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A LARGE TIME-STEPPING SCHEME FOR BALANCE EQUATIONS

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Abstract. We present a well-balanced, large time stepping method for conservation laws with source terms. The numerical method is based on a local reformulation of the balance law as a conservation law with a discontinuous flux function, and the approximate solution of this equation by a front tracking method. This yields an unconditionally stable method which is particularly well suited to calculate stationary states. We demonstrate the viability of this approach by several numerical examples.

1. Introduction

In this paper we propose a numerical scheme for conservation laws with source terms, often referred to as balance laws, a prototype of which is given by

$$\begin{cases} u_t + f(u)_x = A(x,u) & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where u is the (scalar) unknown, f is the flux function, and A is the source term. Frequently the source term takes the form

$$A(x,u) = z'(x)b(u), \tag{1.2}$$

in which case (1.1) can be seen as a model equation for the Saint-Venant (shallow water) equations. We remark that the coefficient z in (1.2) can be discontinuous, which would correspond to a discontinuous bottom topography.

Formally (1.1) with the source (1.2) is equivalent to

$$U_t + AU_x = 0, \tag{1.3}$$

where $U = (u,z)$ and the matrix A is given by

$$A = \begin{pmatrix} f'(u) & -b \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues of the above matrix (wave speeds) are $f'(u)$ and 0, which can coincide and thereby result in so-called "resonance".

Independently of the smoothness of the initial data and of the flux or the source terms, solutions to (1.1) are in general discontinuous, and must therefore be interpreted in the weak sense. Consequently so-called entropy conditions are used to select a unique weak solution to the initial-value problem. This solution is referred to as an entropy solution. Weak and entropy solutions of (1.1)-(1.2) are well defined when $z' \in L^\infty$.

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One of the key issues in designing numerical schemes for (1.1) is the resolution of steady states. If we assume that the solution is smooth at a steady state $\bar{u} = \bar{u}(x)$, the flux function f and the source term A are balanced, i.e., $\bar{u}$ satisfies the equation

$$f(u)_x = A(x, u). \tag{1.4}$$

More detailed forms of (1.4) can be derived for (1.2). The usual strategy of devising numerical methods for (1.1) is to use a Godunov type numerical flux in a finite volume method coupled with a centered differencing of the source term. It is well known, see [1], that this does not preserve discrete steady states. Another alternative is provided by the so-called splitting or fractional steps method, which is based on separating the updates for the flux and the source [2]. This method is also deficient with regard to preserving discrete steady states.

Due to these difficulties, so-called *well-balanced* schemes have been proposed. These schemes are designed to preserve steady states. A variety of well-balanced schemes can be found in literature, see [1, 3, 4, 5, 6] and the references cited therein. For a partial overview, see also the introductory part of [7].

In many applications the goal is to calculate steady states both accurately and quickly. Accurate transient values are not needs, as these are seen merely as intermediate steps in a time marching algorithm to compute the steady states. In such cases, it is desirable to relax the CFL condition (i.e., the relation between the spatial and temporal discretization parameters) to reach the steady states as quickly as possible. One such class of problems is provided by the so-called quasi-steady problems (perturbations of steady states).

Our aim in this paper is to devise a well-balanced finite volume scheme for (1.1) without an intrinsic CFL condition, thereby permitting the calculation of steady state solutions with a minimal computational effort. Our finite volume scheme is designed to find and preserve discrete steady states, and therefore we will refer this scheme as well-balanced. The key element of our strategy will be a "local" transformation of the balance law (1.1) to a conservation law with a space-time dependent discontinuous coefficient:

$$u_t + \tilde{f}(k(x,t), u))_x = 0, \tag{1.5}$$

where $\tilde{f}$ is the flux modified locally by the source. Equations of this type are by now mathematically and computationally well understood within a proper framework of entropy solutions, and various types of numerical methods have been devised and analyzed for these equations (see the list of references given above and for (1.5) in particular reference [8]). Our strategy is to employ numerical schemes designed for conservation laws with discontinuous coefficients (1.5) to approximate solutions of (1.1). Furthermore, since we concentrate on rapidly finding the stationary solutions, we propose a method in which the size of the time step is not limited by the spatial discretization, i.e., no CFL condition is needed.

The main features of the scheme are demonstrated by numerical experiments in Section 3. We believe that the approach of using a local discontinuous flux formulation for designing well-balanced schemes will lead to alternative numerical schemes for systems of conservation laws as well, and plan to address the extension to systems in a future work.

In a recent paper [9] we analyzed the convergence of a variant of the well-balanced scheme proposed herein. However, this scheme, being based on a Godunov-type finite volume discretization of conservation laws with discontinuous flux, is restricted by the usual CFL condition. The purpose of the present paper is to suggest and demonstrate a large time-stepping extension of the scheme from [9]. In addition, by means of numerical experiments, we want to compare the large time-stepping scheme with other schemes from the literature. We will present a convergence analysis of our scheme elsewhere.

