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Planes in Cubic 3-folds

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Abstract

We show that singular complex cubic 3-folds with only isolated singularities contain at most 15 planes, realized by the Clebsch-Segre cubic. We do this by using the classification of singular complex cubic surfaces and by counting lines and (possibly reducible) conics in a curve in $\mathbb{P}^3$ associated to singular complex cubic 3-fold with only isolated singularities.

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Chapter 1

Introduction

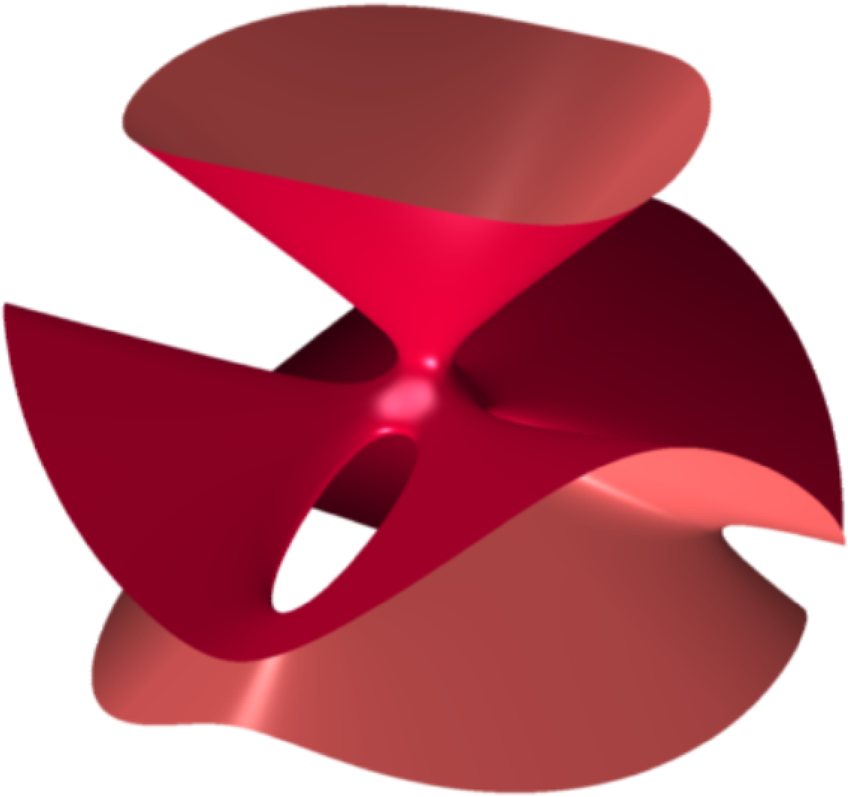

Figure 1.1: Visualization of the Clebsh Cubic by User: Fly by Night [Wik12]. This file is licensed under the Creative Commons Attribution-Share Alike 3.0 Unported license. To view a copy of this license, visit https://creativecommons.org/licenses/by-sa/3.0/deed.en. No changes made.

1.1 The Story So Far

Counting the number of lines in a surface is a classical problem in algebraic geometry. The most classical result about the number of lines on a surface is that any smooth complex cubic surface always contains exactly 27 lines, which was discovered by A. Cayley in 1849[Cay09], and by G. Salmon in 1915[Lon15]. A. Clebsch also produced a proof in 1861[Cle61].

If the complex cubic surface is singular, the maximal number of lines is 21. Moreover, there is a correspondence between types of singularities and the number of lines in a singular complex cubic surface. This correspondence was classified by L. Schläfli[Sch63] and A. Cayley[Cay69]. A more modern approach is given by J. W. Bruce and C. T. C. Wall in 1979, using Dynkin diagrams.[BW79]

For complex quartic surfaces the maximal number of lines is 64, which is given by the F. Schur quartic. This is a result by Segre, but his proof contains a mistake. But the idea by Segre was later used in [RS15] and [DIS17] to prove this fact. For a quintic surface, an upper bound of 127 was shown by S. Rams and M. Schütt[RS20]. But it is unknown if this bound is strict, although the Fermat quintic and the Barth quintic are both examples of quintic surfaces with 75 lines. This suggests moving up in degree leads to more lines. For a smooth complex surface of degree $d \geq 3$, T. Bauer and S. Rams gave an upper bound of $11d^2 - 32d + 24$ lines in 2020[BR20]. This is the best upper bound found so far for $d \geq 6$.

In 1972 C. H. Clemens and P. A. Griffiths used the intermediate Jacobian of a cubic 3-fold to show, among other results, that a general cubic 3-fold is irrational[CG72]. This included the study of lines on a cubic 3-fold. This gave another example of a varieties that was unirational, but not rational.

A new world of problems arises when the dimensions is increased. In 2023 A. Degtyarev, I. Itenberg and J. C. Ottem showed that for a complex smooth cubic 4-fold the maximal number of planes is 450. Moreover they proved that for cubic 4-folds with strictly more than 350 planes, there are only three projectively equivalent cubics.

Theorem 1.1. *Let $X \subset \mathbb{P}^5$ be a complex smooth cubic 4-fold with ≥ 351 planes, then X is projectively equivalent to one of the following:*

- *the Fermat cubic with 405 planes, or*
- *the Clebsch-Segre cubic with 357 planes, or*
- *the 351-cubic with 351 planes.*

Where the Fermat cubic is given by

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0, \tag{1.1}$$

the Clebsh-Segre cubic is given by

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 = x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0 \tag{1.2}$$

And the 351-cubic is given by

$$g(x_0, x_1, x_2) = g(x_3, x_4, x_5), \quad g(t_0, t_1, t_2) := t_0 t_1^2 - t_2^3 - t_0 t_2^2. \tag{1.3}$$

Moreover they gave an upper bound for singular complex 4-folds.

Proposition 1.2. *The number of planes contained in a nodal complex cubic 4-fold is at most 302.*

[DIO23]. In the nodal case they approach the problem by counting the maximal number of lines and conics in a $K3$ surface related to the nodal cubic.

A smooth complex cubic 3-fold contains no planes, but for a singular complex 3-fold a strict upper bound is not known in the literature. This is what motivates the goal of this thesis, which is to prove the following theorem.

Theorem 1.3. *Let $X \subset \mathbb{P}^4$ be a complex singular cubic 3-fold, then the maximal number of planes in X is 15. Moreover this bound is strict and is realized by the Clebsch-Segre cubic given by*

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 0 \tag{1.4}$$

1.2 Outline

In chapter two we will do most of the heavy lifting. To be precise, we will start with a short introduction to the classification of singular complex cubic surfaces. Singular cubic surfaces are fully classified with regards to types of singularities and the number of lines contained in them. This will later help us reduce the objects of study in this thesis to singular complex cubic 3-fold with only isolated singularities. Next we will prove that any complex cubic 3-fold X containing at least one plane must be singular. We then show that if X contains a singular line of multiplicity three, then X is a double cone. And if X is contains a singular line of multiplicity 2, then it must contain a finite number of planes, bounded by the maximal number of lines on a singular cubic surface. Moreover we show that if X is reducible, then it always contains an infinite number of planes. This motivates us to introduce normal the form for the equations for singular cubic 3-folds,

$$f = c(x_0, x_1, x_2, x_3) + q(x_0, x_1, x_2, x_3)x_4. \tag{1.5}$$

Any cubic 3-fold given by a polynomial on this form has a singularity at the point $O = (0 : 0 : 0 : 0 : 1)$, and by a linear change of coordinates we may always move a singular point to O. In particular $q(x_0, x_1, x_2, x_3)$ will enable us to reduce the main problem to three main cases by considering $Q = v(q) \subset \mathbb{P}^3$. The first case is when Q is reducible, so we may assume

$$q = x_0 x_1. \tag{1.6}$$

The second case is when Q is singular in a point, we may then assume

$$q = x_0 x_1 - x_2^2. \tag{1.7}$$

The third case is when Q is smooth, we may then assume

$$q = x_0 x_1 - x_2 x_3. \tag{1.8}$$

One of the main tools for studying planes in a cubic 3-fold will be by projecting planes in X down to $\mathbb{P}^3$. With this in mind we remind the reader of the blowup of $\mathbb{P}^4$ in a point. We then blowup a generic singular cubic 3-fold X in O, and show that $X - V(c, q)$ is in bijection to $\mathbb{P}^3 - V(c, q)$, so X is in fact rational. Next our goal is the find a one to one correspondence with planes in X and lines and (possibly reducible) conics in

$$C = V(c, q) \subset \mathbb{P}^3 \tag{1.9}$$

First we will divide planes contained in X into two classes, namely planes containing O, which we call type 1, and planes not containing O, which we call type 2. Then we will spend some time to show that lines in C is in bijection with planes of type 1. Moreover we show that under certain conditions (possibly reducible) conics in C is in bijection with planes of type 2. We are then done with the preliminaries.

In chapter three the goal is to use the classification of singular cubic surfaces and the corresponding lines to give a strict upper bound when $Q = V(q) \subset \mathbb{P}^3$ is reducible. We will do this by showing that there exits a linear hyperplane $H \subset \mathbb{P}^4$ containing the node $O = (0 : 0 : 0 : 0 : 1)$, such that intersecting H with X induces an injection of planes in X to lines in $S = X \cap H$. We then have a singular cubic surface in $\mathbb{P}^3$. To get an upper bound on the number of planes in X, we then show that S contains a singularity of type A_k with $k \geq 2$, which implies that S contains at most 15 lines, and thus X contains at most 15 planes.

