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On stochastic control for Volterra type dynamics

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So long, and thanks for all the fish
- Hitchhiker's Guide to the Galaxy

Abstract

In this thesis we study stochastic optimal control problems for Volterra type dynamics. To this end, we consider two different approaches: maximum principle and dynamic programming. We use the maximum principle when working with time-changed Lévy noise drivers and obtain a sufficient and necessary optimal condition. We also discuss the two players game setting. We present the dynamic programming approach in the case of a simplified continuous Volterra forward equation, where the dependence from the past is obtained through a convolution kernel. In this case we also provide a numerical approach for the linear-quadratic case.

Abstract

I denne avhandlingen betrakter vi stokastiske optimale kontrollproblemer for dynamikk bestemt av Volterralikninger. For å gjennomføre dette benytter vi to forskjellige tilnærminger: maksimumsprinsippet og dynamisk programmering. Maksimumsprinsippet blir anvendt for å betrakte tidsendrede Lévy-støydrivere og vi oppnår med dette en nødvendig og tilstrekkelig optimal betingelse. Videre diskuterer vi også tilfellet med to spillere. Til slutt presenterer vi den dynamiske programmeringstilnærmingen under en forenklet kontinuerlig Volterra-foroverligning, der fortidsavhengigheten oppnås gjennom en konvolusjonskjerne. I dette siste tilfellet gir vi også en numerisk tilnærming for det lineær-kvadratiske tilfellet.

List of Papers

Paper I

G. Di Nunno, M. Giordano *Stochastic Volterra equations withtime-changed Lévy noise and maximum principles.*
Published in Annals of Operations Research (2023)

Paper II

G. Di Nunno, M. Giordano *Maximum principles for stochastic time-changed Volterra games.*
Submitted for publication. Arxiv : 2012.06449

Paper III

G. Di Nunno, M. Giordano *Lifting of Volterra processes: optimal control and HJB equations.*
Submitted for publication. ArXiv : 2306.14175

Paper IV

M. Giordano, A. Yurchenko-Tytarenko *Optimal control in linear stochastic advertising models with memory.*
Published in Decisions Economics and Finance (2023)

Contents

Chapter 1

Introduction

This thesis is a collection of four related papers following this introductory chapter, where we provide an overview of all the covered topics. All the papers have the common goal of studying classes of optimal control problems with Volterra-type forward dynamics, where one aims to find

$$J(\hat{u}) = \sup_{u \in \mathcal{A}} J(u), \tag{1.0.1}$$

where $J(u)$ is a performance functional depending on a forward equation which describes the state of the system, and $\mathcal{A}$ is a set of admissible controls. These problems naturally arise e.g. when one studies financial markets, portfolio optimization, optimal advertising, recursive utility, and mean-variance selection problems (see e.g. [1, 2, 3, 24, 33, 36, 43]). We consider general performance functionals of the form

$$J(u) = \mathbb{E}\left[\int_0^T F(t, X^u(t), u(t))dt + G(X^u(T))\right], \tag{1.0.2}$$

where the underlying forward process has dynamics given by

$$\begin{aligned} X^u(t) = X_0 &+ \int_0^t b(t, s, X^u(s), u(s))ds + \int_0^t \sigma(t, s, X^u(s), u(s))dW(s) \\ &+ \int_0^t \int_{\mathbb{R}_0} \gamma(t, s, z, X^u(s), u(s))\widetilde{H}(ds, dz), \end{aligned} \tag{1.0.3}$$

for some real-valued functions b, σ and γ, and where W and $\widetilde{H}$ are the continuous and discontinuous parts of the noise respectively. Ideally, one would like to explicitly find both $\hat{u}$ and $J(\hat{u})$ but, as we will see in the following sections, this is not always possible.

Two main approaches for the study of optimal control problems of the type (1.0.1)-(1.0.3) are the dynamic programming principle (DPP) and the maximum principle (MP). In the case where W is a standard Brownian motion and $\widetilde{H}$ is a compensated Poisson random measure, the MP approach for (1.0.1)-(1.0.3) has been studied by many authors (see [2, 3, 36, 43, 44] to cite a few), whereas the DPP for the same problem has been developed only in some particular cases (see e.g. [1, 6, 29]). This gap between the two approaches is due to the fundamental need for the Markov property in the DPP approach, usually lacking when dealing with Volterra dynamics such as the ones in (1.0.3).

Historically the DPP and MP approaches have been developed separately and independently even though they are strictly related. The case where (1.0.3) does not depend on the past is well known and studied e.g. in [34, 45] and it can give us some insight on the relation between those apparently different techniques.
In order to illustrate this connection, we consider the following simplified version of the optimal control problem (1.0.1)-(1.0.3), where

$$J(\hat{u}) = \sup_{u \in \mathcal{A}} J(u) = \sup_{u \in \mathcal{A}} \mathbb{E}\left[\int_0^T F(t, X^u(t), u(t))dt + G(X^u(T))\right], \tag{1.0.4}$$

and the dynamics for the forward process X^u is continuous and does not contain any dependence from the past, i.e.

$$X^u(t) = X_0 + \int_0^t b(X^u(s), u(s))ds + \int_0^t \sigma(X^u(s), u(s))dW(s), \tag{1.0.5}$$

for a standard Brownian motion $W(t)$.

The MP approach, first formulated and derived by Pontryagin, states that any optimal control along with the optimal state trajectory must solve the so-called Hamiltonian system (i.e. a forward-backward differential equation) plus a maximum condition of a function called Hamiltonian. The mathematical importance of the MP lies in that maximizing the Hamiltonian is much easier than solving the original control problem and allows us to get a closed-form solution in certain particular cases. For the optimal control problem (1.0.4)-(1.0.5), the associated Hamiltonian function is given by the function

$$\mathcal{H}(t, x, u, p, q) : [0, T] \times \mathbb{R} \times \mathbb{U} \times \mathbb{R}^2 \longrightarrow \mathbb{R},$$

where $\mathbb{U}$ is a closed convex subset of $\mathbb{R}$ and where

$$\mathcal{H}(t, x, u, p, q) = F(t, x, u) + b(x, u)p + \sigma(x, u)q,$$

for p and q two adjoint variables solving the backward stochastic differential equation (BSDE)

$$\begin{cases} dp(t) &= -\frac{\partial \mathcal{H}}{\partial x}(t)dt + q(t)dW(t) \\ p(T) &= G(X^u(T)). \end{cases} \tag{1.0.6}$$

One can then proceed in using a variational approach, derived from the deterministic counterpart of this maximum principle, in order to derive necessary conditions for the optimality.

On the other hand, the DPP approach, formulated by Bellman, consists in considering a family of optimal controls with different initial times and states and establishing relationships among these via the Hamilton-Jacobi-Bellman (HJB) equation. In order to write such equations, one starts by assuming that the control $u(t) = u(X(t))$ is Markovian and writes the generator A^u of the diffusion X^u as

$$A^u\phi(x) = b\Big(x, u(x)\Big)\frac{\partial \phi}{\partial x}(x) + \frac{1}{2}\sigma^2\Big(x, u(x)\Big)\frac{\partial^2 \phi}{\partial x^2}(x).$$

One can then proceed to formulate the HJB theorem (see Theorem [34, Theorem 5.1]) and obtain an optimal feedback control by taking the maximizer of the Hamiltonian and using some verification techniques.

The Hamiltonian systems associated with MP are stochastic differential equations, whereas the HJB equations associated with the DPP are partial differential equations of second order. In some simple cases such as the one presented in this section, one is actually able to recover the relationship between those two. In fact, we have the following result, whose proof can be found in [34].

Theorem 1.0.1. *Define*

$$J(u, s, X_0) = \mathbb{E}\left[\int_0^{T-s} f(s+t, X^{X_0}(t), u(t))dt + g(X^{X_0}(T))\right], \tag{1.0.7}$$

where X^{X_0} is the solution of equation (1.0.5) with initial condition $X(0) = X_0$ and put

$$V(s, x) = \sup_{u \in \mathcal{A}} J(u, s, x).$$

We remark that $V(s, x)$ solves the HJB equation associated to the problem (1.0.5), (1.0.7). Assume $V(s, x) \in C^{1,3}(\mathbb{R}^2)$, and there exists an optimal Markovian control $\hat{u}$ for problem (1.0.4), with the corresponding solution $\hat{X}$ to (1.0.5). If we define

$$\begin{aligned} p(t) &= \frac{\partial V}{\partial x}(t, \hat{X}(t)) \\ q(t) &= \sigma(t, \hat{X}(t), \hat{u}(t))\frac{\partial^2 V}{\partial x^2}(t, \hat{X}(t)), \end{aligned}$$

then $(p(t), q(t))$ solves the adjoint equation (1.0.6).

We want to remark that when dealing with Volterra dynamics, such as the ones treated in this thesis, is usually very difficult to be able to obtain the HJB equations due to the non-Markovianity of the system. For this reason the first step towards an application of the DPP to the optimal control problem (1.0.1)-(1.0.3) is to recover some form of Markovianity. This approach is feasible thanks to the recent results in [1, 6, 12, 13, 14] that provide the foundation to the *lift* theory for Volterra processes. This consists in rewriting the finite dimensional forward equation X^u as an element of an infinite dimensional Banach space in order to recover the Markov property.

Our goal throughout this thesis is to provide new contributions to the optimal control problem (1.0.1)-(1.0.3) in two different directions. On the one hand, we want to include time-change in the setting as it allows us to gain more flexibility (as suggested in the study of volatility modelling[4, 9, 27, 38, 39], energy markets [7], and default models[32]). This is done by considering a driving noise which is a conditional Lévy process and that can be regarded as a time-changed Lévy noise with time change given by

$$\Lambda_t(\omega) = \int_0^t \lambda_s(\omega)ds, \quad (t,\omega) \in [0,T] \times \Omega, \quad T > 0.$$

For more details on the time-changed processes at play we refer to Subsection 1.1. On the other hand, we would like to formulate a DPP for (1.0.1)-(1.0.3) via an infinite dimensional lift, similarly to what has been done in [6] and in [1] for the linear-quadratic case.

Of course, the above described goals are not achieved without tackling some challenges. Introducing time-changed Lévy noises as drivers in (1.0.3) means that one is not able to use the Malliavin calculus to formulate a MP approach (like in [2, 3, 36]). Our original contribution in that sense, comes then from considering the non-anticipating derivative (as introduced in [18] and presented in Subsection 1.2). This approach allows us to circumvent the issues arising from a form of Malliavin calculus for conditionally independent increments processes (i.e. the dominion of the derivative) that had been solved e.g. in [3] by taking the Hida-Malliavin derivative.

By considering a process μ with conditional independent increments in (1.0.3), the first filtration that one considers is $\mathbb{F}$, the smallest right-continuous filtration generated by μ, $\mathbb{F} = \{\mathcal{F}_t,\ t \in [0,T]\}$. Unfortunately, when working under $\mathbb{F}$ we are not able to exploit the *perfect* stochastic integral representation property (see e.g. [31] and Paper II for more details). We thus introduce another filtration, namely $\mathbb{G} := \{\mathcal{G}_t,\ t \in [0,T]\}$, where $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}^\Lambda$ is generated by μ and the entire history $\mathcal{F}^\Lambda$ of the time-change processes Λ. This allows us exploit the conditional Lévy structure of the noise μ to derive a MP for the optimal control problem (1.0.1) - (1.0.3). Notice that the stochastic control problems are aimed to be studied under $\mathbb{F}$, and that the enlarged filtration $\mathbb{G}$, which introduced anticipated information, is used to take advantage of the structure of the noise. Nevertheless we have also studied the stochastic control under anticipated information, which is an interesting mathematical problem in itself.

This particular structure of the information at play has been crucial when considering the 2-players stochastic games in Paper II. There is, in fact a vast literature on stochastic games among two or more players, see [8, 10, 15, 34, 35, 40, 41, 46] to cite a few and different models. Our model, though, is different from all of the above cited ones as we take conditional Lévy processes, which are square integrable martingale random fields both with respect to the filtration $\mathbb{G}$ (and $\mathbb{F}$). Then we exploit the NA-derivative and the filtrations' interplay like we did in Paper I to obtain a sufficient MP approach.

When considering the DPP approach to solve (1.0.1), (1.0.3), we have to exploit a *lift* approach to solve optimal control for Volterra dynamics. The lift to infinite dimensions, in the setting presented here, was first introduced in [13, 14] in order to recover the Markov property from a finite dimensional uncontrolled forward Volterra integral equation (FVIE) with dependence from the past obtained via a deterministic convolution kernel K. This is achieved by assuming a particular shape for K (see Definition 1.0.2 below) and rewriting the FVIE in an infinite-dimensional Banach setting. This lift was later exploited in [1] to solve a linear-quadratic Volterra-type stochastic optimal control problem where the kernel K can be expressed as Laplace transform of a measure. A similar approach is also presented in [6] to solve a Volterra-type stochastic optimal control problem with a Lévy driver by *lifting* a controlled FVIE to a Hilbert space.

In the present work, we deviate from the setting in [1] by taking a wider class of liftable kernels K, which allows us to consider a broader class of FFVIEs and performance functionals. At the same time, by considering a lift to *Unconditional Martingale Differences* (UMD) Banach spaces, we also have a setting which is not only limited to Hilbert spaces, like in [6]. To the best of our knowledge, this setting was never considered before for an optimal control problem.

We thus deal with a type of the forward dynamics (1.0.3) that can be written as

$$\begin{aligned} X^u(t) = x(t) &+ \int_0^t K(t-s)b(s, X^u(s), u(s))ds \\ &+ \int_0^t K(t-s)\sigma(s, X^u(s))dW(s) \quad t \in [0,T], \end{aligned} \tag{1.0.8}$$

where K is a *liftable* deterministic convolution kernel (see Definition 1.0.2 below), W is a standard Brownian motion and the control u only appears in the drift term of (1.0.8). Notice that the dependence from the past (like in [13]), is obtained through convolution with the kernel K.

To better explain what are our contribution in this framework, we recall the definition of a *liftable* kernel (due to [13]).

Definition 1.0.2. Let Y be a Banach space with dual Y^* and denote with $\langle \cdot, \cdot \rangle_{Y \times Y^*}$ the pairing between Y and Y^*. We say that a kernel $K \in L^2_{loc}(\mathbb{R}_+, \mathbb{R})$ is *liftable* if there exist $g \in Y$, $\nu \in Y^*$ and a uniformly continuous semigroup $\mathcal{S}^*_t$, $t \in [0,T]$ with generator $\mathcal{A}^*$, acting on Y^*, such that

- $K(t) = \langle g, \mathcal{S}^*_t \nu \rangle_{Y \times Y^*}$
- $\mathcal{S}^*_t \nu \in Y^*$ for all $t > 0$
- $\int_0^t \|\mathcal{S}^*_s \nu\|^2_{Y^*} ds < \infty$ for all $t > 0$.

We will also write $\langle \cdot, \cdot \rangle_{Y \times Y^*} = \langle \cdot, \cdot \rangle$ when no confusion arises.

We can now rewrite $X^u(t)$ in (1.0.8) as

$$X^u(t) = \langle g, \mathcal{Z}_t^u \rangle,$$

where

$$\mathcal{Z}_t^u = \zeta_0 + \int_0^t \mathcal{A}^* \mathcal{Z}_\tau^u d\tau + \int_0^t \nu b^g(\tau, \mathcal{Z}_\tau^u, u) + \int_0^t \nu \sigma^g(\tau, \mathcal{Z}_\tau^u) dW_\tau, \qquad (1.0.9)$$

for some functions b^g and σ^g defined in Subsection 1.4. Our goal becomes to solve an infinite dimensional optimal control problem with forward dynamics given by $\mathcal{Z}$ taking values from a Banach space.

Of course, there exists a vast literature on optimal control in infinite dimensions (see [25, 26, 33] to cite a few). In our work, though we manage to introduce some novelty on the optimal control side by working in UMD Banach spaces. To the best of our knowledge, in fact, an optimal control problem such as (1.0.1), (1.0.9) had been solved only in Hilbert spaces (see [26]) and solving it in Banach spaces required to consider a σ not depending on X^u (see [33]) in (1.0.3). In Paper III, we present a new framework, which allows us to consider a σ depending on X^u and a particular class of Banach spaces (UMD Banach spaces) instead of Hilbert spaces.

Our contribution in Paper IV, is then to provide an approach to compute explicitly the optimal value and optimal control for the problem (1.0.8) where the optimal performance functional (1.0.1) takes the form

$$J(u) = \mathbb{E}\left[-\int_0^T a_1 u^2(s) ds + a_2 X^u(T) \right],$$

for a_1, a_2 in $\mathbb{R}_{>0}$. This is achieved thanks to a simple yet efficient approximation scheme, which allows us to explicitly determine the convergence rate of the approximated solution to the exact one.

Open questions and future work

One question comes to mind when reading this first part of the introduction:

"Is it possible to use a lift to solve Volterra equations with a time-changed Lévy process as a driver?"

Due to the nature of the lift approach and thanks to the extension presented in [12] and the work of [6], we know that, in general, it is possible to recover some form of Markovianity for rather general Volterra forward equations with Lévy drivers. This means, in particular, that considering Lévy subordinators for the time-change process also works. At this time, to the best of our knowledge, there are no studies on how to work with an infinite dimensional process with conditionally independent increments. This is thus a first interesting topic that would need some additional research to be carried on. Due to the nature of the lift presented in [12, 13, 14], though, it is not hard to believe that such a process could be lifted to a Hilbert or even a Banach space.

When it comes to the optimal control part, there seem to be more issues to consider. On one hand, [6] managed to solve an optimal control problem for Lévy driven forward equations lifted to Hilbert spaces. On the other hand, having a time change such as the one presented in Paper I and Paper II it is difficult to say if one would need to use techniques of partial/enlarged filtrations to tackle this problem. Nonetheless, this solving such an optimal control problem with a lift approach seems particularly engaging, and the answer to weather it is possible to exploit this approach or not, does not seem obvious.

This introductory chapter is divided as follows: Subsections 1.1 and 1.2 contain technical results on time-changed processes and NA-derivatives. Subsection 1.3 presents an introduction to stochastic games, where we explain how we need to re-define the concept already introduced here in order to deal with two players competing against each other. Subsection 1.4 introduces the background for the infinite dimensional lift in the case of uncontrolled Volterra integral equations of the form (1.0.8). The scope of those subsections is to provide the reader with the theoretical tools exploited in the following part. Some of the results presented here will also be repeated in the papers. Lastly, Subsection 1.5 provides a more detailed summary of the Papers.

1.1 Processes with conditionally stationary and independent increments

This subsection aims to introduce the class of processes with conditionally stationary and independent increments used in Paper I and Paper II as driving noise. Some of the notions presented here will be repeated in the following part.

We consider a complete probability space $(\Omega, \mathcal{F}, P)$ and a time horizon $T < \infty$ and define the time-space

$$\mathbb{X} := [0,T] \times \mathbb{R} := \Big([0,T] \times \{0\}\Big) \cup \Big([0,T] \times \mathbb{R}_0\Big),$$

where $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$, and endowed with the Borel σ-algebra $\mathcal{B}_{\mathbb{X}}$.
On this space, we take a noise μ, defined as a random field with conditional stationary independent increments. To this end, we recall the following definitions due to [28] and [37], respectively.

Definition 1.1.1. Given a càdlàg stochastic real-valued process $\{X_t,\ t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ adapted to the σ-algebra $\mathcal{F} = \{\mathcal{F}_t \quad t \geq 0\}$, we say that X has conditionally independent increments relative to the σ-algebra $\mathcal{G} \subset \mathcal{F}$ if, for almost all $\mathscr{B} \in \mathcal{B}(\mathbb{R})$, and $0 \leq s < t$,

$$P[X_t - X_s \in \mathscr{B} | \mathcal{F}_s \vee \mathcal{G}] = P[X_t - X_s \in \mathscr{B} | \mathcal{G}].$$

Definition 1.1.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and Λ_t, $t \geq 0$, a non-negative real-valued stochastic process on $(\Omega, \mathcal{F}, P)$, with non-decreasing right continuous paths and such that $\Lambda_0 = 0$. Consider $\mathcal{F}^\Lambda = \sigma(\Lambda_u,\ u \geq 0)$, the smallest σ-field that makes Λ measurable.
Given a measurable real-valued process X_t, $t \in [0, T]$, defined on $(\Omega, \mathcal{F}, P)$, we say that X_t is a process with conditionally stationary independent increments with respect to Λ_t if

1. For any $s_1 < t_1 < ... < s_n < t_n \in [0, T]$, and $x_1, ..., x_n \in \mathbb{R}$

$$P[X_{t_1} - X_{s_1} \leq x_1, ..., X_{t_n} - X_{s_n} \leq x_n | \mathcal{F}^\Lambda] = \prod_{k=1}^{n} P[X_{t_k} - X_{s_k} \leq x_k | \mathcal{F}^\Lambda] \quad a.s.$$

2. For any $0 \leq s \leq t \leq T$ and any $\zeta \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left\{i\zeta(X_t - X_s)\right\} | \mathcal{F}^\Lambda\right] = \phi(\zeta)^{\Lambda_t - \Lambda_s},$$

   where ϕ is an infinitely divisible characteristic function.

If, for example, we take $\phi(\zeta) = \exp\{e^{i\zeta} - 1\}$, the process X_t is a conditionally non-homogeneous Poisson process. If $\phi(\zeta) = e^{-\zeta/2}$, then X_t is a conditional Wiener process. With this approach we can consider some general processes and being able to trace them back to some well known cases if we know $\mathcal{F}^\Lambda$. The following result due to [37] holds:

Theorem 1.1.3. *Let Λ_t, $t \geq 0$ be a stochastic process on $(\Omega, \mathcal{F}, P)$ as in Definition 1.1.2. Let $Y(t)$, $t \geq 0$ be a measurable real-valued process on $(\Omega, \mathcal{F}, P)$ with stationary increments, independent of Λ_t, and such that Y_t has characteristic function ϕ^t where ϕ is an infinitely divisible characteristic function.*
Then $X_t = Y(\Lambda_t)$, $t \geq 0$, is a process with conditional stationary independent increments with respect to Λ_t.
Vice versa, every process with conditional stationary independent increments is equal in distribution to a process of the above form.

In order to define random fields as in Definition 1.1.2, we thus need to specify the process Λ. In our case, we consider the space $\mathcal{L}$ of the two dimensional stochastic processes $\lambda = (\lambda^B, \lambda^H)$ such that, for each component $k = B, H$, we have that

1. $\lambda_t^k \geq 0$ $P - a.s.$ for all $t \in [0, T]$,
2. $\lim_{h \to 0} P\left(|\lambda_{t+h}^k - \lambda_t^k| \geq \epsilon\right) = 0$ for all $\epsilon > 0$ and almost all $t \in [0, T]$,
3. $\mathbb{E}\left[\int_0^T \lambda_t^k dt\right] < \infty$.

Let ν be a σ-finite measure on the Borel sets of $\mathbb{R}_0$ satisfying $\int_{\mathbb{R}_0} z^2\nu(dz) < \infty$. Define the random measure Λ on $\mathcal{B}_{\mathbb{X}}$ by

$$\Lambda(\Delta) := \int_0^T \mathbb{1}_{\{(t,0)\in\Delta\}}(t)\lambda_t^B dt + \int_0^T\int_{\mathbb{R}_0} \mathbb{1}_\Delta(t,z)\nu(dz)\lambda_t^H dt,\ \ \Delta \subseteq \mathbb{X}, \quad (1.1.1)$$

and denote with Λ^B and Λ^H the restrictions of Λ to $[0,T]\times\{0\}$ and $[0,T]\times\mathbb{R}_0$ respectively. As in Definition 1.1.2, we introduce the filtration

$$\mathbb{F}^\Lambda = \left\{\mathcal{F}_t^\Lambda, \quad t\in[0,T]\right\},$$

where $\mathcal{F}_t^\Lambda$ is generated by the values of Λ on the Borel sets of $[0,t]\times\mathbb{R}$. Lastly we put $\mathcal{F}^\Lambda := \mathcal{F}_T^\Lambda$. At this point, we can thus define the conditional Gaussian and the conditional Poisson random measures in the following way.

Definition 1.1.4. The *conditional Gaussian measure B* (given $\mathcal{F}^\Lambda$) is a signed random measure on the Borel sets of $[0,T]\times\{0\}$ satisfying

H1. $P\left(B(\Delta)\le x|\mathcal{F}^\Lambda\right) = P\left(B(\Delta)\le x|\Lambda^B(\Delta)\right) = \Phi\left(\frac{x}{\sqrt{\Lambda^B(\Delta)}}\right)$, $x\in\mathbb{R}, \Delta\subseteq[0,T]\times\{0\}$. Here Φ is the cumulative probability distribution function of a standard normal random variable.

H2. For all disjoint $\Delta_1,\Delta_2\subseteq[0,T]\times\{0\}$, $B(\Delta_1)$ and $B(\Delta_2)$ are conditionally independent given $\mathcal{F}^\Lambda$.

The *conditional Poisson measure H* (given $\mathcal{F}^\Lambda$) is a random measure on the Borel sets of $[0,T]\times\mathbb{R}_0$ satisfying

H3. $P\left(H(\Delta){=}k|\mathcal{F}^\Lambda\right){=}P\left(H(\Delta){=}k|\Lambda^H(\Delta)\right){=}\frac{\Lambda^H(\Delta)^k}{k!}e^{-\Lambda^H(\Delta)}$, $k\in\mathbb{N}$, $\Delta\subseteq[0,T]\times\mathbb{R}_0$.

H4. For all disjoint $\Delta_1,\Delta_2\subseteq[0,T]\times\{\mathbb{R}_0\}$, $H(\Delta_1)$ and $H(\Delta_2)$ are conditionally independent given $\mathcal{F}^\Lambda$.

Moreover, we assume

H5. B and H are conditionally independent given $\mathcal{F}^\Lambda$.

And lastly, we define the *conditional centered Poisson random measure* as

$$\widetilde{H}(\Delta) := H(\Delta) - \Lambda^H(\Delta), \quad \Delta\subset\mathbb{X}.$$

Definition 1.1.5. We define the signed random measure μ on the Borel sets $\Delta \subseteq \mathbb{X}$ by

$$\mu(\Delta) := B\left(\Delta \cap [0,T] \times \{0\}\right) + \widetilde{H}\left(\Delta \cap [0,T] \times \mathbb{R}_0\right).$$

The random measure μ has *conditionally independent values.*

Observe that (H1) and (H3) yield

$$\mathbb{E}[\mu(\Delta)|\mathcal{F}^\Lambda] = 0, \qquad \mathbb{E}[\mu(\Delta)^2|\mathcal{F}^\Lambda] = \Lambda(\Delta), \qquad \Delta \subseteq \mathbb{X}. \tag{1.1.2}$$

In this settings there are two different types of information flows which arise naturally. The first one is represented by the filtration

$$\mathbb{F} := \{\mathcal{F}_t,\ t \in [0,T]\}, \qquad \mathcal{F}_t := \bigcap_{r>t} \mathcal{F}^\mu_r,$$

where $\mathbb{F}^\mu := \{\mathcal{F}^\mu_t,\ t \in [0,T]\}$ is generated by the values $\mu(\Delta)$, $\Delta \subset [0,t] \times \mathbb{R}$, $t \in [0,T]$. The second information flow of interest is

$$\mathbb{G} := \{\mathcal{G}_t,\ t \in [0,T]\}, \quad \mathcal{G}_t := \mathcal{F}^\mu_t \vee \mathcal{F}^\Lambda.$$

The filtration $\mathbb{G}$ is right-continuous. Moreover, we notice that $\mathcal{G}_T = \mathcal{F}_T$, $\mathcal{G}_0 = \mathcal{F}^\Lambda$, and $\mathcal{F}_0$ is trivial. Namely, $\mathbb{G}$ includes information on the future values of Λ. For more details on this framework we refer to [22].