In this paper we use front tracking as a basis for a finite volume type approximation to the solution of the balance law (1.1). This is in contrast with the approach taken in [7]. In that paper we used the reformulation (1.3) to design a front tracking algorithm to solve (1.1), with the source in the form (1.2), directly. Although this also gives a very efficient method, the drawback of this method is that the solution of the Riemann problem for (1.3) is quite complicated. Furthermore this approach is limited to source terms on the form (1.2). The finite volume approach used in the present paper also uses front tracking, but it is based on another reformulation of (1.1). This leads to Riemann problems that are much easier to solve. See Section 2 for details.

Although we will not perform a rigorous analysis of our scheme in this paper, it seems appropriate to make a few remarks regarding convergence analysis of well-balanced schemes in general. First of all, if $f' \neq 0$, it is possible to work within the standard BV (bounded variation) framework, see, e.g., [3, 10]. If $f'(u) = 0$ for some u, the situation becomes more complicated. As is the case with conservation laws with discontinuous flux, there is generally no BV bound for the conserved variable u itself. In order to prove the convergence of approximate solutions (and existence of solutions) the so-called singular mapping approach has been used in the last twenty years to achieve compactness of sequences of approximate solutions, in particular for problems with discontinuous coefficients, cf. [11, 12, 13, 14, 15, 16, 17, 18]. More recently, other analytical tools have been utilized for discontinuous flux problems, including compensated compactness [8, 19] and entropy process solutions/kinetic solutions [20]. Regarding convergence analysis for conservation laws with source terms, there are only a few papers that deal with the resonant case where BV estimates are not available, see [21, 6, 14, 9].

We have organized this paper as follows: In Section 2 we define the scheme. The scheme is based on front tracking schemes for conservation laws with discontinuous coefficients, therefore we explain how this numerical method (front tracking) works. Front tracking in turn, depends on the solution of Riemann problems, and we devote a subsection to explaining how the Riemann problems arising in our setting are solved. In Section 3 we show how the scheme performs in various settings, and compare it with other schemes found in the literature. Finally, we summarize our findings in Section 4.

2. The large time stepping scheme

In this section we describe and define the large time stepping scheme. The starting point is the following idea. Let $B(x;u)$ be the function defined by

$$B(x;u) = \int^{x} A(x, u(x,t))\, dx. \tag{2.1}$$

Fix $\Delta t > 0$, and set $t^n = n\Delta t$ for $n = 0, 1, 2, \ldots$. Define $u^0 = u_0$ and u^n, $n > 0$, to be the (entropy) weak solution of

$$
\begin{aligned}
u_t^n + \left(f\left(u^n\right) - B\left(x; u^{n-1}(\cdot, t^{n-1})\right)\right)_x = 0, \quad t \in (t^{n-1}, t^n] \\
u^n\left(\cdot, t^{n-1}\right) = u^{n-1}(\cdot, t^{n-1}).
\end{aligned}
\tag{2.2}
$$

It is obvious that this semi-discrete scheme conserves steady states since these are given by

$$
f(u) - B(x; u) = \text{constant}. \tag{2.3}
$$

Note that the discrete steady state (2.3) reflects the flux-source balance that should characterize a steady state.

In order to use this idea to calculate approximate solutions, we need to choose a numerical method for the following conservation law with a spatially varying (discontinuous) coefficient:

$$
u_t + F^n(x, u)_x = 0 \quad t \in (0, \Delta t], \quad u(x, 0) = u^{n-1}(x), \tag{2.4}
$$

where $F^n(x, u) := f(u) - B(x; u^{n-1})$. There are many methods to choose from, like the aligned Godunov type schemes of [11, 12] and Staggered Enquist-Osher type schemes of [17, 18, 22], but to build an unconditionally stable (large time-stepping) method, we shall use a front tracking method to solve (2.4). We now briefly describe the front tracking method.

2.1. **Front tracking.** Front tracking is a numerical method for (2.4) that have no fixed time step, and is related to the method of characteristics.

To be concrete, consider the equation (2.4) where we suppress the index n, i.e.,

$$
u_t + F(x, u)_x = 0, \quad F(x, u) = f(u) - B(x),
$$

for some piecewise smooth function $B(x)$. The conservation law is assumed to hold for $t > 0$, while we initially at $t = 0$ prescribe $u(x, 0) = u_0(x)$.

