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ON THE UNIQUENESS OF DISCONTINUOUS SOLUTIONS TO THE DEGASPERIS-PROCESI EQUATION

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ABSTRACT. We prove uniqueness within a class of discontinuous solutions to the nonlinear and third order dispersive Degasperis-Procesi equation

$$\partial_t u - \partial^3_{txx} u + 4u\partial_x u = 3\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u.$$

In a recent paper [3], we proved for this equation the existence and uniqueness of $L^1 \cap BV$ weak solutions satisfying an infinite family of Kružkov-type entropy inequalities. The purpose of this paper is to replace the Kružkov-type entropy inequalities by an Oleĭnik-type estimate and to prove uniqueness via a nonlocal adjoint problem. An implication is that a shock wave in an entropy weak solution to the Degasperis-Procesi equation is admissible only if it jumps down in value (like the inviscid Burgers equation).

1. INTRODUCTION

We are interested in the uniqueness problem for discontinuous solutions to the Degasperis-Procesi equation

$$\partial_t u - \partial^3_{txx} u + 4u\partial_x u = 3\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u, \qquad (t,x) \in (0,\infty)\times\mathbb{R}, \tag{1.1}$$

which we augment with the initial condition

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}. \tag{1.2}$$

Degasperis and Procesi [8] considered a family of third order dispersive nonlinear equations, indexed over six constants $c_0, \gamma, \alpha, c_1, c_2, c_3 \in \mathbb{R}$,

$$\partial_t u + c_0\partial_x u + \gamma\partial^3_{xxx} u - \alpha^2 \partial^3_{txx} u = \partial_x \left(c_1 u^2 + c_2(\partial_x u)^2 + c_3 u \partial^2_{xx} u \right).$$

They found that only three equations from this family were asymptotically integrable up to third order: the Korteweg-deVries (KdV) equation ($\alpha = c_2 = c_3 = 0$), the Camassa-Holm equation ($c_1 = -\frac{3c_3}{2\alpha^2}$, $c_2 = \frac{c_3}{2}$) [1], and one new equation ($c_1 = -\frac{2c_3}{\alpha^2}$, $c_2 = c_3$), which properly scaled reads

$$\partial_t u + \partial_x u + 6u\partial_x u + \partial^3_{xxx} u - \alpha^2 \left(\partial^3_{txx} u + \frac{9}{2}\partial_x u \partial^2_{xx} u + \frac{3}{2} u \partial^3_{xxx} u \right) = 0. \tag{1.3}$$

By rescaling, shifting the dependent variable, and finally applying a Galilean boost, equation (1.3) can be transformed into the form (1.1), see [6, 7].

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The Degasperis-Procesi equation (1.1) was considered for the first time in [8]. Then Degasperis, Holm, and Hone [7] proved the exact integrability of (1.1) by constructing a Lax pair and showed the existence of "non-smooth" solutions that are superpositions of multipeakons and described the integrable finite-dimensional peakon dynamics, which were compared with the multipeakon dynamics of the Camassa-Holm equation. An explicit solution was also found in the perfectly antisymmetric peakon-antipeakon collision case. Lundmark and Szmigielski [17, 18] used an inverse scattering approach to determine a completely explicit formula for the general n-peakon solution of the Degasperis-Procesi equation (1.1). Mustafa [19] proved that smooth solutions to (1.1) have infinite speed of propagation. We refer to [10] for a discussion of Camassa-Holm, Degasperis-Procesi, and other related equations, along with many numerical examples.

Regarding the well-posedness of the initial value problem for the Degasperis-Procesi equation, Yin has studied this within certain functional classes in a series of papers [25, 26, 27, 28]. In particular, Yin [27] proved for (1.1), (1.2) the following global existence result: *Suppose $u_0 \in H^1(\mathbb{R})$ and $(1-\partial_{xx}^2)u_0$ is a nonnegative bounded Radon measure on $\mathbb{R}$, i.e., $(1-\partial_{xx}^2)u_0 \in \mathcal{M}_+(\mathbb{R})$. Then (1.1), (1.2) possesses a unique weak solution u belonging to $W^{1,\infty}((0,\infty)\times\mathbb{R})\cap L^\infty_{\mathrm{loc}}((0,\infty);H^1(\mathbb{R}))$. Furthermore, $(1-\partial_{xx}^2)u(t,\cdot)$ belongs to $\mathcal{M}_+(\mathbb{R})$ for a.e. $t \geq 0$ and $I(u) = \int_{\mathbb{R}} u\,dx$, $E(u) = \int_{\mathbb{R}} u^3\,dx$ are two conservation laws.*

All solutions encompassed by Yin's well-posedness theory are regular, that is, they are no worse than H^1, a fact that is reminiscent of the Camassa-Holm equation (see for example [5]). Recently [3] we advocated the view that the Degasperis-Procesi equation could admit discontinuous (shock wave) solutions, which means that it would behave radically different from the Camassa-Holm equation and its kink solutions (peaked solitons), but similar to the inviscid Burgers equation. Consequently, a well-posedness theory should rely on functional spaces containing discontinuous functions. Indeed, in [3] we proved the existence and uniqueness of so-called entropy weak solutions in the class $L^1 \cap BV$. The relevance of these solutions in the present context is supported by Lundmark [16], who found some explicit shock solutions of the Degasperis-Procesi equation that are entropy weak solutions in the sense of [3]. Numerical schemes for computing entropy weak solutions of the Degasperis-Procesi equation is developed and analyzed in [4].

Next we discuss [3] in some more detail. First, what do we mean by a weak solution to the Degasperis-Procesi equation (1.1)? Formally, (1.1) is equivalent to the hyperbolic-elliptic system

$$\partial_t u + \partial_x\left(\frac{u^2}{2}\right) + \partial_x P = 0, \qquad -\partial_{xx}^2 P + P = \frac{3}{2}u^2. \tag{1.4}$$

For any $\lambda > 0$ the operator $(\lambda^2 - \partial_{xx}^2)^{-1}$ has a convolution structure:

$$(\lambda^2 - \partial_{xx}^2)^{-1}(f)(x) = (G_\lambda \star f)(x) = \frac{1}{2\lambda}\int_{\mathbb{R}} e^{-\lambda|x-y|} f(y)\,dy, \qquad x \in \mathbb{R}, \tag{1.5}$$

where $G_\lambda(x) := \frac{1}{2\lambda}e^{-|x|/\lambda}$, so that $P = G_1 \star \left(\frac{3}{2}u^2\right)$. Consequently, (1.4) can be written as a conservation law with a nonlinear and nonlocal source term:

$$\partial_t u + \partial_x\left(\frac{u^2}{2}\right) + \frac{3}{4}\int_{\mathbb{R}} e^{-|x-y|}\mathrm{sign}\,(y-x)\,(u(t,y))^2\,dy = 0. \tag{1.6}$$

By a *weak solution* of the initial value problem (1.1), (1.2) we mean a function $u \in L^\infty((0,\infty); L^2(\mathbb{R}))$ that satisfies (1.6), (1.2) in $\mathcal{D}'([0,\infty)\times\mathbb{R})$.

