps under the connecting map for the cofiber sequence $c \to S_{(p)} \to ju$ to the mod p Hurewicz image of $\beta_1 \in \pi_{pq-2}(c)$, so the letter b is chosen to correspond to β.)

(c) There is a square-zero extension of A_*-comodule algebras

$$0 \to A_* \,\square_{A_{2*}}\, \Sigma^7 K_* \to H_*(j;\mathbb{F}_2) \xrightarrow{\kappa} (A/\!/A_2)_* \to 0\,,$$

where κ is split as an algebra homomorphism. As an $(A/\!/A_2)_*$-module,

$$\ker(\kappa) = A_* \,\square_{A_{2*}}\, \Sigma^7 K_* \cong (A/\!/A_2)_* \otimes \Sigma^7 K_*$$

is free of rank 17. There is an algebra isomorphism

$$H_*(j;\mathbb{F}_2) \cong (A/\!/A_2)_* \otimes (\mathbb{F}_2 \oplus \Sigma^7 K_*)$$

where $\mathbb{F}_2 \oplus \Sigma^7 K_*$ is the split square-zero extension of $\mathbb{F}_2$ with kernel $\Sigma^7 K_*$ of rank 17.

Proof. (a) Let $x \in H^3(ju;\mathbb{F}_2)$ be the class that maps to $\Sigma^3(1)$ in the uniquely split A-module extension of lemma 7.10(a). Then $\psi(x) = x \otimes 1 + 1 \otimes x$ since $H^*(ju;\mathbb{F}_2) = 0$ for $0 < * < 3$, so $E(x) = \mathbb{F}_2\{1, x\}$ (no algebra structure is implied) is a sub-coalgebra of $H^*(ju;\mathbb{F}_2)$ and there a surjective composite A-module coalgebra homomorphism

$$A \otimes E(x) \to A \otimes H^*(ju;\mathbb{F}_2) \to H^*(ju;\mathbb{F}_2)\,.$$

Since the A-module extension is split, the generators Sq^1 and Sq^2 of A_1 act trivially on 1 and x in $H^*(ju;\mathbb{F}_2)$, so the surjection factors through a surjection

$$A/\!/A_1 \otimes E(x) \to H^*(ju;\mathbb{F}_2)\,,$$

which by a dimension count must be an A-module coalgebra isomorphism.

Dually, let $b \in H_3(ju;\mathbb{F}_2)$ be the image of $\Sigma^3(1)$ in the split A_*-comodule extension

$$0 \to \Sigma^3(A/\!/A_1)_* \to H_*(ju;\mathbb{F}_2) \to (A/\!/A_1)_* \to 0\,.$$

Then b is dual to x, and the dual of the above isomorphism is an A_*-comodule algebra isomorphism

$$H_*(ju;\mathbb{F}_2) \cong (A/\!/A_1)_* \otimes E(b)\,.$$

In particular, the unique A_*-comodule splitting $(A/\!/A_1)_* \to H_*(ju;\mathbb{F}_2)$ is an algebra map, b is an A_*-comodule primitive, and $b^2 = 0$.

To determine some Dyer–Lashof operations in $H_*(ju;\mathbb{F}_2)$ we shall use the Nishida relations and the known A_*-comodule structure. Some of the Nishida relations that we shall use are

$$Sq^1_* Q^s = \begin{cases} Q^{s-1} & \text{for } s \text{ even} \\ 0 & \text{for } s \text{ odd,} \end{cases}$$

and

$$Sq^2_* Q^s = \begin{cases} Q^{s-2} + Q^{s-1} Sq^1_* & \text{for } s \equiv 0, 1 \mod 4 \\ Q^{s-1} Sq^1_* & \text{for } s \equiv 2, 3 \mod 4. \end{cases}$$

See [BMMS86, III.1.1(8)].

The Dyer–Lashof operation $Q^4(b)$ lands in $H_7(ju;\mathbb{F}_2) \cong \mathbb{F}_2\{\bar\xi_3, \bar\xi_1^4 b\}$. From the A_*-comodule structure we can read off the dual Steenrod operations $Sq^1_*(\bar\xi_3) = \bar\xi_2^2$ and $Sq^4_*(\bar\xi_1^4 b) = b$, since Sq^i is dual to ξ_1^i. These are linearly independent, so $Q^4(b)$ is determined by its images under Sq^1_* and Sq^4_*. By a Nishida relation we get that $Sq^1_* Q^4(b) = Q^3(b)$, and $Q^3(b) = b^2 = 0$ since $|b| = 3$ and b is an exterior class. By another Nishida relation $Sq^4_* Q^4(b) = Q^2 Sq^2_*(b)$, and $Sq^2_*(b) = 0$ since b is A_*-comodule primitive. Thus $Sq^1_* Q^4(b) = 0$ and $Sq^4_* Q^4(b) = 0$, and the only possibility is that $Q^4(b) = 0$.

The operation $Q^5(\bar\xi_1^4)$ lands in $H_9(ju;\mathbb{F}_2) \cong \mathbb{F}_2\{\bar\xi_2^2 b\}$. By a Nishida relation $Sq^2_* Q^5(\bar\xi_1^4) = (Q^3 + Q^4 Sq^1_*)(\bar\xi_1^4) = 0$, while $Sq^2_*(\bar\xi_2^2 b) = \bar\xi_1^4 b \neq 0$. So $Q^5(\bar\xi_1^4)$ must be zero.

The operation $Q^7(\bar\xi_2^2)$ lands in $H_{13}(ju;\mathbb{F}_2) \cong \mathbb{F}_2\{\bar\xi_2^2\bar\xi_3, \bar\xi_1^4\bar\xi_2^2 b\}$. Its image under κ in $H_*(ku;\mathbb{F}_2) \subset A_*$ must vanish, by the Cartan formula, so in fact $Q^7(\bar\xi_2^2) \in \mathbb{F}_2\{\bar\xi_1^4\bar\xi_2^2 b\}$. By a Nishida relation $Sq^2_* Q^7(\bar\xi_2^2) = Q^6 Sq^1_*(\bar\xi_2^2) = 0$, while $Sq^2_*(\bar\xi_1^4\bar\xi_2^2 b) = (\bar\xi_1^4)^2 b \neq 0$, so $Q^7(\bar\xi_2^2)$ must be zero.

The claim that $Q^{2^k}(\bar\xi_k) = \bar\xi_{k+1}$ follows in the same way as in $A_* = H_*(H\mathbb{F}_2;\mathbb{F}_2)$, see [BMMS86, §III.6]. It suffices to show that $Sq^{2^m}_* Q^{2^k}(\bar\xi_k) = Sq^{2^m}_*(\bar\xi_{k+1})$ for $0 \le m \le k$, because the only A_*-comodule primitives in $H_*(ju;\mathbb{F}_2)$ are 1 and b. The right hand side equals $\bar\xi_k^2$ for $m = 0$, and is zero otherwise, by the formula for $\psi(\bar\xi_{k+1})$. The Nishida relations imply that $Sq^1_* Q^{2^k}(\bar\xi_k) = Q^{2^k-1}(\bar\xi_k) = \bar\xi_k^2$, while $Sq^2_* Q^{2^k}(\bar\xi_k) = Q^{2^k-1} Sq^1_*(\bar\xi_k) = Q^{2^k-1}(\bar\xi_{k-1}^2) = 0$ by the Cartan formula. For $2 \le m \le k$ we have $Sq^{2^m}_* Q^{2^k}(\bar\xi_k) = Q^{2^k-2^{m-1}} Sq^{2^{m-1}}_*(\bar\xi_k) = Q^{2^k-2^{m-1}}(0) = 0$ by the formula for $\psi(\bar\xi_k)$ in A_*. Hence $Q^{2^k}(\bar\xi_k) = \bar\xi_{k+1}$ in $H_*(ju;\mathbb{F}_2)$ is the only possibility.

(b) Let $x \in H^{pq-1}(ju;\mathbb{F}_p)$ be the class that maps to $\Sigma^{pq-1}(1)$ in the A-module extension of lemma 7.10(b). Then $\psi(x) = x \otimes 1 + 1 \otimes x$ since $H^*(ju;\mathbb{F}_p) = 0$ for $0 < * < pq-1$, so $E(x) = \mathbb{F}_p\{1, x\}$ is a sub-coalgebra of $H^*(ju;\mathbb{F}_p)$ and there a surjective composite A-module coalgebra homomorphism

$$A \otimes E(x) \to A \otimes H^*(ju;\mathbb{F}_p) \to H^*(ju;\mathbb{F}_p)\,.$$

Dually, let $b \in H_{pq-1}(ju;\mathbb{F}_p)$ be the image of $\Sigma^{pq-1}(1)$ in the A_*-comodule extension

$$0 \to \Sigma^{pq-1}(A/\!/A_1)_* \to H_*(ju;\mathbb{F}_p) \to (A/\!/A_1)_* \to 0\,.$$

Then $E(b)$ is a quotient algebra of $H_*(ju;\mathbb{F}_p)$ and the dual of the surjection above is an injective A_*-comodule algebra homomorphism

$$H_*(ju;\mathbb{F}_p) \to A_* \otimes E(b)\,.$$

We shall describe $H_*(ju;\mathbb{F}_p)$ in terms of its image under this injection.

Since $(A/\!/A_1)_*$ is a free graded commutative algebra, we can choose an algebra section $s\colon (A/\!/A_1)_* \to H_*(ju;\mathbb{F}_p)$ to the surjection $\kappa\colon H_*(ju;\mathbb{F}_p) \to (A/\!/A_1)_*$. We write $\tilde\xi_1^p = s(\bar\xi_1^p)$, $\tilde\xi_k = s(\bar\xi_k)$ and $\tilde\tau_k = s(\bar\tau_k)$ for the lifted classes in $H_*(ju;\mathbb{F}_p)$. Since $\Sigma^{pq-1}(A/\!/A_1)$ vanishes in the degrees of $\bar\xi_1^p$ and $\bar\tau_2$, the respective lifts $\tilde\xi_1^p$ and $\tilde\tau_2$ are unique, and we can use the commutative S-algebra structure on ju and the resulting Dyer–Lashof operations on $H_*(ju; \mathbb{F}_p)$ to fix the other lifts by the formulas $\tilde{\xi}_k = \beta(\tilde{\tau}_k)$ and $\tilde{\tau}_{k+1} = Q^{p^k}(\tilde{\tau}_k)$ for $k \geq 2$. This specifies the algebra section s uniquely. (But s is not an A_*-comodule homomorphism.)

