A representation theorem and a sensitivity result for functionals of jump diffusions

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Abstract

We use white noise calculus for Lévy processes to obtain a representation formula for the functionals of a jump diffusion. Then we use this to find an explicit formula for the Donsker delta function of a jump diffusion and we suggest its application to sensitivity analysis in mathematical finance for the computation of the Greeks.

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1 Introduction

A difficult, but crucial, task in the analysis of option prices is the prediction of their variation. To this aim it is important to locate which are the factors contributing to the fluctuation of prices and their effect. The *sensitivity analysis* is carried over the parameters appearing in the models for the price dynamics and the so-called *Greeks* represent a form of measure for the price sensitivity to some factors. For example, the "delta" is related to the initial price of the option, the "theta" is related to the time until maturity, the "rho" to the interest rate, the "vega" is the sensitivity to the volatility, etc. Efficient techniques for the computation of the Greeks rely on numerical finite difference methods and simulation. See [GY], [G], for example and references therein.

However, too often some restriction on the regularity of the price processes has to be imposed. In the recent years high attention was dedicated to finding more efficient and more general methods to apply numerics and simulation for the computation of the Greeks. The papers [FLLLT] and [FLLL]

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proved that, with a preliminary application of sophisticated tools of stochastic analysis, some better formulae could be derived which would ease a direct application of Monte Carlo simulation. Their method, based on Malliavin calculus, applies to price dynamics driven by Brownian motion only. See also [B], [GM], [K-HM], [MT], for example, and reference therein.

Several forms of generalization or extension to include dynamics driven by Poisson processes or combinations of independent Brownian motions and Poisson processes have been suggested. We can refer to [BM], [DJ], [E-KP], [PW], for example.

In this paper we present a representation formula for functionals of jump diffusions (see Theorem 3.2) which, if applied to the sensitivity analysis context, gives a computational efficient formula for the Greek "delta". We frame our method in the setting of white noise analysis. A short introduction to this framework with the preliminary results is given in Section 2.

Section 3 presents the representation formula for functionals of a jump diffusion. Moreover, we apply this result to give an explicit representation of the Donsker delta function. Our approach is in the same line as [MØP]. This results gain importance in view of the applications of the Donsker delta function for the computation of hedging portfolios in mathematical finance. See [AØU] for the Brownian motion setting and [DØ] for the pure jump Lévy processes case.

Section 4 is dedicated to the sensitivity analysis.

2 Framework

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $L_2(P)$ the standard (complex) L_2-space of the random variables ξ with finite norm $\|\xi\| := (E|\xi|^2)^{1/2} < \infty$. On the given probability space we consider the real *Lévy process* $\eta(t)$, $t \geq 0$, characterized by the Kolmogorov-deFinetti law

$$\log E\big[e^{iu\eta(t)}\big] = t\Big[i\alpha u - \tfrac{1}{2}\sigma^2 u^2 + \int\limits_{\mathbb{R}_0} (e^{iuz} - 1 - iuz)\nu(dz)\Big], \qquad u \in \mathbb{R},$$

where $\alpha \in \mathbb{R}$, $\sigma^2 > 0$ are constants and $\nu(dz)$, $z \in \mathbb{R}_0$, is a σ-finite Borel measure on $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. Note that

$$\int\limits_{\mathbb{R}_0} z^2 \nu(dz) < \infty\,,$$

namely we assume that $\eta(t) \in L_2(P)$ for all $t \geq 0$.

Throughout this paper we will always consider the càdlàg modification $\eta(t)$, $t \geq 0$, of the stochastic process above. We can refer to [BK], [Be], [S], for example, for general and detailed information about Lévy processes.

In particular we recall that, for every t, the random variable $\eta(t)$ admits a representation in the form

$$\eta(t) = \alpha t + \sigma B(t) + \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz) \tag{2.1}$$

where the standard Brownian motion $B(t)$, $t \geq 0$, and the compensated Poisson random measure

$$\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt, \qquad t \geq 0,\ z \in \mathbb{R},$$

are independent – cf. [I].

