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Variational solutions of semilinear wave equations driven by multiplicative fractional Brownian noise

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Abstract

This paper focuses on variational solutions of the Cauchy problem for a nonlinear wave equation with space-time fractional Brownian noise driving force of Hurst index $H \in (1/(\gamma+1), 1)$ and random initial data. γ is the Hölder exponent of the differentiated nonlinearity in the stochastic term of the equation. It is shown that this problem has a unique solution which depends continuously on the random initial data. Moreover, stability with respect to truncation of the infinite dimensional noise is also established.

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1 Introduction

Gaussian processes with independent increments and a certain self-similarity property were first studied by [10] and [11] in which they were called "Wiener spirals". They were later renamed as fractional Brownian motion in [15] where a representation in terms of a stochastic integral with respect to a standard Brownian motion was given. For an encyclopedic review of the intrinsic properties of the process see the forthcoming book [3]. These processes has now found applications in such diverse fields as finance, see e.g. [1] and the references therein, climatology and hydrology [19], temperature modelling [4] and traffic networks [12] to name a few.

In many applications of these processes, the mathematical model is a differential equation in time, possibly also depending on spatial coordinates, in which case the model is a stochastic partial differential equation perturbed by fractional Brownian noise in some sense. An elliptic equation is treated [9] in a white noise setting but more often parabolic equations are on the menu. Some papers are [16] and [18]. To the best of authors knowledge, the only two papers dealing with hyperbolic equations are [7] which considers a 1-dimensional wave equation without diffusion term, and [6], on a classical linear wave equation, both with additive space/time noise.

In general, hyperbolic equations are known for their notorious difficulty due to the fact that the fundamental solution is not smoothing, as in the parabolic case. Moreover, it is not even a function in dimensions greater than two but a distribution. In case the noise is not fractional but Brownian, some works exist, see e.g. [14] for an equation appearing in relativistic quantum mechanics, and an effort has been made to extend the work on martingale measures in [22] to allow for distributional fundamental solutions which are then applicable to wave equations, see [5]. However, since a fractional Brownian process is never a martingale that approach is not applicable here.

The chosen method in this paper is a variational one, using finite-dimensional Galerkin approximations to generate a sequence of functions, converging in a suitable space to a solution of the original equation. This paves the way for a numerical treatment which, however, is lacking in the present paper, in which focus is on existence, uniqueness, and continuity with respect to input data and truncation of the infinite dimensional noie.

The purpose of this paper is to study stochastic wave equations with random initial values formally written as

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= \mathscr{L}u(x,t) + f(x,t,u(x,t),u'(x,t),Du(x,t)) + \sigma(u(x,t))\frac{dB^H}{dt}(x,t), \\ u(x,0) &= g(x), \\ \frac{\partial u}{\partial t}(x,0) &= h(x), \end{aligned} \tag{1.1}$$

with Neumann boundary condition

$$\langle Du(x,t),\, A(x)N(x)\rangle_{\mathbb{R}^d} = 0 \quad (x,t) \in \partial U \times I. \tag{1.2}$$

Here $U \subset \mathbb{R}^d$ is open and bounded, $I = (0,T]$ for some finite T, and $N(x)$ is the exterior unit normal at $x \in \partial U$. The random force, B^H, is a vector valued fractional Brownian process.

Existence will be proved in a variational setting to this Cauchy problem. Continuous dependence on initial data will also be shown.

In Section 2 the fractional Brownian noise is described. In Section 3 the equation is properly formulated. In Section 4 a unique solution to the Galerkin approximated problem is shown to exist and the existence of a solution to the original equation is the goal of Section 5. In Section 6 we prove uniqueness and continuity with respect to initial data. The final Section 7 is on a continuity property with respect to truncation of the noise.

2 The infinite-dimensional noise

The infinite-dimensional noise is the time derivative of the following $H^1(U)$-valued process

$$B^H(x,t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} e_j(x) \beta_j^H(t)$$

where $\{e_j\}_1^\infty$ is an orthonormal basis of $H^1(U)$ such that $\|e_j\|_{H^{1,\infty}(U)} < \infty$ and $\{\beta_j^H\}_{j=1}^\infty$ is a sequence of independent, zero mean fractional Brownian motions on $\mathbb{R}$ with covariance given by

$$r(t,s) = \mathbf{E}\beta^H(t)\beta^H(s) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$$

and Hurst index $H \in (1/(1+\gamma),1)$. The significance of γ will be descussed in the next section. We require the following hypothesis to hold, regarding the continuity properties of the covariance operator:

$$\text{(C)} \qquad \sum_{j=1}^{\infty} \sqrt{\lambda_j} \|e_j\|_{H^{1,\infty}} \quad < \quad \infty.$$

The noise is white in in time and correlated in space which is in agreement with the suspicion that, in many real-world processes, the correlation in time is often of a much smaller magnitude than the spatial correlation, see [2] and [13]. Due to the continuous imbedding $H^{1,\infty}(U) \mapsto H^1(U)$ we have $\{\sqrt{\lambda_j}\}_{j=1}^{\infty} \in \ell^1$:

$$\sum_{j=1}^{\infty} \sqrt{\lambda_j} \quad = \quad \sum_{j=1}^{\infty} \sqrt{\lambda_j} \|e_j\|_{H^1(U)} \quad \leq \quad K \sum_{j=1}^{\infty} \sqrt{\lambda_j} \|e_j\|_{H^{1,\infty}(U)} \quad < \quad \infty.$$

Hence the covariance operator, C, is not only trace class but also its square root is: $Tr\, C^{1/2} < \infty$.

2.1 The pathwise integral with respect to β^H

Assume $\alpha \in (1-H, 1/2)$. The following space will be needed.

Definition 2.1. *Let $a < b$ and denote by $W^{\alpha,1}(a,b)$ the Banach space of measurable functions $f : (a,b) \mapsto \mathbb{R}^{\mathrm{d}}$ such that*

$$\|f\|_{\alpha,1,a,b} \quad = \quad \int_a^b \frac{|f(\tau)|}{(\tau-a)^{\alpha}}\, d\tau + \int_a^b \int_a^{\tau} \frac{|f(\tau)-f(\theta)|}{|\tau-\theta|^{1+\alpha}}\, d\theta\, d\tau \; < \; \infty.$$

If $\{u_t\}_{t\in I}$ is a process with trajectories in $W^{\alpha,1}(I)$, then its pathwise integral with respect to fractional Brownian motion, β^H, exists (see [21]), and we have the estimate

$$\left| \int_I u(t)\, d\beta^H(t) \right| \leq G \|u\|_{\alpha,1,I}, \tag{2.1}$$

where G is a random variable only depending on β and having finite moments of all orders. The estimate is a result from [18] and we will use it frequently. Since we will be dealing with infinitely many fractional Brownian motions, G_j will be the random variable associated with β_j^H via (2.1). A random variable that appear often in this context is the following

$$\widehat{G} \quad = \quad \sum_{j=1}^{\infty} \sqrt{\lambda_j} \|e_j\|_{H^{1,\infty}}\, G_j$$

which is a.s. finite because of condition (C) and since the G_j's are independent and identically distributed with a finite moment.

3 The equation

The operator $\mathscr{L} = \mathscr{L}(x)$ is a second order differential operator in divergence form defined by

$$\mathscr{L}u \quad = \quad \sum_{k,l=1}^{d} \frac{\partial}{\partial x_k} \left(a_{k,l}(x) \frac{\partial u}{\partial x_l} \right) \quad = \quad \mathrm{div} ADu$$

The matrix $A = \{a_{k,l}\}$ has measurable components and satisfies the conditions

$$(\Delta) \begin{cases} a_{k,l} = a_{l,k} & \text{symmetry} \\ a_0|\xi|^2 \leq \sum_{k,l=1}^{d} a_{k,l}(x)\xi_k\xi_l & \text{uniform ellipticity} \\ \sum_{k,l=1}^{d} a_{k,l}(x)\xi_k\xi_l \leq A_0|\xi|^2 & \text{boundedness} \end{cases}$$

where $0 < a_0 \leq A_0 < \infty$. Du denotes the gradient of u. The drift term f is Lipschitz continuous in its last three variables with a Lipschitz coefficient L_f:

$$(D) \quad |f(x,t,u_1,u_2,u_3) - f(x,t,v_1,v_2,v_3)| \quad \leq \quad L_f\left(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|\right).$$

As for the diffusion coefficient, σ is differentiable with a bounded Hölder continuous derivative of order γ: $\sigma \in C^{1+\gamma}(\mathbb{R})$. In particular, σ is Lipschitz contiuous with Lipschitz coefficient L_σ

$$(S) \quad |\sigma(y) - \sigma(x)| \quad \leq \quad L_\sigma|y - x|$$

By (Δ), the matrix norm of A is bounded by

$$\|A(x)\| \quad \leq \quad A_0. \tag{3.1}$$

The initial condition will be the following: g and h are random fields on U such that

$$(I) \quad \|g\|_{H^1(U)} \quad \text{and} \quad \|h\|_{L^2(U)} \quad \text{are finite a.s.}$$

By proceeding formally with (1.1) and (1.2) we arrive at a weak formulation

$$\begin{aligned} \langle u'(\cdot,t), v\rangle_2 \quad = \quad & \langle h, v\rangle_2 \\ & - \int_0^t \langle Du(\cdot,\tau),\ ADv\rangle_2\, d\tau \\ & + \int_0^t \langle f(\cdot,\tau,u(\cdot,\tau),u'(\cdot,\tau),Du(\cdot,\tau)), v\rangle_2\, d\tau \\ & + \sum_{j=1}^{\infty} \sqrt{\lambda_j} \int_0^t \left\langle \sigma(u(\cdot,\tau)),\ ve_j \right\rangle_2 \beta_j^H(d\tau). \end{aligned} \tag{3.2}$$

In view of (3.2) it can be considered natural to adopt the following solution concept:

Definition 3.1. *An $L^2(U)$-valued random field $u(t)$, $t \in I$, is a weak solution to (1.1) if*

(1) $u \in H^1(I \times U)$ *a.s.*

(2) $u(0) = g$ *a.s.*

(3) *The integral relation* (3.2) *holds a.s. for every $v \in H^1(U)$ and every $t \in I$.*

One should check that all terms in (3.2) are well defined and finite in the chosen function space and this is the topic of the following Lemma.

Lemma 3.2. *The terms appearing in (3.2) are well defined and finite a.s.*

Proof: Estimating the diffusion term gives, by (3.1)

$$\left|\int_0^\tau \langle Du(\cdot,\theta),\ ADv\rangle_2\, d\theta\right| \;\le\; A_0\|v\|_{H^1}\int_0^\tau \|Du(\cdot,\theta)\|_2\, d\theta \;\le\; \sqrt{T}A_0\|v\|_{H^1}\|u\|_{H^1(I\times U)}.$$

As for the drift term we use Hölder's inequality to get

$$\begin{aligned}
&\int_0^t |\langle f(\cdot,\theta,u(\cdot,\theta),u'(\cdot,\theta),Du(\cdot,\theta)),\ v\rangle_2|\, d\theta\\
&\le C\int_0^t \langle 1+|u(\cdot,\theta)|+|u'(\cdot,\theta)|+|Du(\cdot,\theta)|,\ |v|\rangle_2\, d\theta\\
&\le C\int_0^t \left(1+\|u(\cdot,\theta)\|_2+\|u'(\cdot,\theta)\|_2+\|Du(\cdot,\theta)\|_2\right)\|v\|_2\, d\theta\\
&\le C\sqrt{T}\left(1+\|u\|_{H^1(I\times U)}\right)\|v\|_2.
\end{aligned} \tag{3.3}$$

The one-dimensional stochastic integrals are bounded by (2.1) as

$$\left|\int_\theta^\tau \langle \sigma(u(\cdot,s)),\ ve_j\rangle_2\, \beta_j^H(s)\right| \;\le\; G_j\|s\mapsto \langle \sigma(u(\cdot,s)),\ ve_j\rangle_2\|_{\alpha,1,\theta,\tau} \tag{3.4}$$

and estimating the $W^{\alpha,1}(\theta,\tau)$ norm yields

$$\begin{aligned}
&\int_\theta^\tau \left[\frac{|\langle \sigma(u(\cdot,s)),\ ve_j\rangle_2|}{(s-\theta)^\alpha}+\int_\theta^s \frac{|\langle \sigma(u(\cdot,s))-\sigma(u(\cdot,y)),\ ve_j\rangle_2|}{(s-y)^{1+\alpha}}\, dy\right] ds\\
&\le c\,\|v\|_{H^1(U)}\,\|e_j\|_{1,\infty}\\
&\quad\times\left[\int_\theta^\tau \frac{(1+\|u(\cdot,s)\|_2)}{(s-\theta)^\alpha}\, ds + L_\sigma\int_\theta^\tau\int_\theta^s \frac{\|u(\cdot,s)-u(\cdot,y)\|_2}{(s-y)^{1+\alpha}}\, dy\, ds\right].
\end{aligned} \tag{3.5}$$

The first term in square brackets is bounded by

$$\begin{aligned}
&c\int_\theta^\tau \frac{1+\|g\|_2+\int_0^s \|u'(\cdot,\xi)\|_2\, d\xi}{(s-\theta)^\alpha}\, ds\\
&\le C\,(\tau-\theta)^{1-\alpha}\left(a+b\|g\|_2+b\int_0^\tau \|u'(\cdot,\xi)\|_2\, d\xi\right)
\end{aligned} \tag{3.6}$$

and the second is bounded by

$$\begin{aligned}
L_\sigma\int_\theta^\tau\int_\theta^s\int_y^s \frac{\|u'(\cdot,\xi)\|_2}{(s-y)^{1+\alpha}}\, d\xi\, dy\, ds &= L_\sigma\int_\theta^\tau \|u'(\cdot,\xi)\|_2\int_\xi^\tau\int_\theta^\xi \frac{dy}{(s-y)^{1+\alpha}}\, ds\, d\xi\\
&\le C\,L_\sigma\int_\theta^\tau (\xi-\theta)^{1-\alpha}\|u'(\cdot,\xi)\|_2\, d\xi\\
&\le C\,L_\sigma(\tau-\theta)^{1-\alpha}\int_0^\tau \|u'(\cdot,\xi)\|_2\, d\xi
\end{aligned} \tag{3.7}$$

