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On twisted Fourier analysis and convergence of Fourier series on discrete groups

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Abstract

We study convergence and summation processes of Fourier series in reduced twisted group C^*-algebras of discrete groups.

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1 Introduction

Groups are, by definition, algebraic objects. However, it didn't take long time to realize that they hide a deep analytic structure, too. Ever since the pioneering work of Murray and von Neumann, group von Neumann algebras and group C^*-algebras of discrete groups have been an important source of examples within the theory of operator algebras. More recently, the study of geometric properties of groups started to receive increasing attention, so it should not come as a surprise that the operator algebras attached to groups have also inspired several concepts, results and conjectures in noncommutative geometry, e.g. the Baum-Connes conjecture [30, 84, 49]. In many situations, it appears that it is useful to consider also the twisted versions of these algebras. This is not only of academic interest, as the twisted algebras seem often to require the development of new methods to deal with, but even unavoidable for instance when dealing with the study of electrons in solids when a magnetic field is turned on (see e.g. [8, 9, 64]). Twisted group algebras associated with discrete groups may also be viewed as twisted compact quantum groups, being the twisted dual objects to the discrete groups themselves. The basic examples in this picture are then the so-called noncommutative tori [30], which are nothing but the twisted duals of free finitely generated abelian groups.

A major part of classical harmonic analysis on compact abelian groups amounts to the study of Fourier series. One may therefore wonder about which of the classical results in this part of Fourier analysis do survive in the twisted setting. In order to present our work on this theme, it will be useful to recall first some of the basic concepts. For the ease of the reader, we have included in the preliminary section (Section 2) a more detailed introduction to the different twisted operator algebras which may be associated with discrete groups.

Let $A = C_r^*(G,\sigma)$ (resp. $B = vN(G,\sigma)$) denote the reduced twisted group C^*-algebra (resp. the twisted group von Neumann algebra) acting on $\ell^2(G)$ associated with a discrete group G and a 2-cocycle σ on G with values in the unit circle $\mathbb{T}$. The canonical tracial state of B (and A) will be denoted by τ. To each element x in B one may attach its (formal) Fourier series $\sum_{g\in G}\hat{x}(g)\Lambda_\sigma(g)$, where Λ_σ denotes the (left) σ-projective regular representation of G on $\ell^2(G)$ and the Fourier coefficient $\hat{x}(g)$ is given by $\hat{x}(g) = \tau(x\Lambda_\sigma(g)^*)$. While this series is not necessarily convergent in the weak operator topology on B (even if σ is trivial ; see [65]), it converges to x in the $\|\cdot\|_2$-norm on B induced by τ. If one identifies B as a subspace of $\ell^2(G)$ (via the linear injection $x \to \hat{x}$) this norm is just the ℓ^2-norm and the statement is almost trivial. When one restricts to the case where x belongs to A the Fourier series of x will not necessarily be convergent in operator norm. Indeed, if G is abelian and σ is trivial, we may identify A as a C^*-algebra with $C(\widehat{G})$, the continuous complex functions on the dual group of G, and this classical phenomena and its pointwise companions are discussed in most introductory books on harmonic analysis. The usual way, at least when $G = \mathbb{Z}$, to remedy for this defect is either to consider elements having Fourier coefficients satisfying some suitable decay property (e.g. belonging to $\ell^1(\mathbb{Z})$), or to follow ideas of Abel, Cesaro, Poisson and Fejér, and introduce other kind of summation processes which produce generalized Fourier series converging uniformly to the initial function.

To be more specific about these summation processes, let $f \in C(\mathbb{T})$, set $e_k(z) = z^k$ ($z \in \mathbb{T}$) and $\hat{f}(k) = \int_{\mathbb{T}} f\, \overline{e_k} d\mu, k \in \mathbb{Z}$, where μ denotes the normalized Haar measure on $\mathbb{T}$. The (formal) Fourier series of f is then given by $\sum_{k\in\mathbb{Z}} \hat{f}(k)e_k$. Now, let (φ_n) be a sequence in $\ell^1(\mathbb{Z})$ and for each $n \in \mathbb{N}$ set

$$M_n(f) = \sum_{k\in\mathbb{Z}} \varphi_n(k)\hat{f}(k)e_k,$$

this series being absolutely convergent with respect to the uniform norm $\|\cdot\|_\infty$ on $C(\mathbb{T})$. Then we have $\|M_n(f)\|_\infty \leq \|\varphi_n\|_1\|f\|_\infty$, hence each M_n is a bounded linear map on $(C(\mathbb{T}), \|\cdot\|_\infty)$ satisfying $\|M_n\| \leq \|\varphi_n\|_1$. Further, elementary functional analysis shows that $M_n(f)$ converges uniformly (necessarily to f) for all $f \in C(\mathbb{T})$ if and only if (i) $\varphi_n \to 1$ pointwise on $\mathbb{Z}$ and (ii) $\sup_n \|M_n\|$ is finite, in which case one could say that $C(\mathbb{T})$ has the summation property with respect to (φ_n). Many obvious candidates for (φ_n) satisfy (i) and the main difficulty is to compute the operator norms $\|M_n\|$, or at least to get good estimates for them. The usual convergence problem of Fourier series consists of looking at $\varphi_n(k) = d_n(k) := 1$ if $|k| \leq n$ and 0 otherwise. In this case, one can show with a little effort that $\|M_n\| \to \infty$ and thereby deduce that there must exist a continuous function on $\mathbb{T}$ with divergent Fourier series. In the case of Fejér summation, one considers instead $\varphi_n(k) = f_n(k) := 1 - \frac{|k|}{n}$ if $|k| \leq n-1$ and 0 otherwise. Then one has $\|M_n\| = 1$ for all n, hence one deduces that the Fourier series of any f in $C(\mathbb{T})$ is uniformly Fejér summable to f. For Abel-Poisson summation, one picks a sequence (r_n) in the interval $(0,1)$ converging to 1 and considers

$\varphi_n(k) = p_n(k) := r_n^{|k|}$. (Of course one could here just consider $p_r(k) = r^{|k|}$ for $r \in (0,1)$, introduce the operator M_r defined in the obvious way and let $r \to 1$, as is usually done classically. We will indeed use nets instead of sequences in the sequel to accomodate for such situations). Now again $\|M_n\| = 1\,(= \|M_r\|)$ for all n (and all r), hence the Fourier series of any f in $C(\mathbb{T})$ is uniformly Abel-Poisson summable to f. The classical proofs of these statements invoke the Fejér kernel F_n (writing $M_n(f) = F_n * f$) and the Poisson kernel P_r (writing $M_r(f) = P_r * f$) which have "nicer" behaviour than the classical Dirichlet kernel D_n. As it is surely known to experts, it appears that F_n and P_r are indeed "nicer" mainly because they are non-negative functions on $\mathbb{T}$. This implies that their Fourier transforms $\widehat{F_n} = f_n$ and $\widehat{P_r} = p_r$ are positive definite functions on $\mathbb{Z}$ (while $\widehat{D_n} = d_n$ is not), a fact which will be an important motivation for our approach.

The main purpose of this paper is to study convergence and summation processes of Fourier series in reduced twisted group C^*-algebras of discrete groups. Although these aspects are not discussed as such in U. Haagerup's seminal paper [45] dealing with the reduced group C^*-algebras associated with the nonabelian free group on two generators $\mathbb{F}_2$, it should be said that many of the relevant tools can be found there and in some of its follow-ups (like [17, 46, 53, 56, 25]).

Inspired by the work of P. Jolissaint [53] on groups with the Rapid Decay property (with respect to some length function) and its twisted version [19], we first illustrate in Section 3 how convergence in operator norm of the Fourier series of an element in a twisted reduced group C^*-algebra may be established under some quite general decay assumptions.

When the group is amenable (e. g. abelian), it has been known since the work of G. Zeller-Meier [87] that some analogue of Fejér summation for Fourier series exists. As we will explain in Section 6, which is devoted to summation processes, the direct analogue of Fejér summation may be obtained after picking a Følner net for the group (the existence of such a net being equivalent to the amenability of the group [74, 77]). On the other hand the direct analogue of Abel-Poisson summation is more troublesome, unless the group is $\mathbb{Z}^N$ for some $N \in \mathbb{N}$, this special case still being of interest since the associated twisted group C^*-algebra is then a noncommutative N-torus. However, one can push the analysis further and establish a twisted version of the Abel-Poisson summation theorem for a certain subclass of the class of groups having the so-called Haagerup property [25]. This subclass includes for example all finitely generated free groups and all Coxeter groups [52]. In the final section (Section 7), we relate our work to the main result of [45] by exhibiting some sufficient conditions for a reduced twisted group C^*-algebra to have the so-called Metric Approximation Property.

Our approach to all of the above mentioned summation results relies on the twisted version of Haagerup's result [45, 46] saying that any positive definite complex function on a group G induces a completely positive linear map from the reduced group C^*-algebra $C_r^*(G)$ into itself, which is also a "multiplier" of $C_r^*(G)$. This twisted version is established in Section 4 as a corollary to Fell's absortion property of regular projective unitary representations [7], the Stinespring decomposition of the completely positive map being produced as a part of the proof. The same kind of argument may also be used to give a simple proof of the well known fact [87] that the full and the reduced twisted group C^*-algebras of a discrete amenable group are canonically *-isomorphic. We include this proof, not only for the benefit of all readers which may feel some reluctance against twisting, but also because it looks even more natural to us than the proof of this fact presented in [32] in the untwisted case. We have devoted a section (Section 5) to a brief study of twisted multipliers, where we have skipped most of the technical details which are not really relevant to our main issue in this paper. Anyhow, we feel that it could be worth in the future to investigate the structure of twisted multiplier spaces more thoroughly than presented here, as it seems to be some intriguing connection between these spaces.

Finally, we ought to point out that Zeller-Meier deals in [87] with the more general setting of twisted C^*-crossed products by discrete groups (with 2-cocycles taking values in the center of the algebra). One can also find a proof of the direct analogue of Fejér summation for usual C^*-crossed products by an action of $\mathbb{Z}$ in [32]. To keep this paper at a reasonable length, we have chosen not to discuss here the more general case of summation processes for Fourier series in reduced twisted C^*-crossed products by discrete groups, which obviously also deserves a study for its own sake. On the other side, we hope that the present work will serve as a basis for further studies and so we have deliberately included many open questions.

2 Preliminaries

Throughout this article G will denote a discrete group and e its identity element. For the ease of the reader and to fix our notation, we begin with an elementary introduction to the different operator algebras which may be associated to G and a 2-cocycle on G with values in the circle group $\mathbb{T}$. One basic reference on this subject is [87], which deals with a much more general setting. Hopefully, our presentation should be more accessible. A few more recent aspects which are not covered in [87] are also treated.

We recall first that a *2-cocycle* on G with values in $\mathbb{T}$ is a map $\sigma : G\times G \to \mathbb{T}$ such that

$$\sigma(g,h)\sigma(gh,k) = \sigma(h,k)\sigma(g,hk) \quad (g,h,k \in G)$$

(see e.g. [15, Chapter IV]). We will consider only *normalized* 2-cocycles, that is, satisfying

$$\sigma(g,e) = \sigma(e,g) = 1 \qquad (g \in G),$$

which implies that $\sigma(g,g^{-1}) = \sigma(g^{-1},g) \quad (g \in G)$. The set of all normalized 2-cocycles, which we denote by $Z^2(G,\mathbb{T})$, becomes an abelian group under pointwise product, the inverse operation corresponding to conjugation: $\sigma^{-1} = \overline{\sigma}$, where $\overline{\sigma}(g,h) = \overline{\sigma(g,h)}$, and the identity element being the trivial cocycle on G denoted by 1.

An element $\beta \in Z^2(G,\mathbb{T})$ is called a *coboundary* whenever one may write $\beta(g,h) = b(g)b(h)\overline{b(gh)}$ for all $g,h \in G$, for some $b : G \to \mathbb{T}$, $b(e) = 1$; in this case we write $\beta = \mathrm{d}b$ (such a b is uniquely determined up to multiplication by a character of G). The set of all coboundaries, which we denote by $B^2(G,\mathbb{T})$, is a subgroup of $Z^2(G,\mathbb{T})$. We denote elements in the quotient group $H^2(G,\mathbb{T}) := Z^2(G,\mathbb{T})/B^2(G,\mathbb{T})$ by $[\sigma]$ and write $\tilde{\sigma} \sim \sigma$ when $[\tilde{\sigma}] = [\sigma]$ $(\sigma, \tilde{\sigma} \in Z^2(G,\mathbb{T}))$.

To have a concrete example at hand, let $N \in \mathbb{N}$ and set $G = \mathbb{Z}^N$. To each $N \times N$ real matrix Θ, one may associate $\sigma_\Theta \in Z^2(\mathbb{Z}^N,\mathbb{T})$ by $\sigma_\Theta(x,y) = e^{ix\cdot(\Theta y)}$. Note that $\sigma_\Theta \in B^2(\mathbb{Z}^N,\mathbb{T})$ whenever $\Theta \in M_N(\mathbb{R})$ is symmetric : indeed, in this case, $\sigma_\Theta = \mathrm{d}b_\Theta$ where $b_\Theta(x) := e^{-i\frac{1}{2}x\cdot(\Theta x)}$. It follows that in general $[\sigma_\Theta] = [\sigma_{\tilde{\Theta}}]$ where $\tilde{\Theta}$ denotes the skew-symmetric part of Θ. In fact, every element in $H^2(\mathbb{Z}^N,\mathbb{T})$ may be written as $[\sigma_\Omega]$ for some skew-symmetric Ω. (See [4, 5] for more information). For some other examples of 2-cocycles on discrete groups we refer e.g. to [59] (for abelian groups), [68] (for the integer

Heisenberg group), [51] (for Coxeter groups), [16] (for restricted direct sums), [63] (for Fuchsian groups).

We also recall [62, 58] that a *σ-projective unitary representation* U of G on a (non-zero) Hilbert space $\mathcal{H}$ is a map from G into the group $\mathcal{U}(\mathcal{H})$ of unitaries on $\mathcal{H}$ such that

$$U(g)U(h) = \sigma(g,h)U(gh) \qquad (g,h \in G).$$

We then have $U(e) = I_{\mathcal{H}}$ (the identity operator on $\mathcal{H}$) and

$$U(g)^* = \overline{\sigma(g,g^{-1})}U(g^{-1}), \quad g \in G.$$

If we pick some $b : G \to \mathbb{T}$ satisfying $b(e) = 1$ and set $\tilde{U} = bU$, then $\tilde{U}$ becomes a $\tilde{\sigma}$-projective unitary representation of G on $\mathcal{H}$ with 2-cocycle $\tilde{\sigma} \sim \sigma$ given by $\tilde{\sigma} = (\mathrm{d}b)\sigma$. Such a $\tilde{U}$ is called a *perturbation* of U (by b). If $\omega \in Z^2(G,\mathbb{T})$ and V is some ω-projective unitary representation of G on $\mathcal{K}$, one may form the tensor product representation $U \otimes V$ acting on $\mathcal{H} \otimes \mathcal{K}$ in the obvious way, which is then $\sigma\omega$-projective. Further, letting $\overline{U}$ denote the conjugate of U, which acts as U on the conjugate Hilbert space $\overline{\mathcal{H}}$ of $\mathcal{H}$, one sees easily that $\overline{U}$ is $\overline{\sigma}$-projective.

To each $\sigma \in Z^2(G,\mathbb{T})$ one may associate a left (resp. right) regular σ-projective unitary representation λ_σ (resp. ρ_σ) of G on $\ell^2(G)$ defined by

$$(\lambda_\sigma(g)\xi)(h) = \sigma(h^{-1},g)\xi(g^{-1}h),$$

$$(\rho_\sigma(g)\xi)(h) = \sigma(h,g)\xi(hg),$$

$\xi \in \ell^2(G),\ g,h \in G$. One readily checks that

$$\rho_\sigma(g) = U\lambda_\sigma(g)U, \quad g \in G$$

where U is the involutive unitary operator on $\ell^2(G)$ given by

$$U\xi(g) = \xi(g^{-1}), \quad \xi \in \ell^2(G), g \in G.$$

Choosing $\sigma = 1$ gives the left and right regular representations of G, which are usually just denoted by λ and ρ. The above definitions of regular σ-projective unitary representations of G seem to be the most frequently used in the literature. However, as illustrated in [69], it turns out to be useful for some aspects to introduce their unitarily equivalent versions Λ_σ and R_σ, also acting on $\ell^2(G)$, given by

$$\Lambda_\sigma(g) = S_\sigma \lambda_\sigma(g) S_\sigma^*, \quad R_\sigma(g) = S_\sigma \rho_\sigma(g) S_\sigma^*, \quad g \in G,$$

where S_σ is the unitary multiplication operator on $\ell^2(G)$ defined by

$$(S_\sigma \xi)(g) = \sigma(g, g^{-1})\xi(g), \quad \xi \in \ell^2(G), g \in G.$$

In fact, one could just assume that $\sigma(g, g^{-1}) = 1$ for all $g \in G$, which would not be a real loss of generality as this may be achieved by pertubing with a coboundary (see [58]). But in some cases it seems undesirable to "regularize" the given cocycle in this way and we have therefore chosen to work mainly with Λ_σ. Letting $\{\delta_h\}_{h \in G}$ denote the canonical basis of $\ell^2(G)$, this representation has the nice feature that

$$\Lambda_\sigma(g)\delta_h = \sigma(g, h)\delta_{gh}, \quad g, h \in G,$$

so, especially, we have $\Lambda_\sigma(g)\delta_e = \delta_g$. For the convenience of the reader, we record that

$$(\Lambda_\sigma(g)\xi)(h) = \sigma(g, g^{-1}h)\xi(g^{-1}h), \;\; \xi \in \ell^2(G), \;\; g, h \in G.$$

We also remark that the following commutation relations

$$\Lambda_\sigma(g)\rho_{\overline{\sigma}}(h) = \rho_{\overline{\sigma}}(h)\Lambda_\sigma(g),$$

$$\lambda_\sigma(g)R_{\overline{\sigma}}(h) = R_{\overline{\sigma}}(g)\lambda_\sigma(h),$$

hold for all $g, h \in G$ (this is a simple exercise in cocycling). Hence the "right" companion of Λ_σ is $\rho_{\overline{\sigma}}$ (while $R_{\overline{\sigma}}$ is the one for λ_σ).

