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# A CONVERGENT FINITE DIFFERENCE SCHEME FOR THE CAMASSA-HOLM EQUATION WITH GENERAL $H^1$ INITIAL DATA

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Abstract. We suggest a finite dfference scheme for the Camassa-Holm equation that can handle general $H^1$ initial data. The form of the difference scheme is judiciously chosen to ensure that it satisfies a total energy inequality. We prove that the difference scheme converges strongly in $H^1$ towards an exact dissipative weak solution of Camassa-Holm equation.

Contents



## 1. Introduction

In this paper we present and analyze a finite difference scheme for the Camassa-Holm partial differential equation [7]

$$\partial_t u - \partial^3_{txx} u + 3u\partial_x u = 2\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{1.1}$$

which we augment with an initial condition:

$$u|_{t=0} = u_0 \in H^1(\mathbb{R}). \tag{1.2}$$

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Rewriting equation (1.1) as

$$(1-\partial_{xx}^2)\left[\partial_t u + u\partial_x u\right] + \partial_x\left(u^2 + \frac{1}{2}(\partial_x u)^2\right) = 0,$$

we see that (for smooth solutions) (1.1) is equivalent to the elliptic-hyperbolic system

$$\partial_t u + u\partial_x u + \partial_x P = 0, \qquad -\partial_{xx}^2 P + P = u^2 + \frac{1}{2}(\partial_x u)^2. \tag{1.3}$$

Recalling that $e^{-|x|}/2$ is the Green's function of the operator $1-\partial_{xx}^2$, (1.3) can be written as

$$\partial_t u + \partial_x F(u,\partial_x u) = 0, \quad F(u,\partial_x u) = \frac{1}{2}\left[u^2 + e^{-|x|}\star\left(u^2 + \frac{1}{2}(\partial_x u)^2\right)\right], \tag{1.4}$$

which can be viewed as a conservation law with nonlocal flux function. In this paper the relevant formulation of the Camassa-Holm equation (1.1) is the one provided by the hyperbolic-elliptic system (1.3).

The Camassa-Holm equation (1.1) can be viewed as a model for the propagation of unidirectional shallow water waves [7, 32]. The equation is a member of the class of weakly nonlinear and weakly dispersive shallow water models, a class which already contains the Korteweg-de Vries (KdV) and Benjamin-Bona-Mahony (BBM) equations. The Camassa-Holm equation contains higher order nonlinear dispersive/nonlocal balances not present in the KdV and BBM equations. As is the case with the BBM equation but not in the KdV equation, the linear dispersion relation in the Camassa-Holm equation remains bounded for large wave numbers.

In another interpretation the Camassa-Holm equation models finite length, small-amplitude radial deformation waves in cylindrical compressible hyperelastic rods [21]. It arises also in the context of differential geometry as an equation for geodesics of the $H^1$-metric on the diffeomorphism group, see [17, 18, 30, 36].

The Camassa-Holm equation possesses several extraordinary properties such as an inifinite number of conserved integrals, a bi-Hamiltonian structure, and complete integrability [2, 7, 19, 13, 26]. Moreover, it enjoys an infinite number of non-smooth solitary wave solutions, called peakons, of the form

$$u(t,x) = ce^{-|x-ct|}, \qquad c\in\mathbb{R},$$

which have to be interpreted as weak solutions of (1.4).

From a mathematical point of view the Camassa-Holm equation has by now become rather well-studied. While it is impossible to give a complete overview of the mathematical literature, we shall here mention a few typical results, starting with local(-in-time) existence results [14, 35, 37]. For $u_0\in H^s(\mathbb{R})$ with $s>\frac{3}{2}$ there exists a unique solution $u\in C([0,T];H^s(\mathbb{R}))\cap C^1([0,T];H^{s-1}(\mathbb{R}))$ of (1.1)-(1.2) for some $T$ that depends on $\|u_0\|_{H^s(\mathbb{R})}$. Furthermore, the flow-map is continuous from $H^s(\mathbb{R})$ to the class defined above. The proof of this result is based on the "momentum" formulation of the Camassa-Holm equation,

$$\partial_t m + u\partial_x m + 2m\partial_x u = 0, \qquad m := (1-\partial_{xx}^2)u. \tag{1.5}$$

to which one applies Kato's theory for quasilinear hyperbolic equations. For local well-posedness results based on Besov spaces, see [23, 22].

The Camassa-Holm equation posseses an infinite number of conservation laws, but neither of them control the $H^s$-norm for $s>1$. Hence these local existence results cannot (in general) be turned into global ones. Indeed, it is well-known that global solutions do not exist and wave-breaking occurs [7]. Wave-breaking means that the solution itself stays bounded while the spatial derivative $\partial_x u$ tends to $-\infty$ as $t \uparrow T^*$, where $T^*$ denotes the maximal time of existence. More precisely, the following results are proved in [14, 16]. Assume that $u_0 \in H^3(\mathbb{R})$ is odd with $\partial_x u_0(0) < 0$. Then the solution of of (1.1)-(1.2) does not exist globally, and $T^*$ is estimated above by $1/(2\,|\partial_x u_0(0)|)$. Another result says that if the initial function $u_0 \in H^3(\mathbb{R})$ has at some point a slope which is less than $-(1/\sqrt{2})\,\|u_0\|_{H^1(\mathbb{R})}$, then $T^*$ is finite and wave-breaking occurs. It was observed in [14] that the solutions are global if $m_0 := (1-\partial_{xx}^2)u_0$, cf. (1.5), is a bounded measure with definitive sign.

In view of what we have said so far (peakon solutions/wave-breaking) it is clear that a theory based on weak solutions is essential. In the literature there are a number of results on weak solutions of the Camassa-Holm equation. Here we will mention only a few of them, starting with the results obtained in [15, 20]. Suppose $u_0 \in H^1(\mathbb{R})$ with $m_0 := (1-\partial_{xx}^2)u_0 \in \mathcal{M}(\mathbb{R})$. Then the authors prove that there exists a final time $T = T(\|m_0\|_{\mathcal{M}}) > 0$ and a unique weak solution

$$u \in C([0,T]; H^1(\mathbb{R})) \cap L^\infty(0,T; W^{1,1}(\mathbb{R})), \quad \partial_x u \in L^\infty(0,T; BV(\mathbb{R}))$$

of (1.1)-(1.2), i.e., $u$ is a distributional solution of (1.4)-(1.2). Additionally, the following time-dependent quantities remain constant:

$$E(u) := \int_{\mathbb{R}} \left[u^2 + (\partial_x u)^2\right]\,dx, \qquad F(u) := \int_{\mathbb{R}} \left[u^3 + u(\partial_x u)^2\right]\,dx.$$

In particular, this weak solution is total energy conserving, i.e., $E(u(t,\cdot)) = E(u_0)$. Finally, if $m_0$ has a definite sign then $u$ is global in time. The sign of $m_0$ is maintained by $m(t,\cdot)$ at all times $t$. It is possible to prove existence of local weak solutions without the sign assumption on $m_0$, see [22]. The proofs in [15, 20] are based on the momentum formulaton (1.5).

For other approaches to conservative weak solutions, we refer to [4, 5, 29].

More relevant from the point of view of the present paper is the result of Xin and Zhang [38], which states the existence of a global (dissipative) weak solution for any $H^1$ initial data (see [11, 12] for similar results for a generalized Camassa-Holm equation). These solutions are global in the sense that they are defined even past the blow-up time (wave-breaking). More precisely, suppose $u_0 \in H^1(\mathbb{R})$. Then there exists a global weak solution

$$u \in C(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(0,T; H^1(\mathbb{R}))$$

of (1.1)-(1.2), satisfying the following properties:

$$\begin{aligned} &\|u(t,\cdot)\|_{H^1(\mathbb{R})} \le \|u_0\|_{H^1(\mathbb{R})}, \\ &\partial_x u \in L^p_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}), \quad p < 3, \\ &\partial_x u(t,x) \le \frac{2}{t} + C, \quad t > 0, \end{aligned} \tag{1.6}$$

where $C$ is a positive constant that depends only on $\|u_0\|_{H^1(\mathbb{R})}$.

We remark that the last item in (1.6) serves as an "entropy condition" that singles out a (presumably) unique weak solution after the occurrence of wave-breaking. This solution is often referred to as a *dissipative* weak solution as the total energy is merely nonincreasing in time: $E(u(t,\cdot)) \le E(u_0)$. The entropy condition is (formally) seen to hold by inspecting the equation satisfied by the spatial derivative $q := \partial_x u$ (cf. [38] for details), which reads

$$\partial_t q + u\partial_x q + \frac{q^2}{2} - u^2 + P = 0, \qquad -\partial_{xx}^2 P + P = u^2 + \frac{q^2}{2}. \tag{1.7}$$

The proof of the existence result is based on the vanishing viscosity method, which amounts to justifying the limit $\varepsilon \downarrow 0$ of a sequence of smooth solutions $u_\varepsilon$ to the parabolic-elliptic system

$$\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon = \varepsilon \partial_{xx}^2 u_\varepsilon, \quad -\partial_{xx}^2 P_\varepsilon + P_\varepsilon = u_\varepsilon^2 + \frac{1}{2}\left(\partial_x u_\varepsilon\right)^2, \tag{1.8}$$

which is not straightforward, however, due to the nonlinear nature of (1.8), see [38].

Currently there is no uniqueness result for weak solutions of type constructed in [38]. The problem appears to be connected to a lack of temporal integrability (of the $L^\infty$ norm) of the spatial derivative. Indeed, if one furthermore knows the existence of a function $b \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ such that

$$\left\|\partial_x u(t,\cdot)\right\|_{L^\infty(\mathbb{R})} \leq b(t),$$

then the weak solution of Xin and Zhang is unique (in a particular class) [39]. For example, if $m_0$ is a positive bounded Radon measure, then $\partial_x u$ is pointwise bounded [20] and uniqueness thus holds.

For a different approach to dissipative weak solutions, see the recent work [3].

Let us now turn to the main topic of the present paper, namely, convergent numerical schemes for the Camassa-Holm equation. Although there are few works on convergent numerical schemes, there are several authors that employ numerical schemes to obtain approximate solutions. The first numerical results are presented in [8] where a pseudo-spectral scheme is utilized. Additional numerical simulations with pesudo-spectral schemes are reported in [25, 31]. Numerical schemes based on multipeakons (thereby exploiting the Hamiltonian structure of the Camassa-Holm equation) are examined in [6, 9, 10]. In a different direction, an adaptive high-resolution finite volume scheme is deveoped and used in [1].

Regarding works that provide numerical schemes with some sort of theoretical foundation, we know only of the papers [27, 28, 33]. In [28], the authors prove that the multipeakon algorithm from [9, 10] converges to the solution of the Camassa-Holm equation (1.1) as the number of peakons tends to infinity (in an appropriate way). This convergence result applies to the situation where the initial function $u_0 \in H^1$ is such that $(1-\partial_{xx}^2)u_0$ is a positive measure. In [33], the authors establish error estimates for a spectral projection scheme, though under the (unrealistic) assumption of smooth solutions.

It seems rather difficult to construct numerical schemes for which one can prove rigorously the convergence to a solution of the Camassa-Holm equation, a fact that is related to the nonlinear and nonlocal features of the equation. It has been observed in [27] that certain "natural" schemes either diverge or converge to a wrong solution. Indeed, a priori it is not even clear which one of the three formulations of the Camassa-Holm equation, (1.1), (1.3), or (1.5), should be used as a starting point for discretization. Nevertheless, in [27] the authors commence from the momentum formulation (1.5), and thereby restricting themselves to initial data $u_0$ in $H^1$ for which $m_0 = (1 - \partial_{xx}^2)u_0$ is a positive measure, in which case also $m(t,\cdot)$ remains positive and consequently so does $u$. They prove that the following difference scheme converges strongly in $H^1$ to the weak solution identified in [15, 20]:

$$\frac{d}{dt}m_j + D_-(m_j u_j) + m_j D u_j = 0, \quad m_j = u_j - D_- D_+ u_j, \quad t > 0,\, j \in \mathbb{Z},$$

where $D_-$, $D$, and $D_+$ denote respectively the backward, central, and forward difference operators, and $m_j(t) \approx m(t, x_j)$, $u_j(t) \approx u(t, x_j)$, $x_j = j\Delta x$, and $\Delta x > 0$.

The main aim of this paper is to provide a convergent finite difference scheme that works for any $H^1$ initial data and not merely the subclass considered in [27]. Neither the scheme nor the analysis presented in [27] work in the general case.

At variance with [27], we shall herein take as a starting point the hyperbolic-elliptic formulation (1.3). From the point of view of conservation laws (e.g., the inviscid Burgers' equation) and their shock wave (discontinuous) solutions, it might seem natural to employ a conservative finite difference scheme of the upwind type [34] to the $u$-equation in (1.3). As is well-known, the upwinding will render a scheme stable since the difference stencil utilizes information only from the side where the (discontinuous) waves are coming from. However, here one should keep in mind that solutions to the Camassa-Holm equation are continuous, and that prospective discontinuities occur only in the variable $q = \partial_x u$, which satisfies the transport equation in (1.7). Thus, herein we will not opt for this strategy.

Instead we will device a tailored difference scheme for the $u$-equation in (1.3) that yields an upwind difference scheme for the $q$-equation in (1.7). A key feature of the scheme is the satisfaction of a total energy inequality in which only the $q$-part of the total energy is dissipated (not the $u$-part!). To avoid complicating further the convergence analysis, we restrict our attention to a semi-discrete finite difference scheme. To turn the difference scheme into a fully discrete one we can rely on a variety of different time-discretization techniques, see Section 12 for more details.

Now we outline the finite difference scheme (here only briefly since the details can be found in Section 3). To this end, we start with discretizing the spatial domain $\mathbb{R}$ by specifying the mesh points $x_j = j\Delta x$, $x_{j+1/2} = (j+1/2)\Delta x$ for $j = 0, \pm 1, \pm 2, \dots$, where $\Delta x > 0$ is the length between two consecutive mesh points (the mesh size). Our numerical scheme will generate approximations

$$u_{j+1/2}(t) \approx u(t, x_{j+1/2}), \quad P_j(t) \approx P(t, x_j), \quad \text{for } t \geq 0 \text{ and } j \in \mathbb{Z},$$

where we remark that the discretization of $P$ is shifted (staggered) one half-cell compared that of $u$. Our finite difference scheme for $\left\{u_{j+1/2}(t)\right\}_{j\in\mathbb{Z}}$ reads

$$\frac{d}{dt}u_{j+1/2} + \left(u_{j+1/2} \vee 0\right) D_- u_{j+1/2} + \left(u_{j+1/2} \wedge 0\right) D_+ u_{j+1/2} + D_+ P_j = 0, \tag{1.9}$$

while the difference scheme for $\{P_j(t)\}_{j\in\mathbb{Z}}$ takes the form

$$-D_- D_+ P_j + P_j = \left(u_{j+1/2} \vee 0\right)^2 + \left(u_{j-1/2} \wedge 0\right)^2 + \frac{1}{2}\left(D_- u_{j+1/2}\right)^2.$$

Of course, as we have already alluded to above, from the point of view of the inviscid Burgers' equation, (1.9) is not a reasonable discretization. However, the quantity $q_j := D_- u_{j+1/2}$ automatically satisfies the difference scheme

$$\begin{aligned} q_j' + \left(u_{j-1/2} \vee 0\right) D_- q_j + \left(u_{j+1/2} \wedge 0\right) D_+ q_j + \frac{q_j^2}{2} \\ - \left(u_{j+1/2} \vee 0\right)^2 - \left(u_{j-1/2} \wedge 0\right)^2 + P_j = 0, \end{aligned} \tag{1.10}$$

which contains proper upwinding of the transport term in (1.7). In our situation, compare with [27], $u$ does not have a definite sign, hence the splitting of $u$ into positive and negative parts. As with the "pressure" $P$, the discretization of $q$ is staggered compared to that of the "velocity" $u$.