Since $|b| = pq - 1$ is odd we have $b^2 = 0$ and the exterior algebra $E(b)$ is a subalgebra of $H_*(ju; \mathbb{F}_p)$. Writing $i \colon E(b) \to H_*(ju; \mathbb{F}_p)$ for the inclusion, we obtain an algebra map

$$s \otimes i \colon (A/\!/A_1)_* \otimes E(b) \to H_*(ju; \mathbb{F}_p)\,,$$

which we claim is an isomorphism. By a dimension count it suffices to show that its composite with the injection $H_*(ju; \mathbb{F}_p) \to A_* \otimes E(b)$ is injective. We have a diagram of algebra maps

$$\begin{tikzcd}
(A/\!/A_1)_* \otimes E(b) \arrow[r, "s\otimes i"] \arrow[d] & H_*(ju; \mathbb{F}_p) \arrow[r] \arrow[d, "\kappa"] & A_* \otimes E(b) \arrow[d] \\
(A/\!/A_1)_* \arrow[r, "="] & (A/\!/A_1)_* \arrow[r] & A_*
\end{tikzcd}$$

where the vertical maps take b to zero. The lower map takes $\xi \in (A/\!/A_1)_*$ to $\xi \in A_*$, so the upper composite takes $\xi \otimes 1$ to $\xi \otimes 1$ (mod b). Hence the latter takes $\xi \otimes b$ to $\xi \otimes b$ (mod $b^2 = 0$), and it follows that the upper composite indeed is injective.

In low degrees, $H^*(ju; \mathbb{F}_p) \cong \mathbb{F}_p\{1, x, P^p(1), P^1P^p(1), \beta P^1 P^p(1), \dots\}$ is dual to $H_*(ju; \mathbb{F}_p) \cong \mathbb{F}_p\{1, b, -\tilde{\xi}_1^p, \tilde{\xi}_2, \tilde{\tau}_2, \dots\}$. By lemma 7.10(b) we have $\beta(x) = P^p(1)$, so dually the A_*-comodule coactions are given by

$$\begin{aligned}
\nu(b) &= 1 \otimes b \\
\nu(\tilde{\xi}_1^p) &= 1 \otimes \tilde{\xi}_1^p - \tau_0 \otimes b + \bar{\xi}_1^p \otimes 1 \\
\nu(\tilde{\xi}_2) &= 1 \otimes \tilde{\xi}_2 + \bar{\xi}_1 \otimes \tilde{\xi}_1^p + \tau_1 \otimes b + \bar{\xi}_2 \otimes 1 \\
\nu(\tilde{\tau}_2) &= 1 \otimes \tilde{\tau}_2 + \bar{\tau}_0 \otimes \tilde{\xi}_2 + \bar{\tau}_1 \otimes \tilde{\xi}_1^p - \tau_0\tau_1 \otimes b + \bar{\tau}_2 \otimes 1\,.
\end{aligned}$$

In particular, the images in $A_* \otimes E(b)$ of these classes are $1 \otimes b$, $-\tau_0 \otimes b + \bar{\xi}_1^p \otimes 1$, $\tau_1 \otimes b + \bar{\xi}_2 \otimes 1$ and $-\tau_0\tau_1 \otimes b + \bar{\tau}_2 \otimes 1$, respectively.

The Dyer–Lashof operation $Q^{pq/2}(b)$ lands in $H_{p^2q-1}(ju; \mathbb{F}_p) = \mathbb{F}_p\{\tilde{\xi}_1^{pq/2} b\}$. Here $P_*^{pq/2}(\tilde{\xi}_1^{pq/2} b) = b$ is non-zero, in view of the formula above for $\nu(\tilde{\xi}_1^p)$. By a Nishida relation $P_*^{pq/2} Q^{pq/2}(b) = Q^{q/2} P_*^{q/2}(b) = 0$, so $Q^{pq/2}(b) = 0$.

(c) Let $x \in H^7(j; \mathbb{F}_2)$ be the class that maps to $\Sigma^7(1)$ in the A-module extension of lemma 7.10(c). Then $\psi(x) = x \otimes 1 + 1 \otimes x$ since $H^*(j; \mathbb{F}_2) = 0$ for $0 < * < 7$, so $E(x) = \mathbb{F}_2\{1, x\}$ is a sub-coalgebra of $H^*(j; \mathbb{F}_2)$ and there is a surjective composite A-module coalgebra homomorphism

$$A \otimes E(x) \to A \otimes H^*(j; \mathbb{F}_2) \to H^*(j; \mathbb{F}_2)\,.$$

Dually, let $b \in H_7(j; \mathbb{F}_2)$ be the image of $\Sigma^7(1)$ in the A_*-comodule extension

$$0 \to A_* \,\square_{A_{2*}}\, \Sigma^7 K_* \to H_*(j; \mathbb{F}_2) \xrightarrow{\kappa} (A/\!/A_2)_* \to 0\,.$$

Here $K_* \subset A_{2*}$ has rank 17, and contains $1 \in K_* \subset A_{2*}$ in degree 0. The dual of the surjection above is an injective A_*-comodule algebra homomorphism

$$H_*(j; \mathbb{F}_2) \to A_* \otimes E(b)$$

that may otherwise be described as the composite of the A_*-comodule coaction $H_*(j; \mathbb{F}_2) \to A_* \otimes H_*(j; \mathbb{F}_2)$ and the algebra surjection $H_*(j; \mathbb{F}_2) \to E(b)$.

We obtain a vertical map of A_*-comodule extensions

$$\begin{tikzcd}
A_* \,\square_{A_{2*}}\, \Sigma^7 K_* \arrow[r] \arrow[d] & H_*(j; \mathbb{F}_2) \arrow[r, "\kappa"] \arrow[d] & (A/\!/A_2)_* \arrow[d] \\
A_*\{b\} \arrow[r] & A_* \otimes E(b) \arrow[r] & A_*
\end{tikzcd}$$

where the right-hand square consists of A_*-comodule algebra homomorphisms, and the vertical maps are injective. At the left hand side we find the composite map

$$A_* \,\square_{A_{2*}}\, \Sigma^7 K_* \to A_* \,\square_{A_{2*}}\, \Sigma^7 A_{2*} \cong \Sigma^7 A_* \cong A_*\{b\}$$

that is dual to the surjection $A\{x\} \cong \Sigma^7 A \to A \otimes_{A_2} \Sigma^7 K$.

In the lower row the ideal $A_*\{b\}$ has square zero, since $b^2 = 0$, so also the ideal $\ker(\kappa) = A_* \,\square_{A_{2*}}\, \Sigma^7 K_*$ is a square-zero ideal [CE56, XIV.2] in $H_*(j; \mathbb{F}_2)$. Its module action by $H_*(j; \mathbb{F}_2)$ therefore descends to one by $(A/\!/A_2)_*$. It can be described in terms of the algebra product ϕ on A_* by the following commutative diagram with injective vertical maps:

$$\begin{tikzcd}
(A/\!/A_2)_* \otimes (A_* \,\square_{A_{2*}}\, \Sigma^7 K_*) \arrow[r] \arrow[d] & (A_* \,\square_{A_{2*}}\, \Sigma^7 K_*) \arrow[d] \\
A_* \otimes \Sigma^7 A_* \arrow[r, "\Sigma^7 \phi"] & \Sigma^7 A_*
\end{tikzcd}$$

The proof is easy given the algebra embedding of $H_*(j; \mathbb{F}_2)$ into $A_* \otimes E(b)$.

In fact, the square-zero ideal $A_* \,\square_{A_{2*}}\, \Sigma^7 K_*$ is a free $(A/\!/A_2)_*$-module of rank 17. For $A_2 \subset A$ is a direct summand as an A_2-module, so dually $A_* \to A_{2*}$ admits an A_{2*}-comodule section $s \colon A_{2*} \to A_*$. For example, the image of s may be $\mathbb{F}_2\{\bar{\xi}_1^i \bar{\xi}_2^j \bar{\xi}_3^k \mid i < 8, j < 4, k < 2\} \subset A_*$. We then have a map

$$id \,\square\, s \colon \Sigma^7 K_* \cong A_{2*} \,\square_{A_{2*}}\, \Sigma^7 K_* \to A_* \,\square_{A_{2*}}\, \Sigma^7 K_* \,.$$

Its composite with the inclusion $A_* \,\square_{A_{2*}}\, \Sigma^7 K_* \to A_* \,\square_{A_{2*}}\, \Sigma^7 A_{2*} \cong \Sigma^7 A_*$ factors as the two inclusions $\Sigma^7 K_* \to \Sigma^7 A_{2*} \to \Sigma^7 A_*$.

Combining $id \,\square\, s$ with the $(A/\!/A_2)_*$-module action on $A_* \,\square_{A_{2*}}\, \Sigma^7 K_*$ we obtain the left hand map f in a commuting diagram

$$\begin{tikzcd}
(A/\!/A_2)_* \otimes \Sigma^7 K_* \arrow[r] \arrow[d, "f"] & (A/\!/A_2)_* \otimes \Sigma^7 A_{2*} \arrow[d, "\cong"] \\
A_* \,\square_{A_{2*}}\, \Sigma^7 K_* \arrow[r] & A_* \,\square_{A_{2*}}\, \Sigma^7 A_{2*} \,.
\end{tikzcd}$$

Here $A_* \,\Box_{A_{2*}}\, \Sigma^7 A_{2*} \cong \Sigma^7 A_*$ and the right hand isomorphism exhibits $\Sigma^7 A_*$ as a free $(A/\!/A_2)_*$-module on the generators given by the section $s\colon \Sigma^7 A_{2*} \to \Sigma^7 A_*$. (It is a case of the Milnor–Moore comodule algebra theorem [MiMo65, 4.7].) The upper map is injective, hence so is

$$f\colon (A/\!/A_2)_* \otimes \Sigma^7 K_* \to A_* \,\Box_{A_{2*}}\, \Sigma^7 K_* \,.$$

But both sides have the same, finite dimension over $\mathbb{F}_2$ in each degree, so in fact f is an isomorphism of $(A/\!/A_2)_*$-modules.

