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Partial Information Linear Quadratic Control for Jump Diffusions

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Abstract

We study a stochastic control problem where the state process is described by a stochastic differential equation driven by a Brownian motion and a Poisson random measure, being affine in both the state and the control. The performance functional is quadratic in the state and the control. All the coefficients are allowed to be random and non-Markovian. Moreover, we may allow the control to be predictable to a given subfiltration of the filtration of the Brownian motion and the random measure (partial information control).

1 Introduction

The problem of stochastic control is always a hard one. Only in few cases is there an explicit solution. There are two important approaches to the general stochastic optimal control problem. One is the Bellman dynamic programming principle, which results in the Hamilton-Jacobi-Bellman equation. This approach is applicable when the controlled system

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is Markovian. Another important approach is the maximum principle. For detailed account of the approaches to systems driven by Brownian motions see the books [7], [18], and the references therein.

In this paper we will consider the stochastic optimal control problems when the controlled system is a jump-diffusion. If the controlled system is Markovian, there are also some developments recently. See the book [15] and the references therein. Some explicit control problems arising from finance and their solutions are also presented in this book.

Let $(W_t, t \geq 0)$ be a Brownian motion and $(N(ds,dz)\,, s \geq 0\,, z \in \mathbb{R})$ be a Poisson random measure with the intensity measure given by $\nu(dz)$. The compensated Poisson random measure is denoted by $\tilde{N}(ds,dz)$. We will consider only the case when the state x_t at time t is described by a linear controlled jump-diffusion of the form

$$\begin{aligned} dx_t &= [A_t x_t + B_t u_t + \alpha_t]\, dt + [C_t x_t + D_t u_t + \beta_t]\, dW_t \\ &\quad + \int_{\mathbb{R}} [E_t(z) x_{t-} + F_t(z) u_t + \gamma_t(z)]\, \tilde{N}(dt,dz)\,; \quad t \in [0,T] \\ x_0 &= x \in \mathbb{R}\,. \end{aligned} \tag{1.1}$$

Here u_t is our control process and A_t, B_t, α_t, C_t, D_t, β_t, $E_t(z)$, $F_t(z)$ and $\gamma_t(z)$ are given $\mathcal{F}_t$-predictable processes, where $\mathcal{F}_t$ is the filtration generated by the Brownian motion $W(s)$, $s \leq t$, and the Poisson random measures $\tilde{N}(ds,dz)$, $s \leq t$. The control u_t is required to be $\mathcal{E}_t$-predictable, where $\mathcal{E}_t \subseteq \mathcal{F}_t$ is a given filtration representing the information available to the controller at time t. For example, we could have

$$\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+}\,; \quad t \in [0,T]\,,$$

where $\delta > 0$ is a fixed *delay of information*.

The performance functional is assumed to have the form

$$\begin{aligned} J(x,u) &= \mathbb{E}\,\left\{H_1 x_T^2 + H_2 x_T\right\} \\ &\quad + \mathbb{E}\,\left\{\int_0^T \left[Q_{11}(t)x_t^2 + 2Q_{12}(t)x_t u_t + Q_{22}(t)u_t^2 + R_1(t)x_t + 2R_2(t)u_t\right] dt\right\}\,, \end{aligned} \tag{1.2}$$

where $Q_{ij}(t)$ and $R_i(t)$ are given bounded $\mathcal{F}_t$-adapted processes and H_i are given $\mathcal{F}_T$-measurable bounded random variables satisfying certain conditions (see Section 2). Even in the absence of jumps, namely,

$$E_t(z) = F_t(z) = \gamma_t(z) = 0$$

(diffusion case), the theory of classical linear quadratic control only deals with the case that

$$\mathcal{E}_t = \mathcal{F}_t \qquad \text{(complete information case)}$$

and

$$H_2 = 0\,, \quad \alpha_t = 0\,, \quad \beta_t = 0\,, \quad R_1(t) = R_2(t) = 0\,.$$

Namely, there are no first order terms in the utility functional and there are no constant terms in the system. If the coefficients are random (but predictable) and/or $\mathcal{E}_t \subset \mathcal{F}_t$, then the system is no longer Markovian. The most effective method is the technique of completing squares.

However, even if $\mathcal{E}_t = \mathcal{F}_t$ the classical technique of completing squares is not directly applicable to the system we consider because of the appearance of the first order terms in the utility functional and the constant terms in the controlled system. The appearance of such terms is important when we apply the results to minimum variance portfolio selection, for example.

In this paper we introduce an additional auxiliary backward Riccati equation to handle the extra terms. Thus we will have two (coupled) Riccati equations. Fortunately, they are only weakly coupled in the sense that we can solve one equation first and then substitute the solution into the other. This introduction of an additional equation which handles the linear and constant terms was done earlier in [17] for the constant term and in [13] for both linear and constant terms. There is a rich literature on stochastic linear quadratic control and associated Riccati equations. See e.g. [1], [2], [5], [10], [17].

We will apply our results to minimum variance portfolio selection problems with or without partial information [3], [8]. The results extend the ones in [9] (which use the Hamilton-Jacobi-Bellman dynamic programming principle) to the case of random coefficients.

It should be pointed out that the approach of the dynamic programming principle or the maximum principle cannot be applied directly here, both because of the general random coefficients in the controlled system and in the utility functional and because of partial information. Moreover, the technique of completing the square also leads us to the solution of the partial information problem.

2 The Complete Information Case

Let us first consider the case with complete information, i.e. $\mathcal{E}_t = \mathcal{F}_t$. Let the system be described by a one dimensional stochastic differential equation, driven both by Brownian white noise and Poissonian random measure, as follows:

$$\begin{aligned} dx_t = dx_t^{(u)} &= [A_t x_t + B_t u_t + \alpha_t]\,dt + [C_t x_t + D_t u_t + \beta_t]\,dW_t \\ &\quad + \int_{\mathbb{R}} [E_t(z) x_{t-} + F_t(z) u_t + \gamma_t(z)]\,\tilde{N}(dt,dz)\,; \quad 0 \le t \le T \\ x_0 &= x \in \mathbb{R}\,. \end{aligned} \tag{2.1}$$

We assume that A_t, C_t, $E_t(z)$ B_t, D_t, $F_t(z)$, α_t, β_t and $\gamma_t(z)$ are bounded $\mathbb{R}$-valued $\mathcal{F}_t$-predictable processes (they can be random). The goal is to minimize the following cost functional.

$$\begin{aligned} J(x,u) &= \mathbb{E}\left\{H_1 x_T^2 + H_2 x_T\right\} \\ &\quad + \mathbb{E}\left\{\int_0^T \left[Q_{11}(t)x_t^2 + 2Q_{12}(t)x_t u_t + Q_{22}(t)u_t^2 + R_1(t)x_t + 2R_2(t)u_t\right] dt\right\}, \end{aligned} \tag{2.2}$$

where $Q_{ij}(t)$ and $R_i(t)$, $i, j = 1, 2$, are given bounded $\mathcal{F}_t$-adapted (real valued) stochastic processes and H_1 and H_2 are $\mathcal{F}_T$-measurable bounded random variables.

We assume throughout this paper that

$$Q_{22}(t) + \Theta_3(t) \geq 0 \quad \text{for a.a.} \quad t\,, \omega\,,$$

where $\Theta_3(t)$ is defined by (2.14). This is a linear system with a quadratic utility functional. We say that the control u_t is *admissible* and write $u_t \in \mathcal{A}_{\mathcal{F}}$ if u_t is $\mathcal{F}_t$-predictable and Equation (2.1) has a unique strong solution $x_t = x_t^{(u)}$ for $0 \leq t \leq T$ and

$$\mathbb{E}\left[\int_0^T \left\{u^2(t) + \left(x_t^{(u)}\right)^2\right\} dt\right] < \infty\,.$$