By properly extending $\left\{u_{j+1/2}\right\}_{j\in\mathbb{Z}}$, $\left\{q_j\right\}_{j\in\mathbb{Z}}$ to functions $u_{\Delta x}$, $q_{\Delta x}$ defined at all points $(t,x)$ in the domain, we prove that $u_{\Delta x}$ converges strongly in $H^1$ to a dissipative weak solution of the Camassa-Holm equation, which constitute the main result of the present paper. Regarding the proof, we derive several a priori estimates in Lebesgue and Sobolev spaces as well as a uniform upper bound on $q_j$ serving as a discrete version of the "entropy condition", among which a discrete total energy inequality constitutes the key building block. The total energy inequality only ensures weak compactness of the sequence $\left\{q_{\Delta x}^2\right\}_{\Delta x>0}$. However, it is crucial to know that this sequence is strongly compact. Strong compactness is neeed if we want to recover the original equation (1.4) when sending $\Delta x \downarrow 0$ in the finite difference scheme. To establish the strong compactness property we apply ideas from the theory of renormalized solutions (in the sense of DiPerna and Lions) to the finite difference scheme (1.10). As a part of establishing strong compactness, a higher integrability estimate for $q_{\Delta x}$ is needed to ensures that weak limit points of $q_{\Delta x}^2$ do not contain singular measures. Our convergence proof can be best understood as a discrete variant of the proof used in [38] for the vanishing viscosity method.

This paper is organized as follows: In Section 2 we introduce relevant notations and recall a few mathematical results needed for the analysis. The finite difference scheme is presented in Section 3, while the main convergence theorem is stated in Section 4. The main theorem is a consequence of the results stated and proved in Sections 5-11. Finally, in Section 12 we present a few numerical examples.

Throughout this paper we use $C$ to denote a generic constant; The actual value of $C$ may change from one line to the next in a calcuation.

## 2. Preliminaries

In this section we introduce some notations to be used throughout this paper and a few basic mathematical results that will be relevant to the convergence analysis of the numerical scheme.

The following notations will be used frequently:

$$a \vee 0 = \max\{a,0\} = \frac{a+|a|}{2}, \qquad a \wedge 0 = \min\{a,0\} = \frac{a-|a|}{2}.$$

In what follows, unless otherwise stated, the index $j$ will run over $\mathbb{Z}$. For such an index we set $x_{j+1/2} = (j+1/2)\Delta x$ and introduce the grid cells

$$I_j = [x_{j-1/2}, x_{j+1/2}),$$

where $\Delta x$ is a small positive number ("the discretization parameter"). The grid cells $I_j$ are centered around the points $x_j = j\Delta x$. For any sequence $\left\{v_j\right\}_{j\in\mathbb{Z}}$ we introduce the following difference operators:

$$D_+ v_j := \frac{v_{j+1}-v_j}{\Delta x}, \qquad D_- v_j := \frac{v_j - v_{j-1}}{\Delta x},$$
$$D v_j := \frac{D_+ v_j + D_- v_j}{2} = \frac{v_{j+1}-v_{j-1}}{2\Delta x}.$$

We also use the notations

$$\left\|\{v_j\}_j\right\|_{\ell^p} := \left(\Delta x \sum_{j\in\mathbb{Z}} |v_j|^p\right)^{\frac{1}{p}}, \quad \left\|\{v_j\}_j\right\|_{\ell^\infty} := \sup_j |v_j|,$$

$$\left\|\{v_j\}_j\right\|_{h^1} := \left(\Delta x \sum_{j\in\mathbb{Z}} \left[v_j^2 + (D_- v_j)^2\right]\right)^{\frac{1}{2}}.$$

Let $\{v_j\}_{j\in\mathbb{Z}}$ be a sequence such that $\left\|\{v_j\}_j\right\|_{\ell^1\cap\ell^2} < \infty$. Then

$$\left\|\{v_j\}_j\right\|_{\ell^\infty} \leq \frac{1}{\sqrt{\Delta x}} \left\|\{v_j\}_j\right\|_{\ell^2} \leq \frac{1}{\Delta x} \left\|\{v_j\}_j\right\|_{\ell^1}. \tag{2.1}$$

Let $\{v_j\}_{j\in\mathbb{Z}}$ be a sequence such that $\left\|\{v_j\}_j\right\|_{h^1} < \infty$. It is easy to see that the following discrete Sobolev inequality holds:

$$\left\|\{v_j\}_j\right\|_{\ell^\infty} \leq \frac{1}{\sqrt{2}} \left\|\{v_j\}_j\right\|_{h^1}. \tag{2.2}$$

Let $\{v_j\}_{j\in\mathbb{Z}}, \{w_j\}_{j\in\mathbb{Z}}$ be two sequences. Then the discrete Leibniz rule reads

$$D_\pm (v_j w_j) = v_j D_\pm w_j + D_\pm v_j w_{j\pm1}, \tag{2.3}$$

while the discrete chain rule states that for any $C^2$ function $f$ there holds

$$D_\pm f(v_j) = f'(v_j) D_\pm v_j \pm \frac{\Delta x}{2} f''(\xi_j^\pm)(D_\pm v_j)^2, \tag{2.4}$$

for some $\xi_j^\pm$ between $v_{j\pm1}$ and $v_j$. A key difficulty in designing converging difference schemes for nonlinear equations is that there is no exact chain rule for discrete derivatives, but merely the formula (2.4) showing that the chain rule only holds up to a certain error term.

Later we routinely use some well-known results related to weak convergence, which we collect in the remaining part of this section (for proofs, see, e.g., [24]). Throughout the paper we use overbars to denote weak limits.

**Lemma 2.1.** *Let $O$ be a bounded open subset of $\mathbb{R}^M$, with $M \geq 1$.*

*Let $\{v_n\}_{n\geq1}$ be a sequence of measurable functions on $O$ for which*

$$\sup_{n\geq1} \int_O \Phi(|v_n(y)|)\, dy < \infty,$$

*for some given continuous function $\Phi : [0,\infty) \to [0,\infty)$. Then along a subsequence as $n \uparrow \infty$*

$$g(v_n) \rightharpoonup \overline{g(v)} \text{ in } L^1(O)$$

*for all continuous functions $g : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\lim_{|v|\to\infty} \frac{|g(v)|}{\Phi(|v|)} = 0.$$

*Let $g \colon \mathbb{R} \to (-\infty,\infty]$ be a lower semicontinuous convex function and $\{v_n\}_{n\geq1}$ a sequence of measurable functions on $O$, for which*

$$v_n \rightharpoonup v \text{ in } L^1(O),\ g(v_n) \in L^1(O) \text{ for each } n,\ g(v_n) \rightharpoonup \overline{g(v)} \text{ in } L^1(O).$$

*Then*

$$g(v) \leq \overline{g(v)} \text{ a.e. on } O.$$

*Moreover,* $g(v) \in L^1(O)$ *and*

$$\int_O g(v)\,dy \leq \liminf_{n\to\infty} \int_O g(v_n)\,dy.$$

*If, in addition,* $g$ *is strictly convex on an open interval* $(a,b) \subset \mathbb{R}$ *and*

$$g(v) = \overline{g(v)} \text{ a.e. on } O,$$

*then, passing to a subsequence if necessary,*

$$v_n(y) \to v(y) \text{ for a.e. } y \in \{y \in O \mid v(y) \in (a,b)\}.$$

## 3. Finite difference scheme

In this section we present a semi-discrete upwind difference scheme for generating approximate solutions to the Camassa-Holm equation. A fully discrete version of this difference scheme will be presented and examined numerically in Section 12.

For $t > 0$, we let $\left\{u_{j+1/2}(t)\right\}_{j\in\mathbb{Z}}$, where $u_{j+1/2}(t) \approx u(t, x_{j+1/2})$, solve the following system of ODEs:

$$\frac{d}{dt}u_{j+1/2} + \left(u_{j+1/2} \vee 0\right) D_- u_{j+1/2} + \left(u_{j+1/2} \wedge 0\right) D_+ u_{j+1/2} + D_+ P_j = 0, \tag{3.1}$$

where we specify the initial values as follows:

$$u_{j+1/2}(0) = u_0(x_{j+1/2}). \tag{3.2}$$

For $t \geq 0$, we let $\{P_j(t)\}_{j\in\mathbb{Z}}$, where $P_j(t) \approx P(t, x_j)$, solve

$$- D_- D_+ P_j + P_j = \left(u_{j+1/2} \vee 0\right)^2 + \left(u_{j-1/2} \wedge 0\right)^2 + \frac{1}{2}\left(D_- u_{j+1/2}\right)^2. \tag{3.3}$$

Since $\{P_j\}_{j\in\mathbb{Z}}$ be expressed solely in terms of $\left\{u_{j+1/2}(t)\right\}_{j\in\mathbb{Z}}$, cf. the proof of Lemma 6.1 below, we see that (3.1) constitutes an infinite dimensional system of ODEs of the form

$$\frac{d}{dt}u_{j+1/2}(t) = F\left(\left\{u_{j+1/2}(t)\right\}_{j\in\mathbb{Z}}\right). \tag{3.4}$$

**Lemma 3.1.** *For each fixed* $\Delta x > 0$*, the ODE system* (3.1) *has a continuously differentiable solution defined for all* $t > 0$*.*

*Proof.* We view $F$ as a function from $\ell^2$ to $\ell^2$, and momentarily use the notations $F = \{F_j\}_{j\in\mathbb{Z}}$, $u = \left\{u_{j+1/2}\right\}_{j\in\mathbb{Z}}$, and $v = \left\{v_{j+1/2}\right\}_{j\in\mathbb{Z}}$.

For each fixed $\Delta x$, we claim that $F$ is locally Lipschitz continuous, i.e.,

$$\|F(u) - F(v)\|_{\ell^2} \leq C\left(\|u\|_{\ell^2} + \|v\|_{\ell^2}\right)\|u - v\|_{\ell^2}\,, \tag{3.5}$$

for some constant $C = C(\Delta x)$ depending on $\Delta x$.

To show (3.5) we write $F = -F^1 - F^2$, where the two sequences $F^1 = \left\{F^1_j\right\}_{j\in\mathbb{Z}}$ and $F^2 = \left\{F^2_j\right\}_{j\in\mathbb{Z}}$ are defined by

$$\begin{aligned} F^1_j(u) &= \left(u_{j+1/2} \vee 0\right) D_- u_{j+1/2} + \left(u_{j+1/2} \wedge 0\right) D_+ u_{j+1/2}, \\ F^2_j(u) &= D_+ P_j. \end{aligned}$$

We will show that both $F^1$ and $F^2$ are locally Lipschitz. We calculate

$$\begin{aligned}\left|F_j^1(u)-F_j^1(v)\right| &= \Big|\left(u_{j+1/2}\vee 0\right)D_-\left(u_{j+1/2}-v_{j+1/2}\right)\\ &\qquad + \left(u_{j+1/2}\wedge 0\right)D_+\left(u_{j+1/2}-v_{j+1/2}\right)\\ &\qquad\quad + \left[\left(u_{j+1/2}\vee 0\right)-\left(v_{j+1/2}\vee 0\right)\right]D_-v_{j+1/2}\\ &\qquad\qquad + \left[\left(u_{j+1/2}\wedge 0\right)-\left(v_{j+1/2}\vee 0\right)\right]D_+v_{j+1/2}\Big|\\ &\le \frac{\|u\|_{\ell^\infty}}{\Delta x}\Big(\left|u_{j-1/2}-v_{j-1/2}\right|\\ &\qquad\qquad + 2\left|u_{j+1/2}-v_{j+1/2}\right| + \left|u_{j+3/2}-v_{j+3/2}\right|\Big)\\ &\quad + \frac{4\|v\|_{\ell^\infty}}{\Delta x}\left|u_{j+1/2}-v_{j+1/2}\right|.\end{aligned}$$

Hence, there is a constant $C$ such that

$$\begin{aligned}&\left|F_j^1(u)-F_j^1(v)\right|^2\\ &\le \frac{C}{\Delta x}\left(\|u\|_{\ell^\infty}^2+\|v\|_{\ell^\infty}^2\right)\\ &\quad\times\left(\left|u_{j-1/2}-v_{j-1/2}\right|^2+\left|u_{j+1/2}-v_{j+1/2}\right|^2+\left|u_{j+3/2}-v_{j+3/2}\right|^2\right).\end{aligned}$$

Multiplying with $\Delta x$ and summing over $j\in\mathbb{Z}$, we get

$$\left\|F^1(u)-F^1(v)\right\|_{\ell^2}^2 \le \frac{C}{\Delta x}\left(\|u\|_{\ell^\infty}^2+\|v\|_{\ell^\infty}^2\right)\|u-v\|_{\ell^2}^2,$$

which, thanks to (2.1), implies

$$\left\|F^1(u)-F^1(v)\right\|_{\ell^2} \le \frac{C}{\Delta x}\|u-v\|_{\ell^2}.$$

We proceed by demonstrating the local Lipschitz continuity of $F^2$. Let

$$f_j(u) = \left(u_{j+1/2}\vee 0\right)^2 + \left(u_{j-1/2}\wedge 0\right)^2 + \frac{1}{2}\left(D_-u_{j+1/2}\right)^2.$$

Then we have that

$$\begin{aligned}|f_j(u)-f_j(v)| &\le C\left(1+\frac{1}{(\Delta x)^2}\right)\left(\|u\|_{\ell^\infty}+\|v\|_{\ell^\infty}\right)\\ &\qquad\times\left(\left|u_{j+1/2}-v_{j+1/2}\right|+\left|u_{j-1/2}-v_{j-1/2}\right|\right),\end{aligned}$$

from which it follows that

$$\begin{aligned}|f_j(u)-f_j(v)|^2 &\le C\left(1+\frac{1}{(\Delta x)^2}\right)^2\left(\|u\|_{\ell^\infty}+\|v\|_{\ell^\infty}\right)^2\\ &\qquad\times\left(\left|u_{j+1/2}-v_{j+1/2}\right|^2+\left|u_{j-1/2}-v_{j-1/2}\right|^2\right).\end{aligned}$$

Hence, making use of (2.1),

$$\|f(u)-f(v)\|_{\ell^2} \le \frac{C}{\sqrt{\Delta x}}\left(1+\frac{1}{(\Delta x)^2}\right)\|u-v\|_{\ell^2}. \tag{3.6}$$

Next, in view of (6.4) and (6.5) (cf. the proof of Lemma 6.1 below),

$$\left|F_j^2(u)-F_j^2(v)\right| \le C\Delta x\sum_i e^{-\kappa|i-j|}g_i, \qquad g_j := |f_j(u)-f_j(v)|.$$

Therefore

$$\begin{aligned}\left|F_j^2(u)-F_j^2(v)\right|^2 &\le C\Delta x^2 \sum_{i,k} e^{-\kappa(|i-j|+|k-j|)} g_i g_k \\ &\le \frac{C}{2}\Delta x^2 \sum_{i,k} e^{-\kappa(|i-j|+|k-j|)} \left(g_i^2+g_k^2\right).\end{aligned}$$

We multiply with $\Delta x$ and sum over $j\in\mathbb{Z}$. This yields

$$\begin{aligned}\left\|F^2(u)-F^2(v)\right\|_{\ell^2}^2 &\le C\Delta x^3 \sum_{i,j,k} e^{-\kappa(|i-j|+|k-j|)} \left(g_i^2+g_k^2\right) \\ &= C\Delta x^3 \sum_{i,j,k} e^{-\kappa(|i-j|+|k-j|)} g_i^2 + C\Delta x^3 \sum_{i,j,k} e^{-\kappa(|i-j|+|k-j|)} g_k^2 \\ &= C\Delta x^2 \sum_{i,j} e^{-\kappa|i-j|} g_i^2 + C\Delta x^2 \sum_{k,j} e^{-\kappa|k-j|} g_k^2 \\ &= C\Delta x \sum_i g_i^2 + C\Delta x \sum_k g_k^2 = C\left\|g\right\|_{\ell^2}^2.\end{aligned}$$

Combining this with (3.6) gives the local Lipschitz continuity of $F^2$. This concludes the proof of (3.5).