For $\Delta \subseteq (t,T] \times \mathbb{R}$, the conditional independence in (H2) and (H4) allows us to write

$$\mathbb{E}[\mu(\Delta)|\mathcal{G}_t] = \mathbb{E}[\mu(\Delta)|\mathcal{F}_t \vee \mathcal{F}^\Lambda] = \mathbb{E}[\mu(\Delta)|\mathcal{F}^\Lambda] = 0. \tag{1.1.3}$$

Thus (1.1.2) yields the martingale property of μ with respect to $\mathbb{G}$. Moreover, (H5) gives us

$$\mathbb{E}[\mu(\Delta_1)\mu(\Delta_2)|\mathcal{G}_t] = \mathbb{E}[\mu(\Delta_1)|\mathcal{F}^\Lambda]\mathbb{E}[\mu(\Delta_2)|\mathcal{F}^\Lambda] = 0,$$

for disjoint $\Delta_1, \Delta_2 \subseteq (t,T] \times \mathbb{R}$. From the above we see that μ is a martingale random field with respect to $\mathbb{G}$, see e.g. [19, 22].

Working under $\mathbb{G}$, though, is just a technical choice as $\mathbb{G}$ includes information about the future values of the time-change process Λ. For this reason, we look at $\mathbb{G}$ as the full-information filtration and consider the information available at time t (namely the filtration $\mathbb{F}$) as partial with respect to $\mathbb{G}$.

Definition 1.1.6. A square integrable *martingale random field μ with conditionally orthogonal values* is a stochastic set function $\mu(\Delta)$, $\Delta \subseteq \mathbb{X}$, on the Borel sets of $\mathbb{X}$ such that

- $m(\Delta) := \mathbb{E}[\mu(\Delta)^2] = \mathbb{E}[\Lambda(\Delta)]$, $\Delta \subseteq \mathbb{X}$,

- μ is $\mathbb{G}$-adapted,
- μ satisfies the martingale property, (1.1.3)
- μ has conditionally orthogonal values: $\mathbb{E}[\mu(\Delta_1)\mu(\Delta_2)|\mathcal{G}_t] = 0$, for every disjoint $\Delta_1, \Delta_2 \in (t, T] \times \mathbb{R}$.

Thanks to the tower rule for conditional expectation and (1.1.3) we get that μ is also a martingale random field with respect to the partial information flow $\mathbb{F}$.

With the above structures, we access the framework of non-anticipating (Itô type) stochastic integration. For this, we introduce $\mathcal{I}^{\mathbb{G}} \subseteq L^2(\mathbb{X} \times \Omega, \mathcal{B}_X \times \mathcal{F}, \Lambda \times P)$ representing the subspace of the random fields admitting a $\mathbb{G}$-predictable modification and $\mathcal{I}^{\mathbb{F}} \subset \mathcal{I}^{\mathbb{G}}$, the one of $\mathbb{F}$-predictable random fields. Observe that, for all $\phi \in \mathcal{I}^{\mathbb{G}}$, we have that

$$\mathbb{E}\Big[\Big(\iint_{\mathbb{X}} \phi(s,z)\mu(ds,dz)\Big)^2\Big] = \mathbb{E}\Big[\iint_{\mathbb{X}} \phi(s,z)^2\Lambda(ds,dz)\Big], \qquad (1.1.4)$$

due to H5 and the martingale property of μ.

Lastly, notice that the random measures B and H are related to a time-changed Brownian motion and time-changed pure jump Lévy process. To illustrate this connection, we consider the following processes on $[0,T]$:

$$B_t := B([0,t] \times \{0\}), \qquad \Lambda_t^B := \int_0^t \lambda_s^B ds,$$
$$\eta_t := \int_0^t\int_{\mathbb{R}_0} z\widetilde{H}(dsdz), \qquad \Lambda_t^H := \int_0^t \lambda_s^H ds,$$

and compute the characteristic functions of B and η. From H1 we get

$$\mathbb{E}\left[e^{icB_t}\right] = \int_{\mathbb{R}} \mathbb{E}\left[e^{icB_t}|\Lambda_t^B = x\right] P_{\Lambda_t^B}(dx) = \int_{\mathbb{R}} e^{\frac{1}{2}c^2x} P_{\Lambda_t^B}(dx), \quad c \in \mathbb{R},$$

where $P_{\Lambda_t^B}$ is the probability distribution of the time-change process Λ_t^B. Correspondingly, thanks to H3, we have that

$$\mathbb{E}\left[e^{ic\eta_t}\right] = \int_{\mathbb{R}} \exp\left\{\int_{\mathbb{R}_0}[e^{iczx} - 1 - iczx]\nu(dz)\right\} P_{\Lambda_t^H}(dx), \quad c \in \mathbb{R},$$

where $P_{\Lambda_t^H}$ is the probability distribution of the time-change process Λ_t^H. Indeed, we recall the following characterization due to [37, Theorem 3.1]:

Theorem 1.1.7. *Let W_t, $t \in [0,T]$, be a Brownian motion independent of Λ^B and N_t, $t \in [0,T]$, be a centered pure jump Lévy process with Lévy measure ν independent of Λ^H.*
Then B satisfies (H1)-(H2) if and only if $B_t = W_{\Lambda_t^B}$ for any $t \geq 0$ and η satisfies (H3)-(H4) if and only if $\eta_t = N_{\Lambda_t^H}$ for any $t \geq 0$, where the previous inequalities are in distribution sense.

With this interpretation we have that the processes $\lambda \in \mathcal{L}$ represent the *stochastic time-change rate*.

1.2 The non-anticipating derivative

In order to obtain the sufficient and necessary optimality conditions in the MP framework with forward equation driven by time-changed Lévy process, we would like to exploit some form of Malliavin calculus similarly to what has been done in [2, 3]. Unfortunately, we cannot directly apply the classical Malliavin calculus as this is developed for the Brownian motion and Poisson random measures. As a way to circumvent this issue, one could consider a form of conditional Malliavin calculus, such as the one presented in [23, 42] and developed for processes such as the ones introduced in Subsection 1.1.
When applying Malliavin calculus (or the conditional Malliavin calculus) to optimal control, though, the domain of the Malliavin derivative $\mathbb{D}^{1,2} \subsetneq L^2(P)$ constitutes a serious restriction as one does not know in advance whether the optimal control is going to belong to the correct space or not. In [3], for example, the authors overcame such issue by using the Hida-Malliavin calculus for the Brownian motion and centered Poisson random measure. The Hida-Malliavin calculus is an extension of the classical Malliavin calculus to the white noise framework (stochastic distributions) but, in our case, we cannot use this approach since stochastic derivative is not developed for time-changed processes like the ones in Section 1.1 and is currently a topic for research.

In order to deal with these issues while maintaining the highest possible generality, we will use NA-derivative $\mathscr{D}$, which can be seen as the adjoint linear operator to the non-anticipating (Itô) integral and is defined on the whole $L^2(P)$. The advantage of considering the NA-derivative lies in the possibility of considering general martingale random fields such as the ones presented in Section 1.1 and at the same time avoiding any domain restrictions.

In [19], the authors introduced the NA-derivative for martingale random fields such as the ones presented in Subsection 1.1 by generalizing the work in [18], where the NA-derivative for $L^2(P)$-martingales was first introduced.

Definition 1.2.1. The NA-derivative $\mathscr{D}$ with respect to a martingale random field μ such as the ones in Definition 1.1.6 is a linear operator defined for *all* the elements $\zeta \in L^2(P)$ as the limit in $L^2(P \times \Lambda)$

$$\mathscr{D}\zeta := \lim_{n\to\infty} \varphi_n, \tag{1.2.1}$$

of simple $\mathbb{G}$-predictable random fields φ_n, $n \in \mathbb{N}$, defined as:

$$\varphi_n(t,x) := \sum_{k=1}^{K_n} \mathbb{E}\left[\zeta \frac{\mu(\Delta_{nk})}{\mathbb{E}[\Lambda(\Delta_{nk})|\mathcal{G}_{s_{nk}}]}\middle|\mathcal{G}_{s_{nk}}\right] \mathbb{1}_{\Delta_{nk}}(t,x), \quad (t,x) \in \mathbb{X}.$$

Here, the Borel sets Δ_{nk} take the form $\Delta_{nk} := (s_{nk}, u_{nk}] \times B_{nk}$, $k = 1, ..., K_n$, with $0 \le s_{nk} \le u_{nk} \le T$, and $B_{nk} \in \mathfrak{B}$ where $\mathfrak{B}$ is any countable semi-ring that generates the Borel σ-algebra $\mathcal{B}(\mathbb{R})$. Then $\bigcup_{n\in\mathbb{N}}\bigcup_{k=1}^{K_n}\Delta_{nk}=\mathbb{X}$. With a slight abuse of terminology we call the sets Δ_{nk}, $k=1,...,K_n$, a *partition* of $\mathbb{X}$ with refinement n. Clearly all the sets Δ_{nk}, $k=1,...,K_n$, $n\in\mathbb{N}$ constitute a semi-ring generating $\mathcal{B}(\mathbb{X})$. For more details we refer to e.g. [18, 23] and the references therein.

Notice that the NA-derivative allows for an explicit integral representation. Namely, the integrand is characterized in terms of the very random variable to represent, the integrator, and the filtration.

Theorem 1.2.2. *For any $\xi\in L^2(P)$ the NA-derivative $\mathscr{D}\xi$ is well defined and the following stochastic integral representation holds*

$$\xi=\xi^0+\iint_{\mathbb{X}}\mathscr{D}_{t,z}\xi\ \mu(dtdz), \tag{1.2.2}$$

where $\xi^0=\mathbb{E}\left[\xi|\mathcal{F}^{\Lambda}\right]$ satisfies $\mathscr{D}\xi^0\equiv 0$.

The existence and unicity of a stochastic integral representation is well-known from the Kunita-Watanabe Theorem. Theorem 1.2.2 provides an explicit representation to the integrand. The spirit of this result is in line with representations à la Clark-Haussman-Ocone (CHO), see, e.g. [20], however in that case the noise is either a Brownian motion or a centered Poisson random measure and the integrand is characterized in terms of the Malliavin derivative.

We remark that an extension of the Malliavin calculus and CHO representations to the conditional Brownian and the conditional Poisson cases is provided in [42].

From (1.2.2), we can see that $\mathscr{D}$ is actually the dual of the Itô integral:

Proposition 1.2.3. *For all ϕ in $\mathcal{I}^{\mathbb{G}}$ and all ξ in $L^2(P)$, we have*

$$\mathbb{E}\left[\xi\iint_{\mathbb{X}}\phi(t,z)\mu(dt,dz)\right]=\mathbb{E}\left[\iint_{\mathbb{X}}\phi(t,z)\mathscr{D}_{t,z}\xi\ \Lambda(dt,dz)\right].$$

Also we have the martingale representation theorem:

Theorem 1.2.4. *For any square integrable $\mathbb{G}$-martingale $M_t, t\in[0,T]$, the following representation holds true*

$$M(t)=\mathbb{E}[M(T)|\mathcal{F}^{\Lambda}]+\int_0^t\int_{\mathbb{R}}\mathscr{D}_{s,z}M(T)\mu(ds,dz).$$

Remark 1.2.5. Notice that the NA-derivative is continuous in the sense that

$$\xi=L^2(P)-\lim_{n\to\infty}\xi_n,$$

which implies that

$$\mathscr{D}\xi=L^2(P\times\Lambda)-\lim_{n\to\infty}\mathscr{D}\xi_n.$$

1.3 Stochastic games

In Paper II we deal with optimal control for stochastic games between two players. Similarly to what we do in Paper I, we work with time-changed Lévy drivers and obtain a sufficient maximum principle in this setting. In Paper II, similarly to the case presented e.g. in [2] for forward-backward stochastic Volterra integral equations and in [15] for stochastic games, we add some extra complexity to the system. This is done by allowing the players to have their own risk evaluation Y_i, $i = 1, 2$ modeled as a BSDE depending on both the forward equation X^u and the control u. The players will thus take the current state of the market and their perceived risk into account when maximizing their performance functional $J_i(u)$, $i = 1, 2$.

The aim of this subsection is to provide a small introduction to stochastic games between N players. Hereafter we consider a simple case where we assume that at each time $t \in [0, T]$ the players act on a system whose state X can be influenced through their actions. For more details on stochastic games we refer to, e.g., [8] and [34]. The main difference with the setup of a stochastic control problem (which can be regarded as a 1-player game) is the fact that the set of admissible strategies and the performance functionals are of a different nature and need to be re-introduced. In this case, in fact, players have to consider also the choices of others when trying to maximize their own functionals, which leads to the introduction of equilibria in the game. To this end we start by defining what an admissible strategy is, and give different definitions of optimality following the work in [8].

Definition 1.3.1. Given a N-players game, denote with $A^1, ..., A^N$ the sets of actions that players $1, ..., N$ can take at any point in time. Typically A^i, $i = 1, ..., N$ is a compact metric space or a subset of an Euclidean space. We also denote with $\mathcal{A}^i$, $i = 1, ..., N$ its corresponding Borel σ-field.
The set of admissible strategies is denoted with $\mathbb{A}$. The elements $u \in \mathbb{A}$ are N-tuples $u := (u^1, ..., u^N)$ where each $u^i = (u^i_t)_{0 \le t \le T}$, $i = 1, ..., N$ is an A^i-valued process satisfying some measurability and integrability conditions.

In these games, each player is associated with his own performance functional J^i defined as

$$J^i(u) = \mathbb{E}\left[\int_0^T F^i(t, X^u(t), u(t))dt + G^i(X^u(T))\right], \quad i = 1, ..., N.$$

The goal of each player is to maximize his own performance functional.

Definition 1.3.2. A set of admissible strategies $\hat{u} := (\hat{u}^1, ..., \hat{u}^N) \in \mathbb{A}$ is said to be Pareto optimal if there is no $u = (u^1, ..., u^N) \in \mathbb{A}$ such that

$$\begin{aligned} &\forall\ i \in \{1, ..., N\} \quad J^i(\hat{u}) \ge J^i(u), \\ &\exists\ i_0 \in \{1, ..., N\} \quad J^{i_0}(\hat{u}) > J^{i_0}(u). \end{aligned}$$

Definition 1.3.3. A set of admissible strategies $\hat{u} := (\hat{u}^1, ..., \hat{u}^N) \in \mathbb{A}$ is said to be a Nash equilibrium for the game if

$$\forall\ i \in \{1, ..., N\} \quad \forall\ u \in \mathbb{A}, \quad J^i(\hat{u}) \leq J^i(\hat{u}^{-i}, u^i),$$

where $(\hat{u}^{-i}, u^i) := (\hat{u}^1, ..., \hat{u}^{i-1}, u^i, \hat{u}^{i+1}, ..., \hat{u}^N)$.

The notion of Pareto optimality is natural in problems of optimal allocation of resources. A strategy is Pareto optimal if there is no strategy that makes every player at least as well off and at least one player strictly better off. On the other hand the Nash equilibrium definition implies that no player has any incentive in changing his current strategy as long as the other players do not change their choices. In Paper II we assume that the Nash equilibria for our games exist and propose a maximum principle approach to find them.

Notice also that, when dealing with several players, the structure of the information available at time $t \in [0, T]$ may vary from player to player. In general, in fact, there is no reason why different players should have access to the same information. In light of that, we consider subfiltrations $\mathcal{E}_t^{(i)} \subseteq \mathcal{F}_t$, $i = 1, ..., N$, $t \in [0, T]$ where $\mathcal{F}_t$ is the filtration generated by the driving noise μ introduced in Subsection 1.1. Each of those sub-filtration represents the information available at time $t \in [0, T]$ to player i, $i = 1, ..., N$. We will assume that the strategies u^i are $\mathcal{E}_t^{(i)}$-measurable and exploit the relations between our filtrations in order to derive optimality conditions based on the available information.

A particular case of stochastic games is the class of zero-sum games. A zero-sum stochastic game among $N = 2$ players is a stochastic game where $J(u) := J^1(u) = -J^2(u)$, i.e. a gain for the first player is a loss for the second one. In other words this means that player 1 wants to maximize J whereas player 2 will try to minimize it. In this setting, a strategy $u = (u^1, u^2) \in \mathbb{A}$ is a Nash equilibrium for the game if

$$\sup_{u^1 \in \mathbb{A}^1} \inf_{u^2 \in \mathbb{A}^2} J(u^1, u^2) = \inf_{u^2 \in \mathbb{A}^2} \sup_{u^1 \in \mathbb{A}^1} J(u^1, u^2),$$

i.e. u is a saddle point.

1.4 Volterra lift

As we try to recover the Markov property to apply the DPP for a forward equation with Volterra dynamics, we need to introduce the infinite dimensional lift that will allow us to formulate an infinite dimensional problem equivalent to the finite dimensional one given by (1.0.1), (1.0.8). To this end we introduce here the infinite dimensional lift first presented in [13] for uncontrolled affine Volterra processes and then generalized in [12] for uncontrolled Lévy driven processes and, later on, we present the lift for controlled processes exploited in Paper III and

Paper IV. This approach, which widens the setting of [1] and exploited there for the optimal control of a class of linear-quadratic optimal control problems, allows us to recover the Hamilton-Jacobi-Bellman (HJB) equations associated with (1.0.1)-(1.0.3) and derive the optimal value for such optimal control problem. Contrarily to [1] we do not restrict ourselves to the class of kernels that can be represented as Laplace transform of a measure and we consider performance functionals of the form (1.0.1) instead of linear-quadratic ones.
The same inifinite-dimensional lift approach is also presented in [6], where the authors consider a lift of Lévy-driven processes to Hilbert spaces. In our case, thanks to [12], we are able to consider a lift to Banach spaces, which allows us to consider a wider class of kernels. We also notice that, while lifting Lévy driven processes is possible in the current setting, the optimal control of Banach-lifted Lévy-driven forward equations is a topic for future research.

We start by presenting the lift approach in the context of [13]. We thus take uncontrolled linear dynamics for the process X in (1.0.3), i.e.

$$X(t) = x(t) + \int_0^t K(t-s)dV(s), \tag{1.4.1}$$

with

$$dV(t) := \beta(t, X(t))dt + \sigma(t, X(t))dW(t).$$

Here $\beta : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\sigma : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ are affine functions with respect to the second variable and $K(t) = \langle g, \mathcal{S}_t^* \nu \rangle$ is a liftable kernel (see Definition 1.0.2).

In this framework we have that equation (1.4.1) can be rewritten as

$$X(t) = \langle g, \mathcal{Z}(t) \rangle \tag{1.4.2}$$

where the dynamics of $\mathcal{Z}$ are given by

$$\begin{aligned} d\mathcal{Z}^u(t) &= \mathcal{A}^* \mathcal{Z}^u(t)dt + \nu\beta(t, \langle g, \mathcal{Z}^u(t)\rangle) + \nu\sigma(t, \langle g, \mathcal{Z}^u(t)\rangle)dW(t) \\ &:= \Big(\mathcal{A}^* \mathcal{Z}^u(t) + \nu\beta^g(t, \mathcal{Z}_t^u))\Big)dt + \nu\sigma^g(t, \mathcal{Z}_t^u)dW(t) \end{aligned} \tag{1.4.3}$$

with initial condition $\mathcal{Z}(0) = \zeta_0$, where $x(t) := \langle g, \mathcal{S}_t^* \zeta_0 \rangle$. Here, for $t \in [0,T]$ and $z \in Y^*$ we define $\beta^g(t,z) := \beta^g(t, \langle g, z \rangle)$ and $\sigma^g(t,z) := \sigma(t, \langle g, z\rangle)$. Recall now the following definition due to [13]:

Definition 1.4.1. Let Y be a completely regular Hausdorff topological space, Z a Banach space with norm $\|\cdot\|_Z$, and $\rho : Y \longrightarrow (0,\infty)$ an admissible weight function, i.e. a function such that the sets $K_R := \{y \in Y : \ \rho(y) \leq R\}$ are compact for all $R > 0$. We define the Banach space $\mathcal{B}^\rho(Y)$ as the space

$$B^\rho(Y; Z) := \left\{ f : Y \longrightarrow Z : \ \sup_{y \in Y} \rho^{-1}(y) \|f(y)\|_Z < \infty \right\}, \tag{1.4.4}$$

equipped with the norm

$$\|f\|_\rho := \sup_{y \in Y} \rho^{-1}(y) \|f(y)\|_Z.$$

A family of bounded linear operators $P_t : \mathcal{B}^\rho(Y) \longrightarrow \mathcal{B}^\rho(Y)$ for $t \geq 0$ is called *generalized Feller semigroup* if

- $P_0 = I$, the identity on $\mathcal{B}^\rho(Y)$,
- $P_{t+s} = P_t P_s$, for all $t, s \geq 0$
- for all $f \in \mathcal{B}^\rho(Y)$, and $y \in Y$, $\lim_{t \to 0} P_t f(y) = f(y)$,
- there exists a constant $C \in \mathbb{R}$ and $\varepsilon > 0$ such that, for all $t \in [0, \varepsilon]$, $\|P_t\|_{L(\mathcal{B}^\rho(Y))} \leq C$
- P_t is positive for all $t \geq 0$, i.e. for $f \in \mathcal{B}^\rho(Y)$, $f \geq 0$, we have $P_t f \geq 0$.

Then the following result holds

Theorem 1.4.2.

- *The stochastic partial differential equation* (1.4.3) *admits a unique Markovian solution* $(\mathcal{Z}_t)_{t \geq 0}$.
- *This generalized Feller process allows to construct a probabilistically weak and analytically mild càg solution of* (1.4.1)*, i.e. in particular the point evaluations satisfy*

$$\mathcal{Z}_t(x) = \zeta_0(t + x) + \int_0^t K(t - s + x) dV(s),$$

are càg, and for every initial value the semimartingale V can be constructed on an appropriate probabilistic basis.

- *For all $\zeta_0 \in Y^*$ the corresponding jump diffusion stochastic Volterra equation,*

$$\mathcal{Z}_t(0) = \zeta_0(t) + \int_0^t K(t - s) dV(s)$$

admits a probabilistically weak solution with càg trajectories.

Suppose now that the dynamics for the forward process X includes a control u and follows:

$$\begin{aligned} X^u(\tau) &= x(\tau) + \int_t^\tau K(\tau - s)\Big[\beta(s, X^u(s)) + \sigma(s, X^u(s))R(s, X^u(s), u(s))\Big]ds \\ &\quad + \int_t^\tau K(\tau - s)\sigma(s, X^u(s))dW(s) \\ &:= x(\tau) + \int_t^\tau K(\tau - s)dV^u(s), \quad \tau \in [t, T] \end{aligned} \tag{1.4.5}$$

where

$$dV^u(s) = \Big[\beta(s, X^u(s)) + \sigma(s, X^u(s))R(s, X^u(s), u(s))\Big]ds + \sigma(s, X^u(s))dW(s). \tag{1.4.6}$$

Defining ζ as an element in Y^* such that $x(\tau) =: \langle g, \mathcal{S}^*_\tau \zeta\rangle$ (see Remark 1.4.3 below) we can now rewrite (1.4.5) as follows:

$$\begin{aligned} X^u(\tau) &= x(\tau) + \int_t^\tau K(\tau - s)dV^u(s) \\ &= \langle g, \mathcal{S}^*_\tau \zeta\rangle + \int_t^\tau \langle g, \mathcal{S}^*_{\tau-s}\nu\rangle dV^u(s) \\ &= \left\langle g, \mathcal{S}^*_\tau \zeta + \int_t^\tau \mathcal{S}^*_{\tau-s}\nu dV^u(s) \right\rangle \\ &=: \langle g, \mathcal{Z}^u_\tau\rangle, \end{aligned} \tag{1.4.7}$$

where $\mathcal{Z}^u_\tau := \mathcal{S}^*_\tau \zeta + \int_t^\tau \mathcal{S}^*_{\tau-s}\nu dV^u(s)$. One can then check that $\mathcal{Z}^u_\tau$ follows the dynamics:

$$\mathcal{Z}^u_\tau = \mathcal{S}^*_t \zeta + \int_t^\tau \mathcal{A}^* \mathcal{Z}^u_s ds + \int_t^\tau \nu dV^u(s), \tag{1.4.8}$$

In fact, we have that

$$\begin{aligned} \int_t^\tau \mathcal{A}^* \mathcal{Z}^u_s ds &= \int_t^\tau \mathcal{A}^* \left[\mathcal{S}^*_s \zeta + \int_t^s \mathcal{S}^*_{s-v}\nu dV^u(v)\right] ds \\ &= e^{\mathcal{A}^*\tau}\zeta - e^{\mathcal{A}^* t}\zeta + \int_t^\tau \int_v^\tau \mathcal{A}^* e^{\mathcal{A}^*(s-v)}\nu ds dV^u(v) \\ &= e^{\mathcal{A}^*\tau}\zeta - e^{\mathcal{A}^* t}\zeta + \int_t^\tau e^{\mathcal{A}^*(\tau-v)}\nu dV^u(v) - \int_t^\tau \nu dV^u(v) \\ &= \mathcal{Z}^u_\tau - e^{\mathcal{A}^* t}\zeta - \int_t^\tau \nu dV^u(v), \end{aligned}$$

and, rearranging the terms, we obtain (1.4.8). One can then show that, under suitable hypothesis, (1.4.8) (and thus (1.4.5)) admits a unique solution and that the Markov property can be retrieved (see Paper III for more details).

Remark 1.4.3. By defining

$$B(t, X^u(t), u(t)) := \beta(s, X^u(s))\sigma(s, X^u(s))R(s, X^u(s), u(s)),$$

and exploiting (1.4.7), we actually get that the function $x(\tau) = \langle g, \mathcal{S}_t^*\zeta\rangle$ is given by the expression

$$\begin{aligned} x(\tau) &= \mathbb{E}\left[X^u(\tau) - \int_t^\tau K(\tau - s)B(s, X^u(s), u(s))ds\right] \\ &= \left\langle g, \mathbb{E}\left[\mathcal{Z}_\tau^u - \int_t^\tau \mathcal{S}_{\tau-s}^*\nu B(s, X^u(s), u(s))ds\right]\right\rangle \end{aligned}$$

We point out that we can also lift forward equations of the form:

$$X^u(t) = X(0) + \int_0^t K(t-s)dV^u(s),$$

for some liftable kernel K and a semimartingale V^u with specific hypothesis. This is done in Paper IV and follows an approach similar to the one above.

