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ORIGINAL ARTICLE

ADJUSTMENT COEFFICIENTS AND EXACT RATIONAL EXPECTATIONS IN COINTEGRATED VECTOR AUTOREGRESSIVE MODELS

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In this article, we consider the cointegrated vector autoregressive model with adjustment parameters α and cointegration vectors β. We discuss estimation of the model under the exact linear rational expectations, when we also have linear restrictions on the adjustment parameters α. In particular we consider the same restriction on all vectors in α and the hypothesis that some vectors in α are known.

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1. INTRODUCTION

In cointegrated vector autoregressive models CVAR, or reduced rank VAR models, the parameters of the cointegration vectors and the adjustment parameters have an important role. The adjustment coefficients describe the speed of convergence toward the equilibrium defined by the cointegration vectors, and are therefore essential in specifying the error correcting properties of these models. A particular important restriction on the adjustment parameters is weak exogeneity see Engle *et al.* (1983). This implies that the distribution factorizes in a conditional and marginal part such that inference on the parameters of interest can be performed in the conditional part without any loss of information. In the CVAR model weak exogeneity is equivalent to one or several rows in the matrix of adjustment coefficients, α, being zero. Tests for this hypothesis have been developed. If the adjustment coefficients corresponding to particular variables vanish, the error correcting effect of the cointegration vector will not be present for these particular variables.

Expectations are fundamental in economics. For interpretation, a crucial question is how they are defined and implemented. There have been several proposals for how this can be done. In the approach introduced by Muth (1961), which is usually called rational expectations, expectations are essentially the same as predictions of the relevant economic theory. In the setup of this article this is summarized in the CVAR and the predictions are the conditional expectations given the available observations at the time.

Consider for example a present value model. In a simplification of an example from Campbell and Shiller (1987) the value of stocks at the beginning of period t, Y_t is expressed as a discounted sum of expected present and future dividends, $Y_t = \delta\sum_{j=0}^{\infty}\delta^j E_t[y_{t+j}]$, where y_t is the dividend from the period t. Present value models can often be expressed as a linear combination of observed values and conditional expectations of others. In the example an implication is the relation $S_t = [\delta/(1-\delta)]E_t[\Delta Y_{t+1}]$ where the spread S_t equals $Y_t - [\delta/(1-\delta)]y_t$.

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Such linear combinations of observed values and expectations are typical for the rational expectations we consider. Assuming that the data generating process can be expressed as a CVAR, rational expectations imply that certain linear restrictions on the coefficients must be satisfied.

In this article, we will consider how one can test simultaneously for linear rational expectations relations of the type described and that adjustment coefficients satisfy the same restrictions, that is, $\alpha = A\psi$ where A is a known matrix and ψ is unknown. Also the case where some of the adjustment coefficients are known in addition to the restrictions from the exact rational expectation hypothesis, will be considered. In both cases we discuss a convenient reparameterization. In the first case this leads to a rather general procedure for maximum likelihood estimation and testing. The second case is more complicated and here we suggest a switching procedure for a particular case.

The article is a follow up of three related articles dealing with various aspects of the theme: Johansen and Swensen (1999) where the CVAR contained an unrestricted constant, Johansen and Swensen (2004) where the constant and trend could be restricted and in particular Johansen and Swensen (2011) where simultaneous tests for restrictions on the parameters in the cointegration vectors and rational expectations were considered.

We point out that the relations involving the conditional expectations we consider are exact as defined by Hansen and Sargent (1981, 1991), that is, do not contain additional stochastic terms. To see the problems occurring for non-exact specifications in VAR models one can consult Boug *et al.* (2010) or Swensen (2014). An alternative class of models is focused on the solution of a system involving conditional expectations of future variables and random errors. The question of existence and uniqueness of the solutions then arises, for more details one can consult Juselius (2011) and Al-Sadoon (2017).

The organization of the article is as follows. In Section 2 the rational expectation models are explained. In Section 3 we consider the case where the columns of the matrix of adjustment coefficients belong to a subspace. In Section 4 the case where parts of the adjustment parameters are known is treated. Section 5 contains some numerical results and Section 6 an application.

2. THE RESTRICTIONS IMPLIED BY EXACT RATIONAL EXPECTATIONS

This section defines the cointegrated vector autoregressive model as the statistical model which is assumed to generate the data and formulates the parameter restrictions implied by the exact rational expectation hypothesis.

2.1. The Cointegrated Vector Autoregressive Model

Let the p-dimensional vectors of observations be generated according to the vector autoregressive model

$$\Delta X_t = \Pi X_{t-1} + \sum_{i=1}^{k} \Gamma_i \Delta X_{t-i} + \mu + \varepsilon_t,\ t = 1, \dots, T \tag{1}$$

where $X_{-k}, \dots, X_0$ are fixed and $\varepsilon_1, \dots, \varepsilon_T$ are independent, identically distributed Gaussian vectors, with mean zero and positive definite covariance matrix Ω. We assume that $\{X_t\}_{t=1,2,\dots}$ is $I(1)$ and that $\Pi = \alpha\beta'$ where the $p \times r$ matrices α and β have full column rank r, $0 < r < p$. This implies that X_t is non-stationary, ΔX_t is stationary, and that $\beta' X_t$ is stationary. It is the stationary relations between non-stationary processes and the interpretation as long-run relations, that has created the interest in this type of model in economics. Also note that the columns of α and $\Pi = \alpha\beta'$ span the same space. As in Johansen (1996) we define the following models.

- $\mathcal{H}(r)$: The model is defined by (1), where α and β are $p \times r$ matrices and otherwise no further restrictions on the parameters. The number of identified parameters in the matrix $\alpha\beta'$ is $\#(\alpha\beta') = pr + r(p-r)$.
- $\mathcal{H}_1(r)$: The model is defined by (1) and the restriction $\alpha = A\psi$, where A is a known $p \times s$ matrix of rank s, and ψ is an $s \times r$ matrix of parameters, $r \leq s \leq p$. In this case the number of parameters is $\#(\alpha\beta') = sr + r(p-r)$.

$\mathcal{H}_2(r)$: The model is defined by (1) and the restriction $\alpha = (a, a_\perp \phi)$ where a is a known $p \times m$ matrix of rank $m > 0$, and ϕ is a $(p-m) \times (r-m)$ matrix of parameters, $m \leq r \leq p$ such that $\alpha\beta' = a\beta_1' + a_\perp \phi \beta_2'$. In this model $\#(\alpha\beta') = mp + (r-m)(2p-r)$.

All models $\mathcal{H}(r)$, $\mathcal{H}_1(r)$, and $\mathcal{H}_2(r)$ are analyzed in Johansen (1996). It is shown, under the Gaussian assumption, how they can be estimated using the reduced rank technique introduced by Anderson (1951). Furthermore, likelihood ratio tests can be used to test the submodels $\mathcal{H}_1(r)$ or $\mathcal{H}_2(r)$ against the general model $\mathcal{H}(r)$.

In the following, if a is an $n \times m$, $0 < m < n$ matrix of full rank, $a_\perp$ is the orthogonal complement, that is an $n \times (n-m)$ matrix of rank $n-m$ such that $a'a_\perp = 0$. We use the notation $\bar{a} = a(a'a)^{-1}$ and $\bar{a}_\perp = a_\perp(a_\perp' a_\perp)^{-1}$.