Inspired by the stochastic integral representation (2.1) it is natural to consider stochastic processes $\xi(t)$ $t \geq 0$, of the form

$$\xi(t) = \xi(0) + \int_0^t \alpha(s)ds + \int_0^t \beta(s)dB(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s, z)\tilde{N}(ds, dz) \tag{2.2}$$

where $\alpha(t), \beta(t)$ and $\gamma(t, z)$, $t \geq 0$, $z \in \mathbb{R}_0$, are *deterministic* functions satisfying

$$\int_0^\infty \Big[|\alpha(t)| + \beta^2(t) + \int_{\mathbb{R}_0} \gamma^2(t, z)\nu(dz)\Big]dt < \infty\,. \tag{2.3}$$

On the other side, in line with the approach suggested in [AØPU], we could consider a representation of type (2.2) embedded in a multidimensional framework as follows. Let us consider the probability space $(\Omega, \mathcal{F}, P)$ as a product of two complete probability spaces, i.e.

$$\Omega = \Omega_1 \times \Omega_2, \quad \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \quad P = P_1 \otimes P_2. \tag{2.4}$$

In such a framework we could consider stochastic processes $\xi(t)$, $t \geq 0$, on $(\Omega, \mathcal{F}, P)$ such that

$$\begin{aligned} \xi(t, \omega_1, \omega_2) = y &+ \int_0^t \alpha(s)ds + \int_0^t \beta(s)dB(s, \omega_1) \\ &+ \int_0^t \int_{\mathbb{R}_0} \gamma(s, z)\tilde{N}(ds, dz, \omega_2) \end{aligned} \tag{2.5}$$

for $y \in \mathbb{R}$ constant and $\alpha(t), \beta(t)$ and $\gamma(t, z)$, $t \geq 0$, $z \in \mathbb{R}_0$, deterministic functions satisfying (2.3).

We equip the probability space $(\Omega_1, \mathcal{F}_1, P_1)$ with the filtration $\mathcal{F}^1_t$, $t \geq 0$, $(\mathcal{F}^1_\infty = \mathcal{F}_1)$ generated by $B(t)$, $t \geq 0$, augmented of all P_1-null sets and the space $(\Omega_2, \mathcal{F}_2, P_2)$ with the filtration $\mathcal{F}^2_t$, $t \geq 0$ $(\mathcal{F}^2_\infty = \mathcal{F}_2)$ generated by the values of $\tilde{N}(dt, dz)$, $t \geq 0$, $z \in \mathbb{R}_0$, augmented of all P_2-null sets. Then on the product $(\Omega, \mathcal{F}, P)$ we fix the filtration

$$\mathcal{F}_t := \mathcal{F}^1_t \otimes \mathcal{F}^2_t, \qquad t \geq 0.$$

In the sequel we apply white noise analysis and techniques. Thus we choose to set $(\Omega_1, \mathcal{F}_1, P_1)$ to be a *Gaussian white noise probability space* and $(\Omega_2, \mathcal{F}_2, P_2)$ a *Poissonian white noise probability space.*

General references to white noise theory for Gaussian processes are e.g. [H], [HKPS], [HØUZ], [Ku], [O]. As for a white noise theory to non-Gaussian analysis we can refer to e.g. [AKS], [DØP], [KDS], [KDSU], [ØP], [P]. In order to keep this presentation moderate in size we recall here only the Poisson white noise framework in the approach and notation of [DØP] and [ØP].

To ease the notation we drop the index of $(\Omega_2, \mathcal{F}_2, P_2)$ and we write $(\Omega_2, \mathcal{F}_2, P_2) = (\Omega, \mathcal{F}, P)$ from now up to the end of this section.

From now on we assume that for every $\varepsilon > 0$ there exists $\rho > 0$ such that

$$\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} e^{\rho|z|}\nu(dz) < \infty. \tag{2.6}$$

This condition implies that the polynomials are dense in $L_2(\mu)$ where $\mu(dz) = z^2\nu(dz)$. It also garantees that the measure ν integrates all polynomials of degree greater than or equal to 2.

Let $\mathcal{A}$ denote the set of all multi-indices $\alpha = (\alpha_0, \alpha_1, ...)$ which have only finitely many non-zero values $\alpha_i \in \mathbb{N} \setminus \{0\}$. In the space $L_2(\Omega, \mathcal{F}, P) = L_2(\Omega_2, \mathcal{F}_2, P_2)$ we construct the orthogonal basis K_α, $\alpha \in \mathcal{A}$, as follows.