We define

$$\begin{aligned}
\rho_1(t) &= \textstyle\int_{\mathbb{R}} E_t(z)^2 \nu(dz)\,, & \rho_2(t) &= \int_{\mathbb{R}} \mu_t(z)\left[E_t(z)^2 + 2E_t(z)\right]\nu(dz)\\
\rho_3(t) &= \textstyle\int_{\mathbb{R}} E_t(z)F_t(z)\nu(dz)\,, & \rho_4(t) &= \int_{\mathbb{R}} \mu_t(z)\left[E_t(z)F_t(z) + 2F_t(z)\right]\nu(dz)\\
\rho_5(t) &= \textstyle\int_{\mathbb{R}} F_t(z)^2\nu(dz)\,, & \rho_6(t) &= \int_{\mathbb{R}} \mu_t(z)F_t(z)^2\nu(dz)\\
\rho_7(t) &= \textstyle\int_{\mathbb{R}} \gamma_t(z)E_t(z)\nu(dz)\,, & \rho_8(t) &= \int_{\mathbb{R}} \mu_t(z)\gamma_t(z)\left[1 + E_t(z)\right]\nu(dz)\\
\rho_9(t) &= \textstyle\int_{\mathbb{R}} \gamma_t(z)F_t(z)\nu(dz)\,, & \rho_{10}(t) &= \int_{\mathbb{R}} \gamma_t(z)\mu_t(z)F_t(z)\nu(dz)\\
\rho_{11}(t) &= \textstyle\int_{\mathbb{R}} \gamma_t(z)^2\nu(dz)\,, & \rho_{12}(t) &= \int_{\mathbb{R}} \gamma_t(z)^2\mu_t(z)\nu(dz)\\
\rho_{13}(t) &= \textstyle\int_{\mathbb{R}} \tilde{\mu}_t(z)E_t(z)\nu(dz)\,, & \rho_{14}(t) &= \int_{\mathbb{R}} \tilde{\mu}_t(z)F_t(z)\nu(dz)\\
\rho_{15}(t) &= \textstyle\int_{\mathbb{R}} \tilde{\mu}_t(z)\gamma_t(z)\nu(dz)\,.
\end{aligned}$$

We introduce the following system of backward Riccati / backward linear stochastic differential equations in the two unknown processes p_t and $\tilde{p}_t$:

$$\begin{aligned}
&dp_t + \left[2p_tA_t + p_tC_t^2 + 2\eta_tC_t + \rho_1(t)p_t + \rho_2(t) + Q_{11}(t)\right]dt\\
&-\left[Q_{22}(t) + p_tD_t^2 + \rho_5(t)p_t + \rho_6(t)\right]^{-1}\left[p_tB_t + p_tC_tD_t + \eta_tD_t + \rho_3(t)p_t + \rho_4(t) + Q_{12}(t)\right]^2 dt\\
&-\eta_t dW_t - \int_{\mathbb{R}} \mu_t(z)\tilde{N}(dt, dz) = 0 && (2.3)\\
&p_T = H_1\,; && (2.4)
\end{aligned}$$

$$\begin{aligned}
&d\tilde{p}_t + \left[2p_t\alpha_t + 2\beta_tp_tC_t + 2\beta_t\eta_t + 2p_t\rho_7(t) + 2\rho_8(t)\right]dt\\
&+\left[\tilde{p}_tA_t + C_t\tilde{\eta}_t + \rho_{13}(t) + R_1(t)\right]dt\\
&-2\left[Q_{22}(t) + p_tD_t^2 + \rho_5(t)p_t + \rho_6(t)\right]^{-1}\left[p_tB_t + p_tC_dD_t + \eta_tD_t + \rho_3(t)p_t + \rho_4(t) + Q_{12}(t)\right]
\end{aligned}$$

$$\Big[p_t\beta_t D_t + p_t\rho_9(t) + \rho_{10}(t) + \frac{1}{2}\tilde{p}_t B_t + \frac{1}{2}\tilde{\eta}_t D_t + \frac{1}{2}\rho_{14}(t) + R_2(t)\Big]\,dt$$

$$-\tilde{\eta}_t dW_t - \int_{\mathbb{R}} \tilde{\mu}_t(z)\tilde{N}(dt,dz) = 0 \tag{2.5}$$

$$\tilde{p}_T = H_2\,. \tag{2.6}$$

Here the $\mathcal{F}_t$-predictable, square integrable processes ξ_t, η_t, $\mu_t(z)$ and $\tilde{\xi}_t$, $\tilde{\eta}_t$, $\tilde{\mu}_t(z)$ are (implicitly) determined from p_t and $\tilde{p}_t$, respectively, through the semimartingale representations

$$dp_t = \xi_t dt + \eta_t dW_t + \int_{\mathbb{R}} \mu_t(z)\tilde{N}(dt,dz) \tag{2.7}$$

and

$$d\tilde{p}_t = \tilde{\xi}_t dt + \tilde{\eta}_t dW_t + \int_{\mathbb{R}} \tilde{\mu}_t(z)\tilde{N}(dt,dz)\,. \tag{2.8}$$

We now state the first main theorem of this paper.

Theorem 2.1 *Suppose the system of backward Riccati equations (2.3)-(2.6) has a solution p_t and $\tilde{p}_t$. Define*

$$\begin{aligned} u_t \;=\; & -\Big[Q_{22}(t) + p_t D_t^2 + \rho_5(t)p_t + \rho_6(t)\Big]^{-1} \\ & \Big\{ [p_t B_t + p_t C_t D_t + \eta_t D_t + \rho_3(t)p_t + \rho_4(t) + Q_{12}(t)]\, x_{t-} + \\ & p_t\beta_t D_t + p_t\rho_9(t) + \rho_{10}(t) + \frac{1}{2}\left(\tilde{p}_t + \tilde{\eta}_t D_t + \rho_{14}(t)\right) + R_2(t)\Big\}\,. \end{aligned} \tag{2.9}$$

Suppose $u_t \in \mathcal{A}_{\mathcal{F}}$ and

$$\mathbb{E}\,\left[\int_0^T \left\{x_t^4\eta_t^2 + (x_t^4 + u_t^4)(p_t^2 + \int_{\mathbb{R}} \mu_t^2(z)\nu(dz)\right\} dt\right] < \infty\,. \tag{2.10}$$

Then u_t is the unique solution of the complete information linear quadratic control problem (2.1)-(2.2). The corresponding value function is also quadratic and it is given by

$$\begin{aligned} & \mathbb{E}\,(p_0)x^2 + \mathbb{E}\,(\tilde{p}_0)x \\ & + \mathbb{E} \int_0^T \left\{\Theta_6(t) + \Theta_9(t) - [Q_{22}(t) + \Theta_3(t)]^{-1}\,[\Theta_5(t) + \Theta_8(t) + R_2(t)]^2\right\} dt\,, \end{aligned} \tag{2.11}$$

where p_t and $\tilde{p}_t$ are found from solving the above backward equation and $\Theta_i(t)$, $i = 3,5,6,8,9$ are defined by (2.12)-(2.20).

Remark 2.2 *The existence of a solution to (2.3) has been proved recently by Hu and Song. See [11].*

If all the parameters are deterministic, then we can take η_t, $\tilde{\eta}_t$, $\mu_t(z)$ and $\tilde{\mu}_t(z)$ to be 0. In this case the stochastic Riccati equation reduces to the usual (deterministic) Riccati equation.

If at least one of them are stochastic and all of them depend only on Brownian white noise W, then we may choose $\mu_t(z)$ and $\tilde{\mu}_t(z)$ to be 0, but η_t and $\tilde{\eta}_t$ cannot both be 0. If at least one of them are stochastic and all of them depend only on Poisson noise $N(\cdot, dz)$, then we may choose η_t and $\tilde{\eta}_t$ to be 0. But $\mu_t(z)$ and $\tilde{\mu}_t(z)$ cannot both be 0.

Proof of Theorem 2.1. We shall use the technique of completing squares.

Applying (2.7) and the integration by parts formula we have

$$\begin{aligned} dx_t^2 &= 2x_{t-}dx_t + \left[C_tx_t + D_tu_t + \beta_t\right]^2 dt + \int_{\mathbb{R}} \left[E_t(z)x_{t-} + F_t(z)u_t + \gamma_t(z)\right]^2 N(dt, dz) \\ &= 2x_{t-}\Big\{ \left[A_tx_t + B_tu_t + \alpha_t\right] dt + \left[C_tx_t + D_tu_t + \beta_t\right] dW_t \\ &\quad + \int_{\mathbb{R}} \left[E_t(z)x_{t-} + F_t(z)u_t + \gamma_t(z)\right] \tilde{N}(dt, dz) \Big\} + \left[C_tx_t + D_tu_t + \beta_t\right]^2 dt \\ &\quad + \int_{\mathbb{R}} \left[E_t(z)x_{t-} + F_t(z)u_t + \gamma_t(z)\right]^2 \tilde{N}(dt, dz) + \int_{\mathbb{R}} \left[E_t(z)x_t + F_t(z)u_t + \gamma_t(z)\right]^2 \nu(dz)dt\,. \end{aligned}$$