2.2. The Model for Exact Rational Expectations and Some Examples

The model formulates a set of restrictions on the conditional expectation of X_{t+1} given the information $\mathcal{O}_t$ in the variables up to time t, which, as explained in Johansen and Swensen (2008) can be written in the form

$\mathcal{RE}$: The model based exact rational expectations formulates relations for conditional expectations

$$E[c'\Delta X_{t+1} | \mathcal{O}_t] = \tau d' X_t + \sum_{i=1}^{\ell} \tau_i d_i' \Delta X_{t+1-i} + d_\mu. \tag{2}$$

Here $E_t = E[\cdot | \mathcal{O}_t]$ denotes the conditional expectation in the probabilistic sense of model (1), given the variables $X_1, \dots, X_t$. The matrices c of dimensions $p \times q$, d of dimensions $p \times n$ and d_i of dimensions $p \times n_i$, $i = 1, \dots, \ell$ are known full rank matrices and $\tau(q \times n)$, $\tau_i(q \times n_i), i = 1, \dots, \ell$ are parameters. The elements of the $q \times 1$ vector d_μ are either known or parameters. We assume that $n \leq q$ and $\ell \leq k$.

We give next two examples of exact rational expectations models of the form (2).

Example 1. The variables real consumption, CP_t, real labor income, YL_t, and real capital income YK_t, are fundamental in models for aggregate consumption, both for those in the Keynesian tradition and for versions building on a permanent income hypothesis.

Campbell (1987) studied a permanent income hypothesis for consumption of the form:

$$CP_t = \gamma [YK_t + \frac{\rho}{1+\rho} \sum_{i=0}^{\infty} \left(\frac{\rho}{1+\rho}\right)^i E_t[YL_{t+i}]]$$

where ρ is the expected real interest rate and $\gamma \leq 1$ is a proportionality factor. Current consumption is therefore a fraction of present and estimated future labor income and present capital income.

Savings is defined as $S_t = (YL_t + YK_t) - CP_t/\gamma$. Campbell showed that

$$S_t - \Delta YL_t - (1+\rho)S_{t-1} = -\rho e_t \tag{3}$$

where

$$e_t = \frac{1}{1+\rho} \sum_{i=0}^{\infty} \left(\frac{1}{1+\rho}\right)^i (E_t[YL_{t+i}] - E_{t-1}[YL_{t+i}]).$$

Then e_t is a martingale difference, such that $E_t[e_{t+1}] = 0$. Using iterated expectations (3) therefore implies

$$E_t[S_{t+1} - \Delta YL_{t+1}] - (1+\rho)S_t = 0.$$

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Expressed by the variables $X_t = (CP_t, YL_t, YK_t)'$ this can be written

$$E_t[\Delta YK_{t+1} - \Delta CP_{t+1}/\gamma] = -\frac{\rho}{\gamma}CP_t + \rho(YL_t + YK_t).$$

When the proportionality factor γ and the real interest rate ρ are known, this has the form (2) with $c = (-1/\gamma, 0, 1)'$ and $d = \rho(-1/\gamma, 1, 1)'$ known matrices.

An alternative to the permanent income hypothesis is that consumption is determined by current income as suggested by Keynes. This can be modeled using a VAR model for $(CP_t, YL_t, YK_t)'$ of type $\mathcal{H}_1(r)$ with $\alpha = (1, 0, 0)'$ if the reduced rank is 1.

Example 2. In Boug *et al.* (2017) the following model for inflation dynamics was studied

$$\Delta p_t = \gamma_f E_t[\Delta p_{t+1}] - \lambda(p_t - \delta_1 ulc_t - \delta_2 uic_t) + \gamma_b \Delta p_{t-1} + \delta_0$$

where p_t is the consumer price index and ulc_t and uic_t denote unit labor cost and unit import cost, all in logarithms. Define $X_t = (p_t, ulc_t, uic_t)'$, $c = (1, 0, 0)'$ and $d = (1, -\delta_1, -\delta_2)'$. Dividing by γ_f one gets

$$E_t[\Delta p_{t+1}] = (\lambda/\gamma_f)(p_t - \delta_1 ulc_t - \delta_2 uic_t) + (1/\gamma_f)\Delta p_t - (\gamma_b/\gamma_f)\Delta p_{t-1} - \delta_0/\gamma_f$$

which can be expressed as

$$c' E_t[\Delta X_{t+1}] = \tau d' X_t + \tau_1 d_1' \Delta X_t + \tau_2 d_2' \Delta X_{t-1} + \mu. \quad (4)$$

This is of the form (2) with $\ell = 2$ and c, d, $d_1 = d_2 = e_1$ are known vectors.

As mentioned in Boug *et al.* (2017), Aukrust (1977) pointed out that for Norway the direct effect on consumer prices of a proportionate increase in import prices is around 0.33 percent. A reasonable specification where the matrix d is known is therefore $d = (1, -2/3, -1/3)$.

2.3. Combining the Exact Rational Expectations and the Vector Autoregressive Models

We combine the exact rational expectations and the vector autoregressive models, $\mathcal{H}_1(r)$ and $\mathcal{H}_2(r)$ and express the exact rational expectations model (2) as restrictions on the coefficients of the statistical model (1). As indicated in the introduction the arguments are similar to those presented in Johansen and Swensen (2008).

Taking the conditional expectation of $c'\Delta X_{t+1}$ given $X_1, \dots, X_t$, we get by using (1),

$$c' E_t[\Delta X_{t+1}] = c'\alpha\beta' X_t + \sum_{i=1}^{k} c'\Gamma_i \Delta X_{t+1-i} + c'\mu.$$

Equating this expression to (2) implies that the following conditions must be satisfied

$$c'\alpha\beta' = \tau d', \quad (5)$$

and

$$c'\Gamma_i = \tau_i d_i', \ i = 1, \dots, \ell, c'\Gamma_i = 0, \ i = \ell + 1, \dots, k, \ c'\mu = d_\mu. \quad (6)$$

Note that (5) implies that $\tau d'\beta_\perp = 0$, so that when $n \le q$ we find $\bar{\tau}'\tau d'\beta_\perp = d'\beta_\perp = 0$, and hence $sp(d) \subseteq sp(\beta)$ and therefore $n \le r$.

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So far the restrictions from the exact rational expectations are the same as in Johansen and Swensen (2008). Whereas the focus there was on models with restrictions on the cointegration vectors defined by β, we now consider restrictions on the adjustment parameters in α. Consider the following two specifications of restriction (5) on α

$$\alpha = A\psi,\ c'\alpha\beta' = \tau d',\ \text{and}\ \ rank(\alpha\beta') = r, \tag{7}$$

and

$$\alpha = (a, a_{\perp}\phi),\ c'\alpha\beta' = \tau d',\ \text{and}\ \ rank(\alpha\beta') = r. \tag{8}$$

Thus, ψ and ϕ describe restrictions on the adjustment coefficients α and we can now define two submodels of $\mathcal{H}_1(r)$ and $\mathcal{H}_2(r)$ respectively which satisfy the restrictions in $\mathcal{RE}$.