To be able to apply infinite dimensional optimal control techniques and solve the control problem (1.0.1), (1.0.8), we also need to lift the performance functional $J(u)$ in order for it to take values from Y^*. This operation is performed through a change of notation, which actually embodies a shift from a finite to an infinite dimensional approach. This is performed exploiting (1.4.2) as follows:

$$\begin{aligned} J(u) &= \mathbb{E}\left[\int_0^T F(t, X^u(t), u(t))dt + G(X^u(T))\right] \\ &= \mathbb{E}\left[\int_0^T F(t, \langle g, \mathcal{Z}^u(t)\rangle, u(t))dt + G(\langle g, \mathcal{Z}^u(t)\rangle)\right] \\ &:= \mathbb{E}\left[\int_0^T F^g(t, \mathcal{Z}^u(t), u(t))dt + G^g(\mathcal{Z}^u(t))\right] = J^g(u) \end{aligned} \tag{1.4.9}$$

where we defined $F^g(\cdot, z, \cdot) := F(\cdot, \langle g, z\rangle, \cdot)$ and $G^g(z) := G(\langle g, z\rangle)$.
Even though from (1.4.9) we have that $J(u) = J^g(u)$ we write $J^g(u)$ in order to point out the dependence of the performance functional J from the Banach-valued process $\mathcal{Z}$.

With this approach we can thus consider the problem of finding

$$J(\hat{u}) = \sup_{u\in\mathcal{A}} J(u) = \sup_{u\in\mathcal{A}} J^g(u) = J^g(\hat{u}),$$

where the forward dynamics are provided by (1.4.8). We notice that, due to the nature of the lift presented above, maximizing the finite dimensional process (1.0.8) with respect to (1.0.1) is equivalent to maximizing the infinite dimensional process (1.4.8) with respect to (1.4.9). In particular this means that, if we can solve the HJB equations associated with the lifted problem, we are also able to find the optimal control $\hat{u}$ of the original problem.

1.5 Summary of papers

Paper I Our starting point is an optimal harvesting problem from a population, the growth of which is modelled by Volterra time dynamics of the type

$$X(t) = X_0 + \int_0^t \left(r(t,s) - Ku(s)\right) X(s)ds + \int_0^t \sigma(s)X(s)dB(s), \quad t \in [0,T], \tag{1.5.1}$$

and where the performance functional can be regarded as the aggregated net discounted revenue (see [5]):

$$J(u) = \mathbb{E}\left[\int_0^T e^{-\delta(T-t)}X(t)u(t)dt\right]. \tag{1.5.2}$$

Keeping our motivation in mind, we treat here problems of stochastic control for general Volterra type dynamics, allowing also for jumps:

$$\begin{aligned} X^u(t) = X_0 &+ \int_0^t b(t,s,\lambda,u(s),X^u(s))ds \\ &+ \int_0^t\int_{\mathbb{R}} \kappa(t,s,z,\lambda,u(s),X^u(s))\mu(dsdz), \quad t \in [0,T], \end{aligned}$$

where our goal is to find the optimal control $\hat{u}$ such that

$$J(\hat{u}) = \sup_{u\in\mathcal{A}^{\mathbb{F}}} J(u) = \sup_{u\in\mathcal{A}^{\mathbb{F}}} \mathbb{E}\left[\int_0^T F(t,\lambda_t,u(t),X^u(t))dt + G(X^u(T))\right], \tag{1.5.3}$$

among the set $\mathcal{A}^{\mathbb{F}}$ of admissible $\mathbb{F}$-adapted controls. We recall that $\mathbb{F}$ is the smallest right continuous filtration to which μ is adapted and that the noise μ is defined in Definition 1.1.5.

Here the filtration $\mathbb{F}$ is regarded as partial with respect to $\mathbb{G} := \mathbb{F} \vee \mathbb{F}^\Lambda$, where $\mathbb{F}^\Lambda$ is the filtration generated by the time-change process Λ. In order to find the optimal controls, similarly to the results in e.g. [2, 3, 36], we introduce the Hamiltonian function related to the problem and exploit some sort of stochastic derivative. Of course, having a conditional Lévy process, we exploit the NA-derivative. We study the adjoint BSDE associated with such a control problem and formulate both a sufficient and a necessary maximum principle with respect to the information flow $\mathbb{F}$.

We show that in the case of (1.5.1)-(1.5.2), we are able to provide a characterization of the optimal control $\hat{u}$.

Paper II In this second paper we present a model for stochastic games between two players. In this setting we have a forward equation given by

$$\begin{aligned} X(t) = X_0 &+ \int_0^t b(t, s, \lambda_s, u(s), X(s))ds \\ &+ \int_0^t \int_{\mathbb{R}} \kappa(t, s, z, \lambda_s, u(s), X(s))\mu(dsdz), \end{aligned}$$

and two backward equations associated to player $i = 1, 2$

$$\begin{aligned} Y_i(s) = h_i(X(T)) &- \int_t^T g_i(s, \lambda_s, u(s), X(s), Y_i(s), \Theta_i(t, s, \cdot))ds \\ &+ \int_t^T \int_{\mathbb{R}} \Theta_i(s, z)\mu(dsdz) + \int_t^T dM_i(s) \quad i = 1, 2. \end{aligned} \tag{1.5.4}$$

The above term M_i, $(i = 1, 2)$ is a martingale orthogonal to μ naturally appearing in the martingale representation theorem when working with noises and information flows that do not have the perfect stochastic integral representation property, see e.g. [31]. Indeed we recall that Kunita-Watanabe result shows that for a square integrable martingale $\mathcal{M}$ as integrator, and the associated filtration generated

$$\mathbb{F}^{\mathcal{M}} = \left\{ \mathcal{F}^{\mathcal{M}_t, \ t \in [0,T]} \right\},$$

and any square integrable $\mathcal{F}_T^{\mathcal{M}}$-measurable random variable ξ admits representation

$$\xi = \xi^{\perp} + \int \varphi \, d\mathcal{M} \tag{1.5.5}$$

by means of a unique stochastic integrand φ and where $\xi^{\perp}$ is a stochastic remainder orthogonal to the stochastic integrals with respect to $\mathcal{M}$. It is well known that $\xi^{\perp}$ is a constant (naturally given by $\mathbb{E}[\xi]$) whenever $\mathcal{M}$ is a Gaussian or a centered Poisson random measure, or mixture of the two and the reference filtration is generated by $\mathcal{M}$, see e.g. [11, 16, 17, 21]. We say that the representation (1.5.5) is perfect if $\xi^{\perp}$ is a constant. In this case the space of all stochastic integrals with respect to $\mathcal{M}$ coincides with $L^2(\Omega, \mathcal{F}_T^{\mathcal{M}}, P)$. This is tightly connected with chaos expansions (see e.g. [17, 20]). Above $u(t) = (u_1(t), u_2(t))$, where $u_i(t)$ is the control associated to player $i = 1, 2$. The goal of each player is to maximize their own performance functional, defined as

$$J_i(u) := \mathbb{E}\left[\int_0^T F_i(t, u(t), X(t), Y_i(t))dt + \varphi_i(X(T)) + \psi_i(Y_i(0)) \right],$$

for $i = 1, 2$. This model can be interpreted as a game on a financial market, where the forward equation $X^u(t)$ represents the state of the market and the two players interact with the forward equation both via their personal control u_i and via their own risk evaluation $Y^i(t)$, $i = 1, 2$.

In this setting we are also going to assume that each player has a different level of information available. This translates into the model having sub-filtrations of $\mathcal{E}_t^{(i)} \subseteq \mathcal{F}_t$ representing the level of knowledge of each player. Like in Paper I, we exploit the technical interplay between the different filtrations at hand in order to derive sufficient conditions for the optimality of the control $\hat{u}$. Here we do not restrict ourselves to the zero-sum games, which are going to be regarded as a particular case. We also present two possible applications of our model both in the zero and non-zero sum case.

Lastly, we present some conclusive remarks on the use of Volterra-type BSDE of the form:

$$\begin{aligned} Y_i(t) = h_i(X(T)) - \int_t^T g_i(t, s, \lambda_s, u(s), X(s\text{-}), Y_i(s\text{-}), \Theta_i(t, s, \cdot))ds \\ + \int_t^T \int_{\mathbb{R}} \Theta_i(t, s, z)\mu(dsdz). \quad t \in [0, T], \end{aligned} \tag{1.5.6}$$

in place of (1.5.4). Due to the nature of the considered information flows, this is only possible whenever the time-change rates λ^B and λ^H in (1.1.1) are deterministic. We study conditions for the existence and uniqueness of (1.5.6) in the case where g_i is linear, reformulate our optimality conditions and provide an example.

Paper III In this paper we consider the Volterra optimal control problem of maximizing the performance functional

$$J(t, u) = \mathbb{E}\left[\int_t^T F(t, X^u(t), u(t))dt + G(X^u(T))\right], \tag{1.5.7}$$

where the underlying forward process has dynamics given by

$$\begin{aligned} X^u(\tau) = x(\tau) + \int_t^\tau K(\tau - s)\Big[\beta(s, X_s^u)ds + \sigma(s, X_s^u)R(s, X_s^u, u_s)ds\Big] \\ + \int_t^\tau K(\tau - s)\sigma(s, X_s^u)dW_s, \quad \tau \in [t, T] \end{aligned} \tag{1.5.8}$$

Notice also that in this case we assume that the memory effect for our forward process is obtained by means of a convolution kernel K both in the drift and volatility.

As anticipated this control problem cannot usually be solved by means of a DPP due to the non-Markovianity of the system. In order to get around this issue we exploit the infinite dimensional lift presented in Section 1.4 in order to recover some Markov properties of the system. We are thus going to suppose that our kernel is liftable and rewrite the optimal control problem as

$$J^g(t,u) := \mathbb{E}\left[\int_t^T F^g(t,\mathcal{Z}_t^u,u_t)dt + G^g(\mathcal{Z}_T^u)\right]$$

where the dynamics for $\mathcal{Z}$ are given by

$$\begin{aligned} d\mathcal{Z}_\tau^u &= \mathcal{A}^*\mathcal{Z}_\tau^u d\tau + \nu\Big(\beta^g(\tau,\mathcal{Z}_\tau^u) + \sigma(\tau,\mathcal{Z}_\tau^{u,g})R(\tau,\mathcal{Z}_\tau^{u,g},u_\tau)d\tau\Big) \\ &\quad + \nu\sigma(\tau,\mathcal{Z}_\tau^{u,g})dW_\tau, \quad \tau\in[t,T], \end{aligned}$$

where the definition of $g,\nu,\mathcal{A}^*,\beta^g$ and σ^g can be found Subsection 1.4. Having that $X^u(t) = \langle g, \mathcal{Z}^u(t)\rangle$, we study the Markovian lifted optimal control problem in place of the finite-dimensional one.

Inspired by [26, 33], we propose a dynamic programming principle approach in the form of the Hamilton-Jacobi-Bellman equations in infinite dimension, and proceed to derive a method to maximize the infinite dimensional problem. To this end, being our setting more general then the Hilbert-space valued problem presented in [26], but slightly less general than the one introduced in [33], we introduce the *Unconditional Martingale Differences* (UMD) Banach spaces (as presented in [30] and the references therein) in order to obtain some crucial results.

Once we have found the solution to the infinite dimensional problem, we are able to go back to the original control problem by exploiting (1.4.2). Due to the nature of our lift for the performance funcional we notice that the optimal value for the lifted problem and the original one coincide. Moreover, thanks to a verification theorem, we are able to implicitly characterize the optimal value $\hat{u}$ for the lifted infinite-dimensional problem and derive the corresponding optimal value for the original problem.

Paper IV In this paper we study an optimal advertising problem. We consider the following simplified version of (1.5.7)-(1.5.8): we take the performance functional

$$J(u) = \mathbb{E}\left[-\int_0^T a_1 u^2(s)ds + a_2 X^u(T)\right]$$

where the dynamics for $X^u(t)$ are provided by

$$\begin{aligned} X^u(t) &= X_0 + \int_0^t K(t-s)\Big(\alpha u(s) - \beta X^u(s)\Big)ds \\ &\quad + \sigma\int_0^t K(t-s)dW(s), \quad t\in[0,T] \end{aligned}$$

where $\alpha,\beta,\sigma > 0$ are constants the kernel K is in $L^2[0,T]$. In this framework we provide an approximation result that allows us to provide a lift approach for general h-Hölder continuous kernels. The main novelty of this paper is in fact the approximation technique that allows us to consider the approximated forward optimal control problem

$$X_n^u(t) = X_0 + \int_0^t K_n(t-s)\Big(\alpha u(s) - \beta X_n^u(s)\Big)ds$$
$$+ \sigma \int_0^t K_n(t-s)dW(s), \quad t \in [0,T],$$
$$J_n(u) = \sup_{u \in L^2([0,T]\times\Omega)} \mathbb{E}\left[-\int_0^T a_1 u^2(s)ds + a_2 X_n^u(T)\right].$$

and show that both the optimal value and the optimal control of the original problem are "close" to the optimal value and the optimal control of the approximated one.

Exploiting the lift approach like we did in Paper III, we reformulate our problem as an infinite dimensional equivalent one and, due to the simplicity of this model, we are able to explicitly solve the HJB equations for the lifted problem. We then obtain that the optimal value $\hat{J}_n(\hat{u}_n)$ of the approximated problem converges to the optimal value of the original one. This, in turn, allows us to compute explicitly the optimal control for the approximated problem $\hat{u}_n$ and show that $J_n(\hat{u}_n)$ converges to the optimal value of the non approximated problem. We then provide simulations in several interesting cases, in order to show the effectiveness of our approach in different scenarios.

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Papers

I

Paper I

Stochastic Volterra equations with time-changed Lévy noise and maximum principles

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Abstract

Motivated by a problem of optimal harvesting of natural resources, we study a control problem for Volterra type dynamics driven by time-changed Lévy noises, which are in general not Markovian. To exploit the nature of the noise, we make use of different kind of information flows within a maximum principle approach. For this we work with backward stochastic differential equations (BSDE) with time-change and exploit the *non-anticipating* stochastic derivative introduced in [16]. We prove both a sufficient and necessary stochastic maximum principle.

Keywords: time-change; conditionally independent increments; backward stochastic Volterra integral equation; maximum principle; stochastic Volterra equations; non-anticipating stochastic derivative
MSC 2020: 60H10; 60H20; 93E20; 60G60; 91B70;

I.1 Introduction

Optimal harvesting is a fairly classical problem in control theory and it is still a timely question to address when thinking of sustainability in the management of natural resources. In this work we deal with a problem of optimal harvesting from a population, the growth of which is modelled by Volterra time dynamics of the type

$$X(t) = X_0 + \int_0^t \left(r(t,s) - Ku(s)\right) X(s)ds + \int_0^t \sigma(s)X(s)dB(s), \quad t \in [0,T]. \tag{I.1.1}$$

The term r represents the growth rate, the constant K is the catchability coefficient, and the control u is the fishing effort. The Volterra structure is inherited from the deterministic analogous models that can be found, e.g., in [10, 23, 24]. As we can see, this form of time dependence is often used in the description of fish populations. When considering fish as a commodity, the modelling of fish population is representing the possible dynamics of offer, in the interplay between offer and demand. In our work, however, we consider Volterra *stochastic* integral equations, which represent a natural extension including the uncertainty of the environment influencing the population growth. For this we are motivated by [4, 11].

Our model has an element of novelty with respect to the others presented. This is given by the nature of the noise B which is associated to a time-changed Brownian motion. This is well motivated by the clustering effects that such noises can described. For the description on how time-change helps to described clustering, we can refer to a first discussion in [25, Chapter IV, 3e] and a more recent study [35, Chapter 3] in the context of market microstructure. Within population dynamics the evidence of clustering is largely discussed in the recent literature in biology and ecology. See just as example [27].

We remark that in the literature of mathematical finance, dynamics of the form (I.1.1), but with Lévy type noises were used in models [2]. On the other side, time-change has been suggested in the study of volatility modelling, e.g. [6, 12, 21, 37, 38], energy markets, e.g. [9], and default models, e.g.[29]. Also it is used in kinetic theory, see e.g. [30].

Keeping our motivation in mind, we treat here stochastic control for general Volterra type dynamics, allowing also for jumps:

$$\begin{aligned} X^u(t) = X_0 &+ \int_0^t b(t,s,\lambda_s,u(s),X^u(s\text{-}))ds \\ &+ \int_0^t\int_{\mathbb{R}} \kappa(t,s,z,\lambda_s,u(s),X^u(s\text{-}))\mu(dsdz), \end{aligned} \tag{I.1.2}$$

where the driving noise μ is given by the random measure

$$\mu(\Delta) = B(\Delta \cap [0,T]\times\{0\}) + \widetilde{H}(\Delta \cap [0,T]\times\mathbb{R}_0), \quad \Delta \in \mathcal{B}([0,T]\times\mathbb{R}), \tag{I.1.3}$$

which is the mixture of a *conditional Gaussian measure* B on $[0,T]\times\{0\}$ and a *conditional centered Poisson measure* $\widetilde{H}$ on $[0,T]\times\mathbb{R}_0$. Here $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$ and $\mathcal{B}$ represents the Borel σ-algebra. Both B and $\widetilde{H}$ are set in relationship with a time-changed Brownian motion and time-changed Poisson measure, respectively, via Theorem 3.1 in [36] (see also [22]). Note that the coefficients in (I.1.2) may also depend on the time-change via the process λ.

The time-change processes involved are of the form

$$\Lambda_t(\omega) = \int_0^t \lambda_s(\omega)ds, \quad (t,\omega)\in[0,T]\times\Omega,$$

$(T > 0)$. Thus the driving noises (which include jumps) are actually beyond the Brownian and the pure Lévy framework. We abandon noises with independent increments and effectively deal with quite general but still treatable martingales.

Our goal is to find the optimal control $\hat{u}$ such that

$$J(\hat{u}) = \sup_{u \in \mathcal{A}^{\mathbb{F}}} J(u) = \sup_{u \in \mathcal{A}^{\mathbb{F}}} \mathbb{E}\left[\int_0^T F(t, \lambda_t, u(t), X^u(t))dt + G(X^u(T))\right], \quad \text{(I.1.4)}$$

among the set $\mathcal{A}^{\mathbb{F}}$ of admissible $\mathbb{F}$-adapted controls, where $\mathbb{F} = \{\mathcal{F}_t,\ t \in [0, T]\}$ represents the smallest right-continuous filtration generated by μ.

Optimization problems such as (I.1.2), (I.1.4) are studied, e.g. in [2, 3, 8]. In [2, 3] the authors present also a sufficient maximum principle and the dynamics include jumps making use of Malliavin calculus. However, being the restrictions on the domain of the Malliavin derivative extremely serious in the context of optimal control, the authors have lifted the study into the white noise framework and work with the Hida-Malliavin calculus on the space of stochastic distributions. The Hida-Malliavin calculus is taylored for Brownian and for centered Poisson random noises, hence this approach cannot be taken in our work since our driving noises are *not* of the required nature. On the other hand, in [8], the authors propose a backward SDE approach to solve (I.1.4). This is possible due to the introduction of memory in (I.1.2) by means of convolution with a completely monotone kernel which allows for a Markovian representation of the solution of (I.1.2).

Note that a Malliavin/Skorokhod calculus extension to noises with conditional independent increments, is proposed in [20] and [39]. By this, however, we cannot solve the critical issue of the natural restriction of the domains of the involved operators and a Hida-Malliavin type extension is yet not available in the literature. Our approach is then to make use of the *non-anticipating* (NA) derivative. The NA derivative, introduced in [15] for general martingales and then extended to martingale random fields in [16], is the dual of the Itô integral and has an explicit representation in terms of limit of simple integrands in the Itô framework. Also, the NA derivative provides explicit stochastic integral representations. We stress that, contrarily to the Malliavin derivative, the domain of the NA derivative is the whole $L^2(dP)$, thus not creating problems in the context of optimal controls. To the best of our knowledge this is the first time that the non-anticipating derivative is used in optimal control problems such as (I.1.4).

Our approach to the optimization problem (I.1.4) is based on the analysis of the noise and the information flows associated. Indeed, we observe that there are two filtrations of interest. The first one is the already mentioned $\mathbb{F}$ and the second is the filtration $\mathbb{G} := \{\mathcal{G}_t,\ t \in [0, T]\}$, where $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}^\Lambda$ generated by μ and the entire history $\mathcal{F}^\Lambda$ of the time-change processes. Note that while $\mathcal{F}_0$ is substantially trivial, $\mathcal{G}_0 = \mathcal{F}^\Lambda$. We can regard $\mathbb{G}$ as the initial enlargement of $\mathbb{F}$ or, we can see $\mathbb{F}$ as partial information with respect to $\mathbb{G}$. With this observation in hands, we work out the solution to problem (I.1.4) as an optimization problem under partial information. In this we have taken inspiration from [32], where the concept of partial information is however not associated to the properties of the noise, and from [19], where the dynamics are however not of Volterra type. Also, for completeness, we show that our techniques provide necessary and sufficient conditions for the optimization problem

$$J(\hat{u}) = \sup_{u \in \mathcal{A}^{\mathbb{G}}} J(u) = \sup_{u \in \mathcal{A}^{\mathbb{G}}} \mathbb{E}\left[\int_0^T F(t, \lambda_t, u(t), X^u(t))dt + G(X^u(T))\right] \quad \text{(I.1.5)}$$

on the set $\mathcal{A}^{\mathbb{G}}$ of admissible $\mathbb{G}$-adapted controls, where $\mathcal{A}^{\mathbb{F}} \subset \mathcal{A}^{\mathbb{G}}$.

The study of maximum principles is associated to a stochastic Hamiltonian map of the so-called dual variables, which in turn are obtained from the solution of a backward stochastic equation. In the sequel, we deal with backward stochastic differential equations (BSDEs) of type

$$p(t) = \xi(t) + \int_t^T g(s, \lambda_s, p(s\text{-}), q(s, \cdot))ds - \int_t^T\int_{\mathbb{R}} q(s, z)\mu(dsdz), \quad \text{(I.1.6)}$$

under the filtration $\mathbb{G}$. Notice that, these backward equations are not of Volterra type. This is because our Hamiltonian functional is going to involve also the NA-derivatives of the adjoint process p. A different approach could have been to follow the work in [2], where the authors deal with a backward stochastic Volterra integral equation (BSVIE) of the form:

$$p(t) = \xi(t) + \int_t^T g(s, \lambda_s, p(s\text{-}), q(t, s, \cdot))ds - \int_t^T\int_{\mathbb{R}} q(t, s, z)\mu(dsdz).$$

Even though this approach would allow us to work with simpler Hamiltonian functionals (in the sense that the NA-derivative of $p(t)$ would not be involved) we would need to assume smoothness conditions with respect to t on $q(t, s, z)$ and, to the best of our knowledge, is not clear to what extent those properties are satisfied.

Existence and uniqueness of (I.1.6) can be retrieved from [19]. The study of the BSDE under $\mathbb{G}$ is in itself critically based on the stochastic integral representation in the form

$$\xi = \xi^0 + \int_0^T\int_{\mathbb{R}} \phi(s, z)\mu(dsdz), \quad \text{(I.1.7)}$$

where ξ^0 is $\mathcal{G}_0$-measurable and the integrand ϕ is $\mathbb{G}$-predictable. These results are readily available in terms of their existence in the classical Kunita-Watanabe Theorem, while the *explicit* form of ϕ is given by means of the NA derivative in [15] Theorem 3.1 and [16] Theorem 3.1.

The paper is organized as follows. In the next section we give a presentation of the framework providing the necessary details for the random measure μ and the information flows that we are going to use. In Section 3 we prove a sufficient maximum principle and in Section 4 the corresponding necessary maximum principle. Lastly, we show how the results obtained can be applied to characterise the solution in the optimal harvesting problem associated to the dynamics (I.1.1).

I.2 The noise and the non-anticipating derivative

Let us consider a complete probability space $(\Omega, \mathcal{F}, P)$ and a time horizon $T < \infty$. We shall consider the noise on the time-space

$$\mathbb{X} := [0,T] \times \mathbb{R} := \Big([0,T] \times \{0\}\Big) \cup \Big([0,T] \times \mathbb{R}_0\Big),$$

where $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$. The Borel σ-algebra on $\mathbb{X}$ is denoted $\mathcal{B}_{\mathbb{X}}$. Let $\mathcal{L}$ be the space of the two dimensional stochastic processes $\lambda = (\lambda^B, \lambda^H)$ such that, for each component $k = B, H$, we have that

1. $\lambda_t^k \geq 0$ $P-a.s.$ for all $t \in [0,T]$,
2. $\lim_{h\to 0} P\left(|\lambda_{t+h}^k - \lambda_t^k| \geq \epsilon\right) = 0$ for all $\epsilon > 0$ and almost all $t \in [0,T]$,
3. $\mathbb{E}\left[\int_0^T \lambda_t^k dt\right] < \infty$.

The processes $\lambda \in \mathcal{L}$ represent the *stochastic time-change rate*. Let ν be a σ-finite measure on the Borel sets of $\mathbb{R}_0$ satisfying $\int_{\mathbb{R}_0} z^2\nu(dz) < \infty$. We define the random measure Λ on $\mathcal{B}_{\mathbb{X}}$ by

$$\Lambda(\Delta) := \int_0^T \mathbb{1}_{\{(t,0)\in\Delta\}}(t)\lambda_t^B dt + \int_0^T\int_{\mathbb{R}_0} \mathbb{1}_\Delta(t,z)\nu(dz)\lambda_t^H dt, \quad \Delta \subseteq \mathbb{X}. \quad \text{(I.2.1)}$$

Furthermore, denote the restrictions of Λ to $[0,T]\times\{0\}$ and $[0,T]\times\mathbb{R}_0$ by Λ^B and Λ^H, respectively. For later use we also introduce the filtration

$$\mathbb{F}^\Lambda = \{\mathcal{F}_t^\Lambda, t \in [0,T]\},$$

where $\mathcal{F}_t^\Lambda$ is generated by the values of Λ on the Borelian sets of $[0,t]\times\mathbb{R}$. Set $\mathcal{F}^\Lambda := \mathcal{F}_T^\Lambda$. We recall the following definitions.

Definition I.2.1. The *conditional Gaussian measure* (given $\mathcal{F}^\Lambda$) B is a signed random measure on the Borel sets of $[0,T]\times\{0\}$ satisfying

A1. $P\left(B(\Delta) \leq x|\mathcal{F}^\Lambda\right) = P\left(B(\Delta) \leq x|\Lambda^B(\Delta)\right) = \Phi\left(\frac{x}{\sqrt{\Lambda^B(\Delta)}}\right)$,
$x \in \mathbb{R}, \Delta \subseteq [0,T]\times\{0\}$. Here Φ is the cumulative probability distribution function of a standard normal random variable.

A2. For all disjoint $\Delta_1, \Delta_2 \subseteq [0,T] \times \{0\}$, $B(\Delta_1)$ and $B(\Delta_2)$ are conditionally independent given $\mathcal{F}^\Lambda$.