We need to explain why weak solutions $u(t,\cdot)$ ought to be L^2 bounded (this bound is at the heart of the matter in [3]). Our starting point is that if we introduce the quantity $v := G_2 \star u$, then formally the following conservation law holds [7]:

$$\begin{aligned}\partial_t&\Big((\partial_{xx}^2 v)^2 + 5(\partial_x v)^2 + 4v^2\Big)\\ &+ \partial_x\left(\frac{2}{3}u^3 + 4v\, G_1 \star (u^2) + \partial_x v\, \partial_x\left[G_1 \star (u^2)\right] - 4u^2 v\right) = 0,\end{aligned}$$

from which it follows $v(t,\cdot) \in H^2(\mathbb{R})$ and thereby also $u(t,\cdot)\in L^2(\mathbb{R})$, for any $t \geq 0$. The L^2 bound on u implies other bounds as well:

$$P(t,\cdot) \in W^{1,\infty}(\mathbb{R}),\ \partial_{xx}^2 P(t,\cdot) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \text{ for any } t \geq 0, \tag{1.7}$$

and

$$u(t,\cdot) \in L^1(\mathbb{R}) \cap BV(\mathbb{R}), \text{ for any } t > 0.$$

One may wonder why the functional space BV (bounded variation) is relevant for the Degasperis-Procesi equation, but, at least formally, it follows from (1.4) that

$$\frac{d}{dt}\int_{\mathbb{R}} |\partial_x u|\, dx \leq \int_{\mathbb{R}} \left|\partial_{xx}^2 P\right|\, dx.$$

If $\partial_{xx}^2 P(t,\cdot) \in L^1(\mathbb{R})$, then $u(t,\cdot) \in BV(\mathbb{R})$ (and thus also an $u(t,\cdot) \in L^\infty(\mathbb{R})$), for any $t \geq 0$. But an $L^1(\mathbb{R})$ bound on $\partial_{xx}^2 P(t,\cdot)$ is expected in view of (1.7). We refer to [3] for details regarding the above (formal) bounds.

To establish the existence of a weak solution one must construct approximate solutions for which bounds similar to those above can be derived rigorously. In [3], we did this for the following fourth order viscous approximation of (1.1):

$$\partial_t u_\varepsilon - \partial_{txx}^3 u_\varepsilon + 4u_\varepsilon \partial_x u_\varepsilon = 3\partial_x u_\varepsilon \partial_{xx}^2 u_\varepsilon + u_\varepsilon \partial_{xxx}^3 u_\varepsilon + \varepsilon\left(\partial_{xx}^2 u_\varepsilon - \partial_{xxxx}^4 u_\varepsilon\right), \tag{1.8}$$

for $\varepsilon > 0$. Assuming $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, we proved $\{u_\varepsilon\}_{\varepsilon>0} \subset L^\infty((0,\infty); L^2(\mathbb{R}))$ and $\{u_\varepsilon\}_{\varepsilon>0} \subset L^\infty((0,T); L^1(\mathbb{R}) \cap BV(\mathbb{R}))$, for any $T > 0$. Consequently, the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is strongly convergent (at least along a subsequence) to a limit function u satisfying the formal bounds discussed above. Additionally, the limit u is a weak solution of the Degasperis-Procesi equation (1.1), (1.2).

Regarding the constructed weak solution, we point out that $I(u) = \int_{\mathbb{R}} u\, dx$ is a conservation law but $E(u) := \int_{\mathbb{R}} u^3\, dx$ is not. Indeed, a simple calculation will reveal that

$$\frac{d}{dt}\int_{\mathbb{R}} u^3\, dx = -6\varepsilon \int_{\mathbb{R}} u_\varepsilon \left(\partial_x u_\varepsilon\right)^2\, dx, \qquad \varepsilon > 0,$$

and if a shock wave solution persists then the right-hand side of this equation will be non-zero in the limit as $\varepsilon \to 0$.

To account for possibile discontinuities in our weak solutions and thus the loss uniqueness, in [3] we restored the uniqueness by imposing an infinite family of entropy inequalities [12]. For any convex C^2 function $\eta : \mathbb{R} \to \mathbb{R}$ and corresponding function $q : \mathbb{R} \to \mathbb{R}$ defined by $q'(u) = \eta'(u)\, u$, the following inequality holds in $\mathcal{D}'([0,\infty)\times\mathbb{R})$:

$$\partial_t \eta(u) + \partial_x q(u) + \eta'(u)\left[\frac{3}{4}\int_{\mathbb{R}} e^{-|x-y|}\operatorname{sign}(y-x)\,(u(t,y))^2\, dy\right] \leq 0. \tag{1.9}$$

We refer to (1.9) to as *Kružkov-type entropy inequalities* [12]. A weak solution u of (1.1), (1.2) satisfying (1.9) is called an *entropy weak solution*.

We proved in [3] that the weak solution constructed by letting $\varepsilon \to 0$ in (1.8) is indeed an entropy weak solution. Moreover, we proved the L^1 stability and thus the uniqueness of entropy weak solutions.

The purpose of the present paper is to point out that uniqueness still holds if we replace the (infinite family of) Kružkov-type entropy inequalities (1.9) by the Oleĭnik-type (one-sided Lipschitz) estimate $\partial_x u(t,x) \leq K_T \left(\frac{1}{t}+1\right)$, where K_T is a finite constant. The relevance of this estimate comes from [3], where we proved that the constructed entropy weak solution mentioned above satisfies the estimate

$$\partial_x u(t,x) \leq \frac{1}{t} + \left[6\left\|u_0\right\|^2_{L^2(\mathbb{R})} + \frac{3}{2}\left(|u_0|_{BV(\mathbb{R})} + 24T\left\|u_0\right\|^2_{L^2(\mathbb{R})}\right)^2\right]^{1/2}, \qquad x \in \mathbb{R},$$

for each $t \in (0,T]$ with $T > 0$ (fixed). This estimate and the uniqueness result are to some extent in accordance with what we know for the inviscid Burgers equation, see for example [20, 22]. A chief difference is however that the right-hand side of the estimate depends on the total variation of the solution (which can be estimated in terms of the initial data as displayed in the inequality above), whereas for the inviscid Burgers equation the so-called Oleĭnik E-condition reads $\partial_x u(t,x) \leq 1/t$.

Before ending this introduction, we remind the reader of a couple of examples of hyperbolic-elliptic systems that bear some resemblance to the Degasperis-Procesi equation (1.4). The first system reads

$$\partial_t u + \partial_x\left(\frac{u^2}{2}\right) + \partial_x P = 0, \qquad -\partial^2_{xx} P + P = -\partial_x u, \tag{1.10}$$

and it serves as a simplified model for radiating gases, see for example [11, 13, 15, 21] for more details. Observe that while (1.4) can be viewed as a conservation law with a nonlocal convective flux, $\partial_t u + \partial_x\left[\frac{u^2}{2} + G_1 \star \left(\frac{3}{2}u^2\right)\right] = 0$, the radiating gas system can be viewed as a conservation law perturbed by a nonlocal diffusion flux, that is, (1.10) can be written as $\partial_t u + \partial_x\left(\frac{u^2}{2}\right) = G_1 \star (\partial^2_{xx} u)$ $(= G_1 \star u - u)$. Another related system is the Whitman model for shallow water waves [24], which reads

$$\partial_t u + \partial_x\left(\frac{u^2}{2}\right) + \partial_x P = 0, \qquad -\partial^2_{xx} P + P = u,$$

This system was analyzed recently in [9].