We define the *reduced twisted group C^*-algebra $C_r^*(G, \sigma)$* (resp. the *twisted group von Neumann algebra $vN(G, \sigma)$*) as the C^*-subalgebra (resp. von Neumann subalgebra) of $B(\ell^2(G))$ generated by the set $\Lambda_\sigma(G)$, that is, as the closure in the operator norm (resp. weak operator) topology of the *-algebra $\mathbb{C}(G, \sigma)$:=Span($\Lambda_\sigma(G)$). As usual, we set $\delta = \delta_e$, which is clearly a cyclic (= generating) vector for all these algebras. The (normal) state τ on these algebras given by restricting the vector state ω_δ associated to δ is easily seen to be tracial: by density and continuity, it suffices to check traciality on $\mathbb{C}(G, \sigma)$ where it follows from the formula

$$\tau(\Lambda_\sigma(g)\Lambda_\sigma(h)) = \sigma(g, h)\delta_{gh}(e), g, h \in G.$$

Further, τ is faithful as δ is separating for $vN(G,\sigma)$: indeed, if $x \in vN(G,\sigma)$ and $x\delta = 0$, then, as $\rho_{\overline{\sigma}}(h)\delta_h = \delta$, we get

$$x\delta_h = x\rho_{\overline{\sigma}}(h)^*\delta = \rho_{\overline{\sigma}}(h)^* x\delta = 0$$

for all $h \in G$, hence $x = 0$. (Here we have used that x and $\rho_{\overline{\sigma}}(h)^*$ commute for each $h \in G$, which follows from the commutation relations.) Hence $vN(G,\sigma)$ is finite as a von Neuman algebra. We note that there exists a simple characterization of the case where it is a factor [69, Proposition 1.3] (see also [58, Corollary 1]): $vN(G,\sigma)$ is a factor if and only if the conjugacy class of each non-trivial σ-regular element in G is infinite in cardinality ($g \in G$ being σ-regular by definition whenever $\sigma(g,h) = \sigma(h,g)$ for all $h \in G$ which commutes with g). We also mention for completeness that the commutant of $vN(G,\sigma)$ is the von Neumann algebra generated by $\rho_{\overline{\sigma}}(G)$, that is, we have $vN(G,\sigma)' = \rho_{\overline{\sigma}}(G)''$, or equivalently $vN(G,\sigma) = \rho_{\overline{\sigma}}(G)'$. One inclusion follows readily from the commutation relations, while the converse inclusion can also be shown by going through some elementary, but somewhat more involved considerations (see [36]). A cheap way to deduce equality directly is to apply (pre-)Tomita-Takesaki theory to the pair $(vN(G,\sigma),\delta)$: the J-operator is easily seen to be given by $(J_\sigma\xi)(g) = \overline{\sigma}(g,g^{-1})\overline{\xi(g^{-1})}$ and one computes that $J_\sigma\Lambda_\sigma(g)J_\sigma = \rho_{\overline{\sigma}}(g), g \in G$. Thus

$$vN(G,\sigma)' = J_\sigma vN(G,\sigma)J_\sigma = (J_\sigma\Lambda_\sigma(G)J_\sigma)'' = \rho_{\overline{\sigma}}(G)''.$$

Following the same line of ideas, we may consider $vN(G,\sigma)$ as a Hilbert algebra [34] with respect to the inner product $< x,y >:= \tau(y^*x) = (x\delta, y\delta)$. Denoting by $\|\cdot\|_2$ the associated norm, the linear map $x \to \hat{x} := x\delta$ is then an isometry from $(vN(G,\sigma), \|\cdot\|_2)$ to $(\ell^2(G), \|\cdot\|_2)$, which sends $\Lambda_\sigma(g)$ to δ_g for each $g \in G$. (This map is the analogue of the Fourier transform when G is abelian, $\sigma = 1$, and one identifies $vN(G)$ with $L^\infty(\widehat{G})$).

The value $\hat{x}(g)$ is called the *Fourier coefficient* of $x \in vN(G,\sigma)$ at $g \in G$. Considering τ as the normalized "Haar functional" on $vN(G,\sigma)$, we have indeed

$$\hat{x}(g) = (x\delta, \delta_g) = (x\delta, \Lambda_\sigma(g)\delta) = \tau(x\Lambda_\sigma(g)^*).$$

Further, we have

$$\|\hat{x}\|_\infty \leq \|\hat{x}\|_2 = \|x\|_2 \leq \|x\|.$$

The (formal) *Fourier series* of $x \in vN(G,\sigma)$ is defined as $\sum_{g\in G} \hat{x}(g)\Lambda_\sigma(g)$. As remarked in the Introduction, this series does not necessarily converge in the weak operator topology. However, we have

$$x = \sum_{g\in G} \hat{x}(g)\Lambda_\sigma(g) \quad \text{(convergence w.r.t.} \|\cdot\|_2).$$

Indeed, for any finite subset F of G, set $x_F = \sum_{g\in F} \hat{x}(g)\Lambda_\sigma(g)$. Then we have

$$\|x - x_F\|_2 = \|\hat{x} - \sum_{g\in F} \hat{x}(g)\widehat{\Lambda_\sigma(g)}\|_2 = \|\hat{x} - \sum_{g\in F} \hat{x}(g)\delta_g\|_2 = \|\hat{x} - \hat{x}\chi_F\|_2,$$

where χ_F denotes the characteristic function of F. As $\hat{x} \in \ell^2(G)$, the last expression converges to 0 and the assertion follows.

The Fourier series representation of $x \in vN(G,\sigma)$ is unique. For later use, we record the following slightly more general fact :

Let $\xi : G \to \mathbb{C}$ and suppose that $\sum_{g\in G} \xi(g)\Lambda_\sigma(g)$ converges to some $x \in vN(G,\sigma)$ w.r.t. $\|\cdot\|_2$. Then $\xi \in \ell^2(G)$ and $\xi = \hat{x}$.

Indeed, for any finite subset F of G, set $a_F = \sum_{g\in F} \xi(g)\Lambda_\sigma(g)$. Then we have $\widehat{a_F} = \xi\chi_F =: \xi_F$. Now the assumption says that $a_F \to x$ w.r.t. $\|\cdot\|_2$, which implies that $\xi_F \to \hat{x}$ in ℓ^2-norm. This implies that $\sum_{g\in F} |\xi(g)|^2 \le \|\hat{x}\|_2^2$ for all finite subset F of G, hence $\xi \in \ell^2(G)$. But then $\xi_F \to \xi$ in ℓ^2-norm and we get $\hat{x} = \xi$.

Let $f \in \ell^1(G)$. The series $\sum_{g\in G} f(g)\Lambda_\sigma(g)$ is clearly absolutely convergent in operator norm and we shall denote its sum by $\pi_\sigma(f)$. Then we have $\|\pi_\sigma(f)\| \le \|f\|_1$ and

$$\widehat{\pi_\sigma(f)} = (\sum_{g\in G} f(g)\Lambda_\sigma(g))\delta = \sum_{g\in G} f(g)\delta_g = f.$$

Let now $x \in vN(G,\sigma)$ and assume that $\hat{x} \in \ell^1(G)$. Then we get $\widehat{\pi_\sigma(\hat{x})} = \hat{x}$, hence $\pi_\sigma(\hat{x}) = x$. Therefore, *in this case*, we have $\|x\| = \|\pi_\sigma(\hat{x})\| \le \|\hat{x}\|_1$ and

$$x = \sum_{g\in G} \hat{x}(g)\Lambda_\sigma(g) \quad \text{(convergence w.r.t. } \|\cdot\| \text{)},$$

which especially shows that $x \in C_r^*(G,\sigma)$. Hence, setting

$$CF(G,\sigma) := \{x \in C_r^*(G,\sigma) \mid \sum_{g\in G} \hat{x}(g)\Lambda_\sigma(g) \text{ is convergent in operator norm}\},$$

we have $\pi_\sigma(\ell^1(G)) \subseteq CF(G,\sigma)$.

As in classical Fourier analysis, one may consider other type of decay properties on the Fourier transform to ensure convergence of Fourier series in operator norm. We will discuss these aspects in the next section.

The subspace of $\ell^2(G)$ defined by

$$\mathcal{U}(G,\sigma) := \{\hat{x} \mid x \in vN(G,\sigma)\}$$

becomes a Hilbert algebra when equipped with the involution $\hat{x}^* := \widehat{x^*}$ and the product $\hat{x} *_\sigma \hat{y} := \widehat{xy}$. We have $\hat{x}^*(g) = \overline{\sigma}(g,g^{-1})\overline{\hat{x}(g^{-1})}$. Further, as our notation indicates, the product $\hat{x} *_\sigma \hat{y}$ may be expressed as a twisted convolution product.

To see this, let $\xi, \eta \in \ell^2(G)$. The *σ-convolution product* $\xi *_\sigma \eta$ is defined as the complex function on G given by

$$(\xi *_\sigma \eta)(h) = \sum_{g\in G} \xi(g)\sigma(g,g^{-1}h)\eta(g^{-1}h), \ h \in G.$$

As $|(\xi *_\sigma \eta)(h)| \leq (|\xi| * |\eta|)(h), \ h \in G$, it is straightforward to check that $\xi *_\sigma \eta$ is a well defined bounded function on G satisfying

$$\|\xi *_\sigma \eta\|_\infty \leq \||\xi| * |\eta|\|_\infty \leq \|\xi\|_2\|\eta\|_2.$$

We notice that $\delta_a *_\sigma \delta_b = \sigma(a,b)\delta_{ab}, \ a,b \in G$.

Now, if $x \in vN(G,\sigma)$ and $\eta \in \ell^2(G)$, we have $x\eta = \hat{x} *_\sigma \eta$. Indeed,

$$x\eta = \sum_{g\in G} \eta(g)x\delta_g = \sum_{g\in G} \eta(g)\sigma(g,g^{-1})\rho_{\overline{\sigma}}(g^{-1})\hat{x}.$$

Hence,

$$(x\eta)(h) = \sum_{g\in G} \eta(g)\sigma(g,g^{-1})\overline{\sigma}(h,g^{-1})\hat{x}(hg^{-1})$$

$$= \sum_{g\in G} \eta(g)\sigma(hg^{-1},g)\hat{x}(hg^{-1}) = \sum_{b\in G} \hat{x}(b)\sigma(b,b^{-1}h)\eta(b^{-1}h) = (\hat{x} *_\sigma \eta)(h)$$

for all $h \in G$. This implies that $\widehat{xy} = xy\delta = x\hat{y} = \hat{x} *_\sigma \hat{y}$ for all $x, y \in vN(G,\sigma)$, where the last expression is defined through the σ-convolution product, thus justifying our comment above.

For completeness we also mention that one can show that $\mathcal{U}(G,\sigma)$ may be described as the space of all $\xi \in \ell^2(G)$ which are such that $\xi *_\sigma \eta \in \ell^2(G)$ for all $\eta \in \ell^2(G)$ and such that the resulting linear map $\eta \to \xi *_\sigma \eta$ from $\ell^2(G)$ into itself is bounded.

Since $\widehat{\pi_\sigma(f)} = f$ for all $f \in \ell^1(G)$, we have $\ell^1(G) \subseteq \mathcal{U}(G,\sigma)$. Further, one verifies without difficulty that $\ell^1(G)$ is a *-subalgebra of $\mathcal{U}(G,\sigma)$ which becomes a unital Banach *-algebra with respect to the ℓ^1-norm $\|\cdot\|_1$, the unit being given by δ. This Banach *-algebra is usually denoted by $\ell^1(G,\sigma)$. Its involution is explicitely given by $f^*(g) = \overline{\sigma}(g,g^{-1})\overline{f(g^{-1})}, g \in G$.

Consider the map $\pi_\sigma : \ell^1(G) \to C_r^*(G,\sigma) \subseteq B(\ell^2(G))$ defined by $f \to \pi_\sigma(f)$. Clearly we have

$$\pi_\sigma(f)\eta = f *_\sigma \eta,\ f \in \ell^1(G),\ \eta \in \ell^2(G).$$

Further, π_σ is easily seen to be a faithful *-representation of $\ell^1(G,\sigma)$ on $\ell^2(G)$. Hence, the enveloping C*-algebra [33] of $\ell^1(G,\sigma)$ is just the completion of $\ell^1(G,\sigma)$ w.r.t. the norm

$$\|f\|_{max} := \sup_\pi\{\|\pi(f)\|\}$$

where the supremum is taken over all non-degenerate *-representations of $\ell^1(G,\sigma)$ on Hilbert spaces. This C*-algebra is denoted by $C^*(G,\sigma)$ and called the *full twisted group C^*-algebra associated to* (G,σ). We will identify $\ell^1(G,\sigma)$ with its canonical image in $C^*(G,\sigma)$, which is then generated as a C*-algebra by its canonical unitaries δ_g.

The twisted group C^*-algebras of the form $C^*(\mathbb{Z}^N,\sigma_\Theta)$ are often called noncommutative N-tori (since $C^*(\mathbb{Z}^N,\sigma_\Theta)$ is $*$-isomorphic to $C(\mathbb{T}^N)$ in the case where Θ is symmetric).

Any non-degenerate $*$-representation of $\ell^1(G,\sigma)$ extends uniquely to a non-degenerate $*$-representation of $C^*(G,\sigma)$, and we will always use the same symbol to denote the extension. There is a bijective correspondence $U \to \pi_U$ between σ-projective unitary representations of G and non-degenerate $*$-representations of $C^*(G,\sigma)$ determined by

$$\pi_U(f) = \sum_{g\in G} f(g)U(g),\ f \in \ell^1(G),$$

(the series above being obviously absolutely convergent in operator norm), the inverse correspondence being simply given by $U_\pi(g) = \pi(\delta_g), g \in G$. As $\pi_{\Lambda_\sigma} = \pi_\sigma$ we have

$$C_r^*(G,\sigma) = \overline{\pi_\sigma(\ell^1(G,\sigma))}^{\|\cdot\|} = \pi_\sigma(C^*(G,\sigma)).$$

When G is amenable, then π_σ is faithful [87]. We will give a proof of this fundamental result in Section 4.

The dual space of $C^*(G,\sigma)$ may be identified as a subspace $B(G,\sigma)$ of $\ell^\infty(G)$ through the linear injection $\Phi : \phi \to \tilde{\phi}$ where $\tilde{\phi}(g) := \phi(\delta_g), g \in G$. We equip $B(G,\sigma)$ with the transported norm $\|\Phi(\phi)\| := \|\phi\|$. Now, if ϕ is a positive linear functional on $C^*(G,\sigma)$, then $\tilde{\phi}$ is σ-positive definite according to the following definition : a complex function φ on G is *σ-positive definite* (σ-p.d.) whenever we have

$$\sum_{i,j=1}^{n} \overline{c_i} c_j \varphi(g_i^{-1} g_j) \overline{\sigma}(g_i, g_i^{-1} g_j) \geq 0$$

for all $n \in \mathbb{N}, c_1, \dots c_n \in \mathbb{C}, g_1, \dots g_n \in G$. Mimicking the untwisted case, one checks readily that φ is σ-p.d. if and only if there exists a σ-projective unitary representation U of G on some Hilbert space $\mathcal{H}$ and a $\xi \in \mathcal{H}$ (which may be chosen to be cyclic for U) such that $\varphi(g) = (U(g)\xi, \xi)$, $g \in G$, which implies that φ is then bounded with $\|\varphi\|_\infty = \|\xi\|^2 = \varphi(e)$. Further, as we then have $(\pi_U(f)\xi, \xi) = \sum_{g \in G} f(g)\varphi(g)$ for all $f \in \ell^1(G)$, we also get an unambiguously defined positive linear functional L_φ on $C^*(G,\sigma)$ via $L_\varphi(x) := (\pi_U(x)\xi, \xi)$, which satisfies that $\Phi(L_\varphi) = \varphi$. Denoting by $P(G,\sigma)$ the cone of all σ-p.d. functions on G, we now see that $B(G,\sigma) = \mathrm{Span}(P(G,\sigma))$. By considering the universal *-representation of $C^*(G,\sigma)$, one deduces further that $B(G,\sigma)$ consists precisely of all coefficient functions associated to σ-projective unitary representations of G.

We remark that if φ is σ-p.d. and ψ is ω-p.d. for some $\omega \in Z^2(G,\mathbb{T})$ then $\varphi\psi$ is $\sigma\omega$-p.d. Hence we have $B(G,\sigma)B(G,\omega) \subseteq B(G,\sigma\omega)$. Especially, $B(G,\sigma)$ is not a priori an algebra w.r.t. to pointwise multiplication (unless we have $\sigma = 1$, in which case it is usually called the Fourier-Stieltjes algebra of G). It is not a priori closed under complex conjugation either : if $\varphi \in P(G,\sigma)$, then $\overline{\varphi} \in P(G,\overline{\sigma})$. Similarly, if $\tilde{\varphi}(g) := \sigma(g,g^{-1})\varphi(g^{-1})$, then $\tilde{\varphi} \in P(G,\overline{\sigma})$. Hence $\varphi^* \in P(G,\sigma)$, where $\varphi^*(g) := \overline{\sigma}(g,g^{-1})\overline{\varphi}(g^{-1})$. (This just corresponds to the fact that $L_{\varphi^*} = (L_\varphi)^*$ is then also positive linear functional on $C^*(G,\sigma)$).

As $C^*_r(G,\sigma)$ is a quotient of $C^*(G,\sigma)$, we may identify its dual space as a closed subspace $B_r(G,\sigma)$ of $B(G,\sigma)$. It consists of the span of all σ-p.d. functions on G associated to unitary representations of G which are weakly contained in Λ_σ (that is, such that the associated representation of $C^*(G,\sigma)$ is weakly contained in π_σ [33]). Further, the predual of $vN(G,\sigma)$ can be regarded as a closed subspace of the dual of $C^*_r(G,\sigma)$, hence as a closed subspace $A(G,\sigma)$ of $B_r(G,\sigma)$, and of $B(G,\sigma)$, which may be descrided as the set of all coefficient functions of Λ_σ.

Assume $\tilde\sigma \in Z^2(G,\mathbb{T})$ is such that $[\tilde\sigma] = [\sigma]$, that is $\tilde\sigma = (\mathrm{d}b)\sigma$ for some $b : G \to \mathbb{T}, b(e) = 1$. Then one readily checks that $\Lambda_{\tilde\sigma}$ is unitarily equivalent to $b\Lambda_\sigma$, and it follows that $C^*_r(G,\tilde\sigma)$ (resp. $vN(G,\tilde\sigma)$) is (spatially) isomorphic to $C^*_r(G,\sigma)$ (resp. $vN(G,\sigma)$). Further, we have $C^*(G,\tilde\sigma) \simeq C^*(G,\sigma)$, the isomorphism being given at the ℓ^1-level by the map $f \to bf$. The general problem of deciding when two (full or reduced) twisted group C^*-algebras associated to the same group are *-isomorphic is undoubtedly hard. For some results in this direction based on K-theoretical considerations, see e.g. [37, 71, 72, 18, 63] and references therein. The von Neumann algebraic version of this problem is essentially open, with one notable exception. As follows from Connes' work on injective factors [28], if G is countably infinite and amenable, and $vN(G,\sigma)$ is a factor, then it is the hyperfinite II_1-factor; especially this means that $vN(G,\sigma)$ and $vN(G,\omega)$ are then *-isomorphic whenever σ and ω both satisfy the factor condition. The case of nonamenable groups seems untouched so far.