The fact that κ splits as an algebra homomorphism is clear since $(A/\!/A_2)_*$ is a free graded commutative algebra over $\mathbb{F}_2$. However, the splitting is not an A_*-comodule homomorphism, and the $(A/\!/A_2)_*$-module isomorphism f is not an A_*-comodule isomorphism. The A_*-comodule algebra structure on $H_*(j;\mathbb{F}_2)$ may, if desired, be obtained by describing the image of the algebra generators under the A_*-comodule algebra embedding $H_*(j;\mathbb{F}_2) \to A_* \otimes E(b)$. $\square$

Proposition 7.13. *(a) For $p = 2$ the Bökstedt spectral sequence $E^r_{**}(ju)$ collapses at the E^2-term, with*

$$E^\infty_{**}(ju) \cong H_*(ju;\mathbb{F}_2) \otimes E(\sigma\bar\xi_1^4, \sigma\bar\xi_2^2, \sigma\bar\xi_k \mid k \ge 3) \otimes \Gamma(\sigma b) \,.$$

(b) For p odd the Bökstedt spectral sequence $E^r_{**}(ju)$ collapses at the E^p-term, with

$$E^\infty_{**}(ju) \cong H_*(ju;\mathbb{F}_p) \otimes E(\sigma\tilde\xi_1^p, \sigma\tilde\xi_2, \sigma\tilde\tau_k \mid k \ge 2) \otimes \Gamma(\sigma b) \,.$$

(c) The Bökstedt spectral sequence for j at $p = 2$ has E^2-term

$$E^2_{**}(j) \cong HH_*((A/\!/A_2)_*) \otimes HH_*(\mathbb{F}_2 \oplus \Sigma^7 K_*)$$

where

$$HH_*((A/\!/A_2)_*) \cong (A/\!/A_2)_* \otimes E(\sigma\bar\xi_1^8, \sigma\bar\xi_2^4, \sigma\bar\xi_3^2, \sigma\bar\xi_k \mid k \ge 4)$$

and

$$HH_q(\mathbb{F}_2 \oplus \Sigma^7 K_*) \cong [(\Sigma^7 K_*)^{\otimes q}]^{C_q} \oplus [(\Sigma^7 K_*)^{\otimes q+1}]_{C_{q+1}} \,.$$

In particular, $E^2_{**}(j)$ is not flat as a module over $H_*(j;\mathbb{F}_2)$.

Proof. (a) The Bökstedt spectral sequence for ju at $p = 2$ begins

$$E^2_{**}(ju) = H_*(ju;\mathbb{F}_2) \otimes E(\sigma\bar\xi_1^4, \sigma\bar\xi_2^2, \sigma\bar\xi_k \mid k \ge 3) \otimes \Gamma(\sigma b)$$

Proposition 4.7 applies, so a shortest non-zero differential must map from an algebra indecomposable to a coalgebra primitive and A_*-comodule primitive. Here we are referring to the A_*-comodule $H_*(ju;\mathbb{F}_2)$-Hopf algebra structure on $E^2_{**}(ju)$. The only possible algebra indecomposables are the $\gamma_{2^k}(\sigma b)$ in degrees 2^{k+2}, for $k \ge 2$. The coalgebra primitives are $H_*(ju;\mathbb{F}_2)\{\sigma\bar\xi_1^4, \sigma\bar\xi_2^2, \sigma\bar\xi_k \mid k \ge 3, \sigma b\}$, all in filtration $s = 1$, and contain the A_*-comodule primitives $E(b) \otimes \mathbb{F}_2\{\sigma\bar\xi_1^4, \sigma\bar\xi_k \mid k \ge 4, \sigma b\}$. These are in degrees 4, 5, 7, 8, 2^k and $2^k + 3$, for $k \ge 4$. The image of a differential on $\gamma_{2^k}(\sigma b)$ must be in total degree $2^{k+2} - 1$, for $k \ge 2$, but these degrees do not contain any simultaneous coalgebra- and comodule primitives. Therefore there are no nonzero differentials, and the spectral sequence collapses at the E^2-term.

(b) For p odd the spectral sequence begins

$$E^2_{**}(ju) = H_*(ju; \mathbb{F}_p) \otimes E(\sigma\tilde{\xi}_1^p, \sigma\tilde{\xi}_k \mid k \geq 2) \otimes \Gamma(\sigma b, \sigma\tilde{\tau}_k \mid k \geq 2)\,.$$

By proposition 5.6 we have $E^2 = E^{p-1}$ and there are differentials

$$d^{p-1}(\gamma_p(\sigma\tilde{\tau}_k)) = \sigma\tilde{\xi}_{k+1}$$

for $k \geq 2$. This uses the relation $\beta Q^{p^k}(\tilde{\tau}_k) = \beta(\tilde{\tau}_{k+1}) = \tilde{\xi}_{k+1}$. There is also a potential differential

$$d^{p-1}(\gamma_p(\sigma b)) = \sigma(\beta Q^{pq/2}(b))\,,$$

but $\beta Q^{pq/2}(b)$ is in degree $p^2q - 2$ of $H_*(ju; \mathbb{F}_p)$, which is a trivial group, so this differential is zero. Hence

$$E^p_{**}(ju) = H_*(ju; \mathbb{F}_p) \otimes E(\sigma\tilde{\xi}_1^p, \sigma\tilde{\xi}_2) \otimes P_p(\sigma\tilde{\tau}_k \mid k \geq 2) \otimes \Gamma(\sigma b)\,.$$

Proposition 4.7 applies again, so a shortest differential must map from one of the algebra indecomposables $\gamma_{p^k}(\sigma b)$ in degrees $p^{k+1}q$, for $k \geq 2$. Its target must be lie among the coalgebra primitives, which are $H_*(ju; \mathbb{F}_p)\{\sigma\tilde{\xi}_1^p, \sigma\tilde{\xi}_2, \sigma\tilde{\tau}_k \mid k \geq 2, \sigma b\}$, all in filtration $s = 1$. The target must also be A_*-comodule primitive. The formulas for the A_*-comodule structure on $H_*(ju; \mathbb{F}_p)$ imply the following formulas:

$$\begin{aligned}
\nu(\sigma b) &= 1 \otimes \sigma b \\
\nu(\sigma\tilde{\xi}_1^p) &= 1 \otimes \sigma\tilde{\xi}_1^p - \tau_0 \otimes \sigma b \\
\nu(\sigma\tilde{\xi}_2) &= 1 \otimes \sigma\tilde{\xi}_2 + \bar{\xi}_1 \otimes \sigma\tilde{\xi}_1^p + \tau_1 \otimes \sigma b \\
\nu(\sigma\tilde{\tau}_2) &= 1 \otimes \sigma\tilde{\tau}_2 + \bar{\tau}_0 \otimes \sigma\tilde{\xi}_2 + \bar{\tau}_1 \otimes \sigma\tilde{\xi}_1^p - \tau_0\tau_1 \otimes \sigma b\,.
\end{aligned}$$

The $\sigma\tilde{\tau}_k$ are A_*-comodule primitives for $k \geq 3$, in view of the relations $(\sigma\tilde{\tau}_k)^p = \sigma\tilde{\tau}_{k+1}$ for $k \geq 2$, the formula for $\nu(\sigma\tilde{\tau}_2)$, and the fact that $\bar{\tau}_0^p = \bar{\tau}_1^p = (\tau_0\tau_1)^p = 0$.

Thus the simultaneous coalgebra- and comodule primitives are contained in $E(b) \otimes \mathbb{F}_p\{\sigma\tilde{\xi}_1^p, \sigma\tilde{\tau}_k \mid k \geq 3, \sigma b\}$. These are in degrees pq, $pq + 1$, $2pq - 1$, $2pq$, $2p^k$ and $2p^k + pq - 1$ for $k \geq 3$. The image of a differential $d^r(\gamma_{p^k}(\sigma b))$ is in degree $p^{k+1}q - 1$ for $k \geq 2$, which contains none of the possible target classes. Hence there are no further differentials, and the spectral sequence collapses at the E^p-term.

(c) By the Künneth formula

$$E^2_{**}(j) = HH_*(H_*(j; \mathbb{F}_2)) \cong HH_*((A/\!/A_2)_*) \otimes HH_*(\mathbb{F}_2 \oplus \Sigma^7 K_*)\,.$$

Here the first tensor factor was identified in the discussion of tmf in section 6. By the following lemma 7.14,

$$HH_q(\mathbb{F}_2 \oplus \Sigma^7 K_*) \cong [\Sigma^7 K_*^{\otimes q}]^{C_q} \oplus [\Sigma^7 K_*^{\otimes(q+1)}]_{C_{q+1}}\,,$$

and e.g. $HH_1(\mathbb{F}_2 \oplus \Sigma^7 K_*)$ is not flat as an $\mathbb{F}_2 \oplus \Sigma^7 K_*$-module. $\square$

Lemma 7.14. *Let k be a field and V a graded k-vector space. The Hochschild homology of the split square zero extension $k \oplus V$, with unit $(1, 0)$ and multiplication $(k_1, v_1) \cdot (k_2, v_2) = (k_1k_2, k_1v_2 + k_2v_1)$, is*

$$HH_q(k \oplus V) \cong [V^{\otimes q}]^{C_q} \oplus [V^{\otimes (q+1)}]_{C_{q+1}}$$

where $[V^{\otimes q}]^{C_q} \subset V^{\otimes q}$ denotes the invariants of the cyclic group C_q of order q acting by cyclic permutations on $V^{\otimes q}$, and $[V^{\otimes q}]_{C_q}$ denotes the coinvariants of this action. When $\dim_k V \geq 2$ the Hochschild homology $HH_*(k \oplus V)$ is not flat as a module over $k \oplus V$.