The remaining part of this paper is organized as follows: In Section 2 we state the uniqueness result (Theorem 2.1). The proof of this result is based on [20, 22] and uses a nonlocal adjoint problem, which is introduced in Section 3. We will not solve the adjoint problem with the method of characteristics, but rather the method of vanishing viscosity/smoothing of the coefficient, which is introduced and analyzed in Section 4. Finally, in Section 5 we conclude the proof of Theorem 2.1.

2. Statement of main result

In this section we state the uniqueness result. We start however with collecting the notions of weak and entropy weak solutions in a couple of definitions.

Definition 2.1 (Weak solution)**.** *We call a function $u : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ a weak solution of the initial value problem* (1.1), (1.2) *provided* *i)* $u \in L^\infty((0,T)\times\mathbb{R})$, *for any* $T>0$*;*

ii) *there holds the identity*

$$\int_{(0,\infty)}\int_{\mathbb{R}}\left(u\partial_t\phi+\frac{u^2}{2}\partial_x\phi-\partial_x P^u\phi\right)dx\,dt+\int_{\mathbb{R}}u_0(x)\phi(0,x)\,dx=0, \tag{2.1}$$

for every $\phi\in C^\infty([0,\infty)\times\mathbb{R})$ *with compact support, where*

$$P^u(t,x)=G_1\star\left(\frac{3}{2}u^2\right)(t,x)=\frac{3}{4}\int_{\mathbb{R}}e^{-|x-y|}(u(t,y))^2\,dy.$$

Remark 2.1. Due to *i)* we have $P^u\in L^\infty((0,T);W^{1,1}_{\mathrm{loc}}(\mathbb{R}))\cap L^\infty((0,T);W^{1,\infty}(\mathbb{R}))$ for any $T>0$, hence (2.1) makes sense.

By requiring the fulfillment of an estimate of Oleĭnik-type, we arrive at the notion of an entropy weak solution for the Degasperis-Procesi equation (see [3] for a notion based on Kružkov-type entropy inequalities).

Definition 2.2 (Entropy weak solution)**.** *We call a function* $u:(0,\infty)\times\mathbb{R}\to\mathbb{R}$ *an entropy weak solution of the Cauchy problem* (1.1), (1.2) *provided*

i) *u is a weak solution in the sense of Definition 2.1;*

ii) *for each $T>0$ there exists a positive constant K_T such that the estimate*

$$\frac{u(t,x)-u(t,y)}{x-y}\le K_T\left(\frac{1}{t}+1\right) \tag{2.2}$$

holds for any x, $y\in\mathbb{R}$, $x\neq y$, $0<t<T$.

Our main result is contained in

Theorem 2.1. *Suppose $u_0\in L^\infty(\mathbb{R})$. Then there exists at most one entropy weak solution to the initial value problem* (1.1), (1.2)*.*

Remark 2.2. The existence of an entropy weak solution is proved in [3].

3. The nonlocal adjoint problem

As a preliminary step in the proof of Theorem 2.1, let u, $\widetilde{u}$ be two entropy weak solutions of (1.1), (1.2). Then we have to prove

$$u=\widetilde{u}\text{ a.e. in }(0,\infty)\times\mathbb{R}. \tag{3.1}$$

Define

$$\omega:=u-\widetilde{u},\qquad b:=\frac{u+\widetilde{u}}{2}. \tag{3.2}$$

For later use, observe that the following estimates hold (cf. Definitions 2.1, 2.2):

$$\|b\|_{L^\infty((0,T)\times\mathbb{R})}\le\frac{\|u\|_{L^\infty((0,T)\times\mathbb{R})}+\|\widetilde{u}\|_{L^\infty((0,T)\times\mathbb{R})}}{2}=:\Lambda_T,\quad T>0, \tag{3.3}$$

$$\frac{b(t,x)-b(t,y)}{x-y}\le K_T\left(\frac{1}{t}+1\right),\quad x\neq y,\ 0<t<T,\ T>0, \tag{3.4}$$

where

$$K_T:=\frac{K_T^u+K_T^{\widetilde{u}}}{2}$$

and $K_T^u,K_T^{\widetilde{u}}$ denote the constants appearing in Definition 2.2 for $u,\widetilde{u}$, respectively.

By part $ii)$ of Definition 2.1 and $(u-\widetilde{u})|_{t=0}=0$, we see that (3.1) is equivalent to

$$\int_{(0,\infty)\times\mathbb{R}} \omega\psi\,dt\,dx=0, \qquad \forall\psi\in C_c^\infty((0,\infty)\times\mathbb{R}), \tag{3.5}$$

where C_c^∞ denotes the set of compactly supported C^∞ functions.

The starting point for proving (3.5) is a classical method that employs the adjoint problem, see [20] and [22, Theorem 16.10]. We derive our adjoint problem next.

Let $\varphi\in C_c^\infty((0,\infty)\times\mathbb{R})$. Since $u,\widetilde{u}$ satisfy (2.1),

$$\int_{(0,\infty)\times\mathbb{R}}\left[(u-\widetilde{u})\partial_t\varphi+\frac{u^2-\widetilde{u}^2}{2}\partial_x\varphi+\left(P^u-P^{\widetilde{u}}\right)\partial_x\varphi\right]dt\,dx=0. \tag{3.6}$$

Using the notations introduced in (3.2),

$$\int_0^\infty\int_{\mathbb{R}}\left[(u-\widetilde{u})\,\partial_t\varphi+\frac{u^2-\widetilde{u}^2}{2}\partial_x\varphi\right]dt\,dx=\int_0^\infty\int_{\mathbb{R}}\omega\left[\partial_t\varphi+b\partial_x\varphi\right]dt\,dx. \tag{3.7}$$

Next,

$$\begin{aligned}
&\int_{(0,\infty)\times\mathbb{R}}\left(P^u-P^{\widetilde{u}}\right)\partial_x\varphi\,dt\,dx\\
&=\frac{3}{4}\int_{(0,\infty)\times\mathbb{R}\times\mathbb{R}}e^{-|x-y|}\left[(u(t,y))^2-(\widetilde{u}(t,y))^2\right]\partial_x\varphi(t,x)\,dt\,dx\,dy\\
&=\frac{3}{2}\int_{(0,\infty)\times\mathbb{R}}\left(\int_{\mathbb{R}}e^{-|x-y|}\partial_x\varphi(t,x)dx\right)\omega(t,y)b(t,y)\,dt\,dy\\
&=3\int_{(0,\infty)\times\mathbb{R}}\omega b\partial_x\Phi\,dt\,dx,
\end{aligned} \tag{3.8}$$

where

$$\Phi(t,x):=\frac{1}{2}\int_{(0,\infty)\times\mathbb{R}}e^{-|x-y|}\varphi(t,y)\,dy,$$

that is, Φ is the unique solution of the elliptic equation

$$-\partial_{xx}^2\Phi+\Phi=\varphi. \tag{3.9}$$

In view of (3.7) and (3.8), we can rewrite (3.6) as

$$\int_{(0,\infty)\times\mathbb{R}}\omega\left[\partial_t\varphi+b\partial_x\varphi+3b\partial_x\Phi\right]dt\,dx=0. \tag{3.10}$$

Finally, fix $\psi\in C_c^\infty((0,\infty)\times\mathbb{R})$ and let $\tau>0$ be such that

$$\operatorname{supp}(\psi)\subset(0,\tau)\times\mathbb{R}. \tag{3.11}$$

Consider then the following linear hyperbolic-elliptic terminal value problem:

$$\begin{cases}\partial_t\varphi+b\partial_x\varphi+3b\partial_x\Phi=\psi, & (t,x)\in(0,\tau)\times\mathbb{R},\\ -\partial_{xx}^2\Phi+\Phi=\varphi, & (t,x)\in(0,\tau)\times\mathbb{R},\\ \varphi(\tau,x)=0, & x\in\mathbb{R}.\end{cases} \tag{3.12}$$

We coin (3.12) the *adjoint problem* associated with (1.4).