To define a numerical method we choose (for simplicity) a uniform grid in the x direction:

$$
x_j = j\Delta x, \quad x_{j+1/2} = (j + 1/2)\Delta x, \quad j \in \mathbb{Z},
$$

where the spatial discretization parameter $\Delta x > 0$ is a given (small) number. Let I_j denote the interval $(x_{j-1/2}, x_{j+1/2}]$, and set

$$
B_j = \frac{1}{\Delta x} \int_{I_j} B(x)\, dx.
$$

Next, fix a small parameter $\delta > 0$, let $u_i = i\delta$ for $i \in \mathbb{Z}$, and define the piecewise linear interpolation

$$
f^\delta(u) = f\left(u_i\right) + \left(u - u_i\right) \frac{f\left(u_{i+1}\right) - f\left(u_i\right)}{u_{i+1} - u_i} \quad \text{for } u \in [u_i, u_{i+1}]. \tag{2.5}
$$

Then define the approximate flux function $F_{\Delta x}^\delta$ by

$$
F_{\Delta x}^\delta(x, u) = f^\delta(u) + \sum_j B_j \mathbf{1}_{I_j}(x), \tag{2.6}
$$

nwhere $\mathbf{1}_\Omega$ denotes the characteristic function of a set Ω, i.e., $\mathbf{1}_\Omega(x) = 1$ if $x \in \Omega$ and zero otherwise.

Next, let $u_{\Delta x,0}$ be an approximation to the initial function u_0 defined by

$$u_{\Delta x,0}(x) = \frac{1}{\Delta x}\int_{I_j} u_0(x)\,dx, \quad x \in I_j, \tag{2.7}$$

if $|j| \leq 1/\Delta x$, while we set $u_{\Delta x,0}(x) = 0$ otherwise.

Now we claim that we can construct the *exact* (entropy) solution $u^{\delta,\Delta x}$ to the initial value problem

$$\begin{aligned} u_t^{\delta,\Delta x} + F_{\Delta x}^{\delta}\left(x, u^{\delta,\Delta x}\right)_x &= 0, \quad t > 0, \\ u^{\delta,\Delta x}(\cdot, 0) &= u_{\Delta x,0}, \end{aligned} \tag{2.8}$$

by a finite number of operations. How this is done is explained hereunder.

We observe that initially at each location $x = x_{j-1/2}$ we have a Riemann problem of the type

$$\begin{gathered} \begin{cases} u_t + \left(f^{\delta}(u) - B_{j-1}\right)_x = 0, & x < x_{j-1/2}, \\ u_t + \left(f^{\delta}(u) - B_j\right)_x = 0, & x > x_{j-1/2}, \end{cases} \\ u(x,0) = \begin{cases} u_{j-1}, & x < x_{j-1/2}, \\ u_j, & x > x_{j-1/2}. \end{cases} \end{gathered} \tag{2.9}$$

The solution of this Riemann problem (see Section 2.2 for details) is a piecewise constant function of the form

$$u(x,t) = \begin{cases} u_{j-1}, & x - x_{j-1/2} < \sigma_0 t, \\ \hat{u}_k, & \sigma_{k-1} t \leq x - x_{j-1/2} < \sigma_k t, \quad k = 1, \dots, m, \\ u_j, & \sigma_m t \leq t, \end{cases} \tag{2.10}$$

where $\{\sigma_k\}_{k=1}^m$ is an increasing sequence of numbers. This formula is valid for small t and for $\left|x - x_{j-1/2}\right|$ small. We can piece together the solutions of the (finitely many) Riemann problems to obtain an entropy solution for $t \leq t_1$, where $t_1 > 0$ is defined to be the first time two discontinuities collide. The resulting function we call $u^{\delta,\Delta x}$. For a fixed t, $u^{\delta,\Delta x}$ is a piecewise constant function. We also see that the discontinuities in $u^{\delta,\Delta x}$ move at constant speeds, and we call these discontinuities *fronts*.

Assume that the two (or more) fronts collide at $t = t_1$ at a point $\hat{x}$. Since $u^{\delta,\Delta x}$ is piecewise constant, the collision defines a new Riemann problem of the type (2.9). Of course, the left and right initial values are no longer u_{j-1} and u_j, and it may happen that the B's to the left and right are equal. Nevertheless, we can solve this Riemann problem and the solution is defined by a fan of fronts moving with finite speeds. This means that we can define $u^{\delta,\Delta x}$ until the next time two fronts collide. In this way we propagate the solution in time.

Since we initially have only a finite number of fronts, for a large class of f's[1] it turns out that there will only be a finite number of collisions between fronts for all positive times t. In other words, for t larger than a collision time t_M, $u^{\delta,\Delta x}$ will have fronts that are moving apart, or are stationary. Thus the exact solution to (2.8) can be computed by a finite number of operations. For a proof of this, see [15], while for a thorough discussion of front tracking in general, see [23].

[1] It is sufficient that for large $|u|$, $|f(u)| > C\log(|u|+1)$.