$\mathcal{H}_1^{\dagger}(r)$: The model is a submodel of $\mathcal{H}_1(r)$ which satisfies the restrictions (6) and (7),
$\mathcal{H}_2^{\dagger}(r)$: The model is a submodel of $\mathcal{H}_2(r)$ which satisfies the restrictions (6) and (8).

When estimating models $\mathcal{H}_1^{\dagger}(r)$ and $\mathcal{H}_2^{\dagger}(r)$ it is convenient to use a parametrization of freely varying parameters. Such a parametrization is given for the first model in Section 3 together with an analysis of the estimation problem. In Section 4 we discuss the model $\mathcal{H}_2^{\dagger}(r)$.

3. THE SAME RESTRICTIONS ON ALL α, THAT IS, $\mathcal{H}_1^{\dagger}(r)$

We first give a representation in terms of freely varying parameters of the matrix $\alpha\beta'$, when restricted by $\alpha = A\psi$ and $c'\alpha\beta' = \tau d'$, see (7). Then the relation between the $sp(c)$ and $sp(A)$ must be taken into account. Here $sp(c)$ and sp(A) denote the linear subspaces of $\mathbb{R}^p$ spanned by the columns of c and A respectively.

3.1. A Reparameterization of $\mathcal{H}_1^{\dagger}(r)$

Following Johansen and Swensen (2008) it is convenient to define the $s \times o$ matrix u and $(p-q) \times o$ matrix v such that $A'\overline{c}_{\perp} = uv'$ where o is equal to the rank of $A'\overline{c}_{\perp}$. The space $\mathbb{R}^p$ has the orthogonal decomposition $(c, \overline{c}_{\perp}\overline{v}, c_{\perp}v_{\perp})$. Also, when $\overline{c}_{\perp}'A \neq 0$,

$$A = c\overline{c}'A + c_{\perp}\overline{c}_{\perp}'A = c\overline{c}'A + c_{\perp}vu'$$

so $sp(c, A) = sp(c, c_{\perp}v)$ and $c_{\perp}v_{\perp}$ spans the orthogonal complement of $sp(c, A)$. In particular it follows that if $sp(c) \subseteq sp(A)$, then $sp(A) = sp(c, A) = sp(c, c_{\perp}v)$ such that

$$sp(c_{\perp}v) \subseteq sp(A)\ \ \text{and}\ \ o = s - q \tag{9}$$

since A and c have rank s and q respectively. One can then prove the following generalization of Proposition 2 in Johansen and Swensen (1999). The proof can be found in Appendix A.

Proposition 1. Consider the matrix $\Pi = \alpha\beta'$ of the model defined in (1). Let c and d be known matrices of full rank and dimensions $p\times q$ and $p\times n$ respectively where $n \leq q$. Let A be a known $p\times s$ matrix such that $rank(A'\overline{c}_{\perp}) = o \leq s$ and $A'\overline{c}_{\perp} = uv'$ for matrix u of rank o and dimension $s \times o$ and matrix v of rank o and of dimension $(p-q) \times o$.

Assume $A'\overline{c}_{\perp} \neq 0$ and consider two sets of restrictions on the parameters of the matrix Π.

The first set of restrictions is formulated as

$$\alpha = A\psi, c'\alpha\beta' = \tau d'\ \ \text{and rank}(\alpha\beta') = r, \tag{10}$$

see (7).

Table I. Summary of the coefficient and parameter matrices and their dimensions, as used in the formulation of the model and its reparameterization in Proposition 1

	Model			Reparameterization		
Coefficients	$A_{p\times s}$	$c_{p\times q}$	$d_{p\times n}$	$u_{s\times o}$	$v_{(p-q)\times o}$	
Parameters	$\psi_{s\times r}$	$\beta_{p\times r}$	$\tau_{q\times n}$	$\kappa_{o\times(r-n)}$	$\theta_{o\times n}$	$\zeta_{(p-n)\times(r-n)}$

The second set of restrictions is formulated as

$$\alpha\beta' = \overline{c}\tau d' + c_\perp v\theta d' + c_\perp v\kappa\zeta' d'_\perp \tag{11}$$

where $rank(A'\overline{c}_\perp) = o \geq r-n$, and there exist matrices κ and ζ of full rank and dimensions $o \times (r-n)$ and $(p-n)\times(r-n)$ respectively and where θ is $o \times n$.

Then it holds that (10) implies (11), and if further $sp(c) \subseteq sp(A)$ then (11) implies (10).

We summarize the dimensions of the matrices introduced in Table I.

Remark 1. Note that the $p \times n$ matrix d contains the known cointegration relations such that $r-n$ is the number of freely varying cointegration relations. Similarly s is the dimension of the range of α, so $s-r$ is a measure of the indeterminacy of the adjustment vectors α.

Remark 2. From Proposition 1 one gets conditions where a reparameterization using τ, κ, θ and ζ is possible instead of $\alpha = A\psi$ and β restricted as described in (10). To be more specific: in (11) $\alpha\beta'$ is expressed by the parameters τ, θ and $\kappa\zeta'$. Conversely, to express τ, θ and $\kappa\zeta'$ by $\alpha = A\psi$ and β one can use the following equation from the proof of Proposition 1

$$\begin{pmatrix} c' \\ \overline{c}'_\perp \end{pmatrix} A\psi\beta'(\overline{d}, \overline{d}_\perp) = \begin{pmatrix} \tau & 0 \\ v\theta & v\kappa\zeta' \end{pmatrix}.$$

Hence $\tau = c'A\psi\beta'\overline{d}$ and $\theta = (v'v)^{-1}v'\overline{c}'_\perp A\psi\beta'\overline{d} = u'\psi\beta'\overline{d}$. Also $v\kappa\zeta' = \overline{c}'_\perp A\psi\beta'\overline{d}_\perp = vu'\psi\beta'\overline{d}_\perp$ such that $\kappa\zeta' = u'\psi\beta'\overline{d}_\perp$ after multiplication with $\overline{v}'$.

Remark 3. The number of parameters in the matrix Π restricted as in Model $H_1^\dagger(r)$: $\Pi = A\psi\beta'$ and by (7): $c'\Pi = c'A\psi\beta' = \tau d'$, can be found from the representation (11) in Proposition 1. The number is given by

$$\#\tau_{q\times n} + \#\theta_{o\times n} + \#\kappa_{o\times(r-n)}\zeta'_{(r-n)\times(p-n)} = qn + on + (r-n)(p-r+o),$$

which for $sp(c) \subseteq sp(A)$, where $o = s-q$, reduces to $sr + (r-n)(p-r-q)$.

The case when $A'\overline{c}_\perp = 0$ represents a particular case. An important example is when the matrices A and c are equal, $A = c$. In Proposition 2 the situation is described. Details of the proof, using arguments similar to those used to prove Proposition 1, can be found in Appendix B.

Proposition 2. Assume $A'\overline{c}_\perp = 0$. Then the following statements are equivalent.

$$c'A\psi\beta' = \tau d' \tag{12}$$

and

$$A\psi\beta' = \overline{c}\tau d'. \tag{13}$$

In both cases $r = rank(A\psi\beta') = n$.