Proof. We compute $HH_*(k \oplus V)$ as the homology of the normalized Hochschild complex $NC_*(k \oplus V)$ with

$$NC_q(k \oplus V) = (k \otimes V^{\otimes q}) \oplus (V \otimes V^{\otimes q}) \cong V^{\otimes q} \oplus V^{\otimes (q+1)} .$$

Since V is a square zero ideal, the Hochschild boundary ∂ is the direct sum over $q \geq 1$ of the operators

$$1 + (-1)^q t_q \colon V^{\otimes q} \to V^{\otimes q} ,$$

i.e., $1 + (-1)^q t_q$ on the $V^{\otimes q}$-summand and zero on the $V^{\otimes (q+1)}$-summand, where $t_q(v_1 \otimes \cdots \otimes v_q) = (-1)^\epsilon v_q \otimes v_1 \otimes \cdots \otimes v_{q-1}$ with $\epsilon = |v_q|(|v_1| + \cdots + |v_{q-1}|)$. Let the generator $T \in C_q$ act on $V^\otimes$ as $(-1)^{q+1} t_q$, so ∂ is the sum of the operators $1 - T$ (and T^q acts as the identity). Clearly, then, the Hochschild homology is the direct sum over $q \geq 1$ of the kernels

$$\ker(1 - T) = [V^{\otimes q}]^{C_q}$$

in degree q, and the cokernels

$$\operatorname{cok}(1 - T) = [V^{\otimes q}]_{C_q}$$

in degree $(q - 1)$, plus the term $k = V^{\otimes 0}$ in degree 0.

For an example of the failure of flatness, let $V = k\{x, y\}$ with x, y in odd degrees and $q = 1$. Then $HH_1(k \oplus V) \cong V \oplus V^{\otimes 2}_{C_2} \cong k\{\sigma x, \sigma y, x\sigma x, x\sigma y \equiv y\sigma x, y\sigma y\}$ is not flat over $k \oplus V$. ☐

Theorem 7.15. *(a) For $p = 2$ there is an isomorphism*

$$H_*(THH(ju); \mathbb{F}_2) \cong H_*(ju; \mathbb{F}_2) \otimes E(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3) \otimes \Gamma(\sigma b)$$

of $H_*(ju; \mathbb{F}_2)$-Hopf algebras.

(b) For p odd there is an isomorphism

$$H_*(THH(ju); \mathbb{F}_p) \cong H_*(ju; \mathbb{F}_p) \otimes E(\sigma\tilde{\xi}_1^p, \sigma\tilde{\xi}_2) \otimes P(\sigma\tilde{\tau}_2) \otimes \Gamma(\sigma b)$$

of $H_*(ju; \mathbb{F}_p)$-Hopf algebras.

Proof. (a) In view of proposition 7.13(a) we must identify the possible algebra extensions between the E^∞-term

$$E^\infty_{**}(ju) \cong H_*(ju;\mathbb{F}_2) \otimes E(\sigma\bar\xi_1^4, \sigma\bar\xi_2^2, \sigma\bar\xi_k \mid k \ge 3) \otimes \Gamma(\sigma b)$$

and the abutment $H_*(THH(ju);\mathbb{F}_2)$. Here $H_*(ju;\mathbb{F}_2) \cong (A/\!/A_1)_* \otimes E(b)$ by proposition 7.12.

In $H_*(THH(ju);\mathbb{F}_2)$ we have $(\sigma b)^2 = Q^4(\sigma b) = \sigma Q^4(b) = 0$, $(\sigma\bar\xi_1^4)^2 = 0$, $(\sigma\bar\xi_2^2)^2 = 0$ and $(\sigma\bar\xi_k)^2 = \sigma\bar\xi_{k+1}$ for all $k \ge 3$, by proposition 5.9 and the statement about Dyer–Lashof operations in proposition 7.12(a). It remains to prove that we can find classes

$$\gamma_{2^k} \in H_{4\cdot 2^k}(THH(ju);\mathbb{F}_2)$$

that are represented by $\gamma_{2^k}(\sigma b)$ in $E^\infty_{**}(ju)$ and satisfy $\gamma_{2^k}^2 = 0$, for all $k \ge 0$. We have just seen that we can take $\gamma_1 = \sigma b$. So fix a number $k \ge 1$, and assume inductively that we have chosen classes γ_{2^m} for $0 \le m < k$ that are represented by $\gamma_{2^m}(\sigma b)$ and satisfy $\gamma_{2^m}^2 = 0$.

We shall prove below that a class γ_{2^k} representing $\gamma_{2^m}(\sigma b)$ can be chosen so that its square $\gamma_{2^k}^2$ is both an $H_*(ju;\mathbb{F}_2)$-coalgebra primitive and an A_*-comodule primitive. All such coalgebra- and comodule primitives are included among

$$E(b) \otimes \mathbb{F}_2\{\sigma\bar\xi_1^4, \sigma\bar\xi_2^2, \sigma\bar\xi_m \mid m \ge 3, \sigma b) \,.$$

When $k \ge 1$, the only such class in the degree of $\gamma_{2^k}^2$ is $\sigma\bar\xi_{k+3} = (\sigma\bar\xi_{k+2})^2$. So either $\gamma_{2^k}^2 = 0$ or $\gamma_{2^k}^2 = (\sigma\bar\xi_{k+2})^2$. In the latter case we change γ_{2^k} by subtracting $\sigma\bar\xi_{k+2}$, which does not alter the representative at the E^∞-term. Thereby we have achieved $\gamma_{2^k}^2 = 0$, which will complete the inductive step.

To show that $\gamma_{2^k}^2$ can be arranged to be a coalgebra- and comodule primitive, we make use of the maps of E^∞-terms and abutments induced by the commutative S-algebra homomorphism $\kappa\colon ju \to ku_{(2)}$. The target E^∞-term is

$$E^\infty_{**}(ku) \cong H_*(ku;\mathbb{F}_2) \otimes E(\sigma\bar\xi_1^2, \sigma\bar\xi_2^2, \sigma\bar\xi_k \mid k \ge 3)$$

with abutment

$$H_*(THH(ku);\mathbb{F}_2) \cong H_*(ku;\mathbb{F}_2) \otimes E(\sigma\bar\xi_1^2, \sigma\bar\xi_2^2) \otimes P(\sigma\bar\xi_3) \,.$$

Here $H_*(ku;\mathbb{F}_2) \cong (A/\!/E_1)_*$. Note that in degrees less than that of γ_{2^k}, κ maps $H_*(THH(ju);\mathbb{F}_2)$ modulo classes that square to zero, which by the inductive hypothesis is $(A/\!/A_1)_* \otimes P(\sigma\bar\xi_3)$, injectively into $H_*(THH(ku);\mathbb{F}_2)$ modulo classes that square to zero, which is $(A/\!/E_1)_* \otimes P(\sigma\bar\xi_3)$. We shall refer to this property as the "near-injectivity of κ".

So choose any class γ_{2^k} in $H_*(THH(ju);\mathbb{F}_2)$ that is represented by $\gamma_{2^k}(\sigma b)$ in $E^\infty_{**}(ju)$. We shall arrange that its image $\kappa\gamma_{2^k}$ squares to zero in $H_*(THH(ku);\mathbb{F}_2)$. If not, we can write

$$\kappa\gamma_{2^k} \equiv c \cdot (\sigma\bar\xi_3)^\ell$$

with $c \in (A/\!/E_1)_*$ not equal to zero, modulo classes in $H_*(THH(ku);\mathbb{F}_2)$ that square to zero, and modulo similar terms with c of lower degree (or equivalently, with higher exponent ℓ). So $c \cdot (\sigma\bar{\xi}_3)^\ell$ is the "leading term" in $H_*(THH(ku);\mathbb{F}_2)$ modulo classes that square to zero.

We divide into three cases. First, if $\ell = 0$ and $\kappa\gamma_{2^k} \equiv c$ with $c \in (A/\!/E_1)_*$, we can apply the Hopf algebra counit $\epsilon\colon THH(R) \to R$ to see that $c = \epsilon(\kappa\gamma_{2^k}) = \kappa(\epsilon\gamma_{2^k})$ is in the image of $\kappa\colon H_*(ju;\mathbb{F}_2) \to H_*(ku;\mathbb{F}_2)$, so that in fact $c \in (A/\!/E_1)_*$. We can then replace γ_{2^k} by $\gamma_{2^k} - c$ without altering its representative in E^∞, and thus eliminate the case $\ell = 0$.

Second, if $\ell = 1$ and $\kappa\gamma_{2^k} \equiv c \cdot \sigma\bar{\xi}_3$ with $c \in (A/\!/E_1)_*$, we need to argue that $c \in (A/\!/A_1)_*$. If not, we can write $c = \bar{\xi}_1^2 \cdot d$ with $d \in (A/\!/A_1)_*$. Here $\psi(c) \equiv c \otimes 1 + d \otimes \bar{\xi}_1^2$ modulo terms with first tensor factor in degree less that $|d|$. Note from the structure of $H_*(THH(ku);\mathbb{F}_2)$ that $\kappa\gamma_{2^k} \equiv c \cdot \sigma\bar{\xi}_3$ modulo classes that square to zero, and similar terms $e \cdot (\sigma\bar{\xi}_3)^\ell$ with $\ell \geq 2$, so $|e| < |d|$. So $\nu\kappa\gamma_{2^k} \equiv c \otimes \sigma\bar{\xi}_3 + d \otimes \bar{\xi}_1^2\sigma\bar{\xi}_3$ modulo classes that square to zero, and terms with first factor in lower degree. This must be the image of $\nu\gamma_{2^k}$ under $1 \otimes \kappa$, so

$$\nu\gamma_{2^k} \equiv c \otimes \sigma\bar{\xi}_3 + d \otimes x$$

for some class $x \in H_*(THH(ju);\mathbb{F}_2)$ with $\kappa(x) \equiv \bar{\xi}_1^2\sigma\bar{\xi}_3$. But there exists no such class x, since $H_*(THH(ju);\mathbb{F}_2)$ modulo classes that square to zero is trivial in degree 10. (This is clear by inspection of the Bökstedt spectral sequence, where the group that could cause a problem, $E^2_{2,8}(ju)$, is in fact zero.) This contradiction shows that $c \in (A/\!/A_1)_*$, and we can alter γ_{2^k} by $c \cdot \sigma\bar{\xi}_3$ without altering the representative at E^∞, and thus eliminate the case $\ell = 1$.