2.2. **The solution of the Riemann problems.** To complete our description of the front tracking algorithm, we now detail how the Riemann problems are solved. We start with the simpler case where $B_j = B_{j-1}$. In this case we have the Riemann problem

$$u_t + f^\delta(u)_x = 0, \quad u(x,0) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases} \tag{2.11}$$

Recall that f^δ is piecewise linear. The algorithm for solving this problem depends on whether $u_l < u_r$ or not. If $u_l < u_r$ we let $\hat{f}^\delta$ denote the lower convex envelope of f^δ between u_l and u_r, while if $u_l > u_r$ we let $\hat{f}^\delta$ denote the upper concave envelope of f^δ between u_r and u_l. Then $\hat{f}^\delta$ will be a piecewise linear function. We call the discontinuity points of $d\hat{f}^\delta(u)/du$ breakpoints, and we have $\{\hat{u}_i\}_{i=1}^{N-1}$ of these inside the interval with endpoints u_l and u_r. If $u_l < u_r$ we set $\hat{u}_0 = u_l$ and $\hat{u}_N = u_r$, otherwise we set $\hat{u}_0 = u_r$ and $\hat{u}_N = u_l$, and arrange the breakpoints so that $i \mapsto \hat{u}_i$ is monotone. Define

$$\sigma_i = \frac{f^\delta(\hat{u}_i) - f^\delta(\hat{u}_{i-1})}{\hat{u}_i - \hat{u}_{i-1}}, \quad i = 1, \dots, N.$$

Then the solution of (2.11) is given by

$$u(x,t) = \begin{cases} \hat{u}_0, & x < \sigma_1 t, \\ \hat{u}_i, & \sigma_i t \le x < \sigma_{i+1} t, \quad i = 1, \dots, N-1, \\ \hat{u}_N, & \sigma_N t \le x. \end{cases} \tag{2.12}$$

Next, we turn to the more complicated situation where also B has a discontinuity at $x = 0$. Since the solution only depends on the difference in B to the left and right, there is no loss of generality in considering the initial value problem

$$\begin{cases} u_t + \left(f^\delta(u) + B\right)_x = 0, & u(x,0) = u_l, \quad x < 0, \\ u_t + f^\delta(u)_x = 0, & u(x,0) = u_r, \quad x > 0, \end{cases} \tag{2.13}$$

for some (constant) $B \neq 0$. Let

$$u_l' = \lim_{x\to 0-} u(x,t), \quad \text{and} \quad u_r' = \lim_{x \to 0+} u(x,t).$$

The Rankine-Hugoniot condition implies that

$$f^\delta(u_l') + B = f^\delta(u_r'). \tag{2.14}$$

The solution to (2.13) consists in finding $u_{l,r}'$ and then solving the Riemann problem (2.11) with $u_r = u_l'$ using only waves with non-positive speeds, and finally solving the Riemann problem (2.11) with $u_l = u_r'$ using only wave with non-negative speeds.

How this can be done depends on f^δ. The simplest case is when f, and consequently f^δ is monotone. For definiteness we assume that $u \mapsto f(u)$ is increasing. In this case the solution of (2.11) will never contain fronts with negative speeds. Thus $u_l' = u_l$, and u_r' solves (2.14) with $u_l' = u_l$. Since f^δ is monotone, there exists a unique solution.

The case where f is not monotone is more complicated. For simplicity, we detail the solution in the case where f is an even convex function. In this case also f^δ will be even and convex.

In order to find a solution, we first find possible candidates for u_l' and u_r'. It is clear that u_l' must be sought among those values there the Riemann problem (2.11) with $u_r = u_l'$ has waves of non-positive speed only, label this set $H_l(u_l)$. We have that H_l is given by

$$H_l\,(u_l) = \begin{cases} (-\infty, 0], & \text{if } u_l \le 0, \\ \{u_l\} \cup (-\infty, -u_l], & \text{if } u_l > 0. \end{cases} \tag{2.15}$$

Similarly, we let H_r be the set of left states such that the Riemann problem (2.11) with $u_l = u_r'$ is solved by waves with non-negative speed. In our case we have that

$$H_r\,(u_r) = \begin{cases} [0, \infty), & \text{if } u_r \ge 0, \\ \{u_r\} \cup [-u_r, \infty), & \text{if } u_r < 0. \end{cases} \tag{2.16}$$

Let now

$$f_l(u) = \begin{cases} f^\delta(u) + B, & \text{if } u \in H_l(u_l), \\ \min_{H_l} \left\{ f^\delta(u) \right\} + B, & \text{otherwise,} \end{cases}$$

and

$$f_r(u) = \begin{cases} f^\delta(u), & \text{if } u \in H_r(u_r), \\ \min_{H_r} \left\{ f^\delta(u) \right\}, & \text{otherwise.} \end{cases}$$