Thanks to (3.5), there exists a continuously differentiable solution to (3.4) for $t$ in some open interval $(0,t_0)$, where $t_0$ is such that

$$\lim_{t\uparrow t_0}\left\|u(t)\right\|_{\ell^2}=\infty.$$

Lemma 5.1 below shows that $\left\|u(t)\right\|_{\ell^2}$ remains bounded for all $t>0$, and thus the proof of the lemma is completed. □

Next, let us derive the difference scheme satisfied by

$$q_j = D_- u_{j+1/2}. \tag{3.7}$$

This will be done by applying the difference operator $D_-$ to the $u$-equation (3.1). To this end applying the discrete Leibniz rule we get

$$D_-\left[\left(u_{j+1/2}\vee 0\right)D_- u_{j+1/2}\right] = \left(u_{j-1/2}\vee 0\right)D_- q_j + D_-\left(u_{j+1/2}\vee 0\right)q_j$$

and

$$D_-\left[\left(u_{j+1/2}\wedge 0\right)D_+ u_{j+1/2}\right] = \left(u_{j+1/2}\wedge 0\right)D_+ q_j + D_-\left(u_{j+1/2}\wedge 0\right)q_j,$$

so that

$$\begin{aligned}&D_-\left[\left(u_{j+1/2}\vee 0\right)D_- u_{j+1/2} + \left(u_{j+1/2}\wedge 0\right)D_+ u_{j+1/2}\right] \\ &\qquad = \left(u_{j-1/2}\vee 0\right)D_- q_j + \left(u_{j+1/2}\wedge 0\right)D_+ q_j + q_j^2.\end{aligned} \tag{3.8}$$

The $P$-equation (3.3) rephrased in terms of $q$ reads

$$-D_+D_-P_j + P_j = \left(u_{j+1/2}\vee 0\right)^2 + \left(u_{j-1/2}\wedge 0\right)^2 + \frac{1}{2}q_j^2. \tag{3.9}$$

Employing (3.8) and (3.9) when applying $D_-$ to the $u$-equation in (3.1) yields

$$\begin{aligned}&q_j' + \left(u_{j-1/2}\vee 0\right)D_- q_j + \left(u_{j+1/2}\wedge 0\right)D_+ q_j \\ &\qquad + \frac{q_j^2}{2} + P_j - \left(u_{j+1/2}\vee 0\right)^2 - \left(u_{j-1/2}\wedge 0\right)^2 = 0.\end{aligned} \tag{3.10}$$

Regarding the initial values, it easy to see that

$$q_j(0) = \frac{1}{\Delta x}\int_{I_j} \partial_x u_0(x)\,dx, \quad j \in \mathbb{Z}.$$

Inasmuch as $q$ can be discontinuous, (3.10) is a reasonable discretization of (1.7).

## 4. Main convergence result

The main aim of this paper is to prove that the numerical scheme defined in Section 3 converges to a solution of the Camassa-Holm equation. Before we can do that we need to define what we mean by a "solution".

**Definition 4.1.** *We call a function $u = u(t,x) : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ a weak solution of the Cauchy problem* (1.1)-(1.2) *provided*

*(i)* $u \in L^\infty(\mathbb{R}_+; H^1(\mathbb{R})) \cap C([0,\infty) \times \mathbb{R})$;

*(ii)* $u$ *satisfies* (1.4) *in the sense of distributions, that is,* $\forall \phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$)

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} u\partial_t\phi + F(u,\partial_x u)\partial_x\phi\,dxdt = 0; \tag{4.1}$$

*(iii)* $u(0,x) = u_0(x)$, *for every* $x \in \mathbb{R}$;

*(iv)* $\|u(t,\cdot)\|_{H^1(\mathbb{R})} \le \|u_0\|_{H^1(\mathbb{R})}$, *for each* $t > 0$.

*If, in addition, there is a constant $C \ge 0$ depending only on $\|u_0\|_{H^1(\mathbb{R})}$ such that*

$$u_x(t,x) \le \frac{2}{t} + C, \qquad (t,x) \in (0,\infty) \times \mathbb{R}, \tag{4.2}$$

*then we call $u$ a dissipative weak solution of the Cauchy problem* (1.1)-(1.2)*.*

Supplied with the sequences $\{u_{j+1/2}(t)\}_{j\in\mathbb{Z}}$, $\{q_j(t)\}_{j\in\mathbb{Z}}$ defined by (3.1)-(3.7), we introduce the function

$$u_{\Delta x}(t,x) = q_j(t)(x - x_{j+1/2}) + u_{j-1/2}(t), \quad t \ge 0,\, x \in I_j,\, j \in \mathbb{Z}, \tag{4.3}$$

Observe that $u_{\Delta x}(t,\cdot)$ is piecewise linear and continuous. Besides,

$$\begin{aligned} u_{\Delta x}(t, x_{j\pm1/2}) &= u_{j\pm1/2}(t), \quad t \ge 0,\, j \in \mathbb{Z},\\ \partial_x u_{\Delta x}(t,x) &= q_j(t), \qquad t \ge 0,\, x \in I_j\, j \in \mathbb{Z}. \end{aligned}$$

We are now in a position to state our main result.

**Theorem 4.1.** *Suppose* (1.2) *holds. Let $\{u_{\Delta x}\}_{\Delta x>0}$ be a sequence of difference solutions defined by* (4.3) *and* (3.1)-(3.7)*. Then, along a subsequence as $\Delta x \downarrow 0$,*

$$u_{\Delta x} \to u \text{ in } H^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}),$$

*where $u$ is a dissipative weak solution of the Cauchy problem* (1.1)-(1.2)*.*

This theorem is a consequence of the results stated and proved in Sections 5-11.

## 5. Discrete total energy estimate

The finite difference scheme (3.1)-(3.7) is designed to admit the discrete total energy estimate stated below, which contains a dissipation term resulting from the upwind nature of the scheme for the $q$-variable (notice that there is no dissipation associated with the $u$-variable).

**Lemma 5.1.** *For each $t \geq 0$,*

$$\begin{aligned} &\left\| \left\{ u_{j+1/2}(t) \right\}_j \right\|_{h^1}^2 + \Delta x^2 \sum_j \int_0^t \left| u_{j+1/2}(s) \right| \left( D_+ D_- u_{j+1/2}(s) \right)^2 \, ds \\ &\qquad = \left\| \left\{ u_{j+1/2}(0) \right\}_j \right\|_{h^1}^2 . \end{aligned} \tag{5.1}$$

*Proof.* We multiply the $u$-equation in (3.1) by $u_{j+1/2}$ and use $q_j = D_- u_{j+1/2}$ to obtain

$$\begin{aligned} &\frac{d}{dt}\left( \frac{u_{j+1/2}^2}{2} \right) + \left( u_{j+1/2} \vee 0 \right)^2 q_j \\ &\qquad + \left( u_{j+1/2} \wedge 0 \right)^2 q_{j+1} + \left( D_+ P_j \right) u_{j+1/2} = 0, \end{aligned} \tag{5.2}$$

while multiplying the $q$-equation in (3.10) by $q_j$ yields

$$\begin{aligned} &\frac{d}{dt}\left( \frac{q_j^2}{2} \right) + \left( u_{j-1/2} \vee 0 \right) (D_- q_j) q_j + \left( u_{j+1/2} \wedge 0 \right) (D_+ q_j) q_j \\ &\qquad + \frac{q_j^3}{2} - \left( u_{j+1/2} \vee 0 \right)^2 q_j - \left( u_{j-1/2} \wedge 0 \right)^2 q_j + P_j q_j = 0. \end{aligned} \tag{5.3}$$

Adding (5.2) and (5.3) and multiplying the result with $\Delta x$ and summing over $j$ yields

$$\frac{d}{dt}\left[ \Delta x \sum_j \left( \frac{u_{j+1/2}^2}{2} + \frac{q_j^2}{2} \right) \right] + \mathrm{I} + \mathrm{II} + \mathrm{III} = 0,$$

where

$$\begin{aligned} \mathrm{I} &= \Delta x \sum_j \left( u_{j-1/2} \vee 0 \right) D_- q_j q_j + \Delta x \sum_j \left( u_{j+1/2} \wedge 0 \right) D_+ q_j q_j + \Delta x \sum_j \frac{q_j^3}{2}, \\ \mathrm{II} &= \Delta x \sum_j \left( u_{j+1/2} \vee 0 \right)^2 q_j + \Delta x \sum_j \left( u_{j+1/2} \wedge 0 \right)^2 q_{j+1} \\ &\quad - \Delta x \sum_j \left( u_{j+1/2} \vee 0 \right)^2 q_j - \Delta x \sum_j \left( u_{j-1/2} \wedge 0 \right)^2 q_j \equiv 0 \quad \text{(by shifting indices)}, \\ \mathrm{III} &= \Delta x \sum_j D_+ P_j u_{j+1/2} + \Delta x \sum_j P_j q_j \equiv 0 \quad \text{(by summation by parts)}. \end{aligned}$$

Let us now deal with term I. The discrete chain rule implies that

$$D_\pm q_j q_j = D_\pm \left( \frac{q_j^2}{2} \right) \mp \frac{\Delta x}{2} (D_\pm q_j)^2.$$

Hence

$$\begin{aligned}
\mathrm{I} &= \Delta x \sum_j \left(u_{j-1/2} \vee 0\right) \left[D_- \left(\frac{q_j^2}{2}\right) + \frac{\Delta x}{2} (D_- q_j)^2\right] \\
&\qquad + \Delta x \sum_j \left(u_{j+1/2} \wedge 0\right) \left[D_+ \left(\frac{q_j^2}{2}\right) - \frac{\Delta x}{2} (D_+ q_j)^2\right] + \Delta x \sum_j \frac{q_j^3}{2} \\
&= \mathrm{I}_1 + \mathrm{I}_2,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I}_1 &= \Delta x \sum_j \left(u_{j-1/2} \vee 0\right) D_- \left(\frac{q_j^2}{2}\right) + \Delta x \sum_j \left(u_{j+1/2} \wedge 0\right) D_+ \left(\frac{q_j^2}{2}\right) + \Delta x \sum_j \frac{q_j^3}{2}, \\
\mathrm{I}_2 &= \frac{\Delta x^2}{2} \sum_j \Big[\left(u_{j-1/2} \vee 0\right) (D_- q_j)^2 - \left(u_{j+1/2} \wedge 0\right) (D_+ q_j)^2\Big] \\
&= \frac{\Delta x^2}{2} \sum_j \Big[\left(u_{j+1/2} \vee 0\right) (D_- q_{j+1})^2 - \left(u_{j+1/2} \wedge 0\right) (D_+ q_j)^2\Big] \\
&= \frac{\Delta x^2}{2} \sum_j |u_{j+1/2}| (D_+ q_j)^2 \geq 0.
\end{aligned}$$

To handle the $\mathrm{I}_1$-term, we use the discrete Leibniz rule, which implies

$$\begin{aligned}
D_- \left[\left(u_{j+1/2} \vee 0\right) \frac{q_j^2}{2}\right] &= \left(u_{j-1/2} \vee 0\right) D_- \left(\frac{q_j^2}{2}\right) + D_- \left(u_{j+1/2} \vee 0\right) \frac{q_j^2}{2}, \\
D_+ \left[\left(u_{j-1/2} \wedge 0\right) \frac{q_j^2}{2}\right] &= \left(u_{j+1/2} \wedge 0\right) D_+ \left(\frac{q_j^2}{2}\right) + D_+ \left(u_{j-1/2} \vee 0\right) \frac{q_j^2}{2} \\
&= \left(u_{j+1/2} \wedge 0\right) D_+ \left(\frac{q_j^2}{2}\right) + D_- \left(u_{j+1/2} \vee 0\right) \frac{q_j^2}{2},
\end{aligned}$$

to obtain

$$\begin{aligned}
\mathrm{I}_1 =& \Delta x \sum_j D_- \left[\left(u_{j+1/2} \vee 0\right) \frac{q_j^2}{2}\right] - \Delta x \sum_j D_- \left(u_{j+1/2} \vee 0\right) \frac{q_j^2}{2} \\
&+ \Delta x \sum_j D_+ \left[\left(u_{j-1/2} \wedge 0\right) \frac{q_j^2}{2}\right] - \Delta x \sum_j D_- \left(u_{j+1/2} \vee 0\right) \frac{q_j^2}{2} + \Delta x \sum_j \frac{q_j^3}{2} \\
=& - \Delta x \sum_j D_- u_{j+1/2} \frac{q_j^2}{2} + \Delta x \sum_j \frac{q_j^3}{2} = -\Delta x \sum_j \frac{q_j^3}{2} + \Delta x \sum_j \frac{q_j^3}{2} = 0.
\end{aligned}$$

Summarizing our findings, the following discrete energy estimate holds:

$$\frac{d}{dt} \left[\Delta x \sum_j \left(\frac{u_{j+1/2}^2}{2} + \frac{q_j^2}{2}\right)\right] + \frac{\Delta x^2}{2} \sum_j \left|u_{j+1/2}\right| (D_+ q_j)^2 = 0.$$

Finally, integrating over $[0, t]$ we get (5.1). $\square$

**Remark 5.1.** In view of (5.1) and (2.2)

$$\left\| \left\{ u_{j+1/2}(t) \right\}_j \right\|_{\ell^\infty} \leq C \left\| u_0 \right\|_{H^1(\mathbb{R})}, \qquad t \geq 0, \tag{5.4}$$

where $C > 0$ is a constant that is independent of $\Delta x$.