Another integration by parts yields

$$\begin{aligned} d(p_tx_t^2) &= 2p_{t-}x_{t-}\Big\{ \left[A_tx_t + B_tu_t + \alpha_t\right] dt + \left[C_tx_t + D_tu_t + \beta_t\right] dW_t \\ &\quad + \int_{\mathbb{R}} \left[E_t(z)x_{t-} + F_t(z)u_t + \gamma_t(z)\right] \tilde{N}(dt, dz) \Big\} \\ &\quad + p_t \left[C_tx_t + D_tu_t + \beta_t\right]^2 dt + \int_{\mathbb{R}} p_{t-} \left[E_t(z)x_{t-} + F_t(z)u_t + \gamma_t(z)\right]^2 N(dt, dz) \\ &\quad + x_{t-}^2 \left[\xi_t dt + \eta_t dW_t + \int_{\mathbb{R}} \mu_t(z)\tilde{N}(dt, dz)\right] \\ &\quad + 2\eta_t x_t \left[C_tx_t + D_tu_t + \beta_t\right] dt + \int_{\mathbb{R}} \mu_t(z) \left[E_t(z)x_{t-} + F_t(z)u_t + \gamma_t(z)\right]^2 N(dt, dz) \\ &\quad + 2\int_{\mathbb{R}} \mu_t(z)x_{t-} \left[E_t(z)x_{t-} + F_t(z)u_t + \gamma_t(z)\right] N(dt, dz)\,. \end{aligned}$$

Denote

$$\begin{aligned} d\eta_1(t) &= x_{t-}^2 \left[\eta_t dW_t + \int_{\mathbb{R}} \mu_t(z)\tilde{N}(dt, dz)\right] + 2p_tx_t \left[C_tx_t + D_tu_t + \beta_t\right] dW_t \\ &\quad + \int_{\mathbb{R}} \Big\{ \mu_t(z)x_{t-}^2 + 2p_{t-}x_{t-} \left[E_t(z)x_{t-} + F_t(z)u_t + \gamma_t(z)\right] \\ &\quad + (p_{t-} + \mu_t(z)) \left[E_t(z)x_{t-} + F_t(z)u_t + \gamma_t(z)\right]^2 \Big\} \tilde{N}(dt, dz) \\ &\quad + 2\int_{\mathbb{R}} \mu_t(z)x_{t-} \left[E_t(z)x_{t-} + F_t(z)u_t + \gamma_t(z)\right] \tilde{N}(dt, dz) \end{aligned}$$

and $\eta_1(0) = 0$. Then we see from (2.10) that $\mathbb{E}\ \eta_1(t) = 0$ for all $t \geq 0$. We can rewrite

$$\begin{aligned} d(p_t x_t^2) &= x_t^2 \xi_t dt + 2p_t x_t \left[A_t x_t + B_t u_t + \alpha_t\right] dt \\ &\quad + p_t \left[C_t x_t + D_t u_t + \beta_t\right]^2 dt + 2x_t \eta_t \left[C_t x_t + D_t u_t + \beta_t\right] dt \\ &\quad + \int_{\mathbb{R}} \left\{ \left[p_t + \mu_t(z)\right] \left[E_t(z) x_t + F_t(z) u_t + \gamma_t(z)\right]^2 \right\} \nu(dz) dt \\ &\quad + 2 \int_{\mathbb{R}} \mu_t(z) x_t \left[E_t(z) x_t + F_t(z) u_t + \gamma_t(z)\right] \nu(dz) dt + d\eta_1(t) \,. \end{aligned}$$

Introduce the notations

$$\begin{aligned} \Theta_1(t) &= \xi_t + 2p_t A_t + p_t C_t^2 + 2\eta_t C_t \\ &\quad + \int_{\mathbb{R}} \Big[p_t E_t(z)^2 + \mu_t(z) E_t(z)^2 + 2\mu_t(z) E_t(z)\Big] \nu(dz) \,; \end{aligned} \tag{2.12}$$

$$\begin{aligned} \Theta_2(t) &= p_t B_t + p_t C_t D_t + \eta_t D_t \\ &\quad + \int_{\mathbb{R}} \left\{p_t E_t(z) F_t(z) + \mu_t(z) E_t(z) F_t(z) + \mu_t(z) F_t(z)\right\} \nu(dz) \,; \end{aligned} \tag{2.13}$$

$$\Theta_3(t) = p_t D_t^2 + \int_{\mathbb{R}} \Big\{p_t F_t(z)^2 + \mu_t(z) F_t(z)^2\Big\} \nu(dz) \,; \tag{2.14}$$

$$\begin{aligned} \Theta_4(t) &= 2p_t \alpha_t + 2\beta_t p_t C_t + 2\beta_t \eta_t \\ &\quad + 2 \int_{\mathbb{R}} \left[(p_t + \mu_t(z)) \gamma_t(z) E_t(z) + \mu_t(z) \gamma_t(z)\right] \nu(dz) \,; \end{aligned} \tag{2.15}$$

$$\Theta_5(t) = p_t \beta_t D_t + \int_{\mathbb{R}} (p_t + \mu_t(z)) \gamma_t(z) F_t(z) \nu(dz) \,; \tag{2.16}$$

and

$$\Theta_6(t) = p_t \beta_t^2 + \int_{\mathbb{R}} (p_t + \mu_t(z)) \gamma_t^2(z) \nu(dz) \,. \tag{2.17}$$

Then we have

$$\begin{aligned} \mathbb{E}\ \Big\{p_T x_T^2\Big\} &= \mathbb{E}\ \Big\{p_0 x^2\Big\} + \mathbb{E}\ \int_0^T \Bigg\{\Theta_1(t) x_t^2 + 2\Theta_2(t) x_t u_t + \Theta_3(t) u_t^2 \\ &\qquad + \Theta_4(t) x_t + 2\Theta_5(t) u_t + \Theta_6(t)\Bigg\} dt \,. \end{aligned} \tag{2.18}$$

To deal with the first order terms which appeared above (2.18) we combine (2.8) with the integration by parts formula and get

$$\begin{aligned} d(\tilde{p}_t x_t) &= x_{t-} \left[\tilde{\xi}_t dt + \tilde{\eta}_t dW_t + \int_{\mathbb{R}} \tilde{\mu}_t(z) \tilde{N}(dt, dz)\right] \\ &\quad + \tilde{p}_{t-} \{ \left[A_t x_t + B_t u_t + \alpha_t\right] dt + \left[C_t x_t + D_t u_t + \beta_t\right] dW_t \end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}} \left[E_t(z)x_{t-} + F_t(z)u_t + \gamma_t(z)\right] \tilde{N}(dt,dz)\} \\
&+\tilde{\eta}_t \left[C_t x_t + D_t u_t + \beta_t\right] dt \\
&+\int_{\mathbb{R}} \tilde{\mu}_t(z) \left[E_t(z)x_t + F_t(z)u_t + \gamma_t(z)\right] \nu(dz)dt \\
&+\int_{\mathbb{R}} \tilde{\mu}_t(z) \left[E_t(z)x_{t-} + F_t(z)u_t + \gamma_t(z)\right] \tilde{N}(dt,dz)\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{E}\ \left[\tilde{p}_T x_T\right] &= \mathbb{E}\ \Bigg[\tilde{p}_0 x + \int_0^T \{x_t \tilde{\xi}_t + \tilde{p}_t \left[A_t x_t + B_t u_t + \alpha_t\right] \\
&\quad +\tilde{\eta}_t \left[C_t x_t + D_t u_t + \beta_t\right] + \int_{\mathbb{R}} \tilde{\mu}_t(z) \left[E_t(z)x_t + F_t(z)u_t + \gamma_t(z)\right] \nu(dz)\}dt\Bigg] \\
&= \mathbb{E}\ \left[\tilde{p}_0 x + \int_0^T \{\Theta_7(t)x_t + 2\Theta_8(t)u_t + \Theta_9(t)\}dt\right]\,, \qquad (2.19)
\end{aligned}
$$

where

$$
\begin{aligned}
\Theta_7(t) &= \tilde{\xi}_t + \tilde{p}_t A_t + C_t \tilde{\eta}_t + \int_{\mathbb{R}} \tilde{\mu}_t(z) E_t(z) \nu(dz) \qquad (2.20)\\
\Theta_8(t) &= \frac{1}{2}\left\{\tilde{p}_t B_t + \tilde{\eta}_t D_t + \int_{\mathbb{R}} \tilde{\mu}_t(z) F_t(z) \nu(dz)\right\} \qquad (2.21)\\
\Theta_9(t) &= \tilde{p}_t \alpha_t + \tilde{\eta}_t \beta_t + \int_{\mathbb{R}} \tilde{\mu}_t(z) \gamma_t(z) \nu(dz)\,. \qquad (2.22)
\end{aligned}
$$

