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A weighted random walk approximation to fractional Brownian motion

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Abstract

We present a random walk approximation to fractional Brownian motion where the increments of the fractional random walk are defined as a weighted sum of the past increments of a Bernoulli random walk.

Keywords: Fractional Brownian motion, random walks, discrete approximations, weak convergence

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The purpose of this brief note is to describe a discrete approximation to fractional Brownian motion. The approximation works for all Hurst indices H, but take slightly different forms for $H \leq \frac{1}{2}$ and $H > \frac{1}{2}$. There are already several discrete approximations to fractional Brownian motion in the literature (see, e.g., [11], [1], [3], [10], [4], [2], [5], [8] for this and related topics), and the advantage of the present approach is that the increments of the fractional random walk is given as a weighted sum of past increments of an ordinary (Bernoulli) random walk. This gives an excellent understanding of the dynamics of the process and is a good starting point for stochastic calculus with respect to fractional Brownian motion. A similar idea is exploited in much greater generality by Konstantopoulos and Sakhanenko in [5], but they assume that $H > \frac{1}{2}$, while the present paper is mainly of interest when $H < \frac{1}{2}$.

The discrete approximation is based on Mandelbrot and Van Ness' [6] moving frame representation of fractional Brownian motion:

$$x_t = c_H \int_{-\infty}^{t} \left((t-r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}} \right) \, db_r$$

where the scaling constant c_H is given by

$$c_H = \left(\int_0^\infty \left((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}} \right)^2 \, du + \frac{1}{2H} \right)^{-\frac{1}{2}} = \frac{\sqrt{\Gamma(2H+1)\sin(\pi H)}}{\Gamma(H+\frac{1}{2})}$$

(see also [9]). This representation will be used to establish the convergence.

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1 The main theorem

To state the main result, we need some notation. For each natural number N, let $\Delta t_N = \frac{1}{N}$ and think of

$$T_N = \{k\Delta t_N \mid k \in \mathbb{Z}\}$$

as a timeline. We let T_N^+ denote the nonnegative part of T. It is convenient to use the following convention for sums over elements in T_N:

$$\sum_{r=s}^{t} f(r) = f(s) + f(s + \Delta t) + \cdots + f(t - \Delta t_N)$$

Note that the lower limit s is included in the sum, but the upper limit t is not. We shall also write $\Delta f(t) = f(t + \Delta t_N) - f(t)$ for the forward increment of f at t.

For all $t \in T_N$, let $\omega_N(t)$ be independent random variables taking values ± 1 with probability $\frac{1}{2}$. We shall write $\Delta B_N(t) = \sqrt{\Delta t_N}\omega_N(t)$ and think of B_N as a Bernoulli random walk approximating Brownian motion. For $0 < H < 1$ and $N \in \mathbb{N}$, define a process $X_{H,N} : \Omega_N \times T_N^+ \to \mathbb{R}$ by $X_{N,H}(0) = 0$ and

$$\Delta X_{H,N}(s) = K_H \Delta t_N^{H-\frac{1}{2}} \Delta B_N(s) + \sum_{r=-\infty}^{s} (H - \frac{1}{2})(s - r)^{H-\frac{3}{2}} \Delta t_N \Delta B_N(r)$$

(using, e.g., Kolmogorov's one series theorem, see [12], one easily checks that the sum converges a.s.) where the constant K_H is defined by

$$K_H = \begin{cases} -(H - \frac{1}{2})\zeta(\frac{3}{2} - H) & \text{for } H < \frac{1}{2} \\ 1 & \text{for } H \geq \frac{1}{2} \end{cases}$$

(as usual, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ when $s > 1$). Except for the Mandelbrot-Van Ness scaling factor c_H, $X_{H,N}$ will be our random walk approximation to fractional Brownian motion. For convergence puposes it will be convenient to think of $X_{H,N}$ as a càdlàg process defined on $[0, \infty)$, and we do this simply by assuming that $X_{H,N}$ is constant between points in T_N.