Third, for the remaining cases $\ell \geq 2$ we consider the coalgebra coproduct. We find

$$\psi(\kappa\gamma_{2^k}) \equiv \sum_{i+j=\ell} c \cdot (\sigma\bar{\xi}_3)^i \otimes (\sigma\bar{\xi}_3)^j$$

in $H_*(THH(ku);\mathbb{F}_2) \otimes_{H_*(ku;\mathbb{F}_2)} H_*(THH(ku);\mathbb{F}_2)$, modulo classes that square to zero and similar terms with c of lower degree. Since $\ell \geq 2$ this sum includes some terms $c \cdot (\sigma\bar{\xi}_3)^i \otimes (\sigma\bar{\xi}_3)^j$ with both i and j positive, so that $c \cdot (\sigma\bar{\xi}_3)^i$ and $(\sigma\bar{\xi}_3)^j$ are both in degree less than that of γ_{2^k}. This is the image under $\kappa \otimes \kappa$ of $\psi(\gamma_{2^k})$ in $H_*(THH(ju);\mathbb{F}_2) \otimes_{H_*(ju;\mathbb{F}_2)} H_*(THH(ju);\mathbb{F}_2)$, so it follows by the near-injectivity of κ that c in fact lies in $(A/\!/A_1)_* \subset (A/\!/E_1)_*$.

Then we can change the chosen γ_{2^k} in $H_*(THH(ju);\mathbb{F}_2)$ by subtracting $c \cdot (\sigma\bar{\xi}_3)^\ell$ from it, and thus remove the "leading" term $c \cdot (\sigma\bar{\xi}_3)^\ell$ from $\kappa\gamma_{2^k}$. By repeating this process we can arrange that γ_{2^k} has be chosen so that $\kappa\gamma_{2^k}$ is zero modulo classes that square to zero, i.e., that $\kappa\gamma_{2^k}^2 = 0$.

Then $\psi(\gamma_{2^k}) \equiv \gamma_{2^k} \otimes 1 + 1 \otimes \gamma_{2^k}$ modulo classes that square to zero. For $\psi(\kappa\gamma_{2^k}^2) = 0$ so any other terms in $\psi(\gamma_{2^k})$ must map under κ to classes that square to zero, hence square to zero themselves by the near-injectivity of κ. Hence $\gamma_{2^k}^2$ is a coalgebra primitive.

Next consider the A_*-comodule coaction. If $\nu(\gamma_{2^k}) \equiv 1 \otimes \gamma_{2^k}$ modulo classes that square to zero, then $\nu(\gamma_{2^k}^2) = 1 \otimes \gamma_{2^k}^2$ and $\gamma_{2^k}^2$ is an A_*-comodule primitive, as desired. Otherwise, we can write

$$\nu(\gamma_{2^k}) \equiv a \otimes (\sigma\bar{\xi}_3)^\ell$$

with $a \in A_*$ in positive degree, modulo classes in $A_* \otimes H_*(THH(ju); \mathbb{F}_2)$ that square to zero, and modulo similar terms with a of lower degree. Then $\nu(\gamma_{2^k}^2) \equiv a^2 \otimes (\sigma\bar{\xi}_3)^{2\ell}$ modulo terms with a of lower degree. Applying κ yields $a^2 \otimes (\sigma\bar{\xi}_3)^{2\ell} = 0$, since $\kappa\gamma_{2^k}^2 = 0$, so $a^2 = 0$. This is impossible for $a \neq 0$, so we conclude that $\gamma_{2^k}^2$ is indeed an A_*-comodule primitive.

This completes the proof.

(b) Also in the odd primary case we must identify the algebra extensions between the E^∞-term from proposition 7.13(b)

$$E^\infty_{**}(ju) = H_*(ju; \mathbb{F}_p) \otimes E(\sigma\tilde{\xi}_1^p, \sigma\tilde{\xi}_2, \sigma\tilde{\tau}_k \mid k \geq 2) \otimes \Gamma(\sigma b)$$

and the abutment $H_*(THH(ju); \mathbb{F}_p)$. In the latter we have $(\sigma b)^p = Q^{pq/2}(\sigma b) = \sigma Q^{pq/2}(b) = 0$ and $(\sigma\tilde{\tau}_k)^p = Q^{p^k}(\sigma\tilde{\tau}_k) = \sigma Q^{p^k}(\tilde{\tau}_k) = \sigma\tilde{\tau}_{k+1}$, by propositions 5.9 and 7.12(b). The classes $\sigma\tilde{\xi}_1^p$ and $\sigma\tilde{\xi}_2$ are in odd degree, and therefore have square zero, since p is odd.

It remains to prove that we can find classes $\gamma_{p^k} \in H_{p^{k+1}q}(THH(ju); \mathbb{F}_p)$ that are represented by $\gamma_{p^k}(\sigma b)$ in $E^\infty_{**}(ju)$ and satisfy $\gamma_{p^k}^p = 0$, for all $k \geq 0$. We have just verified this for $k = 0$.

The remaining inductive proof follows exactly the same strategy as in the $p = 2$ case. Instead of working modulo classes that square to zero we work modulo classes with p-th power equal to zero. The S-algebra map $\kappa\colon ju \to \ell$ induces an algebra homomorphism in homology that has the required "near–injectivity" property, etc. In the tricky case when $\kappa\gamma_{p^k} \equiv c \cdot \sigma\bar{\tau}_2$, modulo classes in $H_*(THH(\ell); \mathbb{F}_p)$ that have trivial p-th power, and modulo similar terms with c of lower degree, we have $c \in (A//E_1)_*$ and must argue that in fact $c \in (A//A_1)_*$. Writing $c = \bar{\xi}_1^f \cdot d$, with $d \in (A//A_1)_*$ and $0 \leq f < p$ we assume $0 < f < p$ and reach a contradiction. The coaction $\nu\kappa\gamma_{p^k}$ contains a term $f\bar{\xi}_1^{f-1}d \otimes \bar{\xi}_1\sigma\bar{\tau}_2$, and we must check that there is no class $x \in H_*(THH(ju); \mathbb{F}_p)$ with $\kappa x = \bar{\xi}_1\sigma\bar{\tau}_2$. This is again clear by the Bökstedt spectral sequence. The rest of the proof can safely be omitted. □

8. Topological K-theory revisited

We now wish to pass from the homology of the spectra $THH(ku)$ and $THH(ko)$ to their homotopy. The first case is the analogue for $p = 2$ of the discussion in [MS93, §§5–7], where McClure and Staffeldt compute the mod p homotopy groups $\pi_*(THH(\ell); \mathbb{Z}/p)$ of the Adams summand $\ell \subset ku_{(p)}$ for odd primes p.

The idea is to compute homotopy groups using the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_p, H_*(X; \mathbb{F}_p)) \Longrightarrow \pi_{t-s}(X)^\wedge_p\,,$$

which is strongly convergent for bounded below spectra X of $\mathbb{Z}_p$-finite type. This is difficult for $X = THH(ku)$ or $THH(ko)$ at $p = 2$, just as for $THH(\ell)$ at p odd, but becomes manageable after introducing suitable finite coefficients, e.g. after smashing $THH(ku)$ with the mod 2 Moore spectrum $M = C_2$, or smashing $THH(ko)$ with the Mahowald spectrum $Y = C_2 \wedge C_\eta$ [Ma82]. Here C_f denotes the mapping cone of a map f. Neither of these finite CW spectra are ring spectra, so an extra argument is needed to have products on $THH(ku) \wedge M$ and $THH(ko) \wedge Y$. We shall manage with the following weak version of a ring spectrum:

Definition 8.1. A μ-spectrum is a spectrum R with a unit $\eta\colon S \to R$ and multiplication $\mu\colon R \wedge R \to R$ that is left and right unital, but not necessarily associative or commutative.

The following result is similar to [Ok79, 1.5], but slightly easier.

Lemma 8.2. *Let R be a μ-spectrum and let $S^k \xrightarrow{f} S^0 \xrightarrow{i} C_f \xrightarrow{\pi} S^{k+1}$ be a cofiber sequence such that $id_R \wedge id_{C_f} \wedge f\colon R \wedge \Sigma^k C_f \to R \wedge C_f$ is null-homotopic. Then there exists a multiplication $\mu\colon (R \wedge C_f) \wedge (R \wedge C_f) \to (R \wedge C_f)$ that makes $R \wedge C_f$ a μ-spectrum and $id_R \wedge i\colon R \to R \wedge C_f$ a map of μ-spectra.*

Proof. A choice of null-homotopy provides a splitting $m\colon R \wedge C_f \wedge C_f \to R \wedge C_f$ in the cofiber sequence

$$R \wedge \Sigma^k C_f \xrightarrow{id \wedge id \wedge f} R \wedge C_f \xrightarrow{id \wedge id \wedge i} R \wedge C_f \wedge C_f \xrightarrow{id \wedge id \wedge \pi} R \wedge \Sigma^{k+1} C_f$$

which satisfies $m(id \wedge id \wedge i) \simeq id$ (right unitality). The difference $id - m(id \wedge i \wedge id)\colon R \wedge C_f \to R \wedge C_f$ restricts trivially to R, so extends over a map $\phi\colon R \wedge S^{k+1} \to R \wedge C_f$. Using that the Spanier–Whitehead dual of C_f is $\Sigma^{-(k+1)} C_f$, together with the null-homotopy hypothesis again, shows that ϕ extends further to a map $\psi\colon R \wedge \Sigma^{k+1} C_f \to R \wedge C_f$ such that we can alter m by $\psi \circ (id \wedge id \wedge \pi)$ so as to also make $m(id \wedge i \wedge id) \simeq id$ (left unitality), without destroying the right unitality. Then the required multiplication is the composite

$$(R \wedge C_f) \wedge (R \wedge C_f) \xrightarrow{id \wedge \gamma \wedge id} R \wedge R \wedge C_f \wedge C_f \xrightarrow{\mu \wedge id \wedge id} R \wedge C_f \wedge C_f \xrightarrow{m} (R \wedge C_f)\,.$$

□

In the case $f = 2\colon S^0 \to S^0$, the Moore spectrum $M = C_2$ has cohomology $H^*(M; \mathbb{F}_2) = E(Sq^1)$, which equals $E_1 /\!/ E(Q_1)$ as an E_1-module. By the Cartan formula Sq^2 acts nontrivially on $H^*(M \wedge M; \mathbb{F}_2)$, and therefore M does not split off from $M \wedge M$. So M is not a μ-spectrum, and the map $2\colon M \to M$ is not null-homotopic. It must therefore factor as the composite

$$M \xrightarrow{\pi} S^1 \xrightarrow{\eta} S^0 \xrightarrow{i} M\,. \tag{8.3}$$

See e.g. [AT65, 1.1].