First of all we consider the orthonormal basis φ_i, $i \in \mathbb{N}$, in $L_2(\lambda)$ constituted by the Laguerre functions (order 1/2). Here and in the sequel $\lambda(dt) = dt$ denotes the Lebesgue measure on the real line. Moreover we take an orthonormal basis ψ_j, $j \in \mathbb{N}$, in $L_2(\nu)$ of polynomial type. See e.g. [ØP] for further details.

Then we can consider the products

$$\zeta_k(t, z) = \varphi_i(t)\psi_j(z) \tag{2.7}$$

for $k = k(i, j)$ as a bijective mapping $k : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ (e.g. the diagonal counting of the Cartesian product $\mathbb{N} \times \mathbb{N}$).

For any $\alpha \in \mathcal{A}$ with $\max\{i : \alpha_i \neq 0\} = j$ and $|\alpha| := \sum_i \alpha_i = m$, we can define

$$\zeta^{\otimes\alpha}\big((t_1, z_1), ..., (t_m, z_m)\big) := \zeta_1^{\otimes\alpha_1} \otimes \cdots \otimes \zeta_j^{\otimes\alpha_j}\big((t_1, z_1), ..., (t_m, z_m)\big)$$
$$= \zeta_1(t_1, z_1) \cdots \zeta_1(t_{\alpha_1}, z_{\alpha_1}) \cdots \zeta_j(t_{\alpha_1+...+\alpha_{j-1}+1}, z_{\alpha_1+...+\alpha_{j-1}+1}) \cdots \zeta_j(t_m, z_m)$$

and $\zeta^{\otimes 0} = 1$. Moreover, we denote the corresponding *symmetrized tensor product* by $\zeta^{\hat{\otimes}\alpha}$. We can now construct an *orthogonal basis* K_α, $\alpha \in \mathcal{A}$, in $L_2(P)$ as follows:

$$K_\alpha := I_{|\alpha|}(\zeta^{\hat{\otimes}\alpha}), \quad \alpha \in \mathcal{A}, \tag{2.8}$$

where

$$I_n(f) = n! \int_0^\infty \int_{\mathbb{R}} \cdots \int_0^{t_2} \int_{\mathbb{R}} f(t_1, z_1, ..., t_n, z_n) \widetilde{N}(dt_1, dz_1) \cdots \widetilde{N}(dt_n, z_n)$$

is the Itô iterated integral with respect to the centered Poisson stochastic measure. See [I]. Here $f \in L_2(\lambda \times \nu)^n$ is symmetric in the n pairs $(t_1, z_1), \ldots, (t_n, z_n)$. Note that

$$E\big[I_m(g) \cdot I_n(f)\big] = 0, \quad m \neq n$$

and

$$E\big[I_n(f)^2\big] = n! ||f||^2_{L_2(\lambda \times \nu)^n}$$

for all symmetric $g \in L_2(\lambda \times \nu)^m$ and $f \in L_2(\lambda \times \nu)^n$ $(m, n \in \mathbb{N})$.

Hence every $\xi \in L_2(P)$ admits the chaos expansion

$$\xi = \sum_{\alpha \in \mathcal{A}} c_\alpha K_\alpha \quad (c_\alpha \in \mathbb{R}) \tag{2.9}$$

and

$$||\xi||^2_{L_2(P)} = \sum_{\alpha \in \mathcal{A}} c_\alpha^2 ||K_\alpha||^2_{L_2(P)} = \sum_{\alpha \in \mathcal{A}} c_\alpha^2 \alpha! \tag{2.10}$$

where $\alpha! := \alpha_1! \alpha_2! ...$ for $\alpha = (\alpha_1, \alpha_2, ...) \in \mathcal{A}$.

Thanks to these chaos expansions we can characterize the following spaces and chain of embeddings.