The *conditional Poisson measure* (given $\mathcal{F}^\Lambda$) H is a random measure on the Borel sets of $[0,T] \times \mathbb{R}_0$ satisfying

A3. $P\left(H(\Delta)=k|\mathcal{F}^\Lambda\right) = P\left(H(\Delta)=k|\Lambda^H(\Delta)\right) = \frac{\Lambda^H(\Delta)^k}{k!}e^{-\Lambda^H(\Delta)}$, $k \in \mathbb{N}$, $\Delta \subseteq [0,T] \times \mathbb{R}_0$.

A4. For all disjoint $\Delta_1, \Delta_2 \subseteq [0,T] \times \{\mathbb{R}_0\}$, $H(\Delta_1)$ and $H(\Delta_2)$ are conditionally independent given $\mathcal{F}^\Lambda$.

Moreover,

A5. B and H are conditionally independent given $\mathcal{F}^\Lambda$.

Also the *conditional centered Poisson random measure* is defined as

$$\widetilde{H}(\Delta) := H(\Delta) - \Lambda^H(\Delta), \quad \Delta \subset \mathbb{X}.$$

Observe that if λ^B and λ^H were deterministic, then B would be a Gaussian process and H a Poisson random measure. Furthermore, B would be a Wiener process if $\lambda^B \equiv 1$ and H a homogeneous Poisson random measure for $\lambda^H \equiv 1$.

Definition I.2.2. We define the signed random measure μ on the Borel sets $\Delta \subseteq \mathbb{X}$ by

$$\mu(\Delta) := B\left(\Delta \cap [0,T] \times \{0\}\right) + \widetilde{H}\left(\Delta \cap [0,T] \times \mathbb{R}_0\right).$$

The random measure μ has *conditionally independent values*, see [22, 36]. Observe that (A1) and (A3) yield

$$\mathbb{E}[\mu(\Delta)|\mathcal{F}^\Lambda] = 0, \qquad \mathbb{E}[\mu(\Delta)^2|\mathcal{F}^\Lambda] = \Lambda(\Delta), \quad \Delta \subseteq \mathbb{X}. \tag{I.2.2}$$

The random measures B and H are related to a time-changed Brownian motion and time-changed pure jump Lévy process. To illustrate, consider the processes on $[0,T]$:

$$B_t := B([0,t] \times \{0\}), \qquad \Lambda_t^B := \int_0^t \lambda_s^B ds,$$

$$\eta_t := \int_0^t\int_{\mathbb{R}_0} z\widetilde{H}(dsdz), \qquad \Lambda_t^H := \int_0^t \lambda_s^H ds,$$

and compute the characteristic functions of B and η. From (A1) and (A3) we have that

$$\mathbb{E}\left[e^{icB_t}\right] = \int_{\mathbb{R}} \mathbb{E}\left[e^{icB_t}|\Lambda_t^B = x\right] P_{\Lambda_t^B}(dx) = \int_{\mathbb{R}} e^{\frac{1}{2}c^2 x} P_{\Lambda_t^B}(dx), \quad c \in \mathbb{R},$$

where $P_{\Lambda_t^B}$ is the probability distribution of the time-change Λ_t^B. Correspondingly, we have that

$$\mathbb{E}\left[e^{ic\eta_t}\right] = \int_{\mathbb{R}} \exp\left\{\int_{\mathbb{R}_0} [e^{iczx} - 1 - iczx]\nu(dz)\right\} P_{\Lambda_t^H}(dx), \quad c \in \mathbb{R},$$

where $P_{\Lambda_t^H}$ is the probability distribution of the time-change Λ_t^H. Indeed we recall the following characterization [36, Theorem 3.1] :

Theorem I.2.3. *Let W_t, $t \in [0,T]$, be a Brownian motion independent of Λ^B and N_t, $t \in [0,T]$, be a centered pure jump Lévy process with Lévy measure ν independent of Λ^H. Then B satisfies (A1)-(A2) if and only if, for any $t \geq 0$, $B_t \overset{d}{=} W_{\Lambda_t^B}$ and η satisfies (A3)-(A4) if and only if, for any $t \geq 0$, $\eta_t \overset{d}{=} N_{\Lambda_t^H}$.*

In the sequel we shall consider two types of information flows. The first one is represented by the filtration

$$\mathbb{F} := \{\mathcal{F}_t,\ t \in [0,T]\}, \qquad \mathcal{F}_t := \bigcap_{r>t} \mathcal{F}_r^\mu,$$

where $\mathbb{F}^\mu := \{\mathcal{F}_t^\mu,\ t \in [0,T]\}$ is generated by the values $\mu(\Delta)$, $\Delta \subset [0,t] \times \mathbb{R}$, $t \in [0,T]$. Correspondingly, let $\mathbb{F}^B := \{\mathcal{F}_t^B, t \in [0,T]\}$ denote the filtration generated by $B(\Delta \cap [0,t] \times \{0\})$, and $\mathbb{F}^H := \{\mathcal{F}_t^H, t \in [0,T]\}$ the filtration generated by $H(\Delta \cap [0,t] \times \mathbb{R}_0)$. We remark that, for any $t \in [0,T]$, $\mathcal{F}_t^\mu = \mathcal{F}_t^B \vee \mathcal{F}_t^H \vee \mathcal{F}_t^\Lambda$. See [20].

The second information flow of interest is

$$\mathbb{G} := \{\mathcal{G}_t,\ t \in [0,T]\}, \quad \mathcal{G}_t := \mathcal{F}_t^\mu \vee \mathcal{F}^\Lambda.$$

The filtration $\mathbb{G}$ is right-continuous, see [19]. Moreover we note that $\mathcal{G}_T = \mathcal{F}_T$, $\mathcal{G}_0 = \mathcal{F}^\Lambda$, and $\mathcal{F}_0$ is substantially trivial. Namely, $\mathbb{G}$ includes information on the future values of Λ^B and Λ^H. In the sequel we shall technically exploit the interplay between the two filtrations.

For $\Delta \subseteq (t,T] \times \mathbb{R}$, the conditional independence in (A2) and (A4), together with (I.2.2) yield

$$\mathbb{E}[\mu(\Delta)|\mathcal{G}_t] = \mathbb{E}[\mu(\Delta)|\mathcal{F}_t \vee \mathcal{F}^\Lambda] = \mathbb{E}[\mu(\Delta)|\mathcal{F}^\Lambda] = 0. \tag{I.2.3}$$

Moreover, (A5) gives us

$$\mathbb{E}[\mu(\Delta_1)\mu(\Delta_2)|\mathcal{G}_t] = \mathbb{E}[\mu(\Delta_1)|\mathcal{F}^\Lambda]\mathbb{E}[\mu(\Delta_2)|\mathcal{F}^\Lambda] = 0,$$

for disjoint $\Delta_1, \Delta_2 \subseteq (t,T] \times \mathbb{R}$. Hence, μ is a martingale random field with respect to $\mathbb{G}$, see e.g. [16] Definition 2.1:

Definition I.2.4. A square integrable *martingale random field μ with conditionally orthogonal values* is a stochastic set function $\mu(\Delta)$, $\Delta \subseteq \mathbb{X}$ such that

- $m(\Delta) := \mathbb{E}[\mu(\Delta)^2] = \mathbb{E}[\Lambda(\Delta)]$, $\Delta \subseteq \mathbb{X}$, defines a variance measure
- μ is $\mathbb{G}$-adapted
- μ satisfies the martingale property (I.2.3)
- μ has conditionally orthogonal values: $\mathbb{E}[\mu(\Delta_1)\mu(\Delta_2)|\mathcal{G}_t] = 0$, for every disjoint $\Delta_1, \Delta_1 \in (t,T] \times \mathbb{R}$.

It is immediate to see that μ is also a martingale random field with respect to $\mathbb{F}$.

With the above structures, we access the framework of Itô stochastic integration. For this we introduce $\mathcal{I}^{\mathbb{G}} \subseteq L^2(d\Lambda \times dP)$ representing the subspace of the random fields admitting a $\mathbb{G}$-predictable modification and $\mathcal{I}^{\mathbb{F}} \subset \mathcal{I}^{\mathbb{G}}$, the one of $\mathbb{F}$-predictable random fields. Observe that, for all $\phi \in \mathcal{I}^{\mathbb{G}}$, we have that

$$\mathbb{E}\Big[\Big(\iint_{\mathbb{X}} \phi(s,z)\mu(dsdz)\Big)^2\Big] = \mathbb{E}\Big[\iint_{\mathbb{X}} \phi(s,z)^2\Lambda(dsdz)\Big] \tag{I.2.4}$$

thanks to (A5) and the martingale property of μ.

In this work we shall make use of the non-anticipating derivative introduced in [16] for martingale random fields.

Definition I.2.5. The *non-anticipating derivative* (NA-derivative) $\mathscr{D}$ is a linear operator defined for *all* the elements $\zeta \in L^2(dP)$ as the limit in $L^2(d\Lambda \times dP)$

$$\mathscr{D}\zeta := \lim_{n\to\infty} \varphi_n, \tag{I.2.5}$$

of simple $\mathbb{G}$-predictable random fields φ_n, $n \in \mathbb{N}$, defined as:

$$\varphi_n(t,x) := \sum_{k=1}^{K_n} \mathbb{E}\left[\zeta \frac{\mu(\Delta_{nk})}{\mathbb{E}[\Lambda(\Delta_{nk})|\mathcal{G}_{s_{nk}}]}\middle|\mathcal{G}_{s_{nk}}\right] \mathbb{1}_{\Delta_{nk}}(t,x), \quad (t,x) \in \mathbb{X}.$$

Here the Borel sets Δ_{nk} take the form $\Delta_{nk} := (s_{nk}, u_{nk}] \times B_{nk}$, $k = 1, ..., K_n$, with $0 \le s_{nk} \le u_{nk} \le T$, and $B_{nk} \in \mathfrak{B}$ where $\mathfrak{B}$ is any countable semi-ring that generates the Borel σ-algebra $\mathcal{B}(\mathbb{R})$. Then $\bigcup_{n\in\mathbb{N}} \bigcup_{k=1}^{K_n} \Delta_{nk} = \mathbb{X}$. With a slight abuse of terminology we call the sets Δ_{nk}, $k = 1, ..., K_n$, a *partition* of $\mathbb{X}$ with refinement n. Clearly all the sets Δ_{nk}, $k = 1, ..., K_n$, $n \in \mathbb{N}$ constitute a semiring generating $\mathcal{B}(\mathbb{X})$ see, e.g. [16] and the references therein.

The NA-derivative allows for an explicit integral representation. Namely the integrand is characterized in terms of the inputs: the very random variable to represent, the integrator, and the filtration. See Theorem 3.1 in [16].

Theorem I.2.6. *For any $\xi \in L^2(dP)$ the NA-derivative $\mathscr{D}\xi$ is well defined and the following stochastic integral representation holds*

$$\xi = \xi^0 + \iint_{\mathbb{X}} \mathscr{D}_{t,z}\xi \ \mu(dtdz), \tag{I.2.6}$$

where $\xi^0 = \mathbb{E}\left[\xi|\mathcal{F}^\Lambda\right]$ satisfies $\mathscr{D}\xi^0 \equiv 0$.

The existence and unicity of a stochastic integral representation is well-known from the Kunita-Watanabe Theorem. Theorem I.2.6 provides an explicit representation to the integrand. The spirit of this result is in line with representations à la Clark-Haussman-Ocone (CHO), see, e.g. [18]. However in that case the noise is either a Brownian motion or a centered Poisson random measure and the integrand is characterized in terms of the Malliavin derivative. We remark that an extension of the Malliavin calculus and CHO representations to the conditional Brownian and the conditional Poisson cases is provided in [39] and [19]. When applying Malliavin calculus to optimal control, the domain of the Malliavin derivative constitutes a serious restriction as the variables depend on a control yet to be found. In [1] this was overcome for the Brownian and centered Poisson cases by using the Hida-Malliavin extension which is an extension of Malliavin calculus to the white noise framework (stochastic distributions), see [18]. At present there is no such an extension for time-changed noises hence the method cannot be used. In this paper we suggest to use the NA-derivative, which has no restrictions on the domain and it is well defined for all martingales in $L^2(dP)$ as integrators. Furthermore, from Theorem I.2.6 we can see that $\mathscr{D}$ is actually the dual of the Itô integral:

Proposition I.2.7. *For all ϕ in $\mathcal{I}^{\mathbb{G}}$ and all ξ in $L^2(dP)$, we have*

$$\mathbb{E}\left[\xi \iint_{\mathbb{X}} \phi(t,z)\mu(dtdz)\right] = \mathbb{E}\left[\iint_{\mathbb{X}} q(t,z)\mathscr{D}_{t,z}\xi \ \Lambda(dtdz)\right].$$

Also we have the martingale representation theorem:

Theorem I.2.8. *For any square integrable $\mathbb{G}$ martingale, $M(t), t \in [0,T]$, the following representation holds true*

$$M(t) = \mathbb{E}[M(T)|\mathcal{F}^\Lambda] + \int_0^t\int_{\mathbb{R}} \mathscr{D}_{s,z}M(T)\mu(dsdz).$$

For future use we also introduce the space S of the $\mathbb{G}$-adapted stochastic processes $p(t,\omega)$, $t \in [0,T]$, $\omega \in \Omega$ such that

$$\|p\|_S := \mathbb{E}\left[\sup_{0\le t\le T} |p(t)|^2\right]^{1/2} < \infty.$$

I.3 A sufficient maximum principle with time-change

We are now ready to study the optimization problem (I.1.4) with performance functional

$$J(u) = \mathbb{E}\left[\int_0^T F(t, \lambda_t, u(t), X^u(t))dt + G(X^u(T))\right], \tag{I.3.1}$$

where

$$\begin{aligned} &F : [0, T] \times [0, \infty)^2 \times \mathcal{U} \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}, \\ &G : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}, \end{aligned}$$

with $\mathcal{U}$ a closed convex subset of $\mathbb{R}$. For all $\lambda \in [0, \infty)^2$, $u \in \mathcal{U}$, $x \in \mathbb{R}$ the process $F(\cdot, \lambda, u, x, \cdot)$ is $\mathbb{F}$-adapted and the mapping $F(t, \lambda, u, x)$ is C^1 in x P-a.s. uniformly w.r.t. $t \in [0, T]$, $\lambda \in [0, \infty)^2$, $u \in \mathcal{U}$. Also for all $x \in \mathbb{R}$, $G(x, \cdot)$ is $\mathcal{F}_T$-measurable and G is C^1 in x P-a.s. uniformly w.r.t. $t \in [0, T]$, $\lambda \in [0, \infty)^2$, $u \in \mathcal{U}$. The controlled dynamics of X are given by the equation

$$\begin{aligned} X^u(t) = X_0 &+ \int_0^t b(t, s, \lambda_s, u(s), X^u(s\text{-}))ds \\ &+ \int_0^t\int_{\mathbb{R}} \kappa(t, s, z, \lambda_s, u(s), X^u(s\text{-}))\mu(dsdz), \end{aligned} \tag{I.3.2}$$

where $X_0 \in \mathbb{R}$ and the coefficients are given by the mappings

$$\begin{aligned} &b : [0, T] \times [0, T] \times [0, \infty)^2 \times \mathcal{U} \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}, \\ &\kappa : [0, T] \times [0, T] \times \mathbb{R} \times [0, \infty)^2 \times \mathcal{U} \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}. \end{aligned}$$

We assume $b(t, \cdot, \lambda, u, x, \cdot)$ and $\kappa(t, \cdot, z, \lambda, u, x, \cdot)$ to be $\mathbb{F}$-predictable for all $t \in [0, T], \lambda \in [0, \infty)^2, u \in \mathcal{U}, x \in \mathbb{R}$ and $z \in \mathbb{R}$. We also require them to be C^2 with respect to t and to x with partial derivatives L^2-integrable with respect to $dt \times dP$ and $d\Lambda \times dP$, respectively. Notice also that we will often drop the superscript u when it is clear the dependence of X on u.

Later on we can see the coefficients b and κ in a functional setup:

$$\begin{aligned} &b : [0, T] \times [0, T] \times \Xi_{\mathbb{R}^2_+} \times \Xi_{\mathcal{U}} \times \Xi_{\mathbb{R}} \times \Omega \longrightarrow \mathbb{R}, \\ &\kappa : [0, T] \times [0, T] \times \mathbb{R} \times \Xi_{\mathbb{R}^2_+} \times \Xi_{\mathcal{U}} \times \Xi_{\mathbb{R}} \times \Omega \longrightarrow \mathbb{R}, \end{aligned}$$

where we denoted by Ξ_S the space of measurable function on $[0, T]$ with values in S. Then we can interpret the coefficients in (I.3.2) via the evaluation at the point $s \in [0, T]$:

$$\begin{aligned} b(t, \cdot, \lambda_\cdot, u(\cdot), X^u(\cdot))(s) &= b(t, s, \lambda_s, u(s), X^u(s\text{-})) \\ \kappa(t, \cdot, z, \lambda_\cdot, u(\cdot), X^u(\cdot))(s) &= \kappa(t, s, z, \lambda_s, u(s), X^u(s\text{-})). \end{aligned}$$

We assume that b and κ are Fréchet differentiable (in the standard topology of càdlàg paths) with C^2 regularity in t, x and u (with the corresponding derivatives).

In the sequel we assume existence and uniqueness of a solution for (I.3.2). Sufficient conditions for this are provided in the next result, which is in line with the study in [2], though there the driving noises are the Brownian motion and Poisson random measure.

Theorem I.3.1. *Assume that:*

1. $b(t,\cdot,\lambda,u,x,\cdot)$ *and* $\kappa(t,\cdot,z,\lambda,u,x,\cdot)$ *are* $\mathbb{F}$*-predictable for all* $t \in [0,T], z \in \mathbb{R}, \lambda \in [0,\infty)^2, u \in \mathcal{U}$ *and* $x \in \mathbb{R}$.

2. $b(t,s,\lambda,u,\cdot)$ *and* $\kappa(t,s,\cdot,\lambda,u,\cdot)$ *are Lipschitz continuous with respect to* x*, uniformly in* $t,s \in [0,T]^2$*,* $u \in \mathcal{U}$*,* $\lambda \in [0,\infty)^2$*, i.e., for all* $x_1, x_2 \in \mathbb{R}$*,*

$$|b(t,s,\lambda,u,x_1) - b(t,s,\lambda,u,x_2)| + |\kappa(t,s,0,\lambda,u,x_1) - \kappa(t,s,0,\lambda,u,x_2)|\sqrt{\lambda^B}$$
$$+ \left(\int_{\mathbb{R}_0} |\kappa(t,s,z,\lambda,u,x_1) - \kappa(t,s,z,\lambda,u,x_2)|^2 \nu(dz)\right)^{1/2} \sqrt{\lambda^H} \le C|x_1 - x_2|,$$

   $P-a.s.$

3. $b(t,s,\lambda,u,\cdot)$ *and* $\kappa(t,s,z,\lambda,u,\cdot)$ *have linear growth with respect to* x*, i.e., for all* $t,s \in [0,T]^2$*,* $u \in \mathcal{U}$*,* $\lambda \in [0,\infty)^2$*,* $x \in \mathbb{R}$*, we have*

$$|b(t,s,\lambda,u,x)| + |\kappa(t,s,0,\lambda,u,x)|\sqrt{\lambda^B}$$
$$+ \left(\int_{\mathbb{R}_0} |\kappa(t,s,z,\lambda,u,x)|^2 \nu(dz)\right)^{1/2} \sqrt{\lambda^H} \le C(1+|x|),$$

   $P-a.s.$

Then there exists a unique $\mathbb{F}$*-adapted solution to* (I.3.2) *in* $L^2(dt \times dP)$.

Proof. The proof follows a classical Picard iteration scheme. Here we provide the main ideas. Fix $u \in \mathcal{A}^{\mathbb{F}}$ and define inductively

$$X^0(t) := X_0$$
$$X^n(t) := X_0 + \int_0^t b(t,s,\lambda_s,u(s),X^{n-1}(s))ds$$
$$+ \int_0^t\int_{\mathbb{R}} \kappa(t,s,z,\lambda_s,u(s),X^{n-1}(s\text{-}))\mu(dsdz), \qquad t \in [0,T],\ n \ge 1$$

Then, for all $t \in [0,T]$ and for all $n \geq 1$, we have the following estimate

$$\begin{aligned}
&\mathbb{E}\left[|X^{n+1}(t) - X^n(t)|^2\right] \\
&\quad \leq 2\mathbb{E}\left[t\int_0^t |b(t,s,\lambda_s,u(s),X^n(s\text{-})) - b(t,s,\lambda_s,u(s),X^{n-1}(s\text{-}))|^2 ds\right] \\
&\quad + 2\mathbb{E}\left[\int_0^t\int_{\mathbb{R}} |\kappa(t,s,z,\lambda_s,u(s),X^n(s\text{-})) \right. \\
&\quad \left. - \kappa(t,s,z,\lambda_s,u(s),X^{n-1}(s\text{-}))|^2 \Lambda(dsdz)\right].
\end{aligned}$$

By (I.2.1) and using the Lipschitz condition on b and κ, we get

$$\mathbb{E}\left[|X^{n+1}(t) - X^n(t)|^2\right] \leq 2C^2\mathbb{E}\left[t\int_0^t 2|X^n(s\text{-}) - X^{n-1}(s\text{-})|ds\right],$$

which leads to

$$\mathbb{E}\left[|X^{n+1}(t) - X^n(t)|^2\right] \leq K\mathbb{E}\left[\int_0^t |X^n(s\text{-}) - X^{n-1}(s\text{-})|^2 ds\right], \tag{I.3.3}$$

for $K := 4TC^2$. Also, by the linear growth condition *3.* on b and κ, we get that

$$\mathbb{E}\left[|X^1(t) - X^0(t)|\right] \leq Kt(1+X_0)^2. \tag{I.3.4}$$

Combining now (I.3.3) and (I.3.4), we have that

$$\mathbb{E}\left[|X^{n+1}(t) - X^n(t)|^2\right] \leq \frac{2(1+X_0)^2(Kt)^{n+1}}{(n+1)!}.$$

Thus we have that $\{X(t)^n\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(dP)$ and $\{X(t)^n\}_{n=1}^\infty$ is in $L^2(dP \times dt)$ Taking the limit on $n \to \infty$ gives the solution to (I.3.2). The uniqueness is obtained by standard arguments and estimates similar to the ones above. ■

Before moving forward, we need to state a fundamental result that will allow us to rewrite X in (I.3.2) in differential form. This is due to [34] and it is known as transformation rule. Hereafter we state the result within our setting.

Lemma I.3.2. *(Transformation rule) Assume that for all $z \in \mathbb{R}$, $\lambda \in [0,\infty)^2$, $u \in \mathcal{U}$, $x \in \mathbb{R}$ the partial derivative of κ with respect to t (denoted with $\partial_t\kappa(t,s,z,\lambda,u,x)$) is locally bounded (uniformly in t) and satisfies*

$$|\partial_t\kappa(t_1,s,z,\lambda,u,x) - \partial_t\kappa(t_2,s,z,\lambda,u,x)| \leq K|t_1 - t_2|, \tag{I.3.5}$$

for some $K > 0$ and for each fixed $s \leq t$, $\lambda \in [0,\infty)^2$, $u \in \mathcal{U}$, $x \in \mathbb{R}$.

Then, the forward equation (I.3.2) *can be rewritten in differential notation as*

$$
\begin{aligned}
dX(t) = \Big(&b(t,t,\lambda_t,u(t),X(t)) + \int_0^t \partial_t b(t,s,\lambda_s,u(s),X(s))ds \\
&+ \int_0^t\int_{\mathbb{R}} \partial_t \kappa(t,s,z,\lambda_s,u(s),X(s))\mu(dsdz)\Big)dt \\
&+ \int_{\mathbb{R}} \kappa(t,t,z,\lambda_t,u(t),X(t))\mu(dtdz).
\end{aligned} \tag{I.3.6}
$$

Proof. The proof follows the one in [34]. We report it here for completeness. Observe that

$$
\begin{aligned}
X(t) &= \int_0^t b(t,s,\lambda_s,u(s),X(s))ds + \int_0^t\int_{\mathbb{R}} \kappa(t,s,z,\lambda_s,u(s),X(s))\mu(dsdz) \\
&= \int_0^t b(t,s,\lambda_s,u(s),X(s))ds + \int_0^t\int_{\mathbb{R}} \kappa(s,s,z,\lambda_s,u(s),X(s))\mu(dsdz) \\
&\quad + \int_0^t\int_{\mathbb{R}} \kappa(t,s,z,\lambda_s,u(s),X(s)) - \kappa(s,s,z,\lambda_s,u(s),X(s))\mu(dsdz)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\kappa(t,s,z,\lambda_s,u(s),X(s)) - \kappa(s,s,z,\lambda_s,u(s),X(s)) \\
&\quad = \int_s^t \partial_r \kappa(r,s,z,\lambda_s,u(s),X(s))dr \\
&\quad = \int_0^t \mathbb{1}_{s\le r}\partial_r \kappa(r,s,z,\lambda_s,u(s),X(s))dr
\end{aligned}
$$

Then we can apply the Fubini theorem for stochastic integration as in [26] and we obtain that

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}} \kappa(t,s,z,\lambda_s,u(s),X(s)) - \kappa(s,s,z,\lambda_s,u(s),X(s))\mu(dsdz) \\
&\quad = \int_0^t\int_{\mathbb{R}} \left\{\int_0^t \mathbb{1}_{s\le r}\partial_r \kappa(r,s,z,\lambda_s,u(s),X(s))dr\right\}\mu(dsdz) \\
&\quad = \int_0^t \left\{\int_0^r\int_{\mathbb{R}} \partial_r \kappa(r,s,z,\lambda_s,u(s),X(s))\mu(dsdz)\right\}dr.
\end{aligned}
$$

The well posedness and the Lebesgue integrability of

$$
\int_0^r\int_{\mathbb{R}} \partial_r \kappa(r,s,z,\lambda_s,u(s),X(s))\mu(dsdz),
$$

$r \in [0,t]$ is achieved in Theorem 3.2 [34] thanks to (I.3.5). ■

Remark I.3.1. *(A link with functional SDEs)* Lemma I.3.2, suggests a link between the Volterra integral equations of the kind (I.3.2) and functional SDEs (FSDEs). It is in fact clear that, by defining

$$B(t,\lambda.,u.,X.,Z.) := \Big(b(t,t,\lambda_t,u(t),X(t)) + \int_0^t \partial_t b(t,s,\lambda_s,u(s),X(s))ds + Z(t)\Big),$$

where

$$Z(t) = \int_0^t\int_{\mathbb{R}} \partial_t \kappa(t,s,z,\lambda_s,u(s),X(s))\mu(dsdz),$$

we have that (I.3.2) can be rewritten as

$$X(t) = X_0 + \int_0^t B(t,\lambda.,u.,X.,Z.)dt + \int_0^t\int_{\mathbb{R}} \kappa(t,t,z,\lambda_t,u(t),X(t))\mu(dtdz). \tag{I.3.7}$$

We notice that (I.3.7) is a functional SDE, so we could have tried to state an existence result for functional SDEs instead of using Theorem I.3.1. Some existence results for SDEs such as (I.3.7) are available (see e.g. [5, 13, 14, 28, 33]), but no one of those deals with noises such as μ. While some of those results (e.g. [13, 28, 33]) present condition that would be too restrictive for the current setting, we also point out that the results presented in [5, 14] could possibly be extended to the current framework. Nonetheless, this would require to impose some Lipschitz and linear growth conditions on b and κ (like in Theorem I.3.1) and, additionally, to impose a Lipschitzianity condition on $\partial_t b$, not required in the hypothesis of Theorem I.3.1.