The idea is to solve (3.12) and then pass from (3.10) to (3.5). Unfortunately, due to the low regularity of the coefficient b, we cannot solve directly (3.12). Hence, we regularize the first equation by smoothing the coefficient b by convolution and adding an artificial viscosity term. The next section is devoted to studying this "approximate" adjoint problem.

Remark 3.1. The use of an adjoint problem to prove uniqueness is rather common in the context of first order conservation laws, see for example [20, 22, 23, 14].

4. The approximate nonlocal adjoint problem

Let $\{\rho_\varepsilon(t,x)\}_{\varepsilon>0}$ be a sequence of standard mollifiers. Define

$$b_\varepsilon := b \star \rho_\varepsilon, \qquad \varepsilon > 0.$$

Clearly, from (3.3) and (3.4),

$$b_\varepsilon \to b \quad \text{in } L^2((0,T)\times\mathbb{R}),\ T>0, \tag{4.1}$$

$$\|b_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \le \frac{\|u\|_{L^\infty((0,T)\times\mathbb{R})} + \|\widetilde{u}\|_{L^\infty((0,T)\times\mathbb{R})}}{2} =: \Lambda_T, \quad T,\, \varepsilon > 0, \tag{4.2}$$

$$\partial_x b_\varepsilon(t,x) \le K_T\left(\frac{1}{t}+1\right), \quad x\in\mathbb{R},\ 0<t<T,\ \varepsilon>0. \tag{4.3}$$

Now we approximate (3.12) with the following parabolic-elliptic terminal value problem:

$$\begin{cases} \partial_t\varphi + b_\varepsilon\partial_x\varphi + 3b_\varepsilon\partial_x\Phi = \psi - \varepsilon\partial_{xx}^2\varphi, & (t,x)\in(0,\tau)\times\mathbb{R}, \\ -\partial_{xx}^2\Phi + \Phi = \varphi, & (t,x)\in(0,\tau)\times\mathbb{R}, \\ \varphi(\tau,x) = 0, & x\in\mathbb{R}. \end{cases} \tag{4.4}$$

Arguing as in [2, Theorem 2.3] we obtain

Lemma 4.1. *Let $\varepsilon > 0$ and suppose $\psi \in C^\infty([0,\infty)\times\mathbb{R}) \cap C([0,\infty); H^1(\mathbb{R}))$ obeys (3.11). There exists a unique solution $\varphi \in C^\infty([0,\infty)\times\mathbb{R}) \cap C([0,\infty); H^3(\mathbb{R}))$ to the terminal value problem (4.4).*

Since we feel more comfortable with initial value problem problems, we define

$$v(t,x) := \varphi(\tau-t,x), \qquad Q(t,x) := \Phi(\tau-t,x), \tag{4.5}$$

$$\beta_\varepsilon(t,x) := b_\varepsilon(\tau-t,x), \qquad \widetilde{\psi}(t,x) := -\psi(\tau-t,x), \tag{4.6}$$

for $(t,x)\in(0,\tau)\times\mathbb{R}$. Due to Lemma 4.1, v is then the unique smooth solution of the initial value problem

$$\begin{cases} \partial_t v - \beta_\varepsilon\partial_x v - 3\beta_\varepsilon\partial_x Q = \widetilde{\psi} + \varepsilon\partial_{xx}^2 v, & (t,x)\in(0,\tau)\times\mathbb{R}, \\ -\partial_{xx}^2 Q + Q = v, & (t,x)\in(0,\tau)\times\mathbb{R}, \\ v(0,x) = 0, & x\in\mathbb{R}, \end{cases} \tag{4.7}$$

and, thanks to (4.2) and (4.3),

$$\|\beta_\varepsilon\|_{L^\infty((0,\tau)\times\mathbb{R})} \le \frac{\|u\|_{L^\infty((0,\tau)\times\mathbb{R})} + \|\widetilde{u}\|_{L^\infty((0,\tau)\times\mathbb{R})}}{2} =: \Lambda_\tau, \quad \varepsilon > 0, \tag{4.8}$$

$$\partial_x\beta_\varepsilon(t,x) \le K_\tau\left(\frac{1}{\tau-t}+1\right), \quad (t,x)\in(0,\tau)\times\mathbb{R}, \quad \varepsilon>0. \tag{4.9}$$

4.1. **A priori estimates.** The following estimates constitute the key to the success of our adjoint problem approach.

Lemma 4.2. *Let $\psi \in C^\infty([0,\infty)\times\mathbb{R})\cap C([0,\infty);H^1(\mathbb{R}))\cap L^\infty((0,\infty);H^1(\mathbb{R}))$ be a function satisfying* (3.11). *Then, using the notations introduced in* (4.5) *and* (4.6), *for each $\varepsilon > 0$ and $t \in (0,\tau)$*

$$\begin{aligned}&\|v(t,\cdot)\|^2_{H^1(\mathbb{R})} + 2\varepsilon \int_0^t \|\partial_x v(s,\cdot)\|^2_{H^1(\mathbb{R})}\,ds\\ &\qquad\le e^{C_\tau \tau}\left(\frac{\tau}{\tau - t}\right)^{C_\tau}\int_0^t \left\|\widetilde{\psi}(s,\cdot)\right\|^2_{H^1(\mathbb{R})}\,ds,\end{aligned}\tag{4.10}$$

$$\begin{aligned}&\|Q(t,\cdot)\|^2_{H^3(\mathbb{R})} + 2\varepsilon \int_0^t \|\partial_x Q(s,\cdot)\|^2_{H^3(\mathbb{R})}\,ds\\ &\qquad\le e^{C_\tau \tau}\left(\frac{\tau}{\tau - t}\right)^{C_\tau}\int_0^t \left\|\widetilde{\psi}(s,\cdot)\right\|^2_{H^1(\mathbb{R})}\,ds,\end{aligned}\tag{4.11}$$

where C_τ is a constant independent of ε but dependent of τ.