Now the Rankine-Hugoniot condition says that u_l' and u_r' must solve the equation

$$f_l\,(u_l') = f_r\,(u_r')\,. \tag{2.17}$$

Since f_l will be a convex non-increasing function and f_r will be a convex non-decreasing function, (2.17) will always have an infinite number of solutions. Indeed, for any value $\phi \ge \max\{f_l(u_l), f_r(u_r)\}$ we can find a unique pair (u_l', u_r') satisfying (2.14) with $f(u_l') = \phi$. We use the so-called *minimal jump entropy condition* which states that among all possible solutions, we choose the pair minimizing $|u_l' - u_r'|$. In our case this is the same as choosing the minimal possible flux across $x = 0$. Once u_l' and u_r' are determined, we can solve the Riemann problems to the left and right of $x = 0$ and piece together the solutions to form $u(x,t)$ as in (2.10).

The procedure for finding $u_{l,r}'$ in the general case is similar, but the formulae are more complicated, see [24].

2.3. **The large time-stepping scheme.** Now we are in a position to define a fully discrete scheme based of the semi-discrete scheme (2.2). To this end fix three independent (small) parameters δ, Δx, and Δt. Use δ to define the piecewise linear approximation f^δ from f by (2.5). The approximate initial data is defined as in (2.7). For an integrable function g, let Pg be defined as mapping to the piecewise constant functions by taking cell averages over I_j, i.e.,

$$(Pg)(x) = \frac{1}{\Delta x} \sum_j \left(\int_{I_j} g(x)\,dx \right) \mathbf{1}_{I_j}(x).$$

Next, define the sequence $\left\{ u_{\Delta x,n}^\delta \right\}_{n>0}$ by solving

$$\begin{aligned} &\partial_t u_{\Delta x,n}^\delta + \partial_x \left(f^\delta \left(u_{\Delta x,n}^\delta \right) + B^{n-1}(x) \right) = 0, \\ &u_{\Delta x,n}^\delta(x, t^{n-1}) = (P u_{\Delta x,n-1}^\delta(\cdot, t^{n-1}))(x), \end{aligned} \tag{2.18}$$

while $u_{\Delta x,0}$ is defined by (2.7). The "coefficient" B^n is found by defining

$$B_j^n = B_{j-1}^n + \frac{\Delta x}{4}\Big(A\left(x_{j-1/2}+,\bar{u}_{j-1}^n\right) + A\left(x_j-,\bar{u}_{j-1}^n\right) + A\left(x_j+,\bar{u}_j^n\right) + A\left(x_{j+1/2}-,\bar{u}_j^n\right)\Big), \tag{2.19}$$

where we have set $\bar{u}_j^n = P(u^\delta_{\Delta x,n}(\cdot,t^n))|_{I_j}$. Note that since $A(x,0)=0$ and that by finite speed of propagation, $B_j^n = 0$ for sufficiently negative j. Finally, define

$$B^n(x) = \sum_j B_j^n \mathbf{1}_{I_j}(x). \tag{2.20}$$

This is an approximation of the function $B(x, u^\delta_{\Delta x,n-1})$ defined in (2.1). Other approximations are also possible, and these would result in slightly different discrete steady states.

Note that the discrete steady state preserved by the fully-discrete scheme satisfy

$$f(u_j^n) - B_j^n = \text{constant}, \tag{2.21}$$

which is the fully discrete version of (2.3). The solution to (2.18) is computed using front tracking. We plan to return to the issue of theoretical convergence (as the three parameters δ, Δx, and Δt vanish) of this scheme in a forthcoming paper.

We remark that this scheme can be interpreted as a finite volume scheme. To do this we observe that $\bar{u}_j^n$ is the cell average over I_j of the solution of (2.18). Thus

$$\bar{u}_j^n = \bar{u}_j^{n-1} - \frac{1}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right),$$

where the numerical flux function is given by

$$F_{j+1/2}^n = \int_{t^{n-1}}^{t^n} f^\delta\left(u^\delta_{\Delta x,n}\left(x_{j+1/2},t\right)\right)\,dt + \Delta t B_j^{n-1}.$$

Of course, we do *not* need to compute these integrals, but they can be interpreted as numerical fluxes.

3. Numerical experiments

In this section we present two numerical experiments where we have used the large time step method. In order to have fewer parameters, we have set $\delta = \Delta x/2$, and used the CFL number and Δx to parametrize the method. In this context, the CFL number λ is defined as

$$\lambda = \frac{\Delta t}{\Delta x}\max_u |f'(u)|. \tag{3.1}$$

We remark that by taking a CFL number of 0.5, the scheme proposed here is equivalent to the Godunov-type scheme proposed in our recent paper [9]

3.1. **Numerical Experiment** 1.