Adding up (3.6) and (3.7) we get

$$\begin{aligned}
&\left|\int_\theta^\tau \langle \sigma(u(\cdot,s)),\ ve_j\rangle_2\, \beta_j^H(s)\right|\\
&\le C\,G_j\,\|e_j\|_{1,\infty}\,\|v\|_{H^1(U)}\,(\tau-\theta)^{1-\alpha}\left(1+\|g\|_2+\int_0^\tau \|u'(\cdot,\xi)\|_2\, d\xi\right).
\end{aligned} \tag{3.8}$$

Hence, for the stochastic forcing term we have, letting $\theta = 0$ and $\tau = t$ in (3.8),

$$\begin{aligned}
&\left| \sum_{j=1}^{\infty} \sqrt{\lambda_j} \int_0^t \langle \sigma(u(\cdot,\tau)), v e_j \rangle_2 \, \beta_j^H(d\tau) \right| \\
&\quad \leq \; C\, \widehat{G}\, \|v\|_{H^1(U)}\, t^{1-\alpha} \left(1 + \|g\|_2 + \int_0^t \|u'(\cdot,\tau)\|_2 \, d\tau \right) \qquad (3.9)
\end{aligned}$$

which is a.s. finite by (1). □

4 The finite-dimensional solution

We will consider variational solutions and shall therefore assume given a sequence of supposedly easily computable functions, the "elements", $\{w_n\}_{n=1}^{\infty}$ with each w_n belonging to $H^1(U)$ and such that

$$\{w_n\}_{n=1}^{\infty} \text{ is an orthonormal basis in } L^2(U)$$

together with

$$\{w_n\}_{n=1}^{\infty} \text{ is an orthogonal basis in } H^1(U).$$

By the former Lemmas we can now prove a simple result which will be the basis of all further investigations

Corollary 4.1. *Let u satisfy the regularity requrement* (1) *and initial data* (2) *of Definition 3.1. Then u is a weak solution to (1.1) if and only if*

$$\begin{aligned}
\left\langle u'(\cdot,t),\, w_n \right\rangle_2 \;=\; & \langle h,\, w_n \rangle_2 \\
& - \int_0^t \langle Du(\cdot,\tau),\, ADw_n \rangle_2 \, d\tau \\
& + \int_0^t \left\langle f(\cdot,\tau,u(\cdot,\tau),u'(\cdot,\tau),Du(\cdot,\tau)),\, w_n \right\rangle_2 d\tau \\
& + \sum_{j=1}^{\infty} \sqrt{\lambda_j} \int_0^t \left\langle \sigma(u(\cdot,\tau)),\, w_n e_j \right\rangle_2 \beta_j^H(d\tau) \qquad (4.1)
\end{aligned}$$

holds a.s. for every $n \in \mathbb{Z}_+$ and every $t \in I$. In this case u is also called a variational solution to (1.1).

Proof: Any weak solution is clearly a solution to (4.1) so we need only show the if part. Let $v \in H^1(U)$ have the orthogonal decomposition

$$v(x) \;=\; \sum_{j=1}^{\infty} v_n w_n(x). \qquad (4.2)$$

By using the properties (1)-(2) it is then trivial, except perhaps for the stochastic integral term, to note that the finite sums of (4.2) together with (4.1) will give us a sequence of equations with each term converging a.s. in $H^1(I \times U)$ to the corresponding one in (3.2). To verify this for the stochastic integral, let $v_N(x) = \sum_1^N v_n w_n(x)$ and replace v with $v - v_N$ in (3.9). By the general assumptions, convergence follows. □

4.1 Galerkin approximation

Let V_N be the linear span of $w_1, \ldots, w_N$. Since V_N is finite dimensional the norms on $L^2(V_N)$ and $H^1(V_N)$ are equivalent. In particular, if $u(x) = \sum_{n=1}^{N} c_n w_n(x)$,

$$\|Du\|_2^2 \;=\; \sum_{n=1}^{N} |c_n|^2 \|Dw_n\|_2^2 \;\leq\; C_N^2 \sum_{n=1}^{N} |c_n|^2 \;=\; C_N^2 \|u\|_2. \tag{4.3}$$

Let φ_N denote the orthonormal projection of $\varphi \in L^2(U)$ onto V_N.

Definition 4.2. *A random field u_N is an N'th order Galerkin approximation to (4.1) if*

(1) $u_N \in H^1(I \times U)$ *a.s.*

(2) $u_N(0) = g_N$ *a.s.*

(3) The following equation holds a.s. for every $n \in \{1, \ldots, N\}$ and every $t \in I$:

$$\begin{aligned}\left\langle u_N'(\cdot,t),\, w_n \right\rangle_2 \;=\;& \langle h_N,\, w_n \rangle_2 \\ &- \int_0^t \langle Du_N(\cdot,\tau),\, ADw_n \rangle_2 \, d\tau \\ &+ \int_0^t \left\langle f(\cdot,\tau,u_N(\cdot,\tau),u_N'(\cdot,\tau),Du_N(\cdot,\tau)),\, w_n \right\rangle_2 d\tau \\ &+ \sum_{j=1}^{\infty} \sqrt{\lambda_j} \int_0^t \left\langle \sigma(u_N(\cdot,\tau)),\, w_n e_j \right\rangle_2 \beta_j^H(d\tau).\end{aligned} \tag{4.4}$$

Integrating the equation (4.4) gives

$$\begin{aligned}\langle u_N(\cdot,t),\, w_n \rangle_2 \;=\;& t\langle h_N,\, w_n \rangle_2 + \langle g_N,\, w_n \rangle_2 \\ &- \int_0^t \int_0^\tau \langle Du_N(\cdot,\theta),\, ADw_n \rangle_2 \, d\theta \, d\tau \\ &+ \int_0^t \int_0^\tau \langle f(\cdot,\theta,u_N(\cdot,\theta),u_N'(\cdot,\tau),Du_N(\cdot,\theta)),\, w_n \rangle_2 \, d\theta \, d\tau \\ &+ \sum_{j=1}^{\infty} \sqrt{\lambda_j} \int_0^t \int_0^\tau \langle \sigma(u_N(\cdot,\theta)),\, w_n e_j \rangle_2 \, \beta_j^H(d\theta) \, d\tau\end{aligned} \tag{4.5}$$

and because of the assumption (1) this equation can be differentiated (termwise) to yield (4.4). Hence we may as well consider (4.5).

Introduce the V_N-valued mapping

$$\Phi_N(u)(x,t) \;=\; \sum_{n=1}^{N} \langle \Phi_N(u)(\cdot,t), w_n \rangle_2 w_n(x)$$

by specifying the fourier coefficients a.s. as the right hand side of (4.5) with u_N replaced by u:

$$
\begin{aligned}
&\langle \Phi_N(u)(\cdot,t),\, w_n\rangle_2 \\
&\quad= \; t\langle h_N,\, w_n\rangle_2 \;+\; \langle g_N,\, w_n\rangle_2 \\
&\qquad - \int_0^t \int_0^\tau \langle Du(\cdot,\theta),\, ADw_n\rangle_2 \, d\theta\, d\tau \\
&\qquad + \int_0^t \int_0^\tau \big\langle f(\cdot,\theta,u(\cdot,\theta),u'(\cdot,\theta),D(\cdot,\theta)),\, w_n\big\rangle_2 \, d\theta\, d\tau \\
&\qquad + \sum_{j=1}^\infty \sqrt{\lambda_j} \int_0^t \int_0^\tau \langle \sigma(u(\cdot,\theta)),\, w_n e_j\rangle_2 \, \beta_j^H(d\theta)\, d\tau \\
&\quad= \; t\langle h_N,\, w_n\rangle_2 \;+\; \langle g_N,\, w_n\rangle_2 \;-\; E_n(u)(t) \;+\; F_n(u)(t) \;+\; S_n(u)(t)
\end{aligned}
$$

for every $n \in \{1,\dots,N\}$ and every $t \in I$. To solve the N'th order Galerkin approximation problem we will show existence of a fixpoint

$$
\Phi_N(u_N) \quad = \quad u_N \tag{4.6}
$$

in the space shortly written as $H^1(L^2_N)$ and defined by the set of functions

$$
\Big\{ u : I \mapsto V_N \,\Big|\, \sup_{t\in I} \big(\|u(\cdot,t)\|_2 + \|u'(\cdot,t)\|_2 + \|Du(\cdot,t)\|_2 \big) \;<\; \infty \Big\}.
$$

Because of (4.3) and the fact that

$$
\|u(\cdot,t)\|_2 \quad = \quad \left\| g + \int_0^t u'(\cdot,\xi)\, d\xi \right\|_2 \quad \le \quad \|g\|_2 + \int_0^t \|u'(\cdot,\xi)\|_2\, d\xi
$$

$H^1(L^2_N)$ can be more economically written as

$$
H^1(L^2_N) \quad = \quad \big\{ u : I \mapsto V_N \,\big|\, u' \,\exists \text{ and } u' \in L^\infty(I;L^2(U)) \big\}.
$$

We endow it with the following set of eqiuvalent norms:

$$
\|u\|_\beta \quad = \quad \sup_{t\in I} e^{-\beta t} \|u'(\cdot,t)\|_2.
$$

Equip also the space $H^{1,\infty}(I)$ with the equivalent norms

$$
|u|_\beta \quad = \quad \sup_{t\in I} e^{-\beta t} |u'(t)|.
$$

To establish the fixpoint we need some results concerning Lipschitz continuity with respect to u in $H^1(L^2_N)$. That is, we need to consider the differentiated version of (4.6). Introduce the notation $\Delta E_n(t) = E_n(u)(t) - E_n(u^*)(t)$ and similarly for $F_n(t)$ and $S_n(t)$.

Lemma 4.3. *Let $u \in H^1(L^2_N)$. Then $E_n(u), F_n(u) \in L^\infty(I;\mathbb{R})$. In particular,*

$$
|E_n(u)|_\beta \quad \le \quad C\beta^{-1} \|u\|_\beta, \tag{4.7}
$$

and similarly for F_n. Moreover, for every $\beta \in [1,\infty)$, the mappings

$$
E_n, F_n : H^1_N(I \times U) \mapsto L^\infty(I;\mathbb{R})
$$

are Lipschitz continuous, i.e., if $\beta \ge 1$ then there is some $C = C(N)$ such that

$$
|E_n(u) - E_n(v)|_\beta \quad \le \quad C\beta^{-1} \|u - v\|_\beta \tag{4.8}
$$

and similarly for $F_n(t)$.

Proof: Starting with the diffusion term E_n we get, by (3.1),

$$\begin{aligned}
|E_n(u)'(t) - E_n(v)'(t)| &= \left| \int_0^t \langle Du(\cdot,\tau),\ ADw_n\rangle_2 \, d\tau - \int_0^\tau \langle Dv(\cdot,\tau),\ ADw_n\rangle_2 \, d\tau \right| \\
&\leq \|w_n\|_{H^1}\ A_0 \int_0^t \|Du(\cdot,\tau) - Dv(\cdot,\tau)\|_2 \, d\tau \\
&\leq C_N\ \|w_n\|_{H^1}\ A_0 \int_0^t \|u(\cdot,\tau) - v(\cdot,\tau)\|_2 \, d\tau \\
&\leq C_N \int_0^t \int_0^\tau \|u'(\cdot,\xi) - v'(\cdot,\xi)\|_2 \, d\xi \, d\tau
\end{aligned}$$

Using this estimate in the β-norm gives

$$\begin{aligned}
|E_n(u) - E_n(v)|_\beta &= \sup_{t\in I} e^{-\beta t} |E_n'(u)(t) - E_n'(v)(t)| \\
&\leq C_N \sup_{t\in I} \int_0^t \int_0^\tau e^{-\beta\xi} \|u'(\cdot,\xi) - v'(\cdot,\xi)\|_2 \, d\xi \, e^{-\beta(t-\tau)} \, d\tau \\
&\leq C_N\ \|u - v\|_\beta \sup_{t\in I} \int_0^t \tau e^{-\beta(t-\tau)} \, d\tau \\
&\leq C_N\ \|u - v\|_\beta \beta^{-1}. \qquad (4.9)
\end{aligned}$$

Coming to the drift term F_n, note that, by Hölder's inequality, (D), and since V_N is finite dimensional

$$\begin{aligned}
&| \left\langle f(\cdot,\tau,u,u',Du),\ w_n \right\rangle_2 - \left\langle f(\cdot,\tau,v,v',Dv),\ w_n \right\rangle | \\
&\quad \leq C_N \left(\|u(\cdot,\tau) - v(\cdot,\tau)\|_2 + \|u'(\cdot,\tau) - v'(\cdot,\tau)\|_2 \right). \\
&\quad \leq C_N \left(\int_0^\tau \|u'(\cdot,\theta) - v'(\cdot,\theta)\|_2 \, d\theta + \|u'(\cdot,\tau) - v'(\cdot,\tau)\|_2 \right).
\end{aligned}$$

Hence, again by Hölder's inequality,

$$\begin{aligned}
&e^{-\beta t} |F_n'(u)(t) - F_n'(v)(t)| \\
&\quad \leq \int_0^t e^{-\beta(t-\tau)} e^{-\beta\tau} \left(\int_0^\tau \|u'(\cdot,\theta) - v'(\cdot,\theta)\|_2 \, d\theta + \|u'(\cdot,\tau) - v'(\cdot,\tau)\|_2 \right) d\tau \\
&\quad \leq C_N\ \|u - v\|_\beta \int_0^t e^{-\beta(t-\tau)} \, d\tau \\
&\quad \leq C_N\ \|u - v\|_\beta \beta^{-1}.
\end{aligned}$$

By chosing $u^* = 0$ in (4.9) we obtain the special case

$$|E_n(u)|_\beta \quad \leq \quad C_N\ \beta^{-1} \|u\|_\beta$$

by linearity which proves (4.7) for this term. Because of the nonlinearity, that argument does not work for F_n. Instead we estimate the β-norm of $F_n(u)(t)$ at $u = u' = Du = 0$ as follows:

$$|e^{-\beta t} F_n'(0)(t)| \quad \leq \quad C e^{-\beta t} \int_0^t d\tau \quad \leq \quad C\beta^{-1}$$

By the triangle inequality we now obtain

$$|F_n(u)|_\beta \ \leq\ |F_n(u) - F_n(0)|_\beta \ +\ |F_n(0)|_\beta \ \leq\ C\beta^{-1}(1\ +\ \|u\|_\beta) \quad < \quad \infty. \qquad \square$$

We will now prove an analogue of (4.7) and (4.8) for S_n.