By $(\mathcal{S})_\rho$ $(0 \leq \rho \leq 1)$ we denote the space of all random variables $f = \sum_{\alpha \in \mathcal{A}} c_\alpha K_\alpha \in L_2(P)$ such that

$$||f||^2_{\rho,k} = \sum_{\alpha \in \mathcal{A}} (\alpha!)^{1+\rho} c_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty \quad \text{for all} \quad k \in \mathbb{N}$$

where $(2\mathbb{N})^{k\alpha} := (2 \cdot 1)^{k\alpha_1} (2 \cdot 2)^{k\alpha_2} \cdots (2 \cdot j)^{k\alpha_j}$ if $j = \max\{i : \alpha_j \neq 0\}$. And by $(\mathcal{S})_{-\rho}$ we denote the space of all random variables $F = \sum_{\alpha \in \mathcal{A}} c_\alpha K_\alpha \in L_2(P)$ such that

$$||F||^2_{-\rho,-k} = \sum_{\alpha \in \mathcal{A}} (\alpha!)^{1-\rho} c_\alpha^2 (2\mathbb{N})^{-k\alpha} < \infty \quad \text{for some} \quad k \in \mathbb{N}.$$

The subspaces $(\mathcal{S})_\rho$ and $(\mathcal{S})_{-\rho}$ are respectively equipped with the projective topology and the inductive topology induced by the above seminorms. Note that for any $F = \sum_{\alpha\in\mathcal{A}} a_\alpha K_\alpha \in (\mathcal{S})_{-\rho}$ and $f = \sum_{\alpha\in\mathcal{A}} b_\alpha K_\alpha \in (\mathcal{S})_\rho$ the action

$$\langle F, f\rangle := \sum_{\alpha\in\mathcal{A}} a_\alpha b_\alpha \alpha!$$

is well-defined and thus the space $(\mathcal{S})_{-\rho}$ is the dual of $(\mathcal{S})_\rho$, i.e. $(\mathcal{S})_{-\rho} = (\mathcal{S})^*_\rho$. We remark that, for $\rho = 0$, the spaces $(\mathcal{S}) := (\mathcal{S})_0$ and $(\mathcal{S})^* = (\mathcal{S})^*_0 = (\mathcal{S})_{-0}$ appear respectively as a Lévy version for the *Hida test function space* and *Hida distribution space* for pure jump Lévy processes. See e.g. [HKPS], [HØUZ], [Ku], [O]. For $\rho = 1$, the spaces $(\mathcal{S})_1$ and $(\mathcal{S})_{-1}$ are the Lévy version of the *Kondratiev test function space* and the *Kondratiev distribution space* respectively. See [K] and also [HØUZ], for example.

The following relationships hold true

$$(\mathcal{S})_1 \subset (\mathcal{S})_\rho \subset (\mathcal{S}) \subset L_2(P) \subset (\mathcal{S})^* \subset (\mathcal{S})_{-\rho} \subset (\mathcal{S})_{-1}.$$

The relevance of these spaces will be clarified in the sequel. For instance $(\mathcal{S})^*$ is rich enough to contain the white noise of the centered Poisson stochastic measure and of the pure jump Lévy process as its elements. In fact, let us consider the random variable $\xi = \widetilde{N}(t, B) \in L_2(P)$, for any Borel set $B \in \mathcal{B}(\mathbb{R}\setminus\{0\})$, then the following chaos expansion can be written:

$$\widetilde{N}(t, B) = \sum_{i,j\geq 1} \int_0^t \int_B \varphi_i(s)\psi_j(z)\nu(dz)ds \cdot K_{\epsilon^{k(i,j)}}$$

via the application of the orthogonal basis in $L_2(\lambda)$ and $L_2(\nu)$ and the diagonal counting – cf. (2.7). Moreover, for any $k \in \mathbb{N}$, the multi-index $\epsilon^k = (\epsilon^k_1, \epsilon^k_2, ...)$ is defined by

$$\epsilon^k_i := \begin{cases} 1, & i = k \\ 0, & \text{otherwise.} \end{cases}$$

Here we refer to [ØP] for all the details. Then we can define the *white noise for the centered Poisson stochastic measure* as the element

$$\overset{\bullet}{\widetilde{N}}(t, z) := \sum_{i,j\geq 1} \varphi_i(t)\psi_j(z) \cdot K_{\epsilon^{k(i,j)}} \tag{2.11}$$

in $(\mathcal{S})^*$, for almost all $t \geq 0$, $z \in \mathbb{R}$. Naturally it appears as the Radon-Nikodym derivative

$$\overset{\bullet}{\widetilde{N}}(t, z) = \frac{\widetilde{N}(dt, dz)}{dt \times \nu(dz)} \quad \text{in} \quad (\mathcal{S})^*.$$