Proof. From (4.7), we get the following equation for Q:

$$\partial_t\left(Q - \partial^2_{xx}Q\right) - \beta_\varepsilon\left(4\partial_x Q - \partial^3_{xxx}Q\right) = \widetilde{\psi} + \varepsilon\partial^2_{xx}\left(Q - \partial^2_{xx}Q\right).\tag{4.12}$$

Multiplying (4.12) by $Q - \partial^2_{xx}Q$ and integrating on $\mathbb{R}$ we get

$$\begin{aligned}&\int_{\mathbb{R}} \partial_t\left(Q - \partial^2_{xx}Q\right)\left(Q - \partial^2_{xx}Q\right)dx - \varepsilon\int_{\mathbb{R}}\partial^2_{xx}\left(Q - \partial^2_{xx}Q\right)\left(Q - \partial^2_{xx}Q\right)dx\\ &= \int_{\mathbb{R}}\beta_\varepsilon\left(4\partial_x Q - \partial^3_{xxx}Q\right)\left(Q - \partial^2_{xx}Q\right)dx + \int_{\mathbb{R}}\widetilde{\psi}\left(Q - \partial^2_{xx}Q\right)dx.\end{aligned}\tag{4.13}$$

On the left-hand side we use the chain rule and do an integration by parts:

$$\begin{aligned}&\int_{\mathbb{R}} \partial_t\left(Q - \partial^2_{xx}Q\right)\left(Q - \partial^2_{xx}Q\right)dx - \varepsilon\int_{\mathbb{R}}\partial^2_{xx}\left(Q - \partial^2_{xx}Q\right)\left(Q - \partial^2_{xx}Q\right)dx\\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(Q - \partial^2_{xx}Q\right)^2 dx + \varepsilon\int_{\mathbb{R}}\left(\partial_x Q - \partial^3_{xxx}Q\right)^2 dx\\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left[Q^2 + 2(\partial_x Q)^2 + (\partial^2_{xx}Q)^2\right]dx\\ &\qquad + \varepsilon\int_{\mathbb{R}}\left[(\partial_x Q)^2 + 2(\partial^2_{xx}Q)^2 + (\partial^3_{xxx}Q)^2\right]dx.\end{aligned}\tag{4.14}$$

We estimate the right-hand side of (4.13) using (4.8):

$$\begin{aligned}&\left|\int_{\mathbb{R}}\beta_\varepsilon\left(4\partial_x Q - \partial^3_{xxx}Q\right)\left(Q - \partial^2_{xx}Q\right)dx + \int_{\mathbb{R}}\widetilde{\psi}\left(Q - \partial^2_{xx}Q\right)dx\right|\\ &\le \frac{1}{2}\int_{\mathbb{R}}\beta_\varepsilon^2\left(4\partial_x Q - \partial^3_{xxx}Q\right)^2 dx + \int_{\mathbb{R}}\left(Q - \partial^2_{xx}Q\right)^2 dx + \frac{1}{2}\int_{\mathbb{R}}\widetilde{\psi}^2\,dx\\ &\le L_{1,\tau}\int_{\mathbb{R}}\left[Q^2 + (\partial_x Q)^2 + (\partial^2_{xx}Q)^2 + (\partial^3_{xxx}Q)^2\right]dx + \frac{1}{2}\int_{\mathbb{R}}\widetilde{\psi}^2\,dx,\end{aligned}\tag{4.15}$$

where the constant $L_{1,\tau}$ is independent of ε. Hence, using (4.14) and (4.15) in (4.13),

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left[Q^2+2(\partial_x Q)^2+(\partial_{xx}^2 Q)^2\right]dx\\
&\qquad+\varepsilon\int_{\mathbb{R}}\left[(\partial_x Q)^2+2(\partial_{xx}^2 Q)^2+(\partial_{xxx}^3 Q)^2\right]dx\\
&\le L_{1,\tau}\int_{\mathbb{R}}\left[Q^2+(\partial_x Q)^2+(\partial_{xx}^2 Q)^2+(\partial_{xxx}^3 Q)^2\right]dx+\frac{1}{2}\int_{\mathbb{R}}\widetilde{\psi}^2\,dx.
\end{aligned}\tag{4.16}$$

We continue by x-differentiating (4.12) to obtain

$$\begin{aligned}
&\partial_t\left(\partial_x Q-\partial_{xxx}^3 Q\right)-\beta_\varepsilon\partial_x\left(4\partial_x Q-\partial_{xxx}^3 Q\right)\\
&\qquad-\partial_x\beta_\varepsilon\left(4\partial_x Q-\partial_{xxx}^3 Q\right)=\partial_x\widetilde{\psi}+\varepsilon\partial_{xx}^2\left(\partial_x Q-\partial_{xxx}^3 Q\right).
\end{aligned}\tag{4.17}$$

Multiplying (4.17) by $4\partial_x Q-\partial_{xxx}^3 Q$ and integrating on $\mathbb{R}$ we get

$$\begin{aligned}
&\int_{\mathbb{R}}\partial_t\left(\partial_x Q-\partial_{xxx}^3 Q\right)\left(4\partial_x Q-\partial_{xxx}^3 Q\right)dx\\
&\qquad-\varepsilon\int_{\mathbb{R}}\partial_{xx}^2\left(\partial_x Q-\partial_{xxx}^3 Q\right)\left(4\partial_x Q-\partial_{xxx}^3 Q\right)dx\\
&=\int_{\mathbb{R}}\beta_\varepsilon\partial_x\left(4\partial_x Q-\partial_{xxx}^3 Q\right)\left(4\partial_x Q-\partial_{xxx}^3 Q\right)dx\\
&\qquad+\int_{\mathbb{R}}\partial_x\beta_\varepsilon\left(4\partial_x Q-\partial_{xxx}^3 Q\right)^2dx+\int_{\mathbb{R}}\partial_x\widetilde{\psi}\left(4\partial_x Q-\partial_{xxx}^3 Q\right)dx.
\end{aligned}\tag{4.18}$$

On the left-hand side we integrate by parts to produce

$$\begin{aligned}
&\int_{\mathbb{R}}\left(\partial_{tx}^2 Q-\partial_{txxx}^4 Q\right)\left(4\partial_x Q-\partial_{xxx}^3 Q\right)dx\\
&\qquad-\varepsilon\int_{\mathbb{R}}\left(\partial_{xxx}^3 Q-\partial_{xxxxx}^5 Q\right)\left(4\partial_x Q-\partial_{xxx}^3 Q\right)dx\\
&=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left[4(\partial_x Q)^2+5(\partial_{xx}^2 Q)^2+(\partial_{xxx}^3 Q)^2\right]dx\\
&\qquad+\varepsilon\int_{\mathbb{R}}\left[4(\partial_{xx}^2 Q)^2+5(\partial_{xxx}^3 Q)^2+(\partial_{xxxx}^4 Q)^2\right]dx.
\end{aligned}\tag{4.19}$$

On the right-hand side of (4.18) we integrate by parts and use (4.9) to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}} \beta_\varepsilon \partial_x \left(4\partial_x Q - \partial^3_{xxx} Q\right)\left(4\partial_x Q - \partial^3_{xxx} Q\right) dx \\
&\qquad + \int_{\mathbb{R}} \partial_x \beta_\varepsilon \left(4\partial_x Q - \partial^3_{xxx} Q\right)^2 dx + \int_{\mathbb{R}} \partial_x \widetilde{\psi} \left(4\partial_x Q - \partial^3_{xxx} Q\right) dx \\
&= \frac{1}{2}\int_{\mathbb{R}} \partial_x \beta_\varepsilon \left(4\partial_x Q - \partial^3_{xxx} Q\right)^2 dx + \int_{\mathbb{R}} \partial_x \widetilde{\psi} \left(4\partial_x Q - \partial^3_{xxx} Q\right) dx \\
&\le \frac{K_\tau}{2}\left(\frac{1}{\tau - t} + 1\right)\int_{\mathbb{R}} \left(4\partial_x Q - \partial^3_{xxx} Q\right)^2 dx \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}} \left(\partial_x \widetilde{\psi}\right)^2 dx + \frac{1}{2}\int_{\mathbb{R}} \left(4\partial_x Q - \partial^3_{xxx} Q\right)^2 dx \\
&\le L_{2,\tau}\left(\frac{1}{\tau - t} + 1\right)\int_{\mathbb{R}} \left[(\partial_x Q)^2 + (\partial^2_{xx} Q)^2 + (\partial^3_{xxx} Q)^2\right] dx \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}} \left(\partial_x \widetilde{\psi}\right)^2 dx,
\end{aligned} \tag{4.20}
$$

where $L_{2,\tau}$ is a constant independent of ε.