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Remark 4. If $X_t = (X'_{1t}, X'_{2t})'$ of dimensions q and $p-q$ and $A = (I, 0)'$, $sp(c) \subset sp(A)$ means that the elements in the lower rows of a matrix c belonging to $sp(c)$ are equal to 0. Generally, let $c = (c'_1, c'_2)'$ where c_1 and c_2 are $s \times q$ and $(p-s) \times q$ matrices respectively. Then $c'A = (c'_1, c'_2)A = c'_1$ such that the restriction $c'A\psi\beta' = \tau d'$ does not involve c_2 which represents the part of c not in $sp(A)$. Thus, the restrictions on $\Pi = A\psi\beta'$ only involve c_1, whereas the restrictions $c'\Gamma_i = \tau_i d'_i$ involve both c_1 and c_2.

In the following section, we consider estimating and testing where we assume $sp(c) \subseteq sp(A)$. Then the results simplify and a likelihood ratio test can be found.

3.2. Estimating and Testing Model $\mathcal{H}_1^{\dagger}(r)$ when $sp(c) \subseteq sp(A)$

We now show how estimation of $\mathcal{H}_1^{\dagger}(r)$ defined in Section 2.2 can be performed by reduced rank regression and regression under the assumption that $sp(c) \subseteq sp(A)$. Then $o = s - q$, see (9). Two cases need to be distinguished.

First consider the case where $rank(A'\bar{c}_\perp) = o < r-n$. The restricted model can then be estimated by first translating it by premultiplying the model (1) with $(c, \bar{c}_\perp)'$ and incorporating the restrictions. Then one can reparameterize by conditioning $\bar{c}'_\perp \Delta X_t$ on $c'\Delta X_t$ and the past. The conditional equation can be estimated by a combination of reduced rank and ordinary least squares (OLS) regressions, and the parameters in marginal equation for $c'\Delta X_t$ can be estimated by OLS regressions. Details can be found in Appendix D.

Next consider the case where $o \geq r-n$. Then a more elaborate argument is needed. The matrix $\alpha\beta'$ can according to Proposition 1 be reparameterized as

$$\alpha\beta' = \bar{c}\tau d' + c_\perp v\theta d' + c_\perp v\kappa\zeta' d'_\perp, \tag{14}$$

and $c'\Gamma_i,\ i = 1 \ldots, k$, μ are restricted as

$$c'\Gamma_i = \tau_i d'_i,\ i = 1, \ldots, \ell, c'\Gamma_i = 0,\ i = \ell+1, \ldots, k,\ c'\mu = d_\mu, \tag{15}$$

where the parameters to be estimated are $\tau, \theta, \kappa, \zeta, \tau_i,\ i = 1, \ldots, \ell, d_\mu$. Remark that it follows from Proposition 1 that $rank(A'\bar{c}_\perp) = o \geq r-n$ when $c'A\psi\beta' = \tau d'$ and $rank(A\psi\beta') = r$.

To avoid a complicated notation we assume that $k = \ell = 1$ when describing the estimation procedure in this case. This is not a serious restriction and we denote $\Gamma_1 = \Gamma$. The extension to $k = \ell > 1$ presents no problem, and the case $1 \leq \ell < k$ corresponds to $d_{\ell+1} = \cdots = d_k = 0$.

We first define the three processes $X^*_{1t}, X^*_{2t}, X^*_{3t}$, by

$$X^*_t = \begin{pmatrix} X^*_{1t} \\ X^*_{2t} \\ X^*_{3t} \end{pmatrix} = \begin{pmatrix} \bar{v}'\bar{c}'_\perp X_t \\ v'_\perp \bar{c}'_\perp X_t \\ c'X_t \end{pmatrix} = GX_t, \tag{16}$$

of dimensions $(s-q, p-s, q)$ respectively. Define $\alpha^*, \Gamma^*, \mu^*$ and ε^*_t similarly, that is, $\alpha^* = (\alpha^{*\prime}_1, \alpha^{*\prime}_2, \alpha^{*\prime}_3)' = G\alpha, \Gamma^* = G\Gamma$, $\mu^* = G\mu$ and $\varepsilon^*_t = G\varepsilon_t$ and $\Sigma = G\Omega G'$. The equations then become

$$\Delta X^*_t = \alpha^*\beta' X_{t-1} + \Gamma^*\Delta X_{t-1} + \mu^* + \varepsilon^*_t. \tag{17}$$

We next want the conditional equations and define $\varepsilon^{**}_t = K\varepsilon^*_t$ where

$$K = \begin{pmatrix} I_{s-q} & -\omega_{12.3} & -\omega_{13.2} \\ 0 & I_{p-s} & -\omega_{2.3} \\ 0 & 0 & I_q \end{pmatrix} \tag{18}$$

with $\omega_{12.3} = \Sigma_{12.3}\Sigma_{22.3}^{-1}$, $\omega_{13.2} = \Sigma_{13.2}\Sigma_{33.2}^{-1}$ and $\omega_{2.3} = \Sigma_{23}\Sigma_{33}^{-1}$. Then $\varepsilon_{1t}^{**}, \varepsilon_{2t}^{**}, \varepsilon_{3t}^{*}$ are independent. In particular $\varepsilon_{3t}^{**} = c'\varepsilon_t$. Furthermore, we find the equation for ΔX_{1t}^* given $(\Delta X_{2t}^*, \Delta X_{3t}^*)$, the equation for ΔX_{2t}^* given ΔX_{3t}^* and the equation for ΔX_{3t}^* by premultiplying (17) by the matrix K.

For the situation where τ is known we can summarize the estimation procedure in the following proposition. More details can be found in Appendix C.

Proposition 3. Estimation of the model $\mathcal{H}_1^{\dagger}(r)$ when $rank(A'\overline{c}_\perp) > r - n$ and $sp(c) \subseteq sp(A)$ can be conducted in three steps when τ is known:

1. By reduced rank regression of ΔX_{1t}^* on $d_\perp' X_{t-1}$ corrected for the regressors $1, \Delta X_{2t}^*, \Delta X_{3t}^*, d'X_{t-1}, \Delta X_{t-1}, \dots, \Delta X_{t-k}$ find estimates for $\mu_1^* - \omega_{12.3}\mu_2^* - \omega_{13.2}d_\mu, \omega_{12.3}, \omega_{13.2}, (\theta - \omega_{13.2}\tau), \kappa, \zeta, (\overline{v}' - \omega_{12.3}v_\perp')\overline{c}_\perp'\Gamma_i - \omega_{13.2}\tau_i d_i', i = 1, \dots, k$ and $\Sigma_{11.23}$.
2. For fixed value of τ introduce the variable $Y_t^* = \Delta X_{3t}^* - \tau d' X_{t-1}$ and find the system

$$\Delta X_{2t}^* = \mu_2^* - \omega_{2.3}d_\mu + \omega_{2.3}Y_t^* + \sum_{i=1}^{\ell}(v_\perp'\overline{c}_\perp'\Gamma_i - \omega_{2.3}\tau_i d_i')\Delta X_{t-i} + \sum_{i=\ell+1}^{k} v_\perp'\overline{c}_\perp'\Gamma_i\Delta X_{t-i} + \varepsilon_{2t}^{**},$$

$$Y_t^* = d_\mu + \sum_{i=1}^{\ell}\tau_i d_i'\Delta X_{t-i} + \varepsilon_{3t}^{**}.$$