Remark: Note that the increment $\Delta X_{H,N}(s)$ is a weighted sum of increments of the Bernoulli random walk B_N — it is a linear combination of the current coin toss $\omega_N(s)$ and all previous coin tosses $\omega_N(r)$, $r < s$. Observe also that since $\lim_{H\uparrow\frac{1}{2}} -(H - \frac{1}{2})\zeta(\frac{3}{2} - H) = \lim_{s\downarrow 1}(s - 1)\zeta(s) = 1$, the two cases meet continuously at $H = \frac{1}{2}$. For $H > \frac{1}{2}$, we may actually choose K_H as we please since the term will vanish in the limit (see below), but $K_H = 1$ is the natural value and probably the one that gives best results in numerical work.

We are now ready to state the main result. Note that when $H = \frac{1}{2}$, $\Delta X_{\frac{1}{2},N}(t) = \Delta B_N(t)$ and the theorem just reduces to the classical convergence of a Bernoulli random walk to Brownian motion.

Theorem 1 (Main Theorem) *For all real numbers H, $0 < H < 1$, the processes $c_H X_{H,N}$ converge weakly in $D([0,\infty))$ to fractional Brownian motion with Hurst index H.*

Notation: In the rest of the paper, we drop the notational dependence on N and H, and write simply X, B, T, Δt for $X_{H,N}$, B_N, T_N, Δt_N etc. when no confusion can arise.

As we are interested in understanding the dynamics of fractional Brownian motion, we have defined X by specifying its increments $\Delta X(s)$. To prove the main theorem, we need an expression for $X(t)$. This is just a small calculation:

$$X(t) = \sum_{s=0}^{t} \Delta X(s) = \sum_{s=0}^{t} K_H \Delta t^{H-\frac{1}{2}} \Delta B_s + \sum_{s=0}^{t} \sum_{r=-\infty}^{s} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t \Delta B_r$$

Changing the order of summation, we have

$$X(t) = K_H \Delta t^{H-\frac{1}{2}} B_t + \sum_{r=0}^{t} \sum_{s=r+\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t \Delta B_r$$

$$+ \sum_{r=-\infty}^{0} \sum_{s=0}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t \Delta B_r$$

where $B_t = \sum_{r=0}^{t} \Delta B_r$ is a random walk converging to Brownian motion. Observe that when $H > \frac{1}{2}$, the first term $K_H \Delta t^{H-\frac{1}{2}} B_t$ vanishes when $N \to \infty$ (this is why the choice of K_H is irrelevant in this case), but when $H < \frac{1}{2}$, the term explodes. In this case we have a delicate balance between two terms going to infinity, and a correct choice of K_H is crucial.

The idea is now to simplify the expression for X by replacing the sums $\sum (H-\frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t$ by the corresponding integrals $\int (H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\, ds$, and then performing the integration. This works nicely for $H > \frac{1}{2}$, but when $H < \frac{1}{2}$, one of the integrals diverges, and we have to be more careful. Put crudely, it is the divergence of this integral that will cancel the divergence of the term $K_H \Delta t^{H-\frac{1}{2}} B_t$.

We are ready to prove the main theorem, and start with the simplest case.

2 The case $H > \frac{1}{2}$

We start from the expression

$$X(t) = \Delta t^{H-\frac{1}{2}} B_t + \sum_{r=0}^{t} \sum_{s=r+\Delta t}^{t} (H - \frac{1}{2})(s-r)^{H-\frac{3}{2}} \Delta t \Delta B_r +$$

$$+\sum_{r=-\infty}^{0}\sum_{s=0}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t\Delta B_r$$

above (remember that $K_H = 1$ in this case). Since $H > \frac{1}{2}$, we have no problem with convergence, and if we let $\epsilon_N(r,t)$ be the error term:

$$\epsilon_N(r,t) := \sum_{s=r+\Delta t}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t - \int_{r+\Delta t}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\,ds,$$