Lemma 4.4. *Let $u \in H^1(L^2_N)$. Then $S_n(u) \in L^\infty(I;\mathbb{R})$ and the following estimate holds for all $\beta \geq 1$*

$$|S_n(u)|_\beta \quad \leq \quad C\,\widehat{G}\,\beta^{-1/p}(1+\|g\|_2+\|u\|_\beta).$$

Proof: By (3.4), (3.5) and parts of (3.6) and (3.7), taking $v = w_n$, and using Hölder's inequality

$$\begin{aligned}
&e^{-\beta t}|S_n'(u)(t)| \\
&\leq \quad C\,\widehat{G}\left[\int_0^t \frac{e^{-\beta s}(1+\|g\|_2)}{s^\alpha}e^{-\beta(t-s)}\,ds + \int_0^t \frac{\int_0^s e^{-\beta\xi}\|u'(\cdot,\xi)\|_2\,d\xi}{s^\alpha}e^{-\beta(t-s)}\,ds\right. \\
&\qquad \left.\int_0^t e^{-\beta(t-\xi)}\xi^{1-\alpha}e^{-\beta\xi}\|u'(\cdot,\xi)\|_2\,d\xi\right] \\
&\leq \quad C\,\widehat{G}\,\beta^{-1/p}(1+\|g\|_2+\|u\|_\beta)
\end{aligned}$$

for some $p \geq 1$ depending on α. □

In order to prove existence of a fixpoint to Φ_N, we need the following invariance result.

Lemma 4.5. *Let $u \in H^1(L^2_N)$. Then*

$$\Phi_N(u) \in H_N \text{ a.s.}$$

and there exists a large enough random variable β_0 taking values in $[1,\infty)$, and a constant C_N, such that the closed (random) ball

$$B_N \quad = \quad \left\{u \in H_N : \|u\|_{\beta_0} \leq 1+2C\|g_N\|_2\right\}$$

is invariant a.s. with respect to Φ_N, i.e., $\Phi_N(B_N) \subset B_N$ a.s.

Proof: We have

$$\|\Phi_N(u)(t)\|_2 \quad \leq \quad \sum_{n=1}^N |\langle \Phi_N(u)(\cdot,t),\, w_n\rangle_2|$$

By a trivial maximization procedure the linear term has the β-norm

$$|t\,\langle h_N,\, w_n\rangle_2\,|_\beta \quad = \quad C|\,\langle h_N,\, w_n\rangle_2\,|\beta^{-1} \quad \leq \quad C\|h_N\|_2\beta^{-1}.$$

Using this estimate together with Lemmas 4.3 and 4.4 we obtain, since $\beta \geq 1$,

$$\begin{aligned}
&\|\Phi_N(u)\|_\beta \\
&\quad \leq \quad C\,\left(\|h_N\|_2|\beta^{-1}+\|g_N\|_2+|E_n(u)|_\beta+|F_n(u)|_\beta+|S_n(u)|_\beta\right) \\
&\quad \leq \quad C\|g_N\|_2\ +C(1+\widehat{G})\beta^{-1/p}\left(\|h_N\|_2+\|g_N\|_2+\|u\|_\beta\right).
\end{aligned}$$

Hence, a.s., $\Phi_N(u) \in H_N$. Chosing the random variable β_0 to take values in the interval

$$\Big(\max(1,[(1+\widehat{G})(1+\|h_N\|_2+\|g_N\|_2)2C]^p),\infty\Big)$$

ensures $C\beta_0^{-1/p}(1+\widehat{G}_0)(1+\|h_N\|_2+\|g_N\|_2) \leq \frac{1}{2}$ and we obtain

$$\|\Phi_N(u)\|_{\beta_0} \quad \leq \quad C\,\|g_N\|_2+\frac{1}{2}(1+\|u\|_{\beta_0}).$$

If $u \in B_N$, then $\Phi_N(u) \in B_N$ since

$$\|\Phi_N(u)\|_{\beta_0} \leq C\,\|g_N\|_2+\frac{1}{2}\,(1+1+2C\|g_N\|_2) \;=\; 1+2C\|g_N\|_2.$$ □

Lemma 4.6. *If $u, v \in H^1(L^2_N)$ then*

$$|S_n(u) - S_n(v)|_\beta \quad \leq \quad C\,\widehat{G}\,b\,\|u-v\|_\beta \beta^{2\alpha-1}. \tag{4.10}$$

Proof:

$$\begin{aligned}
&\left|\sum_{j=1}^{\infty}\sqrt{\lambda_j}\int_0^t \left\langle \sigma(u(\cdot,\tau)) - \sigma(v(\cdot,\tau)),\ w_n e_j\right\rangle \beta_j^H(d\tau)\right| \\
&\leq\ C\ \sum_{j=1}^{\infty}\sqrt{\lambda_j}G_j\ \|e_j\|_\infty\ \int_0^t \left[\frac{\|u(\cdot,\tau)-v(\cdot,\tau)\|_2}{\tau^\alpha}\right. \\
&\quad +\alpha\int_0^\tau \left.\frac{\|\sigma(u(\cdot,\tau))-\sigma(v(\cdot,\tau))-\sigma(u(\cdot,\theta))+\sigma(v(\cdot,\theta))\|_2}{(\tau-\theta)^{1+\alpha}}dy\right]d\tau \\
&\leq\ C\,\widehat{G}\int_0^t\left[\frac{1}{\tau^\alpha}\int_0^\tau \|u'(\cdot,\theta)-v'(\cdot,\theta)\|_2\,d\theta\right. \\
&\quad +\int_0^\tau \left.\frac{\|\sigma(u(\cdot,\tau))-\sigma(v(\cdot,\tau))-\sigma(u(\cdot,\theta))+\sigma(v(\cdot,\theta))\|_2}{(\tau-\theta)^{1+\alpha}}dy\right]d\tau
\end{aligned}$$

Multiplying by $e^{-\beta t}$ and taking the sup over all $t \in I$ gives the following bound on the first term

$$\begin{aligned}
&\widehat{G}\,b\,\sup_{t\in I}\int_0^t \frac{e^{-\beta(t-\tau)}}{\tau^\alpha}\int_0^\tau e^{-\beta\theta}\|u(\cdot,\theta)-v(\cdot,\theta)\|_2\,d\theta\,d\tau \\
&\leq\ \widehat{G}\,b\,\|u-v\|_\beta\ \sup_{t\in I}\int_0^t \tau^{1-\alpha}e^{-\beta(t-\tau)}\,d\tau \\
&\leq\ C\,\widehat{G}\,b\,\|u-v\|_\beta\ \beta^{-1}.
\end{aligned} \tag{4.11}$$

As for the second term we need Lemma 4, 5, and parts of Proposition 2 of [18] to conclude that it is bounded by

$$C\widehat{G}\beta^{2\alpha-1}\|u-v\|_\beta. \tag{4.12}$$

Adding (4.11) and (4.12) gives the result. □

The next Lemma is on a contraction property of Φ_N, crucial in the fixpoint argument which will provide the N'th order Galerkin approximation to (4.1).

Lemma 4.7. *There exists a random variable $\beta_1 \in [1,\infty)$ such that the map Φ_N is a contraction on $\Phi_N(B_N)$ with respect to the norm $\|\cdot\|_{\beta_1}$: if $u, v \in B_N$ then*

$$\|\Phi_N(u) - \Phi_N(v)\|_{\beta_1} \quad \leq \quad \frac{1}{2}\|u-v\|_{\beta_1} \tag{4.13}$$

Proof: Let $u, v \in H(L^2_N)$. Then, by the Lipschitz continuity of the terms E_n, F_n and S_n (Lemmas 4.3 and 4.10) we find that

$$\begin{aligned}
\|\Phi_N(u)-\Phi_N(v)\|_\beta \quad &\leq \quad \sum_{n=1}^{N}\|\left\langle\Phi_N(u)(\cdot,\cdot),\ w_n\right\rangle_2 w_n - \left\langle\Phi_N(v)(\cdot,\cdot),\ w_n\right\rangle_2 w_n\|_\beta \\
&\leq \quad \sum_{n=1}^{N}\left(|\Delta E_n|_\beta + |\Delta F_n|_\beta + |\Delta S_n|_\beta\right) \\
&\leq \quad C_N(1+\widehat{G})\|u-v\|_\beta\beta^{2\alpha-1}
\end{aligned} \tag{4.14}$$

Let $u, v \in B_N$. Then, chosing the random variable $\beta_1^{2\alpha-1}$ to take any value in the interval

$$\Big(\max\Big(1,\ 2C_N(1+\widehat{G})\Big),\ \infty\Big)$$

ensures the conclusion (4.13). □

Proposition 4.8. *The map Φ_N has a fix point $u_N \in H^1(L_N^2)$ for every positive integer N. Moreover, $u_N \in B_N$.*

Proof: The argument is identical to the existence part of Proposition 2 in [18]. □

This far we have shown that the Galerkin approximation has a unique solution. Note how all arguments are done pathwisely, for a fixed, but arbitrary path ω.

We have the following apriori smoothness of the Galerkin approximation u_N.

Proposition 4.9. *$u_N \in C^{1+1/2}(I\ ;\ H^{-1}(U))$ a.s. and*

$$\begin{aligned}&\|{u_N}'(\cdot,\tau)-{u_N}'(\cdot,\theta)\|_{H^{-1}(U)}\\ &\quad\leq\ C\,(1+\widehat{G})(\tau-\theta)^{1/2}\Big[1+\|g\|_2+\|{u_N}'\|_{L^2(\theta,\tau;L^2(U))}+\|Du_N\|_{L^2(\theta,\tau;L^2(U))}\Big].\end{aligned}$$

Proof: Write

$$r_\theta(\cdot,t)\ =\ {u_N}'(\cdot,t)-{u_N}'(\cdot,\theta).$$

Then, by (4.4), for every $n\in\{1,\ldots,N\}$,

$$\begin{aligned}\langle r(\cdot,\tau),\ w_n\rangle\ &=\ -\int_\theta^\tau \langle Du_N(\cdot,s),\ ADw_n\rangle_2\,ds\\ &\quad+\int_\theta^\tau \big\langle f(\cdot,s,u_N(\cdot,s),{u_N}'(\cdot,s),Du_N(\cdot,s)),\ w_n\big\rangle_2\,ds\\ &\quad+\sum_{j=1}^\infty\sqrt{\lambda_j}\int_\theta^\tau\big\langle\sigma(u_N(\cdot,s)),\ w_ne_j\big\rangle_2\,\beta_j^H(ds).\end{aligned}$$

By linearity, this extends to all V_N-valued $v\in H^1(U)$:

$$\begin{aligned}\langle r(\cdot,\tau),\ v\rangle\ &=\ -\int_\theta^\tau \langle Du_N(\cdot,s),\ ADv\rangle_2\,ds\\ &\quad+\int_\theta^\tau \big\langle f(\cdot,s,u_N(\cdot,s),{u_N}'(\cdot,s),Du_N(\cdot,s)),\ v\big\rangle_2\,ds\\ &\quad+\sum_{j=1}^\infty\sqrt{\lambda_j}\int_\theta^\tau\big\langle\sigma(u_N(\cdot,s)),\ ve_j\big\rangle_2\,\beta_j^H(ds).\end{aligned}$$

By Hölder's inequality and the calculations leading to (3.9),

$$\begin{aligned}|\,\langle r_\theta(\cdot,\tau),\ v\rangle\,|\ &\leq\ \|A_0\|\ \|Dv\|_2\int_\theta^\tau\|Du_N(\cdot,s)\|_2\,ds\\ &\quad+C\ \|v\|_2\left[(\tau-\theta)+\int_\theta^\tau\Big(\|{u_N}'(\cdot,s)\|_2+\|u_N(\cdot,s)\|_{H^1(U)}\Big)\,ds\right]\\ &\quad+C\ \widehat{G}\|v\|_2(\tau-\theta)^{1-\alpha}\left(1+\|g\|_2+\int_0^\tau\|{u_N}'(\cdot,\xi)\|_2\,d\xi\right)\\ &\leq\ C\,(1+\widehat{G})\|v\|_{H^1}(\tau-\theta)^{1/2}\\ &\quad\times\Big[1+\|g\|_2+\|{u_N}'\|_{L^2(\theta,\tau;L^2(U))}+\|Du_N\|_{L^2(\theta,\tau;L^2(U))}\Big].\end{aligned}$$