Concerning the concept of amenability, we refer to [33, 74, 77, 85]. We recall that G is called amenable if there exists a (left or/and right) translation invariant state on $\ell^\infty(G)$. Amenability of G can be formulated in a huge number of equivalent ways. We will make use of the following equivalent characterizations :

1) G has a *Følner net* $\{F_\alpha\}$, that is, each F_α is a finite non-empty subset of G and we have
$$\frac{|gF_\alpha \triangle F_\alpha|}{|F_\alpha|} \underset{\alpha}{\to} 0, \quad g \in G \ . \tag{1}$$

2) there exists a net $\{\varphi_\alpha\}$ of normalized positive definite functions on G with finite support such that $\varphi_\alpha \to 1$ pointwise on G.

   (As usual, a complex function on G is called *normalized* when it takes the value 1 at e).

3) there exists a net $\{\psi_\alpha\}$ of normalized positive definite functions in $\ell^2(G)$ such that $\psi_\alpha \to 1$ pointwise on G.

4) $\Big| \sum_{g\in G} f(g) \Big| \leq \| \sum_{g\in G} f(g)\lambda(g)\| \ (= \|\pi_1(f)\|)$ for all $f \in \ell^1(G)$.

For most aspects of this paper, the reader may take 1) as the definition of the amenability of G, and regard 2), 3) and 4) as properties.

Indeed, assume 1) holds and set $\xi_\alpha := |F_\alpha|^{-1/2}\chi_{F_\alpha}$, which is a unit vector in $\ell^2(G)$. Then 2) is satisfied with $\varphi_\alpha(g) := (\lambda(g)\xi_\alpha, \xi_\alpha) = \frac{|gF_\alpha \cap F_\alpha|}{|F_\alpha|}$: each φ_α is clearly p.d., has finite support given by $\mathrm{supp}(\varphi_\alpha) = F_\alpha \cdot F_\alpha^{-1}$ and the Følner condition (1) is equivalent to $\varphi_\alpha \to 1$ pointwise. Condition 3) is then trivially satisfied with $\psi_\alpha = \varphi_\alpha$. Further, letting ϵ being the state on $B(\ell^2(G))$ obtained by picking any weak*-limit point of the net of vector states $\{\omega_{\xi_\alpha}\}$, we get $\epsilon(\lambda(g)) = 1$ for all $g \in G$, hence

$$\Big| \sum_{g\in G} f(g) \Big| = \Big| \epsilon(\sum_{g\in G} f(g)\lambda(g)) \Big| \leq \| \sum_{g\in G} f(g)\lambda(g)\|$$

for all $f \in \ell^1(G)$, which shows that 4) holds.

We finally review some facts about groups having the so-called Haagerup property, negative definite functions and length functions.

We will say that G has the *Haagerup property* if there exists a net $\{\varphi_\alpha\}$ of normalized positive definite functions on G, vanishing at infinity on G (that is, $\varphi_\alpha \in c_0(G)$ for all α), and converging pointwise to 1. Clearly, all amenable groups have the Haagerup property (by 3)). When G is countable, this property is equivalent to the fact that there exists a negative definite function $h : G \to [0, \infty)$ which is proper, that is, $\lim_{g\to\infty} h(g) = \infty$, or, equivalently, $(1 + h)^{-1} \in c_0(G)$. We will call such a function h a *Haagerup function* on G. We refer to [25] for many other characterizations of Haagerup property for groups, and for a long list of examples and properties of this class of groups, which includes all nonabelian free groups, as first established by

U. Haagerup in [45]. For completeness, we recall that a function $\psi : G \to \mathbb{C}$ is called *negative definite* (also called conditionally negative definite by some authors) whenever ψ is Hermitian, that is $\psi(g^{-1}) = \overline{\psi(g)}$ for all $g \in G$, and

$$\sum_{i,j=1}^{n} \overline{c_i} c_j \psi(g_i^{-1} g_j) \leq 0$$

for all $n \in \mathbb{N}, g_1, \ldots, g_n \in G$ and all $c_1, \ldots, c_n \in \mathbb{C}$ satisfying $\sum_{i=1}^{n} c_i = 0$. For an introduction to this concept, its relationship to the theory of positive definite functions and its relevance to harmonic analysis on semigroups, the reader may consult [11]. We quote the following theorem due to Schoenberg (see [11, Theorem 2.2] for a more general statement), which we will use several times : a function $\psi : G \to \mathbb{C}$ is negative definite if and only if $e^{-t\psi}$ is p.d. for all $t > 0$ (equivalently, r^{ψ} is p.d for all $0 < r < 1$). Another result in the same vein is that $(t + \psi)^{-1}$ is p.d. for all $t > 0$ whenever $\psi : G \to \{z \in \mathbb{C}, \mathrm{Re}(z) \geq 0\}$ is negative definite (see [11, p.75]). We will also need the following fact : if $\psi : G \to \{z \in \mathbb{C}, \mathrm{Re}(z) \geq 0\}$ is negative definite and satisfies $\psi(e) \geq 0$, then so is $\psi^{1/2}$ (see [11, Corollary 2.10]).

To give a simple example, assume that there exists a homorphism b from G into some Hilbert space $\mathcal{H}$ (regarded as a group w.r.t. addition). Then $\psi(g) := \|b(g)\|^2$ is easily seen to be negative definite on G. Especially, $|\cdot|_2$ denoting the Euclidean norm-function on $\mathbb{Z}^N, N \in \mathbb{N}$, it follows that $|\cdot|_2^2$, and therefore also $|\cdot|_2$ (taking the square root), are negative definite on $\mathbb{Z}^N$. The $|\cdot|_1$-norm function on $\mathbb{Z}^N$ is also negative definite. This can be proved by induction : the inductive step being straightforward, it suffices to show this when $N = 1$. Instead of proving this directly (which can be done), one may appeal to Schoenberg's theorem : it suffices then to show that $\varphi(m) := r^{|m|}$ is p.d. on $\mathbb{Z}$ for all $0 < r < 1$. As mentioned in the Introduction, this is well known. Perhaps the simplest way to prove this fact is as follows. Let U denote the unitary representation of $\mathbb{Z}$ on $L^2(\mathbb{T})$ associated to the unitary operator on $L^2(\mathbb{T})$ given by multiplication with the function $z \to z^{-1}, z \in \mathbb{T}$. With $\xi_r := \sum_{k=-\infty}^{\infty} r^{|k|} e_k \in L^2(\mathbb{T})$ for $r \in (0, 1)$, one checks easily that $\varphi(k) = r^{|k|} = (U(k)\xi_r, \xi_r)$ for all $k \in \mathbb{Z}$, and the assertion is then clear.

Haagerup functions can be used to decide whether a countable group G with Haagerup property is amenable. Indeed, let G be such a group and h be a Haagerup function for G, so that we have $(1 + h)^{-1} \in c_0(G)$. Then G is amenable whenever $(1 + h)^{-1} \in \ell^p(G)$ for some $p \geq 1$. More generally, if for some $p \geq 1$ we have $r^h \in \ell^p(G)$ for all $0 < r < 1$, then G is amenable.

To see this, assume first that $(1+h)^{-1} \in \ell^p(G)$ for some $p \geq 1$. Then one deduces without difficulty using the facts mentioned above, that $\varphi_s := (1+sh)^{-([p]+1)}$ is p.d. and summable for all $s > 0$. Obviously, this net converges pointwise to 1 on G as $s \to 0^+$, so the amenability of G follows. Similarly, under the assumption of the second statement, $\psi_r := (r^p)^h$ gives a net of p.d. summable functions converging pointwise to 1 on G as $r \to 1^-$.

An interesting class of functions on a group G is the class of length functions [28, 53, 56]. We recall that a function $L : G \to [0, \infty)$ is called a *length* function if $L(e) = 0, L(g^{-1}) = L(g)$ and $L(gh) \leq L(g) + L(h)$ for all $g, h \in G$. If G acts isometrically on a metric space (X, d) and $x_0 \in X$, then $L(g) := d(g \cdot x_0, x_0)$ gives a *geometric* length function on G. If G is finitely generated and S is a finite generator set for G, then the obvious word-length function $g \to |g|_S$ (w.r.t. to the letters from $S \cup S^{-1}$) is an *algebraic* length function on G. All such algebraic length functions are equivalent in a natural way [53]. Any algebraic length function is clearly proper. Another fact which we will use is the following : for any $t > 0$ and any algebraic length function L on G, the "Gaussian" function e^{-tL^2} is summable (see the proof of [28, Prop. 24]; this corresponds to the fact that the naturally associated unbounded Fredholm module $(\ell^2(G), D_L)$ is θ-summable in Connes' terminology).

Length functions may be used to define growth conditions. Let L be a length function on G and set $B_{r,L} := \{g \in G | L(g) \leq r\}, r \in \mathbb{R}, r \geq 0$. Then G is said to be of *polynomial growth* (w.r.t. L) [53] if there exist some constants $K, p > 0$ such that $|B_{r,L}| \leq K(1+r)^p$ for all $r \geq 0$. Another growth condition on G which is clearly weaker than polynomial growth is the following : G is said to be *exponentially bounded* (w.r.t. L) if for any $b > 1$, there is some $r_0 \in \mathbb{R}, r_0 \geq 0$, such that $|B_{r,L}| < b^r$ for all $r \geq r_0$. This terminology is a generalization of the one used in the case of algebraic case [85, 74], that is, when one considers some algebraic length function on a finitely generated group G. We cite here some facts from these references and from [53]. If G is finitely generated, one just says that G has polynomial growth (resp. is exponentially bounded) if the property holds w.r.t. some or, equivalently, any algebraic length on G. Any exponentially bounded is necessarily amenable. A famous result of M. Gromov says that G is of polynomial growth if (and only if) G is almost nilpotent (the only if part being due to W. Woess). Further, R. I. Grigorchuk has produced examples of exponentially bounded groups which are not of polynomial growth. Finally, if G is finitely generated and has polynomial growth (resp. is exponentially bounded) w.r.t. to some length function L on G, then G has polynomial growth (resp. is exponentially bounded).

Algebraic length functions on finitely generated groups have been used to define (formal) growth series of the type $\sum_{g\in G} z^{L_S(g)}$; these series are known to produce rational functions and to be related to the Euler characteristic in many cases (see e.g. [86, 82, 43, 73, 42]). Our interest will be in summability aspects of this kind of series (for real z between 0 and 1) in the more general case where the length function is not necessarily algebraic.

Length functions also show up in connection with the Haagerup property. Indeed, assume that h is a Haagerup function for some (countable) G. Then $L := h^{1/2}$ is negative definite, and it is also a length function on G : this follows from [11, Proposition 3.3] (the standing assumption that G is abelian is not used in the proof of this proposition). Hence L is a Haagerup length function on G. This means that a countable group has the Haagerup property if and only if it has a Haagerup length function.

In some cases, a Haagerup length function is naturally geometrically given: this is for example the case when G acts isometrically and metrically properly on a tree, or on a $\mathbb{R}$-tree, X (equipped with its natural metric) [12, 83]. In general, one can show [25] that a countable group G has the Haagerup property if and only if there exists an isometric and metrically proper action of G on some metric space (X, d), a unitary representation U of G on some Hilbert space $\mathcal{H}$ and a map $c : X \times X \to \mathcal{H}$ satisfying the following conditions :

$$c(x, z) = c(x, y) + c(y, z), \quad c(g \cdot x, g \cdot y) = U(g)\, c(x, y)$$

$$\|c(x, y)\| \to \infty \quad \text{as} \quad d(x, y) \to \infty, \text{ for all } x, y, z \in X, g \in G.$$

In this case, picking any $x_0 \in X$, $h(g) := d(g \cdot x_0, x_0)^2$ is then a Haagerup function for G, while $L(g) := d(g \cdot x_0, x_0)$ is a Haagerup length function for G.

In the case of finitely generated groups, a Haagerup length function is sometimes algebraically given : this is at least true for finitely generated free groups (see [11] for the abelian case and [45, 25] for the non abelian case) and Coxeter groups [13]. In fact, an interesting question is the following : when does a finitely generated group have an algebraic length function which is negative definite ? One can see from the examples mentioned above that the class of groups having this property, which is clearly a subclass of the class of all finitely generated groups with Haagerup property, contains both amenable and nonamenable groups. The following remark may be worth being pointed out :

Let G be finitely generated and assume that it has an algebraic length function L such that L^2 is negative definite (this implies that L itself is negative definite). Then G is amenable : indeed, the "Gaussian" net of functions on G defined by $\psi_t := e^{-tL^2}, t > 0$ consists then of summable functions which are all normalized and p.d., and it converges pointwise to 1 on G as $t \to 0^+$. One may wonder about which finitely generated amenable groups satisfy this assumption.

3 Convergence of Fourier series and decay properties

Throughout this section, we let $\sigma \in Z^2(G, \mathbb{T})$. In the sequel, $\mathcal{K}(G)$ denotes the set of all complex functions on G having finite support.

Definition 3.1. *Let $\mathcal{L}$ be a subspace of $\ell^2(G)$ which contains $\mathcal{K}(G)$, let $\|\cdot\|'$ be a norm on $\mathcal{L}$ and $\xi \in \mathcal{L}$. When F is finite subset of G, set $\xi_F = \xi\chi_F$.*

We say that $\xi \to 0$ at infinity w.r.t. $\|\cdot\|'$ if for every $\varepsilon > 0$, there exists a finite subset F_0 of G such that $\|\xi_F\|' < \varepsilon$ for all finite subsets F of G which are disjoint from F_0.

Definition 3.2. *Let $\mathcal{L}$ be a subspace of $\ell^2(G)$ which contains $\mathcal{K}(G)$. We say that (G, σ) has the $\mathcal{L}$-decay property (w.r.t. $\|\cdot\|'$) if there exists a norm $\|\cdot\|'$ on $\mathcal{L}$ such that the following two conditions hold:*

i) For each $\xi \in \mathcal{L}$ we have $\xi \to 0$ at infinity w.r.t. $\|\cdot\|'$.

ii) The map $f \to \pi_\sigma(f)$ from $(\mathcal{K}(G), \|\cdot\|')$ to $(C_r^*(G, \sigma), \|\cdot\|)$ is bounded.

We will simply say that G has the $\mathcal{L}$-decay property (w.r.t. $\|\cdot\|'$) if $(G, 1)$ has the $\mathcal{L}$-decay property (w.r.t. $\|\cdot\|'$).

Lemma 3.3. *Assume that (G, σ) has the $\mathcal{L}$-decay property w.r.t. $\|\cdot\|'$ and let $\xi \in \mathcal{L}$. Then the series $\sum_{g\in G} \xi(g)\Lambda_\sigma(g)$ converges in operator norm to some $a \in C_r^*(G, \sigma)$ satisfying $\hat{a} = \xi$. We will denote this a by $\tilde{\pi}_\sigma(\xi)$.*

Proof. Using that ii) holds, we get that there exists $C > 0$ such that

$$\|\sum_{g\in F}\xi(g)\Lambda_\sigma(g)\| = \|\pi_\sigma(\xi_F)\| \leq C\|\xi_F\|'$$

for any finite subset F of G. Now, using that i) holds, we deduce then immediately that the net $\{\sum_{g\in F}\xi(g)\Lambda_\sigma(g)\}_F$, indexed over the finite subsets of G ordered by inclusion, satisfies the Cauchy criterion [35, 9.1.6] w.r.t. operator norm. Hence this net converges in operator norm to some $a \in C_r^*(G,\sigma)$. But then it also converges to a w.r.t. $\|\cdot\|_2$, hence we have $\hat{a} = \xi$ as desired (see Section 2). □

Theorem 3.4. *Assume that (G,σ) has the $\mathcal{L}$-decay property w.r.t. $\|\cdot\|'$. Let $\tilde{\pi}_\sigma : \mathcal{L} \to C_r^*(G,\sigma)$ be the map we obtain from Lemma 3.3. Set $\mathcal{L}^\vee(G,\sigma) = \{x \in vN(G,\sigma) \mid \hat{x} \in \mathcal{L}\}$. Then*

$$\mathcal{L}^\vee(G,\sigma) = \tilde{\pi}_\sigma(\mathcal{L}) \subseteq C_r^*(G,\sigma).$$

Moreover, every $x \in \mathcal{L}^\vee(G,\sigma)$ has a Fourier series which converges to x itself in operator norm, i.e. $\mathcal{L}^\vee(G,\sigma) \subseteq CF(G,\sigma)$.

Proof. Let $x \in \mathcal{L}^\vee(G,\sigma)$. From Lemma 3.3 (with $\xi = \hat{x}$), we get that the Fourier series of x converges in operator norm to $\tilde{\pi}_\sigma(\hat{x}) \in C_r^*(G,\sigma)$ and we have $\widehat{\tilde{\pi}_\sigma(\hat{x})} = \hat{x}$. This implies that $\tilde{\pi}_\sigma(\hat{x}) = x$. Thus we have shown that $\mathcal{L}^\vee(G,\sigma) \subseteq \tilde{\pi}_\sigma(\mathcal{L})$ and also that the last assertion holds. Finally, if $x \in \tilde{\pi}_\sigma(\mathcal{L})$, so $x = \tilde{\pi}_\sigma(\xi)$ for some $\xi \in \mathcal{L}$, then it follows from Lemma 3.3 that $\hat{x} = \xi \in \mathcal{L}$, and the converse inclusion follows. □

It is almost immediate that (G,σ) has the $\ell^1(G)$-decay property w.r.t. $\|\cdot\|_1$. Moreover, we already saw in Section 2 that the assertions in Lemma 3.3 and Theorem 3.4 hold when $\mathcal{L} = \ell^1(G)$.

As another source of examples, we shall now consider weighted spaces. We establish first some notation.