In chapter four the goal is to show that five planes is a strict upper bound in the number of planes contained in any singular cubic 3-fold X when the defining polynomial of X is given by

$$f = c(x_0, x_1, x_2, x_3) + x_4(x_0x_1 - x_2^2). \tag{1.10}$$

We do this first by reducing this to six sub cases to check. We then use the exceptional locus $C = V(c, q)$ and the bijection between lines and (possibly reducible) conics to go through all the six cases and conclude that five planes is the maximal number.

In chapter five the goal is to show that if X is defined by a polynomial on the form

$$f = c(x_0, x_1, x_2, x_3) + x_4(x_0x_1 - x_2x_3), \tag{1.11}$$

then X contains at most 15 planes, and this bound is strict. Our first step here is to note that $Q = V(q) \subset \mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Then we note that $C = V(c, q)$ is isomorphic to some $(3, 3)$-curve for any C. This enables us enumerate the different configurations of $(1, 0)$-, $(0, 1)$- and $(1, 1)$-curves that a $(3, 3)$-curve in $\mathbb{P}^1 \times \mathbb{P}^1$ may contain. As $(1, 0)$- and $(0, 1)$-curves map to lines, $(1, 1)$-curves map to irreducible conics, and the union of a $(1, 0)$-curve and a $(0, 1)$-curve map to a reducible conic. We then enumerate them and show that the maximal number of planes contained in any X given by such an f is 15.

1.3 Notation

We let $\mathbb{P}^4$ have coordinates $(x_0 : x_1 : x_2 : x_3 : x_4)$, and likewise $\mathbb{P}^3$ has coordinates $(x_0 : x_1 : x_2 : x_3)$. We will denote coordinates in $\mathbb{P}^1 \times \mathbb{P}^1$ by $(u_0 : u_1, v_0 : v_1)$. From now on we always assume that a cubic 3-fold X is irreducible and only has isolated singularities unless explicitly stated otherwise. $C = V(c, q)$ will always mean a curve in $\mathbb{P}^3$. We try to strictly use capital letters for the zero locus of some ideal I, while lower case letters will be used to denote polynomials.

Chapter 2

Singular cubics

Cubic hypersurfaces in general, and cubic surfaces in particular are objects that have been of interest in algebraic geometry since its conception. Cubic surfaces on will be of use to us, especially singular cubic surfaces.

2.1 Singular cubic surfaces

In chapter 3 we will use the classification of singular cubic surfaces to show that the upper bound of planes contained in a cubic 3-fold on the form $V(c + x_4x_0x_1)$ is 15. Therefore we introduce some facts about singular cubic surfaces. The table bellow gives a correspondence of types of singularities on a singular cubic surface S and the number of lines contained in S. That the number of lines contained in a singular cubic surface S

Table 2.1: Lines on singular cubic surfaces

Singularity	A_1	$2A_1$	A_1A_2	$3A_1$	A_1A_3	$2A_1A_2$	$4A_1$	A_1A_4	$2A_1A_3$	A_12A_2
No. of lines	21	16	11	12	7	8	9	4	5	5

A_1A_5	A_2	$2A_2$	$3A_2$	A_3	A_4	A_5	D_4	D_5	E_1	$\tilde{E}_6$
2	15	7	3	10	6	3	6	3	1	∞

with at least one singularity of type A_k with $k \geq 2$ is at most 15 will be used in chapter 3. For now we just state that the only case when a singular cubic surface S contains an infinite number of lines is when it contains a singularity of type $\tilde{E}_6$. S is isomorphic to a surface $\tilde{S}$ defined by a polynomial on the form

$$f = x_1^2x_2 - x_0(x_0 - x_2)(x_0 - ax_2), \tag{2.1}$$

where $a \in \mathbb{C} - 0, 1$[SAK10]. In particular S is then a cone.

2.2 Singular cubic 3-folds

The goal of this section is to show what properties the cubic 3-fold we want to study, moreover we want justify why we omit studying any cubic 3-fold which is irreducible or with an infinite number of singularities.

The first observation we make is that any cubic 3-fold containing a plane is singular.

Proposition 2.1. *Let $X \subset \mathbb{P}^4$ be a cubic 3-fold, and let P be some plane such that $P \subset X$ then X contains at least four singular points.*

Proof. Let $X = V(f)$ be a cubic 3-fold and $P = V(l_1, l_2)$, where l_1 and l_2 are polynomials of degree 1. Then P is a plane in $\mathbb{P}^4$. Now assume $P \subset X$. Then by the correspondence $X \supset P \implies I(X) \subset I(P)$, we get that any cubic 3-fold that contains a plane can be written as the zero-set of an ideal on the form $l_1q_1 + l_2q_2$, where q_1, q_2 are quadratic polynomials. We use this to prove that any cubic 3-fold that contains at least one plane must be singular. Let X be a cubic 3-fold as above, then its Jacobian matrix is as follows

$$J = \begin{pmatrix} \frac{\partial l_1}{\partial x_0}q_1 + l_1\frac{\partial q_1}{\partial x_0} + \frac{\partial l_2}{\partial x_0}q_2 + l_2\frac{\partial q_2}{\partial x_0} \\ \frac{\partial l_1}{\partial x_1}q_1 + l_1\frac{\partial q_1}{\partial x_1} + \frac{\partial l_2}{\partial x_1}q_2 + l_2\frac{\partial q_2}{\partial x_1} \\ \frac{\partial l_1}{\partial x_2}q_1 + l_1\frac{\partial q_1}{\partial x_2} + \frac{\partial l_2}{\partial x_2}q_2 + l_2\frac{\partial q_2}{\partial x_2} \\ \frac{\partial l_1}{\partial x_3}q_1 + l_1\frac{\partial q_1}{\partial x_3} + \frac{\partial l_2}{\partial x_3}q_2 + l_2\frac{\partial q_2}{\partial x_3} \\ \frac{\partial l_1}{\partial x_4}q_1 + l_1\frac{\partial q_1}{\partial x_4} + \frac{\partial l_2}{\partial x_4}q_2 + l_2\frac{\partial q_2}{\partial x_4} \end{pmatrix}. \tag{2.2}$$

Observe that for $l_1 = l_2 = q_1 = q_2 = 0$ the rank of the Jacobian matrix drops. Now by Bezout's theorem $V(l_1, l_2, q_1, q_2)$ is four points, counting multiplicity, or infinite. In particular it is non-empty, so X is singular in at least four points. □

As a direct consequence of Proposition 2.1, we can pinpoint that the number of singularities in a plane contained in X is at least four.

Corollary 2.2. *Any plane P in X contains at least four singularities.*

Now as each cubic 3-fold must contain singularities, we introduce normal form for singular cubic 3-folds.

Definition 2.3. A singular cubic 3-fold is said to be on normal form if

$$f = c(x_0, x_1, x_2, x_3) - x_4q(x_0, x_1, x_2, x_3) \tag{2.3}$$

is the defining polynomial, where c is a cubic and q is a quadratic polynomial.

Now let O denote the point $(0 : 0 : 0 : 0 : 1) \in \mathbb{P}^4$. Next we prove a lemma showing that if f is a polynomial on normal form, and $X = V(f)$, then X has a singularity at O. We may then always assume a cubic 3-fold is the zero locus $V(f)$ of a polynomal on normal form, since we may move any singular pont to O by a linear change of coordinates.

Lemma 2.4. *A cubic 3-fold has a singularity at $O = (0 : 0 : 0 : 0 : 1)$ if and only if the defining polynomial f can be written as $f = c(x_0, x_1, x_2, x_3) - x_4q(x_0, x_1, x_2, x_3)$.*

Proof. First assume $f = c(x_0, x_1, x_2, x_3) - x_4q(x_0, x_1, x_2, x_3)$, and $X = V(f)$. Now the Jacobian matrix of f is:

$$J_f = \begin{pmatrix} \frac{\partial c}{\partial x_0} - \frac{\partial q}{\partial x_0}x_4 \\ \frac{\partial c}{\partial x_1} - \frac{\partial q}{\partial x_1}x_4 \\ \frac{\partial c}{\partial x_2} - \frac{\partial q}{\partial x_2}x_4 \\ \frac{\partial c}{\partial x_3} - \frac{\partial q}{\partial x_3}x_4 \\ q \end{pmatrix} \tag{2.4}$$

Let f_i denote the ith row of J_f, now since c is of degree 3, and q is of degree 2, $f_i(O) = 0$, for $i = 0, \ldots, 3$, and clearly $q(O) = 0$. Hence the rank of D_f is 0 at O, and thus X is singular at O. Next assume f is irreducible and not a cone, but that is has a defining polynomial not on normal form. This gives us two cases:

1. Assume
$$f = c(x_0, x_1, x_2, x_3) + l(x_0, x_1, x_2, x_3)x_4^2 \tag{2.5}$$
, where l is of degree 1, and c is of degree 3. Then for at least one i $f_i = \frac{\partial c}{\partial x_i} + \alpha x_4^2$, and thus $f_i(O) = \alpha$. It follows that the Jacobian matrix has rank 1 at O, implying X is non-singular at O.