## 6. Basic estimates on $\{P_j\}_{j\in\mathbb{Z}}$

Next we derive some estimates on $\{P_j\}_{j\in\mathbb{Z}}$ that are all consequences of (5.1).

**Lemma 6.1.** *For each $t \geq 0$,*

$$\left\| \left\{ P_j(t) \right\}_j \right\|_{\ell^\infty}, \left\| \left\{ P_j(t) \right\}_j \right\|_{\ell^1} \leq C \left\| u_0 \right\|_{H^1(\mathbb{R})}^2, \tag{6.1}$$

$$\left\| \left\{ D_+ P_j(t) \right\}_j \right\|_{\ell^\infty}, \left\| \left\{ D_+ P_j(t) \right\}_j \right\|_{\ell^1} \leq C \left\| u_0 \right\|_{H^1(\mathbb{R})}^2, \tag{6.2}$$

*where $C > 0$ is a constant independent of $\Delta x$.*

*Proof.* Introduce the notations

$$f_j = (u_{j+1/2} \vee 0)^2 + (u_{j-1/2} \wedge 0)^2 + \frac{q_j^2}{2},$$

and

$$h = \left( 1 + 2 \frac{1 - e^{-\kappa}}{(\Delta x)^2} \right)^{-1}, \quad \kappa = \ln \left( 1 + \frac{\Delta x^2}{2} + \frac{\Delta x}{2} \sqrt{4 + \Delta x^2} \right). \tag{6.3}$$

Then the solution of (3.3) has the following form:

$$P_j = 2h \sum_i e^{-\kappa |j-i|} f_i, \qquad j \in \mathbb{Z}. \tag{6.4}$$

We observe that

$$h = \frac{\Delta x}{2} + \mathcal{O}(\Delta x^2), \quad \frac{|e^{\kappa} - 1|}{\Delta x} = 1 + \mathcal{O}(\Delta x), \quad \frac{|e^{-\kappa} - 1|}{\Delta x} = 1 + \mathcal{O}(\Delta x). \tag{6.5}$$

We shall need the following estimate (cf. (5.1)):

$$\left\| \left\{ f_j \right\}_j \right\|_{\ell^1} \leq \Delta x \sum_j \left( u_{j+1/2}^2 + q_j^2 \right) \leq \left\| \left\{ u_{j+1/2}(0) \right\}_j \right\|_{h^1}^2. \tag{6.6}$$

For any $t \geq 0$ and $j \in \mathbb{Z}$, using (6.6), we have

$$|P_j(t)| \leq C \left\| \left\{ f_j \right\}_j \right\|_{\ell^1} \leq C \left\| u_0 \right\|_{H^1(\mathbb{R})}^2,$$

for some constant $C > 0$ independent of $\Delta x$. Furthermore, using again (6.6),

$$\left\| \left\{ P_j(t) \right\}_j \right\|_{\ell^1} = 2h \sum_i \left[ \Delta x \sum_j e^{-\kappa |j-i|} \right] f_i \leq C \left\| u_0 \right\|_{H^1(\mathbb{R})}^2,$$

for some constant $C > 0$ independent of $\Delta x$. Hence, we have proved (6.1).

From (6.4),

$$\begin{aligned} D_+ P_j &= \frac{P_{j+1} - P_j}{\Delta x} \\ &= 2h \sum_i \frac{e^{-\kappa |i-j-1|} - e^{-\kappa |i-j|}}{\Delta x} f_i \end{aligned}$$

$$= 2h \sum_{i=j}^{\infty} \frac{e^{-\kappa(i-j-1)} - e^{-\kappa(i-j)}}{\Delta x} f_i + 2h \sum_{i=-\infty}^{j-1} \frac{e^{\kappa(i-j-1)} - e^{\kappa(i-j)}}{\Delta x} f_i$$
$$= 2h \sum_{i=j}^{\infty} e^{-\kappa(i-j)} \frac{e^{\kappa} - 1}{\Delta x} f_i + 2h \sum_{i=-\infty}^{j-1} e^{\kappa(i-j)} \frac{e^{-\kappa} - 1}{\Delta x} f_i.$$

Using (6.5) and (6.6) we acquire from this the following two estimates:

$$|D_+ P_j(t)| \le 2hC \sum_i e^{-\kappa|i-j|} f_i \le C \|u_0\|^2_{H^1(\mathbb{R})}$$

and

$$\left\| \{D_+ P_j(t)\}_j \right\|_{\ell^1} \le 2hC\Delta x \sum_{j,i} e^{-\kappa|i-j|} f_i \le C \|u_0\|^2_{H^1(\mathbb{R})},$$

for some constant $C > 0$ independent of $\Delta x$. Therefore (6.2) holds. □

## 7. Discrete Oleĭnik estimate

The aim of this section is to prove that the quantity $q_j = D_- u_{j+1/2}$ is uniformly upper bounded on $\{t > 0\}$, thereby revealing the dissipative nature of our scheme.

**Lemma 7.1.** *For $t > 0$, $j \in \mathbb{Z}$,*

$$q_j(t) \le \frac{2}{t} + C, \tag{7.1}$$

*for some positive constant $C$ that is independent of $\Delta x$.*

*Proof.* By (5.4) and (6.1), it follows from (1.10) that

$$q_j' + \frac{q_j^2}{2} \le L - \left[ \left(u_{j-1/2} \vee 0\right) D_- q_j + \left(u_{j+1/2} \wedge 0\right) D_+ q_j \right], \quad j \in \mathbb{Z},\ t > 0, \tag{7.2}$$

for some constant $L > 0$. Since $\lim_{j \to \pm\infty} q_j(t) = 0$ there is an index $i(t) \in \mathbb{Z}$ such that

$$q_{i(t)}(t) = \sup_{j \in \mathbb{Z}} q_j(t), \qquad t > 0. \tag{7.3}$$

At $j = i(t)$ for $t > 0$ there holds $D_+ q_{i(t)}(t) \le 0 \le D_- q_{i(t)}(t)$, so that

$$\left(u_{i(t)-1/2}(t) \vee 0\right) D_- q_{i(t)}(t) + \left(u_{i(t)+1/2}(t) \wedge 0\right) D_+ q_{i(t)}(t) \ge 0, \qquad t > 0,$$

which inserted into (7.2) yields

$$q_{i(t)}'(t) + \frac{q_{i(t)}^2(t)}{2} \le L, \qquad t > 0. \tag{7.4}$$

One can check that $f(t) := \frac{2}{t} + \sqrt{2L}$ is a supersolution of the ODE $y' + \frac{y^2}{2} = L$ on $\{t > 0\}$, while (7.4) shows that $q_{i(t)}(t)$ is a subsolution. Hence, by the comparison principle for ODEs and (7.3),

$$q_j(t) \le q_{i(t)}(t) \le \frac{2}{t} + \sqrt{2L}, \qquad j \in \mathbb{Z},\ t > 0.$$

□

## 8. Discrete higher integrability estimate

In view of (5.1) we infer that $(\partial_x u_{\Delta x})^2$ converges (in the sense of measures) along a subsequence as $\Delta x \downarrow 0$. To ensure that the limit does not contain concentration effects (singular measures), we shall derive a discrete higher integrability estimate.

To prepare for the derivation of this estimate (but also for later use), we will derive a "renormalized form" of the finite difference scheme for $q_j$. To this end, let $f$ be a nonlinear function (renormalization) of appropriate regularity and growth. Multiplying (3.10) by $f'(q_j)$ and using the discrete chain rule, which in the present context reads

$$D_\pm q_j f'(q_j) = D_\pm f(q_j) \mp \frac{\Delta x}{2} f''(\xi_j^\pm)\,(D_\pm q_j)^2\,,$$

for some numbers $\xi_j^\pm$ between $q_j$ and $q_{j\pm1}$, we obtain the following renormalized difference scheme:

$$\begin{aligned}\frac{d}{dt} f(q_j) &+ \big(u_{j-1/2} \vee 0\big)\, D_- f(q_j) + \big(u_{j+1/2} \wedge 0\big)\, D_+ f(q_j) + \frac{q_j^2}{2} f'(q_j)\\ &+ \Big[P_j - \big(u_{j+1/2} \vee 0\big)^2 - \big(u_{j-1/2} \wedge 0\big)^2\Big]\, f'(q_j) + I_{\Delta x, f'', j} = 0,\end{aligned} \tag{8.1}$$

where

$$I_{\Delta x, f'', j} := \frac{\Delta x}{2}\Big\{\big(u_{j-1/2} \vee 0\big)\, f''(\xi_j^-)(D_- q_j)^2 - \big(u_{j+1/2} \wedge 0\big)\, f''(\xi_j^+)(D_+ q_j)^2\Big\}.$$

Let us now write (8.1) in divergence-form. To this end, observe that the discrete Leibniz rule (2.3) implies the following relations:

$$\begin{aligned}D_-\left[\big(u_{j+1/2} \vee 0\big)\, f(q_j)\right] &= \big(u_{j-1/2} \vee 0\big)\, D_- f(q_j) + D_-\, \big(u_{j+1/2} \vee 0\big)\, f(q_j),\\ D_+\left[\big(u_{j-1/2} \wedge 0\big)\, f(q_j)\right] &= \big(u_{j+1/2} \wedge 0\big)\, D_+ f(q_j) + D_+\, \big(u_{j-1/2} \wedge 0\big)\, f(q_j)\\ &= \big(u_{j+1/2} \wedge 0\big)\, D_+ f(q_j) + D_-\, \big(u_{j+1/2} \wedge 0\big)\, f(q_j),\end{aligned}$$

and therefore, using that $q_j = D_- u_{j+1/2}$,

$$\begin{aligned}&\big(u_{j-1/2} \vee 0\big)\, D_- f(q_j) + \big(u_{j+1/2} \wedge 0\big)\, D_+ f(q_j)\\ &\qquad = D_-\left[\big(u_{j+1/2} \vee 0\big)\, f(q_j)\right] + D_+\left[\big(u_{j-1/2} \wedge 0\big)\, f(q_j)\right] - q_j f(q_j).\end{aligned}$$

Hence, we end up with the following divergence-form variant of the renormalized difference scheme (8.1):

$$\begin{aligned}\frac{d}{dt} f(q_j) &+ D_-\left[\big(u_{j+1/2} \vee 0\big)\, f(q_j)\right] + D_+\left[\big(u_{j-1/2} \wedge 0\big)\, f(q_j)\right]\\ &+ \frac{q_j^2}{2} f'(q_j) - q_j f(q_j) + \Big[P_j - \big(u_{j+1/2} \vee 0\big)^2 - \big(u_{j-1/2} \wedge 0\big)^2\Big]\, f'(q_j)\\ &+ I_{\Delta x, f'', j} = 0.\end{aligned} \tag{8.2}$$

We are now in a position to prove the following lemma.

**Lemma 8.1.** *Let $\alpha \in (0,1)$, $T > 0$, and $j_a, j_b$ be integers such that $j_a < j_b$. Set $a := j_a \Delta x$ and $b := j_b \Delta x$. There exists a positive constant $C$, depending only on $u_0, \alpha, T, a, b$, such that*

$$\int_0^T \Delta x \sum_{j=j_a}^{j_b} |q_j(t)|^{2+\alpha}\, dt \le C. \tag{8.3}$$

*Proof.* Our proof exploits (7.1). We start by introducing the notations

$$\mathcal{J} = \{j_a, \dots, j_b\},$$
$$\mathcal{N}(t) = \left\{ j \in \mathcal{J} \;\middle|\; q_j(t) < 0 \right\}, \quad \mathcal{P}(t) = \left\{ j \in \mathcal{J} \;\middle|\; q_j(t) \geq 0 \right\},$$
$$I = \int_0^T \Delta x \sum_{j \in \mathcal{J}} |q_j(t)|^{2+\alpha} \, dt,$$
$$I_- = \int_0^T \Delta x \sum_{j \in \mathcal{N}(t)} |q_j(t)|^{2+\alpha} \, dt, \quad I_+ = \int_0^T \Delta x \sum_{j \in \mathcal{P}(t)} |q_j(t)|^{2+\alpha} \, dt,$$

and observing that

$$\mathcal{J} = \mathcal{N}(t) \cup \mathcal{P}(t), \quad I = I_+ + I_-.$$

By (5.1), (7.1), and since $\alpha < 1$,

$$I_+ \leq \int_0^T \Delta x \sum_{j \in \mathcal{P}(t)} |q_j(t)|^2 \left( \frac{2}{t} + C \right)^\alpha dt \leq C(T, \alpha) \left\| \{u_j(0)\}_j \right\|_{h^1}^2.$$

We have to estimate $I_-$. With $f(\xi) = |\xi|^{1+\alpha}$, (8.2) reads

$$\begin{aligned}
&\frac{d}{dt} |q_j|^{1+\alpha} + D_- \left[ \left(u_{j+1/2} \vee 0\right) |q_j|^{1+\alpha} \right] + D_+ \left[ \left(u_{j-1/2} \wedge 0\right) |q_j|^{1+\alpha} \right] \\
&\quad + \frac{\alpha - 1}{2} \operatorname{sign}(q_j) |q_j|^{2+\alpha} + (1+\alpha) P_j \operatorname{sign}(q_j) |q_j|^\alpha \\
&\quad - (1+\alpha) \left[ \left(u_{j+1/2} \vee 0\right)^2 + \left(u_{j-1/2} \wedge 0\right)^2 \right] \operatorname{sign}(q_j) |q_j|^\alpha = -I_{\Delta x, f'', j} \leq 0,
\end{aligned}$$

where we used the convexity of $f$ to conclude the inequality. Let $\chi$ be a smooth cutoff function such that

$$0 \leq \chi \leq 1, \quad \xi \in [a, b+1] \Longrightarrow \chi(\xi) = 1, \quad \xi \notin [a-1, b+2] \Longrightarrow \chi(\xi) = 0.$$

Multiplying by $\Delta x \chi(j\Delta x)$, summing over $j \in \mathbb{Z}$, and integrating over $t \in (0, T)$ we arrive at

$$\begin{aligned}
0 \leq{}& \Delta x \sum_j \left( |q_j(0)|^{1+\alpha} - |q_j(T)|^{1+\alpha} \right) \chi(j\Delta x) \\
&+ \frac{1-\alpha}{2} \int_0^T \Delta x \sum_j \operatorname{sign}(q_j) |q_j|^{2+\alpha} \chi(j\Delta x) \, dt \\
&- \int_0^T \Delta x \sum_j D_- \left[ \left(u_{j+1/2} \vee 0\right) |q_j|^{1+\alpha} \right] \chi(j\Delta x) \, dt \\
&- \int_0^T \Delta x \sum_j D_+ \left[ \left(u_{j-1/2} \wedge 0\right) |q_j|^{1+\alpha} \right] \chi(j\Delta x) \, dt \\
&- (1+\alpha) \int_0^T \Delta x \sum_j |P_j| \, |q_j|^\alpha \chi(j\Delta x) \, dt \\
&+ (1+\alpha) \int_0^T \Delta x \sum_j \left(u_{j+1/2} \vee 0\right)^2 |q_j|^\alpha \chi(j\Delta x) \, dt \\
&+ (1+\alpha) \int_0^T \Delta x \sum_j \left(u_{j-1/2} \wedge 0\right)^2 |q_j|^\alpha \chi(j\Delta x) \, dt.
\end{aligned} \tag{8.4}$$