Similarly, in the case $f = \eta\colon S^1 \to S^0$ the mapping cone C_η has cohomology $H^*(C_\eta; \mathbb{F}_2) = E(Sq^2)$. By the Cartan formula Sq^4 acts nontrivially on $H^*(C_\eta \wedge C_\eta; \mathbb{F}_2)$, and therefore C_η does not split off from $C_\eta \wedge C_\eta$. In particular, the map $\eta\colon \Sigma C_\eta \to C_\eta$ is not null-homotopic, and must factor as the composite

$$\Sigma C_\eta \xrightarrow{\pi} S^3 \xrightarrow{\nu} S^0 \xrightarrow{i} C_\eta\,. \tag{8.4}$$

Here $\nu \in \pi_3(S)$ is the Hopf invariant one class. The Mahowald spectrum $Y = C_2 \wedge C_\eta$ has cohomology $H^*(Y; \mathbb{F}_2) = E(Sq^1, Sq^2)$, which equals $A_1 /\!/ E(Q_1)$ as an A_1-module.

Lemma 8.5. *(a) $THH(ku) \wedge M$ is a μ-spectrum.*

(b) $THH(ko) \wedge C_\eta$ and $THH(ko) \wedge Y$ are μ-spectra.

Proof. (a) Let $T = THH(ku)$. By (8.3) the map $1 \wedge 2 \colon T \wedge M \to T \wedge M$ factors through $1 \wedge \eta \colon \Sigma T \to T$, which in turn factors as

$$\Sigma T \xrightarrow{1\wedge\eta} T \wedge ku \to T\,,$$

since $T = THH(ku)$ is a ku-module spectrum. But $\eta \in \pi_1(S)$ maps to zero in $\pi_1(ku)$, so this map is null-homotopic, and lemma 8.2 applies.

(b) Let $T = THH(ko)$ and $R = THH(ko) \wedge C_\eta$. By (8.4) the map $1\wedge\eta \colon \Sigma R \to R$ factors through $1 \wedge \nu \colon \Sigma^3 T \to T$, which in turn factors as

$$\Sigma^3 T \xrightarrow{1\wedge\nu} T \wedge ko \to T\,,$$

since $T = THH(ko)$ is a ko-module spectrum. But $\nu \in \pi_3(S)$ maps to zero in $\pi_3(ko)$, so $1 \wedge \eta \colon \Sigma R \to R$ is null-homotopic, and lemma 8.2 applies again to prove that R is a μ-spectrum under T.

Using once more that $1\wedge\eta \colon \Sigma R \to R$ is null-homotopic we get that $1\wedge 2 \colon R \wedge M \to R \wedge M$ is null-homotopic by (8.3), and so $R \wedge M = THH(ko) \wedge Y$ is a μ-spectrum under R. □

Lemma 8.6. *Let $B \subset A$ be a sub Hopf algebra and let N be an A_*-comodule algebra. Then there is an isomorphism of A_*-comodule algebras*

$$(A/\!/B)_* \otimes N \cong A_* \,\square_{B_*}\, N\,.$$

Here B_* is the quotient Hopf algebra of A_* dual to B, and $(A/\!/B)_* = A_* \,\square_{B_*}\, \mathbb{F}_p$ is dual to $A \otimes_B \mathbb{F}_p$.

Proof. This is analogous to the usual G-homeomorphism $G/H \times X \cong G \times_H X$ for a G-space X and subgroup $H \subset G$. Let $i \colon (A/\!/B)_* \to A_*$ be the inclusion and $\nu \colon N \to A_* \otimes N$ the coaction. The composite homomorphism

$$(A/\!/B)_* \otimes N \xrightarrow{i\otimes\nu} A_* \otimes A_* \otimes N \xrightarrow{\phi\otimes 1} A_* \otimes N$$

equalizes the two maps to $A_* \otimes B_* \otimes N$, and hence factors uniquely through $A_* \,\square_{B_*}\, N$. An explicit inverse can be constructed using the Hopf algebra conjugation χ on A_*. □

Recall the n-th connective Morava K-theory spectrum $k(n)$, with homotopy $\pi_* k(n) = \mathbb{F}_p[v_n]$ where $|v_n| = 2(p^n - 1)$, and cohomology $H^*(k(n);\mathbb{F}_p) \cong A/\!/E(Q_n)$ [BM72]. Dually, $H_*(k(n);\mathbb{F}_p) \cong (A/\!/E(Q_n))_* \subset A_*$. In particular, for $n = 1$ and $p = 2$ we have $k(1) \simeq ku \wedge M$ with $H_*(k(1);\mathbb{F}_2) \cong (A/\!/E(Q_1))_* \cong P(\bar\xi_1, \bar\xi_2^2, \bar\xi_k \mid k \geq 3)$. The n-th periodic Morava K-theory spectrum $K(n)$ is the telescope $v_n^{-1} k(n)$ of the iterated maps $v_n \colon k(n) \to \Sigma^{2(1-p^n)} k(n)$, with homotopy $\pi_* K(n) = \mathbb{F}_p[v_n, v_n^{-1}]$. These are (non-commutative) S-algebras for all n [Ro89].

Proposition 8.7. *(a) There are A_*-comodule algebra isomorphisms*

$$\begin{aligned} H_*(THH(ku) \wedge M; \mathbb{F}_2) &\cong (A/\!/E(Q_1))_* \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3) \\ &\cong H_*(k(1); \mathbb{F}_2) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu) \,. \end{aligned}$$

Here $\nu(\sigma\bar{\xi}_1^2) = 1 \otimes \sigma\bar{\xi}_1^2$, $\nu(\sigma\bar{\xi}_2^2) = 1 \otimes \sigma\bar{\xi}_2^2$, and $\nu(\sigma\bar{\xi}_3) = 1 \otimes \sigma\bar{\xi}_3 + \bar{\xi}_1 \otimes \sigma\bar{\xi}_2^2$.

The classes $\lambda_1 = \sigma\bar{\xi}_1^2$, $\lambda_2 = \sigma\bar{\xi}_2^2$ and $\mu = \sigma\bar{\xi}_3 + \bar{\xi}_1 \cdot \sigma\bar{\xi}_2^2$ (in degrees 3, 7 and 8, respectively) are A_*-comodule primitives.

(b) There are A_*-comodule algebra isomorphisms

$$\begin{aligned} H_*(THH(ko) \wedge Y; \mathbb{F}_2) &\cong (A/\!/E(Q_1))_* \otimes E(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3) \\ &\cong H_*(k(1); \mathbb{F}_2) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu) \,. \end{aligned}$$

Here $\nu(\sigma\bar{\xi}_1^4) = 1 \otimes \sigma\bar{\xi}_1^4$, $\nu(\sigma\bar{\xi}_2^2) = 1 \otimes \sigma\bar{\xi}_2^2 + \bar{\xi}_1^2 \otimes \sigma\bar{\xi}_1^4$ and $\nu(\sigma\bar{\xi}_3) = 1 \otimes \sigma\bar{\xi}_3 + \bar{\xi}_1 \otimes \sigma\bar{\xi}_2^2 + \bar{\xi}_2 \otimes \sigma\bar{\xi}_1^4$.

The exterior classes $\lambda_1 = \sigma\bar{\xi}_1^4$ and $\lambda_2 = \sigma\bar{\xi}_2^2 + \bar{\xi}_1^2 \cdot \sigma\bar{\xi}_1^4$ are A_*-comodule primitives, while $\mu = \sigma\bar{\xi}_3 + \bar{\xi}_1 \cdot \sigma\bar{\xi}_2^2$ has

$$\nu(\mu) = 1 \otimes \mu + \bar{\xi}_1^2 \otimes \bar{\xi}_1 \cdot \lambda_1 + (\bar{\xi}_2 + \bar{\xi}_1^3) \otimes \lambda_1 \,.$$

The squared class $\mu^2 = (\sigma\bar{\xi}_3)^2$ is A_*-comodule primitive.

In each case, the homology algebra on the left hand side has the unit and product induced by the μ-spectrum structure from lemma 8.5. This product is in fact associative (and graded commutative), in view of the exhibited additive and multiplicative isomorphism with the (associative) algebra on the right hand side.

Proof. (a) By corollary 5.13(a) there is an A_*-comodule algebra isomorphism

$$H_*(THH(ku) \wedge M; \mathbb{F}_2) \cong (A/\!/E_1)_* \otimes (E_1/\!/E(Q_1))_* \otimes E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3)$$

with the diagonal A_*-comodule structure on the first two tensor factors, and the claimed coaction on the remaining generators.

Since $H_*(M; \mathbb{F}_2) \cong (E_1/\!/E(Q_1))_*$ is in fact an A_*-comodule algebra, there is an A_*-comodule algebra isomorphism

$$(A/\!/E_1)_* \otimes (E_1/\!/E(Q_1))_* \cong A_* \,\square_{E_{1*}}\, (E_1/\!/E(Q_1))_* \cong (A/\!/E(Q_1))_*$$

by lemma 8.6. We are free to replace the polynomial generator $\sigma\bar{\xi}_3$ by the primitive class μ, since their difference $\bar{\xi}_1 \cdot \sigma\bar{\xi}_2^2$ has square zero.

(b) By theorem 6.2(a) there is an A_*-comodule algebra isomorphism

$$H_*(THH(ko) \wedge Y; \mathbb{F}_2) \cong (A/\!/A_1)_* \otimes (A_1/\!/E(Q_1))_* \otimes E(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2) \otimes P(\sigma\bar{\xi}_3) \,,$$

with the claimed coaction on the exterior and polynomial generators. Again, by lemma 8.6 there is an isomorphism of A_*-comodule algebras

$$(A/\!/A_1)_* \otimes (A_1/\!/E(Q_1))_* \cong A_* \,\square_{A_{1*}}\, (A_1/\!/E(Q_1))_* \cong (A/\!/E(Q_1))_* \,.$$

The classes λ_1, λ_2 and μ are defined as in the statement of the proposition, and their coactions are obtained by direct calculation. □

Lemma 8.8. *(a) The E_2-term of the Adams spectral sequence for $\pi_*(THH(ku) \wedge M)$ is*

$$E_2^{**} \cong P(v_1) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu)$$

with λ_1, λ_2, μ and $v_1 = [\xi_2]$ in bidegrees $(0,5)$, $(0,7)$, $(0,8)$ and $(1,3)$, respectively.