2. Assume
$$f = c(x_0, x_1, x_2, x_3) + \alpha x_4^3, \tag{2.6}$$
then $f_4 = \frac{\partial c}{\partial x_4} + 3\alpha x_4^2 = 3\alpha x_4^2$. Thus $f_i(O) = 3\alpha$, so the Jacobian matrix has rank 1 at O, implying that X is non-singular at O.

□

From here on we now always assume that X is given by a polynomial f on normal form.

Now we give three examples to justify why we focus on singular complex 3-folds with only isolated singularities.

We begin with an example where X contains a singular line with multiplicity 3, X will then contain an infinite number of planes.

Example 2.5. Let $X \subset \mathbb{P}^4$ be a singular cubic 3-fold such that X contains a singular line of multiplicity 3. Then for any linear hyperplane $H \subset \mathbb{P}^4$ the cubic surface $S = X \cap H$ will be singular and contain a singularity of type $\tilde{E}_6$, which implies that X contains an infinite number of lines.

In the next example we justify why a cubic 3-fold singular in a line of multiplicity 2 will not be studied in this thesis.

Example 2.6. Assume $X \subset \mathbb{P}^4$ is a cubic 3-fold singular in a line of multiplicity 2. Then for any linear hyperplane $H \subset \mathbb{P}^4$, the surface $S \cap H$ wil be singular, and thus be bounded by the number of lines on singular cubic surfaces. The upper bound is then strict with 21 planes. An example of this is any cone with defining polynomial

$$f = (x_0 - ax_1)(-x_0 + (b+1)x_1 - bx_2)(x_1 - cx_2) - x_3(x_0x_2 - x_1^2), \tag{2.7}$$

where a, b, c are distinct elements of $\mathbb{C} - 0, 1$. As $V(f) \subset \mathbb{P}^3$ defines a singular cubic surface with one singularity of type A_1.

Lastly we show that any reducible cubic 3-fold contains an infinite number of planes, thus we will not include reducible cubics in our analysis.

Example 2.7. Assume $X \subset \mathbb{P}^4$ is a reducible cubic 3-fold. Then we may assume $X = V(x_0q)$ by a linear change of coordinates, where q is some quadratic polynomial. Then X contains a $\mathbb{P}^3$ thus and infinite number of lines.

From here on we then assume X contains only isolated singularities and is irreducible unless otherwise stated.

2.3 Projection and blowups

We will study cubic 3-folds by projecting them down to $\mathbb{P}^3$ and observe some properties of this map. The set in $\mathbb{P}^3$ where the projection is not bijective will play a crucial role. We therefore introduce the blowup as it is a handy way to think about projection from a point. Moreover, any birational map factors through a blowup, and we will see that any cubic 3-fold X with only isolated singularities is in fact birational to $\mathbb{P}^3$, that is X is rational. Before we blowup of X at O as a way to study the projection from O, let us recall the definition of the blowup at a point.

Definition 2.8. Let $\mathbb{P}^4$ be projective 4 space, and O be the point $(0:0:0:0:1)$, the blowup $(\mathrm{Bl}_O(\mathbb{P}^4))$ of $\mathbb{P}^4$ at O is then defined to be the zero set of the minors of the two by two matrix

$$A = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ u_0 & u_1 & u_2 & u_3 \end{pmatrix} \tag{2.8}$$

inside $P^4 \times \mathbb{P}^3$

Now let $\widetilde{X} = \mathrm{Bl}_O(\mathbb{P}^3)$, and let π_1 be the map that sends $(x_0 : \dots : x_4) \times (u_0 : \dots : u_3)$ to $(x_0 : \dots : x_4)$. Now $\pi_1^{-1}(O) = E$ is the exceptional divisor of $\widetilde{X}$. The morphism π_1 is bijective outside of E, in other words, π_1 is a birational morphism. Next we look at the morphism π_2 from $\tilde{X}$ to $\mathbb{P}^3$, where we forget the first five coordinates, that is we map $(x_0 : \dots : x_4) \times (u_0 : \dots : u_3)$ to $(u_0 : \dots : u_3)$. Now for a point $p_1 \in \mathbb{P}^3$ we get that $\pi_2^{-1}(p_1)$ are points $(p : x_4) \times (p) \cup (0:0:0:0:1) \times (p)$ by abuse of notation, where $x_4 \neq 0$ but is otherwise free. Thus $\pi_1(\pi_2^{-1}(p_1))$ is a line L through O. Now for any point $p_2 \in L$ not equal to O $\pi_2(\pi_1^{-1}(p_2)) = p_1$. Thus $\pi = \pi_2 \circ \pi_1^{-1}$ corresponds to projecting from $\mathbb{P}^4$ to $\mathbb{P}^3$. As we now have a grasp of how the blowup can be considered a projection, we now want to realize the blowup of a singular cubic, and with the tools at our disposal find the open set where X is birational to $\mathbb{P}^3$.

For a cubic 3-fold on the form $X = V(f)$, where $f = c(x_0, x_1, x_2, x_3) - x_4 q(x_0, x_1, x_2, x_3)$, we will call

$$C = V(c, q) \subset \mathbb{P}^3 \tag{2.9}$$

the exceptional locus. The point of this is that $x_4 = \frac{c}{q}$, and thus we get a one to one morphism from $\mathbb{P}^3 - V(c,q)$ to $X - V(c,q)$, in other words a birational map. That is the map where we send $(x_0 : \dots : x_3)$ to $(qx_0 : \dots : qx_3 : c)$. The exceptional locus will be of great use to us, as we will see later.

Next we prove that any singularity on a cubic 3-fold not equal to O, maps to a singular point on the exceptional locus $C = V(c,q)$.

Lemma 2.9. *Assume $p \in X$ is a singular point, not equal to the point O which we project from. Then any line l from O to p is contained in $S = V(c,q) \subset X \subset \mathbb{P}^4$. Moreover $\pi(p) \in C = V(c,q) \subset \mathbb{P}^3$ is a singular point of C.*

Proof. First assume we project from $O = (0:0:0:0:1)$, and write the equation for X as $f = c - x_4 q$. Then the Jacobian is

$$J_X = \begin{pmatrix} \frac{\partial c}{\partial x_0} - x_4 \frac{\partial q}{\partial x_0} \\ \cdots \\ \frac{\partial c}{\partial x_3} - x_4 \frac{\partial q}{\partial x_3} \\ q \end{pmatrix} \tag{2.10}$$

By a linear change of coordinates, we may assume that $p \in D_+(x_4)$. So as J_X has rank 0 at p, we have that $\frac{\partial c}{\partial x_i} = \frac{\partial q}{\partial x_i}$ for $i = 0 \ldots 3$. We also have that $q(p) = 0$, which implies $c(p) = 0$. So $\pi(p) \in C = V(c, q) \in \mathbb{P}^3$.
The Jacobian of C is

$$J_C = \begin{pmatrix} \frac{\partial c}{\partial x_0} & \frac{\partial q}{\partial x_0} \\ \vdots & \vdots \\ \frac{\partial c}{\partial x_3} & \frac{\partial q}{\partial x_3} \end{pmatrix} \tag{2.11}$$

but, by the above $\frac{\partial c}{\partial x_i} = \frac{\partial q}{\partial x_i}$ at p, so the rank of J_C is strictly less than 2, and hence C is singular at $\pi(p)$. □

2.4 Counting planes by counting lines and conics

We have narrowed down our objects of interest to cubic 3-folds with only isolated singularities, and introduced projecting from O as one tool in our toolbox. We now want to show that by projecting X down to $\mathbb{P}^3$ we may count planes contained in X by counting lines and (possibly reducible) conics in the exceptional locus $C = V(c, q)$. To do this it will be advantageous to split planes contained in a cubic 3-fold into two types. So we introduce two definitions:

Definition 2.10. A Plane P that is contained in a cubic 3-fold X is said to be of type 1 if $O \in P$.

Definition 2.11. A plane P that i contained in a cubic 3-fold X is said to be of type 2 if $O \notin P$:

Next we give an example of a cubic 3-fold that contains one plane of type 1.

Example 2.12. Let $c = x_0^3 + x_1^3 + x_1x_2x_3$, and $q = x_0(x_1 + x_2 + x_3)$. Now let $f = c - x_4q$, and $X = V(f)$.Then X contains one plane $V(x_0, x_1)$, which does not contain O. Observe that by rewriting f as $f = x_0(x_0^2 + (x_4x_1 + x_2 + x_3)) + x_1(x_1^2 + x_2x_3)$, we see clearly that $(f) \in (x_0, x_1)$. Which gives that $V(f) \supset V(x_0, x_1)$.

We also give an example of a cubic 3-fold containing one plane of type 2.

Example 2.13. Let $c = (x_0 - x_1)(x_1^2 + x_2^2 + x_3^2)$, and $q = x_0x_1 - x_2x_3$. Now let $f = c - x_4q$, and $X = V(f)$. Now $(f) \subset (x_0 - x_1, x_4)$, so $P = V(x_0 - x_1, x_4) \subset X = V(f)$. Moreover $O \notin P$.

For the rest if this chapter the goal is to show that planes of type 1 is in bijection with lines in C, and that planes of type 2 is in bijection with (possibly reducible) conics in C. The first step is to prove that for any plane P of type 1 contained in X, $\pi(P)$ is a unique line in $C = V(c, q) \subset \mathbb{P}^3$.