The *Lévy-Wick product* $F \diamond G$ of two elements $F = \sum_{\alpha\in\mathcal{A}} a_\alpha K_\alpha$ and $G = \sum_{\beta\in\mathcal{A}} b_\beta K_\beta$ in $(\mathcal{S})_{-1}$ is defined by

$$F \diamond G = \sum_{\alpha,\beta\in\mathcal{A}} a_\alpha b_\beta K_{\alpha+\beta}. \tag{2.12}$$

It can be shown that the spaces $(\mathcal{S})_1, (\mathcal{S}), (\mathcal{S})^*$ and $(\mathcal{S})_{-1}$ are closed under Wick products.

One of the useful features of the Wick product is the following relationship within Itô stochastic integration and Bochner integration:

$$\int_0^t \int_{\mathbb{R}} Y(s,z)\widetilde{N}(ds,dz) = \int_0^t \int_{\mathbb{R}} Y(s,z) \diamond \overset{\bullet}{\tilde{N}}(s,z)\nu(dz)ds. \tag{2.13}$$

We also mention that for all $F \in (\mathcal{S})_{-1}$ one can define the *Wick exponential* $\exp^\diamond F \in (\mathcal{S})_{-1}$ by

$$\exp^\diamond F := \sum_{n=0}^\infty \frac{1}{n!} F^{\diamond n} \tag{2.14}$$

and the following property

$$E(\exp^\diamond F) = \exp\{EF\} \tag{2.15}$$

holds true.

3 A representation theorem for functionals of a class of jump diffusions

Let $\xi(t) = \xi^y(t)$, $t \in [0,T]$, be a stochastic process on $(\Omega, \mathcal{F}, P)$ – cf. (2.4), of the form

$$\begin{cases} d\xi(t) = \alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz), & t \in [0,T], \\ \xi(0) = y \in \mathbb{R} \end{cases} \tag{3.1}$$

where $\alpha(t), \beta(t)$ and $\gamma(t,z)$; $t \in [0,T]$, $z \in \mathbb{R}$, are deterministic functions satisfying (2.3) – cf. (2.5). For $\lambda \in \mathbb{R}$ define

$$Y(t) = \exp(\lambda\xi(t)), \qquad t \in [0,T]. \tag{3.2}$$

Then by the Itô formula (see e.g. [ØS, Chapter 1]) we have

$$\begin{aligned} dY(t) = Y(t^-)\big[&(\lambda\alpha(t) + \tfrac{1}{2}\lambda^2\beta^2(t))dt + \lambda\beta(t)dB(t) \\ &+ \int_{\mathbb{R}} \{\exp(\lambda\gamma(t,z)) - 1 - \lambda\gamma(t,z)\}\nu(dz)dt \\ &+ \int_{\mathbb{R}} \{\exp(\lambda\gamma(t,z)) - 1\}\tilde{N}(dt,dz)\big]. \end{aligned} \tag{3.3}$$

Using white noise notation and Wick calculus this can be written

$$\frac{dY(t)}{dt} = Y(t^-) \diamond \{\lambda\alpha(t) + \tfrac{1}{2}\lambda^2\beta^2(t) + \lambda\beta(t)\dot{B}(t)$$

$$+ \int_{\mathbb{R}} [\exp\{\lambda\gamma(t,z)\} - 1 - \lambda\gamma(t,z)]\nu(dz)$$

$$+ \int_{\mathbb{R}} \{\exp\{\lambda\gamma(t,z)\} - 1\}\overset{\bullet}{\tilde{N}}(t,z)\nu(dz)\}; \qquad Y(0) = e^{\lambda y}. \tag{3.4}$$

The solution of (3.4) is, using Wick calculus in $(\mathcal{S})^*$,

$$Y(t) = Y(0)\exp^{\diamond}\Big\{\int_0^t \Big\{\lambda\alpha(s) + \tfrac{1}{2}\lambda^2\beta^2(s)$$

$$+ \int_{\mathbb{R}} \{e^{\lambda\gamma(s,z)} - 1 - \lambda\gamma(s,z)\Big\}\nu(dz)\Big\}ds$$

$$+ \int_0^t \lambda\beta(s)dB(s) + \int_0^t \int_{\mathbb{R}} \{e^{\lambda\gamma(s,z)} - 1\}\tilde{N}(ds,dz)\}. \tag{3.5}$$

Comparing (3.5) with (3.2) we get the following formula for the Wick exponential.