Let

$$
p_T = H_1 \quad \text{and} \quad \tilde{p}_T = H_2\,.
$$

Therefore

$$
\begin{aligned}
& J(x,u) \\
=\ & \Bigg\{\int_0^T \Big[Q_{11}(t)x_t^2 + 2Q_{12}(t)x_t u_t + Q_{22}(t)u_t^2 \\
& +R_1(t)x_t + 2R_2(t)u_t\Big]dt + p_T x_T^2 + \tilde{p}_T x_T\Bigg\} \\
=\ & \mathbb{E}\ (p_0 x^2) + \mathbb{E}\ (\tilde{p}_0 x) + \mathbb{E}\ \int_0^T \Big\{ \left[\Theta_1(t) + Q_{11}(t)\right] x_t^2 + 2\left[\Theta_2(t) + Q_{12}(t)\right] x_t u_t \\
& \quad + \left[Q_{22}(t) + \Theta_3(t)\right] u_t^2 + \left[\Theta_4(t) + \Theta_7(t) + R_1(t)\right] x_t \\
& \quad +2\left[\Theta_5(t) + \Theta_8(t) + R_2(t)\right] u_t + \Theta_6(t) + \Theta_9(t)\Big\}dt \\
=\ & \mathbb{E}\ (p_0 x^2) + \mathbb{E}\ (\tilde{p}_0 x) \\
& \quad +\mathbb{E}\ \int_0^T \Bigg\{ \Big[\Theta_1(t) + Q_{11}(t) - \left[Q_{22}(t) + \Theta_3(t)\right]^{-1} \left[\Theta_2(t) + Q_{12}(t)\right]^2\Big] x_t^2 \\
& \quad + \Big[\Theta_4(t) + \Theta_7(t) + R_1(t) - 2\left[Q_{22}(t) + \Theta_3(t)\right]^{-1} \left[\Theta_2(t) + Q_{12}(t)\right] \left[\Theta_5(t) + \Theta_8(t) + R_2(t)\right]\Big] x_t
\end{aligned}
$$

$$+\left[Q_{22}(t)+\Theta_3(t)\right]\Big\{u_t+\left[Q_{22}(t)+\Theta_3(t)\right]^{-1}\left[\Theta_2(t)+Q_{12}(t)\right]x_t$$
$$+\left[Q_{22}(t)+\Theta_3(t)\right]^{-1}\left[\Theta_5(t)+\Theta_8(t)+R_2(t)\right]\Big\}^2$$
$$\Theta_6(t)+\Theta_9(t)-\left[Q_{22}(t)+\Theta_3(t)\right]^{-1}\left[\Theta_5(t)+\Theta_8(t)+R_2(t)\right]^2\Bigg\}dt\,.$$

If

$$\begin{cases}\Theta_1(t)+Q_{11}(t)-\left[Q_{22}(t)+\Theta_3(t)\right]^{-1}\left[\Theta_2(t)+Q_{12}(t)\right]^2=0\\ \Theta_4(t)+\Theta_7(t)+R_1(t)-2\left[Q_{22}(t)+\Theta_3(t)\right]^{-1}\left[\Theta_2(t)+Q_{12}(t)\right]\left[\Theta_5(t)+\Theta_8(t)+R_2(t)\right]=0\end{cases}\tag{2.23}$$

then

$$\begin{aligned}J(x,u) &= \mathbb{E}\,(p_0x^2)+\mathbb{E}\,(\tilde p_0 x)+\mathbb{E}\,\int_0^T J_0(t)dt\\ &\quad+\mathbb{E}\,\int_0^T\left[Q_{22}(t)+\Theta_3(t)\right]\Big\{u_t+\left[Q_{22}(t)+\Theta_3(t)\right]^{-1}\left[\Theta_2(t)+Q_{12}(t)\right]x_{t-}\\ &\quad+\left[Q_{22}(t)+\Theta_3(t)\right]^{-1}\left[\Theta_5(t)+\Theta_8(t)+R_2(t)\right]\Big\}^2dt\,,\end{aligned}\tag{2.24}$$

where

$$J_0(t)=\Theta_6(t)+\Theta_9(t)-\left[Q_{22}(t)+\Theta_3(t)\right]^{-1}\left[\Theta_5(t)+\Theta_8(t)+R_2(t)\right]^2$$

is independent u_t and x_t. This utility functional will achieve its minimum

$$\mathbb{E}\,(p_0x^2)+\mathbb{E}\,(\tilde p_0 x)+\mathbb{E}\,\int_0^T J_0(t)dt$$

when

$$u_t=-\left[Q_{22}(t)+\Theta_3(t)\right]^{-1}\left\{\left[\Theta_2(t)+Q_{12}(t)\right]x_{t-}+\Theta_5(t)+\Theta_8(t)+R_2(t)\right\}\,.\tag{2.25}$$

Thus the optimal control is also a feedback one which is linear and depends only on the state x_t.

Using the notation of $\rho_i(t)$ we may rewrite

$$\Theta_1(t) = \xi_t+2p_tA_t+p_tC_t^2+2\eta_tC_t+\rho_1(t)p_t+\rho_2(t)\,,\tag{2.26}$$
$$\Theta_2(t) = p_tB_t+p_tC_tD_t+\eta_tD_t+\rho_3(t)p_t+\rho_4(t)\,,\tag{2.27}$$
$$\Theta_3(t) = p_tD_t^2+\rho_5(t)p_t+\rho_6(t)\,,\tag{2.28}$$
$$\Theta_4(t) = 2p_t\alpha_t+2\beta_tp_tC_t+2\beta_t\eta_t+2p_t\rho_7(t)+2\rho_8(t)\,,\tag{2.29}$$
$$\Theta_5(t) = p_t\beta_tD_t+p_t\rho_9(t)+\rho_{10}(t)\,,\tag{2.30}$$
$$\Theta_6(t) = p_t\beta_t^2+p_t\rho_{11}(t)+\rho_{12}(t)\,,\tag{2.31}$$
$$\Theta_7(t) = \tilde\xi_t+\tilde p_tA_t+C_t\tilde\eta_t+\rho_{13}(t)\,,\tag{2.32}$$
$$\Theta_8(t) = \frac{1}{2}\left\{\tilde p_tB_t+\tilde\eta_tD_t+\rho_{14}(t)\right\}\,,\tag{2.33}$$
$$\Theta_9(t) = \tilde p_t\alpha_t+\tilde\eta_t\beta_t+\rho_{15}(t)\,.\tag{2.34}$$

The first equation of (2.23) becomes

$$\xi_t + 2p_tA_t + p_tC_t^2 + 2\eta_tC_t + \rho_1(t)p_t + \rho_2(t) + Q_{11}(t)$$
$$+ \Big[Q_{22}(t) + p_tD_t^2 + \rho_5(t)p_t + \rho_6(t)\Big]^{-1}\left[p_tB_t + p_tC_tD_t + \eta_tD_t + \rho_3(t)p_t + \rho_4(t) + Q_{12}(t)\right]^2 = 0\,.$$

Multiplying by dt we get

$$\xi_tdt + \Big[2p_tA_t + p_tC_t^2 + 2\eta_tC_t + \rho_1(t)p_t + \rho_2(t) + Q_{11}(t)\Big]\,dt$$
$$- \Big[Q_{22}(t) + p_tD_t^2 + \rho_5(t)p_t + \rho_6(t)\Big]^{-1}\left[p_tB_t + p_tC_tD_t + \eta_tD_t + \rho_3(t)p_t + \rho_4(t) + Q_{12}(t)\right]^2 dt = 0\,.$$

Substituting

$$\xi_tdt = dp_t - \eta_tdW_t - \int_{\mathbb{R}}\mu_t(z)\tilde{N}(dt,dz)$$

into the equation we have the following backward Riccati equation for p_t

$$dp_t + \Big[2p_tA_t + p_tC_t^2 + 2\eta_tC_t + \rho_1(t)p_t + \rho_2(t) + Q_{11}(t)\Big]\,dt$$
$$- \Big[Q_{22}(t) + p_tD_t^2 + \rho_5(t)p_t + \rho_6(t)\Big]^{-1}\left[p_tB_t + p_tC_tD_t + \eta_tD_t + \rho_3(t)p_t + \rho_4(t) + Q_{12}(t)\right]^2 dt$$
$$-\eta_tdW_t - \int_{\mathbb{R}}\mu_t(z)\tilde{N}(dt,dz) = 0\,.$$

In a similar way we can reduce the second equation of (2.23) to

$$d\tilde{p}_t + \left[2p_t\alpha_t + 2\beta_tp_tC_t + 2\beta_t\eta_t + 2p_t\rho_7(t) + 2\rho_8(t)\right]dt$$
$$+ \left[\tilde{p}_tA_t + C_t\tilde{\eta}_t + \rho_{13}(t) + R_1(t)\right]dt$$
$$-2\left[Q_{22}(t) + p_tD_t^2 + \rho_5(t)p_t + \rho_6(t)\right]^{-1}\left[p_tB_t + p_tC_tD_t + \eta_tD_t + \rho_3(t)p_t + \rho_4(t) + Q_{12}(t)\right]$$
$$\left[p_t\beta_tD_t + p_t\rho_9(t) + \rho_{10}(t) + \frac{1}{2}\tilde{p}_tB_t + \frac{1}{2}\tilde{\eta}_tD_t + \frac{1}{2}\rho_{14}(t) + R_2(t)\right]dt$$
$$-\tilde{\eta}_tdW_t - \int_{\mathbb{R}}\tilde{\mu}_t(z)\tilde{N}(dt,dz) = 0\,.$$

3 The Partial Information Case

We now study the case when our control u_t is required to be $\mathcal{E}_t$-predictable , where

$$\mathcal{E}_t \subseteq \mathcal{F}_t \quad \text{for all} \quad t \in [0,T]$$

is a given sub-filtration representing the information available to the controller at time t. The corresponding family of admissible controls is denoted by $\mathcal{A}_{\mathcal{E}}$.