Dividing by $\|v\|_{H^1}$ and taking the supremum over $v \in H^1 \cap V_N$ yields

$$\begin{aligned}
\|r_\theta(\cdot,\tau)\|_{H^{-1}} &\leq C\,(1+\widehat{G})(\tau-\theta)^{1/2} \\
&\quad \times \left[1+\|g\|_2+\|u_N{}'\|_{L^2(\theta,\tau;L^2(U))}+\|Du_N\|_{L^2(\theta,\tau;L^2(U))}\right] \\
&= C_N\widehat{G}\,(\tau-\theta)^{1/2} \qquad (4.15)
\end{aligned}$$

for some random constant $C_N\widehat{G}$. □

5 Existence of solutions

Proposition 5.1. *Introduce the measurable mappings $p : I \to H^1(U)$ such that $p' : I \mapsto L^2(U)$ and $q' : I \mapsto L^2(U)$. Assume also $e \in H^{1,\infty}(U)$. Define $p_0 = p(0)$. Then*

$$\begin{aligned}
\left|\int_0^t \langle \sigma(p(s)),\, q'(s)e\rangle\, d\beta(s)\right| &\leq C\,G\,\|e\|_{H^{1,\infty}}\Bigg[(1+\|p_0\|_2^2)t^{1-\alpha}+\int_0^t \|p'(y)\|_2^2\,dy \\
&\quad +\int_0^t \frac{\|q'(s)\|_2^2}{s^\alpha}+\int_0^t \|Dp(y)\|_2^2\,dy \\
&\quad +\int_0^t\left(\int_y^t \frac{\|q'(s)-q'(y)\|_{H^{-1}}}{(s-y)^{1+\alpha}}\,ds\right)^2 dy\Bigg]
\end{aligned}$$

Proof: Note first that

$$\|p(s)\|_2 \;=\; \left\|p_0+\int_0^s p'(\xi)\,d\xi\right\|_2 \;\leq\; \|p_0\|_2+\int_0^s \|p'(\xi)\|_2\,d\xi$$

and

$$\begin{aligned}
\left|\int_0^t \langle q'(s),\, e\rangle\, d\beta(s)\right| &\leq G\int_0^t\left[\frac{|\langle q'(s),\, e\rangle|}{s^\alpha}+\alpha\int_0^s \frac{|\langle q'(s)-q'(y),\, e\rangle|}{(s-y)^{1+\alpha}}\,dy\right] ds \\
&\leq G\|e\|_{H^1}\int_0^t\left[\frac{\|q'(s)\|_2}{s^\alpha}+\int_0^s \frac{\|q'(s)-q'(y)\|_{H^{-1}}}{(s-y)^{1+\alpha}}\,dy\right] ds
\end{aligned}$$

By the condition (S)

$$|D\sigma(p(y))| \;\leq\; \|\sigma'\|_\infty |Dp(y)|.$$

By (2.1) we have the following bound on a stochastic integral

$$\begin{aligned}
&\left|\int_0^t \langle \sigma(p(s)),\, q'(s)e\rangle\, d\beta(s)\right| \\
&\leq G\int_0^t\left[\frac{|\langle \sigma(p(s)),\, q'(s)e\rangle|}{s^\alpha}+\alpha\int_0^s \frac{|\langle \sigma(p(s))q'(s)-\sigma(p(y))q'(y),\, e\rangle|}{(s-y)^{1+\alpha}}\,dy\right] ds
\end{aligned}$$

The simple integral can be estimated as

$$\begin{aligned}
&\int_0^t \frac{|\langle \sigma(p(s)),\, q'(s)e\rangle|}{s^\alpha}\,ds \\
&\leq \|e\|_\infty\int_0^t \frac{(1+\|p(s)\|_2)\|q'(s)\|_2}{s^\alpha}\,ds \\
&\leq C\,\|e\|_\infty\left[t^{1-\alpha}(1+\|p_0\|_2^2)+\int_0^t \frac{1}{s^\alpha}\left(\int_0^s \|p'(\xi)\|_2\,d\xi\right)^2 ds+\int_0^t \frac{\|q'(s)\|_2^2}{s^\alpha}\,ds\right] \\
&\leq C\,\|e\|_\infty\left[t^{1-\alpha}(1+\|p_0\|_2^2)+\int_0^t \|p'(\xi)\|_2^2\,d\xi+\int_0^t \frac{\|q'(s)\|_2^2}{s^\alpha}\,ds\right].
\end{aligned}$$

In the estimate of the double integral it is really essential, due to Proposition 4.9, that $e \in H^{1,\infty}$ and not just $L^\infty(U)$. It also displays the difficulty in letting σ depend on derivatives of p (that is u).

$$
\begin{aligned}
& G \int_0^t \int_0^s \frac{|\langle \sigma(p(s))q'(s) - \sigma(p(y))q'(y),\, e\rangle|}{(s-y)^{1+\alpha}}\, dy\, ds \\
&= \int_0^t \int_0^s \frac{|\langle [\sigma(p(s)) - \sigma(p(y))]\, q'(s) + \sigma(p(y))\, [q'(s) - q'(y)]\,,\, e\rangle|}{(s-y)^{1+\alpha}}\, dy\, ds \\
&\leq C\, G\, \|e\|_{H^{1,\infty}} \int_0^t \int_0^s \left[\frac{L_\sigma \|p(s) - p(y)\|_2\, \|q'(s)\|_2}{(s-y)^{1+\alpha}} \right. \\
&\quad \left. + \frac{\|\sigma(p(y))\|_{H^1}\, \|q'(s) - q'(y)\|_{H^{-1}}}{(s-y)^{1+\alpha}} \right] dy\, ds \\
&\leq \|e\|_{H^{1,\infty}} \left[L_\sigma \int_0^t \int_0^s \int_y^s \frac{\|p'(\xi)\|_2\, \|q'(s)\|_2}{(s-y)^{1+\alpha}}\, d\xi\, dy\, ds \right. \\
&\quad \left. + \int_0^t \int_0^s \|\sigma(p(y))\|_{H^1} \frac{\|q'(s) - q'(y)\|_{H^{-1}}}{(s-y)^{1+\alpha}}\, dy\, ds \right] \\
&\leq C\, \|e\|_{H^{1,\infty}} \left[L_\sigma \int_0^t \int_0^s \left(\frac{\|q'(s)\|_2^2 + \|p'(\xi)\|_2^2}{(s-\xi)^\alpha} \right) d\xi\, ds \right. \\
&\quad \left. + \int_0^t \|\sigma(p(y))\|_{H^1} \int_y^t \frac{\|q'(s) - q'(y)\|_{H^{-1}}}{(s-y)^{1+\alpha}}\, ds\, dy \right] \\
&\leq C\, \|e\|_{H^{1,\infty}} \left[L_\sigma \int_0^t \left(\|q'(s)\|_2^2 + \|p'(s)\|_2^2 \right) ds \right. \\
&\quad \left. + \int_0^t \|\sigma(p(y))\|_{H^1}^2\, dy + \int_0^t \left(\int_y^t \frac{\|q'(s) - q'(y)\|_{H^{-1}}}{(s-y)^{1+\alpha}}\, ds \right)^2 dy \right] \\
&\leq C\, \|e\|_{H^{1,\infty}} \left[(1 + \|p_0\|_2^2)t + \int_0^t \left(\|q'(y)\|_2^2 + \|p'(y)\|_2^2 + \|Dp(y)\|_2^2 \right) dy \right. \\
&\quad \left. + \int_0^t \left(\int_y^t \frac{\|q'(s) - q'(y)\|_{H^{-1}}}{(s-y)^{1+\alpha}}\, ds \right)^2 dy \right] \qquad (5.1)
\end{aligned}
$$

□

Substituting p for u_N, q' for u_N', and using (4.15) gives

$$
\begin{aligned}
& \left| \sum_{j=1}^\infty \sqrt{\lambda_j} \int_0^t \langle \sigma(u_N(\cdot,\tau)),\, e_j u_N{}'(\cdot,\tau) \rangle\, d\beta_j(\tau) \right| \\
&\leq C\, \widehat{G} \left\{ (1 + \|g\|_2^2) + \int_0^t \frac{\|u_N{}'(\cdot,s)\|_2^2}{s^\alpha}\, ds + \int_0^t \|Du_N(\cdot,s)\|_2^2\, ds \right. \\
&\quad \left. + (1 + \widehat{G}^2) \left[1 + \|g\|_2^2 + \|u_N{}'\|^2_{L^2(\theta,\tau;L^2(U))} + \|Du_N\|^2_{L^2(\theta,\tau;L^2(U))} \right] \right\} \\
&\leq C\, (1 + \widehat{G}^3) \left[1 + \|g\|_2^2 + \int_0^t \frac{\|u_N{}'(\cdot,y)\|_2^2 + \|Du_N(\cdot,y)\|_2^2}{y^\alpha}\, dy \right]
\end{aligned}
$$

if $\alpha < 1/2$.

The next step is to let $N \to \infty$ in the Galerkin sequence u_N and we will discover that $\{u_N\}_{N=1}^{\infty}$ is suitably bounded and has a subsequence that converges a.s. to a solution of equation (4.1).

Proposition 5.2. *Assume $\alpha \in (1-H, 1/2)$. Then*

$$\{u_N\}_{N=1}^{\infty},\ \{u_N{}'\}_{N=1}^{\infty} \text{ and } \{Du_N\}_{N=1}^{\infty}$$

are a.s. bounded sequences in $L^\infty(I; L^2(U))$.

Proof: Square the equation using Proposition A.1 of [18] to get

$$\begin{aligned}
&\langle u_N{}'(\cdot,t),\ w_n\rangle_2^2 \\
&= \quad \langle h_N,\ w_n\rangle_2^2 \\
&\quad - \int_0^t \left\langle ADu_N(\cdot,\tau),\ \langle u_N{}'(\cdot,\tau),\ w_n\rangle_2 Dw_n\right\rangle_2 d\tau \\
&\quad + 2\int_0^t \left\langle f(\cdot,\tau,u_N(\cdot,\tau),u_N{}'(\cdot,\tau),Du_N(\cdot,\tau)),\ \langle u_N{}'(\cdot,\tau),\ w_n\rangle_2 w_n\right\rangle_2 d\tau \\
&\quad + 2\sum_{j=1}^{\infty}\sqrt{\lambda_j}\int_0^t \left\langle \sigma(u_N(\cdot,\tau)),\ e_j\langle u_N{}'(\cdot,\tau),\ w_n\rangle_2 w_n\right\rangle d\beta_j(\tau).
\end{aligned}$$

Summing over $n \in \{1,\dots,N\}$ gives

$$\begin{aligned}
&\|u_N{}'(\cdot,t)\|_2^2 + 2\int_0^t \left\langle ADu_N(\cdot,\tau),\ Du_N{}'(\cdot,\tau)\right\rangle_2 d\tau \\
&= \quad \|h_N\|_2^2 + 2\int_0^t \left\langle f(\cdot,\tau,u_N(\cdot,\tau),u_N{}'(\cdot,\tau),Du_N(\cdot,\tau)),\ u_N{}'(\cdot,\tau)\right\rangle_2 d\tau \\
&\quad + 2\sum_{j=1}^{\infty}\sqrt{\lambda_j}\int_0^t \left\langle \sigma(u_N(\cdot,\tau)),\ e_j u_N{}'(\cdot,\tau)\right\rangle d\beta_j(\tau).
\end{aligned} \tag{5.2}$$

The second term on the left is bounded from below as

$$\begin{aligned}
&2\int_0^t \left\langle ADu_N(\cdot,\tau),\ Du_N{}'(\cdot,\tau)\right\rangle_2 d\tau \\
&= \quad \int_0^t \frac{d}{d\tau}\|A^{1/2}Du_N(\cdot,\tau)\|_2^2\, d\tau \\
&= \quad \langle ADu_N(\cdot,t),\ Du_N(\cdot,t)\rangle - \langle ADu_N(\cdot,0),\ Du_N(\cdot,0)\rangle \\
&\geq \quad a_0\|Du_N(\cdot,t)\|_2^2 - A_0\|Du_N(\cdot,0)\|_2^2
\end{aligned} \tag{5.3}$$

by the ellipticity condition (Δ). For the second term on the right we use (3.3), Cauchy's inequality and Jensen's inequality to get

$$\begin{aligned}
&\left|2\int_0^t \left\langle f(\cdot,\tau,u_N(\cdot,\tau),u_N{}'(\cdot,\tau),Du_N(\cdot,\tau)),\ u_N(\cdot,\tau)\right\rangle d\tau\, d\tau\right| \\
&\leq \quad C\int_0^t \left(1+\|u_N(\cdot,\tau)\|_2+\|u_N{}'(\cdot,\tau)\|_2+\|Du_N(\cdot,\tau)\|_2\right)\|u_N{}'(\cdot,\tau)\|_2\, d\tau \\
&\leq \quad C\left[t+\int_0^t \left(\|u_N(\cdot,\tau)\|_2^2+\|u_N{}'(\cdot,\tau)\|_2^2+\|Du_N(\cdot,\tau)\|_2^2\right) d\tau\right] \\
&\leq \quad C\left[1+\|g\|_2^2+\int_0^t \left(\|u_N{}'(\cdot,\tau)\|_2^2+\|Du_N(\cdot,\tau)\|_2^2\right) d\tau\right]
\end{aligned} \tag{5.4}$$

The stochastic forcing term is bounded by

$$C\,(1+\widehat{G}^3)\left[1+\|g\|_2^2+\int_0^t \frac{\|u_N{}'(\cdot,\tau)\|_2^2+\|Du_N(\cdot,\tau)\|_2^2}{\tau^\alpha}\,d\tau\right]. \tag{5.5}$$

Using the estimates (5.3), (5.4), and (5.5) together with Hölder's inequality, in (5.2) we get

$$\begin{aligned}&\|u_N{}'(\cdot,t)\|_2^2+a_0\|Du_N(\cdot,t)\|_2^2\\ &\quad\le\ C\,(1+\widehat{G}^3)\left[1\,+\,\|g\|_{H^1(U)}^2\,+\,\|h\|_2^2\,+\,\int_0^t \frac{\|u_N{}'(\cdot,\tau)\|_2^2+\|Du_N(\cdot,\tau)\|_2^2}{\tau^\alpha}\,d\tau\right]\end{aligned}$$

Gronwall's inequality gives then

$$\begin{aligned}\|u_N{}'(\cdot,t)\|_2^2+\|Du_N(\cdot,t)\|_2^2 \ &\le\ C\,(1+\widehat{G}^3)\Big(1\,+\,\|g\|_{H^1(U)}^2\,+\,\|h\|_2^2\Big)\,e^{C\,(1+\widehat{G}^2)t^{1-\alpha}}\\ &\le\ \Big(1\,+\,\|g\|_{H^1(U)}^2\,+\,\|h\|_2^2\Big)\,e^{C\,(1+\widehat{G}^2)(1+t^{1-\alpha})}\end{aligned}$$

Note that the right hand side is finite and independent of the dimension N. The proposition is proved. □

Proposition 5.3. *There is an element $\tilde{u}\in L^2(I;H^1(U))$ with $\tilde{u}'\in L^2(I;L^2(U))$ (i.e., $\tilde{u}\in H^1(I\times U)$) and a subsequence of $\{u_N\}_{N=1}^\infty$ such that, a.s.,*

$$\begin{aligned}&(1)\qquad \int_I\int_U u_N(x,t)\psi(x,t)\,dx\,dt \to \int_I\int_U \tilde{u}(x,t)\psi(x,t)\,dx\,dt\\ &(2)\qquad \int_I\int_U u_N{}'(x,t)\psi(x,t)\,dx\,dt \to \int_I\int_U \tilde{u}'(x,t)\psi(x,t)\,dx\,dt\\ &(3)\qquad \int_I\int_U Du_N(x,t)\Theta(x,t)\,dx\,dt \to \int_I\int_U D\tilde{u}(x,t)\Theta(x,t)\,dx\,dt\end{aligned}$$

as $N\to\infty$, for every $\psi\in L^2(I;L^2(U))$ and $\Theta\in L^2(I;(L^2(U))^{\otimes d})$.