Lemma 3.1 *With $\xi(t)$ as in (3.1) and $\lambda \in \mathbb{R}$ we have*

$$e^{\lambda\xi(t)} = e^{\lambda y}\exp^{\diamond}\Big\{\int_0^t \Big[\lambda\alpha(s) + \tfrac{1}{2}\lambda^2\beta^2(s)$$

$$+ \int_{\mathbb{R}} [e^{\lambda\gamma(s,z)} - 1 - \lambda\gamma(s,z)]\nu(dz)\Big]ds + \int_0^t \lambda\beta(s)dB(s)$$

$$+ \int_0^t \int_{\mathbb{R}} [e^{\lambda\gamma(s,z)} - 1]\tilde{N}(ds,dz)\Big\}. \tag{3.6}$$

Using this we obtain the following result:

Theorem 3.2 (Representation theorem for functionals of a jump diffusion)
Let $g : \mathbb{R} \to \mathbb{R}$ be a function with Fourier transform

$$\hat{g}(\lambda) = \tfrac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda x}g(x)dx, \qquad \lambda \in \mathbb{R},$$

and satisfying the Fourier inversion property

$$g(u) = \int_{\mathbb{R}} e^{i\lambda u}\hat{g}(\lambda)d\lambda, \qquad u \in \mathbb{R}.$$

Then

$$(3.7) \qquad g(\xi^y(t)) = \int_{\mathbb{R}} \hat{g}(\lambda)\exp^{\diamond}\{X_\lambda^y(t)\}d\lambda, \qquad t \in [0,T],$$

where

$$\begin{aligned} X_\lambda^y(t) &= i\lambda y + \int_0^t i\lambda\beta(s)dB(s) \\ &+ \int_0^t\int_{\mathbb{R}} [e^{i\lambda\gamma(s,z)} - 1]\tilde{N}(ds,dz) + \int_0^t \Big\{ i\lambda\alpha(s) - \tfrac{1}{2}\lambda^2\beta^2(s) \\ &+ \int_{\mathbb{R}} [e^{i\lambda\gamma(s,z)} - 1 - i\lambda\gamma(s,z)]\nu(dz)\Big\}ds, \qquad t \in [0,T]. \end{aligned} \tag{3.8}$$

Proof. Applying (3.6) with $i\lambda$ instead of λ we get

$$g(\xi^y(t)) = \tfrac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda\xi(t)}\hat{g}(\lambda)d\lambda = \tfrac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(\lambda)\exp^{\diamond}(X_\lambda^y(t))d\lambda.$$

□

Corollary 3.3 *Let g be a real function as in Theorem 3.2. Then we have*

$$(3.9) \qquad E[g(\xi^y(t))] = \int_{\mathbb{R}} \hat{g}(\lambda)\cdot\exp(i\lambda y + G_\lambda(t))d\lambda,$$

where

$$(3.10) \quad G_\lambda(t) = \int_0^t \Big\{ i\lambda\alpha(s) - \tfrac{1}{2}\lambda^2\beta^2(s) + \int_{\mathbb{R}} (e^{i\lambda\gamma(s,z)} - 1 - i\lambda\gamma(s,z))\nu(dz)\Big\}ds.$$

Proof. This follows from Theorem 3.2 plus the fact that (see (2.15))

$$E[\exp^{\diamond} X_\lambda^y(t)] = \exp(E[X_\lambda^y(t)]).$$

□

In the last part of this section we obtain an explicit formula for the Donsker delta function of $\xi(t) = \xi^y(t)$, $t \geq 0$. This is derived as an application of Theorem 3.2.