Theorem 3.1 *(Partial information linear quadratic control) Suppose the system of Riccati equations (2.3)-(2.6) has a solution p_t and $\tilde{p}_t$. Define*

$$\begin{aligned} u_t^* \;=\; & -\Big(\mathbb{E}\,\Big[\{Q_{22}(t) + \Theta_3(t)\}\Big|\mathcal{E}_t\Big]\Big)^{-1} \\ & \mathbb{E}\,\Big[\{(\Theta_2(t) + Q_{12}(t))x_{t-} + \Theta_5(t) + \Theta_8(t) + R_2(t)\}\Big|\mathcal{E}_t\Big]\,, \end{aligned} \tag{3.1}$$

where $\Theta_i(t)$ are given by (2.26)-(2.34).

Suppose $u_t^* \in \mathcal{A}_{\mathcal{E}}$ and that (2.10) holds. Then u_t^* is the unique solution of the partial information linear quadratic control problem. The value function $J_{\mathcal{E}}(x)$ in the partial observation case is given by

$$J_{\mathcal{E}}(x) = J_{\mathcal{F}}(x) + \mathbb{E}\left[\int_0^T \left\{L_t M_t^2 - \mathbb{E}\left[L_t|\mathcal{E}_t\right]^{-1}\left(\mathbb{E}\left[L_t M_t|\mathcal{E}_t\right]\right)^2\right\} dt\right], \tag{3.2}$$

where $J_{\mathcal{F}}$ is the value function in the complete information case and

$$L_t = Q_{22}(t) + \Theta_3(t) \tag{3.3}$$

and

$$M_t = L_t^{-1}\left[(\Theta_2(t) + Q_{12}(t))x_t + \Theta_5(t) + \Theta_8(t) + R_2(t)\right]. \tag{3.4}$$

Proof We use the computation in the proof of Theorem 2.1. By (2.24) we have

$$J(x,u) = J_{\mathcal{F}}(x) + \mathbb{E}\left[\int_0^T L_t(u_t + M_t)^2 dt\right]. \tag{3.5}$$

Note that L_t does not depend on X_t (or u_t). For each t define the measure Q_t by

$$dQ_t = L_t dP_t \quad \text{on} \quad \mathcal{F}_t\,. \tag{3.6}$$

Then

$$\mathbb{E}\left[\int_0^T L_t(u_t + M_t)^2 dt\right] = \int_0^T \mathbb{E}_{Q_t}\left[(u_t + M_t)^2\right] dt\,.$$

We can minimize this for each t. By the well-known Kallianpur-Striebel formula ([12]) we know that the minimum of $\mathbb{E}_{Q_t}[(u_t + M_t)^2]$ over all $\mathcal{E}_t$-measurable u_t is attained at

$$\begin{aligned} u_t = u_t^* &= -\mathbb{E}_{Q_t}[M_t|\mathcal{E}_t] \\ &= -\frac{\mathbb{E}\left[L_t M_t|\mathcal{E}_t\right]}{\mathbb{E}\left[L_t|\mathcal{E}_t\right]} \\ &= -\frac{\mathbb{E}\left[\left\{(\Theta_2(t) + Q_{12}(t))x_{t-} + \Theta_5(t) + \Theta_8(t) + R_2(t)\right\}|\mathcal{E}_t\right]}{\mathbb{E}\left[\left\{Q_{22}(t) + \Theta_3(t)\right\}|\mathcal{E}_t\right]}\,. \end{aligned} \tag{3.7}$$

This proves (3.1). Substituting (3.7) into (3.6) we get

$$\begin{aligned} J_{\mathcal{E}}(x) &= J_{\mathcal{F}}(x) + \mathbb{E}\left[\int_0^T L_t(u_t^* + M_t)^2 dt\right] \\ &= J_{\mathcal{F}}(x) + \mathbb{E}\left[\int_0^T \left\{L_t M_t^2 - (\mathbb{E}\left[L_t|\mathcal{E}_t\right])^{-1}\left(\mathbb{E}\left[L_t M_t|\mathcal{E}_t\right]\right)^2\right\} dt\right] \end{aligned}$$

which proves (3.2).

Remark 3.2 *We may regard the term*

$$J_{\mathcal{E}}(x) - J_{\mathcal{F}}(x) = \mathbb{E}\left[\int_0^T \left\{L_t M_t^2 - (\mathbb{E}\left[L_t|\mathcal{E}_t\right])^{-1}\left(\mathbb{E}\left[L_t M_t|\mathcal{E}_t\right]\right)^2\right\} dt\right]$$

as the reduction of performance (or cost increase) due to the reduced information flow $\mathcal{E}_t$.

4 Some Particular Cases

4.1 Absence of Poissonian Noise

Let us first consider the case that the system is under the influence of Brownian white noise. In the controlled system (2.1) we let

$$E_t(z) = F_t(z) = \gamma_t = 0\,;$$

and let all the coefficients be adapted with respect to the filtration $\mathcal{F}_t^W = \sigma(W_s, s \le t)$ and H_1, H_2 be $\mathcal{F}_T^W$ measurable. Then

$$\rho_i(t) = 0 \qquad \forall\ \ 1 \le i \le 15\,.$$

We may assume $\mu_t = \tilde{\mu}_t = 0$ and write (2.3)-(2.6) as

$$\begin{aligned}
& dp_t + \left[2p_tA_t + p_tC_t^2 + 2\eta_tC_t + Q_{11}(t)\right] dt \\
& - \left[Q_{22}(t) + p_tD_t^2\right]^{-1} \left[p_tB_t + p_tC_tD_t + \eta_tD_t + Q_{12}(t)\right]^2 dt - \eta_t dW_t = 0 \qquad (4.1)\\
& p_T = H_1 \qquad (4.2)
\end{aligned}$$

$$\begin{aligned}
& d\tilde{p}_t + \left[2p_t\alpha_t + 2\beta_tp_tC_t + 2\beta_t\eta_t\right] dt + \left[\tilde{p}_tA_t + C_t\tilde{\eta}_t + R_1(t)\right] dt \\
& -2\left[Q_{22}(t) + p_tD_t^2\right]^{-1} \left[p_tB_t + p_tC_tD_t + \eta_tD_t + Q_{12}(t)\right] \\
& \left[p_t\beta_tD_t + \frac{1}{2}\tilde{p}_tB_t + \frac{1}{2}\tilde{\eta}_tD_t + R_2(t)\right] dt - \tilde{\eta}_t dW_t = 0 \qquad (4.3)\\
& \tilde{p}_T = H_2\,. \qquad (4.4)
\end{aligned}$$

Theorem 4.1 *Suppose the system of backward Riccati equations (4.1)-(4.4) has a solution p_t and $\tilde{p}_t$. Define*

$$\begin{aligned}
u_t \;\; = \;\; & -\left[Q_{22}(t) + p_tD_t^2\right]^{-1} \Big\{ \left[p_tB_t + p_tC_tD_t + \eta_tD_t + Q_{12}(t)\right] x_{t-} \\
& -p_t\beta_tD_t + \frac{1}{2}\left(\tilde{p}_t + \tilde{\eta}_tD_t - R_2(t)\right) \Big\}\,. \qquad (4.5)
\end{aligned}$$