Let $\kappa : G \to [1,\infty), 1 \leq p \leq \infty$ and define

$$\mathcal{L}^p_\kappa = \{\xi : G \to \mathbb{C} \mid \xi\kappa \in \ell^p(G)\} \subseteq \ell^p(G),$$

which becomes a Banach space w.r.t. the norm $\|\xi\|_{p,\kappa} = \|\xi\kappa\|_p$. Clearly, $\mathcal{L}^p_\kappa$ is the closure of $\mathcal{K}(G)$ w.r.t. $\|\cdot\|_{p,\kappa}$ and $(\mathcal{L}^p_\kappa, \|\cdot\|_{p,\kappa}) \cong (\ell^p(G), \|\cdot\|_p)$ under the map $\xi \to \xi\kappa$. Note also that $\mathcal{L}^p_\kappa \subseteq \mathcal{L}^q_\kappa$ and $\|\cdot\|_{q,\kappa} \leq \|\cdot\|_{p,\kappa}$ whenever $1 \leq p \leq q \leq \infty$, while $\mathcal{L}^p_\gamma \subseteq \mathcal{L}^p_\kappa$ and $\|\cdot\|_{p,\kappa} \leq \|\cdot\|_{p,\gamma}$ whenever $\gamma : G \to [1,\infty)$ is such that $\kappa \leq \gamma$.

We are mostly interested in the case $p = 2$ and recall that $\mathcal{L}^2_\kappa$ becomes a Hilbert space w.r.t. the inner product

$$(\xi,\eta)_\kappa = \sum_{g\in G} \xi(g)\overline{\eta(g)}\kappa(g)^2 = (\xi\kappa,\eta\kappa)_{\ell^2(G)}.$$

Definition 3.5. *We say that (G,σ) is κ-decaying if (G,σ) has the $\mathcal{L}^2_\kappa$-decay property w.r.t. $\|\cdot\|_{2,\kappa}$. We simply say that G is κ-decaying if $(G,1)$ is κ-decaying.*

Remark 3.6. *As $\mathcal{L}^2_\kappa$ obviously satisfies condition i) in Definition 3.2 w.r.t. $\|\cdot\|_{2,\kappa}$, we get that (G,σ) is κ-decaying if and only if the linear map $f \to \pi_\sigma(f)$ from $(\mathcal{K}(G),\|\cdot\|_{2,\kappa})$ to $(C^*_r(G,\sigma),\|\cdot\|)$ is bounded, in which case the norm of this map will be called the* κ-decay constant *of (G,σ). It follows from Theorem 3.4 that if (G,σ) is κ-decaying, then we have*

$$\{x \in vN(G,\sigma) \mid \hat{x} \in \mathcal{L}^2_\kappa\} \subseteq CF(G,\sigma).$$

If $\gamma : G \to [1,\infty)$ is such that $\kappa \leq \gamma$, then (G,σ) is γ-decaying whenever it is κ-decaying. One genuine problem is to determine κ "as small as possible" such that (G,σ) is κ-decaying, as it then gives access to a "biggest possible" decay subspace. Note that if κ is bounded, then $\mathcal{L}^2_\kappa = \ell^2(G)$ and the 2-norms are equivalent; furthermore, in this case, it is not difficult to see that (G,σ) is κ-decaying if and only if G is finite. On the other hand, intuitively, (G,σ) will be κ-decaying if κ grows "rapidly enough".

More concretely, assume that G is countable and $\kappa : G \to [1,\infty)$ satisfies condition (IS), that is, $\kappa^{-1} \in \ell^2(G)$. Then the Cauchy-Schwarz inequality immediately gives that $\mathcal{L}^2_\kappa \subseteq \ell^1(G)$ and $\|f\|_1 \leq \|\kappa^{-1}\|_2\,\|f\|_{2,\kappa}, f \in \mathcal{L}^2_\kappa$. As $\|\pi_\sigma(f)\| \leq \|f\|_1$ for all $f \in \mathcal{K}(G)$, it readily follows that (G,σ) is then κ-decaying.

The assumption that a weight function κ satisfies condition (IS) may be weakened (for nonamenable groups) to a "Haagerup type" assumption still ensuring κ-decay. Before discussing this, it is appropriate to review here some previously known results.

The case $\kappa_{L,s}(g) = (1 + L(g))^s$, L being a length function on G and $s > 0$, has received a lot of attention in the literature. However it seems unnecessary to consider only weight functions of this type, at least from the point of view of this section. Nevertheless, this special case has an obvious geometric flavour and is also an important source of examples. We recall that a (countable) group G is said to have the Rapid Decay property (w.r.t. a length function L on G) according to P. Jolissaint [53] if and only if there exists some $s_0 > 0$ such that G is κ_{L,s_0}-decaying in our terminology. Note that if G is amenable, then G has the RD-property (w.r.t. L) if and only if G has polynomial growth (w.r.t. L) (see [53, Corollary 3.1.8] and [84, 20]). When G is finitely generated, one just talks about the RD-property, having in mind that L is then chosen to be any algebraic length function on G.

According to our remark above, if L is a length function on G such that for some $s_0 > 0$, κ_{L,s_0} satisfies condition (IS), that is, we have

$$(*) \qquad \frac{1}{(1+L)^{s_0}} \in \ell^2(G),$$

then G has the RD-property (w.r.t. L). It is not difficult to see that $(*)$ is equivalent to the fact that the Fredholm module $(\ell^2(G), D_L)$ is finitely summable in the sense of Connes [28]. In the classical case where $G = \mathbb{Z}^N$ and L is the algebraic length function with respect to the canonical generator set, that is, $L = |\cdot|_1$ on $\mathbb{Z}^N$, one has $\frac{1}{(1+L)^{s_0}} \in \ell^2(\mathbb{Z}^N)$ whenever $s_0 > \frac{N}{2}$. Hence $\mathbb{Z}^N$ has the RD-property.

More generally, assume that G is finitely generated. Then condition $(*)$ (w.r.t. any length function) forces G to have polynomial growth (cf. the proof of [28, Prop. 6]). Conversely, if L is any algebraic length function on G and G has polynomial growth, then L satisfies $(*)$ for some $s_0 > 0$ (again, cf. the proof of [28, Prop. 6]), hence G has therefore the RD-property.

Much of the interest around the RD-property is due to the following : when G has the RD-property (w.r.t. L), then the canonical image of the Fréchet space $H_L^\infty := \cap_{s>0}\mathcal{L}^2_{\kappa_{L,s}}$ (w.r.t. the obvious family of seminorms), which is thought as representing a space of "smooth" functions on the "dual" of G, is a dense "spectral" $*$-subalgebra of $C_r^*(G)$. For more about this and the RD-property, see e.g. [54, 56, 57, 84, 19, 20] and references therein. See also the end of this section.

A (countable) group G is said to have the σ-twisted Rapid Decay property (w.r.t. a length function L) according to I. Chatterji [19] if and only if there exists some $s_0 > 0$ such that (G, σ) is κ_{L,s_0}-decaying. She shows [19, Lemma 0.4] that if G has the Rapid Decay property (w.r.t. L) then G has the σ-twisted Rapid Decay property (w.r.t. L). In the general case, (G, σ) will be κ-decaying whenever G is known to be κ-decaying. Indeed, as we trivially have $\| \, |\xi| \, \|_{2,\kappa} = \|\xi\|_{2,\kappa}$ for all $\xi \in \mathcal{L}^2_\kappa$, this follows from our next proposition.

Proposition 3.7. *Assume that G has the $\mathcal{L}$-decay property w.r.t. $\| \cdot \|'$ and that $\| \, |f| \, \|' = \|f\|'$ for all $f \in \mathcal{K}(G)$. Then (G, σ) has the $\mathcal{L}$-decay property w.r.t. $\| \cdot \|'$.*

Proof. Let $C > 0$ denote the κ-decay constant of $(G, 1)$. Let $f \in \mathcal{K}(G)$ and $\eta \in \ell^2(G)$. Then

$$\begin{aligned} \|\pi_\sigma(f)\eta\|_2 = \|f *_\sigma \eta\|_2 &\leq \| \, |f| * |\eta| \, \|_2 = \|\pi_1(|f|)|\eta| \, \|_2 \\ &\leq C\| \, |f| \, \|' \, \| \, |\eta| \, \|_2 = C\|f\|' \, \|\eta\|_2 \end{aligned}$$

Hence, we have $\|\pi_\sigma(f)\| \leq C\|f\|'$ for all $f \in \mathcal{K}(G)$. As the first condition in Definition 3.2 is independent of σ, the assertion follows. □

As all the norms which will be of interest in this paper satisfy the assumption in this proposition, it would suffice henceforth to study only decay properties for G. However, the general question whether decay properties might be sensible to twisting being unanswered, we will still work with a twist.

For finitely generated groups which are not of polynomial growth, one is naturally led to wonder about possible decay-properties w.r.t. exponentiated length functions. Assume that G is finitely generated and let L denote any algebraic length function on G. For $t > 0$, set $\kappa_t := e^{tL^2}$. Then, as pointed out in Section 2, $\kappa_t^{-1} = e^{-tL^2} \in \ell^2(G)$ for all $t > 0$. Hence (G, σ) is κ_t-decaying for all $t > 0$. One may also consider $\gamma_a := a^L, a > 1$. Then γ_a^{-1} is easily seen to belong to $\ell^2(G)$ for all $a > 1$ whenever G is exponentially bounded [85, 74]. For groups which are not exponentially bounded (that is, groups of exponential growth), we can only deduce that there exists some $a_L \geq 1$ such that γ_a^{-1} belongs to $\ell^2(G)$ for all $a > a_L$ (one may for instance choose $a_L = 2|S| + 1$, where S denotes the generator set for G w.r.t. which L is defined). If G is nonamenable and L is negative definite, then a_L can not be chosen to be equal to 1 : the net $\{\gamma_a^{-1}\}_{a>1}$ would then be a net of normalized positive definite functions in $\ell^2(G)$ converging pointwise to 1 (as $a \to 1^+$), contradicting the nonamenability of G (cf. Section 2). Nevertheless, we will soon see that for many groups, even of exponential growth, we can still conclude that (G,σ) is γ_a-decaying for *all* $a > 1$. In fact, all the groups for which we know concretely that this true, will also satisfy the twisted RD-property.

All the following considerations are inspired by Haagerup's treatment of the case $G = \mathbb{F}_2$ [45]. Letting L denote any algebraic length on $\mathbb{F}_2$, it is not difficult to see that for any $s > 0$, $\kappa_{L,s}^{-1} = (1+L)^{-s}$ does not belong to $\ell^2(\mathbb{F}_2)$. However, it follows from [45] (see also [53, 84]) that $\mathbb{F}_2$ still has the RD-property (as it is $\kappa_{L,2}$ -decaying). The clue is that $\kappa_{L,2}$ satisfies the weaker "Haagerup type"condition we alluded to earlier. To formulate it, we first adapt Haagerup's idea to our general setting. It is based on the following simple lemma.

Lemma 3.8. *Let E be a non-empty finite subset of G. Define*

$$c_\sigma(E) := \sup\{\|\pi_\sigma(f)\| \mid f \in \mathcal{K}(G), \operatorname{supp}(f) \subseteq E, \|f\|_2 = 1\}.$$

When $\sigma = 1$ we set $c(E) := c_1(E)$. Then $1 \le c_\sigma(E) \le c(E) \le |E|^{1/2}$.

If G is amenable, then $c(E) = |E|^{1/2}$.

Proof. If $a \in E$, then $\|\delta_a\|_2 = 1$ and $\|\pi_\sigma(\delta_a)\| = \|\Lambda_\sigma(a)\| = 1$, hence it follows that $c_\sigma(E) \ge 1$.

Let $f \in \mathcal{K}(G)$ with $\operatorname{supp}(f) \subseteq E, \|f\|_2 = 1$. Then, for all $\eta \in \ell^2(G)$ such that $\|\eta\|_2 = 1$, as in the proof of Proposition 3.7, we get

$$\|\pi_\sigma(f)\eta\|_2 \le \|\pi_1(|f|)|\eta|\,\|_2 \le \|\pi_1(|f|)\| \le c(E),$$

and it follows that $\|\pi_\sigma(f)\| \le c(E)$. Hence $c_\sigma(E) \le c(E)$.

Next, we have

$$\|\pi_1(f)\| \le \|f\|_1 = \sum_{g\in E} |f(g)| \le |E|^{1/2}(\sum_{g\in E} |f(g)|^2)^{1/2} = |E|^{1/2}\|f\|_2$$

for every $f \in \mathcal{K}(G)$ with $\operatorname{supp} f \subseteq E$. So $c(E) \le |E|^{1/2}$.

Finally, assume that G is amenable. Set $f = \frac{1}{|E|^{1/2}}\chi_E$. Then we have $\|f\|_2 = 1$ and $|E|^{1/2} = \|f\|_1 = \|\pi_1(f)\|$, hence we get $|E|^{1/2} \le c(E)$ and the last assertion follows. □

We leave to the reader to check that $c_\sigma(E) \leq c_\sigma(F)$ whenever $E \subseteq F$ and $c_\sigma(E \cup F) \leq c_\sigma(E) + c_\sigma(F)$ whenever E and F are pairwise disjoint (E, F being finite nonempty subsets of G). The computation of $c_\sigma(E)$, or of $c(E)$, or just finding an upper bound better than $|E|^{1/2}$ for finite subsets E of a nonamenable group G appears to be quite challenging in general. For $c(E)$, it has been dealt with in some special cases (e.g. [61, 1, 40, 45, 39, 27, 56, 41], often in connection with the related problem of estimating the norm $\|\pi_1(f)\|$ for $f \in \mathcal{K}(G)$ (especially when $f = \chi_E$).

Proposition 3.9. *Assume that G is countably infinite and let $\mathcal{E} = \{E_j\}_{j=0}^\infty$ be an indexed partition of G into finite nonempty subsets.*
Set $c_{\sigma,j} := c_\sigma(E_j), j \geq 0$.
Let $\{d_j\}_{j \in \mathbb{N} \cup 0}$ be any sequence in $[1, \infty)$ satisfying $S := \sum_{j=0}^\infty (\frac{c_{\sigma,j}}{d_j})^2 < \infty$

and let $\kappa : G \to [1, \infty)$ be given by $\kappa = \sum_{j=0}^\infty d_j \chi_{E_j}$.

Then (G, σ) is κ-decaying, with decay constant less or equal to $S^{1/2}$.

Proof. Set $\chi_j = \chi_{E_j},\ j \geq 0$. For $f \in \mathcal{K}(G)$, we have

$$\begin{aligned}
\|\pi_\sigma(f)\| &= \|\sum_{j=0}^\infty \pi_\sigma(f\chi_j)\| \leq \sum_{j=0}^\infty \|\pi_\sigma(f\chi_j)\| \\
&\leq \sum_{j=0}^\infty c_{\sigma,j}\|f\chi_j\|_2 = \sum_{j=0}^\infty \frac{c_{\sigma,j}}{d_j} d_j \|f\chi_j\|_2 \\
&\leq \Big(\sum_{j=0}^\infty \Big(\frac{c_{\sigma,j}}{d_j}\Big)^2\Big)^{1/2} \Big(\sum_{j=0}^\infty d_j^2 \|f\chi_j\|_2^2\Big)^{1/2} = C\|f\|_{2,\kappa},
\end{aligned}$$

where $C = S^{1/2}$, as desired.

□

As an immediate corollary we get

Corollary 3.10. *Assume that G is countably infinite and $\kappa : G \to [1, \infty)$ is proper. Let $\mathcal{E}_\kappa = \{E_j\}_{j=0}^\infty$ denote the partition of G into the level sets of κ, indexed according to the enumeration $\{\kappa_j\}_{j=0}^\infty$ of the different values of κ listed in strictly increasing order.*

Set $c_{\sigma,j} := c_\sigma(E_j), j \geq 0,$ *and define* $h_{\sigma,\kappa} : G \to [1, \infty)$ *by*

$$h_{\sigma,\kappa}(g) = \frac{|E_j|^{1/2}}{c_{\sigma,j}}, \ g \in E_j.$$

Further, set $\tilde{\kappa}_\sigma = \kappa\, h_{\sigma,\kappa}$ *(which is clearly* $\geq \kappa$*).*

Then (G, σ) *is* κ*-decaying whenever* κ *satisfies condition* (HC_σ)*:* $\tilde{\kappa}_\sigma^{-1} \in \ell^2(G)$*, that is, whenever* $\sum_{j=0}^{\infty} \left(\frac{c_{\sigma,j}}{\kappa_j}\right)^2 < \infty$.

When $\sigma = 1$ *we just drop the index* σ *in all the notation introduced above.*

Then G *is* κ*-decaying (and therefore* (G, σ) *too for any* σ*) whenever* κ *satisfies condition (HC) :* $\tilde{\kappa}^{-1} \in \ell^2(G)$*, that is, whenever* $\sum_{j=0}^{\infty} \left(\frac{c_j}{\kappa_j}\right)^2 < \infty$.

Remark 3.11. *Let* G *and* κ *be as in Corollary 3.10.*

As $\kappa \leq \tilde{\kappa} \leq \tilde{\kappa}_\sigma$*, it is clear that if* κ *satisfies condition* (IS)*, then it satisfies* (HC)*, which itself implies that it satisfies* (HC_σ)*. Further, if* G *is amenable, then condition (HC) just reduces to condition (IS).*

One may in general try to measure the growth of a (countable) group w.r.t. some proper scale function :

Definition 3.12. *Assume that* G *is countable and let* $F : G \to [0, \infty)$ *be proper.*

For $k \geq 0$ $(k \in \mathbb{Z})$*, set*

$$A_k := \{g \in G \,|\, k \leq F(g) < k + 1\}$$

and

$$C_\sigma(k) := c_\sigma(A_k) \text{ if } A_k \text{ is nonempty, and } 0 \text{ otherwise.}$$

We will say that G *has* polynomial H_σ-growth *(w.r.t.* F*) if there exist some constants* $K, p > 0$ *such that* $C_\sigma(k) \leq K(1 + k)^p$ *for all* $k \geq 0$.

Further, we will say that G *is* exponentially H_σ-bounded *(w.r.t.* F*) if for any* $b > 1$*, there exists some* $j_0 \in \mathbb{N}$ *s.t.* $C_\sigma(j) < b^j$ *whenever* $j \geq j_0$.

When $\sigma = 1$*, we just drop putting an index to* C *and* H *in these definitions.*

Let G and F be as above. Set $B_r := \{g \in G \,|\, F(g) \leq r\}$ for all real $r \geq 0$. Using the properties of c_σ, it is not difficult to check that G has polynomial H_σ-growth if and only if there exist constants $K', p' > 0$ such that $c_\sigma(B_r) \leq K'(1+r)^{p'}$ for all real $r \geq 0$. Similarly, G is exponentially H_σ-bounded if and only if for any $b > 1$, there exists some real $r_1 \geq 0$ such that $c_\sigma(B_r) < b^r$ whenever $r \geq r_1$.