3. This system can be estimated by OLS regression. From the equation for ΔX_{2t}^* a regression gives estimates for $\mu_2^* - \omega_{2.3}d_\mu,\ \omega_{2.3}, (v_\perp'\overline{c}_\perp'\Gamma_i - \omega_{2.3}\tau_i d_i'),\ i = 1, \dots, \ell, v_\perp'\overline{c}_\perp'\Gamma_i,\ i = \ell+1 \dots, k, \Sigma_{22.3}$. From the equation for Y_t^*, regression gives the estimates $\hat{d}_\mu, \hat{\tau}_1, \dots, \hat{\tau}_\ell, \hat{\Sigma}_{33}$ depending on τ.
4. Finally, the maximal value of the likelihood, apart from constants, is

$$L_{\max}^{-2/T}(\tau) = |\hat{\Sigma}_{11.23}||\hat{\Sigma}_{22.3}||\hat{\Sigma}_{33}|/|c'c||\overline{v}'\overline{c}_\perp'\overline{c}_\perp\overline{v}||v_\perp'\overline{c}_\perp'\overline{c}_\perp v_\perp|.$$

Remark 5. We can therefore calculate the maximized likelihood for the three equations for a given τ and maximize with respect to τ by a general maximization algorithm if the parameter τ is not known.

Remark 6. The case $r - n = 0$, which means that the rational expectation hypothesis specifies all cointegrating relations, needs a special comment. Then $\kappa\zeta' = 0$ and the last term in (14) disappears. If also $A'\overline{c}_\perp = 0$, it follows from the proof of Proposition 2 that $A\psi\beta' = \overline{c}\tau d'$. By the eqvivalence in Proposition 2 this means that the condition $c'A\psi\beta' = \tau d'$ is always satisfied and there is nothing to estimate in the reduced rank matrix $\Pi = \alpha\beta'$. When $A'\overline{c}_\perp \neq 0$ and $r - n = 0$ the parameters can be estimated by OLS. For the case where $A = I$ and $n = q$ see Johansen and Swensen (1999) where the procedure is written out in detail.

Remark 7. Another possibility in addition to the one described in Remark 5 for estimating the equations for $\Delta X_{2t}^*, \Delta X_{3t}^*$ when τ is unknown is a constrained regression. Notice that the model

$$\Delta X_{2t}^* = \sum_{i=1}^{k}\Gamma_{2i}^*\Delta X_{t-i}^* + \mu_2^* + \varepsilon_{2t}^*,$$

$$\Delta X_{3t}^* = \tau d' X_{t-1} + \sum_{i=1}^{\ell}\Gamma_{3i}^*\Delta X_{t-i}^* + \mu_3^* + \varepsilon_{3t}^*.$$

with the restriction (15) is linear in the conditional mean and hence can be estimated by generalized least squares for fixed variance matrix. For fixed linear parameters the variance can be estimated from the residuals, such

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that an iteration procedure can be defined. This is an example of a coordinate search method, see for example, Nocedal and Wright (2006). The model is a special case of seemingly unrelated regressions, SUR. In Oberhofer and Kmenta (1974) it is shown that for such models the sequence has a limit point which is a solution of the likelihood equations. In general, for iterative methods for maximization there is no guarantee that they will converge toward the global maximum if there are several local maxima. Drton and Richardson (2004) contains a discussion of multi modularity of the likelihood in bivariate SUR models.

Let the residual from this fit be $R_{23,t}$ and let $S_{23} = \frac{1}{T}\sum_{t=1}^{T} R_{23,t}R'_{23,t}$. Then the maximal value of the likelihood, apart from constants, can be expressed as

$$L_{\max}^{-2/T} = |\hat{\Sigma}_{11.23}||S_{23}|/|c'c||\overline{v}'\overline{c}'_{\perp}\overline{c}_{\perp}\overline{v}||v'_{\perp}\overline{c}'_{\perp}\overline{c}_{\perp}v_{\perp}|.$$

Remark 8. For the case where all elements in the matrix d_μ are known a small modification is necessary in the equation for the variable $\Delta X^*_{3,t}$. Instead of regressing $\Delta X^*_{3,t} - \tau d' X_{t-1}$ on $d'_1\Delta X^*_{t-1}, \dots, d'_\ell \Delta X^*_{t-\ell}$ and 1, regress $\Delta X^*_{3,t} - \tau d' X_{t-1} - d_\mu$ on $d'_1\Delta X^*_{t-1}, \dots, d'_\ell \Delta X^*_{t-\ell}$ only. In particular, if $d_\mu = 0$ the response is $\Delta X^*_{3,t} - \tau d' X_{t-1}$ and the regressor 1 is dropped.

Remark 9. There is an interesting modification of the estimation procedure described above. If the coefficient $-\omega_{2.3}\tau$ of $d'X_{t-1}$ in the equation for ΔX^*_{2t} in the proof of Proposition 3 is replaced by a freely varying parameter the new system will contain $(p-s)n$ extra parameters. It will, however, have a structure so that the expanded parameter set can be estimated by OLS regressions.

4. SOME α ASSUMED KNOWN, THAT IS, $\mathcal{H}_2^{\dagger}(r)$

We consider the situation where the freely varying parameters of the matrix $\alpha\beta'$ are restricted by $\alpha = (a, a_\perp \phi)$ such that $c'\alpha\beta' = c'(a\beta'_1 + a_\perp\phi\beta'_2) = \tau d'$ where a and c are known $p \times m$ and $p \times q$ matrices respectively, both of full rank, see (8). The matrices ϕ, β_1 and β_2 have dimensions $(p-m)\times(r-m), p\times m$ and $p\times(r-m)$. We start with two special cases. Thereafter we show how a more general model can be estimated by combining the reults.

4.1. Estimating Some Special Cases of the Model $\mathcal{H}_2^{\dagger}(r)$

Case 1: $c'a = 0$, $0 < m \le r - n$ and β_1 known.

Then it is possible to apply Proposition 1. The constraints on the matrix Π are now

$$c'\Pi = c'\alpha\beta' = c'(a\beta'_1 + a_\perp\phi\beta'_2) = \tau d' \quad \text{or} \quad c'\alpha\beta' = c'a_\perp\phi\beta'_2 = \tau d'$$

such that the model can be written

$$\Delta X_t - a\beta'_1 X_{t-1} = a_\perp\phi\beta'_2 X_{t-1} + \sum_{i=1}^{k}\Gamma_i \Delta X_{t-i} + \mu + \varepsilon_t$$

with constraints $c'a_\perp\phi\beta'_2 = \tau d'$ and (6).

These are analogous to the restrictions (6) and (7) with $rank(a_\perp\phi\beta'_2) = r_1$ equal to $r-m$ and the known $p\times(p-m)$ matrix $a_\perp$ corresponding to A. In addition $c'a = 0$ is equivalent to $sp(c) \subseteq sp(a_\perp)$. Also, we deal only with the situations where $0 < m \le r - n$. The case where $r - m = n$, that is, $m = r - n$, is special as noted in Remark 6. In particular since $sp(d) \subseteq sp(\beta_2)$, as explained after equation (6), and rank $sp(\beta_2) = r - m = n = rank(d)$, $sp(d) = sp(\beta_2)$ such that β_2 is known up to a normalization in this case.