Suppose $u_t \in \mathcal{A}_{\mathcal{F}}$ and that (2.10) holds. Then u_t is the unique solution of the complete information linear quadratic control problem (2.1)-(2.2). The corresponding value function is also quadratic and it is given by

$$\mathbb{E}\ (p_0)x^2 + \mathbb{E}\ (\tilde{p}_0)x + \mathbb{E}\int_0^T \left\{\Theta_6(t) + \Theta_9(t) - \left[Q_{22}(t) + \Theta_3(t)\right]^{-1} \left[\Theta_5(t) + \Theta_8(t) + R_2(t)\right]^2\right\} dt\,,$$

where p_t and $\tilde{p}_t$ are found from solving the above backward equations and

$$\begin{aligned}
\Theta_3(t) \;\; &= \;\; p_tD_t^2\,, \quad \Theta_5(t) = p_t\beta_tD_t\,, \quad \Theta_6(t) = p_t\beta_t^2 \\
\Theta_8(t) \;\; &= \;\; \frac{1}{2}\left(\tilde{p}_t + \tilde{\eta}_tD_t\right)\,, \Theta_9(t) = \tilde{p}_t\alpha_t + \tilde{\eta}_t\beta_t\,.
\end{aligned}$$

4.2 Absence of Brownian White Noise

If in the controlled system (2.1), $C_t = D_t = \beta_t = 0$ and all the coefficients are adapted to the filtration $\mathcal{F}_t^P = \sigma(N(ds, dz), s \leq t)$ and H_1, H_2 are $\mathcal{F}_T^P$ measurable, then we may consider the system

$$
\begin{aligned}
& dp_t + [2p_t A_t + \rho_1(t)p_t + \rho_2(t) + Q_{11}(t)]\, dt && (4.6)\\
& - [Q_{22}(t) + \rho_5(t)p_t + \rho_6(t)]^{-1} [p_t B_t + \rho_3(t)p_t + \rho_4(t)]^2\, dt - \int_{\mathbb{R}} \mu_t(z)\tilde{N}(dt, dz) = 0 \\
& p_T = H_1 && (4.7)
\end{aligned}
$$

$$
\begin{aligned}
& d\tilde{p}_t + [2p_t\alpha_t + 2\beta_t\eta_t + 2p_t\rho_7(t) + 2\rho_8(t)]\, dt + [\tilde{p}_t A_t + \rho_{13}(t) + R_1(t)]\, dt \\
& -2\, [Q_{22}(t) + \rho_5(t)p_t + \rho_6(t)]^{-1} [p_t B_t + \rho_3(t)p_t + \rho_4(t) + Q_{12}(t)] \\
& \left[p_t\rho_9(t) + \rho_{10}(t) + \frac{1}{2}\tilde{p}_t B_t + \frac{1}{2}\rho_{14}(t) + R_2(t)\right] dt - \int_{\mathbb{R}} \tilde{\mu}_t(z)\tilde{N}(dt, dz) = 0 && (4.8)\\
& \tilde{p}_T = H_2\,. && (4.9)
\end{aligned}
$$

Theorem 4.2 *Suppose the system of backward Riccati equations (2.3)-(2.6) has a solution p_t and $\tilde{p}_t$. Define*

$$
\begin{aligned}
u_t \;\; = \;\; & - [Q_{22}(t) + \rho_5 p_t + \rho_6(t)]^{-1} \Big\{ [p_t B_t + \rho_3(t)p_t + \rho_4(t) + Q_{12}(t)]\, x_{t-} - p_t\rho_9(t) + \rho_{10}(t) \\
& + \frac{1}{2}\,(\tilde{p}_t + \rho_{14}(t)) - R_2(t) \Big\}\,. && (4.10)
\end{aligned}
$$

Suppose $u_t \in \mathcal{A}_{\mathcal{F}}$ and that (2.10) holds. Then u_t is the unique solution of the complete information linear quadratic control problem (2.1)-(2.2). The corresponding value function is also quadratic and it is given by

$$
\mathbb{E}\,(p_0)x^2 + \mathbb{E}\,(\tilde{p}_0)x + \mathbb{E}\int_0^T \left\{\Theta_6(t) + \Theta_9(t) - [Q_{22}(t) + \Theta_3(t)]^{-1}[\Theta_5(t) + \Theta_8(t) + R_2(t)]^2\right\} dt\,,
$$

where p_t and $\tilde{p}_t$ are found from solving the above backward equations and Θ_i are given by corresponding formulas of (2.26)-(2.34).

4.3 Classical Riccati Equations

To obtain the classical Riccati equation, we may assume that in the controlled system (2.1)

$$
\alpha_t = 0\,, \quad \beta_t = 0\,, \quad \gamma_t = 0\,, \quad H_2 = 0\,, \quad Q_{12}(t) = R_1(t) = R_2(t) = 0\,.
$$

In this case we have

$$
\rho_7(t) = \rho_9(t) = \rho_{10}(t) = \rho_{11}(t) = \rho_{12}(t) = 0\,.
$$

The backward stochastic Riccati equation for $\tilde{p}_t$ becomes

$$\begin{aligned}
&d\tilde{p}_t + [\tilde{p}_t A_t + C_t \tilde{\eta}_t + \rho_{13}(t)]\, dt - 2\left[Q_{22}(t) + p_t D_t^2 + \rho_5(t) p_t + \rho_6(t)\right]^{-1} \\
&[p_t B_t + p_t C_t D_t + \eta_t D_t + \rho_3(t) p_t + \rho_4(t)] \\
&\left[\frac{1}{2}\tilde{p}_t B_t + \frac{1}{2}\tilde{\eta}_t D_t + \frac{1}{2}\rho_{14}(t)\right] dt - \tilde{\eta}_t dW_t - \int_{\mathbb{R}} \tilde{\mu}_t(z) \tilde{N}(dt, dz) = 0 \\
&\tilde{p}_T = 0\,.
\end{aligned}$$

Apparently, this equation has a solution 0. Moreover, (2.3) becomes

$$\begin{aligned}
&dp_t + \left[2p_t A_t + p_t C_t^2 + 2\eta_t C_t + \rho_1(t) p_t + \rho_2(t) + Q_{11}(t)\right] dt \\
&- \left[Q_{22}(t) + p_t D_t^2 + \rho_5(t) p_t + \rho_6(t)\right]^{-1} [p_t B_t + p_t C_t D_t + \eta_t D_t + \rho_3(t) p_t + \rho_4(t) + Q_{12}(t)]^2\, dt \\
&-\eta_t dW_t - \int_{\mathbb{R}} \mu_t(z) \tilde{N}(dt, dz) = 0 \qquad (4.11) \\
&p_T = H_1\,. \qquad (4.12)
\end{aligned}$$

Theorem 4.3 *Suppose the system of backward Riccati equations (4.11)-(4.12) has a solution p_t. Define*

$$\begin{aligned}
u_t \;=\; &-\left[Q_{22}(t) + p_t D_t^2 + \rho_5 p_t + \rho_6(t)\right]^{-1} \\
&\Big\{ [p_t B_t + p_t C_t D_t + \eta_t D_t + \rho_3(t) p_t + \rho_4(t) + Q_{12}(t)]\, x_{t-} \\
&-p_t \beta_t D_t + p_t \rho_9(t) + \rho_{10}(t) - R_2(t) \Big\}\,. \qquad (4.13)
\end{aligned}$$

Suppose $u_t \in \mathcal{A}_{\mathcal{F}}$ and that (2.10) holds. Then u_t is the unique solution of the complete information linear quadratic control problem (2.1)-(2.2). The corresponding value function is also quadratic and it is given as before.

If in Theorem 4.3 we further assume

$$E_t(z) = F_t(z) = \gamma_t = 0\,, \quad Q_{12}(t) = R_1(t) = R_2(t) = 0\,,$$

then we have

Corollary 4.4 *Suppose the backward Riccati equation*

$$\begin{aligned}
&dp_t + \left[2p_t A_t + p_t C_t^2 + 2\eta_t C_t + Q_{11}(t)\right] dt \\
&- \left[Q_{22}(t) + p_t D_t^2\right]^{-1} [p_t B_t + p_t C_t D_t + \eta_t D_t]^2\, dt - \eta_t dW_t = 0 \\
&p_T = H_1
\end{aligned}$$

has a solution p_t. Define

$$u_t \;=\; -\left[Q_{22}(t) + p_t D_t^2\right]^{-1} \Big\{ [p_t B_t + p_t C_t D_t + \eta_t D_t]\, x_{t-} - p_t \beta_t D_t + p_t \rho_9(t) + \rho_{10}(t) - R_2(t) \Big\}\,.$$

Suppose $u_t \in \mathcal{A}_{\mathcal{F}}$ and that (2.10) holds. Then u_t is the unique solution of the complete information linear quadratic control problem (2.1)-(2.2). The corresponding value function is also quadratic and it is given as before.