Further, one checks without trouble the following facts (w.r.t. a fixed F): G has polynomial H_σ-growth (resp. is exponentially H_σ-bounded) whenever it has polynomial H-growth (resp. is exponentially H-bounded). Further, G is exponentially H_σ-bounded whenever it has polynomial H_σ-growth.

Finally, if L is length function on G and G has polynomial growth (w.r.t L), then it has polynomial H_σ-growth (w.r.t L); when G is amenable, polynomial H-growth (w.r.t L) reduces to polynomial growth (w.r.t L), while exponential H-boundedness (w.r.t. L) reduces to exponential boundedness (w.r.t. L).

Some interesting examples of groups with polynomial H-growth w.r.t. some appropriate length functions are given in the following list.

Example 3.13. *1) Let $G = \mathbb{F}_n, n < \infty$, denote a free group and let L denote the natural algebraic length on G. Then we have $C(k) \leq k+1$ for all $k \geq 0$ (see [45] for $n = 2$ and [84] for a nice geometric proof of the general case due to T. Steger). Hence G has polynomial H-growth (w.r.t. L).*

2) More generally, let G denote a finitely generated Gromov hyperbolic group [44] and let L denote the natural algebraic length on G (w.r.t. to some finite generator set S). Then G has polynomial H-growth (w.r.t. L). This may be deduced from [53, 50] : in the course of their proof that G has the RD property, they implicitely show that there exists a constant $K > 0$ such that $C(k) \leq K(1+k)$ for all $k \geq 0$. This may also be deduced from [67] and [30], where a stronger form of this Haagerup's type condition is shown.

3) Let (G, S) denote a finitely generated Coxeter group [52] and let L denote the natural algebraic length on G (w.r.t. S). Then $C(k) \leq K(1+k)^{\frac{3}{2}P}$ for some $K > 0$ and $P \in \mathbb{N}$, see [41]. Hence G has polynomial H-growth (w.r.t. L). Note that G is nonamenable whenever it is neither finite nor affine [48].

4) Let $G = G_1 *_A G_2$ be an amalgamated free product of groups and let L denote the "block" length on G induced by some integer-valued length functions L_j on G_j satisfying $L_j = 0$ on A, $j = 1, 2$, (cf. [76, 12]). If A is finite and each G_j has polynomial H-growth (w.r.t. L_j), $j = 1, 2$, then, adapting the proof of [53, Theorem 2.2.2 (1)], one can deduce that G has polynomial H-growth (w.r.t. L).

Let G be a group acting freely and isometrically on a metric space and L denote the geometric length w.r.t. any base point. In [20, Prop. 1.7] (due to V. Lafforgue), some sufficient conditions are given ensuring that G will have the RD property (w.r.t L). It seems reasonable to believe that G will then also have polynomial H-growth (w.r.t. L) as these conditions are of polynomial type, but we have not checked all details. It could also be interesting to formulate (and prove) an exponentially bounded H-growth version of this result. Another related question is whether the conditions in [20, Prop. 1.7] are always satisfied if the space on which the group acts is a "space with walls" [25].

To produce an example of a nonamenable group which is exponentially H-bounded (w.r.t. some length function L) without also having polynomial H-growth (w.r.t. L), one may proceed as follows. Let G_1 be any finitely generated group which is exponentially bounded, but does not have polynomial growth (cf. [85]) and let L_1 denote some algebraic length function on G_1. Further let G_2 denote some group which is exponentially H-bounded w.r.t. some length function L_2 on G_2. Let L be the length function on $G := G_1 \times G_2$ defined by $L(g_1, g_2) = L_1(g_1) + L_2(g_2)$. Then one checks that G is exponentially H-bounded (w.r.t. L), but G can not have polynomial H-growth (w.r.t. L) as this would force G_1 to have it (w.r.t. L_1), hence G_1 would have polynomial growth.

The main reason for introducing these types of growth concepts is that the following result holds.

Corollary 3.14. *Assume that G is countably infinite and $F : G \to [0, \infty)$ is proper.*

1) Assume that G has polynomial H-growth (w.r.t. F), or, more generally, that G has polynomial H_σ-growth (w.r.t. F).

Then there exist some $s_0 > 0$ such that (G, σ) is $(1 + F)^{s_0}$-decaying.

Especially, if $F = L$ is a length function, then G has the σ-twisted RD-property (w.r.t. L).

2) Assume that G is exponentially H-bounded (w.r.t. F), or more generally, that G is exponentially H_σ-bounded (w.r.t. F).

Then (G, σ) is a^F-decaying for all $a > 1$.

Proof. For each $k \geq 0$, let A_k and $C_\sigma(k)$ be defined as in Definition 3.12.

Define $I = \{k \in \mathbb{N} \cup \{0\} \mid A_k \text{ is nonempty}\}$ and let $\{k_j\}_{j=0}^{\infty}$ denote an enumeration of the elements of I, listed in strictly increasing order. Note that $k_j \geq j$ for all j. Further, set $E_j := A_{k_j}, j \geq 0$. Then the family $\{E_j\}_{j\geq 0}$ is a partition of G in finite nonempty subsets.

For $j \geq 0$, set $c_{\sigma,j} := c_\sigma(E_j) = C_\sigma(k_j)$.

We will now prove the first assertion. We assume therefore that there exist some contants $K, p > 0$ such that $C_\sigma(k) \leq K(1 + k)^p$ for all $k \geq 0$. Choose $s_0 > 0$ such that $s_0 > p + \frac{1}{2}$.
Then we have

$$\sum_{j=0}^{\infty} \Big(\frac{c_{\sigma,j}}{(1 + k_j)^{s_0}}\Big)^2 \leq \sum_{j=0}^{\infty} K^2 \Big(\frac{(1 + k_j)^p}{(1 + k_j)^{s_0}}\Big)^2 = K^2 \sum_{j=0}^{\infty} \frac{1}{(1 + k_j)^{2(s_0-p)}}$$
$$\leq K^2 \sum_{j=0}^{\infty} \frac{1}{(1 + j)^{2(s_0-p)}} < \infty$$

as $2(s_0 - p) > 1$.
Hence, defining $\kappa : G \to [1, \infty)$ by $\kappa := \sum_{j=0}^{\infty} (1 + k_j)^{s_0} \chi_{E_j}$, we get from Proposition 3.9 that (G, σ) is κ-decaying. Now, as $\kappa \leq (1+F)^{s_0}$, this implies that (G, σ) is $(1 + F)^{s_0}$-decaying, and assertion 1) follows.

Next, assume that G is exponentially H_σ-bounded (w.r.t. F) and let $a > 1$. Choose $b > 1$ such that $b < a$, and $j_0 \in \mathbb{N}$ such that $C_\sigma(j) < b^j$ whenever $j \geq j_0$.

Then we have

$$\sum_{j=j_0}^{\infty}\left(\frac{c_{\sigma,j}}{a^{k_j}}\right)^2 \leq \sum_{j=j_0}^{\infty}\left(\frac{b^{k_j}}{a^{k_j}}\right)^2 = \sum_{j=j_0}^{\infty}\left(\frac{b^2}{a^2}\right)^{k_j}$$
$$\leq \sum_{k=k_{j_0}}^{\infty}\left(\frac{b^2}{a^2}\right)^{k} < \infty$$

as $b^2/a^2 < 1$.

Hence, defining $\gamma : G \to [1,\infty)$ by $\gamma := \sum_{j=0}^{\infty} a^{k_j}\chi_{E_j}$, we get from Proposition 3.9 that (G,σ) is γ-decaying. Now, as $\gamma \leq a^F$, this implies that (G,σ) is a^F-decaying, and assertion 2) follows.

□

Example 3.15. *Let G be any of the groups listed in Example 3.13, equipped with the length function L introduced there. As G has polynomial H-growth (w.r.t. L), it follows from Corollary 3.14 that G has the σ-twisted RD-property (w.r.t. L), and also that (G,σ) is a^L-decaying for all $a > 1$.*

We conclude this section with some remarks on the interesting class of weight functions κ satisfying

$$\kappa(e) = 1, \quad \kappa(g^{-1}) = \kappa(g), \quad \kappa(gh) \leq \kappa(g)\kappa(h)$$

for all $g, h \in G$. Such functions are called "absolute values" in [11], and just "weights" in [81], so we will call them *absolute* weights here. Note that $\kappa^s, s > 0$ is then also an absolute weight. If L is a length function on G, then $(1 + L)^s, s > 0$ and $a^L, a > 1$ are all examples of such absolute weight functions. Conversely, if κ is an absolute weight function, then $\log_a(\kappa)$ is a length function for any $a > 1$.

It may be worth mentioning that absolute weights are related to certain norms on $\mathcal{K}(G)$. If N is a norm on $\mathcal{K}(G)$ satisfying $N(\delta_e) = 1, N(\xi^*) = N(\xi)$, and $N(\xi *_\sigma \eta) \leq N(\xi)N(\eta)$ for all $\xi, \eta \in \mathcal{K}(G)$, that is, N is a $*$-algebra norm on $\mathcal{K}(G)$ (w.r.t. σ-twisted convolution and involution), then $\kappa_N(g) := N(\delta_g)$ gives an absolute weight on G. Conversely, one may show (using the first inequality in the next paragraph) that if κ is an absolute weight on G, then $N_\kappa := \|\cdot\|_{1,\kappa}$ gives a norm on $\mathcal{K}(G)$ satisfying the above properties (for any σ).

Now, fix an absolute weight κ on G. For $\xi, \eta \in \mathcal{L}^2_\kappa$, it is an easy exercise to verify that

$$|(\xi *_\sigma \eta)\kappa| \leq |\xi\kappa| \, * \, |\eta\kappa|.$$

This clearly implies that

$$\|\xi *_\sigma \eta\|_{1,\kappa} \leq \|\xi\|_{1,\kappa}\|\eta\|_{1,\kappa}$$

whenever $\xi, \eta \in \mathcal{L}^1_\kappa$. Using this, it follows readily that $\mathcal{L}^1_\kappa$ becomes a Banach $*$-algebra w.r.t. σ-twisted convolution and involution.

Further, define $H^\infty_\kappa(G) := \cap_{s>0} \mathcal{L}^2_{\kappa^s}$, which becomes a Fréchet space (w.r.t. the obvious family of seminorms) and contains $\mathcal{K}(G)$. If G is κ-decaying with decay constant C, then we have

$$\|\xi *_\sigma \eta\|_{2,\kappa} \leq C\|\xi\|_{2,\kappa^2}\|\eta\|_{2,\kappa}$$

whenever $\xi \in \mathcal{L}^2_{\kappa^2}, \eta \in \mathcal{L}^2_\kappa$. Indeed, as we have then $\xi\kappa \in \mathcal{L}^2_\kappa$, we get

$$\|\xi *_\sigma \eta\|_{2,\kappa} \leq \||\xi\kappa| * |\eta\kappa|\|_2 \leq \|\tilde{\pi}_1(\xi\kappa)\|\|\eta\kappa\|_2 \leq C\|\xi\kappa\|_{2,\kappa}\|\eta\|_{2,\kappa} = C\|\xi\|_{2,\kappa^2}\|\eta\|_{2,\kappa}.$$

Assume now that G is κ^{s_0}-decaying for some $s_0 > 0$. (For instance, G may be any finitely generated group, L any algebraic length function on G and $\kappa = e^L$, or G may be any exponentially H-bounded group w.r.t. some length function L and again $\kappa = e^L$). Then one deduces easily from the above inequality (considering $\xi, \eta \in H^\infty_\kappa(G)$ and replacing κ with κ^s for any $s \geq s_0$) that $H^\infty_\kappa(G)$ becomes a $*$-algebra under twisted convolution and involution, hence that $\tilde{\pi}_\sigma(H^\infty_\kappa(G))$ is a (dense) $*$-subalgebra of $C^*_r(G, \sigma)$. Quite probably, it is also a "spectral" subalgebra, as it is in the case where $\kappa = 1 + L$ and G has property RD w.r.t. L, cf. our earlier remark on this and [19], but we have not checked all details. For much more about Fréchet subalgebras in the general setting of crossed products, the reader may consult e.g. [81, 57, 24] and references therein.

4 Fell's property and Haagerup's lemma - twisted versions

Throughout this section, we let $\sigma \in Z^2(G, \mathbb{T})$. Let V be any projective unitary representation of G on some Hilbert space $\mathcal{H}$, with associated 2-cocycle

$\omega \in Z^2(G,\mathbb{T})$.Then Fell's absorbing property [33, 13.1.3] for the regular representation λ generalizes to the twisted case : this has been already observed in [7, Prop. 2.2] (where it is phrased in terms of λ_σ and $\lambda_{\sigma\omega}$); the Λ-version is as follows.

Let $\mathcal{V}$ be the unitary operator on $\ell^2(G)\otimes\mathcal{H} \cong \ell^2(G,\mathcal{H})$ given by

$$(\mathcal{V}\psi)(g) = V(g)\psi(g),\ \ g\in G, \psi\in\ell^2(G,\mathcal{H}).$$

Then we have

$$\mathcal{V}^*(\Lambda_\sigma(g)\otimes V(g))\mathcal{V} = \Lambda_{\sigma\omega}(g)\otimes I_{\mathcal{H}}$$

for all $g\in G$, that is,

$$\Lambda_\sigma\otimes V\ \cong\ \Lambda_{\sigma\omega}\otimes I_{\mathcal{H}}.$$

For completeness, we sketch the proof. First, one checks that $\Lambda_\sigma(g)\otimes V(g)$ acts on $\ell^2(G,\mathcal{H})$ by $[(\Lambda_\sigma(g)\otimes V(g))\psi](h) = \sigma(g,g^{-1}h)V(g)\psi(g^{-1}h)$. Next, one computes that for every $g,h\in G$ and $\psi\in\ell^2(G,\mathcal{H})$, we have

$$\begin{aligned}
[(\mathcal{V}^*(\Lambda_\sigma(g)\otimes V(g))\mathcal{V})\psi](h) &= \overline{\omega(h,h^{-1})}V(h^{-1})[(\Lambda_\sigma(g)\otimes V(g))\mathcal{V}\psi](h)\\
&= \overline{\omega(h,h^{-1})}V(h^{-1})\sigma(g,g^{-1}h)V(g)[(\mathcal{V}\psi)(g^{-1}h)]\\
&= \overline{\omega(h,h^{-1})}\sigma(g,g^{-1}h)V(h^{-1})V(g)V(g^{-1}h)\psi(g^{-1}h)\\
&= \overline{\omega(h,h^{-1})}\sigma(g,g^{-1}h)V(h^{-1})\omega(g,g^{-1}h)V(h)\psi(g^{-1}h)\\
&= \overline{\omega(h,h^{-1})}\sigma(g,g^{-1}h)\omega(g,g^{-1}h)\omega(h^{-1},h)\psi(g^{-1}h)\\
&= (\sigma\omega)(g,g^{-1}h)\psi(g^{-1}h) = [(\Lambda_{\sigma\omega}(g)\otimes I_{\mathcal{H}})\psi](h),
\end{aligned}$$

which shows the assertion.

Using this property, we can deduce the following twisted generalization of Haagerup's fundamental lemma [45, Lemma 1.1] (see also [17, Theorem 2.2] and [31, Corollary 1.2.2]):

Lemma 4.1. *Let $\omega\in Z^2(G,\mathbb{T})$ and $\varphi\in B(G,\omega)$. Choose an ω-projective unitary representation V of G on some Hibert space $\mathcal{H}$ and $\eta_1,\eta_2\in\mathcal{H}$ such that*

$$\varphi(g) = (V(g)\eta_1,\eta_2)\ g\in G.$$

1) There exists a completely bounded normal map $M=\tilde{M}_\varphi : vN(G,\sigma\omega)\to vN(G,\sigma)$ such that

$$\tilde{M}_\varphi(\Lambda_{\sigma\omega}(g)) = \varphi(g)\Lambda_\sigma(g),\quad g\in G, \tag{2}$$

and $\|\tilde{M}_\varphi\| \le \|\tilde{M}_\varphi\|_{cb} \le \|\varphi\| \le \|\eta_1\|\|\eta_2\|$.

As $\tilde{M}_\varphi$ *then clearly maps* $C_r^*(G, \sigma\omega)$ *into* $C_r^*(G, \sigma)$, *we obtain by restriction a completely bounded map* M_φ *from* $C_r^*(G, \sigma\omega)$ *into* $C_r^*(G, \sigma)$, *which satisfies* $\|\tilde{M}_\varphi\| = \|M_\varphi\|$ *and* $\|\tilde{M}_\varphi\|_{cb} = \|M_\varphi\|_{cb}$.

2) If $\varphi \in P(G, \omega)$, *so we may assume that* $\eta_1 = \eta_2 =: \eta$, *then* $\tilde{M}_\varphi$ *and* M_φ *are completely positive and*

$$\|\tilde{M}_\varphi\| = \|M_\varphi\| = \varphi(e) = \|\eta\|^2 \ .$$

Proof. Consider the amplification map $A : vN(G, \sigma\omega) \to B(\ell^2(G) \otimes \mathcal{H})$ given by $x \mapsto x \otimes I_{\mathcal{H}}$. Define $\pi : vN(G, \sigma\omega) \to B(\ell^2(G) \otimes \mathcal{H})$ by $\pi = \mathrm{Ad}(\mathcal{V}) \circ A$, that is,

$$\pi(x) = \mathcal{V}(x \otimes I_{\mathcal{H}})\mathcal{V}^*, \quad x \in vN(G, \sigma\omega),$$

where $\mathcal{V}$ is the unitary associated to V as in Fell's property. For $j = 1, 2$, let $T_j : \ell^2(G) \to \ell^2(G) \otimes \mathcal{H}$ be the linear operator given by $T_j(\xi) = \xi \otimes \eta_j$. Then each T_j is bounded with $T_j^*(\xi' \otimes \eta') = (\eta', \eta_j)\xi'$.

Now, define a normal linear map $M : vN(G, \sigma\omega) \to B(\ell^2(G) \otimes \mathcal{H})$ by

$$M(\cdot) = T_2^* \pi(\cdot) T_1.$$

Then M is clearly completely bounded with $\|M\| \le \|M\|_{cb} \le \|\eta_1\|\|\eta_2\|$ (see [75, p. 29]; $T_2^*\pi(\cdot)T_1$ is the "Paulsen-Kirchberg" decomposition of M [75, Theorem 8.4]). We also have $\|\varphi\| \le \|\eta_1\|\|\eta_2\|$. Now, to see that $\|M\|_{cb} \le \|\varphi\|$, it suffices to recall from Section 2 that V, η_1 and η_2 may be chosen so that $\|\varphi\| = \|\eta_1\|\|\eta_2\|$.