In view of (4.19) and (4.20), it follows from (4.18) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \left[4(\partial_x Q)^2 + 5(\partial^2_{xx} Q)^2 + (\partial^3_{xxx} Q)^2\right] dx \\
&\qquad + \varepsilon \int_{\mathbb{R}} \left[4(\partial^2_{xx} Q)^2 + 5(\partial^3_{xxx} Q)^2 + (\partial^4_{xxxx} Q)^2\right] dx \\
&\le L_{2,\tau}\left(\frac{1}{\tau - t} + 1\right)\int_{\mathbb{R}} \left[(\partial_x Q)^2 + (\partial^2_{xx} Q)^2 + (\partial^3_{xxx} Q)^2\right] dx \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}} \left(\partial_x \widetilde{\psi}\right)^2 dx.
\end{aligned} \tag{4.21}
$$

Adding (4.16) and (4.21) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \left[Q^2 + 6(\partial_x Q)^2 + 6(\partial^2_{xx} Q)^2 + (\partial^3_{xxx} Q)^2\right] dx \\
&\qquad + \varepsilon \int_{\mathbb{R}} \left[(\partial_x Q)^2 + 6(\partial^2_{xx} Q)^2 + 6(\partial^3_{xxx} Q)^2 + (\partial^4_{xxxx} Q)^2\right] dx \\
&\le L_{1,\tau}\int_{\mathbb{R}} \left[Q^2 + (\partial_x Q)^2 + (\partial^2_{xx} Q)^2 + (\partial^3_{xxx} Q)^2\right] dx \\
&\qquad + L_{2,\tau}\left(\frac{1}{\tau - t} + 1\right)\int_{\mathbb{R}} \left[(\partial_x Q)^2 + (\partial^2_{xx} Q)^2 + (\partial^3_{xxx} Q)^2\right] dx \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}} \left[\widetilde{\psi}^2 + \left(\partial_x \widetilde{\psi}\right)^2\right] dx \\
&\le \frac{L_{3,\tau}}{2}\left(\frac{1}{\tau - t} + 1\right)\int_{\mathbb{R}} \left[Q^2 + 6(\partial_x Q)^2 + 6(\partial^2_{xx} Q)^2 + (\partial^3_{xxx} Q)^2\right] dx \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}} \left[\widetilde{\psi}^2 + \left(\partial_x \widetilde{\psi}\right)^2\right] dx,
\end{aligned} \tag{4.22}
$$

where $L_{3,\tau}$ is a constant independent of ε. Therefore, introducing the notation

$$
\|\phi\|_{\widetilde{H}^3(\mathbb{R})} := \sqrt{\phi^2 + 6(\partial_x \phi)^2 + 6(\partial^2_{xx} \phi)^2 + (\partial^3_{xxx} \phi)^2},
$$

we can rewrite (4.22) in the following way:

$$\begin{aligned}
&\frac{d}{dt}\left\|Q(t,\cdot)\right\|^2_{\widetilde{H}^3(\mathbb{R})}+2\varepsilon\left\|\partial_x Q(t,\cdot)\right\|^2_{\widetilde{H}^3(\mathbb{R})}\\
&\le L_{3,\tau}\left(\frac{1}{\tau-t}+1\right)\left\|Q(t,\cdot)\right\|^2_{\widetilde{H}^3(\mathbb{R})}+\left\|\widetilde{\psi}(t,\cdot)\right\|^2_{H^1(\mathbb{R})}.
\end{aligned}\tag{4.23}$$

Let $f(t)$ be a nonnegative, absolutely continuous function on $[a,b]$, satisfying for a.e. t the inequality

$$f'(t)+g(t)\le k(t)f(t)+h(t),$$

where $k(t),g(t),h(t)$ are nonnegative, integrable functions on $[a,b]$. Then Grönwall's inequality says that

$$f(t)+\int_a^b e^{\int_s^t k(s')\,ds'}g(s)\,ds\le e^{\int_a^t k(s)\,ds}\left[f(a)+\int_a^t h(s)\,ds\right],\quad a\le t\le b.$$

For (4.23), $k(t)=L_{3,\tau}\left(\frac{1}{\tau-t}+1\right)$ and thus $e^{\int_s^t k(s')\,ds'}=e^{L_{3,\tau}(t-s)}\left(\frac{\tau-s}{\tau-t}\right)^{L_{3,\tau}}$, so we obtain, keeping in mind that $Q(0,\cdot)=0$,

$$\begin{aligned}
&\left\|Q(t,\cdot)\right\|^2_{\widetilde{H}^3(\mathbb{R})}+2\varepsilon\int_0^t e^{L_{3,\tau}(t-s)}\left(\frac{\tau-s}{\tau-t}\right)^{L_{3,\tau}}\left\|\partial_x Q(s,\cdot)\right\|^2_{\widetilde{H}^3(\mathbb{R})}\,ds\\
&\le e^{L_{3,\tau}(t-s)}\left(\frac{\tau}{\tau-t}\right)^{L_{3,\tau}}\int_0^t\left\|\widetilde{\psi}(s,\cdot)\right\|^2_{H^1(\mathbb{R})}\,ds.
\end{aligned}\tag{4.24}$$

Finally, using the facts

$$\begin{aligned}
&\left\|\cdot\right\|_{\widetilde{H}^3(\mathbb{R})}\ge\left\|\cdot\right\|_{H^3(\mathbb{R})},\\
&1\le e^{L_{3,\tau}\tau}\left(\frac{\tau}{\tau-t}\right)^{L_{3,\tau}},\qquad 0\le s\le t<\tau,
\end{aligned}$$

we get, from (4.24),

$$\begin{aligned}
&\left\|Q(t,\cdot)\right\|^2_{H^3(\mathbb{R})}+2\varepsilon\int_0^t\left\|\partial_x Q(s,\cdot)\right\|^2_{H^3(\mathbb{R})}\,ds\\
&\qquad\le e^{L_{3,\tau}\tau}\left(\frac{\tau}{\tau-t}\right)^{L_{3,\tau}}\int_0^t\left\|\widetilde{\psi}(s,\cdot)\right\|^2_{H^1(\mathbb{R})}\,ds.
\end{aligned}$$

Hence, (4.11) is proved.