In this experiment, we consider the following initial value problem:

$$u_t + (\frac{1}{2}u^2)_x = -z'(x)u, \quad z(x) = \begin{cases} \sqrt{4-x^2}, & |x|<2 \\ 0, & \text{otherwise}, \end{cases}$$
$$u(x,0) = \begin{cases} 1, & x<-3 \\ 0, & x>-3. \end{cases} \tag{3.2}$$

We consider (3.2) in the domain $x \in [-3.5, 3.5]$, $t \in [0, 8]$, and impose characteristic boundary conditions at $x = \pm 3.5$.

The exact solution in this case is a right moving shock which starts interacting with the bottom topography z and creates a smooth wave. The steady state is reached after the shock has moved out of the domain and is given by the function

$$\overline{u}(x) = 1 - z(x).$$

In order to test the method for various CFL numbers, we exhibit the results of computations for the CFL numbers 20, 10, 5, and 2.5. These are shown in Figure 1. The errors at the time $t = 8$ are shown in Table 1. We show both the L^1 and L^∞ errors by comparing the computated solutions with the exact steady state. Based on these, it seems that the computations with $\lambda = 5$ and $\lambda = 2.5$ are acceptable, while for larger CFL numbers the results are inaccurate near $x = 2$. In all of these computations we used $\Delta x = 7/100$.

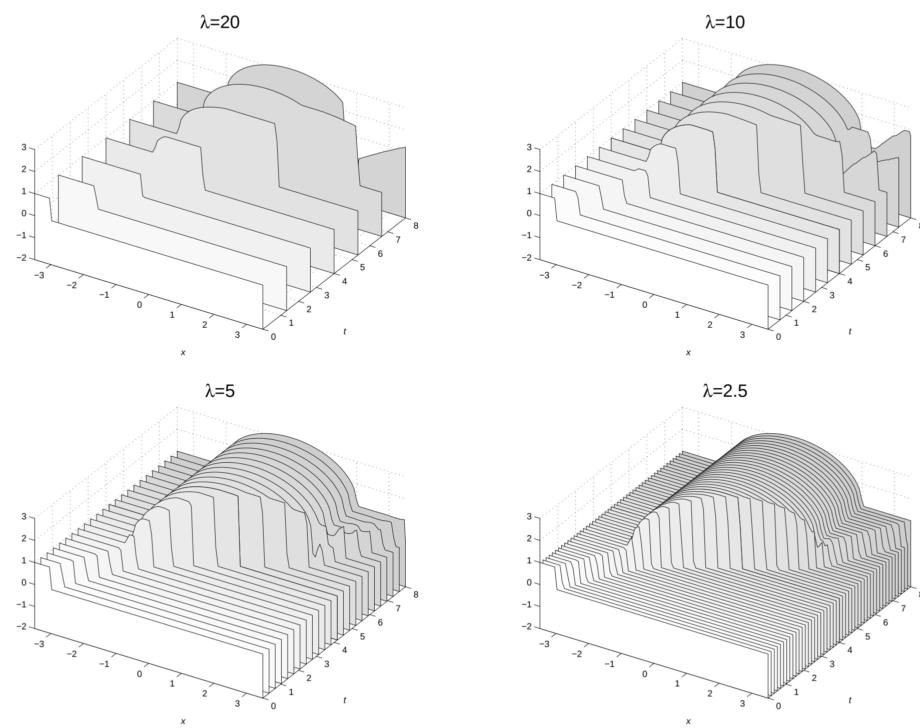

FIGURE 1. Approximations to the solution of (3.2) with different CFL numbers; $\lambda = 20$ (top left), $\lambda = 10$ (top right), $\lambda = 5$ (bottom left) and $\lambda = 2.5$ (bottom right).

We also check how the approximate solutions vary when Δt is fixed and the spatial discretization Δx varies. We show the results of these computations in Figure 2, while the errors are displayed in Table 1. Here the CFL numbers vary from 2.5 in the case where $\Delta x = 7/50$ to 20 in the case where $\Delta x = 7/400$. We see that the approximations are very similar despite the larger difference in CFL

$\lambda\backslash n$	50	100	200
20	54.8247	143.7400	13.8333
10	128.5679	24.4285	0.3032
5	11.2544	0.1918	0.0003
2.5	0.0917	0.0001	0.0002

$\lambda\backslash n$	50	100	200
20	25.2185	13.4698	1.6539
10	13.2428	2.3777	0.0062
5	1.1438	0.0201	0.0001
2.5	0.00464	0.00003	0.0005

TABLE 1. $100 \times L^\infty$ error (left) and $100 \times L^1$ error (right), where λ denotes the CFL number and $\Delta x = 7/n$.

numbers. Hence the quality of the results are largely independent of Δx and the CFL number.