Next, we introduce the notations

$$\widetilde{\mathcal{N}}(t) = \left\{ j \in \mathbb{Z} \;\middle|\; q_j(t) < 0 \right\}, \qquad \widetilde{\mathcal{P}}(t) = \left\{ j \in \mathbb{Z} \;\middle|\; q_j(t) \geq 0 \right\}.$$

and observe that, since $\mathcal{N}(t) \subset \widetilde{\mathcal{N}}(t)$ and $\widetilde{\mathcal{N}}(t) \cup \widetilde{\mathcal{P}}(t) = \mathbb{Z}$, there holds

$$\begin{aligned}
&\int_0^T \Delta x \sum_j \operatorname{sign}(q_j) \left|q_j\right|^{2+\alpha} \chi(j\Delta x)\, dt \\
&\qquad = \Delta x \int_0^T \sum_{j \in \widetilde{\mathcal{P}}(t)} \left|q_j\right|^{2+\alpha} dt - \int_0^T \Delta x \sum_{j \in \widetilde{\mathcal{N}}(t)} \left|q_j\right|^{2+\alpha} dt \\
&\qquad \leq \int_0^T \Delta x \sum_{j \in \widetilde{\mathcal{P}}(t)} \left|q_j\right|^{2+\alpha} dt - I_-.
\end{aligned}$$

Therefore, from (8.4),

$$\frac{1-\alpha}{2} I_- \leq I_1 + I_2 + I_3 + I_4, \tag{8.5}$$

where

$$\begin{aligned}
I_1 &= \frac{1-\alpha}{2} \int_0^T \Delta x \sum_{j \in \widetilde{\mathcal{P}}(t)} \left|q_j\right|^{2+\alpha} dt, \\
I_2 &= \Delta x \sum_j \left( \left|q_j(0)\right|^{1+\alpha} - \left|q_j(T)\right|^{1+\alpha} \right) \chi(j\Delta x), \\
I_3 &= -\int_0^T \Delta x \sum_j D_- \left[ \left(u_{j+1/2} \vee 0\right) \left|q_j\right|^{1+\alpha} \right] \chi(j\Delta x)\, dt \\
&\qquad - \int_0^T Dx \sum_j D_+ \left[ \left(u_{j-1/2} \wedge 0\right) \left|q_j\right|^{1+\alpha} \right] \chi(j\Delta x)\, dt, \\
I_4 &= -(1+\alpha) \int_0^T \Delta x \sum_j \left|P_j\right| \left|q_j\right|^{\alpha} \chi(j\Delta x)\, dt \\
&\qquad + (1+\alpha) \int_0^T \Delta x \sum_j \left(u_{j+1/2} \vee 0\right)^2 \left|q_j\right|^{\alpha} \chi(j\Delta x)\, dt \\
&\qquad + (1+\alpha) \int_0^T \Delta x \sum_j \left(u_{j-1/2} \wedge 0\right)^2 \left|q_j\right|^{\alpha} \chi(j\Delta x)\, dt.
\end{aligned}$$

For $I_1$ we repeat what we did for $I_+$. Indeed, due to (5.1), (7.1), and $\alpha < 1$,

$$I_1 \leq \frac{1}{2} \int_0^T \Delta x \sum_{j \in \widetilde{\mathcal{P}}(t)} \left|q_j\right|^2 \left( \frac{2}{t} + C \right)^{\alpha} dt \leq C(T, \alpha) \left\| \{u_{j+1/2}(0)\}_j \right\|_{h^1}^2.$$

For the other terms we use Hölder's inequality for sums, the discrete Leibniz rule, (5.1), (5.4), and (6.1)

$$\begin{aligned}
I_2 &\leq \left( \left\| \left|q_j(0)\right|^{1+\alpha} \}_j \right\|_{\ell^{\frac{2}{1+\alpha}}} + \left\| \left\{ \left|q_j(T)\right|^{1+\alpha} \right\}_j \right\|_{\ell^{\frac{2}{1+\alpha}}} \right) \left\| \{\chi(j\Delta x)\}_j \right\|_{\ell^{\frac{2}{1-\alpha}}} \\
&= \left( \left\| \{q_j(0)\}_j \right\|_{\ell^2}^{1+\alpha} + \left\| \{q_j(T)\}_j \right\|_{\ell^2}^{1+\alpha} \right) \left\| \{\chi(j\Delta x)\}_j \right\|_{\ell^{\frac{2}{1-\alpha}}}
\end{aligned}$$

$$
\begin{aligned}
&\leq C_1 \left\|\{u_j(0)\}_j\right\|_{h^1}^{1+\alpha} \left\|\{\chi(j\Delta x)\}_j\right\|_{\ell^{\frac{2}{1-\alpha}}}, \\
I_3 &= \int_0^T \Delta x \sum_j \left(u_{j+1/2} \vee 0\right) |q_j|^{1+\alpha} D_-\chi((j-1)\Delta x)\, dt \\
&\quad + \int_0^T \Delta x \sum_j \left(u_{j-1/2} \wedge 0\right) |q_j|^{1+\alpha} D_+\chi((j+1)\Delta x)\, dt \\
&\leq \int_0^T \left\|\left\{u_{j+1/2}(t)\right\}_j\right\|_{\ell^\infty} \left\|\left\{q_j(t)\right\}_j\right\|_{\ell^2}^{1+\alpha} \\
&\qquad \times \left( \left\|\{D_-\chi((j-1)\Delta x)\}_j\right\|_{\ell^{\frac{2}{1+\alpha}}} + \left\|\{D_+\chi((j+1)\Delta x)dt\}_j\right\|_{\ell^{\frac{2}{1+\alpha}}} \right) dt \\
&\leq C_2 T \left\|\{u_j(0)\}_j\right\|_{h^1}^{2+\alpha} \\
&\qquad \times \left( \left\|\{D_-\chi((j-1)\Delta x)\}_j\right\|_{\ell^{\frac{2}{1+\alpha}}} + \left\|\{D_+\chi((j+1)\Delta x)\}_j\right\|_{\ell^{\frac{2}{1+\alpha}}} \right), \\
I_4 &\leq (1+\alpha) \int_0^T \Delta x \sum_j \left( \left\|\{P_j\}_j\right\|_{\ell^\infty} + 2 \left\|\left\{u_{j+1/2}\right\}_j\right\|_{\ell^\infty}^2 \right) |q_j|^\alpha \chi(j\Delta x)\, dt \\
&\leq C_3 \left\|\{u_j(0)\}_j\right\|_{h^1}^2 \int_0^T \left\|\{q_j(t)_j\}_j\right\|_{\ell^2}^\alpha \left\|\{\chi(j\Delta x)\}_j\right\|_{\ell^{\frac{2}{2-\alpha}}}\, dt \\
&\leq C_4 \left\|\{u_j(0)\}_j\right\|_{h^1}^{2+\alpha} T \left\|\{\chi(j\Delta x)\}_j\right\|_{\ell^{\frac{2}{2-\alpha}}},
\end{aligned}
$$

where the constants $C_1, \dots, C_4$ are independent of $\Delta x$. Now a bound on $I_-$ follows from (8.5), and thereby the proof is concluded. □

## 9. Basic convergence results

In this section we present some convergence results that are straightforward consequences of the á priori estimates established earlier.

**Lemma 9.1.** *There exists a limit function*

$$u \in L^\infty(\mathbb{R}_+; H^1(\mathbb{R})) \cap C([0,\infty) \times \mathbb{R}),$$

*such that along a subsequence as* $\Delta x \downarrow 0$

$$u_{\Delta x} \overset{\star}{\rightharpoonup} u \text{ in } L^\infty(\mathbb{R}_+; H^1(\mathbb{R})), \tag{9.1}$$

$$u_{\Delta x} \to u \text{ uniformly on } [0,T] \times [a,b], \tag{9.2}$$

*for each set* $[0,T] \times [a,b] \subset \mathbb{R}^2$. *Additionally,*

$$u(t,x) \overset{t\downarrow 0}{\to} u_0(x) \text{ for each } x \in \mathbb{R}. \tag{9.3}$$

*Proof.* It is not hard to see that (4.3) imply

$$\left| \int_{\mathbb{R}} u_{\Delta x}^2\, dx - \Delta x \sum_j u_{j+1/2}^2 \right| \leq \left( \Delta x \sum_j q_j^2 \right) \Delta x, \qquad t \geq 0,$$

and

$$\int_{\mathbb{R}} (\partial_x u_{\Delta x})^2\, dx = \Delta x \sum_j q_j^2, \qquad t \geq 0,$$

so, by (5.1),

$$
\begin{aligned}
&\int_{\mathbb{R}} \Big[(u_{\Delta x})^2 + (\partial_x u_{\Delta x})^2\Big]\, dx \\
&\le \int_{\mathbb{R}} \Big[(u_{\Delta x}(0,x))^2 + (\partial_x u_{\Delta x}(0,x))^2\Big]\, dx + \mathcal{O}(\Delta x) \\
&\le C\left(\|u_0\|^2_{H^1(\mathbb{R})} + \Delta x\right), \qquad t \ge 0,
\end{aligned} \tag{9.4}
$$

where $C$ is independent of $\Delta x$. Claim (9.1) is an outcome of (9.4).

To verify claim (9.2) we willl show that

$$
\{u_{\Delta x}\}_{\Delta x>0} \text{ is bounded in } W^{1,2+\alpha}((0,T)\times(a,b)), \tag{9.5}
$$

for any (fixed) set $(0,T)\times(a,b)\subset\mathbb{R}^2$, where $\alpha$ is provided by Lemma (8.1).

Without loss generality, let us assume that $a = j_a\Delta x$ and $b = j_b\Delta x$ for some integers $j_a$ and $j_b$. Then Lemma 8.1 tells us that

$$
\int_0^T\int_a^b \left|\partial_x u_{\Delta x}\right|^{2+\alpha}\, dx\, dt = \Delta x \sum_{j=j_a}^{j_b}\int_0^T \left|q_j(t)\right|^{2+\alpha}\, dt \le C, \tag{9.6}
$$

for some constant $C = C(u_0,\alpha,T,a,b)$ independent of $\Delta x$.

Taking into account (4.3), (3.1), and (3.10), there holds for any $x\in I_j$, $j\in\mathbb{Z}$,

$$
\begin{aligned}
\left|\partial_t u_{\Delta x}(t,x)\right| &= \left|q_j'(t)(x - x_{j-1/2}) + u_{j-1/2}'(t)\right| \\
&\le \left|u_{j-1/2}'(t)\right| + \left|q_j'(t)\right|\Delta x \\
&\le \left|u_{j+1/2}q_j\right| + \left|u_{j+1/2}q_{j+1}\right| + \left|D_+P_j\right| \\
&\qquad + \Delta x\Big(\left|u_{j-1/2}D_-q_j\right| + \left|u_{j+1/2}D_+q_j\right| \\
&\qquad\qquad + \frac{q_j^2}{2} + u_{j+1/2}^2 + u_{j-1/2}^2 + \left|P_j\right|\Big).
\end{aligned} \tag{9.7}
$$

Observe that $\Delta x\, q_j = \mathcal{O}(1)$ for all $j$, which is clearly true thanks to (3.7) and (5.4). Using this and (5.4), (6.2) in (9.7) we acquire the pointwise estimate

$$
\left|\partial_t u_{\Delta x}\right| \le C\left(1 + |q_{j-1}| + |q_j| + |q_{j+1}|\right), \quad \text{for each } x\in I_j,\ j\in\mathbb{Z}, \tag{9.8}
$$

for some constant $C$ independent of $\Delta x$. Consequently, in view of (8.3),

$$
\int_0^T\int_a^b \left|\partial_t u_{\Delta x}\right|^{2+\alpha}\, dx\, dt \le C\left(1 + \Delta x\sum_{j=j_a}^{j_b}\int_0^T \left|q_j\right|^{2+\alpha}\, dt\right) \le C,
$$

for some constant $C = C(u_0,\alpha,T,a,b)$ independent of $\Delta x$.

Summarizing, we have proved that (9.5) holds. Since $W^{1,2+\alpha}((0,T)\times(a,b))$ is compactly embedded into $C^{0,\ell}([0,T]\times[a,b])$ with $\ell = 1 - 2/(2+\alpha)$, there exists a continuous function $u:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ such that along a subsequence

$$
u_{\Delta x}\to u \text{ uniformly on } [0,T]\times[a,b] \text{ and pointwise in } \mathbb{R}_+\times\mathbb{R} \text{ as } \Delta x\downarrow 0.
$$

Combining this with a diagonal argument we conclude that claim (9.2) is true.

Finally, let us prove that the limit $u$ satisfies the initial condition. We fix an arbitrary point $x_0\in\mathbb{R}$ and let $t\in(0,1)$. Then we proceed as follows:

$$
|u(t,x_0) - u_0(x_0)| \le |u(t,x_0) - u_{\Delta x}(t,x_0)|
$$

$$
\begin{aligned}
&\quad + |u_{\Delta x}(t,x_0) - u_{\Delta x}(0,x_0)| + |u_{\Delta x}(0,x_0) - u_0(x_0)|\,, \\
&\le |u(t,x_0) - u_{\Delta x}(t,x_0)| \\
&\quad + Ct^{\ell} + |u_{\Delta x}(0,x_0) - u_0(x_0)|\,, \quad \ell = 1 - \frac{2}{2+\alpha},
\end{aligned}
$$

where we used (9.5) to derive the last inequality ($C$ does not depend on $\Delta x$). Equipped with (9.2) and (1.2), (3.2) we deduce (9.3) by sending first $\Delta x \downarrow 0$ and second $t \downarrow 0$. This concludes the proof of the lemma. □

Equipped with the sequence $\{P_j\}_{j\in\mathbb{Z}}$ defined by (3.1)-(3.7), we introduce the piecewise linear and continuous function

$$
P_{\Delta x}(t,x) = P_j(t) + (x - x_j)D_+P_j(t), \quad t \ge 0,\, x \in I_{j+1/2},\, j \in \mathbb{Z}. \tag{9.9}
$$

Observe that $\partial_x P_{\Delta x}(t,x) = D_+P_j(t)$ for $t \ge 0$, $x \in I_{j+1/2}$, $j \in \mathbb{Z}$.