Since

$$v=Q-\partial^2_{xx}Q,\qquad \partial_x v=Q-\partial^3_{xxx}Q,$$

we have

$$\left\|v(t,\cdot)\right\|_{H^1(\mathbb{R})}\le\left\|Q(t,\cdot)\right\|_{H^3(\mathbb{R})},\quad \left\|\partial_x v(t,\cdot)\right\|_{H^1(\mathbb{R})}\le\left\|\partial_x Q(t,\cdot)\right\|_{H^3(\mathbb{R})},$$

so (4.10) is consequence of (4.11). □

Coming back to the terminal value problem, the previous results for the initial value problem translate into following ones for (4.4):

Corollary 4.1. *Let* $\psi \in C^\infty([0,\infty)\times\mathbb{R}) \cap C([0,\infty);H^1(\mathbb{R})) \cap L^\infty((0,\infty);H^1(\mathbb{R}))$ *be a function satisfying* (3.11). *Then for each* $\varepsilon > 0$ *and* $t \in (0,\tau)$

$$\begin{aligned}\|\varphi(t,\cdot)\|^2_{H^1(\mathbb{R})} &+ 2\varepsilon \int_t^\tau \|\partial_x \varphi(s,\cdot)\|^2_{H^1(\mathbb{R})}\,ds\\ &\le e^{C_\tau \tau}\left(\frac{\tau}{t}\right)^{C_\tau}\int_t^\tau \|\psi(s,\cdot)\|^2_{H^1(\mathbb{R})}\,ds,\end{aligned}\tag{4.25}$$

$$\begin{aligned}\|\Phi(t,\cdot)\|^2_{H^3(\mathbb{R})} &+ 2\varepsilon \int_t^\tau \|\partial_x \Phi(s,\cdot)\|^2_{H^3(\mathbb{R})}\,ds\\ &\le e^{C_\tau \tau}\left(\frac{\tau}{t}\right)^{C_\tau}\int_t^\tau \|\psi(s,\cdot)\|^2_{H^1(\mathbb{R})}\,ds,\end{aligned}\tag{4.26}$$

where C_τ *is the constant from Lemma 4.2.*

4.2. **Existence of solutions to the nonlocal adjoint problem.** Although we will not use this fact directly, an interesting consequence of the estimates from the previous subsection is the existence of a solution to (3.12).

Theorem 4.1. *Let* $\psi \in C^\infty([0,\infty)\times\mathbb{R}) \cap C([0,\infty);H^1(\mathbb{R})) \cap L^\infty((0,\infty);H^1(\mathbb{R}))$ *satisfy* (3.11)*, and fix any* $0 < \delta < \tau$. *Then there exists at least one distributional solution* $(\varphi,\Phi) \in L^\infty((\delta,\tau);H^1(\mathbb{R})) \times L^\infty((\delta,\tau);H^3(\mathbb{R}))$ *to the terminal value problem* (3.12).

Proof. For each fixed $\varepsilon > 0$, let $(\varphi_\varepsilon, \Phi_\varepsilon)$ denote the solution of (4.4). Due to Corollary 4.1,

$$\{\varphi_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty((\delta,\tau);H^1(\mathbb{R})), \text{ for } \delta \in (0,\tau),\tag{4.27}$$

$$\{\Phi_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty((\delta,\tau);H^3(\mathbb{R})), \text{ for } \delta \in (0,\tau).\tag{4.28}$$

Then there exist

$$\varphi \in L^\infty((\delta,\tau);H^1(\mathbb{R})),\quad \Phi \in L^\infty((\delta,\tau);H^3(\mathbb{R})),\quad 0<\delta<\tau,$$

and $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\varepsilon_k \to 0$, such that

$$\varphi_{\varepsilon_k} \rightharpoonup \varphi \quad \text{weakly in } L^p((\delta,\tau);H^1(R)), \text{ for } \delta \in (0,\tau),\ p \in (1,\infty),\tag{4.29}$$

$$\Phi_{\varepsilon_k} \rightharpoonup \Phi \quad \text{weakly in } L^p((\delta,\tau);H^3(R)), \text{ for } \delta \in (0,\tau),\ p \in (1,\infty).\tag{4.30}$$

It remains to verify that the limit pair (φ,Φ) is a solution of (3.12) in the sense of distributions. Fix any $\phi \in C_c^\infty((0,\tau)\times\mathbb{R})$. We need to show that

$$\int_0^\tau\int_{\mathbb{R}} \phi b_\varepsilon \partial_x \varphi_\varepsilon\,dt\,dx \to \int_0^\tau\int_{\mathbb{R}} \phi b \partial_x \varphi\,dt\,dx,\tag{4.31}$$

$$\int_0^\tau\int_{\mathbb{R}} \phi b_\varepsilon \partial_x \Phi_\varepsilon\,dt\,dx \to \int_0^\tau\int_{\mathbb{R}} \phi b \partial_x \Phi\,dt\,dx.\tag{4.32}$$

Observe that

$$\begin{aligned}&\int_0^\tau\int_{\mathbb{R}} \phi\left(b_\varepsilon\partial_x\varphi_\varepsilon - b\partial_x\varphi\right)\,dt\,dx\\ &\qquad = \int_0^\tau\int_{\mathbb{R}} \phi\left(b_\varepsilon - b\right)\partial_x\varphi_\varepsilon\,dt\,dx + \int_0^\tau\int_{\mathbb{R}} \phi b\left(\partial_x\varphi_\varepsilon - \partial_x\varphi\right)\,dt\,dx.\end{aligned}\tag{4.33}$$

Since ϕ has compact support in $(0,\tau)\times\mathbb{R}$, we can employ (4.1) and (4.27) to obtain

$$\begin{aligned}(4.34)\qquad &\int_0^\tau\int_{\mathbb{R}}\phi\,(b_\varepsilon-b)\,\partial_x\varphi_\varepsilon\,dt\,dx\\ &\qquad\le\|b_\varepsilon-b\|_{L^2((0,\tau)\times\mathbb{R})}\,\|\phi\|_{L^\infty((0,\tau)\times\mathbb{R})}\,\|\partial_x\varphi_\varepsilon\|_{L^\infty((\delta,\tau);L^2(\mathbb{R}))}\to 0.\end{aligned}$$

Moreover, since $\phi b\in L^2((0,\tau)\times\mathbb{R})$, from (4.29) it follows that

$$\int_0^\tau\int_{\mathbb{R}}\phi b\,(\partial_x\varphi_\varepsilon-\partial_x\varphi)\,dt\,dx\to 0. \tag{4.35}$$

Clearly, (4.33), (4.34), and (4.35) imply (4.31). Since for (4.32) we can use the same argument, the proof is completed. □

5. Proof of Theorem 2.1

In this final section we prove Theorem 2.1.

Proof of Theorem 2.1. We begin by fixing a test function $\psi\in C_c^\infty((0,\infty)\times\mathbb{R})$. Let $0<\tau_0<\tau_1$ be such that

$$\mathrm{supp}\,(\psi)\subset(\tau_0,\tau_1)\times\mathbb{R}. \tag{5.1}$$

From Lemma 4.1, for each $\varepsilon>0$ there exists a unique $\widetilde{\varphi}_\varepsilon\in C^\infty([0,\infty)\times\mathbb{R})\cap C([0,\infty);H^3(\mathbb{R}))$ solving (4.4). Let $\{\varphi_\varepsilon\}_\varepsilon\subset C_c^\infty((0,\tau_1)\times\mathbb{R})$ be such that

$$(5.2)\quad \varepsilon|\mathrm{supp}\,(\varphi_\varepsilon)|\to 0,$$

$$(5.3)\quad \widetilde{\varphi}_\varepsilon-\varphi_\varepsilon\to 0\quad\text{strongly in }\begin{cases}L^1((0,\infty);W^{2,1}(\mathbb{R}))\cap W^{1,1}((0,\infty)\times\mathbb{R})\cap\\ \cap\, W^{1,\infty}((0,\infty);H^1(\mathbb{R}))\cap L^\infty((0,\infty);H^3(\mathbb{R})),\end{cases}$$

and define the family $\{\psi_\varepsilon\}_\varepsilon$ as follows

$$\psi_\varepsilon:=\partial_t\varphi_\varepsilon+b_\varepsilon\partial_x\varphi_\varepsilon+\frac{3}{2}b_\varepsilon\int_{\mathbb{R}}e^{-|x-y|}\partial_x\varphi_\varepsilon(t,y)dy+\varepsilon\partial_{xx}^2\varphi_\varepsilon,\quad \varepsilon>0. \tag{5.4}$$