Proof: Since, by Lemma 5.2, $\{u_N\}$ is, a.s., a bounded sequence in $L^2(I;L^2(U))$, which is the dual of $L^2(I;L^2(U))$, there is a subsequence, also denoted by $\{u_N\}$, and an element $\tilde{u}\in L^2(I;L^2(U))$ such that $u_N\to\tilde{u}$, a.s., in the weak topology of $L^2(I;L^2(U))$. This means that, with probability one,

$$\langle u_N,\ \psi\rangle \to \langle \tilde{u},\psi\rangle, \qquad \forall\psi\in L^2(I;L^2(U)), \tag{5.6}$$

where $\langle f_1,\ f_2\rangle$ is short for the integral of the product f_1f_2 over $U\times I$. Hence (1) holds. Similarly, by passing to still another subsequence we have, for some $v\in L^2(I;(L^2(U))^{\otimes d})$,

$$\langle Du_N,\ \Theta\rangle \to \langle v,\ \Theta\rangle\,, \qquad \forall\Theta\in L^1(I;(L^2(U))^{\otimes d}).$$

We will now identify v. Let $\varphi: U\times I\mapsto\mathbb{R}^d\in(C_c^\infty(U\times I))^{\otimes d}$. By the Gauss' divergence theorem (assuming div acts on the x variable only) and (5.6)

$$\langle v,\ \varphi\rangle \ =\ \lim_{N\to\infty}\langle Du_N,\ \varphi\rangle \ =\ -\lim_{N\to\infty}\langle u_N,\ \mathrm{div}\varphi\rangle_2 \ =\ -\langle\tilde{u},\ \mathrm{div}\varphi\rangle\,.$$

This means $\tilde{u}\in L^2(I;H^1(U))$, $D\tilde{u}=v$, and (3) holds. A similar calculation shows $\tilde{u}'\in L^2(I;L^2(U))$ and that (2) holds. □

A similar reasoning using the fact that the sequence $\{u_N\}$ is really in $L^\infty(I;L^2(U))$ gives an element $\tilde{v} \in L^\infty(I;H^1(U))$ with $v' \in L^\infty(I;L^2(U))$ such that the limits (1), (2), and (3) are valid when $\psi \in L^1(I;L^2(U))$ and $\Theta \in L^1(I;(L^2(U))^{\otimes d})$. If $\psi \in L^2(I;L^2(U))$ then

$$\left\langle v',\ \psi \right\rangle_2 \;=\; \lim_{N\to\infty} \left\langle u_N{}',\ \psi \right\rangle_2 \;=\; \left\langle \tilde{u}',\ \psi \right\rangle_2$$

so that we can identify v with $\tilde{u}$. Hence, we have

$$\tilde{u} \in H^{1,\infty}(I\times U).$$

It is natural to hope $\tilde{u}$ qualifies as a solution to (4.1). We will send N to ∞ in each term of (4.1) separately and discover this very fact.

Theorem 5.4. *There exists a solution to (4.1).*

Proof: With the help of Proposition 5.3 we will now show that each term in (4.4) converges a.s. on $\mathbb{R}$ to the corresponding term in (4.1) from which the theorem follows. Since the test functions are invariant with respect to multiplication by a characteristic function $i_{[0,t]}$ we need only check this for the full time interval I. It is immediate that the terms involving initial data converge to the same ones with u_N replaced by $\tilde{u}$. Coming next to the diffusion term we have similarly

$$\begin{aligned}&\int_I\int_U \langle Du_N(x,\tau),\ A(x)Dw_n(x)\rangle_{\mathbb{R}^d}\, dx\,d\tau \\ &\quad\to\quad \int_I\int_U \langle D\tilde{u}(x,\tau),\ A(x)Dw_n(x)\rangle_{\mathbb{R}^d}\, dx\,d\tau\end{aligned}$$

because $\Gamma = ADw_n \in L^1(I;L^2(U))^{\otimes d}$. As for the drift term we have, by (D),

$$\begin{aligned}&\left|\int_I \left\langle f(\cdot,\tau,u_N(\cdot,\tau),u_N{}'(\cdot,\tau),Du_N(\cdot,\tau)) - f(\cdot,\tau,\tilde{u}(\cdot,\tau),\tilde{u}'(\cdot,\tau),D\tilde{u}(\cdot,\tau)),\ w_n\right\rangle_2 d\tau\right| \\ &\le\quad b\int_I \left\langle |u_N(\cdot,\tau)-\tilde{u}(\cdot,\tau)| + |u_N{}'(\cdot,\tau)-\tilde{u}'(\cdot,\tau)| + |Du_N(\cdot,\tau)-D\tilde{u}(\cdot,\tau)|,\ |w_n|\right\rangle_2 d\tau \\ &\to\quad 0,\end{aligned}$$

since $\psi = |w_n| \in L^2(I\times U)$. Finally, we discuss the noise term. By Lemma 13, 14, 15 in [18] and copying parts of their lemma 16 and Proposition 4 we obtain immediately the required convergence of the stochastic integral. □

6 Uniqueness and stability

We will now prove a general inequality from which both global existence, uniqueness and continuity with respect to initial data and will follow. We start with a variant of Proposition 4.9. In this section it is assumed throughout that σ is an affine function:

$$\sigma(r) \;=\; a+br.$$

Proposition 6.1. *Let u satisfying (1) and (2) be a solution to (1.1) and similarly for u^* but with initial data g^* and h^*. Then*

$$\begin{aligned}
&\int_0^t \frac{\|u_N{}'(\cdot,t) - u_N^*{}'(\cdot,t) - u_N{}'(\cdot,s) + u_N^*{}'(\cdot,s)\|_{H^{-1}}}{(t-s)^{1+\alpha}}\, ds \\
&\leq \; C \left(\|g_N - g_N^*\|_2 t^{1-\alpha} + (1+\widehat{G}) t^{1-\alpha} \int_0^t \frac{\|u_N{}'(\cdot,\tau) - u_N^*{}'(\cdot,\tau)\|_2}{(t-\tau)^\alpha}\, d\tau \right. \\
&\quad \left. + \int_0^t \frac{\|Du_N(\cdot,\tau) - Du_N^*(\cdot,\tau)\|_2}{(t-\tau)^\alpha}\, d\tau \right).
\end{aligned}$$

Proof: Note that, by linearity, (4.4) is satisfied if w_n is replaced by any $v \in V_N$. For such v's define

$$r_s(t) \;=\; \left\langle u_N{}'(\cdot,t) - u_N^*{}'(\cdot,t) - u_N{}'(\cdot,s) + u_N^*{}'(\cdot,s),\; v \right\rangle.$$

Then, by (4.4),

$$\begin{aligned}
r_s(t) \;=\; & -\int_s^t \langle Du_N(\cdot,\tau) - Du^*(\cdot,\theta)\, d\theta,\; ADv\rangle_2\, d\xi \\
& + \int_s^t \langle f(\cdot,\tau,u_N(\cdot,\tau),u_N{}'(\cdot,\tau),Du_N(\cdot,\tau)) \\
& \qquad - f(\cdot,\tau,u_N^*(\cdot,\tau),u_N^*{}'(\cdot,\tau),Du_N^*(\cdot,\tau)),\; v\rangle_2\, d\tau \\
& + b\sum_{j=1}^\infty \sqrt{\lambda_j} \int_s^t \left\langle \sigma(u_N(\cdot,\tau)) - \sigma(u_N^*(\cdot,\tau)),\; ve_j\right\rangle_2 \beta_j^H(d\tau).
\end{aligned}$$

By Hölder's inequality and the calculations in (3.9)

$$\begin{aligned}
&|r_s(t)| \\
&\leq \; \|A\|\, \|v\|_{H^1} \int_s^t \|Du_N(\cdot,\tau) - Du_N^*(\cdot,\tau)\|_2\, d\tau \\
&\quad + L_f\, \|v\|_{H^1} \int_s^t \left(\|u_N(\cdot,\tau) - u_N^*(\cdot,\tau)\|_2 + \|u_N{}'(\cdot,\tau) - u_N^*{}'(\cdot,\tau)\|_2\right) d\tau \\
&\quad + b\sum_{j=1}^\infty \sqrt{\lambda_j} G_j \left[\int_s^t \left| \frac{\left\langle u_N(\cdot,\tau) - u_N^*(\cdot,\tau),\; ve_j\right\rangle}{\tau^\alpha} \right| d\tau \right. \\
&\quad \left. + \int_s^t \left| \int_s^\tau \frac{\left\langle u_N(\cdot,\tau) - u_N^*(\cdot,\tau) - u_N(\cdot,y) + u_N^*(\cdot,y),\; ve_j\right\rangle}{(\tau-y)^{1+\alpha}}\, dy \right| d\tau \right] \qquad (6.1)
\end{aligned}$$

Dividing the first term with $(t-s)^{1+\alpha}$ and integrating s from 0 to t gives the bound

$$\|A\|\, \|v\|_{H^1} \int_0^t \frac{\|Du_N(\cdot,\tau) - Du_N^*(\cdot,\tau)\|_2}{(t-\tau)^\alpha}\, d\tau. \qquad (6.2)$$

The integral on the second line is bounded by

$$\begin{aligned}
&\|g_N - g_N^*\|_2 (t-s) + (t-s)\int_0^t \|u_N{}'(\cdot,\xi) - u_N^*{}'(\cdot,\xi)\|_2\, d\xi \\
&\qquad + \int_s^t \|u_N{}'(\cdot,\tau) - u_N^*{}'(\cdot,\tau)\|_2\, d\tau \qquad (6.3)
\end{aligned}$$

Dividing (6.3) with $(t-s)^{1+\alpha}$ and integrating s from 0 to t gives the bound

$$C\Bigg(\|g_N - g_N^*\|_2 t^{1-\alpha} + t^{1-\alpha}\int_0^t \|u_N{}'(\cdot,\xi) - u_N^*{}'(\cdot,\xi)\|_2\, d\xi \\ + \int_0^t \frac{\|u_N{}'(\cdot,\tau) - u_N^*{}'(\cdot,\tau)\|_2}{(t-\tau)^\alpha}\, d\tau\Bigg) \tag{6.4}$$

The term on the third line is bounded by

$$\begin{aligned} & b\sum_{j=1}^\infty \sqrt{\lambda_j} G_j \int_s^t \|v\|_2\, \|e_j\|_\infty \Bigg(\frac{\|g_N - g_N^*\|_2}{\tau^\alpha} \\ &\qquad + \frac{1}{\tau^\alpha}\int_0^\tau \|u_N{}'(\cdot,\xi) - u_N^*{}'(\cdot,\xi)\|_2\, d\xi\Bigg)\, d\tau \\ \leq\ & C\, b\, \widehat{G}\, \|v\|_2 \Bigg(\|g_N - g_N^*\|_2 \int_s^t \frac{d\tau}{\tau^\alpha} \\ &\qquad + \int_s^t \frac{1}{\tau^\alpha}\int_0^\tau \|u_N{}'(\cdot,\xi) - u_N^*{}'(\cdot,\xi)\|_2\, d\xi\, d\tau\Bigg) \end{aligned} \tag{6.5}$$

Dividing (6.5) by $(t-s)^{1+\alpha}$ and integrating in s from 0 to t gives

$$C\, b\, \widehat{G}\, \|v\|_2 \Bigg(\|g_N - g_N^*\|_2 \int_0^t \frac{1}{(t-s)^{1+\alpha}}\int_s^t \frac{d\tau}{\tau^\alpha}\, ds \\ + \int_0^t \frac{1}{(t-s)^{1+\alpha}}\int_s^t \frac{1}{\tau^\alpha}\int_0^\tau \|u_N{}'(\cdot,\xi) - u_N^*{}'(\cdot,\xi)\|_2\, d\xi\, d\tau\, ds\Bigg), \tag{6.6}$$

where the first integral is

$$\begin{aligned} \int_0^t \frac{1}{(t-s)^{1+\alpha}}\int_s^t \frac{d\tau}{\tau^\alpha}\, ds &= \int_0^t \frac{1}{\tau^\alpha}\int_0^\tau \frac{ds}{(t-s)^{1+\alpha}}\, d\tau \\ &\leq C\int_0^t \frac{d\tau}{\tau^\alpha (t-\tau)^\alpha} \\ &\leq \frac{C}{t^\alpha}\int_0^t \left(\frac{1}{\tau^\alpha} + \frac{1}{(t-\tau)^\alpha}\right) d\tau \\ &\leq C\, t^{1-2\alpha}. \end{aligned} \tag{6.7}$$