we get

$$\sum_{s=r+\Delta t}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t = \int_{r+\Delta t}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\,ds + \epsilon_N(r,t) =$$

$$= (t-r)^{H-\frac{1}{2}} - \Delta t^{H-\frac{1}{2}} + \epsilon_N(r,t)$$

Similarly, with

$$\delta_N(r,t) := \sum_{s=0}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t - \int_{0}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\,ds,$$

we get

$$\sum_{s=0}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t = \int_{0}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\,ds + \delta_N(r,t) =$$

$$= (t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}} + \delta_N(r,t)$$

This means that

$$X(t) = \sum_{r=0}^{t}\left((t-r)^{H-\frac{1}{2}} + \epsilon_N(r)\right)\Delta B_r +$$

$$+\sum_{r=-\infty}^{0}\left((t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}} + \delta_N(r)\right)\Delta B_r =$$

$$= \sum_{r=-\infty}^{t}\left((t-r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}\right)\Delta B_r + \sum_{r=0}^{t}\epsilon_N(r,t)\Delta B_r + \sum_{r=-\infty}^{0}\delta_N(r,t)\Delta B_r$$

We want to prove that X converges weakly to fractional Brownian motion. According to Theorem 1 in [5], it suffices to show that $E(c_H^2 X(t)^2) \to t^{2H}$. This follows immediately from the Mandelbrot-Van Ness representation and the following lemma.

Lemma 2 *For* $\frac{1}{2} < H < 1$*:*

(i) $E\left((\sum_{r=0}^{t}\epsilon_N(r,t)\Delta B_r)^2\right) \leq (H-\frac{1}{2})^2 t\Delta t^{2H-1}$

(ii) $E\left((\sum_{r=-\infty}^{0}\delta_N(r,t)\Delta B_r)^2\right) \leq (H-\frac{1}{2})^2\zeta(3-2H)\Delta t^{2H}$

Proof: (i) We first observe that

$$\epsilon_N(r,t) = \sum_{s=r+\Delta t}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t - \int_{r+\Delta t}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\,ds > 0$$

since $\sum_{s=r+\Delta t}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t$ is an upper Riemann sum for the integral. Since $\sum_{s=r+2\Delta t}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t$ is a lower Riemann sum, we also have

$$0 \leq \epsilon_N(r,t) \leq (H-\frac{1}{2})\Delta t^{H-\frac{3}{2}}\Delta t = (H-\frac{1}{2})\Delta t^{H-\frac{1}{2}}$$

Thus

$$E\left((\sum_{r=0}^{t}\epsilon_N(r,t)\Delta B_r)^2\right) = \sum_{r=0}^{t}\epsilon_N(r,t)^2\Delta t \leq$$

$$\leq \sum_{r=0}^{t}(H-\frac{1}{2})^2\Delta t^{2H-1}\Delta t \leq (H-\frac{1}{2})^2 t\Delta t^{2H-1}$$

(ii) Using approximating Riemann sums as in part (i), we see that

$$0 \leq \delta_N(r,t) \leq (H-\frac{1}{2})(-r)^{H-\frac{3}{2}}\Delta t,$$

and thus

$$E\left((\sum_{r=-\infty}^{0}\delta_N(r,t)\Delta B_r)^2\right) = \sum_{-\infty}^{0}\delta_N(r,t)^2\Delta t \leq \sum_{r=-\infty}^{0}(H-\frac{1}{2})^2(-r)^{2H-3}\Delta t^3$$

Letting $r=-k\Delta t$, we get

$$E\left((\sum_{r=-\infty}^{0}\delta_N(r,t)\Delta B_r)^2\right) \leq \sum_{k=0}^{\infty}(H-\frac{1}{2})^2k^{2H-3}\Delta t^{2H} =$$

$$= (H-\frac{1}{2})^2\zeta(3-2H)\Delta t^{2H}$$

This completes the proof of the lemma (and also the proof of the Main Theorem for the case $H>\frac{1}{2}$). □