Remark 4.5 *This equation coincides with the equation in [13] for example.*

4.4 Deterministic Linear Quadratic Problem

If all the data are deterministic, then we may assume p_t and $\tilde{p}_t$ to be deterministic too. Hence

$$\eta_t = \mu_t(z) = \tilde{\eta}_t = \tilde{\mu}_t(z) = 0$$

and we have the following

Theorem 4.6 *Consider the following system of backward Riccati / backward linear stochastic differential equations*

$$dp_t + \left[2p_tA_t + p_tC_t^2 + \rho_1(t)p_t + \rho_2(t) + Q_{11}(t)\right] dt \tag{4.14}$$
$$- \left[Q_{22}(t) + p_tD_t^2 + \rho_5(t)p_t + \rho_6(t)\right]^{-1} \left[p_tB_t + p_tC_tD_t + \rho_3(t)p_t + \rho_4(t) + Q_{12}(t)\right]^2 dt = 0$$

$$\begin{aligned}
&d\tilde{p}_t + \left[2p_t\alpha_t + 2\beta_tp_tC_t + 2p_t\rho_7(t) + \rho_8(t)\right] dt \\
&+ \left[\tilde{p}_tA_t + \rho_{13}(t) + R_1(t)\right] dt \\
&-2\left[Q_{22}(t) + p_tD_t^2 + \rho_5(t)p_t + \rho_6(t)\right]^{-1} \left[p_tB_t + p_tC_tD_t + \rho_3(t)p_t + \rho_4(t) + Q_{12}(t)\right] \\
&\left[p_t\beta_tD_t + p_t\rho_9(t) + \rho_{10}(t) + \frac{1}{2}\tilde{p}_tB_t + \frac{1}{2}\rho_{14}(t) + R_2(t)\right] dt = 0\,.
\end{aligned} \tag{4.15}$$

The terminal conditions are

$$p_T = H_1 \quad \text{and} \quad \tilde{p}_T = H_2\,.$$

If the Riccati system (4.14)-(4.15) has a solution p_t and $\tilde{p}_t$, then the linear quadratic control problem (2.1)-(2.2) has a solution with the optimal control given by

$$\begin{aligned}
u_t \;=\; &-\left[Q_{22}(t) + p_tD_t^2 + \rho_5p_t + \rho_6(t)\right]^{-1} \\
&\Big\{ \left[p_tB_t + p_tC_tD_t + \rho_3(t)p_t + \rho_4(t) + Q_{12}(t)\right] x_{t-} - \big[p_t\beta_tD_t + p_t\rho_9(t) + \rho_{10}(t) \\
&+\frac{1}{2}\left(\tilde{p}_t + \tilde{\eta}_tD_t + \rho_{14}(t)\right)\big] - R_2(t)\Big\}\,,
\end{aligned} \tag{4.16}$$

provided that $u_t \in \mathcal{A}_{\mathcal{F}}$ and that (2.10) holds. The corresponding value function is given by (2.11) with

$$Q_6(t) \;=\; p_t\beta_t^2 \int_{\mathbb{R}} p_t\gamma_t^2(z)\nu(dz)$$

$$
\begin{aligned}
Q_9(t) &= \tilde{p}_t \alpha_t \\
Q_3(t) &= p_t D_t^2 + p_t^2 \int_{\mathbb{R}} F_t^2(z)\nu(dz) \\
Q_5(t) &= p_t \beta_t D_t + \int_{\mathbb{R}} p_t \gamma_t(z) F_t(z)\nu(dz) \\
Q_8(t) &= \frac{1}{2}\tilde{p}_t B_t \,.
\end{aligned}
$$

5 Partial Information Mean-Variance Portfolio Problem

We now apply our results to a partial information mean-variance portfolio problem in finance.

Suppose we have a market with the following two investment possibilities:

i) a risk free asset, whose unit price $S_0(t)$ at time t is given by

$$
dS_0(t) = \rho_t S_0(t) dt\,, \quad S_0(0) = 1\,; \quad 0 \le t \le T \tag{5.1}
$$

ii) a risky asset, whose unit price $S_1(t)$ at time t is given by

$$
\begin{aligned}
dS_1(t) &= S_1(t-)\left[a_t dt + b_t dW_t + \int_{\mathbb{R}} c_t(z)\tilde{N}(dt,dz)\right], \quad 0 \le t \le T\,. \\
S_1(0) &> 0\,;
\end{aligned} \tag{5.2}
$$

Here ρ_t, a_t, b_t, and $c_t(z)$ are given $\mathcal{F}_t$-predictable processes. We assume that

$$
\mathbb{E}\left[\int_0^T \left\{|\rho_t| + |a_t| + b_t^2 + \int_{\mathbb{R}} c_t(z)^2 \nu(dz)\right\} dt\right] < \infty \tag{5.3}
$$

$$
\text{There exists } \varepsilon > 0 \text{ such that} \quad c_t(z) > -1 + \varepsilon \qquad \text{a.s.} \quad \text{for a.a.} \quad t, z \tag{5.4}
$$

$$
p_t b_t^2 + \int_{\mathbb{R}} (p_t + \mu_t(z)) c_t(z)^2 \nu(dz) > 0 \qquad \text{for a.a.} \quad t, \omega \tag{5.5}
$$

where p_t is the solution of the following (5.14)-(5.15).

Conditions (5.3)-(5.4) ensure that the solution to (5.2) is given by

$$
\begin{aligned}
S_1(t) &= S_1(0)\exp\left\{ \int_0^t (a_s - \frac{1}{2}b_s^2)ds + \int_0^t b_s dW_s \right. \\
&\quad \left. + \int_0^t \int_{\mathbb{R}} \{\log(1 + c_s(z)) - c_s(z)\}\, \nu(dz)ds + \int_0^t \int_{\mathbb{R}} \log(1 + c_s(z))\tilde{N}(ds,dz) \right\}.
\end{aligned} \tag{5.6}
$$

A portfolio in this market is a predictable process $\phi(t) = (\phi_0(t), \phi_1(t)) \in \mathbb{R}^2$ giving the number of units of the risk free and the risky asset, respectively, held at time t. The corresponding *wealth process* $x(t) = x^\phi(t)$ is defined by

$$
x^\phi(t) = \phi_0(t)S_0(t) + \phi_1(t)S_1(t)\,. \tag{5.7}
$$

We say that $\phi(t)$ is *self-financing* if

$$dx^{\phi}(t) = \phi_0(t)dS_0(t) + \phi_1(t)dS_1(t)\,. \tag{5.8}$$

Suppose we are given a subfiltration

$$\mathcal{E}_t \subseteq \mathcal{F}_t\,, \quad t \in [0,T]\,.$$

Let

$$u_t = \phi_1(t)S_1(t)$$

be the *amount* (instead of number of shares) invested in the risky asset at time t. We say that u_t is *admissible* and write $u_t \in \mathcal{A}_{\mathcal{E}}$ if u_t is $\mathcal{E}_t$-predictable , $\phi_1(t) = \dfrac{u_t}{S_1(t)}$ is self-financing and $x^{(u)}(t) := x^{(\phi)}(t)$ is *lower bounded.* Combining the above we see that if $\phi \in \mathcal{A}_{\mathcal{E}}$ then

$$\begin{aligned} dx^{(u)}(t) &= \left\{\rho_t x^{(u)}(t) + (a_t - \rho_t)u_t\right\} dt + b_t u_t dW(t) + u_t \int_{\mathbb{R}} c_t(z)\tilde{N}(dt,dz) \\ x^{(u)}(0) &= x > 0\,. \end{aligned} \tag{5.9}$$

We now consider the *partial information mean-variance portfolio problem*, which is to find the portfolio $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ which minimizes the variance

$$\mathbb{E}\,\left[x^{\phi}(T) - \mathbb{E}\,x^{\phi}(T)\right]^2 \tag{5.10}$$

under the constraint

$$\mathbb{E}\,\left[x^{\phi}(T)\right] = K\,, \tag{5.11}$$

where K is a given constant.

Using the Lagrange multiplier method we see that the problem is equivalent to minimizing

$$\mathbb{E}\,\left[x^{\phi}(T) - \lambda\right]^2 \tag{5.12}$$

for a given constant $\lambda \in \mathbb{R}$, without constraints. We refer to [18] and [8] for more information about the mean-variance portfolio problem.

If $\mathcal{E}_t = \mathcal{F}_t$ and the coefficients are all deterministic, then this problem was solved in [8] by using the maximum principle for jump diffusions.

Subsequently this was extended to the partial information case $\mathcal{E}_t \subseteq \mathcal{F}_t$ (but still with deterministic coefficients) by [3].

We now show how Theorem 3.1 gives us a solution also in the case of stochastic coefficients.