Lemma 2.14. *For any plane P of type 1 , $\pi(P)$ as a unique line in $C = V(c, q)$.*

Proof. Let $P = V(l_1(x_0, x_1, x_2, x_3), l_2(x_0, x_1, x_2, x_3))$, where l_i is of degree 1. By assumption $P \subset X$, thus we may write $X = V(l_1q_1 + l_2q_2)$. But we may write the equation of X as $c + x_4q$, thus we may split q_1 and q_2 into $k_i(x_0, x_1, x_2, x_3) + x_4e_i(x_0, x_1, x_2, x_3)$, where e_i is of degree 1 and k_i is of degree 2. We rewrite this as $X = V(l_1k_1 + l_2k_2 + l_1x_4e_1 + l_2x_4e_2)$, and note that $l_1k_1 + l_2k_2 = c$, and $l_1e_1 + l_2e_2 = q$. This implies that $P \subset S = V(c, q) \subset \mathbb{P}^4$. Thus any line $L \subset P$ contracts to a point when projected to $\mathbb{P}^3$, which implies P contracts to a line when projected to $\mathbb{P}^3$. To see that this line is unique assume that there exists two distinct planes P_1 and P_2 of type 1. Now $\pi(P_1) = \pi(P_2)$ implies that any line in P_1 is also a line in P_2, but then they must be the same plane, contradicting that they are distinct. $\square$

The next lemma will show that given a line in $C = V(c, q)$ there exists a unique plane of type 1 contained in X. Thus lines in $C = V(c, q)$ is in bijection with planes in X.

Lemma 2.15. *Given a line $L \subset C = V(c, q) \subset \mathbb{P}^3$, there is a unique plane $P \subset X \subset \mathbb{P}^4$ of type 1.*

Proof. Assume $L = V(l_1, l_2)$ is a line contained in C. Then $c = l_1k_1 + l_2k_2$ for some quadratic polynomials $k_i(x_0, x_1, x_2, x_3)$ for $i = 1, 2$ and $q = l_1e_1 + l_2e_2$ for some linear polynomials $e_i(x_0, x_1, x_2, x_3)$ for $i = 0, 1$. As $X = V(f)$, where $f = c - x_4q$ and $(f) \subset (l_1, l_2)$, which implies $V(l_1, l_2) \subset V(F) = X$. $\square$

This leads immediately to the following proposition:

Proposition 2.16. *Any plane of type* 1 *is in one to one correspondence with lines in $C = V(c, q)$.*

Our next goal is to show that planes of type 2 is in bijection with (possibly reducible) conics in $C = V(c, q)$, under the condition that if $K \subset C$ is a (possibly reduced) conic, then K intersects no other line in C in more than one point counting multiplicity. The first step on the way will be to show that a plane of type 2 intersects C in a (possibly reducible) conic when projected down to $\mathbb{P}^3$.

Lemma 2.17. *For any plane P of type* 2 *in X, the intersection of $\pi(P)$ and $C = V(c, q)$ is a (possibly reducible) conic. To be clear $\pi(P) \cap C = V(l, \hat{(q)})$, where l is a polynomial of degree one, and q is a polynomial of degree 2.*

Proof. Let $l_1(x_0, x_1, x_2, x_3)$ and $l_2(x_0, x_1, x_2, x_3)$ be distinct linear polynomial. But l_2 may be identically zero, then $P = V(l_1, l_2 + x_4)$ is a plane of type 2. We may write $X = V(l_1k_1 + (l_2 + x_4)k_2)$, for some polynomial $k_i(x_0, x_1, x_2, x_3)$, $i = 0, 1$, of degree 2. Now $c = l_1k_1 + l_2k_2$, and q is simply equal to k_2. We project this plane down, and obtain $\pi(V(l_1, l_2 + x_4)) = V(l_1)$, and intersect this with $V(c, q)$. But we observed above that $c = l_1k_2 + l_2q$, so $V(l_1, q, l_1k_1 + l_2q) = V(l_1, q)$ which is a (possibly reducible) conic, in $\mathbb{P}^3$, which completes the proof. $\square$

Next we show that if $K \subset C$ is a (possibly reducible) conic such that for any line $L \subset C$ that is not contained K, the intersection $K \cap L$ is at most a point counting multiplicity, there is a corresponding plane of type 2 in X.

Lemma 2.18. *If $K = V(l_1, q_1) \subset C$, where l is a polynomial of degree one, and q is a polynomial of degree 2, such that $K.L \leq 1$ for any line L in C not contained in K, then there exists a unique corresponding plane $P \subset X$, of type 2.*

Proof. Assume $K = V(l_1, k_1) \subset C \subset \mathbb{P}^3$ where $l_1(x_0, x_1, x_2, x_3)$ is some linear polynomial, and $k_1(x_0, x_1, x_2, x_3)$ is a quadratic polynomial, in other words K is a (possibly reducible) conic. Moreover assume $K.L \leq 1$ for any line L in C not contained in K. Now $K \subset C = V(c, q)$ implies that $c = l_1k_2 + k_1l_2$ and and $q = l_1l_3 + k_1$, for some quadratic polynomial $k_2(x_0, x_1, x_2, x_3)$ and linear polynomial $l_2(x_0, x_1, x_2, x_3)$. But, this implies that

$$f = l_1k_2 + k_1l_2 + x_4(l_1l_3 + k_1) = l_1(k_2 + x_4l_3) + (l_2 + x_4)k_1. \tag{2.12}$$

$V(l_1, l_2 + x_4) \subset X$ is a plane of type 2 in X as O clearly is not a point in P. □

Remark. The assumption that $K.L \leq 1$ is necessary here. If $K.L = 2$, then K and L lie in the same plane $V(l_1) \subset \mathbb{P}^3$. Assume $K = V(l_1, k_1)$, $L = V(l_1, l_2)$ and $K.L = 2$, then $V(l_1) \cap V(c, q) = V(l_1, l_2k_1)$. But this implies that

$$c = l_1k_2 + l_2k_1, \tag{2.13}$$

and that

$$q = l_1l_2. \tag{2.14}$$

Thus

$$f = l_1l_2 + l_2k_1 + x_4l_1l_3. \tag{2.15}$$

For X to contain a P plane of type 2 such that $\pi(P) \cap C = K$, we must have $k_1 = l_1l_3$, but then X is not irreducible.

Next we want to make sure that no two planes of type 2 project down to the same plane, and thus intersect in the same conic.

Lemma 2.19. *Assume X is a cubic 3-fold that is irreducible, then no two planes of type 2 project down to the same plane in $\mathbb{P}^3$.*

Proof. For contradiction, assume two planes of type 2, say P_1 and P_2 in X both project down to the same plane in $\mathbb{P}^3$. Then as our projection point O has multiplicity 2, and both planes project to the same plane, any line from O to any point on P_1 also intersects P_2. By Bézout's theorem the entire line is contained in X. But this amounts to X containing a $\mathbb{P}^3$, then X is not irreducible, contradicting our assumption. □

Combining Lemmas 2.17 to 2.19 we now have a one to one correspondence between (possibly reducible) conics in $C = V(c, q)$ and planes of type 2.

Proposition 2.20. *Planes of type 2 is in one to one correspondence with (possibly reducible) conics K such that for any $L \subset C$ not contained in K, $K.L \leq 1$.*

We have now introduced most of the heavy machinery we need to prove the main goal of this thesis.

Chapter 3

The maximal number of planes in X when $V(q) \subset \mathbb{P}^3$ is reducible

In this chapter we will use the classification of cubic singular surfaces in $\mathbb{P}^3$ to give an upper bound on the number of planes a cubic 3-fold X may have if $X = V(c + x_4 q)$, where $V(q) \subset \mathbb{P}^3$ is the union of two planes. Now if $V(q)$ is the union of two planes, we may by a linear change of coordinates assume $q = x_0 x_1$.

Cubic singular surfaces in $\mathbb{P}^4$ have been classified in families where types of singularity correspond to the number of lines on said cubic singular surface. In particular we will show that if $X = V(c + x_4 x_0 x_1)$, then there exits a general enough linear hyperplane H such that $X \cap H$ induces an injection from planes to lines. Then using Lemma 3 in [BW79], we will be able to deduce that X has at most 15 planes, giving us the following theorem.

Theorem 3.1. *Let $f = c + x_4 x_0 x_1$, then $V(f)$ contains at most 15 planes.*

3.1 Intersecting planes to unique lines

The goal of this section is to show that we may always find a linear hyperplane H that contains O, and such that if P_i is any finite collection of planes in $\mathbb{P}^4$ then $H \cap P_i$ is a line for all i and $H \cap P_i \neq H \cap P_j$ for $i \neq j$.

We start by setting up a so that if there exits a linear hyperplane H satisfying the two conditions, it will induce and injection from a finite number of planes in $\mathbb{P}^4$ to lines in $\mathbb{P}^3$.

Proposition 3.2. *Let H be a linear hyperplane, $\mathfrak{P} = \cup_0^n P_i$ any finite union of planes in $\mathbb{P}^4$, and assume the following holds:*

1. *For all $P_i \subset \mathfrak{P}$ in $\mathbb{P}^4$, $H \cap P_i$ is a line.*

2. *For all $P_i, P_j \subset \mathfrak{P}$ with $i \neq j$ in $\mathbb{P}^4$, $H \cap P_i \neq H \cap P_j$.*

Then $H \cap \mathfrak{P}$ is is an injection of planes $\mathfrak{P}$ to lines in $\mathbb{P}^3$.