3 The case $H<\frac{1}{2}$

Again we start from the expression

$$X(t) = K_H\Delta t^{H-\frac{1}{2}}B_t + \sum_{r=0}^{t}\sum_{s=r+\Delta t}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t\Delta B_r +$$

$$+\sum_{r=-\infty}^{0}\sum_{s=0}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t\Delta B_r$$

In this case, one of the integrals we worked with above diverges, and we have to be more careful. Let us start with a closer look at the term $\sum_{s=r+\Delta t}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t$. We obviously have

$$\sum_{s=r+\Delta t}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t=$$

$$\sum_{s=r+\Delta t}^{\infty}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t-\sum_{s=t}^{\infty}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t$$

and if we let $r=N\Delta t$, $s=k\Delta t$, we get

$$\sum_{s=r+\Delta t}^{\infty}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t=\sum_{k=N+1}^{\infty}(H-\frac{1}{2})(k\Delta t-N\Delta t)^{H-\frac{3}{2}}\Delta t$$

$$=(H-\frac{1}{2})\Delta t^{H-\frac{1}{2}}\sum_{k=N+1}^{\infty}(k-N)^{H-\frac{3}{2}}=(H-\frac{1}{2})\Delta t^{H-\frac{1}{2}}\sum_{n=1}^{\infty}n^{H-\frac{3}{2}}$$

$$=(H-\frac{1}{2})\Delta t^{H-\frac{1}{2}}\zeta(\frac{3}{2}-H)=-K_H\Delta t^{H-\frac{1}{2}}$$

Substituting this into the expression for $X(t)$, we get

$$X(t)=\sum_{r=0}^{t}\sum_{s=t}^{\infty}-(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t\Delta B_r$$

$$+\sum_{r=\infty}^{0}\sum_{s=0}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t\Delta B_r$$

The two sums in this expression have less dangerous limits than the one we just got rid of, and can be approximated by integrals. If we let

$$\tilde{\epsilon}_N(r,t):=\sum_{s=t}^{\infty}-(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t-\int_t^{\infty}-(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\,ds,$$

we get (remember that $H<\frac{1}{2}$):

$$\sum_{s=t}^{\infty}-(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t=\int_t^{\infty}-(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\,ds+\tilde{\epsilon}_N(r,t)$$

$$=\left[-(s-r)^{H-\frac{1}{2}}\right]_{s=t}^{s=\infty}+\tilde{\epsilon}_N(r,t)=(t-r)^{H-\frac{1}{2}}+\tilde{\epsilon}_N(r,t)$$

Similarly, if we let

$$\tilde{\delta}_N(r,t) := \sum_{s=0}^{t} -(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t - \int_0^t -(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\, ds,$$

we get

$$\sum_{s=0}^{t}(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t = \int_0^t (H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\, ds - \tilde{\delta}_N(r,t)$$

$$= \Big[(s-r)^{H-\frac{1}{2}}\Big]_{s=0}^{s=t} - \tilde{\delta}_N(r,t) = (t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}} - \tilde{\delta}_N(r,t)$$

We thus have

$$X(t) = \sum_{r=0}^{t}\Big((t-r)^{H-\frac{1}{2}} + \tilde{\epsilon}_N(r,t)\Big)\Delta B_r +$$

$$+ \sum_{r=-\infty}^{0}\Big((t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}} - \tilde{\delta}_N(r,t)\Big)\Delta B_r$$

$$= \sum_{r=-\infty}^{t}\Big((t-r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}\Big)\Delta B_r +$$

$$+ \sum_{r=0}^{t}\tilde{\epsilon}_N(r,t)\Delta B_r - \sum_{r=-\infty}^{0}\tilde{\delta}_N(r,t)\Delta B_r$$

To prove that $c_H X$ converges weakly to fractional Brownian motion, we can now longer use Theorem 1 of [5] as in the previous case since this theorem requires that $H > \frac{1}{2}$. However, the first term in the expression above obviously converges weakly to

$$\int_{r=-\infty}^{t}\Big((t-r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}\Big)\, db_r,$$

and the next lemma shows that error terms go uniformly to zero. Using the Mandelbrot-Van Ness representation, we then get the Main Theorem for $H < \frac{1}{2}$.