The second integral in (6.6) is

$$\begin{aligned} & \int_0^t \frac{1}{(t-s)^{1+\alpha}}\int_s^t \frac{1}{\tau^\alpha}\int_0^\tau \|u_N{}'(\cdot,\xi) - u_N^*{}'(\cdot,\xi)\|_2\, d\xi\, d\tau\, ds \\ &= \int_0^t \frac{1}{\tau^\alpha}\int_0^\tau \frac{1}{(t-s)^{1+\alpha}}\int_0^\tau \|u_N{}'(\cdot,\xi) - u_N^*{}'(\cdot,\xi)\|_2\, d\xi\, ds\, d\tau \\ &\leq C\int_0^t \frac{1}{\tau^\alpha (t-\tau)^\alpha}\int_0^\tau \|u_N{}'(\cdot,\xi) - u_N^*{}'(\cdot,\xi)\|_2\, d\xi\, d\tau \\ &\leq C\, t^{1-2\alpha}\int_0^t \|u_N{}'(\cdot,\xi) - u_N^*{}'(\cdot,\xi)\|_2\, d\xi \end{aligned}$$

by the calculations in (6.7). Hence, (6.6) is bounded by

$$C\, b\, \widehat{G}\, \|v\|_2 t^{1-2\alpha}\left(\|g_N - g_N^*\|_2 + \int_0^t \|u_N{}'(\cdot,\xi) - u_N^*{}'(\cdot,\xi)\|_2\, d\xi\right). \tag{6.8}$$

Summing up the terms on the last line gives the bound

$$b\,\widehat{G}\,\|v\|_{H^1}\int_s^t\int_s^\tau \frac{\|u_N(\cdot,\tau)-u_N^*(\cdot,\tau)-u_N(\cdot,y)+u_N^*(\cdot,y)\|_{H^{-1}}}{(\tau-y)^{1+\alpha}}\,dy\,d\tau$$

and dividing by $(t-s)^{1+\alpha}$ and integrating in s from 0 to t gives a factor $b\ \widehat{G}\ \|v\|_{H^1}$ multiplied by

$$\begin{aligned}
&\int_0^t \frac{1}{(t-s)^{1+\alpha}}\int_s^t\int_s^\tau \frac{\|u_N(\cdot,\tau)-u_N^*(\cdot,\tau)-u_N(\cdot,y)+u_N^*(\cdot,y)\|_{H^{-1}}}{(\tau-y)^{1+\alpha}}\,dy\,d\tau\,ds\\
\leq\ &\int_0^t \frac{1}{(t-s)^{1+\alpha}}\int_s^t\int_s^\tau\int_y^\tau \|u_N{}'(\cdot,\xi)-u_N^*{}'(\cdot,\xi)\|_{H^{-1}}\,d\xi\,\frac{1}{(\tau-y)^{1+\alpha}}\,dy\,d\tau\,ds\\
=\ &\int_0^t \|u_N{}'(\cdot,\xi)-u_N^*{}'(\cdot,\xi)\|_{H^{-1}}\int_0^\xi\int_0^y \frac{ds}{(t-s)^{1+\alpha}}\int_\xi^t \frac{d\tau}{(\tau-y)^{1+\alpha}}\,dy\,d\xi \qquad (6.9)
\end{aligned}$$

where

$$\begin{aligned}
\int_0^\xi\int_0^y \frac{ds}{(t-s)^{1+\alpha}}\int_\xi^t \frac{d\tau}{(\tau-y)^{1+\alpha}}\,dy\ &\leq\ C\int_0^\xi \frac{dy}{(t-y)^\alpha(\xi-y)^\alpha}\\
&\leq\ C\,\frac{1}{(t-\xi)^\alpha}\int_0^\xi \frac{dy}{(\xi-y)^\alpha}\\
&\leq\ C\,\frac{\xi^{1-\alpha}}{(t-\xi)^\alpha}.
\end{aligned}$$

Hence, (6.9) is bounded by

$$b\,\widehat{G}\,\|v\|_{H^1}\,t^{1-\alpha}\int_0^t \frac{\|u_N{}'(\cdot,\xi)-u_N^*{}'(\cdot,\xi)\|_{H^{-1}}}{(t-\xi)^\alpha}\,d\xi. \qquad (6.10)$$

Dividing (6.1) by $\|v\|_{H^1}$ and taking the supremum over $v\in H^1$, using (6.2), (6.4), (6.8) and (6.10) yields

$$\begin{aligned}
&\|u_N{}'(\cdot,t)-u_N^*{}'(\cdot,t)-u_N{}'(\cdot,s)+u_N^*{}'(\cdot,s)\|_{H^{-1}}\\
&\leq\ C\Bigg(\|g_N-g_N^*\|_2 t^{1-\alpha}+t^{1-\alpha}\int_0^t \|u_N{}'(\cdot,\xi)-u_N^*{}'(\cdot,\xi)\|_2\,d\xi\\
&\qquad +\ (1+\widehat{G}t^{1-\alpha})\int_0^t \frac{\|u_N{}'(\cdot,\tau)-u_N^*{}'(\cdot,\tau)\|_2+\|Du_N(\cdot,\tau)-Du_N^*(\cdot,\tau)\|_2}{(t-\tau)^\alpha}\,d\tau\Bigg)\\
&\leq\ C\Bigg(\|g_N-g_N^*\|_2 t^{1-\alpha}+(1+\widehat{G})t^{1-\alpha}\int_0^t \frac{\|u_N{}'(\cdot,\tau)-u_N^*{}'(\cdot,\tau)\|_2}{(t-\tau)^\alpha}\,d\tau\\
&\qquad +\int_0^t \frac{\|Du_N(\cdot,\tau)-Du_N^*(\cdot,\tau)\|_2}{(t-\tau)^\alpha}\,d\tau\Bigg). \qquad (6.11)
\end{aligned}$$

which proves the Proposition. □

Theorem 6.2. *Let u and u^* be solutions corresponding to initial data (g,h) and (g^*,h^*) respectively. Then there is a (random) constant, depending on $\widehat{G}$, such that*

$$\|u-u^*\|_{H^1(I\times U)}\ \leq\ C\left(\|g-g^*\|_{H^1(U)}+\|h-h^*\|_2\right),\qquad a.s.$$

Proof: Consider the difference of the equations (4.4) for two different sequences u_n and u_N^* and denote this difference $Z = Z_N(x,t)$. As in the proof of Proposition 5.2 we now square, using Proposition A.1 in [18], and sum over $n \in \{1,\dots,N\}$ to get

$$\begin{aligned}
&\left\|Z_N'(\cdot,t)\right\|_2^2 + 2\int_0^t \left\langle DZ_N(\cdot,\tau),\ ADZ_N'(\cdot,\tau)\right\rangle d\tau \\
&= \quad 2\left\langle h_N - h_N^*,\ Z_N'(\cdot,t)\right\rangle_2 \\
&\quad + 2\int_0^t \langle f(\cdot,\tau,u_N(\cdot,\tau),u_N{}'(\cdot,\tau),Du_N(\cdot,\tau)) \\
&\qquad\qquad -f(\cdot,\tau,u_N^*(\cdot,\tau),u_N^*{}'(\cdot,\tau),Du_N^*(\cdot,\tau)),Z_N'(\cdot,\tau)\rangle_2\, d\tau \\
&\quad + 2b\sum_{j=1}^{\infty}\sqrt{\lambda_j}\int_0^t \left\langle Z_N(\cdot,\tau),\ e_j Z_N'(\cdot,\tau)\right\rangle_2 d\beta_j(\tau)
\end{aligned} \tag{6.12}$$

Hence, integrating the second term to the left making use of the ellipticity condition and boundedness of the elliptic operator,

$$\begin{aligned}
&\left\|Z_N'(\cdot,t)\right\|_2^2 + a_0\left\|DZ_N(\cdot,t)\right\|_2^2 \\
&\leq \quad A_0\left\|DZ_N(\cdot,0)\right\|_2^2 + 2\left\|h_N - h_N^*\right\|_2\left\|Z_N'(\cdot,t)\right\|_2 \\
&\quad + 2\int_0^t \left(\left\|Z_N(\cdot,\tau)\right\|_2 + \left\|Z_N'(\cdot,\tau)\right\|_2 + \left\|DZ_N(\cdot,\tau)\right\|_2\right)\left\|Z_N'(\cdot,t)\right\|_2 d\tau \\
&\quad + 2b\sum_{j=1}^{\infty}\sqrt{\lambda_j}G_j\Bigg\{\int_0^t \left|\frac{\left\langle Z_N(\cdot,\tau),\ e_j\left[Z_N'(\cdot,\tau)\right]\right\rangle_2}{\tau^{\alpha}}\right| d\tau \\
&\quad + \alpha\int_0^t \left|\int_0^{\tau}\frac{\left\langle\left[Z_N(\cdot,\tau) - Z_N(\cdot,\theta)\right]Z_N'(\cdot,\tau),\ e_j\right\rangle_2}{(\tau-\theta)^{1+\alpha}}\,d\theta\right| d\tau \\
&\quad + \alpha\int_0^t \left|\int_0^{\tau}\frac{\left\langle Z_N(\cdot,\theta)\left[Z_N'(\cdot,\tau) - Z_N'(\cdot,\theta)\right],\ e_j\right\rangle_2}{(\tau-\theta)^{1+\alpha}}\,d\theta\right| d\tau\Bigg\}
\end{aligned} \tag{6.13}$$

By Cauchy's inequality with ε the second term to the right is bounded by

$$C\ \left(\frac{1}{\varepsilon}\left\|h_N - h_N^*\right\|_2^2 + \varepsilon\left\|Z_N'(\cdot,t)\right\|_2^2\right). \tag{6.14}$$

The first integral term is, by Cauchy's inequality, bounded by

$$C\ \int_0^t \left(\left\|Z_N(\cdot,\tau)\right\|_2^2 + \left\|Z_N'(\cdot,\tau)\right\|_2^2 + \left\|DZ_N(\cdot,\tau)\right\|_2^2\right) d\tau \tag{6.15}$$

where the integral of the first integrand is bounded by

$$\begin{aligned}
&C\ \left(\int_0^t \left[\left\|g_N - g_N^*\right\|_2^2 + \left(\int_0^{\tau}\left\|Z_N'(\cdot,\xi)\right\|_2 d\xi\right)^2\right] d\tau\right) \\
&\leq \quad C\ \left(\left\|g_N - g_N^*\right\|_2^2 t + \int_0^t \tau\left\|Z_N'(\cdot,\tau)\right\|_2^2 d\xi\right) \\
&\leq \quad C\ \left(\left\|g_N - g_N^*\right\|_2^2 + \int_0^t \left\|Z_N'(\cdot,\tau)\right\|_2^2 d\xi\right).
\end{aligned}$$

Hence, (6.15) is bounded by

$$C\ \left(\left\|g_N - g_N^*\right\|_2^2 + \int_0^t \left(\left\|Z_N'(\cdot,\tau)\right\|_2^2 + \left\|DZ_N(\cdot,\tau)\right\|_2^2\right) d\tau\right). \tag{6.16}$$

The second integral term on the right, i.e., the first one appearing in the sum, is bounded by, using Hölder's and Cauchy's inequalities and changing the order of integration

$$
\begin{aligned}
&\|e_j\|_2 \int_0^t \frac{\|Z_N(\cdot,\tau)\|_2 \, \|Z_N'(\cdot,\tau)\|_2}{\tau^\alpha}\, d\tau \\
&\leq \quad \|e_j\|_2 \int_0^t \left(\|g_N - g_N^*\|_2 + \int_0^\tau \|Z_N'(\cdot,\xi)\|_2 \, d\xi \right) \frac{\|Z_N'(\cdot,\tau)\|_2}{\tau^\alpha}\, d\tau \\
&\leq \quad C\, \|e_j\|_2 \left(\|g_N - g_N^*\|_2^2 t^{1-\alpha} + \int_0^t \frac{\|Z_N'(\cdot,\tau)\|_2^2}{\tau^\alpha}\, d\tau \right. \\
&\qquad \left. + \int_0^t \int_0^\tau \|Z_N'(\cdot,\xi)\|_2 \, d\xi \frac{\|Z_N'(\cdot,\tau)\|_2}{\tau^\alpha}\, d\tau \right) \\
&\leq \quad C\, \|e_j\|_2 \left(\|g_N - g_N^*\|_2^2 t^{1-\alpha} + \int_0^t \frac{\|Z_N'(\cdot,\tau)\|_2^2}{\tau^\alpha}\, d\tau \right. \\
&\qquad \left. + t^{1-\alpha} \int_0^t \|Z_N'(\cdot,\xi)\|_2 \, d\xi \right) \\
&\leq \quad C\, \|e_j\|_2 \left(\|g_N - g_N^*\|_2^2 t^{1-\alpha} + \int_0^t \frac{\|Z_N'(\cdot,\tau)\|_2^2}{\tau^\alpha}\, d\tau \right). \qquad (6.17)
\end{aligned}
$$