**Lemma 9.2.** *There exists a limit function*

$$
P \in L^\infty(\mathbb{R}_+; W^{1,\infty}(\mathbb{R})) \cap L^\infty(\mathbb{R}_+; W^{1,1}(\mathbb{R}))
$$

*such that along a subsequence as* $\Delta x \downarrow 0$

$$
P_{\Delta x} \to P \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}) \text{ for each } 1 \le p < \infty. \tag{9.10}
$$

*Proof.* First of all, it is not difficult to see from (9.9) and Lemma 6.1 that

$$
\{P_{\Delta x}\}_{\Delta x>0} \text{ is bounded in } L^\infty(\mathbb{R}_+; W^{1,\infty}(\mathbb{R})) \cap L^\infty(\mathbb{R}_+; W^{1,1}(\mathbb{R})).
$$

Next, we prove that $\{\partial_t P_{\Delta x}\}_{\Delta x>0}$ is bounded in $L^1([0,T]\times\mathbb{R})$, for each fixed $T > 0$. To this end, we write $P_j = P_{1,j} + P_{2,j}$, cf. (6.4), where

$$
\begin{aligned}
P_{1,j} &= 2h \sum_{i\in\mathbb{Z}} \left(e^{-\kappa|j-i|}\right)\left((u_{i+1/2}\vee 0)^2 + (u_{i-1/2}\wedge 0)^2\right), \\
P_{2,j} &= h \sum_{i\in\mathbb{Z}} \left(e^{-\kappa|j-i|}\right) q_i^2.
\end{aligned}
$$

We shall prove that there is a constant $C$, independent of $\Delta x$, such that

$$
\int_0^T \Delta x \sum_j \left|\frac{d}{dt}P_{1,j}\right| dt \le C, \tag{9.11}
$$

$$
\int_0^T \Delta x \sum_j \left|\Delta x\, D_+\left(\frac{d}{dt}P_{1,j}\right)\right| dt \le C, \tag{9.12}
$$

$$
\int_0^T \Delta x \sum_j \left|\frac{d}{dt}P_{2,j}\right| dt \le C, \tag{9.13}
$$

$$
\int_0^T \Delta x \sum_j \left|\Delta x\, D_+\left(\frac{d}{dt}P_{2,j}\right)\right| dt \le C. \tag{9.14}
$$

Note that (9.12) and (9.14) follow from (9.11) and (9.13) respectively, since if $|a_j| \le C$ for all $j$, then $|\Delta x D_\pm a_j| \le |a_j| + |a_{j\pm1}| \le 2C$ for all $j$.

To prove (9.11) observe that

$$
P'_{1,j} = 4h \sum_{i\in\mathbb{Z}} \left(e^{-\kappa|j-i|}\right)\left((u_{i+1/2}\vee 0)u'_{i+1/2} + (u_{i-1/2}\wedge 0)u'_{i-1/2}\right)
$$

$$\leq 2h\sum_{i\in\mathbb{Z}}\left(e^{-\kappa|j-i|}\right)\left(u_{i+1/2}^2+(u'_{i+1/2})^2+u_{i-1/2}^2+(u'_{i-1/2})^2\right),$$

and thus, using (6.3) and (6.5),

$$\begin{aligned}\Delta x\sum_j\left|P'_{1,j}\right| &\leq 2h\Delta x\sum_{i,j}\left(e^{-\kappa|j-i|}\right)\left(u_{i+1/2}^2+(u'_{i+1/2})^2+u_{i-1/2}^2+(u'_{i-1/2})^2\right)\\ &\leq C\left(\left\|\left\{e^{-\kappa|j|}\right\}_j\right\|_{\ell^1}+1\right)\left(\left\|\left\{u_{j+1/2}\right\}_j\right\|_{\ell^2}^2+\left\|\left\{u'_{j+1/2}\right\}_j\right\|_{\ell^2}^2\right),\end{aligned}$$

where $C>0$ is a constant independent of $\Delta x$. From (9.8) and (5.1) it follows that $\left\{u'_{j+1/2}\right\}_{j\in\mathbb{Z}}$ is bounded in $L^2(0,T;\ell^2)$, which implies (9.11).

To prove (9.13) we use (8.2) with $f(q)=\frac{q^2}{2}$ to obtain

$$\begin{aligned}P'_{2,j} &= 2h\sum_{i\in\mathbb{Z}}\left(e^{-\kappa|j-i|}\right)\frac{d}{dt}\left(\frac{q_i^2}{2}\right)\\ &= -2h\sum_{i\in\mathbb{Z}}\left(e^{-\kappa|j-i|}\right)\left(D_-\left[\left(u_{i+1/2}\vee 0\right)\frac{q_i^2}{2}\right]+D_+\left[\left(u_{i-1/2}\wedge 0\right)\frac{q_i^2}{2}\right]\right)\\ &\quad -2h\sum_{i\in\mathbb{Z}}\left(e^{-\kappa|j-i|}\right)\Bigg[P_iq_i-\left(u_{j+1/2}\vee 0\right)^2q_i-\left(u_{i-1/2}\wedge 0\right)^2q_i\\ &\qquad\qquad +\frac{\Delta x}{2}\left[\left(u_{i-1/2}\vee 0\right)(D_-q_i)^2-\left(u_{i+1/2}\wedge 0\right)(D_+q_i)^2\right]\Bigg]\\ &= -2h\sum_{i\in\mathbb{Z}}\left(D_-\left(e^{-\kappa|j-i|}\right)\right)\left(u_{i-1/2}\vee 0\right)\frac{q_{i-1}^2}{2}\\ &\quad -2h\sum_{i\in\mathbb{Z}}\left(D_+\left(e^{-\kappa|j-i|}\right)\right)\left(u_{i+1/2}\wedge 0\right)\frac{q_{i+1}^2}{2}\\ &\quad -2h\sum_{i\in\mathbb{Z}}\left(e^{-\kappa|j-i|}\right)\Bigg[P_iq_i-\left(u_{i+1/2}\vee 0\right)^2q_i-\left(u_{i-1/2}\wedge 0\right)^2q_i\\ &\qquad\qquad +\frac{\Delta x}{2}\left[\left(u_{i-1/2}\vee 0\right)(D_-q_i)^2-\left(u_{i+1/2}\wedge 0\right)(D_+q_i)^2\right]\Bigg],\end{aligned}$$

and hence (9.13) follows from (5.1), (5.4), (6.3), and (6.5).

Since $\partial_tP_{\Delta x}(t,x)=P'_j(t)+(x-x_j)D_+P'_j(t)$, $t\geq 0$, $x\in I_{j+1/2}$, the bound on $\{\partial_tP_{\Delta x}\}_{\Delta x>0}$ follows from (9.11)-(9.14). As a result the sequence $\{P_{\Delta x}\}_{\Delta x>0}$ is bounded in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R})$. Combining this with the $L^\infty$-bound in Lemma 6.1 yields the existence of a subsequence that converges as claimed in (9.10). □

## 10. Strong convergence result

Endowed with the sequence $\{q_j(t)\}_{j\in\mathbb{Z}}$ defined by (3.1)-(3.7), we introduce the function

$$q_{\Delta x}(t,x)=q_j(t),\quad t\geq 0,\ x\in I_j,\ j\in\mathbb{Z}. \tag{10.1}$$

Observe that

$$\partial_xu_{\Delta x}(t,x)=q_{\Delta x}(t,x),\quad t\geq 0,\ x\in I_j,\ j\in\mathbb{Z}. \tag{10.2}$$

The ensuing lemma is a straightforward consequence of the main estimates established in earlier sections.

**Lemma 10.1.** *Fix any $1 \le p < 3$ and $1 \le r < \frac{3}{2}$. Then there exist two functions $q \in L^p_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R})$, $\overline{q^2} \in L^r_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R})$ such that along a subsequence as $\Delta x \downarrow 0$*

$$q_{\Delta x} \overset{\star}{\rightharpoonup} q \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R})), \qquad q_{\Delta x} \rightharpoonup q \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}), \tag{10.3}$$

$$q^2_{\Delta x} \rightharpoonup \overline{q^2} \text{ in } L^r_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}). \tag{10.4}$$

*Moreover,*

$$q^2(t,x) \le \overline{q^2}(t,x) \text{ for a.e. } (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{10.5}$$

*and*

$$\partial_x u = q \quad \text{in the sense of distributions on } [0,\infty) \times \mathbb{R}. \tag{10.6}$$

*Finally, there is a positive constant $C$ depending only on $\|u_0\|_{H^1(\mathbb{R})}$ such that*

$$q(t,x) \le \frac{2}{t} + C, \qquad t > 0,\ x \in \mathbb{R}. \tag{10.7}$$

*Proof.* Claims (10.3), (10.4) are direct consequences of (10.1), (3.7), and Lemmas 5.1 and 8.1. Claim (10.5) is true thanks to (10.4), cf. Lemma 2.1, while (10.6) is a consequence of (10.2) and Lemma 9.1. Finally, by (7.1),

$$q_{\Delta x}(t,x) \le \frac{2}{t} + C, \qquad t \ge 0, \quad x \in \mathbb{R},$$

and hence, because of (10.3), cf. again Lemma 2.1, claim (10.7) follows. □

In view of the weak convergences stated in (10.3), we can assume that for any function $f \in C^1(\mathbb{R})$ with $f'$ bounded

$$\begin{aligned} f(q_{\Delta x}) &\overset{\star}{\rightharpoonup} \overline{f(q)} \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R})), \\ f(q_{\Delta x}) &\rightharpoonup \overline{f(q)} \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}),\ 1 \le p < 3, \end{aligned} \tag{10.8}$$

where the same subsequence of $\Delta x \downarrow 0$ applies to any $f$ from the specified class.

In what follows, we let $\overline{qf(q)}$ and $\overline{f'(q)q^2}$ denote the weak limits of $q_{\Delta x} f(q_{\Delta x})$ and $f'(q_{\Delta x}) q^2_{\Delta x}$, respectively, in $L^r_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R})$, $1 \le r < \frac{3}{2}$.

**Lemma 10.2.** *For any convex function $f \in C^1(\mathbb{R})$ with $f'$ bounded we have that*

$$\begin{aligned} &\iint_{\mathbb{R}_+ \times \mathbb{R}} \left( \overline{f(q)} \partial_t \varphi + u \overline{f(q)} \partial_x \varphi \right) dx\, dt \\ &\qquad \ge \iint_{\mathbb{R}_+ \times \mathbb{R}} \left( \frac{1}{2} \overline{f'(q) q^2} - \overline{q f(q)} + \left(P - u^2\right) \overline{f'(q)} \right) \varphi \, dx\, dt, \end{aligned} \tag{10.9}$$

*for any nonnegative $\varphi \in C^\infty_c(\mathbb{R}_+ \times \mathbb{R})$.*

*Proof.* Set $\varphi_j(t) = \frac{1}{\Delta x} \int_{I_j} \varphi(x,t)\, dx$. We multiply (8.2) by $\Delta x \varphi_j$, sum over $j \in \mathbb{Z}$, integrate over $t \in \mathbb{R}_+$, and take into account the convexity of $f$. After a partial integration and a partial summation, the final result reads

$$\begin{aligned} &\int_{\mathbb{R}_+} \Delta x \sum_j f(q_j) \varphi_j' \, dt \\ &+ \int_{\mathbb{R}_+} \Delta x \sum_j \Big[ \left(u_{j+1/2} \vee 0\right) f(q_j) D_+ \varphi_j + \left(u_{j-1/2} \wedge 0\right) f(q_j) D_- \varphi_j \Big] dt \end{aligned}$$

$$+\int_{\mathbb{R}_+} \Delta x \sum_j \Big[ q_j f(q_j) - \frac{q_j^2}{2} f'(q_j) \Big] \varphi_j \, dt$$
$$+\int_{\mathbb{R}_+} \Delta x \sum_j \Big[ \big(u_{j+1/2} \vee 0\big)^2 + \big(u_{j-1/2} \wedge 0\big)^2 - P_j \Big] f'(q_j) \varphi_j \, dt \geq 0.$$

We write this inequality as

$$E_1 + E_2 + E_3 + E_4 \geq 0.$$

Clearly, by the choice of $\varphi_j$ and (10.8),

$$E_1 = \int_{\mathbb{R}_+} \int_{\mathbb{R}} f(q_{\Delta x}) \partial_t \varphi \, dx \, dt \to \int_{\mathbb{R}_+} \int_{\mathbb{R}} \overline{f(q)} \partial_t \varphi \, dx dt \quad \text{as } \Delta x \downarrow 0.$$

Next,

$$E_2 = \int_{\mathbb{R}_+} \int_{\mathbb{R}} u_{\Delta x} f(q_{\Delta x}) \partial_x \varphi \, dx \, dt + E_{2,1} + E_{2,2},$$

where

$$E_{2,1} = \int_{\mathbb{R}_+} \sum_j \int_{I_j} \Big[ \big(u_{j+1/2} \vee 0\big) f(q_j) D_+ \varphi_j - (u_{\Delta x} \vee 0) f(q_{\Delta x}) \partial_x \varphi \Big] \, dx \, dt$$

and

$$E_{2,2} = \int_{\mathbb{R}_+} \sum_j \int_{I_j} \Big[ \big(u_{j-1/2} \wedge 0\big) f(q_j) D_- \varphi_j - (u_{\Delta x} \wedge 0) f(q_{\Delta x}) \partial_x \varphi \Big] \, dx \, dt.$$

We decompose $E_{2,1}$ as follows:

$$\begin{aligned} E_{2,1} &= \int_{\mathbb{R}_+} \sum_j \int_{I_j} \Big[ \big(u_{j+1/2} \vee 0\big) - (u_{\Delta x} \vee 0) \Big] f(q_{\Delta x}) \partial_x \varphi \, dx \, dt \\ &\qquad + \int_{\mathbb{R}_+} \sum_j \int_{I_j} \big(u_{j+1/2} \vee 0\big) f(q_{\Delta x}) \Big[ D_+ \phi - \partial_x \varphi \Big] \, dx \, dt \\ &=: E_{2,1,1} + E_{2,1,2}. \end{aligned}$$

For $x \in I_j$, we have by (4.3)

$$\left| \big(u_{j+1/2} \vee 0\big) - (u_{\Delta x} \vee 0) \right| \leq \left| u_{j+1/2} - u_{\Delta x} \right| \leq C_1 \Delta x \, |q_j| \, . \tag{10.10}$$

Hence, keeping in mind that $|f(q)| = \mathcal{O}(1+|q|)$ for all $q \in \mathbb{R}$ and using (10.1), (5.1),

$$|E_{2,1,1}| \leq C_2 \Delta x \, \|\partial_x \varphi\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \iint_{\mathrm{supp}\,(\varphi)} q_{\Delta x}^2 \, dx \, dt \leq C_2 \Delta x,$$

where the constant $C_2$ depends on $\varphi$ but not on $\Delta x$. For $x \in I_j$,

$$|D_+ \phi - \partial_x \varphi| \leq C_3 \Delta x,$$

where $C_3$ depends on $\varphi$ but not $\Delta x$, and thus by (10.1) and (5.1),

$$|E_{2,1,2}| \leq C_4 \Delta x \iint_{\mathrm{supp}\,(\varphi)} q_{\Delta x} \, dx \, dt \leq C_5 \Delta x,$$

where the final constant $C_5$ depends on $\varphi$ but not on $\Delta x$. To summarize, we have proved

$$|E_{2,1}| = \mathcal{O}(\Delta x) \to 0 \text{ as } \Delta x \downarrow 0.$$

Similarly, we can prove

$$|E_{2,2}| = \mathcal{O}(\Delta x) \to 0 \text{ as } \Delta x \downarrow 0.$$

Consequently, by (9.2) and (10.8),

$$E_2 \to \int_{\mathbb{R}_+}\int_{\mathbb{R}} u\overline{f(q)}\partial_x\varphi\,dx\,dt \quad \text{as } \Delta x \downarrow 0.$$