Having discussed the existence of a solution for (I.3.2), we are finally ready to proceed to our optimization results. We start by introducing the notion of admissible controls:

Definition I.3.3. The admissible controls for (I.3.2) in the optimization problems (I.1.4) and (I.1.5) are predictable stochastic processes $u : [0,T]\times\Omega \longmapsto \mathcal{U}$ such that X in (I.3.2) has a unique strong solution and

$$\mathbb{E}\left[\int_0^T F(t,\lambda_t,u(t),X(t))dt + G(X(T)) + |\partial_x G(X(T))|^2\right] < \infty$$

We denote $\mathcal{A}^{\mathbb{F}}$ and $\mathcal{A}^{\mathbb{G}}$ the sets of $\mathbb{F}$- or $\mathbb{G}$-predictable controls, respectively. We say that $(\hat{u},\hat{X})$ is an *optimal pair* if

$$J(\hat{u}) = \sup_{u\in\mathcal{A}^{\cdot}} \mathbb{E}\left[\int_0^T F(t,\lambda_t,u(t),X(t))dt + G(X(T))\right], \tag{I.3.8}$$

where $\hat{X} := X^{\hat{u}}$ is as in (I.3.2), and $\mathcal{A}^{\cdot}$ is either the set $\mathcal{A}^{\mathbb{F}}$ or $\mathcal{A}^{\mathbb{G}}$.

Define $\mathcal{R}^{\mathbb{G}}$ to be the space of $\mathbb{G}$-predictable processes with values in $L^2(dP)$. We remark that, if $y \in \mathcal{R}^{\mathbb{G}}$, then the NA-derivative (I.2.5) is also in $\mathcal{R}^{\mathbb{G}}$ i.e. for all t, z $\mathscr{D}_{t,z}y(\cdot) \in \mathcal{R}^{\mathbb{G}}$. In the sequel, when no confusion arises, we will denote with $\mathscr{D}_{t,0}y(\cdot)$ the NA-derivative with respect to the conditional Brownian motion, and with $\mathscr{D}_{t,z}y(\cdot)$, $z \in \mathbb{R}_0$, the NA-derivative with respect to the conditional Poisson random measure.

In view of the Volterra structure of the dynamics (I.3.2), the system is not Markovian. We tackle the problem (I.3.8) by the maximum principle approach, better suited in this case, see e.g. [40]. We introduce the Hamiltonian function:

$$\mathcal{H} : [0, T] \times \Xi_{\mathbb{R}^2_+} \times \Xi_{\mathcal{U}} \times \Xi_{\mathbb{R}} \times \mathcal{R}^{\mathbb{G}} \times \Xi_{\mathcal{Z}} \times \Omega \longrightarrow \mathbb{R},$$

as the mapping given by the sum

$$\mathcal{H}(t, \lambda, u, x, p, q) := H_0(t, \lambda, u, x, p, q) + H_1(t, \lambda, u, x, p, q) \tag{I.3.9}$$

of the two components

$$\begin{aligned} H_0(t, \lambda, u, x, p, q) :=\ & F(t, \lambda_t, u_t, x_t) + b(t, t, \lambda_t, u_t, x_t)p(t) \\ & + \kappa(t, t, 0, \lambda_t, u_t, x_t)q_t(0)\lambda_t^B \\ & + \int_{\mathbb{R}_0} \kappa(t, t, z, \lambda_t, u_t, x_t)q_t(z)\lambda_t^H \nu(dz) \\ H_1(t, \lambda, u, x, p, q) :=\ & \int_0^t \partial_t b(t, s, \lambda_s, u_s, x_s)ds\ p(t) \\ & + \int_0^t\int_{\mathbb{R}} \partial_t\kappa(t, s, z, \lambda_s, u_s, x_s)\mathscr{D}_{s,z}p(t)\Lambda(dsdz), \end{aligned}$$

where $\mathcal{Z}$ is the space of functions $q : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$|q(0)|^2 + \int_{\mathbb{R}_0} q(z)^2\nu(dz) < \infty.$$

Remark I.3.2. Following up on Remark I.3.1, instead of considering (I.3.2) as a Volterra equation, we could have taken the FSDE (I.3.7) and, following e.g. [14], write the Hamiltonian functional for the functional SDE. We notice that, regardless of the chosen approach, we would still end up with the Hamiltonian functional (I.3.9).

Associated to $\mathcal{H}$ (I.3.9), we introduce a BSDE of the type (I.1.6), which we study under $\mathbb{G}$:

$$\begin{aligned} p(t) =& \partial_x G(X(T)) + \int_t^T \partial_x \mathcal{H}(s, \lambda_s, u(s), X(s\text{-}), p(s\text{-}), q(s, \cdot))ds \\ & - \int_t^T\int_{\mathbb{R}} q(s, z)\mu(dsdz), \quad t \in [0, T], \end{aligned} \tag{I.3.10}$$

where the derivative $\partial_x \mathcal{H}$ is meant in the Fréchet sense.

Sufficient conditions to guarantee the existence of (I.3.10) on $\mathcal{R}^{\mathbb{G}} \times \mathcal{I}^{\mathbb{G}}$ can be found in [19].

Remark I.3.3. Notice that (I.3.10) is actually a BSDE and not a Volterra-type backward SDE. In fact, the term $\partial_x H_1(t, \lambda, u, X, p, q)$ in the driver $\partial_x \mathcal{H}(t, \lambda, u, X, p, q)$, corresponds to

$$\begin{aligned}\partial_x H_1(s, \lambda_s, u, X, p, q) &= \partial_x \int_0^s \partial_s b(s, r, \lambda_r, u(r), X(r))dr\ p(s)\\ &+ \partial_x \int_0^s\int_{\mathbb{R}} \partial_s \kappa(s, r, z, \lambda_r, u(r), X(r))\mathscr{D}_{r,z} p(s)\Lambda(drdz),\end{aligned}$$

which is a function of time s, after integration.

The optimal control problem (I.1.4):

$$J(\hat{u}) = \sup_{u\in\mathcal{A}^{\mathbb{F}}} J(u) = \sup_{u\in\mathcal{A}^{\mathbb{F}}} \mathbb{E}\left[\int_0^T F(t, \lambda_t, u(t), X^u(t))dt + G(X^u(T))\right], \quad \text{(I.3.11)}$$

associated to the performance functional (I.3.1) is treated in the framework of optimization under partial information. This is inspired by [19], where this approach is taken for standard time-changed dynamics. In the Volterra case treated in the present work, the functionals stemming out of (I.3.9) are very different from the ones in [19]. Indeed we introduce the mapping $\mathcal{H}^{\mathbb{F}}$ defined for $t \in [0, T]$, $\lambda \in \Xi_{\mathbb{R}^2_+}$, $u \in \Xi_{\mathcal{U}}$, $x \in \Xi_{\mathbb{R}}$, $p \in \mathcal{R}^{\mathbb{G}}$ and $q \in \mathcal{I}^{\mathbb{G}}$ as

$$\begin{aligned}\mathcal{H}^{\mathbb{F}}(t, \lambda, u, x, p, q) &:= H_0^{\mathbb{F}}(t, \lambda, u, x, p, q) + H_1^{\mathbb{F}}(t, \lambda, u, x, p, q)\\ &:= \mathbb{E}\left[\mathcal{H}(t, \lambda, u, x, p, q)|\mathcal{F}_t\right], \qquad \text{(I.3.12)}\end{aligned}$$

where

$$\begin{aligned}H_0^{\mathbb{F}}(t, \lambda, u, x, p, q) &:= F(t, \lambda_t, u_t, x_t) + b(t, t, \lambda_t, u_t, x_t)\mathbb{E}[p(t)|\mathcal{F}_t]\\ &+ \kappa(t, t, 0, \lambda_t, u_t, x_t)\mathbb{E}[q(t, 0)|\mathcal{F}_t]\lambda_t^B\\ &+ \int_{\mathbb{R}_0} \kappa(t, t, z, \lambda_t, u_t, x_t)\mathbb{E}[q(t, z)|\mathcal{F}_t]\lambda_t^H \nu(dz)\\ H_1^{\mathbb{F}}(t, \lambda, u, x, p, q) &:= \int_0^t \partial_t b(t, s, \lambda_s, u_s, x_s)ds\ \mathbb{E}[p(t)|\mathcal{F}_t]\\ &+ \int_0^t\int_{\mathbb{R}} \partial_t \kappa(t, s, z, \lambda_s, u_s, x_s)\mathbb{E}[\mathscr{D}_{s,z} p(t)|\mathcal{F}_t]\Lambda(dsdz)\end{aligned}$$

Notation I.3.4. Given $u, \hat{u} \in \mathcal{A}^{\cdot}$, $X, \hat{X}$ represent the associated controlled dynamics of (I.3.2) and $(p, q), (\hat{p}, \hat{q})$ are the corresponding solutions of (I.3.10). From now on, if no confusion arises, we will use the compact notation:

$$b(t, s) := b(t, s, \lambda_s, u(s), X(s)), \quad \hat{b}(t, s) := b(t, s, \lambda_s, \hat{u}(s), \hat{X}(s)).$$

Similarly, for κ, $\hat{\kappa}$, F, $\hat{F}$, G, $\hat{G}$, we will also write:

$$\mathcal{H}^u(s) := \mathcal{H}(s, \lambda, u, X, \hat{p}, \hat{q}), \quad \mathcal{H}^{\hat{u}}(s) := \mathcal{H}(s, \lambda, \hat{u}, \hat{X}, \hat{p}, \hat{q})$$

and similarly for $\mathcal{H}^{\mathbb{F},u}$, $\mathcal{H}^{\mathbb{F},\hat{u}}$, H_0^u, $H_0^{\hat{u}}$, $H_0^{\mathbb{F},u}$, $H_0^{\mathbb{F},\hat{u}}$, H_1^u, $H_1^{\hat{u}}$, $H_1^{\mathbb{F},u}$, $H_1^{\mathbb{F},\hat{u}}$.

Theorem I.3.4. (Sufficient maximum principle with respect to $\mathbb{F}$). *Let $\lambda \in \mathcal{L}$. Let $\hat{u} \in \mathcal{A}^{\mathbb{F}}$ and assume that the corresponding solutions $\hat{X}$, $(\hat{p}, \hat{q})$ of* (I.3.2) *and* (I.3.10) *exist. Assume that*

- *$x \longmapsto G(x)$ is concave.*
- *For any t, the map*

$$x \longmapsto \text{ess sup}_{u \in \Xi_{\mathcal{U}}} \mathcal{H}^{\mathbb{F}}(t, \lambda, u, x, \hat{p}, \hat{q}), \quad x \in \Xi_{\mathbb{R}}, \tag{I.3.13}$$

  is concave.
- *For all $t \in [0, T]$,*

$$\text{ess sup}_{u \in \Xi_{\mathcal{U}}} \mathcal{H}^{\mathbb{F}}(t, \lambda, u, x, \hat{p}, \hat{q}) = \mathcal{H}^{\mathbb{F}}(t, \lambda, \hat{u}, \hat{X}, \hat{p}, \hat{q}). \tag{I.3.14}$$

Then $\hat{u}$ is an optimal control for problem (I.3.1) *and $(\hat{u}, \hat{X})$ is an optimal pair.*

Proof. This proof is inspired by both the proof of [3] Theorem 4.1 and [19] Theorem 6.2. The main difference with [3] is the use of the random measure μ instead of a Brownian motion and a compensated random Poisson measure, which requires to abandon the framework of Malliavin calculus. The main difference with [19] is the Volterra structure of the dynamics for the forward equation (I.3.2), which lead to more involved stochastic calculus. Recall that $\hat{u} \in \mathcal{A}^{\mathbb{F}}$ is a candidate to be optimal and $X^{\hat{u}}$ is the corresponding solution of (I.3.2). Choose an arbitrary other $u \in \mathcal{A}^{\mathbb{F}}$ with corresponding controlled dynamics X and consider $J(u) - J(\hat{u}) = I_1 + I_2$, where

$$I_1 := \mathbb{E}\left[\int_0^T F(t, \lambda_t, u(t), X(t)) - F(t, \lambda_t, \hat{u}(t), \hat{X}(t))dt\right], \tag{I.3.15}$$

$$I_2 := \mathbb{E}\left[G(X(T)) - G(\hat{X}(T))\right]. \tag{I.3.16}$$

Considering now I_1, from the definition of $H_0^{\mathbb{F}}$ we get that,

$$I_1 = \mathbb{E}\left[\int_0^T \left\{H_0^{\mathbb{F},u}(t) - H_0^{\mathbb{F},\hat{u}}(t) - [b(t,t) - \hat{b}(t,t)]\mathbb{E}\left[\hat{p}(t)|\mathcal{F}_t\right]\right\} dt \right.$$
$$\left. - \int_0^T \int_{\mathbb{R}} [\kappa(t,t,z) - \hat{\kappa}(t,t,z)]\mathbb{E}\left[\hat{q}(t,z)|\mathcal{F}_t\right] \Lambda(dtdz)\right].$$

By the concavity of G, we have

$$I_2 \leq \mathbb{E}\left[\partial_x G(\hat{X}(T))\left(X(T)-\hat{X}(T)\right)\right] = \mathbb{E}\left[\hat{p}(T)\left(X(T)-\hat{X}(T)\right)\right].$$

We apply the transformation rule (Lemma I.3.2) to rewrite the Volterra forward dynamics of X as

$$dX(t) = \Big(b(t,t)+\int_0^t \partial_t b(t,s)ds+\int_0^t\int_{\mathbb{R}} \partial_t\kappa(t,s,z)\mu(dsdz)\Big)dt+\int_{\mathbb{R}} \kappa(t,t,z)\mu(dtdz).$$

Also the BSDE $\hat{p}$ (I.3.10) associated to the optimal pair $(\hat{u},\hat{X})$ in differential notation is

$$d\hat{p}(t) = -\partial_x\mathcal{H}^{\hat{u}}(t)dt + \int_{\mathbb{R}} \hat{q}(t,z)\mu(dtdz).$$

Using the Itô formula for the product we obtain

$$\begin{aligned} I_2 \leq \mathbb{E}\Bigg[&\int_0^T\Big\{\hat{p}(t)\Big(\Big(b(t,t)-\hat{b}(t,t)\Big)+\int_0^t\Big(\partial_t b(t,s)-\partial_t\hat{b}(t,s)\Big)ds \\ &+\int_0^t\int_{\mathbb{R}}\left(\partial_t\kappa(t,s,z)-\partial_t\hat{\kappa}(t,s,z)\right)\mu(dsdz)\Big)\Big\}dt \\ &-\int_0^T \partial_x\mathcal{H}^{\hat{u}}(t)\Big(X(t)-\hat{X}(t)\Big)dt+\int_0^T\Big\{[\kappa(t,t,0)-\hat{\kappa}(t,t,0)]\hat{q}(t,0)\lambda_t^B \\ &+\int_{\mathbb{R}_0}[\kappa(t,t,z)-\hat{\kappa}(t,t,z)]\hat{q}(t,z)\nu(dz)\lambda_t^H\Big\}dt\Bigg]. \end{aligned} \tag{I.3.17}$$

Now notice that,

$$\begin{aligned} &\mathbb{E}\left[\int_0^T\left(\int_0^t\int_{\mathbb{R}}\partial_t\kappa(t,s,z)\mu(dsdz)\right)\hat{p}(t)dt\right] \\ &= \int_0^T\mathbb{E}\left[\left(\int_0^t\int_{\mathbb{R}}\partial_t\kappa(t,s,z)\mu(dsdz)\right)\hat{p}(t)\right]dt \\ &= \int_0^T\mathbb{E}\left[\int_0^t\int_{\mathbb{R}}\partial_t\kappa(t,s,z)\mathscr{D}_{s,z}\hat{p}(t)\Lambda(dsdz)\right]dt \\ &= \mathbb{E}\left[\int_0^T\int_0^t\int_{\mathbb{R}}\partial_t\kappa(t,s,z)\mathscr{D}_{s,z}\hat{p}(t)\Lambda(dsdz)dt\right] \end{aligned} \tag{I.3.18}$$

where we have used Fubini's theorem and the duality formula (Proposition I.2.7). By substituting (I.3.18) into (I.3.17), and taking the conditional expectation given $\mathcal{F}_t$ we get that

$$
\begin{aligned}
I_2 \leq \mathbb{E}\Bigg[\int_0^T & \Big\{\Big(b(t,t)-\hat{b}(t,t)\Big)\,\mathbb{E}\left[\hat{p}(t)|\mathcal{F}_t\right] \\
&+ \int_0^t \Big(\partial_t b(t,s)-\partial_t\hat{b}(t,s)\Big)\,ds\;\mathbb{E}\left[\hat{p}(t)|\mathcal{F}_t\right] \\
&+ \int_0^t\int_{\mathbb{R}_0} \left(\partial_t\kappa(t,s,z)-\partial_t\hat{\kappa}(t,s,z)\right)\mathbb{E}\left[\mathscr{D}_{s,z}\hat{p}(t)|\mathcal{F}_t\right]\Lambda(dsdz)\Big\}\,dt \\
&- \int_0^T \partial_x\mathcal{H}^{\mathbb{F},\hat{u}}(t)\left(X(t)-\hat{X}(t)\right)dt \\
&+ \int_0^T\int_{\mathbb{R}} \mathbb{E}\left[\hat{q}(t,z)|\mathcal{F}_t\right]\left[\kappa(t,t,z)-\hat{\kappa}(t,t,z)\right]\Lambda(dtdz)\Bigg].
\end{aligned}
$$

Hence

$$
\begin{aligned}
I_1 + I_2 \leq \mathbb{E}\Bigg[\int_0^T & \Big(H_0^{\mathbb{F},u}(t)-H_0^{\mathbb{F},\hat{u}}(t)+H_1^{\mathbb{F},u}(t)-H_1^{\mathbb{F},\hat{u}}(t) \qquad \text{(I.3.19)}\\
&- \partial_x\mathcal{H}^{\mathbb{F},\hat{u}}(t)\left(X(t)-\hat{X}(t)\right)\Big)dt\Bigg] \\
= \mathbb{E}\Bigg[\int_0^T & \Big(\mathcal{H}^{\mathbb{F},u}(t)-\mathcal{H}^{\mathbb{F},\hat{u}}(t)-\partial_x\mathcal{H}^{\mathbb{F},\hat{u}}(t)\left(X(t)-\hat{X}(t)\right)\Big)\,dt\Bigg] \leq 0, \qquad \text{(I.3.20)}
\end{aligned}
$$

$dt \times dP$ a.e. by the maximality of $\hat{u}$ in (I.3.14) and the concavity condition (I.3.13). Hence $J(u) \leq J(\hat{u})$ and $\hat{u}$ is an optimal control for (I.3.1). This conclusion is reached applying a separating hyperplane argument to the concave map (I.3.13). ■

Notice that a result analogous to Theorem I.3.4 can also be obtained when working under the initially enlarged filtration $\mathbb{G}$. Though the next result might not be of direct applicability in view of the anticipated information included in $\mathbb{G}$, the study has mathematical validity.

Remark I.3.5. The transformation rule under (I.3.5) allows for the use of an Itô-type formula in the context of Volterra dynamics. If the equation would not present Volterra structure in the stochastic integral part (i.e. in the coefficient κ), then the requirement (I.3.5) is clearly lifted.

Proposition I.3.5. (Sufficient maximum principle with respect to $\mathbb{G}$). *Let $\lambda \in \mathcal{L}$. Let $\hat{u} \in \mathcal{A}^{\mathbb{G}}$ and assume that the corresponding solutions $\hat{X}(t), (\hat{p}, \hat{q})$ of* (I.3.2) *and* (I.3.10) *exist. Assume that:*

- *$x \longmapsto G(x)$ is concave.*

- *For any t, $\hat{p}$, $\hat{q}$, the function*

$$x \longmapsto ess\,sup_{u\in\Xi_{\mathcal{U}}}\mathcal{H}(t,\lambda,u,x,\hat{p},\hat{q}), \qquad x \in \xi_{\mathbb{R}} \tag{I.3.21}$$

  is concave in x.

- *For all $t \in [0,T]$,*

$$ess\,sup_{v\in\Xi_{\mathcal{U}}}\mathcal{H}(t,\lambda,v,\hat{X},\hat{p},\hat{q}) = \mathcal{H}(t,\lambda,\hat{u},\hat{X},\hat{p},\hat{q}). \tag{I.3.22}$$

Then $\hat{u}$ is an optimal control for the problem (I.1.5)*.*

Proof. Once considering the filtration $\mathbb{G}$, the arguments in the proof of Theorem I.3.4 leading to

$$J(u) - J(\hat{u}) \le \mathbb{E}\left[\int_0^T \left(\mathcal{H}^u(t) - \mathcal{H}^{\hat{u}}(t) - \partial_x\mathcal{H}^{\hat{u}}(t)\left(X(t) - \hat{X}(t)\right)\right)dt\right] \le 0$$

apply directly without conditioning. ∎

I.4 Necessary maximum principles with time-change

Hereafter we study necessary conditions to identify the possible candidates for optimal controls. This can be a useful starting point before applying a verification theorem to ensure optimality. We remark that our results relax the condition of concavity present in Theorem I.3.4 and I.3.5. However, we introduce some other assumptions on the set of admissible controls and the first variation process of the forward dynamics (I.3.2).

In the literature we find a first version of necessary maximum principle for Volterra dynamics in [1]. There the driving noises were the Gaussian and the centered Poisson random measure. Our work goes beyond these noises.
For any $t \in [0,T]$, we consider a random perturbation of the type

$$\beta(s) := \alpha_t \mathbb{1}_{[t,t+h]}(s), \quad s \in [0,T], \tag{I.4.1}$$

where α_t is a bounded $\mathcal{F}_t$ measurable random variable and $h \in [0, T-t]$. We make the following assumptions:

1. The set of admissible controls $\mathcal{A}^{\mathbb{F}}$ is such that, for all $u \in \mathcal{A}^{\mathbb{F}}$,

$$u + \varepsilon\beta \in \mathcal{A}^{\mathbb{F}},$$

   for all perturbations β as in (I.4.1) and all $\varepsilon > 0$ sufficiently small.

2. The first variation process $\mathcal{X}(t)$, $t \in [0,T]$, given by the derivative

$$\mathcal{X}(t) := \partial_\varepsilon X^{(u+\varepsilon\beta)}|_{\varepsilon=0} \tag{I.4.2}$$

(see (I.3.2)) exists and is well defined.

3. $\partial_x b(t,s)$ and $\partial_u b(t,s)$ are well defined and C^1 with respect to t with partial derivatives L^2-integrable with respect to $dt \times dP$. $\partial_x \kappa(t,s,\cdot)$ and $\partial_u \kappa(t,s,\cdot)$ are well defined and C^1 with respect to t with partial derivatives L^2-integrable with respect to $d\Lambda \times dP$.

4. $\partial_x \kappa(t,s,z)$ and $\partial_u \kappa(t,s,z)$ are such that, for all $z \in \mathbb{R}$ $\lambda \in [0,\infty)^2$, $u \in \mathcal{U}$, $x \in \mathbb{R}$, the partial derivative of $\partial_x\kappa + \partial_u\kappa$ with respect to t is locally bounded (uniformly in t) and satisfies

$$|\partial_t(\partial_x\kappa(t_1,s,z,)+\partial_u\kappa(t_1,s,z))-\partial_t(\partial_x\kappa(t_2,s,z,)+\partial_u\kappa(t_2,s,z))| \le K|t_1-t_2|,$$

for some $K > 0$ and for each fixed $s \le t$, $\lambda \in [0,\infty)^2$, $u \in \mathcal{U}$, $x \in \mathbb{R}$.

Assumption 2. above implies that

$$\begin{aligned}\mathcal{X}(t) = &\int_0^t \Big(\partial_x b(t,s)\mathcal{X}(s) + \partial_u b(t,s)\beta(s)\Big)ds \\ &+ \int_0^t\int_{\mathbb{R}} \Big(\partial_x\kappa(t,s,z)\mathcal{X}(s) + \partial_u\kappa(t,s,z)\beta(s)\Big)\mu(dsdz),\end{aligned}$$

exists and is well defined, whereas assumption 4. ensure us to be able to apply the transformation rule for $\mathcal{X}$.

Remark that sufficient conditions that ensure the existence of the first variation process are that b and κ are in $C^1(\mathcal{U})$ uniformly for all $s,t \in [0,T]$ $\lambda \in [0,\infty)^2$, $x \in \mathbb{R}$ and that $(\partial_x b(t,s)\mathcal{X}(s) + \partial_u b(t,s)\beta(s))$ and $(\partial_x\kappa(t,s,z)\mathcal{X}(s) + \partial_u\kappa(t,s,z)\beta(s))$ satisfy the linear growth and lipschitzianity conditions of Theorem I.3.1.

As above we consider the performance functional (I.3.1) with the related conditions on F and G as in Section I.3. We also continue using the compact notation there introduced, see Notation I.3.4.

Theorem I.4.1. (Necessary maximum principle with respect to $\mathbb{F}$). *Let $\lambda \in \mathcal{L}$. Suppose that $\hat{u} \in \mathcal{A}^{\mathbb{F}}$ and the corresponding solutions $\hat{X}, (\hat{p},\hat{q})$ of* (I.3.2) *and* (I.3.10) *exist. Assume also that P-a.s. $F \in C^1(\mathcal{U})$ for all $t \in [0,T]$ $\lambda \in [0,\infty)^2$, $x \in \mathbb{R}$. If, for all perturbations β as in* (I.4.1)*, we have that*

$$\partial_\varepsilon J(\hat{u}+\varepsilon\beta)|_{\varepsilon=0} = 0, \tag{I.4.3}$$

then

$$\partial_u \mathcal{H}^{\mathbb{F},\hat{u}}(t) = 0. \tag{I.4.4}$$

The converse also holds true.