The fifth integral to the right is bounded by, using Hölder's inequality, changing the order of integration, and Cauchy's inequality,

$$
\begin{aligned}
&\|e_j\|_2 \int_0^t \|Z_N'(\cdot,\tau)\|_2 \int_0^\tau \frac{\|Z_N(\cdot,\tau) - Z_N(\cdot,\theta)\|_2}{(\tau-\theta)^{1+\alpha}}\, d\theta\, d\tau \\
&\leq \quad \|e_j\|_2 \int_0^t \|Z_N'(\cdot,\tau)\|_2 \int_0^\tau \int_\theta^\tau \frac{\|Z_N'(\cdot,\xi)\|_2}{(\tau-\theta)^{1+\alpha}}\, d\xi\, d\theta\, d\tau \\
&\leq \quad C\, \|e_j\|_2 \int_0^t \|Z_N'(\cdot,\tau)\|_2 \int_0^\tau \frac{\|Z_N'(\cdot,\xi)\|_2}{(\tau-\xi)^{\alpha}}\, d\xi\, d\tau \\
&\leq \quad C\, \|e_j\|_2 \left[\int_0^t \|Z_N'(\cdot,\tau)\|_2^2 \, d\tau + \int_0^t \left(\int_0^\tau \frac{\|Z_N'(\cdot,\xi)\|_2}{(\tau-\xi)^{\alpha}}\, d\xi \right)^2 d\tau \right] \\
&\leq \quad C\, \|e_j\|_2 \left[\int_0^t \|Z_N'(\cdot,\tau)\|_2^2 \, d\tau + \int_0^t \tau^{1-\alpha} \int_0^\tau \frac{\|Z_N'(\cdot,\xi)\|_2^2}{(\tau-\xi)^{\alpha}}\, d\xi\, d\tau \right] \\
&\leq \quad C\, \|e_j\|_2 \left[\int_0^t \|Z_N'(\cdot,\tau)\|_2^2 \, d\tau + t^{2-2\alpha} \int_0^t \|Z_N'(\cdot,\xi)\|_2^2 \, d\xi \right] \\
&\leq \quad C\, \|e_j\|_2 \int_0^t \|Z_N'(\cdot,\tau)\|_2^2 \, d\tau \qquad (6.18)
\end{aligned}
$$

The last integral in (6.13) is, by changing the order of integration and using Cauchy's and Hölder's inequalities, bounded by

$$
\begin{aligned}
&\|e_j\|_{H^{1,\infty}} \int_0^t \|Z_N(\cdot,\theta)\|_{H^1(U)} \int_\theta^t \frac{\|Z_N'(\cdot,\tau)-Z_N'(\cdot,\theta)\|_{H^{-1}}}{(\tau-\theta)^{1+\alpha}}\,d\tau\,d\theta \\
&\quad\le\; C\,\|e_j\|_{H^{1,\infty}} \int_0^t \left[\|Z_N(\cdot,\theta)\|_{H^1(U)}^2 + \left(\int_\theta^t \frac{\|Z_N'(\cdot,\tau)-Z_N'(\cdot,\theta)\|_{H^{-1}}}{(\tau-\theta)^{1+\alpha}}\,d\tau\right)^2\right] d\theta \\
&\quad\le\; C\,\|e_j\|_{H^{1,\infty}} \left[\int_0^t \|Z_N(\cdot,\theta)\|_{H^1(U)}^2\,d\theta \right.\\
&\qquad\qquad \left. + \int_0^t (t-\theta)^{1-\alpha} \int_\theta^t \frac{\|Z_N'(\cdot,\tau)-Z_N'(\cdot,\theta)\|_{H^{-1}}^2}{(\tau-\theta)^{2+\alpha}}\,d\tau\,d\theta\right] \\
&\quad\le\; C\,\|e_j\|_{H^{1,\infty}} \left[\|Z_N(\cdot,0)\|_2^2 + \int_0^t \|Z_N'(\cdot,\theta)\|_2^2\,d\theta + \int_0^t \|DZ_N(\cdot,\theta)\|_2^2\,d\theta \right.\\
&\qquad\qquad \left. + \int_0^t\int_0^\tau \frac{\|Z_N'(\cdot,\tau)-Z_N'(\cdot,\theta)\|_{H^{-1}}^2}{(\tau-\theta)^{2+\alpha}}\,d\theta\,d\tau\right] \\
&\quad\le\; C\,\|e_j\|_{H^{1,\infty}}(1+\widehat{G}^2) \left[\|Z_N(\cdot,0)\|_2^2 + \int_0^t \frac{\|Z_N'(\cdot,\theta)\|_2^2}{(t-\theta)^\alpha}\,d\theta \right.\\
&\qquad\qquad \left. + \int_0^t \frac{\|DZ_N(\cdot,\theta)\|_2^2}{(t-\theta)^\alpha}\,d\theta\right]
\end{aligned} \tag{6.19}
$$

Chosing $\varepsilon = 1/(2C)$ in (6.14), and putting the estimates (6.14), (6.16), (6.17), (6.18), and (6.19) back into (6.13) gives

$$
\begin{aligned}
&\left\|Z_N'(\cdot,t)\right\|_2^2 + a_0\,\|DZ_N(\cdot,t)\|_2^2 \\
&\quad\le\; C\,(1+\widehat{G}^3)\Big(\|h_N-h_N^*\|_2^2 + \|g_N-g_N^*\|_{H^1(U)}^2 \\
&\qquad + \int_0^t \frac{\|Z_N'(\cdot,\tau)\|_2^2}{\tau^\alpha(t-\tau)^\alpha}\,d\tau + \int_0^t \frac{\|DZ_N(\cdot,\theta)\|_2^2}{(t-\theta)^\alpha}\,d\theta\Big)
\end{aligned} \tag{6.20}
$$

An application of Gronwall's lemma gives now

$$
\left\|Z_N'(\cdot,t)\right\|_2^2 + \|DZ_N(\cdot,t)\|_2^2 \;\le\; C\,(1+\widehat{G}^3)e^{C\widehat{G}^3}\left(\|h_N-h_N^*\|_2^2 + \|g_N-g_N^*\|_{H^1(U)}^2\right)
$$

Now, let $\psi \in H^1(I\times U)$. Then we have

$$
\begin{aligned}
|\langle u_N-u_N^*,\,\psi\rangle| &\le \|u_N-u_N^*\|_{H^1(I\times U)}\;\|\psi\|_{H^1(I\times U)} \\
&\le C\,e^{C\widehat{G}^3}\left(\|g_N-g_N^*\|_{H^1(U)} + \|h_N-h_N^*\|_2\right)\|\psi\|_{H^1(I\times U)}\,.
\end{aligned}
$$

Hence, in the limit $N\to\infty$ we get, by previous results,

$$
|\langle u-u^*,\,\psi\rangle| \;\le\; C\,e^{C\widehat{G}^3}\left(\|g-g^*\|_{H^1(U)} + \|h-h^*\|_2\right)\|\psi\|_{H^1(I\times U)}\,.
$$

Dividing by $\|\psi\|_{H^1(I\times U)}$ and taking the supremum over all functions ψ gives the final estimate. □

The following are immediate consequences.

Corollary 6.3. *Any solution to (4.1) is unique.*

Corollary 6.4. *The solution to (4.1) depends continuously on the initial data.*

7 Stability with respect to truncation

When trying to implement a numerical scheme for (4.4) it would be a natural first step to truncate the infinite sum of (one dimensional) stochastic integrals. One then hopes that the solution corresponding to the truncated sum is, in some sense, close to the original one. To discuss these matters in more detail, let u be the solution to (4.4) and u^{π} a solution of the same equation but with $\lambda_j = 0,\ j \geq \Pi + 1$, for some positive integer Π. With this notation the following result holds.

Theorem 7.1. *Assume $\sigma(x) = a + bx$. Then there is a random variable $C = C(\omega)$, which is finite a.s., such that the following estimate holds:*

$$\|u' - u^{\pi\prime}\|_{L^\infty(I;L^2(U))} + \|Du - Du^{\pi}\|_{L^\infty(I;L^2(U))} \quad \leq \quad C(\omega)\,(\widehat{G} - \widehat{G}^{\pi}).$$

Before proving this we will derive a variant of Proposition 4.9 for the process $Z_N{}' = u_N{}' - u_N^{\pi}{}'$. Let us use the notation $R_\theta(\cdot,t) = Z_N{}'(\cdot,t) - Z_N{}'(\cdot,\theta)$. Similarly as for $u_N{}'$ we get

$$\begin{aligned} |\langle R_\theta(\cdot,\tau),\ \nu\rangle| \quad \leq \quad & \|A_0\|\ \|D\nu\|_2 \int_\theta^\tau \|DZ_N(\cdot,s)\|_2\,ds \\ & + C\ \|nu\|_2 \int_\theta^\tau \Big(\|Z_N(\cdot,s)\|_{H^1(U)} + \|Z_N{}'(\cdot,s)\|_2\Big)\,ds \\ & + \widehat{G}^{\pi}(\tau-\theta)^{1-\alpha} b \int_0^\tau \|Z_N{}'(\cdot,s)\|_2\,ds \\ & + (\widehat{G} - \widehat{G}^{\pi})\|\nu\|_2(\tau-\theta)^{1-\alpha}\left(a + b\|g\|_2 + b\int_0^\tau \|u_N{}'(\cdot,s)\|_2\,ds\right). \end{aligned}$$

Dividing by $\|\nu\|_{H^1(U)}$ gives the following upper bound on $\|R_\theta(\cdot,\tau)\|_{H^{-1}(U)}$:

$$\begin{aligned} & C\ \int_\theta^\tau \Big(\|Z_N(\cdot,s)\|_{H^1(U)} + \|Z_N{}'(\cdot,s)\|_2\Big)\,ds \\ & \quad + \widehat{G}^{\pi}(\tau-\theta)^{1-\alpha} b \int_0^\tau \|Z_N{}'(\cdot,s)\|_2\,ds \\ & \quad + c\ (\widehat{G} - \widehat{G}^{\pi})(\tau-\theta)^{1-\alpha}\left(1 + \|g\|_2 + \int_0^\tau \|u_N{}'(\cdot,s)\|_2\,ds\right). \end{aligned} \tag{7.1}$$

Proof: Let u^{π} be the solution to the noise truncated equation

$$\begin{aligned} \big\langle u'(\cdot,t),\ w_n\big\rangle_2 \quad = \quad & \langle h,\ w_n\rangle_2 \\ & - \int_0^t \langle Du(\cdot,\tau),\ ADw_n\rangle_2\,d\tau \\ & + \int_0^t \big\langle f(\cdot,\tau,u(\cdot,\tau),u'(\cdot,\tau),Du(\cdot,\tau)),\ w_n\big\rangle_2\,d\tau \\ & + \sum_{j=1}^{\pi} \sqrt{\lambda_j} \int_0^t \big\langle \sigma(u(\cdot,\tau)),\ w_n e_j\big\rangle_2\,\beta_j^H(d\tau). \end{aligned}$$

Since $Z_N(0) = 0$ we have

$$\begin{aligned}
\left\langle Z_N{}'(\cdot,t),\, w_n\right\rangle_2 &= \int_0^t \langle DZ_N(\cdot,\tau),\, ADw_n\rangle_2\, d\tau \\
&+ \int_0^t \langle f(\cdot,\tau,u(\cdot,\tau),u'(\cdot,\tau),Du(\cdot,\tau)) \\
&\qquad - f(\cdot,\tau,u^\pi(\cdot,\tau),u^{\pi\prime}(\cdot,\tau),Du^\pi(\cdot,\tau)),\, w_n\rangle_2\, d\tau \\
&+ \sum_{j=1}^{\pi} \sqrt{\lambda_j} \int_0^t \left\langle \sigma(u(\cdot,\tau)) - \sigma(u^\pi(\cdot,\tau)),\, w_n e_j\right\rangle_2 \beta_j^H(d\tau) \\
&+ \sum_{j=\pi+1}^{\infty} \sqrt{\lambda_j} \int_0^t \left\langle \sigma(u(\cdot,\tau)),\, w_n e_j\right\rangle_2 \beta_j^H(d\tau).
\end{aligned}$$

Squaring and summing gives

$$\begin{aligned}
\|Z_N{}'(\cdot,t)\|_2^2 &= -2\int_0^t \left\langle DZ_N(\cdot,\tau),\, ADZ_N{}'(\cdot,\tau)\right\rangle_2 d\tau \\
&+2\int_0^t \langle f(\cdot,\tau,u(\cdot,\tau),u'(\cdot,\tau),Du(\cdot,\tau)) \\
&\qquad - f(\cdot,\tau,u^\pi(\cdot,\tau),u^{\pi\prime}(\cdot,\tau),Du^\pi(\cdot,\tau)),\, Z_N'(\cdot,\tau)\rangle\, d\tau \\
&+2\sum_{j=1}^{\pi} \sqrt{\lambda_j} \int_0^t \left\langle \sigma(u(\cdot,\tau)) - \sigma(u^\pi(\cdot,\tau)),\, Z_N{}'(\cdot,\tau)e_j\right\rangle_2 \beta_j^H(d\tau) \\
&+2\sum_{j=\pi+1}^{\infty} \sqrt{\lambda_j} \int_0^t \left\langle \sigma(u(\cdot,\tau)),\, Z_N{}'(\cdot,\tau)e_j\right\rangle_2 \beta_j^H(d\tau).
\end{aligned}$$

As in (5.3) and (5.4) we get an upper bound of $\|Z_N{}'(\cdot,t)\|_2^2 + a_0\|DZ_N(\cdot,t)\|_2^2$ as

$$\begin{aligned}
&C\int_0^t \left(\|Z_N(\cdot,\tau)\|_2^2 + \|Z_N{}'(\cdot,\tau)\|_2^2 + \|DZ_N(\cdot,\tau)\|_2^2\right) d\tau \\
&\quad +2\sum_{j=1}^{\pi} \sqrt{\lambda_j} G_j \int_0^t \left| \frac{\left\langle Z_N(\cdot,\tau),\, Z_N{}'(\cdot,\tau)e_j\right\rangle_2}{\tau^\alpha} \right. \\
&\qquad\qquad \left. + \int_0^\tau \frac{\left\langle Z_N(\cdot,\tau),\, Z_N{}'(\cdot,\tau)e_j\right\rangle - \left\langle Z_N(\cdot,y),\, Z_N{}'(\cdot,y)e_j\right\rangle}{(\tau-y)^{1+\alpha}}\, dy \right| d\tau \\
&\quad +2\sum_{j=\pi+1}^{\infty} \sqrt{\lambda_j} G_j \int_0^t \left| \frac{\left\langle 1+u_N(\cdot,\tau),\, Z_N{}'(\cdot,\tau)e_j\right\rangle_2}{\tau^\alpha} \right. \\
&\quad \left. + \int_0^\tau \frac{\left\langle a+bu_N(\cdot,\tau),\, Z_N{}'(\cdot,\tau)e_j\right\rangle - \left\langle a+bu_N(\cdot,y),\, Z_N{}'(\cdot,y)e_j\right\rangle}{(\tau-y)^{1+\alpha}}\, dy \right| d\tau
\end{aligned} \tag{7.2}$$