Proof. As $H \simeq \mathbb{P}^3$, now if H satisfies both conditions, any plane in $\mathfrak{P}$ intersects down to a unique line in $\mathbb{P}^3$. $\square$

A priori we do not know if we may find a hyperplane satisfying those conditions. To show that there does exits such a hyperplane, we will show that finding such a linear hyperplane H is equivalent to finding a linear hyperplane H such that for any finite collection of lines $\mathfrak{L} = \cup_i^n L_i$, $H \cap L_i$ is a point for all i.

First we show that condition 1. in Proposition 3.2 is may be replaces by finding a line L such that $H \cap L$ is a point.

Lemma 3.3. *Let P be a plane, and H some a hypersurface in $\mathbb{P}^4$, then $P \cap H$ is a line if and only if there exists a line $L \subset P$ such that $H \cap L$ is a point.*

Proof. First assume $P \cap H$ is a line L_1. Now pick any other line L_2 contained in P. As $P \simeq \mathbb{P}^2$, $L_2 \cap L_1$ is a point. $H \cap L_2$ must be a point.

For the other implication, assume for contradiction that $P \subset H$, then clearly for any line $L \subset P$, $H \cap L = L$. □

Next we want to show that condition 2. in Proposition 3.2 may also be replaced by requiring H intersecting a given line only in a point. We will do this in two steps.

First we show that if we have two planes P_1 and P_2 where $P_1 \cap \mathbb{P}_2$ is either a point or empty, then $H \cap P_1 \neq H \cap P_2$, for any H. This will show that for any pair of planes that intersect in a point or is empty, we now know that we get injectivity for free.

Lemma 3.4. *Assume P_1 and P_2 are two planes in $\mathbb{P}^4$ such that $P_1 \cap P_2$ is either a point or empty. Then $H \cap P_1 \neq H \cap P_2$.*

Proof. Assume P_1 and P_2 are two planes in $\mathbb{P}^4$ such that $P_1 \cap P_2$ is either a point or empty. Now $H \cap P_i$ is either a plane or a line. If $H \cap P_1 = H \cap P_2$, then $P_1 \cap P_2 \cap H$ is either a plane or a line. But $\mathbb{P}_1 \cap P_2$ is assumed to be either a point or empty, so $H \cap P_1 \neq H \cap P_2$. □

What remains to show is that if two planes intersect in a line, we can to pick a hyperplane that does not intersect both planes in the same line. In line with the previous two lemmas we show that this may also be replaced by finding a linear hypersurface H that intersects a given line in only a point.

Lemma 3.5. *Let P_1 and P_2 be planes in $\mathbb{P}^4$ such that $P_1 \cap P_2$ is a line L. Moreover let H be a linear hypersurface in $\mathbb{P}^4$, and assume $P_1, P_2 \not{s}ubset H$. Then $P_1 \cap H \neq P_2 \cap H$ if and only if $P_1 \cap P_2 = L \not\subset H$.*

Proof. For one implication assume $P_1 \cap H \neq P_2 \cap H$, then $(P_1 \cap H)$ and $(P_2 \cap H)$ are two different lines. Moreover $(P_1 \cap H) \cap (P_2 \cap H) = (P_1 \cap P_2) \cap H = L \cap H$ is a point, so $L \not\subset H$.

For the other implication assume $P_1 \cap P_2 = L \not\subset H$. Then $L \cap H$ is a point, which implies that $P_1 \cap H \neq P_2 \cap H$. □

The two conditions on H in Proposition 3.2 now reduces to finding a linear hypersurface H such that for a finite collection of lines in $\mathbb{P}^4$, each line intersects H only in a point.

Our next step is to show that there is an open set in the dual space $(\mathbb{P}^4)^\vee$, where each point corresponds to a hypersurface in $\mathbb{P}^4$ which satisfies this. Above we mentioned that we want a linear hypersurface H such that $O \in H$. We want this to ensure that $X \cap H$ is a singular surface, otherwise it will be smooth, and thus have 27 lines.

Now each point p in P^4 has a corresponding hypersurface $H_p^\vee \subset (\mathbb{P}^4)^\vee$, and for any point $p^\vee \in H_p^\vee$, there is a corresponding hypersurface H_p that contains p. In other words

$H_p^\vee$ parametrizes linear hypersurfaces in $\mathbb{P}^4$ that contain p. We want to make this explicit. $(\mathbb{P}^4)^\vee$ is isomorphic to $\mathbb{P}^4$, so to keep track of where we are let $(\mathbb{P}^4)^\vee$ have coordinates $(y_0 : y_1 : y_2 : y_3 : y_4)$. Now let $p = (a_0 : a_1 : a_2 : a_3 : a_4) \in \mathbb{P}^4$ be a point. Then $H_p^\vee \subset (\mathbb{P}^4)^\vee$ is given by $V(a_0y_0 + a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4) \subset (\mathbb{P}^4)^\vee$. Now pick any point $p^\vee \in H_p$, say $p^\vee = (b_0 : b_1 : b_2 : b_3 : b_4)$. Now $H_p = V(b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4)$, and by construction it contains p.

Conversely if H_p parametrizes hypersurfaces containing $p \in \mathbb{P}^4$, then $(\mathbb{P}^4)^\vee - H_p$ parametrizes hypersurfaces not containing p. We are now ready to show that we may always find some hypersurface H, so that H intersects each line in a finite union of lines only in a point. We will then have shown that we may always find a hypersurface H, such that if X contains a finite number of planes, $H \cap X$ is a singular surface, and each plane intersects down to a unique line.

Proposition 3.6. *Let p be a point, and $\mathfrak{P} = \cup_{i=0}^n P$ be a finite union of lines in $\mathbb{P}^4$, then there exits a linear hypersurface H such that*

1. *$p \in H$.*

2. *For each $i \neq j$ $H \cap P_i \neq H \cap P_j$, moreover $P_i \not\subset H$, for any i.*

Proof. To satisfy 1 we $p \in H$ for for any point $p_h \in H_p \subset V(p^\vee) \subset \mathbb{P}^4$. We have already shown that 2 is equivalent to finding an H such that for any finite union of lines, there is a point on each line not contained in H. Now points corresponds to hypersurfaces in $(\mathbb{P}^4)^\vee$, and so we need to argue that for $n \in \mathbb{N}$,

$$\mathfrak{H} := ((\mathbb{P}^4)^\vee - \cup_{i=0}^n H_{P_i}) \cap H_p \tag{3.1}$$

is non-empty. But this is

$$H_p - \cup_{i=0}^n H_{P_i} \cap H_p. \tag{3.2}$$

Now H_p is isomorphic to $\mathbb{P}^3$, and $H_{P_i} \cap H_p$ is isomorphic to $\mathbb{P}^2$, unless $H_{P_i} = H_p$, but we are free to choose any point on each line, so we may choose a point different from p. Therefore we have a three dimensional space where we cut out a finite union of two dimensional linear hypersurfaces, so by dimension $\mathfrak{H}$ is non empty. □

Remark. $\mathfrak{H}$ is in fact an open subset of $V(H_p^\vee)$. We will use this at the end of the chapter to prove Theorem 3.1 at the end of the chapter.

3.2 Counting lines by way of singularities

Now that we have shown that for any collection of planes in $\mathbb{P}^4$ we may always find a linear hypersurface H, such that intersecting H with any finite collection of planes in $\mathbb{P}^4$, each plane intersects uniquely down to a line in $\mathbb{P}^3$. We want to use a lemma from Bruce and Wall's paper to show that when we take any hypersurface satisfying Proposition 3.2, and intersect it with a cubic 3-fold X, where $X = V(c(x_0, \dots, x_3) + x_4x_0x_1)$, the resulting singular cubic surface has either one singularity of type A_k with $k \geq 2$, or more than one singularity.

Lemma 3.7 (Lemma 3[BW79]). *Let $f = c(x_0, x_1, x_2) + x_3x_0x_1$, $p = (0 : 0 : 0 : 1)$, and $S = V(f) \subset \mathbb{P}^3$. Then*

- *S has a singularity of type A_k for $k \geq 2$.*

- *singularities of S other than that at p correspond with multiple intersections of $V(x_0x_1)$ with $V(c)$ away from $(0:0:1)$.*
- *A k-tuple intersection away from $(0:0:1)$ corresponds to an A_{k-1} singularity.*
- *If $c(0:0:1) \neq 0$ we have an A_2 singularity at p. If $(0:0:1)$ is a k_i-tuple intersection with $x_i = 0$ with $c = 0$, $i = 0, 1$, then p is an $A_{k_0+k_1+1}$ singularity for*

$$k_0, k_1 = 1, 1, \quad 1, 2 \quad 1, 3. \tag{3.3}$$

If k_0 and k_2 are both at least 2, then S has a non isolated singularity.

We omit the proof of this lemma.

Before we prove the main goal of this chapter, we list the number of lines and corresponding singularities for singular cubic surfaces containing at least one A_k with $k \geq 2$. As this will be useful in the proof. We are now ready for the main goal of this

Table 3.1: Lines on singular cubic surfaces

Singularity type	A_2	A_2A_1	$2A_2$	A_22A_1	$2A_2A_1$	$3A_2$	A_3
No. of lines	15	11	7	8	5	3	10

Singularity type	A_3A_1	A_32A_1	A_4	A_4A_1	A_5	A_5A_1	
No. of lines	7	7	5	6	4	3	2

chapter, to prove Theorem 3.1. We will do this by showing that we may find a linear hyperplane H, such that $S = X \cap H$ can be defined by a polynomial on the form in Lemma 3.7.