But there are also two other distinct cases depending on the value of $rank(a'_\perp\overline{c}_\perp)$. If $rank(a'_\perp\overline{c}_\perp) < r_1 - n = r - m - n$, the argument based on the conditional equation of $\overline{c}'_\perp\Delta X_t$ given $c'\Delta X_t$ and the past must be applied, see

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Appendix D. If $rank(a_\perp' \bar{c}_\perp) \geq r_1 - n > 0$, the conditions of Proposition 1 are satisfied, and the parameters can be estimated as described in Proposition 3.

In all cases the estimation can be done either by first keeping τ fixed and then using a general optimizing algorithm to find the maximizing value of τ or using the SUR procedure described in Remark 7.

Case 2: $0 < m = r$, $\alpha = a$.

Then all adjustment parameters are known. After premultiplying the model (1) with $(c, \bar{c}_\perp)'$ and incorporating the restrictions, the parameters of the coefficient of the level do not have a multiplicative structure. The reason is that the parameters in β are the only unknowns. A direct application of Proposition 3 is therefore not possible, and a small modification of the arguments used there is necessary. The details can be found in Appendix E.

4.2. Estimating the Model $\mathcal{H}_2^\dagger(r)$ When $c'a = 0$ and $0 < m \leq r - n$

Now, consider the situation where $c'(a\beta_1' + a_\perp \phi \beta_2') = \tau d'$, $c'a = 0$ and $0 < m \leq r - n$. Then the model with the restrictions imposed can be written

$$\bar{c}_\perp' \Delta X_t = \bar{c}_\perp' a \beta_1' X_{t-1} + \bar{c}_\perp' a_\perp \phi \beta_2' X_{t-1} + \sum_{i=1}^{k} \bar{c}_\perp' \Gamma_i \Delta X_{t-i} + \bar{c}_\perp' \mu + \bar{c}_\perp' \varepsilon_t,$$

$$c' \Delta X_t = \tau d' X_{t-1} + \sum_{i=1}^{\ell} \tau_i d_i' \Delta X_{t-i} + c' \mu + c' \varepsilon_t,$$

where the parameters to be estimated are $\beta_1, \phi, \beta_2, \bar{c}_\perp' \Gamma_i, i = 1, \dots, k, \tau, \tau_i, i = 1, \dots, \ell, \mu$ and Ω.

The model is unidentified if $rank(\bar{c}_\perp' a) < m$ and $rank(\bar{c}_\perp' a) > m$ is not possible since the matrix a has rank m. Hence $rank(\bar{c}_\perp' a) = m$ if we assume that the model is identified. We then propose the following iterative procedure for estimating the parameters by switching between Step 1 and Step 2.

Consider first the situation where $rank(a_\perp' \bar{c}_\perp) \geq r_1 - n$. Assume first that τ is fixed and known.

Step 1. Keep ϕ and β_2 fixed. Writing the model

$$\bar{c}_\perp' (\Delta X_t - a_\perp \phi \beta_2' X_{t-1}) = \bar{c}_\perp' a \beta_1' X_{t-1} + \sum_{i=1}^{k} \bar{c}_\perp' \Gamma_i \Delta X_{t-i} + \bar{c}_\perp' \mu + \bar{c}_\perp' \varepsilon_t,$$

$$c' \Delta X_t - \tau d' X_{t-1} - \sum_{i=1}^{\ell} \tau_i d_i' \Delta X_{t-i} = c' \mu + c' \varepsilon_t$$

estimate the parameters $\beta_1, \bar{c}_\perp' \Gamma_i,\ i = 1, \dots, k,\ \tau_i,\ i = 1, \dots, \ell, \mu$ and Σ as described in Case 2 in the previous subsection.

Step 2. Keep β_1 fixed. Remember that $rank(a_\perp \phi \beta_2') = r_1 = r - m$ and $c'a = 0$ and write the model

$$\bar{c}_\perp' (\Delta X_t - a \beta_1' X_{t-1}) = \bar{c}_\perp' a_\perp \phi \beta_2' X_{t-1} + \sum_{i=1}^{k} \bar{c}_\perp' \Gamma_i \Delta X_{t-i} + \bar{c}_\perp' \mu + \bar{c}_\perp' \varepsilon_t,$$

$$c' \Delta X_t - \tau d' X_{t-1} - \sum_{i=1}^{\ell} \tau_i d_i' \Delta X_{t-i} = c' \mu + c' \varepsilon_t.$$

Estimate the parameters ϕ, β_2 (when they are unknown, i.e., $a_\perp' \bar{c}_\perp \neq 0$), $\bar{c}_\perp' \Gamma_i,\ i = 1, \dots, k,\ \tau_i,\ i = 1, \dots, \ell,\ \mu$ and Σ as described in Case 1 in the previous subsection.

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The value of the likelihood increases for each iteration even though convergence toward a global maximum, or even convergence, cannot be guaranteed. Also convergence, when it occurs, can be slow. But such coordinate search methods can still be useful, see Nocedal and Wright (2006), page 230.

If $\hat\beta_1, \hat\phi, \hat\beta_2, \widehat{\bar c'_\perp\Gamma_i},\ i = 1, \dots, k, \hat\mu, \hat\tau_1, \dots, \hat\tau_\ell$ and $\hat\Sigma$ are the maximum likelihood estimates when τ is fixed and

$$R_t = \begin{pmatrix} \bar c'_\perp \Delta X_t \; - \; \bar c'_\perp a\hat\beta_1 X_{t-1} - \bar c'_\perp a_\perp \hat\phi\hat\beta'_2 X_{t-1} - \sum_{i=1}^k \widehat{\bar c'_\perp\Gamma_i}\Delta X_{t-i} - \bar c'_\perp\hat\mu \\ c'\Delta X_t \; - \; \tau d' X_{t-1} - \sum_{i=1}^{\ell}\hat\tau_i d'_i \Delta X_{t-i} - c'\hat\mu \end{pmatrix},\ t = 1 \dots, T$$

are the residuals, the maximal value of the likelihood is

$$L_{\max}^{-2/T}(\tau) = \left|\sum_{t=1}^{T} R_t R'_t\right| / |c'c\,||\bar c'_\perp \bar c_\perp|.$$

For unknown τ we can find the maximum likelihood estimators using a general numerical optimization procedure. Another possibility is to insert a SUR step to estimate an unknown τ in the conditional distributions described in Case 1 and Case 2 in the previous subsection.

Now consider the other situation, where $rank(a'_\perp \bar c_\perp) < r_1 - n$. One can proceed as follows. In Step 1 one can as before apply the method described in Case 2 in the previous subsection and in more detail in Appendix E. In Step 2 the matrix $\bar c'_\perp a_\perp \phi\beta'_2$ has reduced rank, and the method described in Appendix D can be used.