Clearly

$$\psi_\varepsilon\in C^\infty((0,\infty)\times\mathbb{R})\cap C([0,\infty);H^1(\mathbb{R})),\qquad \varepsilon>0, \tag{5.5}$$

and, due to (4.1), (4.2), (5.3),

$$\psi_\varepsilon\to\psi\ \text{ strongly in } L^1((0,\infty)\times\mathbb{R})\cap L^\infty((0,\infty);H^1(\mathbb{R})). \tag{5.6}$$

In particular, φ_ε and ψ_ε satisfy the two equations (see (4.4) and (5.4))

$$\partial_t\varphi_\varepsilon+b_\varepsilon\partial_x\varphi_\varepsilon+3b_\varepsilon\partial_x\Phi_\varepsilon=\psi_\varepsilon-\varepsilon\partial_{xx}^2\varphi_\varepsilon,\quad -\partial_{xx}^2\Phi_\varepsilon+\Phi_\varepsilon=\varphi_\varepsilon. \tag{5.7}$$

Hence, using (5.1) and (5.7),

$$
\begin{aligned}
\int_{(0,\infty)\times\mathbb{R}} \omega\psi\,dt\,dx &= \int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \omega\psi\,dt\,dx \\
&= \int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \omega\psi_\varepsilon\,dt\,dx + \int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \omega(\psi-\psi_\varepsilon)\,dt\,dx \\
&= \int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \omega\left(\partial_t\varphi_\varepsilon + b_\varepsilon\partial_x\varphi_\varepsilon + 3b_\varepsilon\partial_x\Phi_\varepsilon + \varepsilon\partial_{xx}^2\varphi_\varepsilon\right)\,dt\,dx \\
&\qquad + \int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \omega(\psi-\psi_\varepsilon)\,dt\,dx \\
&= \int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \omega\left(\partial_t\varphi_\varepsilon + b\partial_x\varphi_\varepsilon + 3b\partial_x\Phi_\varepsilon\right)\,dt\,dx \\
&\qquad + \varepsilon\int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \omega\partial_{xx}^2\varphi_\varepsilon\,dt\,dx + \int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \omega(b_\varepsilon-b)\partial_x\varphi_\varepsilon\,dt\,dx \\
&\qquad + 3\int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \omega(b_\varepsilon-b)\partial_x\Phi_\varepsilon\,dt\,dx + \int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \omega(\psi-\psi_\varepsilon)\,dt\,dx.
\end{aligned}
\tag{5.8}
$$

Using the fact that $\varphi_\varepsilon \in C_c^\infty((0,\infty)\times\mathbb{R})$ and (3.10), we find

$$
\int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \omega\left(\partial_t\varphi_\varepsilon + b\partial_x\varphi_\varepsilon + 3b\partial_x\Phi_\varepsilon\right)\,dt\,dx = 0. \tag{5.9}
$$

Employing (4.25), (5.2), part $i)$ of Definition 2.1, and Hölder's inequality, we can estimate as follows:

$$
\begin{aligned}
&\left|\varepsilon\int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \omega\partial_{xx}^2\varphi_\varepsilon\,dt\,dx\right| \le \varepsilon\left\|\omega\right\|_{L^\infty((\tau_0,\tau_1)\times\mathbb{R})}\left\|\partial_{xx}^2\varphi_\varepsilon\right\|_{L^1((\tau_0,\tau_1)\times\mathbb{R})} \\
&\le \varepsilon\left\|\omega\right\|_{L^\infty((\tau_0,\tau_1)\times\mathbb{R})}\sqrt{|\mathrm{supp}\,(\varphi_\varepsilon)|}\left\|\partial_{xx}^2\varphi_\varepsilon\right\|_{L^2((\tau_0,\tau_1)\times\mathbb{R})} \\
&\le \left(\frac{\varepsilon|\mathrm{supp}\,(\varphi_\varepsilon)|}{2}\right)^{\frac{1}{2}}\left\|\omega\right\|_{L^\infty((\tau_0,\tau_1)\times\mathbb{R})} \\
&\qquad \times e^{\frac{C_\mathcal{T}\tau}{2}}\left(\frac{\tau_1}{\tau_0}\right)^{\frac{C_\mathcal{T}}{2}}\left(\int_{\tau_0}^{\tau_1}\left\|\psi_\varepsilon(s,\cdot)\right\|_{H^1(\mathbb{R})}^2\,ds\right)^{\frac{1}{2}} \to 0.
\end{aligned}
\tag{5.10}
$$

By (4.25), (4.26), part $i)$ of Definition 2.1, (4.1) and the Hölder inequality,

$$
\begin{aligned}
&\left|\int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \omega\left(b-b_\varepsilon\right)\left(\partial_x\varphi_\varepsilon + 3\partial_x\Phi_\varepsilon\right)\,dt\,dx\right| \\
&\le \left\|\omega\right\|_{L^\infty((\tau_0,\tau_1)\times\mathbb{R})} \\
&\qquad \times\left\|b-b_\varepsilon\right\|_{L^2((\tau_0,\tau_1)\times\mathbb{R})}\left\|\partial_x\varphi_\varepsilon + 3\partial_x\Phi_\varepsilon\right\|_{L^2((\tau_0,\tau_1)\times\mathbb{R})} \\
&\le 4\left\|\omega\right\|_{L^\infty((\tau_0,\tau_1)\times\mathbb{R})}\sqrt{\tau_1-\tau_0}\left\|b-b_\varepsilon\right\|_{L^2((\tau_0,\tau_1)\times\mathbb{R})} \\
&\qquad \times e^{\frac{C_\mathcal{T}\tau}{2}}\left(\frac{\tau_1}{\tau_0}\right)^{\frac{C_\mathcal{T}}{2}}\left(\int_{\tau_0}^{\tau_1}\left\|\psi_\varepsilon(s,\cdot)\right\|_{H^1(\mathbb{R})}^2\,ds\right)^{\frac{1}{2}} \to 0.
\end{aligned}
\tag{5.11}
$$

Finally, from (5.6) and part $i)$ of Definition 2.1,

$$
\left|\int_{\tau_0}^{\tau_1}\int_{\mathbb{R}} \omega\left(\psi-\psi_\varepsilon\right)\,dt\,dx\right| \le \left\|\omega\right\|_{L^\infty((\tau_0,\tau_1)\times\mathbb{R})}\left\|\psi-\psi_\varepsilon\right\|_{L^1((0,\infty)\times\mathbb{R})} \to 0. \tag{5.12}
$$

Summarizing, using (5.9), (5.10), (5.11), (5.12) in (5.8) yields

$$\int_{(0,\infty)\times\mathbb{R}} \omega\psi\, dt\, dx = 0.$$

Due to the freedom in the choice of ψ, this implies (3.1), and the proof is completed. □

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