Proof. Assume X is a cubic 3-fold with defining polynomial

$$f = c(x_0, x_1, x_2, x_3) + x_0x_1x_4. \tag{3.4}$$

Now let $\mathfrak{H}$ be the open set which parametrizes linear hyperplane satisfying Proposition 3.2, that is for any $p^\vee \in \mathfrak{H}$, there is a linear hyperplane H_p such that $S = H_p \cap X$ is a singular surface, and all planes contained in X intersect down to lines in S. Now as $\mathfrak{H}$ is open $\mathfrak{H} \cap D_+(x_3)$ is non empty. We now pick any point in $\mathfrak{H} \cap D_+(x_3)$, then $H_p = V(a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3)$ and we may assume $a_3 = 1$, while $a_i \in \mathbb{C}$ for $i = 0, 1, 2$. Now we set $x_3 = a_0x_1 + a_1x_1 + a_2x_2$ and substitute x_3 in f so we have

$$f = c(x_0, x_1, x_2, a_0x_1 + a_1x_1 + a_2x_2) + x_0x_1x_4. \tag{3.5}$$

Now we do a coordinate change so $x_3 = x_4$. Then we get a surface in $\mathbb{P}^3$ defined by

$$f = c(x_0, x_1, x_2) + x_0x_1x_3. \tag{3.6}$$

This is on the form as in Lemma 3.7, which implies that it has at least one singularity of type A_k with $k \geq 2$. Intersecting with H was an injection of planes in X to lines in S, and so by the classification of singular cubic surfaces X has at most 15 planes. □

Chapter 4

The maximal number of planes in X when $V(q) \subset \mathbb{P}^3$ is singular

In this chapter we turn our attention to the case where $Q = V(q) \subset \mathbb{P}^3$ is a singular quadratic surface. That is when Q is isomorphic to $V(x_0x_1 - x_2^2)$. The goal of this chapter is then to prove the following theorem

Theorem 4.1. *If $Q = V(q) \subset \mathbb{P}^3$ is a singular quadratic surface, and $X = V(c + x_4q)$ is a singular complex cubic 3-fold, then X contains at most five planes.*

4.1 Reducing cases

To prove the main theorem of this section, the strategy will be to enumerate the possible configurations of lines and (possibly reducible) conics in $C = V(c, q)$. We may then use Propositions 2.16 and 2.20 to prove Theorem 4.1 by counting lines and (possibly reducible) conics in C. Our first step is to prove a lemma about X that gives us a bound on the number of lines in C.

Lemma 4.2. *If X is a cubic 3-fold that contain* 3 *planes that intersect in a line L, then X is singular along L.*

Proof. Let $X = V(f)$ be a cubic 3-fold in $\mathbb{P}^4$ that contains 3 planes P_1, P_2, P_3 such that $P_1 \cap P_2 \cap P_3$ is a line L. We may then assume that $L = V(l_1, l_2, l_3)$ where each l_i is a linear polynomial. Then, we may further assume that $P_1 = V(l_1, l_2)$, $P_2 = V(l_1, l_3)$ and $P_3 = V(l_2, l_3)$. Now $P_1 \cup P_2 \cup P_3 \subset X$, so we have that

$$(f) \subset (l_1, l_2) \cap (l_1, l_3) \cap (l_2, l_3) = V(l_1l_2, l_1l_3, l_2l_3). \tag{4.1}$$

Hence

$$f = l_1l_2A + l_1l_3B + l_2l_3C. \tag{4.2}$$

where A, B, C are linear polynomials. Now each partial derivative of f is of the form

$$\frac{\partial l_1}{x_i}l_2A+l_1\frac{\partial l_2}{x_i}A+l_1l_2\frac{\partial A}{x_i}+\frac{\partial l_1}{x_i}l_3B+l_1\frac{\partial l_3}{x_i}B+l_1l_3\frac{\partial B}{x_i}+\frac{\partial l_2}{x_i}l_3C+l_2\frac{\partial l_3}{x_i}C+l_2l_3\frac{\partial C}{x_i}. \tag{4.3}$$

The point is that for each term either l_1,l_2 or l_3 is a factor. So along the line $L = V(l_1, l_2, l_3)$, the Jacobian matrix of f has rank 0, implying that X is singular along L. □

As a consequence of this lemma observe that if $C = V(c, q)$ contains three lines, all three lines intersect in the singularity $O_3 = (0:0:0:1)$ of $Q = V(q) \subset \mathbb{P}^3$. Moreover as each line corresponds to a plane of type 1, and $\pi^{-1}(O_3)$ is a line contained in all three planes, we have three planes intersecting in a line in the cubic $X = V(c + x_4 q)$, so X is singular in a line. Giving us the corollary below

Corollary 4.3. *Let $Q = V(q) \subset \mathbb{P}^3$ and suppose $X = V(c + x_4 q)$ is not singular in a line, then $C = V(c, q)$ contains a most two lines.*

Thus any $C = V(c, q) \subset \mathbb{P}^3$ containing more than three lines corresponds to a 3-fold that is too singular to be of interest to us.

We have now reduced C to three main cases: either C contains zero lines, one line or two lines. Now we want to pinpoint the possible configuration of lines and (possibly reducible) conics contained in all three cases. Then we get en exhaustive list of what the possible configurations of lines and (possibly reducible) conics C may contain.

We start by showing that if C contains two lines it contains either zero or three conics. Which gives us two different configurations that correspond to cubic 3-folds X containing planes.

Lemma 4.4. *If $q = x_0x_1 - x_2^2$ and $c(x_0, x_1, x_2, x_3)$ is a cubic polynomial such that $C = V(c, q)$ contains two lines, and $X = V(c + x_4 q)$ is not singular in a line then C contains either no conics, or exactly two conics, not excluding the same conic twice.*

Proof. Assume $q = x_0x_1 - x_2^2$ and $c(x_0, x_1, x_2, x_3)$ is a cubic polynomial such that $V(c, q)$ contains two lines, and $X = V(c + x_4 q)$ is not singular in a line. We will show that if C contains at least one conic and two lines, then it contains two conics and two lines. So assume $C = L_1 \cup L_2 \cup M_1 \cup R$, where L_i are lines, M_1 is a conic. Then C is of degree six, and $L_1 \cup L_2 \cup M_1$ is of degree four. It follows that R must be of degree two. R does not contain any lines by assumption, so it is irreducible which implies that R is a conic. $\square$

Next we show that if C contains one line, then it contains zero or one conic. This will in turn give us two different configurations of lines and conics in C corresponding to cubic 3-folds containing planes.

Lemma 4.5. *If $q = x_0x_1 - x_2^2$ and $c(x_0, x_1, x_2, x_3)$ is a cubic polynomial such that $C = V(c, q)$ contains one line, and $X = V(c + x_4 q)$ is not singular in a line, then C contains at most one conic.*

Proof. Assume $C = L \cup K \cup R$, where L is a line K is a conic. C is of degree 6 implies that R is of degree 3, as L is of degree one and K is of degree 2. Now if R also contains a conic, then R is the union of a line and a conic, which contradicts our assumption. So if C contains exactly one line, then it contains at most one conic. $\square$

We now have one more main case to exhaust, before we have found all different possible configurations of lines and conics in C. The next lemma shows that if C contains no lines, it contains zero, one or three conics, completing our list of cases to check.

Lemma 4.6. *If $q = x_0x_1 - x_2^2$ and $c(x_0, x_1, x_2, x_3)$ is a cubic polynomial such that $V(c, q)$ contains no lines, and $X = V(c + x_4 q)$ is not singular in a line, then C contains zero, one or three conics.*

Table 4.1: Configurations of C and number of planes in X

Case	Lines	Conics	#Planes in X
1	0	1	1
2	0	3	3
3	1	0	1
4	1	1	2
5	2	0	3
6	2	2	5

Proof. That C may contain zero conics or one conic is easy to see. Therefore all we have to prove is that if C contains at least two conics, then it contains exactly three. So assume C contains two conics and no lines. Then $C = H_1 \cup H_2 \cup R$, since C has degree 6, and each H_i is of degree 2, R must be of degree 2. But by assumption C contains no lines, so R must be of degree two, hence a conic. □

Now we have full control over the possible configurations of lines and conics in C. We now present a proposition that gives us an upper bound on planes in X assuming C admits the corresponding configuration lines and (possibly reducible) conics. This will then finish the main goal of this chapter by proving Theorem 4.1.

Proposition 4.7. *If $q = x_0x_1 - x_2^2$ and $c(x_0, x_1, x_2, x_3)$ is a cubic polynomial such that $X = V(c + x_4q) \subset \mathbb{P}^4$, then the following table lists the possible configurations of lines and conics in $C = V(c, q)$ and the corresponding number of planes in X:*

We remind the reader that Proposition 2.16 gives a one to one correspondence between lines in $C = V(c, q) \subset \mathbb{P}^3$ and planes of type 1 in the cubic 3-fold $X = V(c - x_4q)$, and that Proposition 2.20 gives a one to one correspondence between $C = V(c, q)$ and planes of type 2 in the cubic 3-fold $X = V(c - x_4q)$. We do this as we will now use these two theorems repeatedly to prove the above proposition. Our hope is that it is clear from context if we use Proposition 2.16 or **??** or both.