For the first finite sum we have

$$\begin{aligned}
&2\sum_{j=1}^{\pi} \sqrt{\lambda_j} G_j \int_0^t \left| \frac{\left\langle Z_N(\cdot,\tau),\, Z_N{}'(\cdot,\tau)e_j\right\rangle_2}{\tau^\alpha} \right| d\tau \\
&\quad \le \; 2\sum_{j=1}^{\pi} \sqrt{\lambda_j} G_j\, \|e_j\|_\infty \int_0^t \frac{\|Z_N(\cdot,\tau)\|_2^2 + \|Z_N{}'(\cdot,\tau)\|_2^2}{\tau^\alpha}\, d\tau \\
&\quad \le \; 2\widehat{G}^\pi \int_0^t \frac{\|Z_N{}'(\cdot,\tau)\|_2^2}{\tau^\alpha}\, d\tau.
\end{aligned}$$

For the first infinite sum

$$
\begin{aligned}
&2\sum_{j=\pi+1}^{\infty}\sqrt{\lambda_j}G_j\int_0^t\left|\frac{\langle 1+u_N(\cdot,\tau),\,Z_N{}'(\cdot,\tau)e_j\rangle_2}{\tau^\alpha}\right|d\tau\\
&\quad\le\; c\left(\sum_{j=\pi+1}^{\infty}\sqrt{\lambda_j}G_j\,\|e_j\|_\infty\right)^2\int_0^t\frac{1+\|u_N(\cdot,\tau)\|_2^2}{\tau^\alpha}\,d\tau+\int_0^t\frac{\|Z_N{}'(\cdot,\tau)\|_2^2}{\tau^\alpha}\,d\tau\\
&\quad\le\; C\,(\widehat{G}-\widehat{G}^\pi)^2\left(1+\|g\|_2^2+\|u_N\|_{L^2(U\times I)}\right)+\int_0^t\frac{\|Z_N{}'(\cdot,\tau)\|_2^2}{\tau^\alpha}\,d\tau
\end{aligned}
$$

so these two terms sum up to

$$
C\,(1+\widehat{G})\int_0^t\frac{\|Z_N{}'(\cdot,\tau)\|_2^2}{\tau^\alpha}\,d\tau+C\,(\widehat{G}-\widehat{G}^\pi)^2\left[1+\left(\|g\|_2^2+\|u_N\|_{L^2(U\times I)}\right)\right].
$$

For the second finite sum we have, as in (5.1), and using (7.1),

$$
\begin{aligned}
&2\sum_{j=1}^{\pi}\sqrt{\lambda_j}G_j\int_0^t\left|\int_0^\tau\frac{\langle Z_N(\cdot,\tau),\,Z_N{}'(\cdot,\tau)e_j\rangle-\langle Z_N(\cdot,y),\,Z_N{}'(\cdot,y)e_j\rangle}{(\tau-y)^{1+\alpha}}\,dy\right|d\tau\\
&\quad\le\; C\,\widehat{G}^\pi\left[\|g\|_2^2t+\int_0^t\left(\|Z_N{}'(\cdot,\tau)\|_2^2+\|DZ_N(\cdot,\tau)\|_2^2\right)d\tau\right.\\
&\qquad\qquad\left.+\int_0^t\left(\int_y^t\frac{\|Z_N{}'(\cdot,s)-Z_N{}'(\cdot,y)\|_{H^{-1}}}{(s-y)^{1+\alpha}}\,ds\right)^2dy\right]\\
&\quad\le\; C\,\widehat{G}^\pi\left(\|g\|_2^2t+\int_0^t\left(\|Z_N{}'(\cdot,\tau)\|_2^2+\|DZ_N(\cdot,\tau)\|_2^2\right)d\tau\right.\\
&\qquad\qquad\left.+(\widehat{G}-\widehat{G}^\pi)^2\left[1+\|g\|_2^2+\|u_N{}'\|^2_{L^\infty(I;L^2(U))}\right]\right)
\end{aligned}
$$

For the second infinite sum

$$
\begin{aligned}
&2\sum_{j=\pi+1}^{\infty}\sqrt{\lambda_j}G_j\int_0^t\left|\int_0^\tau\frac{\langle a+bu_N(\cdot,\tau),\,Z_N{}'(\cdot,\tau)e_j\rangle-\langle a+bu_N(\cdot,y),\,Z_N{}'(\cdot,y)e_j\rangle}{(\tau-y)^{1+\alpha}}\,dy\right|d\tau\\
&=\;2\sum_{j=\pi+1}^{\infty}\sqrt{\lambda_j}G_j\int_0^t\\
&\qquad\left|\int_0^\tau\frac{\langle a(Z_N{}'(\cdot,\tau)-Z_N{}'(\cdot,y))+b(u_N(\cdot,\tau)Z_N{}'(\cdot,\tau)-u_N(\cdot,y)Z_N{}'(\cdot,y)),\,e_j\rangle}{(\tau-y)^{1+\alpha}}\,dy\right|d\tau\\
&=\;2(\widehat{G}-\widehat{G}^\pi)\int_0^t\int_0^\tau\left[a\frac{\|Z_N{}'(\cdot,\tau)-Z_N{}'(\cdot,y)\|_{H^{-1}}}{(\tau-y)^{1+\alpha}}\right.\\
&\qquad\qquad+b\frac{\|u_N(\cdot,\tau)-u_N(\cdot,y)\|_2\,\|Z_N{}'(\cdot,\tau)\|_2}{(\tau-y)^{1+\alpha}}\\
&\qquad\qquad\left.+b\frac{\|Z_N{}'(\cdot,\tau)-Z_N{}'(\cdot,y)\|_{H^{-1}}\,\|u_N(\cdot,y)\|_2}{(\tau-y)^{1+\alpha}}\right]dy\,d\tau\\
&\le\;C\,(\widehat{G}-\widehat{G}^\pi)\left[(1+\|u_N\|_{L^\infty(I;L^2(U))})\int_0^t\int_0^\tau\frac{\|Z_N{}'(\cdot,\tau)-Z_N{}'(\cdot,y)\|_{H^{-1}}}{(\tau-y)^{1+\alpha}}\,dy\,d\tau\right.\\
&\qquad\qquad\left.+\|u_N{}'\|_{L^\infty(I;L^2(U))}\int_0^t\|Z_N{}'(\cdot,\tau)\|_2\,d\tau\right]\\
&\le\;C\,(\widehat{G}-\widehat{G}^\pi)^2\left[1+\|g\|_2^2+\|u_N\|^2_{L^\infty(I;L^2(U))}+\|u_N{}'\|^2_{L^\infty(I;L^2(U))}\right]+\int_0^t\|Z_N{}'(\cdot,\tau)\|_2^2\,d\tau
\end{aligned}
$$

Putting al these estimates back into (7.2) yields

$$
\begin{aligned}
&\|Z_N{}'(\cdot,t)\|_2^2 + a_0\|DZ_N(\cdot,t)\|_2^2 \\
&\leq \; C\int_0^t \left(\|Z_N(\cdot,\tau)\|_2^2 + \|Z_N{}'(\cdot,\tau)\|_2^2 + \|DZ_N(\cdot,\tau)\|_2^2\right) d\tau \\
&\quad +C\,(1+\widehat{G})\int_0^t \frac{\|Z_N{}'(\cdot,\tau)\|_2^2}{\tau^\alpha}\,d\tau + C\,(\widehat{G}-\widehat{G}^\pi)^2\left[1+\|g\|_2^2+\|u_N\|_{L^2(U\times I)}\right] \\
&\qquad\quad + (\widehat{G}-\widehat{G}^\pi)^2\left[1+\|g\|_2^2+\|u_N{}'\|^2_{L^\infty(I;L^2(U))}\right]\Big) \\
&\quad + C\,(\widehat{G}-\widehat{G}^\pi)^2\left[1+\|u_N\|^2_{L^\infty(I;L^2(U))}+\|g\|_2^2+\|u_N{}'\|^2_{L^\infty(I;L^2(U))}\right] \\
&\qquad\quad + \int_0^t \|Z_N{}'(\cdot,\tau)\|_2^2\,d\tau \\
&\leq \; C\,(1+\widehat{G})\int_0^t\left(\frac{\|Z_N{}'(\cdot,\tau)\|_2^2}{\tau^\alpha}+\frac{\|DZ_N(\cdot,\tau)\|_2^2}{\tau^\alpha}\right)d\tau \\
&\quad + C\,(\widehat{G}-\widehat{G}^\pi)^2\left[1+\left(\|g\|_2^2+\|u_N{}'\|^2_{L^\infty(I;L^2(U))}\right)\right].
\end{aligned}
$$

An application of Gronwall's inequality gives then

$$
\|Z_N{}'(\cdot,t)\|_2^2 + \|DZ_N(\cdot,t)\|_2^2 \;\leq\; C\,(\widehat{G}-\widehat{G}^\pi)^2\left[1+\left(\|g\|_2^2+\|u_N{}'\|^2_{L^\infty(I;L^2(U))}\right)\right]e^{C\widehat{G}}.
$$

Taking the square root, the sup over I, and sending N to ∞ now yields

$$
\begin{aligned}
&\|u'-u^{\pi\prime}\|_{L^\infty(I;L^2(U))} + \|Du-Du^\pi\|_{L^\infty(I;L^2(U))} \\
&\leq \; C\,(\widehat{G}-\widehat{G}^\pi)\left[a+b\left(\|g\|_2+\lim\|u_N{}'\|_{L^\infty(I;L^2(U))}\right)\right]e^{C\widehat{G}} \\
&\leq \; C(\omega)\,(\widehat{G}-\widehat{G}^\pi)
\end{aligned}
$$

for some random constant $C(\omega)$. □

References

[1] Barndorff-Nielsen, O.E., Probability and Statistics: Self-decomposability, Finance and Turbulence, *Probability towards 2000*, New York, 45-57, (1995).

[2] Biswas, S.K., Ahmed, N.U., Stabilization of Systems governed by the Wave Equation in the Presence of Distributed White Noise, *IEEE Trans. Automat. Control.* **AC 30**, 1043-1945, (1985).

[3] Biagini, F., Hu, Y., Øksendal, B., Zhang, T., *Fractional Brownian Motion and Applications*, Springer verlag, forthcoming book.

[4] Brody, D., Syroka, J., Zervos, M., Dynamical pricing of weather derivatives, *Quantitative Finance*, **2**, 189-198, (2002).

[5] Dalang, R.C., Extending martingale measure stochastic integral with applications to spatially Homogeneous S.P.D.E's, *El. J. Prob.* **4**, 1-29, (1999).

[6] Duncan, T.E., Maslowski, B., Pasic-Duncan, B., Some Properties of Linear Stochastic Distributed Parameter Systems with Fractional Brownian Motion, *Proc. 40'th IEEE* Conference on Decision and Control, Orlando, Florida, USA, December, (2001).

[7] Erraoui, M., Nualart, N., Ouknine, Y., Hyperbolic stochastic partial differential equations with additive fractional Brownian sheet, *Stochastic Dynamics* **3**, 121-139, (2003).

[8] Folland, B.F., *Modern techniques and their applications*, Wiley Interscience, (1984).

[9] Hu, Y., Øksendal, B., Zhang, T., Stochastic partial differential equations driven by multiparameter fractional noise, *Stochastic Processes, Physics and Geometry: New Interplays II* Ed. Gesztesy et.al AMS 2000, 327-337, (2000).

[10] Kolmogorov, A.N., Curves in a Hilbert Space that are invariant under the one-parameter group of motions, *Dokl. Acad. Nauk USSR* **26**, 6-9, (1940).

[11] Kolmogorov, A.N., Wienersche Spiralen und einige andere interessante Kurven in Hilbertschen Raum, *Dokl. Acad. Nauk USSR* **26**, 115-118, (1940).

[12] Leland, W., Taqqu, M., Willinger, W., Wilson, D., On the self-similar nature of ethernet traffic (extended version), *IEEE7ACM Trans.* **2**, 1-15, (1994).

[13] Miller, R.N., Tropical Data Assimilation with Simulated Data: the Impact of the Tropical Ocean and Global Atmosphere Thermal Array for the Ocean, *J. Geophys. Res.* **95**, 11461-11482, (1990).

[14] Millet, A., Morien, P.-L., On a Nonlinear Stochastic Wave Equation in the Plane: Existence and Uniqueness of the Solution, *Ann. Appl. Prob.* **11**, 3, 922-951, (2001).

[15] Mandelbrot, B.B., Van Ness, J.W., Fractional Brownian motion, fractional noises and applications, *SIAM Rev.* **10**, 422-437, (1968).

[16] Nualart, D., Ouknine, Y., Regularization of quasilinear heat equation equations by a fractional noise, *Stochastics and Dynamics* **4**, 201-221, (2004).

[17] Nualart, D., Rascanu, A., Differential Equations Driven by Fractional Brownian Motion, Preprint.

[18] Nualart, D., Vuillermot, P.-A., Variational Solutions for Partial Differential Equations Driven by a Fractional Noise, *J. Funct. Anal.* **232** , no. 2, 390–454, (2006).

[19] Pelletier, J.D., Turcotte, D.L., Long-range persistence in climatological and hydrological time series: Analysis, modelling and application to drought hazard assessment, *J. Hydrol.* **203**, 198-208, (1997).

[20] Ziemer, W.P., *Weakly differentiable functions*, Springer-Verlag, (1989).

[21] Zähle, M., Integration with respect to Fractional Functions and Stochastic Calculus. I, *Probab. Theory Relat. Fields* **111**, 333-374, (1998).

[22] Walsh, J.B., An introduction to stochastic partial differential equations, *Lecture Note in Mathematics* **1180**, Springer verlag, 266-439, (1984).