Furthermore, using Fell's property, we get

$$\begin{aligned} M(\Lambda_{\sigma\omega}(g))\xi &= T_2^*(\mathcal{V}(\Lambda_{\sigma\omega}(g) \otimes I_{\mathcal{H}})\mathcal{V}^*)T_1\xi \\ &= T_2^*(\Lambda_\sigma(g) \otimes V(g))T_1\xi \\ &= T_2^*(\Lambda_\sigma(g)\xi \otimes V(g)\eta_1) \\ &= (V(g)\eta_1, \eta_2)\Lambda_\sigma(g)\xi \\ &= \varphi(g)\Lambda_\sigma(g)\xi, \quad g \in G, \xi \in \ell^2(G) \ . \end{aligned}$$

So equation (2) holds and $M(vN(G, \sigma\omega)) \subseteq vN(G, \sigma)$ since M is normal. The final assertion in 1) follows easily.

If $\eta_1 = \eta_2 = \eta$, then $T_1^*\pi(\cdot)T_1$ is the Stinespring decomposition of the completely positive map $\tilde{M}_\varphi$ we then get. Further, in this case, we obtain

$$\|\eta\|^2 \geq \|\tilde{M}_\varphi\| = \|M_\varphi\| \geq \|M_\varphi(I)\| = \|T_1^*T_1\| = \|\eta\|^2 = \varphi(e) \ .$$

□

Remark 4.2. *One can go even further and obtain the twisted analogue of [17, Theorem 2.2]. Indeed, consider again $\omega \in Z^2(G, \mathbb{T})$ and assume now that V is an ω-projective uniformly bounded representation in $GL(\mathcal{H})$, the group of invertible operators on some Hilbert space $\mathcal{H}$. Set $\|V\| := \sup_{g\in G} \|V(g)\|$. Let $\eta_1, \eta_2 \in \mathcal{H}$ and define $\varphi(g) := (V(g)\eta_1, \eta_2), g \in G$. Then one can still conclude that there exists a completely bounded normal map $\tilde{M}_\varphi : vN(G, \sigma\omega) \to vN(G, \sigma)$ such that*

$$\tilde{M}_\varphi(\Lambda_{\sigma\omega}(g)) = \varphi(g)\Lambda_\sigma(g), \quad g \in G,$$

and satisfying

$$\|\tilde{M}_\varphi\| \leq \|\tilde{M}_\varphi\|_{cb} \leq \|V\|^2 \|\eta_1\| \|\eta_2\|.$$

The proof goes essentially as the proof of 1) above, using now the generalized version of Fell's property, saying that

$$\mathcal{V}^{-1}(\Lambda_\sigma(g) \otimes V(g))\mathcal{V} = \Lambda_{\sigma\omega}(g) \otimes I_{\mathcal{H}}$$

for all $g \in G$, where $\mathcal{V}$ is the invertible operator on $\ell^2(G, \mathcal{H})$ defined by the same formula as for a unitary V.

As we have for the moment no examples in mind of twisted projective nonunitary representations, except those which are similar to the projective unitary ones, we leave to the reader to check the details. We just add that one can also get the analogue of [17, Theorem 2.3], and further prove that if G is amenable, then any uniformly bounded ω-projective representation of G is similar to a ω-projective unitary representation, using for example the fact that $C^*(G, \omega)$ is then nuclear (see Proposition 6.11).

As a consequence of Lemma 4.1, we present an elementary proof of a well known result of Zeller-Meier [87, Théorème 5.1] for twisted group C^*-algebras. (His result is much more general as it deals with twisted crossed products. For the extension of his result to the case of locally compact groups, see [70]).

Theorem 4.3. *[87] Suppose that G is amenable and let $\omega \in Z^2(G,\mathbb{T})$. Then the full twisted group C^*-algebra $C^*(G,\omega)$ is canonically isomorphic to the reduced twisted group C^*-algebra $C_r^*(G,\omega)$.*

Proof. Clearly, we have to show that $\|f\|_{\max} \leq \|\pi_\omega(f)\|$ for every $f \in \ell^1(G)$.

Letting V be a ω-projective unitary representation of G, one has to show that

$$\|\pi_V(f)\| \leq \|\pi_\omega(f)\|, \quad f \in \ell^1(G)\ . \tag{3}$$

Without loss of generality, by exploiting the C^*-property of these norms we may here assume that $f = f^*$ and so $\pi_V(f)$ is self-adjoint.

Then

$$\|\pi_V(f)\| = \sup\{|\omega_\eta(\pi_V(f))|\ ;\ \eta \in \mathcal{H}_V, \|\eta\| = 1\}$$

and it suffices show that $|\omega_\eta(\pi_V(f))| \leq \|\pi_\omega(f)\|$ for all unit vectors $\eta \in \mathcal{H}_V$.

Let η be such a vector and set $\varphi(g) := (V(g)\eta,\eta), g \in G$. Then we have

$$\omega_\eta(\pi_V(f)) = \sum_{g\in G} f(g)(V(g)\eta,\eta) = \sum_{g\in G}(f\varphi)(g)\,.$$

Now, as G is assumed to be amenable, and $f\varphi$ is in $\ell^1(G)$, we get

$$|\omega_\eta(\pi_V(f))| = \Big|\sum_{g\in G}(f\varphi)(g)\Big| \leq \|\pi_1(f\varphi)\|\ .$$

On the other hand,

$$\begin{aligned}\pi_1(f\varphi) &= \sum_{g\in G}(f\varphi)(g)\lambda(g) = \sum_{g\in G} f(g)\varphi(g)\lambda(g)\\ &= M_\varphi(\sum_{g\in G} f(g)\Lambda_\omega(g)) = M_\varphi(\pi_\omega(f))\ ,\end{aligned}$$

where $M_\varphi : C_r^*(G,\omega) \to C_r^*(G)$ is the completely positive map obtained from Lemma 4.1 with $\sigma = 1$ (and $\eta_1 = \eta_2 = \eta$).

Hence, we get

$$|\omega_\eta(\pi_V(f))| \leq \|\pi_1(f\varphi)\| = \|M_\varphi(\pi_\omega(f))\| \leq \|\pi_\omega(f)\|$$

as $\|M_\varphi\| = \varphi(e) = 1$ and the proof is complete. □

An interesting question is whether the converse holds true. Of course, this is the case if ω is a coboundary, but puzzingly, this seems open in general.

For a twisted C*-algebraic characterization of the amenability of G, see Proposition 6.11.

5 Twisted multipliers

In [45, Def.1.6] Haagerup introduces the concept of a function which *multiplies* $C_r^*(G)$ into itself. We propose a twisted analogue as follows.

Definition 5.1. *Let φ be a complex function on G and $\sigma, \omega \in Z^2(G, \mathbb{T})$. Consider the linear map $M_\varphi : \mathbb{C}(G, \omega) \to \mathbb{C}(G, \sigma)$ given by*

$$M_\varphi(\pi_\omega(f)) = \pi_\sigma(\varphi f), \;\; f \in \mathcal{K}(G).$$

*We say that φ is a (σ, ω)-*multiplier *if M_φ is bounded w.r.t. the operator norms on $\mathbb{C}(G, \omega)$ and $\mathbb{C}(G, \sigma)$, in which case we also denote by M_φ the (unique) extension of M_φ to an element in $B(C_r^*(G, \omega), C_r^*(G, \sigma))$. Note that M_φ is then the unique element in this space satisfying*

$$M_\varphi(\Lambda_\omega(g)) = \varphi(g)\Lambda_\sigma(g), \quad g \in G.$$

We denote by $MA(G, \sigma, \omega)$ the set of all (σ, ω)-multipliers on G. Clearly $MA(G, \sigma, \omega)$ is a subspace of $\ell^\infty(G)$. We set $MA(G, \sigma) := MA(G, \sigma, \sigma)$ and $MA(G) := MA(G, 1)$.

Adapting the arguments of Haagerup-de Cannière given in the proof of [17, Proposition 1.2], one can show the following result.

Proposition 5.2. *Let φ be a complex function on G and $\sigma, \omega \in Z^2(G, \mathbb{T})$. Then the following four conditions are equivalent:*

1) $\varphi\psi \in A(G, \omega)$ for all $\psi \in A(G, \sigma)$.

2) There exists a (unique) normal operator $\tilde{M}_\varphi$ from $vN(G, \omega)$ to $vN(G, \sigma)$ such that

$$\tilde{M}_\varphi(\Lambda_\omega(g)) = \varphi(g)\Lambda_\sigma(g), \quad g \in G.$$

3) $\varphi \in MA(G, \sigma, \omega)$.

4) $\varphi\psi \in B_r(G, \omega)$ for all $\psi \in B_r(G, \sigma)$.

Further, let then m_φ (resp. $\overline{m}_\varphi$) denote the bounded linear map from $A(G, \sigma)$ (resp. $B_r(G, \sigma)$) to $A(G, \omega)$ (resp. $B_r(G, \omega)$) we can associate to an element $\varphi \in MA(G, \sigma, \omega)$. Then $\tilde{M}_\varphi$ is the transpose of m_φ, we have

$$\|M_\varphi\| = \|\tilde{M}_\varphi\| = \|m_\varphi\| = \|\overline{m}_\varphi\|$$

and $MA(G, \sigma, \omega)$ becomes a Banach space under the norm $|||\varphi||| := \|M_\varphi\|$.

Still following Haagerup-de Cannière [17], one may also introduce the twisted analogue of their concept of *completely bounded* multipliers :

$$M_0A(G,\sigma,\omega) := \{\varphi \in MA(G,\sigma,\omega) \mid M_\varphi \text{ is a completely bounded map}\}$$

and equip this space with the norm $\|\varphi\|_{cb} = \|M_\varphi\|_{cb}$.

The existence of (completely bounded) twisted multipliers is guaranteed by the following proposition.

Proposition 5.3. *Let $\sigma, \omega \in Z^2(G,\mathbb{T})$. Then we have*

$$B(G,\omega) \subseteq M_0A(G,\sigma,\sigma\omega) \subseteq MA(G,\sigma,\sigma\omega)$$

and $|||\varphi||| \leq \|\varphi\|_{cb} \leq \|\varphi\|$ for all $\varphi \in B(G,\omega)$. If $\varphi \in P(G,\omega)$ then $|||\varphi||| = \|\varphi\| = \varphi(e)$.

Proof. This follows immediately from Lemma 4.1. □

As a consequence of this result, we have

$$B(G,\overline{\sigma}\omega) \subseteq M_0A(G,\sigma,\omega) \subseteq MA(G,\sigma,\omega).$$

Further, using Proposition 5.2, we can deduce that

$$A(G,\sigma)B(G,\omega) \subseteq A(G,\sigma\omega) \quad \text{and} \quad B_r(G,\sigma)B(G,\omega) \subseteq B_r(G,\sigma\omega)$$

which are the twisted analogues of the facts [38] that $A(G)$ and $B_r(G)$ are ideals in the Fourier-Stieltjes algebra $B(G)$.

Choosing $\omega = 1$ in Proposition 5.3, we get

Corollary 5.4. *Let $\sigma \in Z^2(G,\mathbb{T})$. Then we have*

$$B(G) \subseteq M_0A(G,\sigma) \subseteq MA(G,\sigma)$$

and $|||\varphi||| \leq \|\varphi\|_{cb} \leq \|\varphi\|$ for all $\varphi \in B(G)$. If $\varphi \in P(G)$, then $|||\varphi||| = \|\varphi\| = \varphi(e)$.

Remark 5.5. *A result of C. Nebbia says that [66] that G is amenable if and only if $B(G) = MA(G)$, in which case one has $\|\varphi\| = |||\varphi|||$ for all $\varphi \in B(G)$. One may wonder whether a twisted version of this result exists. At first, if G is amenable, it seems reasonable to ask whether we have $B(G) = MA(G,\sigma)$,* *or, more generally, whether we then have* $B(G,\overline{\sigma}\omega) = MA(G,\sigma,\omega)$ *for any* σ,ω*.*

Here is a proof of G *amenable* $\Rightarrow B(G,\omega) = MA(G,1,\omega)$*, in which case we have* $\|\varphi\| = |||\varphi|||$ *for all* $\varphi \in B(G,\omega)$ *:*

Let $\varphi \in MA(G,1,\omega)$*. Then for any* $f \in \ell^1(G)$*, we have*

$$\Big|\sum_{g\in G} f(g)\varphi(g)\Big| \le \|\pi_1(\varphi f)\| = \|M_\varphi(\pi_\omega(f))\| \le \|M_\varphi\|\,\|\pi_\omega(f)\|$$

(using amenability to get the first inequality above).

This clearly implies that $\varphi \in B(G,\omega)$ *with* $\|\varphi\| \le \|M_\varphi\| = |||\varphi|||$*. The converse inclusion and the converse inequality follow from Proposition 5.3.*

Remark 5.6. *In an analogous way, one may consider multipliers on full twisted group* C^**-algebras. In the untwisted case, G. Pisier [78, Corollary 8.7] shows that all multipliers from* $C^*(G)$ *into itself are completely bounded, and that the set of such multipliers coincides with* $B(G)$*. In the twisted case, it is not immediate that the same description holds. One can show without difficulty that any element of* $B(G,\omega)$ *induces a multiplier from* $C^*(G,\sigma\omega)$ *into* $C^*(G,\sigma)$*, which is completely bounded. However, it is not clear to us how to proceed to deduce that any multiplier, or at least that any completely bounded multiplier, is given in this way. The problem lies in that there is no substitute for the trivial representation of a group in the twisted setting.*

Most of what we have seen so far in this section and the previous one can undoubtedly be formulated and proved in the setting of locally compact groups. We will now make use of the fact that we are dealing with discrete groups to exhibit another important class of multipliers.

Proposition 5.7. *Let* $\sigma,\omega \in Z^2(G,\mathbb{T}), \varphi \in \ell^2(G)$*. Then* $\varphi \in MA(G,\sigma,\omega)$ *with* $|||\varphi||| \le \|\varphi\|_2$ *. Moreover, for each* $x \in C_r^*(G,\omega)$*, we have*

$$M_\varphi(x) = \sum_{g\in G} \varphi(g)\hat{x}(g)\Lambda_\sigma(g)$$

(the sum being absolutely convergent in operator norm).

Proof. For any $f \in \mathcal{K}(G)$, using the Cauchy-Schwarz inequality, we get

$$\|\varphi f\|_1 = \sum_{g\in G} |\varphi(g)||f(g)| \ \le\ \|\varphi\|_2\|f\|_2 \le \|\varphi\|_2\|\pi_\omega(f)\|,$$

hence

$$\|M_\varphi(\pi_\omega(f))\| = \|\pi_\sigma(\varphi f)\| \leq \|\varphi f\|_1 \leq \|\varphi\|_2\|\pi_\omega(f)\|.$$

It follows that φ is a (σ, ω)-multiplier with $|||\varphi||| \leq \|\varphi\|_2$.

Let now $x \in C_r^*(G, \omega)$. As $\hat{x} \in \ell^2(G)$, we have then $\varphi\hat{x} \in \ell^1(G)$ with $\|\varphi\hat{x}\|_1 \leq \|\varphi\|_2 \, \|\hat{x}\|_2$. Hence the series $\sum_{g\in G} \varphi(g)\hat{x}(g)\Lambda_\sigma(g)$ is absolutely operator norm convergent in $C_r^*(G, \sigma)$.

Denoting its sum by $F_\varphi(x)$, we have

$$\|F_\varphi(x)\| \leq \|\varphi\hat{x}\|_1 \leq \|\varphi\|_2 \, \|\hat{x}\|_2 \leq \|\varphi\|_2 \, \|x\| \ .$$

Hence the map F_φ is in $B(C_r^*(G, \omega), C_r^*(G, \sigma))$. Further, by definition, we have $F_\varphi(\Lambda_\omega(g)) = \varphi(g)\Lambda_\sigma(g), g \in G$. But this implies that $F_\varphi = M_\varphi$ and the final assertion clearly follows.

□

Remark 5.8. *Let $\sigma, \omega \in Z^2(G, \mathbb{T})$. By a similar argument to the one given in the proof above, one gets that if $\varphi \in \ell^2(G)$ then $\varphi \in B_r(G, \omega) \subseteq B(G, \omega)$, with $\|\varphi\| \leq \|\varphi\|_2$.*

Indeed, for any $f \in \mathcal{K}(G)$, we have

$$|\sum_{g\in G} \varphi(g)f(g)| \leq \|\varphi f\|_1 \leq \|\varphi\|_2\|f\|_2 \leq \|\varphi\|_2\|\pi_\omega(f)\|.$$

Hence the linear map $\pi_\omega(f) \to \sum_{g\in G} \varphi(g)f(g)$ extends to a bounded linear functional on $C_r^*(G, \omega)$. Identified as an element $\varphi \in B_r(G, \omega) \subseteq B(G, \omega)$, it clearly satisfies $\|\varphi\| \leq \|\varphi\|_2$.

From Proposition 5.3 it now follows that

$$\ell^2(G) \subseteq M_0A(G, \sigma, \sigma\omega) \subseteq MA(G, \sigma, \sigma\omega)$$

with $|||\varphi||| \leq \|\varphi\|_{cb} \leq \|\varphi\|_2$ for all $\varphi \in \ell^2(G)$. Since this holds for arbitrary σ and ω, we deduce that

$$\ell^2(G) \subseteq M_0A(G, \sigma, \omega) \subseteq MA(G, \sigma, \omega).$$

Proposition 5.7 (with $\sigma = \omega$) will be important for our considerations in the next section. To prepare for these, the following discussion will be instructive.

Let $\varphi \in MA(G,\sigma)$ and $x \in C_r^*(G,\sigma)$. Then $\widehat{M_\varphi(x)} = \varphi\hat{x}$. (If $x \in \mathbb{C}(G,\sigma)$, this is trivial; otherwise the statement follows easily by a density argument). Hence the Fourier series of $M_\varphi(x)$ is $\sum_{g\in G}\varphi(g)\hat{x}(g)\Lambda_\sigma(g)$. Of course, this series converges to $M_\varphi(x)$ in $\|\cdot\|_2$-norm, but it does not necessarily converge in operator norm. However, as pointed out in Proposition 5.7, it *does* if one knows that $\varphi \in \ell^2(G)$.