Proof. With (I.4.2), we consider for $u \in \mathcal{A}^{\mathbb{F}}$ and the perturbation (I.4.1),

$$\begin{aligned} &\partial_\epsilon J(u+\varepsilon\beta)|_{\varepsilon=0} \\ &= \mathbb{E}\Bigg[\int_0^T (\partial_x F(t,\lambda_t,u(t),X(T))\chi(t) + \partial_u F(t,\lambda_t,u(t),X(t))\beta(t))\,dt \\ &\quad + \partial_x G(X(T))\chi(T)\Bigg]. \end{aligned} \tag{I.4.5}$$

By considering a suitable increasing family of stopping times converging to T as in [31] Theorem 2.2, we may assume that all the local martingales appearing here are true martingales. From (I.3.12), the transformation rule (Lemma I.3.2) and the Itô formula for the product, we find that

$$\begin{aligned} &\mathbb{E}\left[\partial_x G(X(T))\chi(T)\right] = \mathbb{E}\left[p(T)\chi(T)\right] \\ &= \mathbb{E}\Bigg[\int_0^T p(t)\left(\partial_x b(t,t)\chi(t) + \partial_u b(t,t)\beta(t)\right)dt - \int_0^T \chi(t)\partial_x \mathcal{H}(t)dt \\ &\quad + \int_0^T p(t)\left(\int_0^t \left(\partial_t\partial_x b(t,s)\chi(s) + \partial_t\partial_u b(t,s)\beta(s)\right)ds\right)dt \\ &\quad + \int_0^T p(t)\left(\int_0^t\!\!\int_{\mathbb{R}} \left(\partial_t\partial_x \kappa(t,s,z)\chi(s) + \partial_t\partial_u \kappa(t,s,z)\beta(s)\right)\mu(dsdz)\right)dt \\ &\quad + \int_0^T\!\!\int_{\mathbb{R}} q(s,z)\Big(\partial_x\kappa(t,t,z)\chi(t) + \partial_u\kappa(t,t,z)\beta(t)\Big)\Lambda(dtdz)\Bigg]. \end{aligned}$$

Now, recalling equality (I.3.18), and taking the conditional expectation under $\mathcal{F}_t$ we get that

$$\begin{aligned} &\mathbb{E}\left[p(T)\chi(T)\right] \\ &= \mathbb{E}\Bigg[\int_0^T \Bigg\{\partial_x b(t,t)\mathbb{E}\left[p(t)|\mathcal{F}_t\right] + \int_0^t \partial_x\partial_t b(t,s)ds\ \mathbb{E}\left[p(t)|\mathcal{F}_t\right] \\ &\quad + \int_0^t\!\!\int_{\mathbb{R}} \partial_x\partial_t\kappa(t,s,z)\mathbb{E}\left[\mathscr{D}_{s,z}p(t)|\mathcal{F}_t\right]\Lambda(dsdz)\Bigg\}\chi(t)dt \\ &\quad + \int_0^T \Bigg\{\partial_u b(t,t)\mathbb{E}\left[p(t)|\mathcal{F}_t\right] + \int_0^t \partial_u\partial_t b(t,s)ds\ \mathbb{E}\left[p(t)|\mathcal{F}_t\right] \\ &\quad + \int_0^t\!\!\int_{\mathbb{R}} \partial_u\partial_t\kappa(t,s,z)\mathbb{E}\left[\mathscr{D}_{s,z}p(t)|\mathcal{F}_t\right]\Lambda(dsdz)\Bigg\}\beta(t)dt - \int_0^T \partial_x\mathcal{H}^{\mathbb{F},u}(t)\chi(t)dt \\ &\quad + \int_0^T\!\!\int_{\mathbb{R}} \left(\partial_x\kappa(t,t,z)\chi(t) + \partial_u\kappa(t,t,z)\beta(t)\right)\mathbb{E}\left[q(t,z)|\mathcal{F}_t\right]\Lambda(dtdz)\Bigg]. \end{aligned}$$

So that, from (I.3.12), we can write

$$\mathbb{E}\Bigg[\int_0^T (\partial_x F(t,\lambda_t,u(t),X(t))\chi(t) + \partial_u F(t,\lambda_t,u(t),X(t))\beta(t))\,dt$$
$$+ \partial_x G(X(T))\chi(T)\Bigg]$$
$$= \mathbb{E}\left[\int_0^T \partial_x \mathcal{H}^{\mathbb{F},u}(t)\chi(t)dt - \int_0^T \partial_x \mathcal{H}^{\mathbb{F},u}(t)\chi(t)dt + \int_0^T \partial_u \mathcal{H}^{\mathbb{F},u}(t)\beta(t)dt\right]. \tag{I.4.6}$$

Summarizing, equation (I.4.5) together with (I.4.6) and the perturbations in (I.4.1) give

$$\partial_\varepsilon J(u+\varepsilon\beta)|_{\varepsilon=0} = \mathbb{E}\left[\int_0^T \partial_u \mathcal{H}^{\mathbb{F},u}(t)\beta(t)dt\right] = \mathbb{E}\left[\int_t^{t+h} \partial_u \mathcal{H}^{\mathbb{F},u}(s)ds\ \alpha_t\right], \tag{I.4.7}$$

and, for $\hat{u}$, (I.4.3) gives

$$\partial_\varepsilon J(\hat{u}+\varepsilon\beta)|_{\varepsilon=0} = 0.$$

Applying the Fubini theorem to the right-hand side of (I.4.7) and differentiating at $h=0$ we obtain

$$\mathbb{E}\left[\partial_u \mathcal{H}^{\mathbb{F},\hat{u}}(t)\ \alpha_t\right] = 0,$$

for all α_t bounded and $\mathcal{F}_t$ measurable. Hence

$$\mathbb{E}\left[\partial_u \mathcal{H}^{\mathbb{F},\hat{u}}(t)\big|\mathcal{F}_t\right] = \partial_u \mathcal{H}^{\mathbb{F},\hat{u}}(t) = 0. \tag{I.4.8}$$

Vice versa, if (I.4.8) holds, we can reverse the argument to obtain (I.4.3). ■

As in Section 3, for the sake of completeness, we propose a necessary maximum principle under the information flow $\mathbb{G}$. This refers to the optimization problem (I.1.5). In this case we assume that, for all $u \in \mathcal{A}^{\mathbb{G}}$, $u+\varepsilon\beta \in \mathcal{A}^{\mathbb{G}}$ for all perturbations β as in (I.4.1) and $\varepsilon > 0$ sufficiently small.

Proposition I.4.2. (Necessary maximum principle with respect to $\mathbb{G}$). *Let $\lambda \in \mathcal{L}$. Suppose that $\hat{u} \in \mathcal{A}^{\mathbb{G}}$ and the corresponding solutions $\hat{X}, (\hat{p},\hat{q})$ of* (I.3.2) *and* (I.3.10) *exist. Also assume that $F \in C^1(\mathcal{U})$ for all $t \in [0,T]$ $\lambda \in [0,\infty)^2$, $x \in \mathbb{R}$. If, for all perturbations β,*

$$\partial_\varepsilon J(\hat{u}+\varepsilon\beta)|_{\varepsilon=0} = 0, \tag{I.4.9}$$

then

$$\partial_u \mathcal{H}^{\hat{u}}(t) = 0. \tag{I.4.10}$$

Conversely, if (I.4.10) *holds, then* (I.4.9) *is true.*

Proof. The argument in the proof of Theorem I.4.2 leading to

$$\partial_\epsilon J(\hat{u}+\epsilon\beta)|_{\varepsilon=0} = \mathbb{E}\left[\int_0^T \partial_u \mathcal{H}^{\hat{u}}(t)\beta(t)dt\right]$$

still holds with no need to use conditional expectations. Since $\hat{u}$ and $\hat{u}+\epsilon\beta$ are $\mathbb{G}$-predictable, we obtain

$$\begin{aligned}\mathbb{E}\Bigg[\int_0^T \Bigg\{&\partial_u\hat{b}(t,t)\hat{p}(t) + \int_0^t \partial_u\partial_t\hat{b}(t,s)ds\hat{p}(t)\\ &+ \int_0^t\int_{\mathbb{R}} \partial_u\partial_t\hat{\kappa}(t,s,z)\mathscr{D}_{s,z}\hat{p}(t)\Lambda(dsdz)\Bigg\}\beta(t)dt\\ &+ \int_0^T\int_{\mathbb{R}} \left(\partial_u\hat{\kappa}(t,t,z)\beta(t)\right)\hat{q}(t,z)\Lambda(dtdz)\Bigg]\\ = \mathbb{E}&\left[\int_0^T \partial_u\mathcal{H}^{\hat{u}}(t)\beta(t)dt\right],\end{aligned}$$

where we have used the definition of $\mathcal{H}$ as in (I.3.9). We conclude as in Theorem I.4.1. ■

I.5 A maximum principle approach in optimal harvesting

We now go back to the optimal harvesting problem within fishery, where the population dynamics is given by the dynamics (I.1.1). We recall that our starting point are [10, 23, 24], where the authors consider deterministic Volterra models to model population growth and, following e.g. [4, 11], we introduce some random fluctuations that will affect the population growth. Hence, the dynamics considered are of type (I.1.1):

$$X^u(t) = X_0 + \int_0^t \left(r(t,s) - Ku(s)\right)X^u(s)ds + \int_0^t \sigma(s)X^u(s)dB(s), \quad t\in[0,T], \tag{I.5.1}$$

where $r(t,s): [0,T]^2 \longrightarrow \mathbb{R}$, $\sigma(s): [0,T]\longrightarrow\mathbb{R}$, $K>0$, $X_0>0$. Here, B is the conditional Gaussian measure. We assume that (I.5.1) admits a solution, that $r(t,s)$ is C^2 with respect to both t and s, and that σ is C^1 with respect to t and $\sigma(t) > -1$ for all $s\in[0,T]$, $z\in\mathbb{R}$. Lastly we assume $r(t,s)$, $\partial_t r(t,s)$ and $\sigma(t)$ are in $L^2(dt)$. For sufficient conditions that guarantee the existence of a solution of X we refer to Theorem I.3.1. In the context of optimal harvesting of fish, r represents the growth rate, K the catchability coefficient, and the control u is the fishing effort.

Let us define

$$\tau := \inf\{t\in[0,T],\ \text{such that } X^u(t)=0\}\wedge T.$$

Then we can see that $X^u(t) = 0$ for all $t \geq \tau$. In fact , for $0 \leq \tau \leq t \leq T$, we have that (I.1.1) can be rewritten as

$$\begin{aligned} X(t) = X_0 &+ \int_0^\tau (r(\tau, s) - Ku(s))X(s)ds + \int_0^\tau \sigma(s)X(s)dB(s) \\ &+ \int_\tau^t (r(t,s) - Ku(s))X(s)ds + \int_\tau^t \sigma(s)X(s)dB(s) \\ &+ \int_0^\tau (r(t,s) - r(\tau, s))X(s)ds \\ = X(\tau) &+ \int_\tau^t (r(t,s) - Ku(s))X(s)ds + \int_\tau^t \sigma(s)X(s)dB(s) \\ &+ \int_0^\tau (r(t,s) - r(\tau, s))X(s)ds. \end{aligned}$$

Being $X(0) = X_0 > 0$ and the process X continuous, we have that X is strictly positive, up to restricting ourselves to the interval $[0, \tau]$.

Our goal is to characterise the optimal solution to maximization of the performance functional

$$J(u) = \mathbb{E}\left[\int_0^T e^{-\delta(T-t)}X(t)u(t)dt\right], \tag{I.5.2}$$

where $u \in \mathcal{A}^{\mathbb{F}}$, $\delta > 0$. In the context of oprimal harvesting this can be regarded as the aggregated net discounted revenue, see [7]. Following the approach given in this work, we consider the Hamiltonian functional (I.3.9), which can be here rewritten as

$$\begin{aligned} \mathcal{H}^u(t) = e^{-\delta(T-t)}u(t)X(t) &+ \big(r(t,t) - Ku(t)\big)X(t)p(t) + \sigma(t)X(t)q(t)\lambda_t^B \\ &+ \int_0^t \partial_t r(t,s)X(s) - Ku(s)ds\ p(t), \end{aligned}$$

where the backward dynamics for p are given by

$$\begin{aligned} dp(t) &= e^{-\delta(T-t)}u(t)dt + \left(r(t,t) + \int_0^t \partial_t r(t,s)ds\right)p(t)dt + \sigma(t)q(t)\lambda_t^B dt \\ &\quad + q(t)dB(t) \\ p(T) &= 0. \end{aligned} \tag{I.5.3}$$

Also, we consider the mapping $\mathcal{H}^{\mathbb{F}}$ in (I.3.12):

$$\begin{aligned} \mathcal{H}^{\mathbb{F},u}(t) = e^{-\delta(T-t)}u(t)X(t) &+ \big(r(t,t) - Ku(t)\big)X(t)\mathbb{E}[p(t)|\mathcal{F}_t] \\ &+ \sigma(t)X(t)\mathbb{E}[q(t)|\mathcal{F}_t]\lambda_t^B + \int_0^t \partial_t r(t,s)X(s) - Ku(s)ds\ \mathbb{E}[p(t)|\mathcal{F}_t]. \end{aligned}$$

From Theorem I.4.1 we see that a necessary condition for an admissible control $\hat{u}$ to be optimal is that, for all $t \in [0,T]$, $\partial_u \mathcal{H}^{\mathbb{F},\hat{u}}(t) = 0$. Furthermore, from Theorem I.3.4, being the map (I.3.13) trivially concave, the condition $\partial_u \mathcal{H}^{\mathbb{F},\hat{u}}(t) = 0$ is also sufficient for the maximality. In particular this means that an admissible control $\hat{u}$ is optimal if and only if

$$e^{-\delta(T-t)}\hat{X}(t) = K\hat{X}(t)\mathbb{E}[\hat{p}(t)|\mathcal{F}_t]. \tag{I.5.4}$$

Namely, for all $t \in [0,\tau]$

$$\mathbb{E}[\hat{p}(t)|\mathcal{F}_t] = K^{-1}e^{-\delta(T-t)}. \tag{I.5.5}$$

To find a solution to (I.5.3) with respect to the information flow $\mathbb{G}$, we use a Girsanov change of measure as presented in [17]. Define the measure $\mathbb{Q}$ by $d\mathbb{Q} = \mathcal{M}(T)dP(T)$ on $\mathcal{G}_T$, where

$$\begin{aligned} d\mathcal{M}(t) &= \mathcal{M}(t)\sigma(t)dB(t) \\ \mathcal{M}(0) &= 1. \end{aligned} \tag{I.5.6}$$

An explicit solution for (I.5.6) is obtained by the Itô formula (see [17]) and is given by

$$\mathcal{M}(t) = \exp\left\{\int_0^t \sigma(s)dB(s) - \int_0^t \frac{1}{2}\sigma(s)^2\lambda_s^B ds\right\}, \quad t \in [0,T].$$

We thus have that, under the measure $\mathbb{Q}$,

$$dB^\sigma(t) = dB(t) - \sigma(t)\lambda_t^B dt,$$

is a $\mathbb{G}$-martingale. Equation (I.5.3) can now be rewritten under $\mathbb{Q}$ as

$$\begin{aligned} d\hat{p}(t) &= e^{-\delta(T-t)}\hat{u}(t)dt + \left(r(t,t) + \int_0^t \partial_t r(t,s)ds\right)\hat{p}(t)dt + \hat{q}(t)dB^\sigma(t) \\ \hat{p}(T) &= 0. \end{aligned} \tag{I.5.7}$$

Thanks to [19] we know that (I.5.7) admits a unique solution $(\hat{p},\hat{q})$ and that the process $\hat{p}$ is given by

$$\hat{p}(t) = \mathbb{E}_{\mathbb{Q}}\left[\int_t^T \exp\left\{\int_t^s \tilde{r}(v)dv\right\} e^{-\delta(T-s)}\hat{u}(s)ds\right],$$

where we defined $\tilde{r}(t) := r(t,t) + \int_0^t \partial_t r(t,s)ds$. We thus obtain that

$$\mathbb{E}\left[\hat{p}(t)|\mathcal{F}_t\right] = \mathbb{E}\left[\frac{1}{\mathcal{M}(T)}\int_t^T \exp\left\{\int_t^s \tilde{r}(v)dv\right\} e^{-\delta(T-s)}\hat{u}(s)ds\Big|\mathcal{F}_t\right]. \tag{I.5.8}$$

Substituting (I.5.8) in (I.5.5) we obtain a characterization of $\hat{u}(t)$.

Declarations

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Paper IV

Optimal control in linear-quadratic stochastic advertising models with memory

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Abstract

This paper deals with a class of optimal control problems which arises in advertising models with Volterra Ornstein-Uhlenbeck process representing the product goodwill. Such choice of the model can be regarded as a stochastic modification of the classical Nerlove-Arrow model that allows to incorporate both presence of uncertainty and empirically observed memory effects such as carryover or distributed forgetting. We present an approach to solve such optimal control problems based on an infinite dimensional lift which allows us to recover Markov properties by formulating an optimization problem equivalent to the original one in a Hilbert space. Such technique, however, requires the Volterra kernel from the forward equation to have a representation of a particular form that may be challenging to obtain in practice. We overcome this issue for Hölder continuous kernels by approximating them with Bernstein polynomials, which turn out to enjoy a simple representation of the required type. Then we solve the optimal control problem for the forward process with approximated kernel instead of the original one and study convergence. The approach is illustrated with simulations.

Declarations

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IV.1 Introduction

The problem of optimizing advertising strategies has always been of paramount importance in the field of marketing. Starting from the pioneering works of Vidale and Wolfe [23] and Nerlove and Arrow [18], this topic has evolved into a full-fledged field of research and modeling. Realizing the impossibility of describing all existing classical approaches and results, we refer the reader to the review article of Sethi [21] (that analyzes the literature prior to 1975) and a more recent paper by Feichtinger, Hartl and Sethi [11] (covering the results up to 1994) and references therein.

It is worth noting that the Nerlove–Arrow approach, which was the foundation for numerous modern dynamic advertising models, assumed no time lag between spending on advertising and the impact of the latter on the goodwill stock. However, many empirical studies (see, for example, [15]) clearly indicate some kind of a “memory” phenomenon that is often called the “distributed lag” or “carryover” effect: the influence of advertising does not have an immediate impact but is rather spread over a period of time varying from several weeks to several months. This shortcoming of the basic Nerlove–Arrow model gave rise to many modifications of the latter aimed at modeling distributed lags. For a long time, nevertheless, the vast majority of dynamic advertising models with distributed lags had been formulated in a deterministic framework (see e.g. [21, §2.6] and [11, Section 2.3]).

In recent years, however, there have been several landmark papers that consider the Nerlove-Arrow-type model with memory in a stochastic setting. Here, we refer primarily to the series of papers [13, 14] (see also a more recent work [16]), where goodwill stock is modeled via Brownian linear diffusion with delay of the form

$$\begin{aligned} dX^u(t) = \bigg(\alpha_0 X^u(t) + \int_{-r}^0 \alpha_1(s) X^u(t+s)ds + \beta_0 u(t) \\ + \int_{-r}^0 \beta_1(s) u(t+s) ds\bigg) dt + \sigma dW(t), \end{aligned} \tag{IV.1.1}$$

where X^u is interpreted as the product's goodwill stock and u is the spending on advertising. The corresponding optimal control problem in this case was solved using the so-called lift approach: equation (IV.1.1) was rewritten as a stochastic differential equation (without delay) in a suitable Hilbert space, and then infinite-dimensional optimization techniques (either dynamic programming principle or maximum principle) were applied.

In this article, we present an alternative stochastic model that also takes the carryover effect into account. Instead of the delay approach described above, we incorporate the memory into the model by means of the Volterra kernel $K \in L^2([0,T])$ and consider the controlled Volterra Ornstein-Uhlenbeck process of the form

$$X^u(t) = X(0) + \int_0^t K(t-s)\Big(\alpha u(s) - \beta X^u(s)\Big)ds + \sigma \int_0^t K(t-s)dW(s), \quad \text{(IV.1.2)}$$

where $\alpha, \beta, \sigma > 0$ and $X(0) \in \mathbb{R}$ are constants (see e.g. [1, Section 5] for more details on affine Volterra processes of such type). Note that such goodwill dynamics can be regarded as the combination of deterministic lag models described in [11, Section 2.3] and the stochastic Ornstein-Uhlenbeck-based model presented by Rao [19]. The main difference from (IV.1.1) is the memory incorporated to the noise along with the drift as the stochastic environment (represented by the noise) tends to form "clusters" with time. Indeed, in reality positive increments are likely to be followed by positive increments (if conditions are favourable for the goodwill during some period of time) and negative increments tend to follow negative increments (under negative conditions). This behaviour of the noise cannot be reflected by a standard Brownian driver but can easily be incorporated into the model (IV.1.2).

Our goal is to solve an optimization problem of the form

$$\begin{cases} X^u(t) = X(0) + \int_0^t K(t-s)\Big(\alpha u(s) - \beta X^u(s)\Big)ds + \sigma \int_0^t K(t-s)dW(s), \\ J(u) := \mathbb{E}\left[-\int_0^T a_1 u^2(s)ds + a_2 X^u(T)\right] \to \max, \end{cases} \quad \text{(IV.1.3)}$$

where $a_1, a_2 > 0$ are given constants. The set of admissible controls for the problem (IV.1.3), denoted by $L_a^2 := L_a^2(\Omega \times [0,T])$, is the space of square integrable real-valued stochastic processes adapted to the filtration generated by W. Note that the process X^u is well defined for any $u \in L_a^2$ since, for almost all $\omega \in \Omega$, the equation (IV.1.2) treated pathwisely can be considered as a deterministic linear Volterra integral equation of the second kind that has a unique solution (see e.g. [22]).

The optimization problem (IV.1.3) for underlying Volterra dynamics has been studied by several authors (see, e.g. [3, 24] and the bibliography therein). Contrarily to most of the work in our bibliography, we will not solve such problem by means of a maximum principle approach. Even though this method allows to find necessary and sufficient conditions to obtain the optimal control to (IV.1.3), we cannot directly apply it as we deal with low regularity conditions on the coefficients of our drift and volatility. Furthermore, such method has another notable drawback in the practice. In fact, its application is often associated with computations of conditional expectations that are substantially challenging due to the absence of Markovianity. Another possible method to solve the optimal control problem (IV.1.3) is to get an explicit solution of the forward equation (IV.1.2), plug it into the performance functional and try to solve the maximization problem using differential calculus in Hilbert spaces. But, even though this method seems appealing, obtaining the required explicit representation of X^u in terms of u might be tedious and burdensome. Instead, we will use the approach introduced in Paper III-[2] that is in the same spirit of the one in [13, 14, 16] mentioned above: we will rewrite the original forward stochastic Volterra integral equation as a stochastic differential equation in a suitable Hilbert space and then apply standard optimization techniques in infinite dimensions (see e.g. [9, 12]). Moreover, the shape of the corresponding infinite-dimensional Hamilton-Jacobi-Bellman equation allows to obtain an explicit solution to the latter by exploiting the "splitting" method from [14, Section 3.3].

We notice that, while the optimization problem (IV.1.3) is closely related to the one presented in [2], there are several important differences in comparison to our work. In particular, [2] demands the kernel to have the form

$$K(t) = \int_{\mathbb{R}_+} e^{-\theta t}\mu(d\theta), \tag{IV.1.4}$$

where μ is a signed measure such that $\int_{\mathbb{R}_+}(1 \wedge \theta^{-1/2})|\mu|(d\theta) < \infty$. Although there are some prominent examples of such kernels, not all kernels K are of this type; furthermore, even if a particular K admits such a representation in theory, it may not be easy to find the explicit shape of μ. In contrast, our approach works for *all Hölder continuous kernels* without any restrictions on the shape and allows to get explicit approximations $\hat{u}_n$ of the optimal control $\hat{u}$. The lift procedure presented here is also different from the one used in [2] (although they both are specific cases of the technique presented in [7]).

The lift used in the present paper was introduced in [7], then generalized in [8] for the multi-dimensional case, but the approach itself can be traced back to [6]. It should be also emphasised that this method has its own limitations: in order to perform the lift, the kernel K is required to have a specific representation of the form $K(t) = \langle g, e^{t\mathcal{A}}\nu\rangle_{\mathbb{H}}$, $t \in [0,T]$, where g and ν are elements of some Hilbert space $\mathbb{H}$ and $\{e^{t\mathcal{A}},\ t \in [0,T]\}$ is a uniformly continuous semigroup acting on $\mathbb{H}$ with $\mathcal{A} \in \mathcal{L}(\mathbb{H})$ and, in general, it may be hard to find feasible $\mathbb{H}$, g, ν and $\mathcal{A}$. Here, we work with Hölder continuous kernels K and we overcome this issue by approximating the kernel with Bernstein polynomials (which turn out to enjoy a simple representation of the required type). Then we solve the optimal control problem for the forward process with approximated kernel instead of the original one and we study convergence.

The paper is organised as follows. In section IV.2, we present our approach in case of a *liftable* K (i.e. K having a representation in terms of $\mathbb{H}$, g, ν and $\mathcal{A}$ mentioned above). Namely, we describe the lift procedure, give the necessary results from stochastic optimal control theory in Hilbert spaces as well as derive an explicit representation of the optimal control $\hat{u}$ by solving the associated Hamilton-Jacobi-Bellman equation. In section IV.3, we introduce a liftable approximation for general Hölder continuous kernels, give convergence results for the solution to the approximated problem and discuss some numerical aspects for the latter. In section IV.4, we illustrate the application of our technique with examples and simulations.

IV.2 Solution via Hilbert space-valued lift

IV.2.1 Preliminaries

First of all, let us begin with some simple results on the optimization problem (IV.1.3). Namely, we notice that X^u and the optimization problem (IV.1.3) are well defined for any $u \in L^2_a$.

Theorem IV.2.1. *Let $K \in L^2([0,T])$. Then, for any $u \in L^2_a$,*

1) *the forward Volterra Ornstein-Uhlenbeck-type equation* (IV.1.2) *has a unique solution;*

2) *there exists a constant $C > 0$ such that*

$$\sup_{t\in[0,T]} \mathbb{E}[|X^u(t)|^2] \le C(1 + \|u\|_2^2),$$

where $\|\cdot\|_2$ denotes the standard $L^2(\Omega \times [0,T])$ norm;

3) $|J(u)| < \infty$.

Proof. Item 1) is evident since, for almost all $\omega \in \Omega$, the equation (IV.1.2) treated pathwisely can be considered as a deterministic linear Volterra integral equation of the second kind that has a unique solution (see e.g. [22]). Next, it is straightforward to deduce that

$$\begin{aligned}
\mathbb{E}\left[|X^u(t)|^2\right] \le C\Bigg(1 + \mathbb{E}\left[\left(\int_0^t K(t-s)u(s)ds\right)^2\right] \\
+ \mathbb{E}\left[\left(\int_0^t K(t-s)X^u(s)ds\right)^2\right] + \mathbb{E}\left[\left(\int_0^t K(t-s)dW(s)\right)^2\right]\Bigg) \\
\le C\left(1 + \|K\|_2^2\|u\|_2^2 + \|K\|_2^2 \int_0^t \mathbb{E}\left[|X^u(s)|^2\right]ds + \|K\|_2^2\right) \\
\le C\left(1 + \|u\|_2^2 + \int_0^t \mathbb{E}\left[|X^u(s)|^2\right]ds\right).
\end{aligned}$$

Now, item 2) follows from Gronwall's inequality. Finally, $\mathbb{E}[X^u(t)]$ satisfies the deterministic Volterra equation of the form

$$\mathbb{E}[X^u(t)] = -\beta\int_0^t K(t-s)\mathbb{E}[X^u(s)]ds + X(0) + \alpha\int_0^t K(t-s)\mathbb{E}[u(s)]ds$$

and hence can be represented in the form

$$\begin{aligned}\mathbb{E}[X^u(t)] &= X(0) + \alpha \int_0^t K(t-s)\mathbb{E}[u(s)]ds - \beta \int_0^t R_\beta(t,s)X(0)ds \\ &\quad - \alpha\beta \int_0^t R_\beta(t,s) \int_0^s K(s-v)\mathbb{E}[u(v)]dvds \\ &=: X(0) + \mathcal{L}u,\end{aligned}$$

where R_β is the resolvent of the corresponding Volterra integral equation and the operator $\mathcal{L}$ is linear and continuous. Hence $J(u)$ can be re-written as

$$J(u) = -a_1\langle u, u\rangle_{L^2(\Omega\times[0,T]} + a_2(X(0) + \mathcal{L}u), \tag{IV.2.1}$$

which immediately implies that $|J(u)| < \infty$. ■

IV.2.2 Construction of Markovian lift and formulation of the lifted problem

As anticipated above, in order to solve the optimization problem (IV.1.3) we will rewrite X^u in terms of Markovian Hilbert space-valued process $\mathcal{Z}^u$ using the lift presented in [7] and then apply the dynamic programming principle in Hilbert spaces. We start from the description of the core idea behind the Markovian lifts in case of liftable kernels.