We can take the limit in $E_3$ directly:

$$\begin{aligned} E_3 &= \iint_{\mathbb{R}_+\times\mathbb{R}} \Big[q_{\Delta x} f(q_{\Delta x}) - \frac{q_{\Delta x}^2}{2} f'(q_{\Delta x})\Big]\,dx\,dt \\ &\overset{\Delta x\downarrow 0}{\longrightarrow} \iint_{\mathbb{R}_+\times\mathbb{R}} \Big[\overline{qf(q)} - \overline{f'(q)q^2}\Big]\,dx\,dt. \end{aligned}$$

Finally, let us analyze $E_4$, which we write as the sum of four terms:

$$E_4 = \iint_{\mathbb{R}_+\times\mathbb{R}} \left[u_{\Delta x}^2 - P_{\Delta x}\right] f'(q_{\Delta x})\varphi\,dx\,dt + E_{4,1} + E_{4,2} - E_{4,3},$$

where

$$\begin{aligned} E_{4,1} &= \int_{\mathbb{R}_+}\sum_j \int_{I_j} \Big[\big(u_{j+1/2}\vee 0\big)^2 f'(q_j)\varphi_j - (u_{\Delta x}\vee 0)^2 f'(q_{\Delta x})\varphi\Big]\,dx\,dt, \\ E_{4,2} &= \int_{\mathbb{R}_+}\sum_j \int_{I_j} \Big[\big(u_{j-1/2}\wedge 0\big)^2 f'(q_j)\varphi_j - (u_{\Delta x}\wedge 0)^2 f'(q_{\Delta x})\varphi\Big]\,dx\,dt, \\ E_{4,3} &= \int_{\mathbb{R}_+}\sum_j \int_{I_j} \Big[P_j f'(q_j)\varphi_j - P_{\Delta x} f'(q_{\Delta x})\varphi\Big]\,dx\,dt. \end{aligned}$$

Let us write

$$\begin{aligned} E_{4,1} &= \underbrace{\int_{\mathbb{R}_+}\sum_j \int_{I_j} \Big[\big(u_{j+1/2}\vee 0\big)^2 - (u_{\Delta x}\vee 0)^2\Big] f'(q_{\Delta x})\varphi\,dx\,dt}_{E_{4,1,1}} \\ &\quad + \underbrace{\int_{\mathbb{R}_+}\sum_j \int_{I_j} \big(u_{j+1/2}\vee 0\big)^2 f'(q_{\Delta x})\Big[\varphi_j - \varphi\Big]\,dx\,dt}_{E_{4,1,2}}. \end{aligned}$$

In view of (5.4) and (10.10), we have the following estimate for $x \in I_j$:

$$\Big|\big(u_{j+1/2}\vee 0\big)^2 - (u_{\Delta x}\vee 0)^2\Big| \le \left|u_{j+1/2} - u_{\Delta x}\right| \left|u_{j+1/2} + u_{\Delta x}\right| \le C\Delta x\,|q_j|\,,$$

where the constant $C$ does not depend on $\Delta x$. Hence we infer $|E_{4,1,1}| = \mathcal{O}(\Delta x)$ (cf. the treatment of $E_{2,1,1}$). As $|\varphi_j - \varphi| = \mathcal{O}(\Delta x)$, we can argue as we did with $E_{2,1,2}$ to reach the conclusion $E_{4,1,2} = O(\Delta x)$. Therefore, $E_{4,1} = \mathcal{O}(\Delta x)$.

Along the same lines we can prove that $E_{4,2} = O(\Delta x)$.

Similarly to the estimates of $E_{4,1}$ and $E_{4,2}$ we can show that $E_{4,3} = \mathcal{O}(\Delta x)$, by exploiting (9.9) and (6.2) to conclude that $|P_j - P_{\Delta x}| = \mathcal{O}(\Delta x)$ for $x \in I_{j+1/2}$.

By the previous calculations,

$$E_4 \to \iint_{\mathbb{R}_+\times\mathbb{R}} \left[u^2 - P\right] \overline{f'(q)}\varphi\,dx\,dt \quad \text{as } \Delta x \downarrow 0.$$

This concludes the proof of (10.9). $\square$

We know that $\left\{q_{\Delta x}^2\right\}_{\Delta x>0}$ is a subset of $L^\infty(\mathbb{R}_+;L^1(\mathbb{R}))\cap L^r_{\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R})$, for $1\le r<\frac{3}{2}$. Additionally, from (8.2) with $f(q)=\frac{q^2}{2}$ (cf. the proof of Lemma 10.2), it follows that $\left\{\partial_t q_{\Delta x}^2\right\}_{\Delta x>0}$ is bounded in $L^1(0,T;W^{-1,1}(\mathbb{R}))$ for each $T>0$. The Ascoli-Arzelà theorem then implies the following convergence for each $\varphi\in C_c^\infty(\mathbb{R})$:

$$\int_{\mathbb{R}} q_{\Delta x}^2\varphi\,dx \to \int_{\mathbb{R}} \overline{q^2}\varphi\,dx \quad \text{uniformly on compact subsets of } [0,\infty), \tag{10.11}$$

and thus

$$[0,\infty)\ni t\mapsto \int_{\mathbb{R}} \overline{q^2}\varphi\,dx \quad \text{is continuous on } [0,\infty). \tag{10.12}$$

The statements (10.11) and (10.12) hold with $q_{\Delta x}^2$ and $\overline{q^2}$ replaced respectively by $f(q_{\Delta x})$ and $\overline{f(q_{\Delta x})}$, for any convex function $f\in C^1(\mathbb{R})$ with $f'$ bounded.

**Lemma 10.3.** *Let $q$ and $\overline{q^2}$ be the weak limits identified in Lemma 10.1. Then*

$$\iint_{\mathbb{R}_+\times\mathbb{R}} (q\partial_t\varphi+uq\partial_x\varphi)\,dx\,dt=\iint_{\mathbb{R}_+\times\mathbb{R}}\left(-\frac{1}{2}\overline{q^2}+\left(P-u^2\right)\right)\varphi\,dx\,dt, \tag{10.13}$$

*for any $\varphi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R})$.*

*Proof.* Starting off from (8.2) with $f(q)=q$, we argue as in the proof of Lemma 10.2 to conclude the validity of (10.13). □

The succeeding lemma tells us in which sense the weak limits singled out in Lemma 10.1 satisfy the initial data.

**Lemma 10.4.** *Let $q$ and $\overline{q^2}$ be the weak limits identified in Lemma 10.1. Then*

$$\begin{aligned}\lim_{t\downarrow 0}\int_{\mathbb{R}} q^2(t,x)\,dx&=\int_{\mathbb{R}}(\partial_x u_0)^2\,dx,\\ \lim_{t\downarrow 0}\int_{\mathbb{R}} \overline{q^2}(t,x)\,dx&=\int_{\mathbb{R}}(\partial_x u_0)^2\,dx.\end{aligned} \tag{10.14}$$

*Proof.* In view of (10.6) and (9.3), a couple of integration-by-parts will reveal that

$$\lim_{t\downarrow 0}\int_{\mathbb{R}} q(t,x)\varphi(x)\,dx=\int_{\mathbb{R}}\partial_x u_0\varphi\,dx,\qquad \forall\varphi\in C_c^\infty(\mathbb{R}).$$

Since $q\in L^\infty(\mathbb{R}_+;L^2(\mathbb{R}))$ this translates into the statement

$$q(t,\cdot)\rightharpoonup\partial_x u_0 \text{ in } L^2(\mathbb{R}) \text{ as } t\downarrow 0.$$

Hence, cf. Lemma 2.1,

$$\int_{\mathbb{R}}(\partial_x u_0)^2\,dx\le\liminf_{t\downarrow 0}\int_{\mathbb{R}} q^2(t,x)\,dx. \tag{10.15}$$

On the other hand, (9.1) tells us that $u_{\Delta x}(t,\cdot)\rightharpoonup u(t,\cdot)$ in $H^1(\mathbb{R})$ for a.e. $t>0$, and thereby, using also (9.2), (9.4), (10.4), and Lemma 2.1,

$$\int_{\mathbb{R}}(u(t,x))^2\,dx+\int_{\mathbb{R}}\overline{q^2}(t,x)\,dx\le\int_{\mathbb{R}}u_0^2\,dx+\int_{\mathbb{R}}(\partial_x u_0)^2\,dx. \tag{10.16}$$

Since (10.12) holds, this inequality is valid for all $t>0$. By exploiting the continuity of $u$ (see Lemma 9.1), (10.16) yields

$$\limsup_{t\downarrow 0}\int_{\mathbb{R}}\overline{q^2}(t,x)\,dx\le\int_{\mathbb{R}}\left(\partial_x u_0\right)^2\,dx. \tag{10.17}$$

Clearly, (10.5), (10.15), and (10.17) imply (10.14). □

We are now in a position to conclude the strong convergence of $\{q_{\Delta x}\}_{\Delta x>0}$.

**Lemma 10.5.** *Let $q$ and $\overline{q^2}$ be the weak limits identified in Lemma* 10.1. *Then*

$$\overline{q^2}(t,x) = q^2(t,x) \text{ for a.e. } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{10.18}$$

*Consequently, as $\Delta x \downarrow 0$,*

$$q_{\Delta x} \to q \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}) \text{ and a.e. in } \mathbb{R}_+ \times \mathbb{R}. \tag{10.19}$$

*Proof.* Lemma 10.2 tells us that for any convex function $f \in C^1(\mathbb{R})$ with $f'$ bounded there holds

$$\partial_t \overline{f(q)} + \partial_x \left( u \overline{f(q)} \right) \le \overline{q f(q)} - \frac{1}{2} \overline{f'(q) q^2} + \left( u^2 - P \right) \overline{f'(q)}, \tag{10.20}$$

in the sense of distributions on $\mathbb{R}_+ \times \mathbb{R}$. Moreover, by Lemma 10.3,

$$\partial_t q + \partial_x (uq) = \frac{1}{2} \overline{q^2} + u^2 - P, \tag{10.21}$$

in the sense of distributions on $\mathbb{R}_+ \times \mathbb{R}$. Equipped with (10.20), (10.21), Lemma 10.14, and (10.7), we can argue exactly as in Xin and Zhang [38] to arrive at (10.18). In view of Lemma 2.1, claim (10.19) follows immediately from (10.18) and (9.6). □

## 11. Concluding the proof of Theorem 4.1

Lemma 9.1, Lemma 10.1, and (4.2) show that the strong $H^1$ - limit $u$ satisfies conditions (i), (iii), (iv), and (4.2) of Definition 4.1. It remains to prove that $u$ satisfies condition (ii), i.e., the weak formulation (4.1).

We start by deriving a divergence-form version of the scheme (3.1). To this end, introduce the functions $f_\vee$, $f_\wedge$ defined by

$$f_\vee'(u) = u \vee 0, \quad f_\vee(0) = 0, \qquad f_\wedge'(u) = u \wedge 0, \quad f_\wedge(0) = 0,$$

i.e., $f_\vee(u) = \frac{1}{2}(u \vee 0)^2$ and $f_\wedge(u) = \frac{1}{2}(u \wedge 0)^2$. Observe that $f_\vee$ and $f_\wedge$ are piecewise $C^2$, and the absolute value of the second derivatives are bounded by 1. By the discrete chain rule,

$$\left(u_{j+1/2} \vee 0\right) D_- u_{j+1/2} = D_- f_\vee(u_{j+1/2}) + \mathcal{O}\left( \Delta x \left( D_- u_{j+1/2} \right)^2 \right)$$

and

$$\left(u_{j+1/2} \wedge 0\right) D_+ u_{j+1/2} = D_+ f_\wedge(u_{j+1/2}) + \mathcal{O}\left( \Delta x \left( D_+ u_{j+1/2} \right)^2 \right).$$

Consequently, we can replace (3.1) by

$$\begin{aligned} &u'_{j+1/2} + D_- f_\vee(u_{j+1/2}) + D_+ f_\wedge(u_{j+1/2}) + D_+ P_j \\ &\qquad = \mathcal{O}\left( \Delta x \left\{ \left( D_- u_{j+1/2} \right)^2 + \left( D_+ u_{j+1/2} \right)^2 \right\} \right). \end{aligned} \tag{11.1}$$

Observe that

$$D = \frac{D_- + D_+}{2}, \qquad \Delta x D_- D_+ = D_+ - D_-, \qquad f_\vee + f_\wedge = \frac{u^2}{2}. \tag{11.2}$$

Using these identities, we can restate (11.1) as

$$\begin{aligned} u'_{j+1/2} &+ D_-\left[\frac{u^2_{j+1/2}}{4} + \frac{1}{2}\left(f_\vee(u_{j+1/2}) - f_\wedge(u_{j+1/2})\right)\right] \\ &+ D_+\left[\frac{u^2_{j+1/2}}{4} + \frac{1}{2}\left(f_\wedge(u_{j+1/2}) - f_\vee(u_{j+1/2})\right)\right] + D_+P_j \\ &= \mathcal{O}\left(\Delta x\left\{\left(D_-u_{j+1/2}\right)^2 + \left(D_+u_{j+1/2}\right)^2\right\}\right). \end{aligned} \tag{11.3}$$

Using, cf. (11.2),

$$\begin{aligned} &D_-\left(f_\vee(u_{j+1/2}) - f_\wedge(u_{j+1/2})\right) + D_+\left(f_\wedge(u_{j+1/2}) - f_\vee(u_{j+1/2})\right) \\ &\quad = \Delta x D_-D_+f_\wedge(u_{j+1/2}) - \Delta x D_-D_+f_\vee(u_{j+1/2}), \end{aligned}$$

equation (11.3) becomes

$$\begin{aligned} u'_{j+1/2} &+ D\left(\frac{u^2_{j+1/2}}{2}\right) + D_+P_j \\ &= \mathcal{O}\left(\Delta x\left\{\left(D_-u_{j+1/2}\right)^2 + \left(D_+u_{j+1/2}\right)^2\right\}\right) \\ &\quad + \Delta x\left\{D_-D_+f_\vee(u_{j+1/2}) - D_-D_+f_\wedge(u_{j+1/2})\right\}. \end{aligned} \tag{11.4}$$

Fix $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$, and set $\varphi_j(t) = \frac{1}{\Delta x}\int_{I_j}\varphi(x,t)\,dx$. We multiply (11.4) by $\Delta x\varphi_j$, sum over $j \in \mathbb{Z}$, and integrate over $t \in \mathbb{R}_+$. After a partial integration and a partial summation, the final result reads

$$\begin{aligned} &\underbrace{\int_{\mathbb{R}_+}\Delta x\sum_j u_{j+1/2}\varphi'_j\,dt}_{=:E_1} + \underbrace{\int_{\mathbb{R}_+}\Delta x\sum_j \frac{u^2_{j+1/2}}{2}D\varphi_j\,dt}_{=:E_2} \\ &\qquad + \underbrace{\int_{\mathbb{R}_+}\Delta x\sum_j P_jD_-\varphi_j\,dt}_{=:E_3} = \mathcal{O}(\Delta x), \end{aligned} \tag{11.5}$$

where the right-hand side is a consequence of (5.1).