Remark 10. The number of parameters in the matrix Π satisfying $\mathcal{H}_2^\dagger(r) : \Pi = a\beta'_1 + a_\perp\phi\beta'_2$ and the restriction $c'(a\beta'_1 + a_\perp\phi\beta'_2) = \tau d'$, τ unknown can be found counting the parameters estimated by the recursive procedure. Remember $rank(\bar c'_\perp a) = m$ in an identified model and that $c'a = 0$ and $0 < m$ are assumed. First, β_1 in Step 1 contains $p \cdot rank(\bar c'_\perp a) = pm$ parameters. Second, in Step 2 we have to consider the two cases.

If $rank(a'_\perp \bar c_\perp) \ge r_1 - n$, the results from Section 3.2 can be applied. Then $rank(a_\perp\phi\beta'_2) = r_1 = r - m$ and it follows from Remark 3 that the number of parameters in $a_\perp\phi\beta'_2$ satisfying $c'a_\perp\phi\beta'_2 = \tau d'$ is $(p-m)(r-m) + (r-m-n)(p-r+m-q)$. We have used that s corresponds to $p-m$. and that the rank of $\phi\beta'_2$ is $r-m$.

If $o = rank(a'_\perp \bar c_\perp) < r_1 - n$ the matrix $\bar c'_\perp a_\perp\phi\beta'_2$ of dimension $(p-q)\times p$ and rank o contains $(p-q)o + o(p-o) = o(2p-q-o)$ parameters.

5. SOME NUMERICAL RESULTS

To get an impression of the small sample distribution we carried out a simulation study. 1000 replications of the following 27 CVAR(1) time series were simulated

$$\Delta X_t = f_j \cdot \alpha\beta' + \varepsilon_t,\ t = 1, \dots, T,$$

where $X_0 = 0$, $T = 50{,}100, 200$ and $f_j = 1.0 + j\cdot 0.03,\ j = 0, \dots, 8$. The rank 1 reduced rank matrix is $\alpha\beta' = \bar c\tau d' + c_\perp v\theta d'$ where, inspired by Example 1, $c' = (-1, 0, 0)$, $d' = (-1, 1, 1)$ and $\tau = 0.99$. The vector $v = (1\ 0)'$ and parameter θ is taken as $\theta = -2.0$. Note in particular that $sp(c)$ is contained in $sp(A)$ where $A' = \begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$. Then $A'\bar c_\perp = \begin{pmatrix}0\\1\end{pmatrix}(1\ 0) = uv'$.

Also the 27 specifications satisfy the $I(1)$ requirement, and the moduli of the roots of the determinant of the characteristic polynomial are all larger or equal to one.

To each replication a test of the rational expectation hypothesis described in Proposition 3 was applied with the matrices c and A as specified above. Likelihood ratio tests with level 0.05 were performed against two alternatives. In one no restrictions except reduced rank 1 were imposed. The test has 4 degrees of freedom. In the other also

Table II. Probability of rejection, null hypothesis rational expectation, alternative only reduced rank 1

	Values of f								
	1.0	1.03	1.06	1.09	1.12	1.15	1.18	1.21	1.24
$T = 50$	0.06	0.09	0.09	0.14	0.19	0.27	0.38	0.49	0.65
$T = 100$	0.06	0.08	0.12	0.19	0.36	0.55	0.71	0.86	0.94
$T = 200$	0.06	0.09	0.19	0.37	0.67	0.86	0.95	0.99	1.00

Table III. Probability of rejection, null hypothesis rational expectation, alternative reduced rank 1 and weak exogeneity of last element

	Values of f								
	1.0	1.03	1.06	1.09	1.12	1.15	1.18	1.21	1.24
$T = 50$	0.06	0.09	0.10	0.17	0.21	0.29	0.42	0.55	0.70
$T = 100$	0.06	0.08	0.14	0.26	0.38	0.58	0.78	0.90	0.96
$T = 200$	0.05	0.10	0.22	0.41	0.72	0.89	0.97	1.00	1.00

weak exogeneity of the form $\alpha = A\psi$ was imposed. This test has 3 degrees of freedom. The result of the tests are displayed in Tables II and III. The matrix $\alpha\beta'$ was generated by $f_0 = 1.0$, which represents the null hypothesis whereas $f_1, \dots, f_8$ represent alternatives. They describe sensitivity of the tests to increasing values of the adjustment coefficients. The first column shows the size of the tests and the remaining columns the power. One can see that the size properties are satisfactory in both cases. As f_j increases, the power or the probability of rejection increases. Under the alternatives the power for both test are increasing in T, that is, the tests are consistent. Also the power is larger in for the case where weak exogeneity, $\alpha = A\psi$ is imposed.

The simulations and the calculation of the two likelihood ratio statistics were carried out using the software package `R`, R Core Team (2020). The calculations only take a few seconds on a laptop.

6. AN APPLICATION

We consider the inflation model discussed in Example 2. In Boug *et al.* (2017) a data set covering the period 1982:1-2005:4 was analyzed. A reduced rank vector autoregressive model with three lags, that is, $k = 2$, an unrestricted constant, seasonal dummies and five impulse dummies was fitted to the time series. The inclusion of the non-stochastic dummies makes this model a bit different from those treated earlier in this article. Equation (2) now implies restrictions on the dummies. In the application below we regress on the dummies, which means that these restrictions are dropped and only coefficients on stochastic terms are considered. Thus, the null hypothesis will be larger than specified in (2) for a model containing dummies.

Rank equal to 1 was found to yield a satisfactory fit, the cointegration vector was estimated as

$$p_t = 0.649ulc_t + 0.340uic_t, \ 2\log L_{\max}(\mathcal{H}(r)) = 2536.72$$

and a test that the coefficients are proportional to $(1, -2/3, -1/3)'$ was accepted with a p-value of 0.93. The adjustment parameters are $\hat{\alpha} = (-0.056, 0.175, 0.143)'$. Table II shows the result of testing that $\alpha_3 = 0$, that is whether uic_t is weakly exogenous for α_1, α_2 and β. The p-value is 0.10. We can therefore turn to testing the model $\mathcal{H}_1^{\dagger}(1)$ against the model $\mathcal{H}_1(1)$ using the results from Section 3. We then assume that the matrix d is equal to $d = (1, -2/3, -1/3)'$. As explained in Example 2 this is an interesting case. At the end of this section we point out how the situation where d contains unknown parameters can be treated.

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We estimate the model $\mathcal{H}_1^{\dagger}(1)$ under the assumption $\alpha_3 = 0$ in addition to satisfying the exact rational expectations hypothesis defined in (4). Then

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $c = (1, 0, 0)'$ such that $sp(c) \subseteq sp(A)$. Also

$$A' c_\perp = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1, 0) = uv'.$$

Thus $\overline{v}' \overline{c}'_\perp = (0, 1, 0) = e'_2$ and $v'_\perp \overline{c}'_\perp = (0, 0, 1) = e'_3$ such that $G = (e_2, e_3, e_1)'$.