The following definition seems therefore appropriate. We let $MCF(G,\sigma)$ denote the set of all $\varphi : G \to \mathbb{C}$ such that the series $\sum_{g\in G}\varphi(g)\hat{x}(g)\Lambda_\sigma(g)$ converges in operator norm for all $x \in C_r^*(G,\sigma)$. Then we have

Proposition 5.9. *$MCF(G,\sigma) \subseteq MA(G,\sigma)$. Moreover,*

$$MCF(G,\sigma) = \{\varphi \in MA(G,\sigma) \,|\, M_\varphi \text{ maps } C_r^*(G,\sigma) \text{ into } CF(G,\sigma)\}$$

and

$$M_\varphi(x) = \sum_{g\in G}\varphi(g)\hat{x}(g)\Lambda_\sigma(g) \ \text{(convergence in operator norm)}$$

for all $\varphi \in MCF(G,\sigma)$ and all $x \in C_r^*(G,\sigma)$.

Proof. Let $\varphi \in MCF(G,\sigma)$. Define a linear map $F_\varphi : C_r^*(G,\sigma) \to C_r^*(G,\sigma)$ by

$$F_\varphi(x) = \sum_{g\in G}\varphi(g)\hat{x}(g)\Lambda_\sigma(g).$$

Using the closed graph theorem, one gets that F_φ is bounded. (Indeed, assume $x_n \to x$ and $F_\varphi(x_n) \to y$ in $C_r^*(G,\sigma)$. Then $\widehat{F_\varphi(x_n)} = \varphi\hat{x_n} \to \varphi\hat{x} = \widehat{F_\varphi(x)}$ pointwise on G and also $\widehat{F_\varphi(x_n)} \to \hat{y}$ pointwise on G. Hence, $\widehat{F_\varphi(x)} = \hat{y}$, so $F_\varphi(x) = y$, as desired).

As $F_\varphi(\Lambda_\sigma(g)) = \varphi(g)\Lambda_\sigma(g), g \in G$, this implies that F_φ is a bounded extension of M_φ from $\mathbb{C}(G,\sigma)$ to $C_r^*(G,\sigma)$. Hence $\varphi \in MA(G,\sigma)$. The last assertions follows then from the fact that we have $\widehat{M_\varphi(x)} = \varphi\hat{x}$ for all $x \in C_r^*(G,\sigma)$. □

Proposition 5.7 shows that $\ell^2(G) \subseteq MCF(G,\sigma)$. Inspired by [45, Lemma 1.7], we can produce other examples of multipliers in $MCF(G,\sigma)$ related to the decay subspaces introduced in Section 3.

Proposition 5.10. *Assume that $\kappa : G \to [1,\infty)$ is such that (G,σ) is κ-decaying, with decay constant C. Let $\psi \in \mathcal{L}_\kappa^\infty$ and set $K := \|\psi\kappa\|_\infty$.*
Then $\psi \in MCF(G,\sigma)$ with $|||\psi||| \leq CK$.

Proof. Let $f \in \mathcal{K}(G)$. Then

$$\|\psi f\|_{2,\kappa} = \|\psi f \kappa\|_2 \leq K\|f\|_2 \leq K\|\pi_\sigma(f)\|.$$

Hence

$$\|\pi_\sigma(\psi f)\| \leq C\|\psi f\|_{2,\kappa} \leq CK\|\pi_\sigma(f)\|.$$

Thus the linear map $M_\psi : \mathbb{C}(G,\sigma) \to \mathbb{C}(G,\sigma)$ defined by $M_\psi(\pi_\sigma(f)) = \pi_\sigma(\psi f)$ is bounded with $\|M_\psi\| \leq CK$. It follows that $\psi \in MA(G,\sigma)$ and it remains only to show that $\psi \in MCF(G,\sigma)$. Let $x \in C_r^*(G,\sigma)$. Then we have $\|\psi\hat{x}\|_{2,\kappa} \leq K\|\hat{x}\|_2 < \infty$. Hence $\widehat{M_\psi(x)} = \psi\hat{x} \in \mathcal{L}^2_\kappa$. Applying Theorem 3.4, we get that $\sum_{g\in G} \varphi(g)\hat{x}(g)\Lambda_\sigma(g)$ converges in operator norm, as desired. □

Remark 5.11. *In the setting of Proposition 5.10 (or if $\psi \in \ell^2(G)$) one can deduce that ψ satisfies a slightly stronger property than just being an element of $MCF(G,\sigma)$, namely that $\sum_{g\in G} \psi(g)\hat{x}(g)\Lambda_\sigma(g)$ is operator norm convergent for all $x \in vN(G,\sigma)$. If ψ is p.d. and satisfies this property, one may wonder whether it also has the strong Feller property introduced by J.L. Sauvageot [79, 80], that is, whether one has $M_\psi^{**}(C_r^*(G,\sigma)^{**}) \subseteq C_r^*(G,\sigma)$. We notice that in the case where ψ is p.d. and satisfies the assumptions of Proposition 5.10 (or if $\psi \in \ell^2(G)$), one readily sees that there exists a constant $C' > 0$ such that $\|\pi_\sigma(\psi f)\| \leq C'\|f\|_2$ for all $f \in \mathcal{K}(G)$, and it then follows that ψ has the strong Feller property (cf. [80, Lemma 3.3 and Proposition 5.2]).*

6 Summation properties

Throughout this section we assume that $\sigma \in Z^2(G,\mathbb{T})$. We begin with some definitions.

Definition 6.1. *A net $\{\varphi_\alpha\}$ in $MA(G,\sigma)$ will be called an* approximate multiplier unit *whenever M_{φ_α} converges to the identity map in the strong operator topology (SOT) on $B(C_r^*(G,\sigma))$.*

Definition 6.2. *Let X be a Banach space. A net $\{M_\alpha\}$ in $B(X)$ will be called* bounded *if it is uniformly bounded (that is, $\sup_\alpha \|M_\alpha\| < \infty$) or, equivalently, due to the uniform boundedness principle, if $\sup_\alpha \|M_\alpha(x)\| < \infty$ for all $x \in X$.*

A net $\{\varphi_\alpha\}$ in $MA(G,\sigma)$ will be called bounded *if $\{M_{\varphi_\alpha}\}$ is bounded in $B(C_r^*(G,\sigma))$.*

Remark 6.3. *We record the following simple but useful facts :*

1) *Assume that* $\{\varphi_\alpha\}$ *is an approximate multiplier unit in* $MA(G,\sigma)$. *Then* $\varphi_\alpha \to 1$ *pointwise on* G *and we have* $\sup_\alpha \|M_{\varphi_\alpha}\| \geq 1$. *If* $\{\varphi_\alpha\}$ *is a sequence, then* $\{\varphi_\alpha\}$ *is bounded (as follows from the uniform boundedness principle).*

2) *Let* $\{\varphi_\alpha\}$ *be a net in* $MA(G,\sigma)$. *Using a straightforward* $\varepsilon/3-$*argument, one deduces that* $\{\varphi_\alpha\}$ *is a bounded approximate multiplier unit if and only if* $\varphi_\alpha \to 1$ *pointwise on* G *and* $\{\varphi_\alpha\}$ *is bounded.*

   If the net is a sequence, then it is a bounded approximate multiplier unit if and only if it is an approximate multiplier unit.

3) *If* G *is countable (so* $C_r^*(G,\sigma)$ *is separable), then (mimicking the trick used to produce a countable approximate unit in a separable* C^**-algebra) one can always extract a sequence from a given bounded approximate multiplier unit to produce a (bounded) countable approximate multiplier unit if necessary.*

Example 6.4. *Assume that there exists a net* $\{\varphi_\alpha\}$ *of normalized positive definite functions on* G *which converges pointwise to* 1 *(as it does when* G *has the Haagerup property). Then we deduce from Lemma 4.1 that* $\|M_{\varphi_\alpha}\| = 1$ *for all* α. *Hence* $\{\varphi_\alpha\}$ *is bounded, and 2) above gives that it is an approximate multiplier unit for* $C_r(G,\sigma)$.

In view of our considerations at the end of the previous section, the following definition is natural.

Definition 6.5. *Let* $\{\varphi_\alpha\}$ *be a net of complex functions on* G. *We say that* $C_r^*(G,\sigma)$ *has the* Summation Property (S.P.) w.r.t. $\{\varphi_\alpha\}$, *or, equivalently, that* $\{\varphi_\alpha\}$ *is a* Fourier summing net for (G,σ), *if* $\{\varphi_\alpha\}$ *is an approximate multiplier unit such that* $\varphi_\alpha \in MCF(G,\sigma)$ *for all* α.

In this case, the series $\sum_{g\in G} \varphi_\alpha(g)\hat{x}(g)\Lambda_\sigma(g)$ *is convergent in operator norm for all* α, *and we have*

$$\sum_{g\in G} \varphi_\alpha(g)\hat{x}(g)\Lambda_\sigma(g) \underset{\alpha}{\to} x$$

for all $x \in C_r^*(G,\sigma)$ *(convergence in operator norm).*

Example 6.6. *Let us revisit the case $G = \mathbb{Z}$, $\sigma = 1$ already dealt with in the Introduction. For each $n \in \mathbb{N}$, let $\varphi_n \in \mathcal{K}(\mathbb{Z})$ be defined as*

$$\varphi_n(k) = \begin{cases} 1 - \frac{|k|}{n}, & |k| \leq n-1 \\ 0, & \textit{otherwise} \end{cases}.$$

Then the Fejér summation theorem can be restated as the fact that $C_r^*(\mathbb{Z}, 1)$ has the S.P. w.r.t. $\{\varphi_n\}$.

Moreover, for each $0 < r < 1$, let $\psi_r(k) = r^{|k|}$, $k \in \mathbb{Z}$. Then the Abel-Poisson summation theorem corresponds to the fact that $C_r^*(\mathbb{Z}, 1)$ has the S.P. w.r.t. $\{\psi_r\}_{0<r<1}$ (letting $r \to 1$).

To produce Fourier summing nets, the following remark will be useful:

Remark 6.7. *Let $\{\varphi_\alpha\}$ be a net of complex functions on G. Assume that it satisfies at least one of the following two conditions :*

i) $\{\varphi_\alpha\} \subseteq \ell^2(G)$

ii) for each α there exists some $\kappa_\alpha : G \to [1, \infty)$ such that (G, σ) is κ_α-decaying and $\{\varphi_\alpha\} \in \mathcal{L}^\infty_{\kappa_\alpha}$.

As follows from Section 5 (Proposition 5.7 and Proposition 5.10), either of these conditions ensures that $\{\varphi_\alpha\} \subseteq MCF(G, \sigma)$.

Assume further that $\{\varphi_\alpha\}$ also satisfies the following two conditions :

iii) $\varphi_\alpha \to 1$ *pointwise on* G

iv) $\{\varphi_\alpha\}$ *is bounded*

Then, as a consequence of Remark 6.3, we get that $C_r^*(G, \sigma)$ has the S.P. w.r.t. $\{\varphi_\alpha\}$ (and $\sup_\alpha \|M_{\varphi_\alpha}\| \geq 1$).

It is an open question whether one can always find a Fourier summing net for a general pair (G, σ). When G is finitely generated, there seems to be a good candidate at hand : letting L denote some algebraic length function on G, consider the "Gaussian" net given by $\varphi_t = e^{-tL^2}, t > 0$. Then i) (cf. Section 2) and obviously iii) in the above Remark are satisfied. But it is not clear whether $\{\varphi_t\}$ is always bounded or not. It will be whenever L can be chosen so that L^2 is negative definite (cf. Example 6.4), but this in turn will force G to be amenable (cf. our last comment in Section 2).

One may also wonder about when a Fourier summing net can be picked in $\ell^2(G)$ (like in the Abel-Poisson case), or even in $\mathcal{K}(G)$ (like in the Fejér case). This suggests the following terminology.

Definition 6.8. *We say that (G, σ) has the Fejér property (resp. the Abel-Poisson property) if there exists a net $\{\varphi_\alpha\}$ in $\mathcal{K}(G)$ (resp. in $\ell^2(G)$) such that $C_r^*(G, \sigma)$ has the S.P. w.r.t. $\{\varphi_\alpha\}$.*

If the net $\{\varphi_\alpha\}$ can be chosen to be bounded, then we say that (G, σ) has the bounded Fejér property (resp. the bounded Abel-Poisson property).

Moreover, if this net can be chosen to satisfy $\sup_\alpha \|M_{\varphi_\alpha}\| = 1$, then we say that (G, σ) has the metric Fejér property (resp. the metric Abel-Poisson property).

Clearly "Fejér" is stronger than "Abel-Poisson" in all cases. Further, when $\sigma = 1$, we just talk about the corresponding property for the group G.

As we have seen, $\mathbb{Z}$ has the Fejér property and we will soon see that it in fact has the metric Fejér property. The Abel-Poisson property is the weakest of the properties introduced in the above definition, while the metric Fejér property is the strongest. All pairs (G, σ) having the Abel-Poisson property we know of turn out to have the metric Fejér property. But we do not know whether such a statement holds in general.

Concerning all the Fejér and Abel-Poisson properties introduced above, one may add the letter "s" to each property whenever the net may be chosen as a sequence in the actual definition. For countable groups, the concepts of s-Fejér (resp. s-Abel-Poisson) property, bounded s-Fejér (resp. s-Abel-Poisson) property and bounded Fejér (resp. Abel-poisson) property do then coincide respectively, cf. Remark 6.3. However we will not dwelve further into this.

To motivate the use of the adjective "metric" in the metric Fejér property, we recall that a Banach space X is said to have the *Metric Approximation Property* (M.A.P.) if there exists a net of finite rank contractions on X approximating the identity map in the SOT on $B(X)$.

If (G, σ) has the metric Fejér property then $C_r^*(G, \sigma)$ has the M.A.P. (since M_φ has finite rank whenever $\varphi \in \mathcal{K}(G)$). In [45], Haagerup showed that $C_r^*(\mathbb{F}_2)$ has the M.A.P. (despite the fact that $C_r^*(\mathbb{F}_2)$ is not nuclear; see below). In our terminology, what he really shows is that $\mathbb{F}_2$ has the metric Fejér property. We will generalize his result in the section 7.

On the other hand, (G, σ) will have the metric Fejér property whenever G is an amenable group. This is essentialy well-known : Zeller-Meier shows in [87] that the bounded Fejér property holds in this case, but it is not difficult to deduce our slightly stronger statement. Before explaining how this works, let us point out that in general, the main difficulty when trying to use Remark 6.7 to produce Fourier summing nets is to check the boundedness condition, as illustrated with the case of Gaussian nets on finitely generated groups. However, thanks to Lemma 4.1, this boundedness condition is fullfied when the assumptions in Example 6.4 are satisfied. Hence we get the following proposition, which will be used repeatedly in the sequel as our main tool to produce Fourier summing nets.

Proposition 6.9. *Let $\{\varphi_\alpha\}$ be a net of normalized positive definite functions on G converging pointwise to* 1. *Assume that it satisfies*

i) $\{\varphi_\alpha\} \subseteq \ell^2(G)$,

or

ii) for each α there exists some $\kappa_\alpha : G \to [1, \infty)$ such that (G, σ) is κ_α-decaying and $\varphi_\alpha \in \mathcal{L}^\infty_{\kappa_\alpha}$.

Then $C_r^*(G, \sigma)$ has the S.P. w.r.t. $\{\varphi_\alpha\}$ (with $\|M_{\varphi_\alpha}\| = 1$ for all α).

We note that if the net $\{\varphi_\alpha\}$ satisfies condition i) above, then G must be amenable (cf. Section 2).

On the other hand, assume that G is amenable, $\sigma = 1$ and $\varphi \in \ell^2(G)$ is normalized. Then we have $\|M_\varphi\| = 1$ (or, equivalently, $\|M_\varphi\| \leq 1$) if and only if φ is positive definite. (Indeed, if $\|M_\varphi\| = 1$ then, using the result of Nebbia mentioned in Remark 5.5 and considering φ as an element of the dual of $C^*(G)$, we get $\|\varphi\| = 1 = \varphi(I)$, hence φ is a state on $C^*(G)$, which means that φ is positive definite on G. The converse is clear). Hence, in this case, the existence of a Fourier summing net $\{\varphi_\alpha\}$ for G of normalized elements in $\ell^2(G)$ satisfying $\|M_{\varphi_\alpha}\| \leq 1$ can only be achieved if all φ_α's are assumed to be positive definite.

As an immediate corollary of Proposition 6.9, we get :

Corollary 6.10. *(Zeller-Meier) Let G be amenable. Then (G, σ) has the metric Fejér property.*

Indeed, letting $\{\varphi_\alpha\}$ be any net of normalized positive definite functions in $\mathcal{K}(G)$ converging to 1 pointwise on G, $C_r^*(G, \sigma)$ has the S.P. w.r.t. $\{\varphi_\alpha\}$ and $\|M_{\varphi_\alpha}\| = 1$ for all α.

Any net $\{\varphi_\alpha\}$ as in Corollary 6.10 gives a net $\{M_{\varphi_\alpha}\}$ of finite rank completely positive maps on $C_r^*(G, \sigma)$ converging to the identity in the SOT. Hence we recover the known fact that if G is amenable, then $C_r^*(G, \sigma)$ has the so-called C.P.A.P., a property which is known to be equivalent to nuclearity ([60, 26, 78]). Actually, we have :

Proposition 6.11. *The following four statements are equivalent :*

1) G is amenable.

2) $C^*(G, \sigma)$ is nuclear.

3) $C_r^*(G, \sigma)$ is nuclear.

4) $vN(G, \sigma)$ is injective.

Proof. This result is well known in the untwisted case, see [74] and references therein. The twisted case can also be deduced from the existing literature. Indeed, 1) ⇒ 2) follows from [70, Corollary 3.9] (or from what we just pointed out above, taking Corollary 6.10 into account). The implication 2) ⇒ 3) is a consequence of the fact that quotients of nuclear C*-algebras are nuclear. As a nuclear C*-algebra generates an injective von Neumann algebra in any of its representations, 3) ⇒ 4) is true. Finally, if 4) holds, then $vN(G, \sigma)$ has a hypertrace and G is then amenable, see [6, Corollary 1.7] and its proof. □

Example 6.12. *(About Følner and Fejér)*

Assume that G is amenable and pick a a Følner net $\{F_\alpha\}$ for G. Set

$$\varphi_\alpha(g) = \frac{|gF_\alpha \cap F_\alpha|}{|F_\alpha|}, \quad g \in G.$$

When $G = \mathbb{Z}$, one may choose $F_n = \{0, 1, \ldots, n-1\}$, which gives

$$\varphi_n(g) = \begin{cases} 1 - \frac{|g|}{n}, & |g| \leq n-1 \\ 0, & \text{otherwise} \end{cases}$$

that is, we get the Fejér functions on $\mathbb{Z}$ *used in the classical Fejér summation theorem.*

As pointed out in Section 2, $\{\varphi_\alpha\}$ *satisfies all the conditions of Proposition 6.10, and we have* $\mathrm{supp}(\varphi_\alpha) = F_\alpha \cdot F_\alpha^{-1}$. *Hence the following analogue of Fejér's summation theorem holds : for all* $x \in C_r^*(G,\sigma)$ *we have*

$$\sum_{g \in F_\alpha \cdot F_\alpha^{-1}} \frac{|gF_\alpha \cap F_\alpha|}{|F_\alpha|} \hat{x}(g)\Lambda_\sigma(g) \underset{\alpha}{\to} x$$

(in operator norm).