Here

$$\begin{aligned} &A_t = \rho_t\,, \quad B_t = a_t - \rho_t\,, \quad \alpha_t = 0\,, \quad C_t = 0\,, \quad D_t = b_t\,, \quad \beta_t = 0 \\ &E_t(z) = 0\,, \quad F_t(z) = c_t(z)\,, \quad \gamma_t(z) = 0\,, \quad Q_{ij}(t) = R_i(t) = 0\,, \end{aligned}$$

and

$$H_1 = 1\,, \quad H_2 = -2\lambda\,.$$

Then (3.1) gives the following candidate for the optimal partial information portfolio:

$$u_t^* = -\left(\mathbb{E}\ [\Theta_3(t)|\mathcal{E}_t]\right)^{-1}\mathbb{E}\ \left[\left\{\Theta_2(t)x^{(u)}(t-) + \Theta_5(t) + \Theta_8(t)\right\}|\mathcal{E}_t\right]\,, \tag{5.13}$$

where $\Theta_i(t)$, $i = 2, 3, 5, 8$ are defined by (2.26)-(2.34). Hence

$$\begin{aligned} u_t^* &= -\left(\mathbb{E}\ \left[\left\{p_t b_t^2 + \int_{\mathbb{R}}(p_t + \mu_t(z))c_t(z)^2\nu(dz)\right\}|\mathcal{E}_t\right]\right)^{-1}\mathbb{E}\ \left[\left\{\ (p_t(a_t - \rho_t) + \eta_t b_t)\, x^{(u)}(t-)\right.\right. \\ &\quad \left.\left. + \frac{1}{2}\left(\tilde{p}_t + \tilde{\eta}_t b_t\right) + \int_{\mathbb{R}}\tilde{\mu}_t(z)c_t(z)\nu(dz)\right\}|\mathcal{E}_t\right]. \end{aligned}$$

Here p_t, η_t, $\mu_t(z)$, and $\tilde{p}_t$, $\tilde{\eta}_t$, $\tilde{\mu}_t(z)$ are the solutions of the backward Riccati equations (2.3)-(2.6), *i.e.*

$$\begin{aligned} dp_t &= -\left\{2\rho_t p_t + \left[p_t b_t^2 + \int_{\mathbb{R}}(p_t + \mu_t(z))c_t^2(z)\nu(dz)\right]^{-1}\left[p_t(a_t - \rho_t) + \eta_t b_t\right.\right. \\ &\quad \left.\left. + \int_{\mathbb{R}}\mu_t(z)c_t(z)\nu(dz)\right]^2\right\}dt + \eta_t dW_t + \int_{\mathbb{R}}\mu_t(z)\tilde{N}(dt,dz)\,; \quad t < T \qquad (5.14) \\ p_T &= 1 \qquad (5.15) \end{aligned}$$

and

$$\begin{aligned} d\tilde{p}_t &= -\left\{\rho_t\tilde{p}_t + \frac{1}{2}\left[p_t b_t^2 + \int_{\mathbb{R}}(p_t + \mu_t(z))c_t^2(z)\nu(dz)\right]^{-1}\left[p_t(a_t - \rho_t) + \tilde{\eta}_t b_t\right.\right. \\ &\quad \left.\left. + \int_{\mathbb{R}}\mu_t(z)c_t(z)\nu(dz)\right]\right\}dt + \tilde{\eta}_t dW_t + \int_{\mathbb{R}}\tilde{\mu}_t(z)\tilde{N}(dt,dz)\,; \quad t < T \qquad (5.16) \\ \tilde{p}_T &= -2\lambda\,. \qquad (5.17) \end{aligned}$$

Summarizing the above we get

Theorem 5.1 *Suppose the system of backward Riccati equations (5.14)-(5.17) has a unique solution p_t and $\tilde{p}_t$. Define*

$$\begin{aligned} u_t^* &= -\left(\mathbb{E}\ \left[\left\{p_t b_t^2 + \int_{\mathbb{R}}(p_t + \mu_t(z))c_t(z)^2\nu(dz)\right\}|\mathcal{E}_t\right]\right)^{-1}\mathbb{E}\ \left[\left\{\ (p_t(a_t - \rho_t) + \eta_t b_t)\, x^{(u)}(t-)\right.\right. \\ &\quad \left.\left. + \frac{1}{2}\left(\tilde{p}_t + \tilde{\eta}_t b_t\right) + \int_{\mathbb{R}}\tilde{\mu}_t(z)c_t(z)\nu(dz)\right\}|\mathcal{E}_t\right] \qquad (5.18) \end{aligned}$$

Suppose $u_t^* \in \mathcal{A}_{\mathcal{E}}$ and that (2.10) holds. Then u_t^* is the unique solution to the minimum variance problem (5.12).

Remark 5.2 *Suppose the conditions of Theorem 5.1 hold for each choice of $\lambda \in \mathbb{R}$. Let $x^*_\lambda(T)$ be the optimal terminal wealth determined by the optimal control $u^*_t = u^*_{\lambda,t}$ corresponding to λ. Then, in order to solve the original mean-variance portfolio problem (5.10), it remains to determine λ such that*

$$\mathbb{E}\ [x^*_\lambda(T)] = K\,.$$

We omit the discussion of this equation.

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References

[1] Ait Rami, M.; Moore , J.B. and Zhou, X. Indefinite stochastic linear quadratic control and generalized differential Riccati equation, *SIAM Journal on Control and Optimization*, Vol. 40 (2001), pp. 1296-1311.

[2] Ait Rami, M. and Zhou, X. Linear matrix inequalities, Riccati equations, and indefinite stochastic linear quadratic controls, *IEEE Transactions on Automatic Control*, Vol. AC-45 (2000), pp. 1131-1143.

[3] Baghery F. and Øksendal, B. A maximum principle for stochastic control with partial information. J. Stochastic Anal. Appl. 25 (2007), 705-717.

[4] Benth, F. E.; Di Nunno, G.; Løkka, A.; Øksendal, B.; Proske, F. Explicit representation of the minimal variance portfolio in markets driven by Lévy processes. Math. Finance 13 (2003), no. 1, 55–72.

[5] Chen, S.; Li, X. and Zhou, X. Stochastic linear quadratic regulators with indefinite control weight costs, *SIAM Journal on Control and Optimization*, Vol. 36 (1998), pp. 1685-1702.

[6] Chen, X. and Zhou, X. Stochastic linear quadratic regulators with indefinite control weight costs. II, *SIAM Journal on Control and Optimization*, Vol. 39 (2000), pp. 1065-1081.

[7] Fleming, W. H.; Soner, H. M. Controlled Markov Processes and Viscosity Solutions. Second edition. Stochastic Modelling and Applied Probability, 25. Springer, 2006.

[8] Framstad, N. C.; Øksendal, B.; Sulem, A. Sufficient stochastic maximum principle for the optimal control of jump diffusions and applications to finance. J. Optim. Theory Appl. 121 (2004), no. 1, 77–98.

Errata, J. Optim. Theory Appl. 124 (2005), no. 2, 511–512.

[9] Guo, W. and Xu, C. Optimal portfolio selection when stock prices follow a jump-diffusion process. Math. Methods Oper. Res. 60 (2004), no. 3, 485–496.

[10] Hu, Y. and Zhou, X. Indefinite stochastic Riccati equations, *SIAM Journal on Control and Optimization*, Vol. 42 (2003), pp. 123-137.

[11] Hu, Y. and Song, X.M. Global solution of backward stochastic differential equation with jumps and application to stochastic LQ control. Preprint, 2007.

[12] Kallianpur, G. Stochastic Filtering Theory. Springer, 1980.

[13] Kohlmann, M. and Tang, S. Global adapted solution of one-dimensional backward stochastic Riccati equations, with application to the mean-variance hedging. Stochastic Process. Appl. 97 (2002), no. 2, 255–288.

[14] Lim, A. Mean-variance hedging when there are jumps. SIAM J. Control Optim. 44 (2005), 1893-1922.

[15] Øksendal, B. and Sulem, A. Applied Stochastic Control of Jump Diffusions. Universitext. Springer, Second Edition 2007.

[16] Tang, S. General linear quadratic optimal stochastic control problems with random coefficients: linear stochastic Hamilton systems and backward stochastic Riccati equations. SIAM J. Control Optim. 42 (2003), no. 1, 53–75.

[17] Yao, D.; Zhang, S. and Zhou, X. Stochastic LQ control via semidefinite programming, *SIAM Journal on Control and Optimization*, Vol. 40 (2001), pp. 801-823.

[18] Yong, J.; Zhou, X. Stochastic Controls. Hamiltonian Systems and HJB Equations. Applications of Mathematics 43. Springer, 1999.