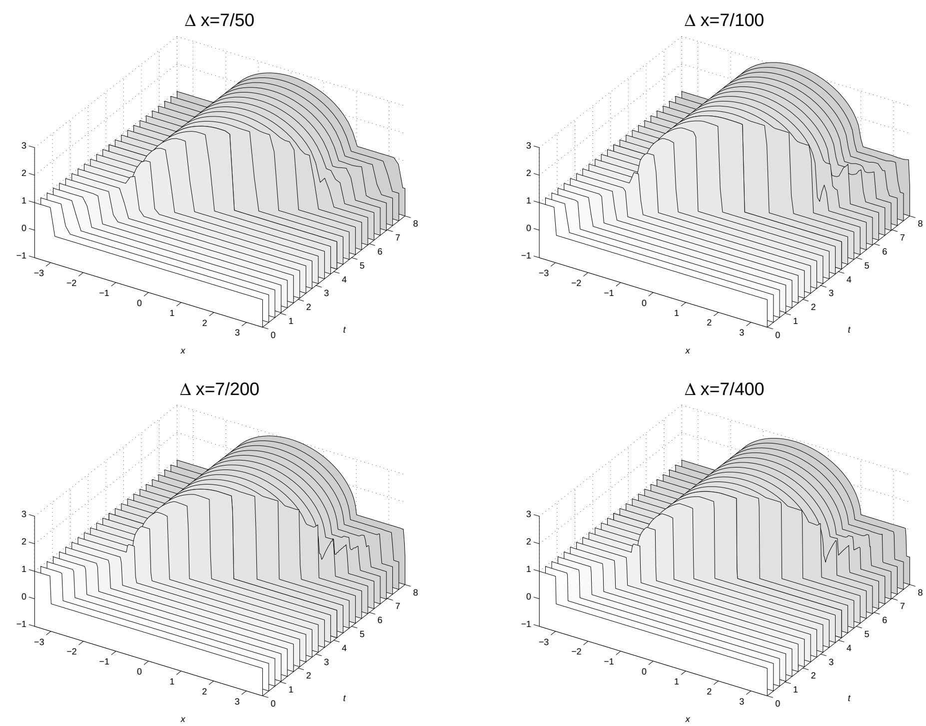

FIGURE 2. Approximations to the solution of (3.2) with Δt constant, $\lambda = 2.5$ (top left), $\lambda = 5$ (top right), $\lambda = 10$ (bottom left) and $\lambda = 20$ (bottom right).

We also compare the small time step version of this method by taking the CFL number to be 0.5. As remarked earlier, this scheme is equivalent to the Godunov-type scheme proposed in [9]. Finally, we compare the above results with the solutions computed by the well-balanced scheme of [6], which is based on a projection to the local steady states. The numerical results are shown in Figure 3. As expected, the scheme resolves the solution very well at this low CFL number and the steady state is approximated to machine precision. On the other hand, the well-known well-balanced scheme of [6] leads to unphysical transients although it also resolves

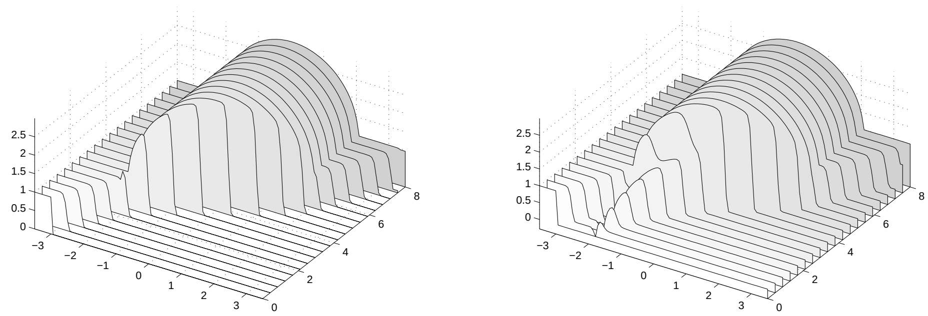

FIGURE 3. Approximations to the solution of (3.2) with $\Delta x = \frac{7}{100}$, $\lambda = 0.5$ (left) and the well-balanced scheme of [6] (right).

the steady state to machine precision. From the above data, the large step method gives very good results at reasonable CFL numbers even with coarse discretizations in space. The results for both transients as well as steady states are very good at a CFL number of 5, thus leading to a order of magnitude speed-up compared to the standard time-step (CFL = 0.5) version of the scheme. For higher CFL numbers, the transients seem to be poorly resolved for coarse mesh discretizations. This is expected as interesting wave phenomena are averaged over longer periods of time. By going to finer meshes in space, the quality of the results improves significantly at even higher CFL numbers.