(b) The E_2-term of the Adams spectral sequence for $\pi_*(THH(ko) \wedge Y)$ is

$$E_2^{**} \cong \big(P(v_1) \otimes E(\lambda_2, \lambda_3) \oplus E(\lambda_2)\{\lambda_1\}\big) \otimes P(\mu^2)$$

with λ_1, λ_2, $\lambda_3 = \lambda_1\mu$, μ^2 and $v_1 = [\xi_2]$ in bidegrees $(0,5)$, $(0,7)$, $(0,13)$, $(0,16)$ and $(1,3)$, respectively. It is the homology of the algebra

$$P(v_1) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu)$$

with respect to the differential $d(\mu) = v_1\lambda_1$, with cycles λ_1, λ_2 and v_1.

Proof. (a) By change of rings, the E_2-term of the Adams spectral sequence for $X = THH(ku) \wedge M$ is

$$\begin{aligned}
E_2^{**} &= \mathrm{Ext}_{A_*}^{**}(\mathbb{F}_2, (A/\!/E(Q_1))_* \otimes E(\lambda_1, \lambda_2) \otimes P(\mu)) \\
&\cong \mathrm{Ext}_{E(Q_1)_*}^{**}(\mathbb{F}_2, E(\lambda_1, \lambda_2) \otimes P(\mu)) \\
&\cong P(v_1) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu)
\end{aligned}$$

as a graded algebra. Here $E(Q_1)_* = E(\xi_2)$ and $\mathrm{Ext}_{E(\xi_2)}^{**}(\mathbb{F}_2, \mathbb{F}_2) = P(v_1)$ with $v_1 = [\xi_2]$ in the cobar complex.

(b) The E_2-term of the Adams spectral sequence for $X = THH(ko) \wedge Y$ is

$$\begin{aligned}
E_2^{**} &= \mathrm{Ext}_{A_*}^{**}(\mathbb{F}_2, (A/\!/E(Q_1))_* \otimes E(\lambda_1, \lambda_2) \otimes P(\mu)) \\
&\cong \mathrm{Ext}_{E(Q_1)_*}^{**}(\mathbb{F}_2, E(\lambda_1, \lambda_2) \otimes P(\mu)) \\
&\cong \mathrm{Ext}_{E(Q_1)_*}^{**}(\mathbb{F}_2, \mathbb{F}_2\{1, \lambda_1, \lambda_2, \mu, \lambda_1\lambda_2, \lambda_1\mu, \lambda_2\mu, \lambda_1\lambda_2\mu\} \otimes P(\mu^2))\,.
\end{aligned}$$

Here

$$\mathbb{F}_2\{1, \lambda_1, \lambda_2, \mu, \lambda_1\lambda_2, \lambda_1\mu, \lambda_2\mu, \lambda_1\lambda_2\mu\} \cong \mathbb{F}_2\{1, \lambda_2, \lambda_1\mu, \lambda_1\lambda_2\mu\} \oplus E(Q_1)_*\{\lambda_1, \lambda_1\lambda_2\}$$

as $E(Q_1)_*$-comodules. For λ_1, λ_2 and μ^2 are A_*-comodule primitives, while $\nu(\mu)$ maps to $1 \otimes \mu + \xi_2 \otimes \lambda_1$ in $E(Q_1)_* \otimes H_*(THH(ko) \wedge Y; \mathbb{F}_2)$. So

$$E_2^{**} = \big(P(v_1) \otimes E(\lambda_2, \lambda_1\mu) \oplus E(\lambda_2)\{\lambda_1\}\big) \otimes P(\mu^2)\,.$$

We let $\lambda_3 = \lambda_1\mu$ to obtain the claimed formula. □

To determine the differentials in the Adams spectral sequence for $\pi_*(THH(ku) \wedge M)$ or $\pi_*(THH(ko) \wedge Y)$ we first compute the v_1-periodic homotopy. This is in turn easy to derive from the $K(1)$-homology. Recall that $BP_*BP \cong BP_*[t_k \mid k \geq 1]$ with $|t_k| = 2(2^k - 1)$ and $K(1)_* = \mathbb{F}_2[v_1, v_1^{-1}]$ with $|v_1| = 2$.

Lemma 8.9. *For $p = 2$ there are isomorphisms of $K(1)_*$-algebras*

$$K(1)_*(ku) \cong K(1)_*[t_k \mid k \geq 1]/(v_1 t_k^2 = v_1^{2^k} t_k) \cong K(1)_*[u_k \mid k \geq 1]/(u_k^2 = u_k)$$

and

$$K(1)_*(ko) \cong K(1)_*[t_k \mid k \geq 2]/(v_1 t_k^2 = v_1^{2^k} t_k) \cong K(1)_*[u_k \mid k \geq 2]/(u_k^2 = u_k)\,.$$

Proof. We first follow the proof of [MS93, 5.3]. We have $K(1)_*(BP) \cong K(1)_* \otimes_{BP_*} BP_*(BP) \cong K(1)_*[t_k \mid k \geq 1]$, where $|t_k| = 2(2^k - 1)$. The spectrum $ku_{(2)} = BP\langle 1\rangle$ can be constructed from BP by Baas–Sullivan cofiber sequences killing the classes v_{k+1} for $k \geq 1$, which map to

$$\eta_R(v_{k+1}) \equiv v_1 t_k^2 + v_1^{2^k} t_k \mod (\eta_R(v_2), \dots, \eta_R(v_k))$$

by [Ra04, 6.1.13]. So each $\eta_R(v_{k+1}) \in K(1)_*(BP)$ is not a zero divisor mod $(\eta_R(v_2), \dots, \eta_R(v_k))$, and

$$K(1)_*(ku) \cong K(1)_*[t_k \mid k \geq 1]/(v_1 t_k^2 = v_1^{2^k} t_k)\,.$$

Substituting $u_k = v_1^{1-2^k} t_k$, the relations become $u_k^2 = u_k$ for each $k \geq 1$.

The cofiber sequence

$$\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku \xrightarrow{\partial} \Sigma^2 ko$$

induces a short exact sequence

$$0 \to K(1)_*(ko) \xrightarrow{c_*} K(1)_*(ku) \xrightarrow{\partial_*} \Sigma^2 K(1)_*(ko) \to 0$$

since multiplication by η is zero in $K(1)$-homology. The connecting map ∂ right multiplies by Sq^2 in mod 2 cohomology, so right "comultiplies" with the dual class $\bar{\xi}_1^2$ in mod 2 homology, which corresponds to t_1 in BP_*BP [Za72, p.488]. From the coproduct formula for $\Delta(t_j)$ [Ra04, A2.1.27(e)] it follows that $\partial_*(t_1) = \Sigma^2(1)$ while $\partial_*(t_k) = 0$ for $k \geq 2$. Hence we can identify $K(1)_*(ko)$ with the claimed subalgebra of $K(1)_*(ku)$, via the $K(1)_*$-algebra homomorphism c_*. □

Lemma 8.10. *The unit maps $R \to THH(R)$ induce isomorphisms*

$$K(1)_*(ku) \xrightarrow{\cong} K(1)_* THH(ku)$$

and

$$K(1)_*(ko) \xrightarrow{\cong} K(1)_* THH(ko)\,.$$

Proof. The proof of [MS93, 5.3] continues as follows. There is a $K(n)$-based Bökstedt spectral sequence

$$E^2_{**} = HH_*^{K(n)_*}(K(n)_*(R)) \Longrightarrow K(n)_* THH(R)$$

for every S-algebra R, derived like the one in (4.1), but by applying $K(n)$-homology to the skeleton filtration of $THH(R)$. The identification of the E^2-term uses the Künneth formula for Morava K-theory. When $K(n)_*(R) \cong K(n)_* \otimes_{\mathbb{F}_p} K(n)_0(R)$ is concentrated in degrees $* \equiv 0 \mod |v_n|$, we can rewrite the E^2-term as

$$E^2_{**} \cong K(n)_* \otimes_{\mathbb{F}_p} HH^{\mathbb{F}_p}_*(K(n)_0(R)) \,.$$

This is the case for $n = 1$, $p = 2$ and $R = ku$, when $K(1)_0(ku)$ is the colimit over m of the algebras $\mathbb{F}_2[u_k \mid 1 \le k \le m]/(u_k^2 = u_k) \cong \prod_{i=1}^{2^m} \mathbb{F}_2$. Then for each m the unit map $\prod_{i=1}^{2^m} \mathbb{F}_2 \to HH^{\mathbb{F}_2}_*(\prod_{i=1}^{2^m} \mathbb{F}_2)$ is an isomorphism, so by passage to the colimit the unit map

$$K(1)_0(ku) \to HH^{\mathbb{F}_2}_*(K(1)_0(ku))$$

is an isomorphism. Hence the $K(1)$-based Bökstedt spectral sequence collapses at the edge $s = 0$, and the unit map $ku \to THH(ku)$ induces the asserted isomorphism.

Likewise, $K(1)_0(ko)$ is the colimit of the algebras $\mathbb{F}_2[u_k \mid 2 \le k \le m]/(u_k^2 = u_k) \cong \prod_{i=1}^{2^{m-1}} \mathbb{F}_2$, and the same argument shows that the unit map $ko \to THH(ko)$ induces an isomorphism in $K(1)$-homology. $\square$

The mod 2 Moore spectrum M admits a degree 8 self-map $v_1^4 \colon M \to \Sigma^{-8} M$ that induces multiplication by v_1^4 in $ku_*(M) = k(1)_* = P(v_1)$. Smashing with C_η yields a self-map $v_1^4 \colon Y \to \Sigma^{-8} Y$, which admits a fourth root $v_1 \colon Y \to \Sigma^{-2} Y$ (up to nilpotent maps). It induces multiplication by v_1 in $ko_*(Y) = k(1)_* = P(v_1)$. See [DM82, 1.2].