Lemma 3 *For each H, $0 < H < \frac{1}{2}$, there is a constant $K_H \in \mathbb{R}_+$ (independent of N and t) such that*

$$\left| X(t) - \sum_{r=-\infty}^{t}\Big((t-r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}\Big)\Delta B_r \right| \le K_H \Delta t^H$$

Proof: It clearly suffices to show that there are constants $C_H, D_H \in \mathbb{R}_+$ (independent of N and t) such that

$$\left|\sum_{r=0}^{t}\tilde{\epsilon}_N(r,t)\Delta B_r\right| \le C_H \Delta t^H \quad \text{and} \quad \left|\sum_{r=-\infty}^{0}\tilde{\delta}_N(r)\Delta B_r\right| \le D_H \Delta t^H$$

We begin with the $\tilde{\epsilon}_N$-case. By definition

$$\tilde{\epsilon}_N(r,t) = \sum_{s=t}^{\infty} -(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t - \int_t^{\infty} -(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\, ds$$

Since $\sum_{s=t}^{\infty} -(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t$ is an upper Riemann sum for the integral $\int_t^{\infty} -(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\, ds$, and $\sum_{s=t+\Delta t}^{\infty} -(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t$ is a lower Riemann sum, we have

$$0 \leq \tilde{\epsilon}_N(r,t) \leq -(H-\frac{1}{2})(t-r)^{H-\frac{3}{2}}\Delta t$$

Hence (remember that $|\Delta B_r| = \Delta t^{\frac{1}{2}}$)

$$\left|\sum_{r=0}^{t} \tilde{\epsilon}_N(r,t)\Delta B_r\right| \leq \sum_{r=0}^{t} -(H-\frac{1}{2})(t-r)^{H-\frac{3}{2}}\Delta t^{\frac{3}{2}}$$

If we let $t = K\Delta t$, $r = k\Delta t$, we can rewrite the last sum as

$$\sum_{k=0}^{K-1} -(H-\frac{1}{2})(K-k)^{H-\frac{3}{2}}\Delta t^H \leq -(H-\frac{1}{2})\zeta(\frac{3}{2}-H)\Delta t^H$$

This completes the $\tilde{\epsilon}_N$-part of the argument.

Turning to the term $\sum_{r=-\infty}^{0} \tilde{\delta}_N(r)\Delta B_r$, we first observe that by definition

$$\tilde{\delta}_N(r) = \sum_{s=0}^{t} -(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t - \int_0^{t} -(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\, ds$$

Again, $\sum_{s=0}^{t} -(H-\frac{1}{2})(s-r)^{H-\frac{3}{2}}\Delta t$ is an upper Riemann sum, and we easily see that

$$0 \leq \tilde{\delta}_N(r) \leq -(H-\frac{1}{2})(-r)^{H-\frac{3}{2}}\Delta t$$

Letting $r = -k\Delta t$, we get

$$E\left|\sum_{r=-\infty}^{0} \tilde{\delta}_N(r)\Delta B_r\right| \leq \sum_{r=-\infty}^{0} -(H-\frac{1}{2})(-r)^{H-\frac{3}{2}}\Delta t^{\frac{3}{2}} \leq$$

$$\leq -(H-\frac{1}{2})\Delta t^H \sum_{k=0}^{\infty} k^{H-\frac{3}{2}} = -(H-\frac{1}{2})\zeta(\frac{3}{2}-H)\Delta t^H$$

This proves the lemma (and hence the Main Theorem for the remaining case $H < \frac{1}{2}$). $\square$

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