First, since $\left|u_{j+1/2} - u_{\Delta x}\right| \le C\Delta x\,|q_j|$, cf. (4.3), and using Lemmas 5.1 and 9.1,

$$\begin{aligned} E_1 &= \int_{\mathbb{R}_+}\int_{\mathbb{R}} u_{\Delta x}\partial_t\varphi\,dt + \int_{\mathbb{R}_+}\sum_j\int_{I_j}\left(u_{j+1/2} - u_{\Delta x}\right)\partial_t\varphi\,dx\,dt \\ &= \iint_{\mathbb{R}_+\times\mathbb{R}} u_{\Delta x}\partial_t\varphi\,dx\,dt + \mathcal{O}(\Delta x) \overset{\Delta x\downarrow 0}{\longrightarrow} \iint_{\mathbb{R}_+\times\mathbb{R}} u\partial_t\varphi\,dx\,dt. \end{aligned}$$

Next,

$$E_2 = \underbrace{\iint_{\mathbb{R}_+\times\mathbb{R}}\frac{u^2_{\Delta x}}{2}u_{\Delta x}\partial_t\varphi\,dt}_{E_{2,1}} + \underbrace{\int_{\mathbb{R}_+}\sum_j\int_{I_j}\left(\frac{u^2_{j+1/2}}{2}D\varphi_j - \frac{u^2_{\Delta x}}{2}\partial_x\varphi\right)dx\,dt}_{E_{2,2}},$$

Let us analyze the term $E_{2,2}$. We have that $E_{2,2} = E_{2,2,1} + E_{2,2,2}$, where

$$E_{2,2,1} = \int_{\mathbb{R}_+} \sum_j \int_{I_j} \left( \frac{u_{j+1/2}^2}{2} - \frac{u_{\Delta x}^2}{2} \right) \partial_x \varphi \, dx \, dt,$$

$$E_{2,2,2} = \int_{\mathbb{R}_+} \sum_j \int_{I_j} \frac{u_{j+1/2}^2}{2} \left( D\varphi_j - \partial_x \varphi \right) \, dx \, dt.$$

Since, by (5.4),

$$\left| \frac{u_{j+1/2}^2}{2} - \frac{u_{\Delta x}^2}{2} \right| \leq C\Delta x \, |q_j| \,, \quad |D\varphi_j - \partial_x \varphi| \leq C\Delta x, \quad x \in I_j, \, j \in \mathbb{Z},$$

where $C$ does not depend on $\Delta x$, we use again (5.1) to conclude $|E_{2,2}| = \mathcal{O}(\Delta x)$.

It remains to analyze $E_3$. We have

$$E_3 = \int_{\mathbb{R}_+} 2h\Delta x \sum_{j,i} \left( e^{-\kappa |j-i|} \right) \left( (u_{i+1/2} \vee 0)^2 + (u_{i-1/2} \wedge 0)^2 + \frac{q_i^2}{2} \right) D_- \varphi_j \, dt,$$

Due to (6.3) and (6.5), we have

$$\lim_{\Delta x \downarrow 0} \frac{h}{\Delta x} = \lim_{\Delta x \downarrow 0} \frac{1}{\Delta x + 2\frac{1-e^{-\kappa}}{\Delta x}} = \frac{1}{2}. \tag{11.6}$$

Moreover, for all $i, j \in \mathbb{Z}$,

$$\begin{aligned} e^{-\kappa |j-i|} &= (e^{\kappa})^{-|j-i|} = \left( 1 + \frac{(\Delta x)^2}{2} + \frac{\Delta x}{2} \sqrt{4 + \Delta x^2} \right)^{-|j-i|} \\ &= \left( 1 + \Delta x + \mathcal{O}(\Delta x^2) \right)^{-|j-i|} = \left( 1 + \mathcal{O}(\Delta x^2) \right) e^{-|x_i - x_j|}, \end{aligned} \tag{11.7}$$

where the final result comes from replacing $1 + \Delta x$ with $e^{\Delta x} + \mathcal{O}(\Delta x^2)$. By (11.6), (11.7), Lemmas 9.1 and 10.1, and arguing along the above lines, we infer

$$\lim_{\Delta x \downarrow 0} E_3 = \frac{1}{2} \iint_{\mathbb{R}_+ \times \mathbb{R}} \left[ \int_{\mathbb{R}} e^{-|x-y|} \left( (u(t,y))^2 + \frac{1}{2} (q(t,y))^2 \right) \, dy \right] \partial_x \varphi(t,x) \, dx \, dt.$$

Summarizing, our calculations show that by sending $\Delta x \downarrow 0$ in (11.5) we obtain (4.1). This concludes the proof of Theorem 4.1.

## 12. Numerical examples

The finite difference scheme we analyzed in previous sections is semi-discrete as well as infinite dimensional, and to use it we must integrate the defining ODE numerically and impose some numerical boundary conditions. We chose to do this by a simple forward Euler method, which results in the scheme

$$\begin{aligned} \frac{u_{j+1/2}^{n+1} - u_{j+1/2}^n}{\Delta t} &+ \left( u_{j+1/2}^n \vee 0 \right) D_- u_{j+1/2}^n \\ &+ \left( u_{j+1/2}^n \wedge 0 \right) D_+ u_{j+1/2}^n + D_+ P_j^n = 0 \quad \text{for } |j| < J_{\Delta x}, \end{aligned} \tag{12.1}$$

where $J_{\Delta x} \Delta x = X$ and the computational domain is $[-X, X]$. We set $u_{j+1/2}^n = u_{J_{\Delta x}+1/2}$ for $j \geq J_{\Delta x}$ and $u_{j+1/2}^n = u_{-J_{\Delta x}+1/2}$ for $j \leq -J_{\Delta x}$. Here $\Delta t$ is a small positive number (the time step), and we used $\Delta t = 0.5\Delta x$. The sequence $\left\{P_j^n\right\}_{j\in\mathbb{Z}}$ is defined by

$$-D_-D_+P_j^n + P_j^n = \left(u_{j+1/2}^n \vee 0\right)^2 + \left(u_{j-1/2}^n \wedge 0\right)^2 + \frac{1}{2}\left(D_-u_{j+1/2}^n\right)^2.$$

The first example uses a two-peakon solution with initial data

$$u_0(x) = 2e^{-|x+4|} + e^{-|x-4|}. \tag{12.2}$$

The exact solution is given by

$$u(t,x) = \sum_{j=1}^{2} p_j(t)e^{-|x-q_j(t)|},$$

where $(p,q)$ solves the system of ODEs

$$\begin{aligned} q_i'(t) &= \sum_{j=1}^{2} p_j(t)p_j(t)e^{-|q_i(t)-q_j(t)|} \\ p_i'(t) &= p_i(t)\sum_{j=1}^{2} p_j(t)\operatorname{sign}\left(q_i(t)-q_j(t)\right)e^{-|q_i(t)-q_j(t)|}. \end{aligned} \tag{12.3}$$

The "exact" solution of (12.3) is calculated using a high-order Runge-Kutta method.

The example is a case of a two-peakon collision, where the faster peakon overtakes the slower peakon. See Figure 1 where we show the exact solution and a numerical approximation with 1024 gridpoints in the interval $[-15,45]$. From Figure 1 it is clear that the quality of the approximate solution is not very good. However, to resolve such a two peakon collision is a difficult numerical problem, see, e.g., [1] and [33]. Our scheme requires a very small mesh size $\Delta x$ to compute reasonable solutions for this example, which however is not surprising and appears to be the case with other schemes in the literature as well. To improve the accuracy, in particular at a wave crest, we could attempt to build a high-order version of our scheme that also utilizes an adaptive mesh strategy, see [1] for a finite volume scheme along these lines, which achieves third order accuracy by employing Marquina's local hyperbolic reconstruction technique.

In what follows, we use the simpler one peakon solution to measure the (rate of) convergence of the scheme (12.1). We measure the relative $H^1$ - error defined as

$$\mathrm{err}_{h^1} = \max_{t\in[0,T]} \frac{\|u_{\Delta x} - u_{\Delta x}^e\|_{h^1}}{\|u^e\|_{h^1}},$$

as well as the $\ell^\infty$ - and $\ell^1$ - errors defined as

$$\begin{aligned} \mathrm{err}_{\ell^\infty} &= \max_{n\Delta t\in[0,T]} \frac{\max_j \left|u_{j+1/2}^n - u^e(n\Delta t, x_{j+1/2})\right|}{\max_j \left|u^e(n\Delta t, x_{j+1/2})\right|}. \\ \mathrm{err}_{\ell^1} &= \max_{n\Delta t\in[0,T]} \frac{\|u_{\Delta x} - u_{\Delta x}^e\|_{\ell^1}}{\|u^e\|_{\ell^1}}. \end{aligned}$$

Here $u^e$ is the piecewise linear function defined by interpolating the exact solution linearly between the points $\left\{x_{j+1/2}\right\}_{j\in\mathbb{Z}}$. As initial data we used $u_0(x) = e^{-|x|}$, which implies $u^e(t,x) = e^{-|x-t|}$. We computed the approximate solutions in the interval $x \in [-15,15]$ for $t \in [0,6.4]$ with mesh sizes $\Delta x = 30/2^n$ for $n = 7,8,9,\ldots$;

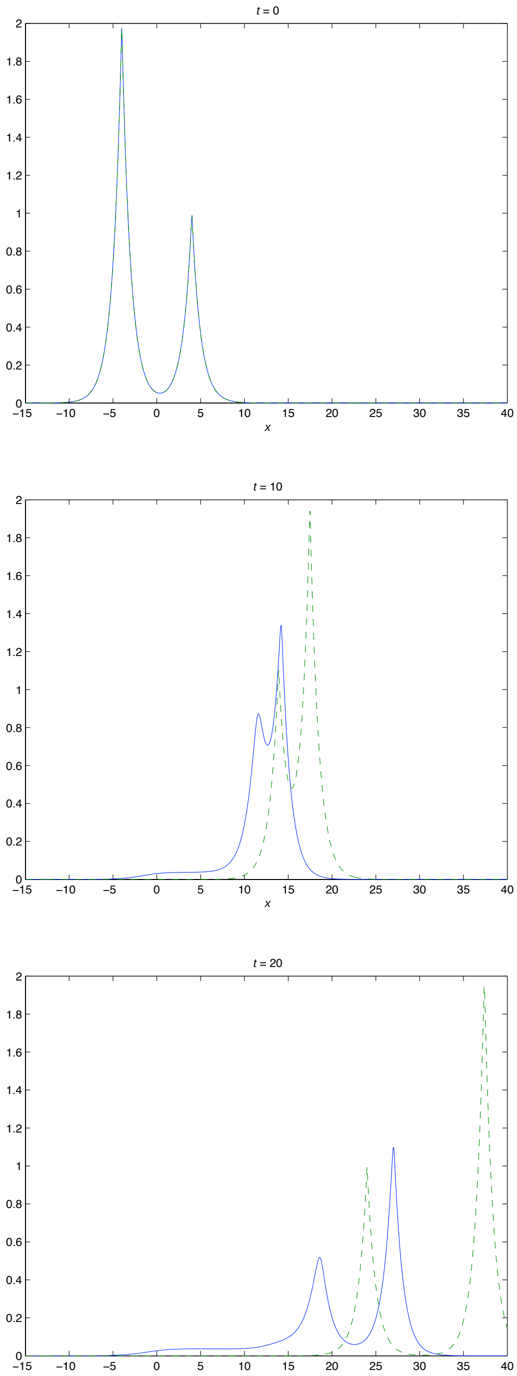


FIGURE 1. The numerical (solid) and exact (dashed) solutions of (12.2), at $t = 0$ (top), $t = 10$ (middle) and $t = 20$ (bottom). For the numerical solution we use $\Delta x = 60/1024$.

The errors are reported in Table 1. This experiment indicate that we do indeed have convergence, but it is not clear whether we have a convergence rate.

| $n$ | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|
| $\mathrm{err}_{h^1}$ | 0.95 | 0.67 | 0.57 | 0.36 | 0.31 | 0.18 | 0.13 | 0.12 |
| $\mathrm{err}_{\ell^\infty}$ | 0.23 | 0.14 | 0.08 | 0.04 | 0.03 | 0.01 | 0.005 | 0.01 |
| $\mathrm{err}_{\ell^1}$ | 0.52 | 0.24 | 0.11 | 0.05 | 0.03 | 0.01 | 0.01 | 0.04 |

| $n$ | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|
| $\mathrm{err}_{h^1}$ | 1.21 | 0.87 | 0.46 | 0.74 | 0.64 | 0.57 | 0.43 | 0.09 |
| $\mathrm{err}_{\ell^\infty}$ | 0.40 | 0.23 | 0.12 | 0.13 | 0.10 | 0.07 | 0.04 | 0.02 |
| $\mathrm{err}_{\ell^1}$ | 1.25 | 0.57 | 0.49 | 0.40 | 0.26 | 0.16 | 0.09 | 0.08 |

TABLE 1. Errors for the single peakon example, for $\Delta x = 30/2^n$, $n = 7, 8, \dots$, $t = 3.2$ (top), $t = 6.4$ (bottom).

In our final example we choose initial data corresponding to a peakon-antipeakon collision:

$$u_0(x) = e^{-|x+4|} - e^{-|x-4|}. \tag{12.4}$$

In this case we have a collision at $t \approx 4.6$. In Figure 2 shows the approximate solution. It is clear that our scheme generates the dissipative solution, and for $t$ larger than the collision time, the approximate solution vanishes.

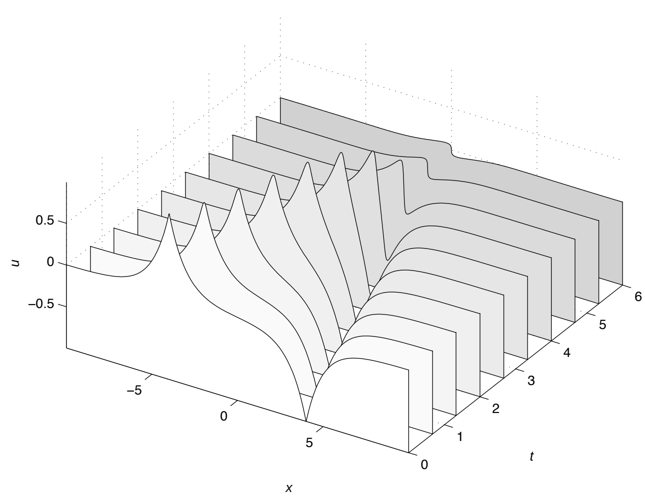


FIGURE 2. The numerical solution to the initial value problem (12.4) for $\Delta x = 20/2^{10}$.

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