Definition IV.2.2. Let $\mathbb{H}$ denote a separable Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$. A kernel $K \in L^2([0,T])$ is called $\mathbb{H}$-*liftable* if there exist $\nu, g \in \mathbb{H}$, $\|\nu\|_{\mathbb{H}} = 1$, and a uniformly continuous semigroup $\{e^{t\mathcal{A}},\ t \in [0,T]\}$ acting on $\mathbb{H}$, $\mathcal{A} \in \mathcal{L}(\mathbb{H})$, such that

$$K(t) = \langle g, e^{t\mathcal{A}}\nu\rangle, \quad t \in [0,T]. \tag{IV.2.2}$$

For examples of liftable kernels, we refer to Section IV.4 and to [7].

Consider a controlled Volterra Ornstein-Uhlenbeck process of the form (IV.1.2) with a liftable kernel $K(t) = \langle g, e^{t\mathcal{A}}\nu\rangle$, $\|\nu\|_{\mathbb{H}} = 1$, and denote $\zeta_0 := \frac{X(0)}{\|g\|^2_{\mathbb{H}}} g$ and

$$dV^u(t) := (\alpha u(t) - \beta X^u(t))dt + \sigma dW(t).$$

Using the fact that $X(0) = \langle g, \zeta_0\rangle$, we can now rewrite (IV.1.2) as follows:

$$\begin{aligned}X^u(t) &= X(0) + \int_0^t K(t-s)dV^u(s) \\ &= \langle g, \zeta_0\rangle + \int_0^t \langle g, e^{(t-s)\mathcal{A}}\nu\rangle dV^u(s) \\ &= \left\langle g, \zeta_0 + \int_0^t e^{(t-s)\mathcal{A}}\nu dV^u(s)\right\rangle \\ &=: \langle g, \widetilde{\mathcal{Z}}^u_t\rangle,\end{aligned}$$

where $\widetilde{\mathcal{Z}}_t^u := \zeta_0 + \int_0^t e^{\mathcal{A}t-s}\nu dV^u(s)$. It is easy to check that, $\widetilde{\mathcal{Z}}^u$ is the unique solution of the infinite dimensional SDE

$$\widetilde{\mathcal{Z}}_t^u = \zeta_0 + \int_0^t \Big(\mathcal{A}(\widetilde{\mathcal{Z}}_s^u - \zeta_0) + (\alpha u(s) - \beta\langle g, \widetilde{\mathcal{Z}}_s^u\rangle)\nu\Big)ds + \int_0^t \sigma\nu dW(s)$$

and thus the process $\{\mathcal{Z}_t^u, t \in [0,T]\}$ defined as $\mathcal{Z}_t^u := \widetilde{\mathcal{Z}}_t^u - \zeta_0$ satisfies the infinite dimensional SDE of the form

$$\mathcal{Z}_t^u = \int_0^t \Big(\bar{\mathcal{A}}\mathcal{Z}_s^u - \nu\beta\langle g, \zeta_0\rangle + \nu\alpha u(s)\Big)ds + \int_0^t \sigma\nu dW(s),$$

where $\bar{\mathcal{A}}$ is the linear bounded operator on $\mathbb{H}$ such that

$$\bar{\mathcal{A}}z := \mathcal{A}z - \beta\langle g, z\rangle\nu, \quad z \in \mathbb{H}. \tag{IV.2.3}$$

These findings are summarized in the following theorem.

Theorem IV.2.3. *Let $\{X^u(t), t \in [0,T]\}$ be a Volterra Ornstein-Uhlenbeck process of the form* (IV.1.2) *with the $\mathbb{H}$-liftable kernel $K(t) = \langle g, e^{t\mathcal{A}}\nu\rangle$, $g, \nu \in \mathbb{H}$, $\|\nu\|_{\mathbb{H}} = 1$, $\mathcal{A} \in \mathcal{L}(\mathbb{H})$. Then, for any $t \in [0,T]$,*

$$X^u(t) = \langle g, \zeta_0\rangle + \langle g, \mathcal{Z}_t^u\rangle, \tag{IV.2.4}$$

where $\zeta_0 := \frac{X(0)}{\|g\|_{\mathbb{H}}^2} g$ and $\{\mathcal{Z}_t^u,\ t \in [0,T]\}$ is the $\mathbb{H}$-valued stochastic process given by

$$\mathcal{Z}_t^u = \int_0^t \Big(\bar{\mathcal{A}}\mathcal{Z}_s^u - \nu\beta\langle g, \zeta_0\rangle + \nu\alpha u(s)\Big)ds + \int_0^t \sigma\nu dW(s) \tag{IV.2.5}$$

and $\bar{\mathcal{A}} \in \mathcal{L}(\mathbb{H})$ is such that

$$\bar{\mathcal{A}}z := \mathcal{A}z - \beta\langle g, z\rangle\nu, \quad z \in \mathbb{H}.$$

Using Theorem IV.2.3, one can rewrite the performance functional $J(u)$ from (IV.1.3) as

$$J^g(u) = \mathbb{E}\left[-\int_0^T a_1 u^2(s)ds + a_2\langle g, \mathcal{Z}_T^u\rangle\right] + a_2\langle g, \zeta_0\rangle, \tag{IV.2.6}$$

where the superscript g in J^g is used to highlight dependence on the $\mathbb{H}$-valued process $\mathcal{Z}^u$. Clearly, maximizing (IV.2.6) is equivalent to maximizing

$$J^g(u) - a_2\langle g, \zeta_0\rangle = \mathbb{E}\left[-\int_0^T a_1 u^2(s)ds + a_2\langle g, \mathcal{Z}_T^u\rangle\right].$$

Finally, for the sake of notation and coherence with literature, we will sometimes write our maximization problem as a minimization one by simply noticing that the maximization of the performance functional $J^g(u) - a_2\langle g, \zeta_0\rangle$ can be reformulated as the minimization of

$$\bar{J}^g(u) := -J^g(u) + a_2\langle g, \zeta_0\rangle = \mathbb{E}\left[\int_0^T a_1 u^2(s)ds - a_2\langle g, \mathcal{Z}_T^u\rangle\right]. \tag{IV.2.7}$$

Remark IV.2.1. Using the arguments similar to the proof of Theorem IV.2.1, it is straightforward to check that J^g and $\bar{J}^g$ are continuous w.r.t. u.

In other words, in case of $\mathbb{H}$-liftable kernel K, the original optimal control problem (IV.1.3) can be replaced by the following one:

$$\begin{cases} \mathcal{Z}_t^u = \int_0^t \left(\bar{\mathcal{A}}\mathcal{Z}_s^u - \beta\langle g, \zeta_0\rangle\nu + \alpha u(s)\nu\right)ds + \int_0^t \sigma\nu dW(s), \\ \bar{J}^g(u) := \mathbb{E}\left[\int_0^T a_1u^2(s)ds - a_2\langle g, \mathcal{Z}_T^u\rangle\right] \to \min, \end{cases} \qquad u \in L_a^2. \tag{IV.2.8}$$

Remark IV.2.2. The machinery described above can also be generalized for strongly continuous semigroups on Banach spaces, see e.g. [7, 8]. However, for our purposes, it is sufficient to consider the case when $\mathcal{A}$ is a linear bounded operator on a Hilbert space.

IV.2.3 Solution to the lifted problem

In order to solve the optimal control problem (IV.2.8), we intend to use the dynamic programming approach as in [10]. A comprehensive overview of this method for more general optimal control problems can also be found in [9] and [12].

Denote by $\widetilde{\sigma}$ an element of $\mathcal{L}(\mathbb{R}, \mathbb{H})$ acting as

$$\widetilde{\sigma}x = x\sigma\nu, \quad x \in \mathbb{R},$$

and consider the *Hamilton-Jacobi-Bellman* (HJB) equation associated with the problem (IV.2.8) of the form

$$\begin{cases} \frac{\partial}{\partial t}v(t,z) &= -\frac{1}{2}\mathrm{Trace}\left(\widetilde{\sigma}\widetilde{\sigma}^*\nabla^2 v(t,z)\right) - \langle\nabla v(t,z), \bar{\mathcal{A}}z\rangle - \mathcal{H}(t,z,\nabla v(t,z)), \\ v(T,z) &= -\langle a_2 g, z\rangle, \end{cases} \tag{IV.2.9}$$

where by ∇v we denote the partial Gateaux derivative w.r.t. the spacial variable z and the Hamiltonian functional $\mathcal{H} : [0,T] \times \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is defined as

$$\mathcal{H}(t,z,\xi) := \inf_{u\in\mathbb{R}}\left\{a_1u^2 + \langle\xi, -\beta\langle g,\zeta_0\rangle\nu + \alpha u\nu\rangle\right\} = -\frac{\alpha^2\langle\xi,\nu\rangle^2}{4a_1} - \beta\langle g,\zeta_0\rangle\langle\xi,\nu\rangle.$$

Proposition IV.2.4. *The HJB equation* (IV.2.9) *associated with the lifted problem* (IV.2.8) *admits a classical solution (in the sense of [10, Definition 3.4]) of the form*

$$v(t,z) = \langle w(t), z\rangle + c(t), \tag{IV.2.10}$$

where

$$w(t) = -a_2 e^{-(t-T)\bar{\mathcal{A}}^*} g, \quad t \in [0,T], \tag{IV.2.11}$$

$\bar{\mathcal{A}}^* = \mathcal{A}^* - \beta\langle \nu, \cdot\rangle g$, *and*

$$c(t) = -\int_t^T \left(\beta X(0)\langle w(s), \nu\rangle + \frac{\alpha^2}{4a_1}\langle w(s), \nu\rangle^2 \right) ds, \quad t \in [0,T]. \tag{IV.2.12}$$

Proof. Let us solve the HJB equation (IV.2.9) explicitly using the approach presented in [14, Section 3.3]. Namely, we will look for the solution in the form (IV.2.10), where $w(t)$ and $c(t)$ are (unknown) functions such that $\frac{\partial}{\partial t} v$ and ∇v are well-defined. In this case,

$$\frac{\partial}{\partial t} v(t,z) = \langle w'(t), z\rangle + c'(t), \quad \nabla v(t,z) = w(t), \quad \nabla^2 v(t,z) = 0,$$

and, recalling that $\langle g, \zeta_0\rangle = X(0)$, we can rewrite the HJB equation (IV.2.9) as

$$\begin{cases} \langle w'(t), z\rangle + \langle z, \bar{\mathcal{A}}^* w(t)\rangle + c'(t) - \beta X(0)\langle w(t), \nu\rangle - \frac{\alpha^2}{4a_1}\langle w(t), \nu\rangle^2 = 0 \\ \langle w(T), z\rangle + c(T) = -\langle a_2 g, z\rangle. \end{cases}$$

Now it would be sufficient to find w and c that solve the following systems:

$$\begin{cases} \langle w'(t), z\rangle + \langle z, \bar{\mathcal{A}}^* w(t)\rangle = 0 \\ \langle w(T), z\rangle + \langle a_2 g, z\rangle = 0 \end{cases} ; \quad \begin{cases} c'(t) - \beta X(0)\langle w(t), \nu\rangle - \frac{\alpha^2}{4a_1}\langle w(t), \nu\rangle^2 = 0 \\ c(T) = 0, \end{cases} \tag{IV.2.13}$$

Noticing that the first system in (IV.2.13) has to hold for all $z \in \mathbb{H}$, we can solve

$$\begin{cases} w'(t) + \bar{\mathcal{A}}^* w(t) = 0, \\ w(T) + a_2 g = 0 \end{cases}$$

instead, which is a simple linear equation and its solution has the form (IV.2.11). Now it is easy to see that c has the form (IV.2.12) and

$$v(t,z) = \langle w(t), z\rangle + c(t), \quad t \in [0,T].$$

It remains to note that (IV.2.10)–(IV.2.12) is indeed a classical solution to (IV.2.9) in the sense of [10, Definition 3.4].

■

Let us now identify v in (IV.2.10)–(IV.2.12) with the value function of the lifted optimal control problem (IV.2.8) using the result presented in [10, Theorem 4.1].

Theorem IV.2.5 (Verification theorem)**.** *Let v be the solution* (IV.2.10)–(IV.2.12) *to the HJB equation* (IV.2.9) *associated with the lifted optimal control problem* (IV.2.8)*. Then*

1) $\inf_{u \in L^2_a} \bar{J}^g(u) = v(0,0)$*;*

2) *the optimal control $\hat{u}$ minimizing $\bar{J}^g$ in* (IV.2.8) *has the form*

$$\hat{u}(t) = -\frac{\alpha}{2a_1}\langle w(t), \nu\rangle = \frac{\alpha a_2}{2a_1}\langle g, e^{(T-t)\bar{\mathcal{A}}}\nu\rangle, \tag{IV.2.14}$$

where $\bar{\mathcal{A}} = \mathcal{A} - \beta\langle g, \cdot\rangle\nu$.

In particular, $\hat{u}$ given by (IV.2.14) *solves the original optimal control problem* (IV.1.2)*.*

Proof. It is straightforward to check that the coefficients of the forward equation in (IV.2.8) satisfy [10, Hypothesis 3.1] whereas the cost functional $\bar{J}^g(u)$ satisfies the conditions of [10, Hypothesis 3.3]. Moreover, the term $-\beta\langle g, \zeta_0\rangle\nu$ in (IV.2.8) satisfies condition (i) of [10, Theorem 3.7] and, since v given by (IV.2.10)–(IV.2.12) is a classical solution to the HJB equation (IV.2.9), condition (ii) of [10, Theorem 3.7] holds automatically. Finally, it is easy to see that v has sufficient regularity as required in [10, Theorem 4.1]. Therefore, both statements of Theorem IV.2.5 immediately follow from [10, Theorem 4.1]. ■

Remark IV.2.3. The approach described above can be extended by lifting to Banach space-valued stochastic processes. See Paper III for more details.

IV.3 Approximate solution for forwards with Hölder kernels

The crucial assumption in section IV.2 that allowed to apply the optimization techniques in Hilbert space was the liftability of the kernel. However, in practice it is often hard to find a representation of the required type for the given kernel, and even if this representation is available, it is not always convenient from the implementation point of view. For this reason, we provide a liftable approximation for the Volterra Ornstein-Uhlenbeck process (IV.1.2) for a general C^h-kernel K, where $C^h([0,T])$ denotes the set of h-Hölder continuous functions on $[0,T]$.

This section is structured as follows: first we approximate an arbitrary C^h-kernel by a liftable one in a uniform manner and introduce a new optimization problem where the forward dynamics is obtained from the original one replacing the kernel K with its liftable approximation. Afterwards, we prove that the optimal value of the approximated problem converges to the optimal value of the original problem and give an estimate for the rate of convergence. Finally, we discuss some numerical aspects that could be useful from the implementation point of view.

Remark IV.3.1. In what follows, by C we will denote any positive constant the particular value of which is not important and may vary from line to line (and even within one line). By $\|\cdot\|_2$ we will denote the standard $L^2(\Omega \times [0,T])$-norm.

IV.3.1 Liftable approximation for Volterra Ornstein-Uhlenbeck processes with Hölder continuous kernels

Let $K \in C([0,T])$, $\mathbb{H} = L^2(\mathbb{R})$, the operator $\mathcal{A}$ be the 1-shift operator acting on $\mathbb{H}$, i.e.

$$(\mathcal{A}f)(x) = f(x+1), \quad f \in \mathbb{H},$$

and denote K_n a Bernstein polynomial approximation for K of order $n \geq 0$, i.e.

$$\begin{aligned} K_n(t) &= \frac{1}{T^n}\sum_{k=0}^{n} K\left(\frac{Tk}{n}\right)\binom{n}{k} t^k (T-t)^{n-k} \\ &=: \sum_{k=0}^{n} \kappa_{n,k} t^k, \quad t \in [0,T], \end{aligned} \tag{IV.3.1}$$

where

$$\kappa_{n,k} := \frac{1}{T^k}\sum_{i=0}^{k}(-1)^{k-i} K\left(\frac{iT}{n}\right)\binom{n}{i}\binom{n-i}{k-i}. \tag{IV.3.2}$$

Observe that

$$(e^{t\mathcal{A}}\mathbb{1}_{[0,1]})(x) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\mathcal{A}^k \mathbb{1}_{[0,1]}\right](x) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\mathbb{1}_{[-k,-k+1]}(x)$$

and hence K_n is $\mathbb{H}$-*liftable* as

$$K_n(t) = \langle g_n, e^{t\mathcal{A}}\nu\rangle_{\mathbb{H}} = \sum_{k=0}^{n}\kappa_{n,k}t^k, \quad t \in [0,T],$$

with $g_n := \sum_{k=0}^{n} k!\kappa_{n,k}\mathbb{1}_{[-k,-k+1]}$ and $\nu := \mathbb{1}_{[0,1]}$.

By the well-known approximating property of Bernstein polynomials, for any $\varepsilon > 0$, there exist $n = n(\varepsilon) \in \mathbb{N}_0$ such that

$$\sup_{t\in[0,T]}|K(t) - K_n(t)| < \varepsilon.$$

Moreover, if additionally $K \in C^h([0,T])$ for some $h \in (0,1)$, [17, Theorem 1] guarantees that for all $t \in [0,T]$

$$|K(t) - K_n(t)| \leq H\left(\frac{t(T-t)}{n}\right)^{h/2} \leq \frac{HT^h}{2^h}n^{-\frac{h}{2}}, \tag{IV.3.3}$$

where $H > 0$ is such that

$$|K(t) - K(s)| \leq H|t-s|^h, \quad s, t \in [0,T]. \tag{IV.3.4}$$

Now, consider a controlled Volterra Ornstein-Uhlenbeck process $\{X^u(t),\ t \in [0,T]\}$ of the form (IV.1.2) with the kernel $K \in C^h([0,T])$ satisfying (IV.3.4). For a given admissible u define also a stochastic process $\{X_n^u(t),\ t \in [0,T]\}$ as a solution to the stochastic Volterra integral equation of the form

$$X_n^u(t) = X(0) + \int_0^t K_n(t-s)\Big(\alpha u(s) - \beta X_n^u(s)\Big)ds + \sigma \int_0^t K_n(t-s)dW(s), \tag{IV.3.5}$$

$t \in [0,T]$, where $K_n(t) = \sum_{k=0}^n \kappa_{n,k} t^k$ with $\kappa_{n,k}$ defined by (IV.3.2), i.e. the Bernstein polynomial approximation of K of degree n.

Remark IV.3.2. It follows from [5, Corollary 4] that both stochastic processes $\int_0^t K(t-s)dW(s)$ and $\int_0^t K_n(t-s)dW(s)$, $t \in [0,T]$, have modifications that are Hölder continuous at least up to the order $h \wedge \frac{1}{2}$. From now on, these modifications will be used.

Now we move to the main result of this subsection.

Theorem IV.3.1. *Let $K \in C^h([0,T])$, $u \in L_a^2$, and X^u, X_n^u are given by* (IV.1.2) *and* (IV.3.5) *respectively. Then there exists $C > 0$ which does not depend on n or u such that for any admissible $u \in L_a^2$:*

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X^u(t) - X_n^u(t)|^2\right] \leq C(1 + \|u\|_2^2)n^{-h}.$$

Proof. First, by Theorem IV.2.1, there exists a constant $C > 0$ such that

$$\sup_{t\in[0,T]} \mathbb{E}[|X^u(t)|^2] \leq C(1 + \|u\|_2^2). \tag{IV.3.6}$$

Consider an arbitrary $\tau \in [0,T]$, and denote $\Delta(\tau) := \sup_{t\in[0,\tau]} \mathbb{E}\left[|X^u(t) - X_n^u(t)|^2\right]$.

Then

$$\begin{aligned}
\Delta(\tau) = \sup_{t\in[0,\tau]} \mathbb{E}\Bigg[\Bigg| \int_0^t K(t-s)\Big(\alpha u(s) - \beta X^u(s)\Big)ds \\
+ \int_0^t K_n(t-s)\Big(\alpha u(s) - \beta X_n^u(s)\Big)ds \\
+ \int_0^t \sigma\Big(K(t-s) - K_n(t-s)\Big)dW(s)\Bigg|^2\Bigg] \\
\le C \sup_{t\in[0,\tau]} \mathbb{E}\left[\int_0^t \left|\Big(K(t-s) - K_n(t-s)\Big)u(s)\right|^2 ds\right] \\
+ C \sup_{t\in[0,\tau]} \mathbb{E}\left[\int_0^t \left|K_n(t-s)\Big(X^u(s) - X_n^u(s)\Big)\right|^2 ds\right] \\
+ C \sup_{t\in[0,\tau]} \mathbb{E}\left[\int_0^t \left|X^u(s)\Big(K(t-s) - K_n(t-s)\Big)\right|^2 ds\right] \\
+ C \sup_{t\in[0,\tau]} \mathbb{E}\left[\left|\int_0^t \Big(K(t-s) - K_n(t-s)\Big)dW(s)\right|^2\right].
\end{aligned}$$

Note that, by (IV.3.3) we have that

$$\sup_{t\in[0,\tau]} \mathbb{E}\left[\int_0^t \left|\Big(K(t-s) - K_n(t-s)\Big)u(s)\right|^2 ds\right] \le Cn^{-h}\|u\|_2^2.$$

Moreover, since $\{K_n,\ n \ge 1\}$ are uniformly bounded due to their uniform convergence to K it is true that

$$\sup_{t\in[0,\tau]} \mathbb{E}\left[\int_0^t \left|K_n(t-s)\Big(X^u(s) - X_n^u(s)\Big)\right|^2 ds\right] \le C\int_0^\tau \Delta(s)ds$$

with C not dependent on n, and from (IV.3.3), (IV.3.6) one can deduce that

$$\sup_{t\in[0,\tau]} \mathbb{E}\left[\int_0^t \left|X^u(s)\Big(K(t-s) - K_n(t-s)\Big)\right|^2 ds\right] \le Cn^{-h}(1 + \|u\|_2^2).$$

Lastly, by the Ito isometry and (IV.3.3),

$$\sup_{t\in[0,\tau]} \mathbb{E}\left[\left|\int_0^t \Big(K(t-s) - K_n(t-s)\Big)dW(s)\right|^2\right] \le Cn^{-h}.$$

Hence

$$\Delta(\tau) \le Cn^{-h}(1 + \|u\|_2^2) + C\int_0^t \Delta(s)ds,$$

where C is a positive constant (recall that it may vary from line to line). The final result follows from Gronwall's inequality. ■

IV.3.2 Liftable approximation of the optimal control problem

As it was noted before, our aim is to find an approximate solution to the the optimization problem (IV.1.3) by solving the liftable problem of the form

$$\begin{cases} X_n^u(t) = X(0) + \int_0^t K_n(t-s)\Big(\alpha u(s) - \beta X_n^u(s)\Big)ds + \sigma \int_0^t K_n(t-s)dW(s), \\ J_n(u) := \mathbb{E}\left[-\int_0^T a_1 u^2(s)ds + a_2 X_n^u(T)\right] \to \max, \end{cases} \tag{IV.3.7}$$

where the maximization is performed over $u \in L_a^2$. In (IV.3.7), K_n is the Bernstein polynomial approximation of $K \in C^h([0,T])$, i.e.

$$K_n(t) = \langle g_n, e^{t\mathcal{A}}\nu\rangle, \quad t \in [0,T],$$

where $\mathcal{A} \in \mathcal{L}(\mathbb{H})$ acts as $(\mathcal{A}f)(x+1)$, $\nu = \mathbb{1}_{[0,1]}$ and $g_n = \sum_{k=0}^n k!\kappa_{n,k}\mathbb{1}_{[-k,-k+1]}$ with $\kappa_{n,k}$ defined by (IV.3.2). Due to the liftability of K_n, the problem (IV.3.7) falls in the framework of section IV.2, so, by Theorem IV.2.5, the optimal control $\hat{u}_n$ has the form (IV.2.14):

$$\hat{u}_n(t) = \frac{\alpha a_2}{2a_1}\langle g_n, e^{(T-t)\bar{\mathcal{A}}_n}\nu\rangle, \quad t \in [0,T], \tag{IV.3.8}$$

where $\bar{\mathcal{A}}_n := \mathcal{A} - \beta\langle g_n, \cdot\rangle\nu$. The goal of this subsection is to prove the convergence of the optimal performance in the approximated dynamics to the actual optimal, i.e.

$$J_n(\hat{u}_n) \to \sup_{u\in L_a^2} J(u), \quad n \to \infty,$$

where J is the performance functional from the original optimal control problem (IV.1.3).

Proposition IV.3.2. *Let the kernel $K \in C^h([0,T])$. Then*

$$\sup_{n\in\mathbb{N}} J_n(u) \to -\infty \qquad as\ \|u\|_2 \to \infty, \tag{IV.3.9}$$

$$J(u) \to -\infty \qquad as\ \|u\|_2 \to \infty, \tag{IV.3.10}$$

where $\|\cdot\|_2$ denotes the standard $L^2(\Omega \times [0,T])$ norm.