Example 6.13. *Let* E *be a finite subset of* G. *An interesting problem is to compute, or at least to obtain an upper bound, for the operator norm of* $M_{\chi_E} \in B(C_r^*(G,\sigma))$. *If* E *is a finite* subgroup *of* G, *then* χ_E *is a normalized positive definite function on* G, *hence we have* $\|M_{\chi_E}\| = 1$.

Moreover, if one writes $G = \cup_\alpha E_\alpha$ *for an increasing net* $\{E_\alpha\}$ *of finite subsets of* G, *it would also be interesting to know conditions ensuring that* $\sup_\alpha \|M_{\chi_{E_\alpha}}\|$ *is finite (resp. infinite). A simple instance of the infinite case is when* $G = \mathbb{Z}$ *and* $E_n = \{-n, \ldots, -1, 0, 1, \ldots, n\}, n \in \mathbb{N}$, *as* χ_{E_n} *is then equal to the "Dirichlet" function* $\{d_n\}$ *mentioned in the Introduction. In the special case where all* E_α*'s may be chosen as finite subgroups of* G *(that is, if* G *is locally finite, hence especially amenable), then the net* $\{\varphi_\alpha\} := \{\chi_{E_\alpha}\}$ *satisfies all the conditions of Proposition 6.10. Hence, in this particular case, we have*

$$\sum_{g \in E_\alpha} \hat{x}(g)\Lambda_\sigma(g) \underset{\alpha}{\to} x$$

(in operator norm) for all $x \in C_r^*(G,\sigma)$.

Example 6.14. *Consider* $G = \mathbb{Z}^N$ $(N \in \mathbb{N})$, *and* $\sigma_\Theta \in Z^2(\mathbb{Z}^N, \mathbb{T})$.
Set $P := (N+1)/2$ *(resp.* $(N+2)/2$*) if* N *is odd (resp. even).*

For $0 < t$, $0 < r < 1$, $j = 1, 2$, *and* $m \in \mathbb{Z}^N$, *set*

$$f_{t,j}(m) := \frac{1}{(1+t|m|_j)^{N+1}}, \qquad s_t(m) := \frac{1}{(1+t|m|_2^2)^P},$$

$$g_t(m) := e^{-t|m|_2^2} \quad \textit{and} \quad p_{r,j}(m) := r^{|m|_j}.$$

Then the noncommutative torus $C_r^*(\mathbb{Z}^N, \sigma_\Theta)$ *has the S.P. w.r.t.* $\{f_{t,j}\}$ *(resp.* $\{s_t\}$*) (resp.* $\{g_t\}$*)(resp.* $\{p_{r,j}\}$*), (where* $j = 1, 2$, $t \to 0^+$, $r \to 1^-$*).*

Indeed, all these nets consist of normalized p.d. square-summable functions on G converging pointwise to 1 (cf. Section 2), hence Proposition 6.9 applies.

In particular, the following analogue of the Abel-Poisson summation theorem holds: for all $x \in C_r^*(\mathbb{Z}^N, \sigma_\Theta)$ *we have*

$$\sum_{m \in \mathbb{Z}^N} r^{|m|_j}\, \hat{x}(m)\Lambda_\sigma(m) \underset{r \to 1^-}{\longrightarrow} x$$

(in operator norm), $j = 1, 2$.

We will now discuss possible analogues of the Abel-Poisson summation theorem for some other groups.

Let $h : G \to \mathbb{R}$ be nonnegative and satisfy $h(e) = 0$ (e.g. h is a length function). One may first try to find conditions ensuring that $r^h \in \ell^2(G)$ for all $r \in (0,1)$. Clearly we must then assume that G is countable. In full generality, this seems difficult. But if we assume for simplicity that h is integer-valued and we set $A_n = \{g \in G \mid h(g) = n\}, n \in \mathbb{N}$, then it is easy to see (using the root test) that a necessary and sufficient condition is $\limsup_n |A_n|^{\frac{1}{n}} \leq 1$. This condition is for instance trivially satisfied whenever G is finitely generated and exponentially bounded, and one chooses $h = L_S$ to be the algebraic length on G with respect to some finite set S of generators for G (cf. Section 2).

Next, assume that we know that $r^h \in \ell^2(G)$ for all $r \in (0,1)$. Obviously, $(r^h)(g) \to 1$ for all $g \in G$ as $r \to 1$. Hence, having in mind to apply Remark 6.7, a natural question is then whether one can find conditions ensuring that $\sup_r \|M_{r^h}\| < \infty$. This seems also to be hard in general. However, if one happens to know that h is negative definite, so that r^h is positive definite for all $r \in (0,1)$, then Proposition 6.9 will apply and give that $\{r^h\}_{0<r<1}$ is then a Fourier summing net for $C_r^*(G, \sigma)$.

Now, one may wonder about when there exists a negative definite function h on G satisfying $h(e) = 0$ and $r^h \in \ell^2(G)$ for all $r \in (0,1)$. Obviously, h has to be proper, hence G must at least be of Haagerup type. Moreover, as follows from Section 2, G must even be amenable. It is conceivable that all finitely generated exponentially bounded groups satisfy the above requirement, but if G is such a group and $h = L_S$ is some algebraic length function on G, it is not clear to us that L_S will necessarily be negative definite for some suitable choice of generator set S.

On the other hand, one may obtain analogues of the Abel-Poisson summation theorem for certain nonamenable groups by exploiting the decay condition in Proposition 6.9 instead of the ℓ^2-condition. The following result concerning the existence of Fourier summing nets for (non necessarily amenable) groups of Haagerup type is now a simple corollary to this proposition, thanks to our efforts in the previous sections.

Theorem 6.15. *Let G be countably infinite with Haagerup property and let h be a Haagerup function on G (especially, h may be a Haagerup length function).*

1) Assume that G has polynomial H-growth (w.r.t. h), or, more generally, that G has polynomial H_σ-growth (w.r.t. h).
Then there exists some $q \in \mathbb{N}$ such that $C_r^*(G, \sigma)$ has the S.P. w.r.t. $\{(1 + th)^{-q}\}_{0<t}$ (as $t \to 0^+$).

2) Assume that G is exponentially H-bounded (w.r.t. h), or, more generally, that G is exponentially H_σ-bounded (w.r.t. h). Then $C_r^*(G, \sigma)$ has the S.P. w.r.t. $\{r^h\}_{0<r<1}$ (as $r \to 1^-$).

3) Finally, set $L = h^{1/2}$ (which is then a Haagerup length function on G) and assume that G is exponentially H_σ-bounded (w.r.t. L). Then $C_r^*(G, \sigma)$ has the S.P. w.r.t. $\{r^L\}_{0<r<1}$ (as $r \to 1^-$) and w.r.t. $\{e^{-tL^2}\}_{t>0}$ (as $t \to 0^+$).

Proof. Set $\psi_{p,t} := (1 + th)^{-p}, p \in \mathbb{N}, t > 0$, and $\varphi_r := r^h, 0 < r < 1$. Then φ_r and $\psi_{p,t}$ are normalized positive definite functions on G (cf. Section 2)

To show 1), assume that G has polynomial H_σ-growth w.r.t. h and pick $s_0 > 0$ such that (G, σ) is $(1 + h)^{s_0}$-decaying (cf. Corollary 3.14, 1)). Set $\kappa = (1 + h)^{s_0}$.

Further, choose $q \in \mathbb{N}$ such that $q \geq s_0$.

Then one easily sees check that $\psi_{q,t} \in \mathcal{L}_\kappa^\infty$ for all $t > 0$. This means that condition ii) in Proposition 6.9 is satisfied (using $\kappa_t = \kappa$ for all $t > 0$). Hence, this proposition applies and assertion 1) follows.

The proof of 2) is similar. Assume that G is exponentially H_σ-bounded (w.r.t. h). Let $0 < r < 1$ and set $\gamma_r := (\frac{1}{r})^h$. Then, according to Corollary 3.14, 2), with $a = 1/r > 1$, (G, σ) is γ_r-decaying. Trivially, we have $\varphi_r \in \mathcal{L}_{\gamma_r}^\infty$. This means that condition ii) in Proposition 6.9 is satisfied (using γ_r as κ_α), and assertion 2) follows.

The proof of 3) is also similar, the first part following directly from 2) (with L instead of h). To prove the last assertion, let $t > 0$ and set $\delta_t = e^{tL}$. Then (G, σ) is then δ_t-decaying and we have $e^{-tL^2}\delta_t = e^{t(L-L^2)} \in c_0(G) \subseteq \ell^\infty(G)$ for all $t > 0$. Hence the conclusion follows again from Proposition 6.9. □

Example 6.16. *1) Let G be a finitely generated free group with generator set S, or let (G, S) be a Coxeter group. Let $L = L_S$ denote the algebraic length on G w.r.t. S. Then L is negative definite (see [25]). Further, G has polynomial H-growth w.r.t. L (see Example 3.13). Hence, both assertions of Theorem 6.15 apply to (G, σ).*

In both cases, G is an example of a group which acts on a "space with walls" (see [25]). For such groups, the natural length function L is known to be negative definite. As alluded to in Section 3, it would be interesting to know whether such a group automatically has polynomial H-growth (w.r.t. L).

2) Theorem 6.15 also applies to finitely generated groups with polynomial growth (resp. which are exponentially bounded) which have a negative definite algebraic length function, e.g. $\mathbb{Z}^N$.

7 A generalized Haagerup theorem

We still let $\sigma \in Z^2(G, \mathbb{T})$ throughout this section. The following result is an elaboration of [45, Theorem 1.8]. It provides a sufficient set of conditions ensuring that (G, σ) has the metric Fejér property.

Theorem 7.1. *Assume that the following three conditions hold:*

(1) There exists an approximate multiplier unit $\{\varphi_\alpha\}$ in $MA(G, \sigma)$ satisfying $\|M_{\varphi_\alpha}\| = 1$ for all α.

(2) For each α there exists a function $\kappa_\alpha : G \to [1, +\infty)$ such that (G, σ) is κ_α-decaying.

(3) We have $\varphi_\alpha \kappa_\alpha \in c_0(G)$ for all α.

Then (G, σ) has the metric Fejér property. (Especially, $C_r^*(G, \sigma)$ has the M.A.P.).

Proof. Clearly we have $\varphi_\alpha \neq 0$ for all α.

Let $\alpha \in \Lambda$, $n \in \mathbb{N}$. Using assumption (3), we can pick a finite subset $A_{\alpha,n}$ of G such that $|\varphi_\alpha \kappa_\alpha| \leq \frac{1}{n}$ outside $A_{\alpha,n}$. If necessary, we enlarge $A_{\alpha,n}$ to include at least one element where φ_α is nonzero.

Define

$$\varphi_{\alpha,n}(g) = \begin{cases} \varphi_\alpha(g), & g \in A_{\alpha,n} \\ 0, & g \notin A_{\alpha,n} \end{cases}.$$

Then

$$(\varphi_\alpha - \varphi_{\alpha,n})(g) = \begin{cases} 0, & g \in A_{\alpha,n} \\ \varphi_\alpha(g), & g \notin A_{\alpha,n} \end{cases}.$$

and so

$$((\varphi_\alpha - \varphi_{\alpha,n})\kappa_\alpha)(g) = \begin{cases} 0, & g \in A_{\alpha,n} \\ (\varphi_\alpha \kappa_\alpha)(g), & g \notin A_{\alpha,n} \end{cases}.$$

Hence

$$\|(\varphi_\alpha - \varphi_{\alpha,n})\kappa_\alpha\|_\infty = \sup\{|(\varphi_\alpha \kappa_\alpha)(g)| \ , \ g \notin A_{\alpha,n}\} \leq \frac{1}{n} \ .$$

Using Proposition 5.10, we get that $(\varphi_\alpha - \varphi_{\alpha,n}) \in MA(G,\sigma)$ and

$$\|M_{(\varphi_\alpha - \varphi_{\alpha,n})}\| \leq \frac{C_\alpha}{n}$$

for all $n \in \mathbb{N}$, where C_α denotes the decay constant of (G,σ) w.r.t. κ_α. Thus $\|M_{\varphi_\alpha} - M_{\varphi_{\alpha,n}}\| = \|M_{\varphi_\alpha - \varphi_{\alpha,n}}\| \leq \frac{C_\alpha}{n} \to 0$ as $n \to +\infty$.

Now, as $\|M_{\varphi_\alpha}\| = 1$, we get $\|M_{\varphi_{\alpha,n}}\| \to 1$ as $n \to +\infty$. So, setting

$$\psi_{\alpha,n} = \frac{1}{\|M_{\varphi_{\alpha,n}}\|}\varphi_{\alpha,n},$$

we have $\|M_{\psi_{\alpha,n}}\| = 1$, and $\|M_{\psi_{\alpha,n}} - M_{\varphi_\alpha}\| \to 0$ as $n \to \infty$.

Further, as $\varphi_\alpha \to 1$ pointwise and $\sup_\alpha \|M_{\varphi_\alpha}\| = 1$, we have $M_{\varphi_\alpha} \to \mathrm{Id}$ in the SOT on $B(C_r^*(G,\sigma))$ (cf. Remark 6.3).

It follows easily that $\mathrm{Id} \in \{M_{\psi_{\alpha,n}} \mid \alpha \in \Lambda, n \in \mathbb{N}\}^{-\mathrm{SOT}}$, and the existence of a net $\{\psi_\beta\}$ in $\mathcal{K}(G)$ converging pointwise to 1 and satisfying $\|M_{\psi_\beta}\| = 1$ for all β is then clear. Hence, it follows that (G,σ) has the metric Fejér property. ∎

One drawback about this result is that a Fourier summing net for (G, σ) is not constructively produced in its proof.

The untwisted version of the first part of the following result may be seen as a variation of a result of Jolissaint and Valette [56] (cf. also [14]). In fact, they consider a stronger property than the M.A.P. for $C_r^*(G)$, which is not relevant for our considerations in this paper.

Corollary 7.2. *Assume that G has the Haagerup property.*

1) If there exists a Haagerup length function L on G such that G has the RD-property (w.r.t. L), or, more generally, such that G has the σ-twisted RD-property (w.r.t. L), then (G, σ) has the metric Fejér property.

2) If there exists a Haagerup function h on G such that G is exponentially H-bounded (w.r.t. h), or, more generally, such that G is exponentially H_σ-bounded (w.r.t. h), then (G, σ) has the metric Fejér property.

Proof. In the first assertion, assumptions (1) and (2) in Theorem 7.1 hold with $\varphi_r := r^L$ $(0 < r < 1, r \to 1^-)$ and $\kappa := (1 + L)^{s_0}$ for some suitably chosen $s_0 > 0$. As L is proper, $\varphi_r \kappa = r^L(1 + L)^{s_0} \in c_0(G)$ for all $0 < r < 1$, i.e. assumption (3) in this theorem also holds (with $\kappa_r = \kappa$ for all r), and the conclusion follows. The second assertion can be handled similarly, now by considering $\varphi_r = r^h$ and $\kappa_r := (r^{-1/2})^h$, $0 < r < 1$. □

We remark that one can recover from Theorem 7.1 the fact that (G, σ) has the metric Fejér property whenever G is a (countable) amenable group: one just pick some net $\{\varphi_\alpha\}$ of normalized functions in $\mathcal{K}(G) \cap \mathcal{P}(G)$ such that $\varphi_\alpha \to 1$ pointwise. Then 1) and 3) are satisfied for *any* κ. On the other hand, 2) holds when choosing $\kappa_\alpha = \kappa : G \to [1, \infty)$ for any κ satisfying condition (IS), that is, $\kappa^{-1} \in \ell^2(G)$.

It is not unlikely that Theorem 7.1 could serve as a basis to prove in the future that for any group G with the Haagerup property, (G, σ) will always have the metric Fejér property.

Following M. Choda [21, 22, 23] (see also [2, 55]), a finite von Neumann algebra N with a faithful normal tracial state τ is said to have the Haagerup property if there exists a net $\{P_\alpha\}$ of normal completely positive linear maps from N to itself such that each P_α is $\|\cdot\|_2$ -compact (in the sense of approximation by finite dimensional linear maps) and $\|P_\alpha(x) - x\|_2 \to 0$ for all $x \in N$, where $\|x\|_2 := \tau(x^*x)^{1/2}, x \in N$. She shows that $vN(G)$ has the Haagerup property (w.r.t. τ) if and only if G has the Haagerup property. It is not difficult to see that her proof may adapted to the twisted case : $vN(G,\sigma)$ has the Haagerup property if and only if G has it. The proof is essentially as hers, except that one now uses the twisted Haagerup Lemma 4.1 and the fact that if P is a completely positve linear map on $vN(G,\sigma)$ to itself, then $\varphi(g) := \tau(P(\Lambda_\sigma(g))\Lambda_\sigma(g)^*)$ gives a positive definite function on G.

There are several other approximation properties for C^*-algebras and von Neumann algebras in the literature than the ones we have mentioned in this paper. In the group operator algebra case, some of them are known to be equivalent to suitably defined properties of G : see e.g. [47] and references therein. Undoubtedly, if one considers twisted group operator algebras instead, these equivalences will remain valid if one just takes the pain to adapt the proofs. It looks like adding a twist does not have any effect on this kind of properties at the operator algebraic level. On the other hand, no characterization of the M.A.P. for reduced group C^*-algebras is known at the group level. Another intriguing approximation property is the strong Feller approximation property considered by Sauvageot [79, 80], which seems somehow related to the Haagerup property in some still unclear way. In this connection, we mention that under the same assumptions as in Proposition 6.9, one immediately deduces that $C_r^*(G,\sigma)$ has the strong Feller approximation property (since each φ_α has then the strong Feller property, cf. Remark 5.11). This observation can be applied to many groups having the Haagerup property (see for example Theorem 6.15 and its proof).

We end this paper with a couple of questions. Assume that G is countably infinite and has property T (see [25] and references therein). Then G has no Haagerup function and one may ask whether G (or (G,σ)) does have the metric Fejér property. More generally, does there always exist a Fourier summing net for G (or (G,σ))?

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