Proof. Consider Table 4.1. As previously explained there are six different possible configurations of lines and conics in $C = V(c, q)$.
In case 1 C contains no lines and one conic, and thus corresponds to a cubic 3-fold containing one plane of type 2
In case 2 C contains no lines and three conics, and thus corresponds to a cubic 3-fold containing three planes of type 2.
In case 3 C contains one line and no conics, and thus corresponds to a cubic 3-fold containing one plane of type 1.
In case 4 C contains one line and one conic, and thus corresponds to a cubic 3-fold containing one plane of type 1 and one plane of type 2.
In case 5 C contains two lines and no conics, moreover any two lines on $Q = V(q)$ intersect, so C also contains a reducible conic, the corresponding cubic 3-fold contains two planes of type 1, and one plane of type 2.
In case 6 C contains two lines and two conics, as in case 5, it must also contain a reducible conics, so the corresponding cubic 3-fold contains two planes of type 1 and three planes of type 2. □

We have now shown that if a $X = V(c - x_4(x_0x_1 - x_2)) \subset \mathbb{P}^4$, then X contains at most five planes, which proves Theorem 4.1.

Chapter 5

The maximal number of planes in X when $V(Q) \subset \mathbb{P}^3$ is smooth

5.1 C as a curve in $\mathbb{P}^1 \times \mathbb{P}^1$

The goal of this chapter is to prove the following theorem

Theorem 5.1. *If $Q = V(q) \subset \mathbb{P}^3$ is smooth and $c \neq 0$ is any cubic polynomial such that the 3-fold $X = V(c + x_4 q) \subset \mathbb{P}^4$ is irreducible and not a cone, then X contains at most 15 planes.*

We assume that $Q = V(q)$ is smooth, we may then assume $q = x_0x_3 - x_1x_2$. We want to use that $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Before we continue we need a definition so we may compare curves on Q with curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

Definition 5.2. A (a, b)-curve in $\mathbb{P}^1 \times \mathbb{P}^1$, where $a, b \in \mathbb{N}$ is defined as the zero set of a polynomial

$$\sum_{i=0}^{a} \sum_{j_0}^{b} \alpha_{ij} u_0^{a-i} u_1^i v_0^{b-j} v_1^j, \tag{5.1}$$

where $\alpha_{ij} \in \mathbb{C}$.

Now C is isomorphic to a $(3, 3)$-curve in $\mathbb{P}^1 \times \mathbb{P}^1$. So we may enumerate the different configurations of $(1, 0)$-, $(0, 1)$- and $(1, 1)$-curves that a $(3, 3)$-curve can contain. As we shall see shortly $(1, 0)$- and $(0, 1)$-curves correspond to lines in C, and so we may count the number of planes of type 1 in X by counting $(1, 0)$- and $(0, 1)$-curves and using Proposition 2.16. Moreover a $(1, 1)$-curve correspond to a plane quadratic curve in C, and the union of a $(1, 0)$- and a $(0, 1)$-curve correspond to the union of two lines in C. In other words if we have a $(1, 1)$-curve, or one $(1, 0)$-curve and a $(0, 1)$-curve in $\mathbb{P}^1 \times \mathbb{P}^1$, we get a conic in C in the first case, and a reducible conic in the second case, so by Proposition 2.20 there is a corresponding plane in X in each case.

Remark. Proposition 2.20 requires that no other line in C intersects the (possibly reducible) conic in C. Since $V(q) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ we will check this in $\mathbb{P}^1 \times \mathbb{P}^1$ where it is straight forward to see.

Now let us set the stage, let $\mathbb{P}^1 \times \mathbb{P}^1$, where $\mathbb{C}[u_0, u_1 : v_0, v_1]$ is the usual bigraded ring. The Segre embedding

$$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3 \tag{5.2}$$

which sends

$$(u_0 : u_1; v_0 : v_1) \mapsto (u_0v_0 : u_0v_1 : u_1v_0 : u_1v_1) \tag{5.3}$$

gives us an isomorphism $V(q) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ [Har77, Ex. I.2.15]. Moreover, if we pick an element $g \in \Gamma(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,3))$, then $\sigma(V(g)) = V(c)$,for some $c \in \Gamma(\mathcal{O}_Q(3))$. We will use this to enumerate, up to symmetry, all the different ways $C = v(c,q)$ can split into lines and plane quadratic curves. To do this, we need to know how bidegree $(1,0)$-, $(0,1)$- and $(1,1)$-curves map into Q. We begin with a lemma showing that $(1,0)$-curves and $(0,1)$-curves map to lines via the Segre embedding.

Lemma 5.3. *If $g \in \Gamma(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0))$ or $g \in \Gamma(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1))$, in other words, if $V(g)$ is a bidegree $(1,0)$- or $(0,1)$-curve, then $V(g)$ maps to a line in $\mathbb{P}^3$ under the Segre embedding.*

Proof. Let $g \in \Gamma(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0))$, then $g = au_0 + bu_1$ with $a, b \in \mathbb{C}$, not all zero. Now $D_+(v_0), D_+(v_1)$ is a cover of $\mathbb{P}^1 \times \mathbb{P}^1$, and on $D_+(v_0)$, $V(au_0 + bu_1) = V(au_0v_0 + bu_1v_0)$. and likewise on $D_+(v_1)$ we have that $V(au_0 + bu_1) = V(au_0v_1 + bu_1v_1)$. So

$$\sigma(V(au_0 + bu_1)) = V(ax_0 + bx_2, ax_1 + bx_3, q) \tag{5.4}$$

Moreover $q = x_0x_3 - x_1x_2$ is contained in $(ax_0 + bx_2, ax_1 + bx_3)$, so:

$$\begin{aligned}(ax_0 + bx_2)(ax_1 + bx_3) - ax_1(ax_0 + bx_2) - bx_2(ax_1 + bx_3) \\ = ab(x_0x_3 - x_1x_2).\end{aligned} \tag{5.5}$$

Thus

$$\sigma(V(au_0 + bu_1)) = V(ax_0 + bx_2, ax_1 + bx_3), \tag{5.6}$$

which is the intersection of two planes in $\mathbb{P}^3$, thus a line. The case where $g \in \Gamma(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1))$ is symmetrical. □

Next we introduce a lemma showing how $(1,1)$-curves map into Q.

Lemma 5.4. *If $g \in \Gamma(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1))$, then $V(g)$ maps to a (possible reducible) conic in Q under the Segre embedding.*

Proof. Let $g \in \Gamma(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1))$, then $g = au_0v_0 + bu_0v_1 + cu_1v_0 + du_1v_1$, for some $a, b, c, d \in \mathbb{C}$, with at least one of a, b not zero, and at least on of c, d not zero. . So we have that

$$\sigma(V(au_0v_0 + bu_0v_1 + cu_1v_0 + du_1v_1)) = V(ax_0 + bx_1 + cx_2 + dx_3, q) \tag{5.7}$$

which is a quadratic curve in $\mathbb{P}^3$. □

Note that we at no point here assumed the $(1,1)$-curve is irreducible, but since the Segre embedding is an isomorphism in this case, if the $(1,1)$-curve is irreducible, it maps into a conic, and if the $(1,1)$-curve is reducible it maps into a reducible conic.

We will now start with a $g \in \Gamma(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,3))$, then map it into $\mathbb{P}^3$, the image $\sigma(V(g))$ will then be something on the form $C = V(c,q) \subset \mathbb{P}^3$, which will then give us our $X \subset \mathbb{P}^4$, by letting $f = c + x_4q$. By Proposition 2.16, any plane of type 1 in X have a corresponding line in C. By Proposition 2.20 any plane of type 2 in X, there is a corresponding (possibly reducible) conic in C which does not intersect any line in more than one point. The reasoning behind starting with a $(3,3)$-curve in $\mathbb{P}^1 \times \mathbb{P}^1$, is that we then have full control over how many different lines C contains, how they intersect and how many ways C contains a (possibly reducible) conic. To find all the ways C may contain lines and (possibly reducible conics) we will enumerate partitions of $(3,3)$, but as, for instance $(1,0)+(2,3)$ is symmetrical to $(0,1)+(3,2)$, we can skip some partitions. By the fundamental theorem of algebra, any curve of bidegree $(a,0)$, where $a \geq 2$, is reducible, and is a sum of a bidegree $(1,0)$ curves, which we will write as $a(1,0)$. To keep track of all the components, we will denote curves of bidegree $(1,0)$ as L_i, with $i = 1,2,3$, and curves of bidegree $(0,1)$ as M_i, while curves of bidegree $(1,1)$ will be denoted H_i. We will also denote any irreducible curves of bidegree (a,b) where $a+b \geq 3$ as R.