3.2. **Numerical Experiment 2.** In this experiment we consider the equation

$$u_t + (\frac{1}{2}u^2)_x = -z'(x)u, \quad z(x) = -\cos(\pi x), \tag{3.3}$$

with the initial data

$$u(x,0) = \cos(\pi x) + \frac{1}{10}\sin(4\pi x).$$

We consider the above problem in the domain $[-1,1]$ with periodic boundary conditions. The exact steady state is given by

$$\overline{u}(x) = \cos(\pi x).$$

Thus the initial data is a periodic perturbation of the steady state and we expect the solution to converge to the steady state. This problem is a prototype for quasi-steady problems. The exact solution consists of small amplitudes waves which decay quickly to the steady state. We have computed the solution to this problem with both the small (CFL = 0.5) time-step version of our method and the well-balanced scheme of [6] and show the solutions in Figure 4. Both schemes perform equally well and resolve the steady state to machine precision. But our interest in such quasi-steady problems is to compute the steady state accurately and quickly. Hence, we increase the CFL number in an attempt to take large time steps. We are not interested in an accurate resolution of the transients in this case. We show the results with a really large time-step ($\lambda = 80$) version of our method in Figure 5. From the figure, it is clear that the steady state is resolved accurately even at such high CFL numbers. As expected, there are some oscillations in the transient as the

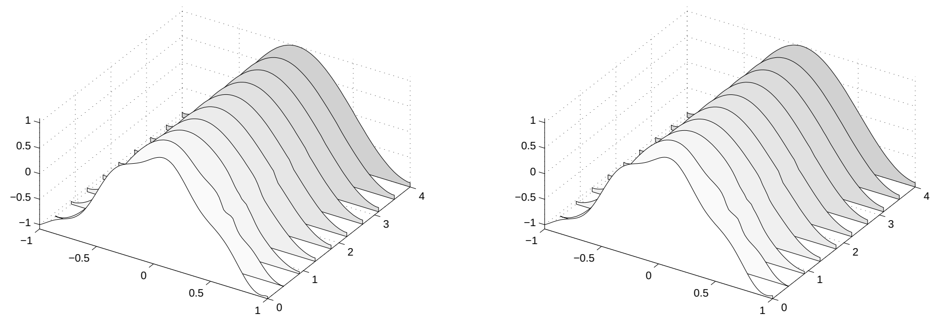

FIGURE 4. Approximations to the solution of (3.3) with $\Delta x = 0.02$, $\lambda = 0.5$ (left) and the well-balanced scheme of [6] (right).

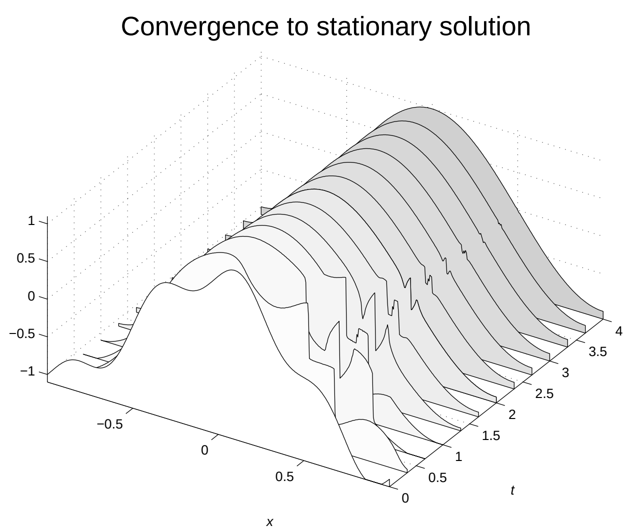

FIGURE 5. The numerical solution to (3.3), $\Delta x = 1/200$, $\lambda = 80$.

averaging is over really large time steps. But we are interested in the resolution of the steady state and see that we can increase the size of time step to more than two orders of magnitude to obtain accurate resolutions of the steady state.

The above examples illustrate the effectiveness of this numerical method. It resolves steady states quite accurately. The method is fast as large time steps can be taken due to high CFL numbers. Taking large time steps can lead to incorrect transients but the solutions improves considerably by refining the space mesh.

4. CONCLUSION

We present a new numerical method for conservation laws with source terms. The main numerical issue is the accurate resolution of steady states. In order to preserve discrete steady states, the numerical method has to reflect the balance between the flux and the source at the steady state.

We propose a numerical scheme based on local reformulation the balance law as a conservation law with discontinuous coefficients. The resulting equations are solved by a front tracking method based on solutions of Riemann problems. The method preserves discrete steady states exactly, and is therefore well-balanced. Since we use front tracking, the method is unconditionally stable and we can take arbitrarily large time steps without blowup.

Numerical examples are presented and they illustrate the effectiveness of the method. In particular, the method resolves discrete steady states to a high degree of accuracy. The method is also fast since we can take very large time steps. We have also compared the method with other existing well-balanced schemes. The method is very effective for quasi steady problems i.e perturbations from steady states.

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