Proof. We prove only (IV.3.9); the proof of (IV.3.10) is the same. Let $u \in L_a^2$ be fixed. For any $n \in \mathbb{N}$ denote

$$G_n(t) := \int_0^t K_n(t-s)dW(s), \quad t \in [0,T],$$

and notice that for any $t \in [0,T]$ we have that

$$
\begin{aligned}
|X_n^u(t)| &\le X(0) + \alpha \int_0^t |K_n(t-s)||u(s)|ds \\
&\quad + \beta \int_0^t |K_n(t-s)||X_n^u(s)|ds + \sigma\,|G_n(t)| \\
&\le C\left(1 + \left(\int_0^T u^2(s)ds\right)^{\frac{1}{2}} + \int_0^t |X_n^u(s)|ds + \sup_{r\in[0,T]}|G_n(r)|\right),
\end{aligned}
$$

where $C>0$ is a deterministic constant that does not depend on n, t or u (here we used the fact that $K_n \to K$ uniformly on $[0,T]$). Whence, for any $n \in \mathbb{N}$,

$$
\mathbb{E}\left[|X_n^u(t)|\right] \le C\left(1 + \|u\|_2 + \int_0^t \mathbb{E}\left[|X_n^u(s)|\right]ds + \mathbb{E}\left[\sup_{r\in[0,T]}|G_n(r)|\right]\right). \tag{IV.3.11}
$$

Now, let us prove that there exists a constant $C>0$ such that

$$
\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{r\in[0,T]}|G_n(r)|\right] < C.
$$

First note that, by Remark IV.3.2, for each $n\in\mathbb{N}$ and $\delta \in \left(0, \frac{h}{2}\wedge\frac{1}{4}\right)$ there exists a random variable $\Upsilon_n = \Upsilon_n(\delta)$ such that

$$
|G_n(r_1) - G_n(r_2)| \le \Upsilon_n |r_1 - r_2|^{h\wedge\frac{1}{2}-2\delta}
$$

and whence

$$
\sup_{r\in[0,T]}|G_n(r)| \le T^{h\wedge\frac{1}{2}-2\delta}\Upsilon_n.
$$

Thus it is sufficient to check that $\sup_{n\in\mathbb{N}}\mathbb{E}\Upsilon_n < \infty$. It is known from [5] that one can put

$$
\Upsilon_n := C_\delta\left(\int_0^T\int_0^T \frac{|G_n(x)-G_n(y)|^p}{|x-y|^{(h\wedge\frac{1}{2}-\delta)p+1}}dxdy\right)^{\frac{1}{p}},
$$

where $p := \frac{1}{\delta}$ and $C_\delta > 0$ is a constant that does not depend on n. Let $p' > p$. Then Minkowski integral inequality yields

$$
\begin{aligned}
\left(\mathbb{E}\Upsilon_n^{p'}\right)^{\frac{p}{p'}} &= C_\delta^p\left(\mathbb{E}\left[\left(\int_0^T\int_0^T \frac{|G_n(x)-G_n(y)|^p}{|x-y|^{(h\wedge\frac{1}{2}-\delta)p+1}}dxdy\right)^{\frac{p'}{p}}\right]\right)^{\frac{p}{p'}} \\
&\le C_\delta^p \int_0^T\int_0^T \frac{\left(\mathbb{E}\left[|G_n(x)-G_n(y)|^{p'}\right]\right)^{\frac{p}{p'}}}{|x-y|^{(h\wedge\frac{1}{2}-\delta)p+1}}dxdy.
\end{aligned} \tag{IV.3.12}
$$

Note that, by [17, Proposition 2], every Bernstein polynomial K_n that corresponds to K is Hölder continuous of the same order h and with the same constant H, i.e.

$$|K_n(t) - K_n(s)| \leq H|t-s|^h, \quad s,t \in [0,T],$$

whenever

$$|K(t) - K(s)| \leq H|t-s|^h, \quad s,t \in [0,T].$$

This implies that there exists a constant C which does not depend on n such that

$$\begin{aligned}\mathbb{E}\left[|G_n(x) - G_n(y)|^{p'}\right] &= C\Bigg(\int_0^{x\wedge y} (K_n(x-s) - K_n(y-s))^2 ds \\ &\quad + \int_{x\wedge y}^{x\vee y} K_n^2(x\vee y - s) ds\Bigg)^{\frac{p'}{2}} \\ &\leq C|x-y|^{p'(h\wedge\frac{1}{2})}.\end{aligned}$$

Plugging the bound above to (IV.3.12), we get that

$$\begin{aligned}\left(\mathbb{E}\left[\Upsilon_n^{p'}\right]\right)^{\frac{p}{p'}} &\leq C\int_0^T\int_0^T |x-y|^{(h\wedge\frac{1}{2})p - (h\wedge\frac{1}{2}-\delta)p - 1} dxdy \\ &= C\int_0^T\int_0^T |x-y|^{-1+\delta p} dxdy \\ &< C,\end{aligned}$$

where $C > 0$ denotes, as always, a deterministic constant that does not depend on n, t, u and may vary from line to line.

Therefore, there exists a constant, again denoted by C not depending on n, t or u such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\Upsilon_n\right] < C$$

and thus, by (IV.3.11),

$$\mathbb{E}\left[|X_n^u(t)|\right] \leq C\left(1 + \|u\|_2 + \int_0^t \mathbb{E}\left[|X_n^u(s)|\right] ds\right).$$

By Gronwall's inequality, there exists $C > 0$ which does not depend on n such that

$$\mathbb{E}\left[|X_n^u(T)|\right] \leq C(1 + \|u\|_2),$$

and so

$$\sup_{n\in\mathbb{N}} J_n(u) \leq C(1 + \|u\|_2) - \|u\|_2^2 \to -\infty, \quad \|u\|_2 \to \infty.$$

■

Theorem IV.3.3. *Let $K \in C^h([0,T])$ and K_n be its Bernstein polynomial approximation of order n. Then there exists constant $C > 0$ such that*

$$\left| J_n(\hat{u}_n) - \sup_{u \in L^2_a} J(u) \right| \leq C n^{-\frac{h}{2}}. \tag{IV.3.13}$$

*Moreover, $\hat{u}_n$ is "*almost optimal*" for J in the sense that there exists a constant $C > 0$ such that*

$$\left| J(\hat{u}_n) - \sup_{u \in L^2_a} J(u) \right| \leq C n^{-\frac{h}{2}}.$$

Proof. First, note that for any $r \geq 0$

$$\sup_{u \in B_r} \left| J_n(u) - J(u) \right| \leq C(1 + r^2)^{\frac{1}{2}} n^{-\frac{h}{2}}, \tag{IV.3.14}$$

where $B_r := \{u \in L^2_a : \ \|u\|_2 \leq r\}$. Indeed, by definitions of J, J_n and Theorem IV.3.1, for any $u \in B_r$:

$$\begin{aligned} \left| J_n(u) - J(u) \right| = \left| \mathbb{E}[X^u_n(T) - X^u(T)] \right| &\leq C(1 + \|u\|^2_2)^{\frac{1}{2}} n^{-\frac{h}{2}} \\ &\leq C(1 + r^2)^{\frac{1}{2}} n^{-\frac{h}{2}}. \end{aligned} \tag{IV.3.15}$$

In particular, this implies that there exists $C > 0$ that does not depend on n such that $J(0) - C < J_n(0)$, so, by Proposition IV.3.2, there exists $r_0 > 0$ that does not depend on n such that $\|u\|_2 > r_0$ implies

$$J_n(u) < J(0) - C < J_n(0), \quad n \in \mathbb{N}.$$

In other words, all optimal controls $\hat{u}_n$, $n \in \mathbb{N}$ must be in the ball B_{r_0} and that $\sup_{u \in L^2_a} J(u) = \sup_{u \in B_{r_0}} J(u)$. This, together with uniform convergence of J_n to J over bounded subsets of L^2_a and estimate (IV.3.14), implies that there exists $C > 0$ not dependent on n such that

$$\left| J_n(\hat{u}_n) - \sup_{u \in L^2_a} J(u) \right| \leq C n^{-\frac{h}{2}}. \tag{IV.3.16}$$

Finally, taking into account (IV.3.14) and (IV.3.16) as well as the definition of B_{r_0},

$$\begin{aligned} \left| J(\hat{u}_n) - \sup_{u \in L^2_a} J(u) \right| &\leq \left| J(\hat{u}_n) - J_n(\hat{u}_n) \right| + \left| J_n(\hat{u}_n) - \sup_{u \in L^2_a} J(u) \right| \\ &\leq \left| J(\hat{u}_n) - J_n(\hat{u}_n) \right| + \left| J_n(\hat{u}_n) - \sup_{u \in B_{r_0}} J(u) \right| \\ &\leq C n^{-\frac{h}{2}}. \end{aligned}$$

which ends the proof. ■

Theorem IV.3.4. *Let $K \in C^h([0,T])$ and $\hat{u}_n$ be defined by* (IV.3.8)*. Then the optimization problem* (IV.1.3) *has a unique solution $\hat{u} \in L^2_a$ and*

$$\hat{u}_n \to \hat{u}, \quad n \to \infty,$$

in the weak topology of $L^2(\Omega \times [0,T])$.

Proof. By (IV.2.1), the performance functional J can be represented in a linear-quadratic form as

$$J(u) = -a_1 \langle u, u \rangle_{L^2(\Omega\times[0,T]} + a_2(X(0) + \mathcal{L}u),$$

where $\mathcal{L}$: $L^2(\Omega \times [0,T]) \to L^2(\Omega \times [0,T])$ is a continuous linear operator. Then, by [4, Theorem 9.2.6], there exists a unique $\hat{u} \in L^2(\Omega \times [0,T])$ that maximizes J and, moreover, $\hat{u}_n \to \hat{u}$ weakly as $n \to \infty$. Furthermore, since all $\hat{u}_n$ are deterministic, so is $\hat{u}$; in particular, it is adapted to filtration generated by W which implies that $\hat{u} \in L^2_a$. ■

IV.3.3 Algorithm for computing $\hat{u}_n$

The explicit form of $\hat{u}_n$ given by (IV.3.8) is not very convenient from the implementation point of view since one has to compute $e^{(T-t)\bar{\mathcal{A}}_n}\nu = e^{(T-t)\bar{\mathcal{A}}_n}\mathbb{1}_{[0,1]}$, where $\bar{\mathcal{A}}_n := \mathcal{A} - \beta\langle g_n, \cdot\rangle\mathbb{1}_{[0,1]}$, $(\mathcal{A}f)(x) = f(x+1)$. A natural way to simplify the problem is to truncate the series

$$\sum_{k=0}^{\infty} \frac{(T-t)^k}{k!}\bar{\mathcal{A}}_n^k \mathbb{1}_{[0,1]} \approx \sum_{k=0}^{M} \frac{(T-t)^k}{k!}\bar{\mathcal{A}}_n^k \mathbb{1}_{[0,1]}$$

for some $M \in \mathbb{N}$. However, even after replacing $e^{(T-t)\bar{\mathcal{A}}_n}$ in (IV.3.8) with its truncated version, we still need to be able to compute $\bar{\mathcal{A}}_n^k \mathbb{1}_{[0,1]}$ for the given $k \in \mathbb{N}$. An algorithm to do so is presented in the proposition below.

Proposition IV.3.5. *For any $k \in \mathbb{N} \cup \{0\}$,*

$$\bar{\mathcal{A}}_n^k \mathbb{1}_{[0,1]} = \sum_{i=0}^{k} \gamma(i,k)\mathbb{1}_{[-i,-i+1]},$$

where, $\gamma(0,0) = 1$ and, for all $k \geq 1$,

$$\gamma(i,k) = \begin{cases} \gamma(i-1,k-1), & i = 1, ..., k \\ \sum_{j=0}^{(k-1)\wedge n}(-\beta)j!\kappa_{n,j}\gamma(j,k-1), & i = 0. \end{cases}$$

Proof. The proof follows an inductive argument. The statement for $\gamma(0,0)$ is obvious. Now let

$$\bar{\mathcal{A}}_n^{k-1} \mathbb{1}_{[0,1]} = \sum_{i=0}^{k-1} \gamma(i,k-1)\mathbb{1}_{[-i,-i+1]}.$$

Then

$$\begin{aligned}
\bar{\mathcal{A}}_n^k \mathbb{1}_{[0,1]} &= \bar{\mathcal{A}}_n \Big(\bar{\mathcal{A}}_n^{k-1} \mathbb{1}_{[0,1]} \Big) \\
&= \sum_{i=0}^{k-1} \gamma(i,k-1) \bar{\mathcal{A}}_n \mathbb{1}_{[-i,-i+1]} \\
&= \sum_{i=1}^{k} \gamma(i-1,k-1) \mathbb{1}_{[-i,-i+1]} \\
&\quad + \mathbb{1}_{[0,1]}(-\beta) \left\langle \sum_{j=0}^{k-1} \gamma(j,k-1)\mathbb{1}_{[-j,-j+1]}, \sum_{j=0}^{n} j!\kappa_{n,j} \mathbb{1}_{[-j,-j+1]} \right\rangle \\
&= \sum_{i=1}^{k} \gamma(i-1,k-1) \mathbb{1}_{[-i,-i+1]} + \mathbb{1}_{[0,1]} \sum_{j=0}^{(k-1)\wedge n} (-\beta) j! \kappa_{n,j} \gamma(j,k-1).
\end{aligned}$$

■

Finally, consider

$$\begin{aligned}
\hat{u}_{n,M}(t) &:= \frac{\alpha a_2}{2a_1} \left\langle g_n, \sum_{k=0}^{M} \frac{(T-t)^k}{k!} \bar{\mathcal{A}}_n^k \mathbb{1}_{[0,1]} \right\rangle \\
&= \frac{\alpha a_2}{2a_1} \left\langle \sum_{i=0}^{n} i!\kappa_{n,i} \mathbb{1}_{[-i,-i+1]}, \sum_{k=0}^{M}\sum_{i=0}^{k} \frac{(T-t)^k}{k!} \gamma(i,k) \mathbb{1}_{[-i,-i+1]} \right\rangle \\
&= \frac{\alpha a_2}{2a_1} \left\langle \sum_{i=0}^{n} i!\kappa_{n,i} \mathbb{1}_{[-i,-i+1]}, \sum_{i=0}^{M} \left(\sum_{k=i}^{M} \frac{(T-t)^k}{k!} \gamma(i,k) \right) \mathbb{1}_{[-i,-i+1]} \right\rangle \\
&= \frac{\alpha a_2}{2a_1} \sum_{i=0}^{n\wedge M} \sum_{k=i}^{M} \frac{i!\kappa_{n,i}\gamma(i,k)}{k!} (T-t)^k \\
&= \frac{\alpha a_2}{2a_1} \sum_{k=0}^{M} \left(\sum_{i=0}^{k\wedge n} \frac{i!\kappa_{n,i}\gamma(i,k)}{k!} \right) (T-t)^k, \qquad \text{(IV.3.17)}
\end{aligned}$$

where $\kappa_{n,i}$ are defined by (IV.3.2) and $\gamma(i,k)$ are from Proposition IV.3.5.

Theorem IV.3.6. *Let $n \in \mathbb{N}$ be fixed and $M \geq (T-t)\|\bar{\mathcal{A}}_n\|_{\mathcal{L}}$, where $\|\cdot\|_{\mathcal{L}}$ denotes the operator norm. Then, for all $t \in [0,T]$,*

$$|\hat{u}_n(t) - \hat{u}_{n,M}(t)| \leq \frac{\alpha a_2}{2a_1} \|g_n\| e^{(T-t)\|\bar{\mathcal{A}}_n\|_{\mathcal{L}}} \left(1 - e^{-\frac{(T-t)\|\bar{\mathcal{A}}_n\|_{\mathcal{L}}}{M+1}} \right).$$

Moreover,

$$\sup_{t\in[0,T]} |\hat{u}_n(t) - \hat{u}_{n,M}(t)| \leq \frac{\alpha a_2}{2a_1} \|g_n\| e^{T\|\bar{\mathcal{A}}_n\|_{\mathcal{L}}} \left(1 - e^{-\frac{T\|\bar{\mathcal{A}}_n\|_{\mathcal{L}}}{M+1}} \right) \to 0, \quad M \to \infty.$$

Proof. One has to prove the first inequality and the second one then follows. It is clear that

$$|\hat{u}_n(t) - \hat{u}_{n,M}(t)| \le \frac{\alpha a_2}{2a_1} \|g_n\| \left\| \sum_{k=M+1}^{\infty} \frac{(T-t)^k}{k!} \bar{\mathcal{A}}_n^k \mathbb{1}_{[0,1]} \right\|$$

and, if $M \ge (T-t)\|\bar{\mathcal{A}}_n\|_{\mathcal{L}}$, we have that

$$\begin{aligned} \left\| \sum_{k=M+1}^{\infty} \frac{(T-t)^k}{k!} \bar{\mathcal{A}}_n^k \mathbb{1}_{[0,1]} \right\| &\le \sum_{k=M+1}^{\infty} \frac{\left((T-t)\left\|\bar{\mathcal{A}}_n\right\|_{\mathcal{L}}\right)^k}{k!} \\ &\le e^{(T-t)\|\bar{\mathcal{A}}_n\|_{\mathcal{L}}} \left(1 - e^{-\frac{(T-t)\|\bar{\mathcal{A}}_n\|_{\mathcal{L}}}{M+1}}\right), \end{aligned}$$

where we used a well-known result on tail probabilities of Poisson distribution (see e.g. [20]). ■

IV.4 Examples and simulations

Example IV.4.1 (*monomial kernel*)**.** Let $N \in \mathbb{N}$ be fixed. Consider an optimization problem of the form

$$\begin{cases} X^u(t) = X(0) + \int_0^t (t-s)^N \Big(u(s) - X^u(s)\Big) ds + \int_0^t (t-s)^N dW(s), \\ \mathbb{E}\left[X^u(T) - \int_0^T u^2(s) ds\right] \to \max, \end{cases} \tag{IV.4.1}$$

where, as always, we optimize over $u \in L_a^2$. The kernel $K(t) = t^N$ is $\mathbb{H}$-liftable,

$$t^N = \langle N! \mathbb{1}_{[-N,-N+1]}, e^{t\mathcal{A}} \mathbb{1}_{[0,1]} \rangle,$$

where $(\mathcal{A}f)(x) = f(x+1)$, $f \in \mathbb{H}$. By Theorem IV.2.5, the optimal control for the problem (IV.4.1) has the form

$$\hat{u}(t) = \frac{N!}{2} \langle \mathbb{1}_{[-N,-N+1]}, e^{(T-t)\bar{\mathcal{A}}} \mathbb{1}_{[0,1]} \rangle,$$

where $\bar{\mathcal{A}} = \mathcal{A} - N!\langle \mathbb{1}_{[-N,-N+1]}, \cdot \rangle \mathbb{1}_{[0,1]}$. In this simple case, we are able to find an explicit expression for $e^{(T-t)\bar{\mathcal{A}}^*} \mathbb{1}_{[-i,-i+1]}$. Indeed, it is easy to see that, for any $i \in \mathbb{N} \cup \{0\}$, $p \in \mathbb{N} \cup \{0\}$ and $q = 0, 1, ..., N$,

$$\bar{\mathcal{A}}^{p(N+1)+q} \mathbb{1}_{[0,1]} = \sum_{j=0}^{p} (-1)^{p-j} (N!)^{p-j} \mathbb{1}_{[-j(N+1)-q,-j(N+1)-q+1]}$$

and whence

$$
\begin{aligned}
&\langle \mathbb{1}_{[-N,-N+1]}, e^{(T-t)\bar{\mathcal{A}}}\mathbb{1}_{[0,1]}\rangle \\
&= \left\langle \mathbb{1}_{[-N,-N+1]}, \sum_{p=0}^{\infty}\sum_{q=0}^{N} \frac{(T-t)^{pN+p+q}}{(pN+p+q)!} \sum_{j=0}^{p}(-1)^{p-j}(N!)^{p-j}\mathbb{1}_{[-j(n+1)-q,-j(N+1)-q+1]} \right\rangle \\
&= \sum_{p=0}^{\infty} \frac{(T-t)^{pN+p+N}}{(pN+p+N)!}(-1)^p(N!)^p \\
&= (T-t)^N E_{N+1,N+1}(-N!(T-t)^{N+1}),
\end{aligned}
$$

where $E_{a,b}(z) := \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(ap+b)}$ is the Mittag-Leffler function. This, in turn, implies that

$$
\hat{u}(t) = \frac{N!(T-t)^N}{2} E_{N+1,N+1}(-N!(T-t)^{N+1}). \tag{IV.4.2}
$$

On Fig. IV.1, the black curve depicts the optimal $\hat{u}$ computed for the problem IV.4.1 with $K(t) = t^2$ and $T = 2$ using (IV.4.2); the othere curves are the approximated optimal controls $\hat{u}_{n,M}$ (as in (IV.3.17)) computed for $n = 1, 2, 5, 10$ and $M = 20$.

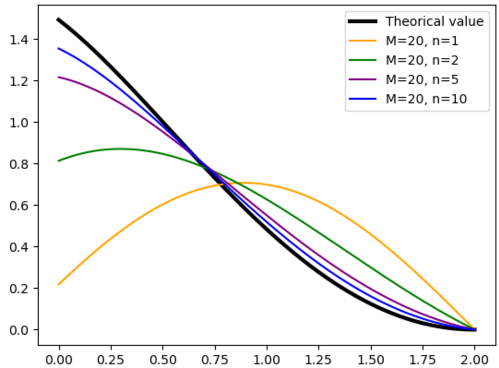

Figure IV.1: Optimal control of Volterra Ornstein-Uhlenbeck process with monomial kernel $K(t) = t^2$ (in black) and control approximants $\hat{u}_{n,M}$.

Remark IV.4.1. The solution of the problem (IV.4.1) described in Example IV.4.1 should be regarded only as an illustration of the optimization technique via infinite-dimensional lift: in fact, the kernel K in this example is degenerate and thus the process X^u in (IV.4.1) is Markovian. This means that other finite dimensional techniques could have been used in this case.

Example IV.4.2 (*fractional and gamma kernels*). Consider three optimization problems of the form

$$\begin{cases} X_i^u(t) = \int_0^t K_i(t-s)\Big(\alpha u(s) - \beta X^u(s)\Big)ds + \int_0^t K_i(t-s)dW(s), \\ \mathbb{E}\left[X_i^u(T) - \int_0^T u^2(s)ds\right] \to \max, \end{cases} \tag{IV.4.3}$$

$i = 1, 2, 3,, u \in L_a^2$, where the kernels are chosen as follows: $K_1(t) := t^{0.3}$ (fractional kernel), $K_2(t) := t^{1.1}$ (smooth kernel) and $K_3(t) := e^{-t}t^{0.3}$ (gamma kernel). In these cases, we apply all the machinery presented in section IV.3 to find $\hat{u}_{n,M}$ for each of the optimal control problems described above. In our simulations, we choose $T = 2$, $n = 20$, $M = 50$; the mesh of the partition for simulating sample paths of X^u is set to be 0.05, $\sigma = 1$, $X(0) = 0$.

Fig. IV.2 depicts approximated optimal controls for different values of α and β. Note that the gamma kernel $K_3(t)$ (third column) is of particularly interest in optimal advertising. This kernel, in fact, captures the peculiarities of the empirical data (see [15]) since the past dependence comes into play after a certain amount of time (like a delayed effect) and its relevance declines as time goes forward.

Remark IV.4.2. Note that the stochastic Volterra integral equation from (IV.4.3) can be sometimes solved explicitly for certain kernels (e.g. via the resolvent method). For instance, the solution X^u which corresponds to the fractional kernel of the type $K(t) = t^h$, $h > 0$, and $\beta = 1$ has the form

$$X^u(t) = \Gamma(h+1) \int_0^t (t-s)^h E_{h+1,h+1}\left(-\Gamma(h+1)(t-s)^{h+1}\right)\left(\alpha u(s)ds + dW(s)\right),$$

$t \in [0, T]$, where $E_{a,b}$ again denotes the Mittag-Leffler function. Having the explicit solution, one could solve the optimization problem (IV.4.3) by plugging in the shape of X^u to the performance functional and applying the standard minimization techniques in Hilbert spaces. However, as mentioned in the introduction, this leads to some tedious calculations that are complicated to implement, whereas our approach allows to get the approximated solution in a relatively simple manner.

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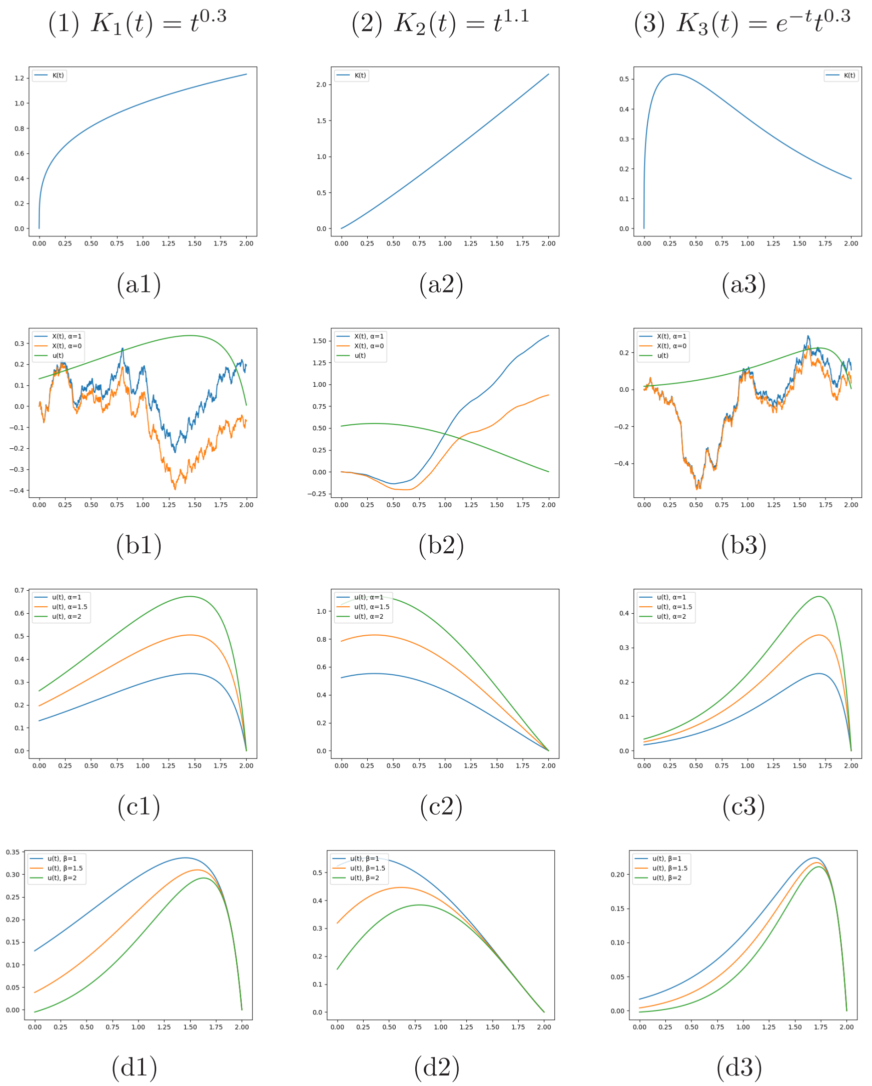

Figure IV.2: Optimal advertising strategies for control problems with kernels K_1–K_3 from Example IV.4.2; plots related to the kernel K_i are contained in the ith column. Panels (a1)–(a3) depict the graphs of kernels K_1–K_3; each of (b1)–(b3) represents a sample path of the corresponding $X_i^u(t)$ under optimal control with $\alpha = 0$ (orange) and $\alpha = 1$ (blue) as well as the approximated optimal control $\hat{u}_{n,M}$ itself (green). Panels (c1)–(c3) show $\hat{u}_{n,M}$ for $\alpha = 1$ (blue), $\alpha = 1.5$ (orange) and $\alpha = 2$ (green; in all three cases $\beta = 1$), whereas (d1)–(d3) plot the behaviour of $\hat{u}_{n,M}$ for $\beta = 1$ (blue), $\beta = 1.5$ (orange) and $\beta = 2$ (green; in all three cases $\alpha = 1$).

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