The Donsker delta function is a generalized white noise functional, we can refer e.g. [H], [HKPS], [Ku], for general information. Here we give its definition within the white noise framework we have introduced, in the line of [AØP] and [MØP]. Note that the Donsker delta function has been used for giving explicit representation formulae for the hedging portfolio in some market models driven by Brownian motion or pure jump Lévy processes, see [AØP] and [DØ].

Definition 3.4 *For a given random variable $X \in (\mathcal{S})_{-1}$ the Donsker delta function of X is a continuous function $\delta_\bullet(X) : \mathbb{R} \to (\mathcal{S})_{-1}$ such that*

$$\int_{\mathbb{R}} h(u)\delta_u(X)du = h(X)$$

for all Borel real functions h on $\mathbb{R}$ for which the integral is well-defined in $(\mathcal{S})_{-1}$. Following [MØP] we consider a measure P_2 in $P = P_1 \otimes P_2$ (see (2.4)) which satisfies the condition: there exists $\epsilon \in (0, 1)$ such that

$$\lim_{|u|\to\infty} |u|^{-(1+\epsilon)} Re\Big\{ \int_{\mathbb{R}_0} (e^{iuz} - 1 - iuz)\nu(dz)\Big\} = \infty\,. \tag{3.11}$$

Using Theorem 3.2 we can obtain an explicit formula for the Donsker delta function of $\xi(t)$:

Theorem 3.5 *Assume that (3.11) holds. Then the Donsker delta function $\delta_u(\xi^y(t))$, $u \in \mathbb{R}$, of $\xi^y(t)$, $t \in [0, T]$, exists in $(\mathcal{S})^*$ and is given by*

$$\begin{aligned}\delta_u(\xi^y(t)) = \tfrac{1}{2\pi} \int_{\mathbb{R}} \exp^\diamond \Big[\int_0^t \int_{\mathbb{R}} (e^{i\lambda\gamma(s,z)} - 1)\tilde{N}(ds, dz) + \int_0^t i\lambda\beta(s)dB(s) \\ + \int_0^t \Big\{ \int_{\mathbb{R}} (e^{i\lambda\gamma(s,z)} - 1 - i\lambda\gamma(s, z))\nu(dz) \\ + i\lambda\alpha(s) - \tfrac{1}{2}\lambda^2\beta^2(s)\Big\}ds + i\lambda y - i\lambda u\Big] d\lambda.\end{aligned} \tag{3.12}$$

Proof (sketch). Formally this follows from (3.7) by using the Fubini theorem in $(\mathcal{S})^*$, as follows. By (3.7) we have

$$\begin{aligned}g(\xi^y(t)) &= \int_{\mathbb{R}} \hat{g}(\lambda) \exp^\diamond(X^y_\lambda(t))d\lambda = \tfrac{1}{2\pi} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} e^{-i\lambda u} g(u)du\Big) \exp^\diamond(X^y_\lambda(t))d\lambda \\ &= \tfrac{1}{2\pi} \int_{\mathbb{R}} g(u) \int_{\mathbb{R}} \exp^\diamond(-i\lambda u + X^y_\lambda(t))d\lambda\, du.\end{aligned}$$

For justification and more details we refer to the proof in [MØP, Theorem 3.1.4]. □

4 Application to sensitivity with respect to the starting point

Let $X(t) = X^x(t)$, $t \in [0,T]$, be a jump diffusion of the form

$$(4.1) \qquad \begin{cases} dX(t) = X(t^-)\left[\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \theta(t,z)\tilde{N}(dt,dz)\right] \\ X(0) = x > 0 \end{cases}$$

where $\mu(t), \sigma(t)$ and $\theta(t,z)$, $t \in [0,T]$, $z \in \mathbb{R}$, are deterministic, $\theta(t,z) > -1$ for a.a. t, z and

$$\int_0^T \Big\{|\mu(t)| + \sigma^2(t) + \int_{\mathbb{R}} \theta^2(t,z)\nu(dz)\Big\}dt < \infty$$