Lemma 8.11. *Let X be a spectrum such that $K(1)_*(X) = 0$. Then $v_1^{-1}\pi_*(X \wedge Y) = 0$ and $v_1^{-4}\pi_*(X \wedge M) = 0$.*

Proof. Recall that j is the homotopy fiber of the map $\psi^3 - 1 \colon ko_{(2)} \to bspin_{(2)}$. The unit map $e \colon S \to j$ induces an equivalence of mapping telescopes

$$v_1^{-1}(S \wedge Y) \xrightarrow{\simeq} v_1^{-1}(j \wedge Y) \,.$$

See e.g. the case $n = 0$ of [M82, 1.4]. Here $v_1^{-1}(ko_{(2)} \wedge Y) \simeq v_1^{-1} k(1) = K(1)$ and likewise $v_1^{-1}(bspin_{(2)} \wedge Y) \simeq K(1)$, so there is a cofiber sequence of spectra

$$v_1^{-1} Y \to K(1) \xrightarrow{\psi} K(1) \,.$$

Furthermore, $v_1^{-1} Y \simeq v_1^{-4} Y$ sits in a cofiber sequence

$$v_1^{-4} \Sigma M \xrightarrow{\eta} v_1^{-4} M \to v_1^{-4} Y$$

where η is nilpotent ($\eta^4 = 0$).

From the first cofiber sequence it follows that if $K(1)_*(X) = 0$, then $\pi_*(X \wedge v_1^{-1} Y) = v_1^{-1}\pi_*(X \wedge Y) = 0$. From the second cofiber sequence it then follows that multiplication by η is an isomorphism on $\pi_*(X \wedge v_1^{-4} M) = v_1^{-4}\pi_*(X \wedge M)$. Since η is nilpotent this implies that $v_1^{-4}\pi_*(X \wedge M) = 0$. $\square$

Corollary 8.12. *The unit maps induce isomorphisms*

$$K(1)_* = v_1^{-4}\pi_*(ku \wedge M) \xrightarrow{\cong} v_1^{-4}\pi_*(THH(ku) \wedge M)$$

and

$$K(1)_* = v_1^{-1}\pi_*(ko \wedge Y) \xrightarrow{\cong} v_1^{-1}\pi_*(THH(ko) \wedge Y)\,.$$

Proof. Apply lemmas 8.10 and 8.11 to the cofiber of the unit map $R \to THH(R)$, for $R = ku$ and $R = ko$, respectively. □

Theorem 8.13. *Consider the Adams spectral sequence*

$$E_2^{**} = P(v_1) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu)$$

for $\pi_*(THH(ku) \wedge M)$, with $\lambda_1 = \sigma\bar\xi_1^2$, $\lambda_2 = \sigma\bar\xi_2^2$, $\mu = \sigma\bar\xi_3 + \bar\xi_1 \cdot \sigma\bar\xi_2^2$ and $v_1 = [\xi_2]$ in bidegrees $(s,t) = (0,3)$, $(0,7)$, $(0,8)$ and $(1,3)$, respectively. Recursively define

$$\lambda_n = \lambda_{n-2}\mu^{2^{n-3}}$$

for $n \geq 3$. Likewise define $r(1) = 2$, $r(2) = 4$, $r(n) = 2^n + r(n-2)$, $s(1) = 3$, $s(2) = 7$ and $s(n) = 2^n + s(n-2)$, for $n \geq 3$. So λ_n has bidegree $(0, s(n))$ and $2r(n) + s(n) = 2^{n+2} - 1$.

Then the Adams spectral sequence has differentials generated by

$$d^{r(n)}(\mu^{2^{n-1}}) = v_1^{r(n)}\lambda_n$$

for all $n \geq 1$. This leaves the E_∞-term

$$E_\infty^{**} = P(v_1)\{1\} \oplus \bigoplus_{n=1}^{\infty} P_{r(n)}(v_1)\{\lambda_n\} \otimes E(\lambda_{n+1}) \otimes P(\mu^{2^n})\,.$$

Hence $\pi_*(THH(ku) \wedge M) = \pi_*(THH(ku); \mathbb{Z}/2)$ is generated as a $P(v_1)$-module by elements 1, $x_{n,m} = \lambda_n\mu^{2^n m}$ and $x'_{n,m} = \lambda_n\lambda_{n+1}\mu^{2^n m}$ for $n \geq 1$ and $m \geq 0$. Here $|x_{n,m}| = s(n) + 2^{n+3}m$ and $|x'_{n,m}| = s(n) + s(n+1) + 2^{n+3}m$. The module structure is generated by the relations $v_1^{r(n)}x_{n,m} = 0$ and $v_1^{r(n)}x'_{n,m} = 0$ for $n \geq 1$, $m \geq 0$.

Proof. The classes λ_1, λ_2 and μ were introduced in proposition 8.7, and the E_2-term was found in lemma 8.8. By corollary 8.12 the abutment $\pi_*(THH(ku) \wedge M)$ is all v_1-torsion, except the direct summand $\pi_*(ku \wedge M) = P(v_1)$, which is included by the unit map. Hence every class λ_n is v_1-torsion, so there is some integer $r(n)$ such that $v_1^{r(n)}\lambda_n$ is hit by a differential.

Suppose by induction that the $d^{r(k)}$-differentials for $1 \leq k < n$ have been found, leaving the term

$$E_{r(n-1)+1}^{**} = P(v_1) \otimes E(\lambda_n, \lambda_{n+1}) \otimes P(\mu^{2^{n-1}}) \oplus \bigoplus_{k=1}^{n-1} P_{r(k)}(v_1)\{\lambda_k\} \otimes E(\lambda_{k+1}) \otimes P(\mu^{2^k})\,.$$

Consider the $P(v_1)$-module generated by λ_n. Let r be minimal such that $v_1^r\lambda_n$ is a boundary. The source x of such a differential cannot be divisible by v_1, since r is minimal, so $d^r(x) = v_1^r\lambda_n$ where x has Adams filtration $s = 0$ and even total degree. Furthermore, $v_1^r\lambda_n$ is not v_1-torsion at this term, so x cannot be v_1-torsion. Likewise, λ_{n+1} does not annihilate $v_1^r\lambda_n$, so λ_{n+1} cannot annihilate x either. This forces $x \in P(\mu^{2^{n-1}})$. By lemma 8.5, $THH(ku) \wedge M$ is a μ-spectrum, so d^r is a derivation and the Leibniz rule shows that $x = \mu^{2^{n-1}}$, since d^r on any higher power of $\mu^{2^{n-1}}$ must be divisible by $\mu^{2^{n-1}}$. Hence $d^r(\mu^{2^{n-1}}) = v_1^r\lambda_n$, and by a degree count we must have $r = r(n)$.

To complete the induction step, we must compute the $E_{r(n)+1}$-term of the Adams spectral sequence. The $d^{r(n)}$-differential does not affect the summands $P_{r(k)}(v_1)\{\lambda_k\} \otimes E(\lambda_{k+1}) \otimes P(\mu^{2^k})$ for $1 \leq k < n$, and is zero on $E(\lambda_{n+1}) \otimes P(\mu^{2^n})$. It acts on $P(v_1)\{1, \lambda_n, \mu^{2^{n-1}}, \lambda_n\mu^{2^{n-1}}\}$, leaving $P_{r(n)}(v_1)\{\lambda_n\} \oplus P(v_1) \otimes E(\lambda_{n+2})$, where by definition $\lambda_{n+2} = \lambda_n\mu^{2^{n-1}}$. This shows that the term $P(v_1) \otimes E(\lambda_n, \lambda_{n+1} \otimes P(\mu^{2^{n-1}})$ at the $E_{r(n-1)+1}$-term gets replaced by the direct sum of $P(v_1) \otimes E(\lambda_{n+1}, \lambda_{n+2}) \otimes P(\mu^{2^n})$ and $P_{r(n)}(v_1)\{\lambda_n\} \otimes E(\lambda_{n+1}) \otimes P(\mu^{2^n})$. □

Theorem 8.14. *Consider the Adams spectral sequence*

$$E_2^{**} = \big(P(v_1) \otimes E(\lambda_2, \lambda_3) \oplus E(\lambda_2)\{\lambda_1\}\big) \otimes P(\mu^2)$$

for $\pi_*(THH(ko) \wedge Y)$, *with* $\lambda_1 = \sigma\bar\xi_1^4$, $\lambda_2 = \sigma\bar\xi_2^2 + \bar\xi_1^2 \cdot \sigma\bar\xi_1^4$, $\lambda_3 = \sigma\bar\xi_1^4(\sigma\bar\xi_3 + \bar\xi_1 \cdot \sigma\bar\xi_2^2)$, $\mu^2 = (\sigma\bar\xi_3)^2$ *and* $v_1 = [\xi_2]$ *in bidegrees* $(s, t) = (0, 5)$, $(0, 7)$, $(0, 13)$, $(0, 16)$ *and* $(1, 3)$, *respectively. Recursively define*

$$\lambda_n = \lambda_{n-2}\mu^{2^{n-3}}$$

for $n \geq 4$. *Likewise define* $r(1) = 1$, $r(2) = 4$, $r(n) = 2^n + r(n-2)$, $s(1) = 5$, $s(2) = 7$ *and* $s(n) = 2^n + s(n-2)$, *for* $n \geq 3$. *So* λ_n *has bidegree* $(0, s(n))$ *and* $2r(n) + s(n) = 2^{n+2} - 1$.

Then the Adams spectral sequence has differentials generated by

$$d^{r(n)}(\mu^{2^{n-1}}) = v_1^{r(n)}\lambda_n$$

for all $n \geq 2$. *This leaves the* E_∞*-term*

$$E_\infty^{**} = P(v_1)\{1\} \oplus \bigoplus_{n=1}^{\infty} P_{r(n)}(v_1)\{\lambda_n\} \otimes E(\lambda_{n+1}) \otimes P(\mu^{2^n})\,.$$

Hence $\pi_*(THH(ko) \wedge Y) = \pi_*(THH(ko); Y)$ *is generated as a* $P(v_1)$*-module by elements* 1, $x_{n,m} = \lambda_n\mu^{2^n m}$ *and* $x'_{n,m} = \lambda_n\lambda_{n+1}\mu^{2^n m}$ *for* $n \geq 1$ *and* $m \geq 0$. *Here* $|x_{n,m}| = s(n) + 2^{n+3}m$ *and* $|x'_{n,m}| = s(n) + s(n+1) + 2^{n+3}m$. *The module structure is generated by the relations* $v_1^{r(n)}x_{n,m} = 0$ *and* $v_1^{r(n)}x'_{n,m} = 0$ *for* $n \geq 1$, $m \geq 0$.

Proof. Starting with an imagined E_1-term

$$E_1^{**} = P(v_1) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu)$$

and differential $d_1(\mu) = v_1\lambda_1$, the proof is the same as for theorem 8.13. □

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