Proposition 5.5. *Up to symmetry the following table are the possible configuration of* $(1,0)$-,$(0,1)$-, *and* $(1,1)$-*curves in a* $(3,3)$-*curve in* $\mathbb{P}^1 \times \mathbb{P}^1$. *Moreover corresponding to each configuration there is the maximal number of planes in a cubic 3-fold given by* $V(c + x_4 q)$ *where* $V(q)$ *is isomorphic to a* $(3,3)$-*curve of this configuration.*

Table 5.1: Partitions of $(3,3)$ and maximal number of planes

Case	Partition	#Planes in X
1	$(1,0)+(2,3)$	1
2	$(1,1)+(2,2)$	1
3	$2(1,0)+(1,3)$	2
4	$(1,0)+(0,1)+(2,2)$	3
5	$3(1,1)$	3
6	$(1,0)+(0,1)+2(1,1)$	5
7	$2(1,0)+(0,1)+(1,2)$	5
8	$2(1,0)+2(0,1)+(1,1)$	9
9	$3(1,0)+3(0,1)$	15

The next section is dedicated to going through all our cases above, and verifying, once we are done we have a proof of Theorem 5.1. From now on, we will assume $g\Gamma(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3))$, and that $V(g)$ is the union of at least two bidegree curves, all of which are irreducible. Moreover $C = V(c,q) \subset \mathbb{P}^3$ will be the image of $V(g)$ under the Segre embedding, and $X = V(c + x_4 q) \subset \mathbb{P}^4$ is the cubic corresponding to each partition. We will also just write (a,b)-curve when we mean a curve of bidegree (a,b). Assume $V(g)$ is the union of a $(1,0)$-curve, and a $(2,3)$-curve. Then

$$V(g) = L_1 \cup R \tag{5.8}$$

, and so $C = \sigma(V(g))$ is the union of a line and a degree 5 curve. Now by Proposition 2.16 X has exactly one plane. Likewise if g is the union of some M_1 and R, in other words, a curve of bidegree $(0,1)$ and a curve of bidegree $(3,2)$, thus X contains exactly one plane.

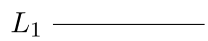

Figure 5.1: A single line corresponding to L_1

1. $(1,1)+(2,2)$ Assume $V(g)$ is the union of a $(1,1)$-curve and a $(2,2)$-curve. Then $V(g) = H_1 \cup R$, so $C = \sigma(V(g))$ is the union of a quadratic curve, and a quntic curve. Now by Proposition 2.20 we know that X contains exactly one plane.

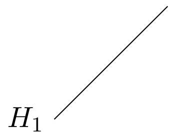

Figure 5.2: A single line corresponding to H_1

2. $2(1,0)+(1,3)$ Assume $V(g)$ is the union of two $(1,0)$-curves and a $(1,3)$-curve. Then $V(g) = L_1 \cup L_2 \cup R$, and so C consists of two lines, that does not intersect, and a curve of degree 4. So by Proposition 2.16 X has exactly two planes.

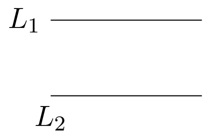

Figure 5.3: Two nonintersecting lines L_1 and L_2

3. $(1,0)+(0,1)+(2,2)$ Assume $V(g)$ is the union of a $(1,0)$-curve, a $(0,1)$-curve and a $(2,2)$-curve, then $V(g) = L_1 \cup M_1 \cup R$. So C now has three components, two intersecting lines, and a curve of degree 4. So by Proposition 2.16 we have two planes corresponding to each line, moreover $\sigma(V(L_1 \cup M_1))$ is now a reducible conic in C, so by Proposition 2.16, there is a corresponding plane in X.

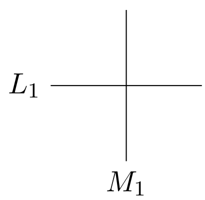

Figure 5.4: Two intersecting lines L_1 and M_1

4. $3(1,1)$ Assume $V(g)$ is the union of three $(1,1)$-curves, then $V(g) = H_1 \cup H_2 \cup H_3$. C then consists of three irreducible conic curves, and by Proposition 2.20 X contains three planes.

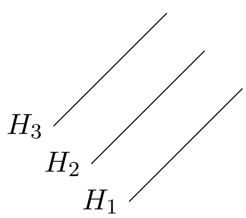

Figure 5.5: Three lines corresponding to H_1,H_2 and H_3

5. $(1,0)+(0,1)+2(1,1)$ Assume $V(g)$ is the union of one $(1,0)$-curve, one $(0,1)$-curve and two $(2,2)$ curves. Then $V(g) = L_1 \cup M_1 \cup H_1 \cup H_2$. So C consists of two intersecting lines, and two irreducible conic curves. As we saw above, the two intersecting lines gives us 3 planes, and since $L_1 \cap H_1$ is just one point, each H_i is in one to one correspondence with a plane in X, so X contains five planes.

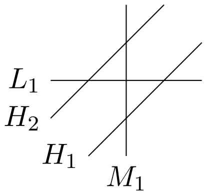

Figure 5.6: The configuration in $\mathbb{P}^1 \times \mathbb{P}^1$

6. $2(1,0) + (0,1) + (1,2)$ Assume $V(g)$ is the union of two $(1,0)$-curves, one $(0,1)$-curve, and one $(1,2)$-curve. Then $V(g) = L_1 \cup L_2 \cup M_1 \cup R$. So C consists of three lines, and a degree 3 curve. Now by Proposition 2.16 we se imediatley that we have at least three planes, moreover as $L_1 \cap M_1$ is a point, and $L_2 \cap M_1$ is a point, we have two reducible conics in C, so by Proposition 2.20 we have two more planes, so X contains exactly five planes.

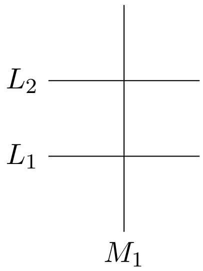

Figure 5.7: The configuration in $\mathbb{P}^1 \times \mathbb{P}^1$

7. $2(1,0) + 2(0,1) + (1,1)$ Assume $V(g)$ is the union of two $(1,0)$-curves, two $(0,1)$-curves and one $(1,1)$-curve. Then $V(g) = L_1 \cup L_2 \cup M_1 \cup M_2 \cup H_1$. So C consists of four lines and an irreducible conic curve. So by **??** we have four planes corresponding to the lines in C. Morover, H_1 and each $L_i \cup M_j$ correspond to a plane by Proposition 2.20, giving five aditional planes, for a total of nine planes in X.

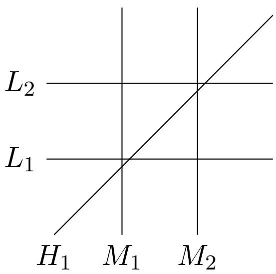

Figure 5.8: The configuration in $\mathbb{P}^1 \times \mathbb{P}^1$

8. $3(1,0) + 3(0,1)$ Assume $V(g)$ is the union of three $(1,0)$-curves, and three $(0,1)$-curves. Then $V(g) = L_1 \cup L_2 \cup L_3 \cup M_1 \cup M_2 \cup M_3$. So C consists of six lines, and by Proposition 2.16 there are six corresponding lines in X, moreover each of the nine unions of lines $L_i \cup M_j$ corresponds to a plane in X by Proposition 2.20, hence there are a total of 15 planes in the corresponding X.

This proves Proposition 5.5 and thus we have shown Theorem 5.1.

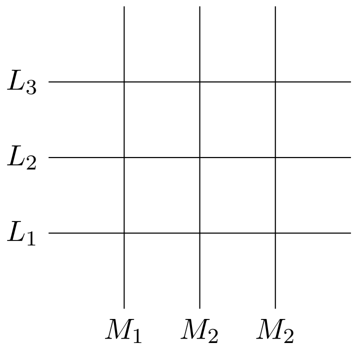

Figure 5.9: The configuration in $\mathbb{P}^1 \times \mathbb{P}^1$

5.2 Clebsch-Segre cubic 3-fold

We stated in our introduction that there is a cubic 3-fold X containing 15 planes, and that this is isomorphic to the Clebsch-Segre cubic 3-fold defined by

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = x_0 + x_1 + x_2 + x_3 + x_4 + x_4 = 0. \tag{5.9}$$

We will do this by giving an example of a cubic 3-fold X containing 15 planes, we will then just state that it has 10 singularities. and use two propositions from [Dol16] showing that X is isomorphic to the Clebsch-Segre cubic 3-fold.

Example 5.6. Let

$$c = -x_0x_1x_2 + x_0x_2x_3 + x_1x_2x_3 - x_2x_3^2, \tag{5.10}$$

and let

$$q = x_0x_1 + x_2x_3. \tag{5.11}$$

Then $X = V(c - x_4q)$ is a cubic 3-fold containing 15 planes. We show this by showing that $C = V(c, q)$ is the union of six lines. Then X is a realization of case 9 in Table 5.1. In fact

$$C = V(c,q) = V(x_3,x_1) \cup V(x_3,x_0) \cup V(x_2,x_1) \cup V(x_2,x_0) \cup V(x_1 - x_3, x_0 + x_2) \cup V(x_1 + x_2, x_0 - x_3). \tag{5.12}$$

By using Maccaulay2[GS] we found that X has 10 isolated singularities. Now we write out proposition 2.1 and 2.2 from [Dol16], which will show that X is isomorphic to the Clebsch-Segre cubic 3-fold.

Proposition 5.7. *Two 10 nodal-cubic hypersurfaces in $\mathbb{P}^4$ are projectively isomorphic.*

and

Proposition 5.8. *The Segre-Clebsh cubic 3-fold contains 10 nodes and 15 planes.*

Chapter 6

Conclusion

In conclusion we have shown that is X is a cubic 3-fold with only isolated singularities, X contains at most 15 planes. Moreover the Clebsch-Segre cubic 3-fold is a realization of such a cubic 3-fold. We conjecture that the bound of 15 from Theorem 3.1 is not strict, and that a strict bound is in fact 9 when $Q = V(q) \subset \mathbb{P}^3$ is reducible.

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