– cf. (2.3). By the Itô formula for Lévy processes (see e.g. [ØS, Theorem 1.14], the solution of this equation is

$$X^x(t) = x\exp\Big[\int_0^t \Big\{\mu(s) - \tfrac{1}{2}\sigma^2(s) + \int_{\mathbb{R}} (\ln(1+\theta(s,z)) - \theta(s,z))\nu(dz)\Big\}ds$$

$$(4.2) \qquad + \int_0^t \sigma(s)dB(s) + \int_0^t \int_{\mathbb{R}} \ln(1+\theta(s,z))\tilde{N}(ds,dz)\Big] = \exp(\xi^y(t))$$

where $d\xi^y(t) = \alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz)$, with

$$\alpha(t) = \mu(t) - \tfrac{1}{2}\sigma^2(t) + \int_{\mathbb{R}} (\ln(1+\theta(t,z)) - \theta(t,z))\nu(dz),$$

$$(4.3) \qquad \beta(t) = \sigma(t), \quad \gamma(t,z) = \ln(1+\theta(t,z)) \quad \text{and} \quad y = \ln x$$

- cf. (3.1). Therefore, if $h : \mathbb{R} \to \mathbb{R}$ then

$$E[h(X^x(T))] = E[h(\exp(\xi^y(T)))] = E[g(\xi^y(T))],$$

where

$$g(u) := h(\exp(u)), \qquad u \in \mathbb{R}.$$

If this g satisfies the conditions of Theorem 3.2 then

$$\tfrac{d}{dx}E[h(X^x(T))] = \tfrac{d}{dx}[Eg(\xi^{\ln x}(T))] = \tfrac{d}{dx}\int_{\mathbb{R}} \hat{g}(\lambda)\exp(i\lambda\ln x + G_\lambda(T))d\lambda$$

$$= \int_{\mathbb{R}} \hat{g}(\lambda)\tfrac{i\lambda}{x}\exp(i\lambda\ln x + G_\lambda(T))d\lambda\,,$$

where $G_\lambda(T)$ is given by (3.10). We have proved

Theorem 4.1 *Suppose $h : \mathbb{R} \to \mathbb{R}$ is such that $g(u) := h(\exp(u))$, $u \in \mathbb{R}$, satisfies the conditions of Theorem 3.2 and that*

$$\int_{\mathbb{R}} \left|\hat{g}(\lambda)\lambda \exp\{\operatorname{Re} G_\lambda(T)\}\right| d\lambda < \infty .$$

Then

$$\tfrac{d}{dx} E[h(X^x(T))] = \int_{\mathbb{R}} \hat{g}(\lambda) \tfrac{i\lambda}{x} \exp(i\lambda \ln x + G_\lambda(T)) d\lambda . \tag{4.4}$$

Example 4.2 Choose $h(u) = \chi_{[H,K]}(u)$, $u \in \mathbb{R}$ $(H, K > 0)$. Then $h(X^x(T))$ may be regarded as the payoff of a digital option on a stock with price $X^x(T)$. In this case

$$g(u) = \chi_{[H,K]}(e^u), \qquad u \in \mathbb{R},$$

and

$$2\pi\hat{g}(\lambda) = \int_{\mathbb{R}} e^{-i\lambda u} g(u) du = \int_{\mathbb{R}} e^{-i\lambda u} \chi_{[H,K]}(e^u) du = \int_{\ln H}^{\ln K} e^{-i\lambda u} du = \frac{H^{-i\lambda} - K^{-i\lambda}}{i\lambda} .$$

Therefore

$$\tfrac{d}{dx} E\left[\chi_{[H,K]}(X^x(T))\right] = \int_{\mathbb{R}} \frac{H^{-i\lambda} - K^{-i\lambda}}{x} \exp(i\lambda \ln x + G_\lambda(T)) d\lambda , \tag{4.5}$$

provided that the integral converges.

A sufficient condition for this is that, for some $\delta > 0$,

$$\lambda^2 \int_0^T \left\{\beta^2(s) + \int_{\mathbb{R}} (1 - \cos(\lambda\gamma(s,z)))\nu(dz)\right\} ds \geq \delta\lambda^2 \quad \text{for all } \lambda \in \mathbb{R}$$

which is a weak form of non-degeneracy of the equation (4.1). Thus, in spite of the fact that h is not even continuous, (4.4) is a computationally efficient formula for $\frac{d}{dx} E^x[h(X^x(T))]$.

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