After taking the restrictions into account the equations from Section 3.2 become

$$\begin{aligned}
\overline{v}'\overline{c}'_\perp \Delta X_t &= \theta d' X_{t-1} + \overline{v}'\overline{c}'_\perp \Gamma_1 \Delta X_{t-1} + \overline{v}'\overline{c}'_\perp \Gamma_2 \Delta X_{t-2} + \overline{v}'\overline{c}'_\perp \Phi D_t + \overline{v}'\overline{c}'_\perp \varepsilon_t, \\
v'_\perp \overline{c}'_\perp \Delta X_t &= v'_\perp \overline{c}'_\perp \Gamma_1 \Delta X_{t-1} + v'_\perp \overline{c}'_\perp \Gamma_2 \Delta X_{t-2} + v'_\perp \overline{c}'_\perp \Phi D_t + v'_\perp \overline{c}'_\perp \varepsilon_t, \\
c' \Delta X_t &= \tau d' X_{t-1} + \tau_1 d'_1 X_{t-1} + \tau_2 d'_2 X_{t-2} + c' \Phi D_t + c' \varepsilon_t
\end{aligned}$$

and the conditional equations become

$$\begin{aligned}
\overline{v}'\overline{c}'_\perp \Delta X_t = \Delta ulc_t &= \omega_{12.3} \Delta uic_t + \omega_{13.2} \Delta p_t \\
&\quad + (\theta - \omega_{13.2}\tau) d' X_{t-1} - \sum_{i=1}^{2} (\omega_{13.2} \tau_i d'_i - e'_2 \Gamma_i) \Delta X_{t-i} + e'_2 \Phi D_t + \varepsilon^{**}_{1t}, \\
v'_\perp \overline{c}'_\perp \Delta X_t = \Delta uic_t &= \omega_{2.3} (\Delta p_t - \tau d' X_{t-1}) \\
&\quad - \sum_{i=1}^{2} (\omega_{13.2} \tau_i d'_i - e'_3 \Gamma_i) \Delta X_{t-i} + e'_3 \Phi D_t + \varepsilon^{**}_{2t}, \\
c' \Delta X_t = \Delta p_t &= \tau d' X_{t-1} + \sum_{i=1}^{2} \tau_i d'_i \Delta X_{t-i} + e'_1 \Phi D_t + \varepsilon^{**}_{3t}.
\end{aligned}$$

When τ is known, the system can be estimated by first regressing Δulc_t on Δuic_t, Δp_t, $d' X_{t-1}$, Δulc_{t-i}, $i = 1, 2$, Δuic_{t-i}, $i = 1, 2$, Δp_{t-i}, $i = 1, 2$ and D_t, then regressing Δuic_t on $\Delta p_t - \tau d' X_{t-1}$, Δulc_{t-i}, $i = 1, 2$, Δuic_{t-i}, $i = 1, 2$, Δp_{t-i}, $i = 1, 2$, and D_t and finally by regressing $\Delta p_t - \tau d' X_{t-1}$ on $d'_1 \Delta X_{t-1}, d'_2 \Delta X_{t-2}, D_t$.

The number of parameters in the reduced rank VAR model is $3 + 2 + 18 = 23$ in addition to the coefficients of constants and dummies. The corresponding number after imposing the restrictions is $1 + 12 + 2 = 15$. The appropriate degrees of freedom is therefore $23 - 15 = 8$ when τ is known and 7 when it must be estimated. When τ is unknown the maximum likelihood estimates can be obtained from the profile likelihood. Alternatively the approach based on generalized least squares can be used to find the maximal value of the likelihood.

The maximal values of 2 log likelihood from fitting the models are displayed in Table IV when d is fixed as $d = (1, -2/3, -1/3)'$. As mentioned, this is a sensible value of d and an interesting choice. We also noted that the p-value of the hypothesis that $\alpha_3 = 0$ is 0.10. The further hypothesis imposing in addition rational expectations, that is (4), is also not rejected. The maximum likelihood estimate for τ is denoted by $\hat{\tau}$. Ignoring the cross equation restrictions, as pointed out in Remark 9, implies that τ is estimated by regression from the marginal equation for

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Table IV. Summary of tests of $\alpha_3 = 0$ and the restriction (4) for $\mathcal{H}_1^{\dagger}(1)$ with $d' = (1, -2/3, -1/3)'$ fixed. LR is the likelihood ratio

Model	$2 \log L_{\max}$(Model)	$-2 \log LR$	df	p-value	$\hat{\tau}$	$\tilde{\tau}$
$\mathcal{H}(1)$	2536.72	–	–	–	–	–
$\mathcal{H}_1(1)$	2534.06	2.66	1	0.10	–	–
$\mathcal{H}_1^{\dagger}(1)$	2531.03	3.03	6	0.81	−0.048	−0.051

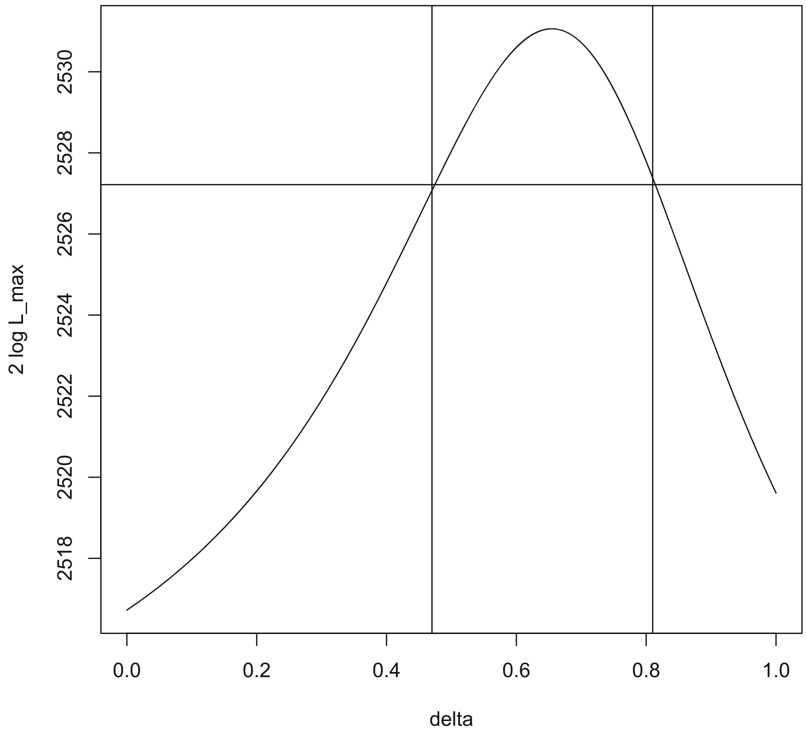

Figure 1. Profile likelihood of δ when $\alpha_3 = 0$. A 95% confidence interval is indicated

$\Delta X_{3,t}$. As one can see this estimate, denoted by $\tilde{\tau}$, is numerically quite similar to the result when the cross equation restrictions are taken into account. The estimated standard error is 0.013 such that an approximate 95% confidence interval is (−0.076, −0.026).

Up to now the matrices c and d have been considered as fixed. Often they contain unknown parameters. To deal with these parameters we suggest treating them first as known quantities. Then the likelihood can be found, as function of these variables from the results in Sections 3 and 4. The maximum of the function will correspond to the maximum likelihood estimate, and the maximum can be found by using a general purpose maximization algorithm. For the situation where $d = (1, -\delta, -(1-\delta))'$ the profile likelihood is shown in Figure 1 using the procedure `optimize` in the software package R, R Core Team (2020). The maximum likelihood estimate of δ corresponds to the maximizing value. This is 2531.06 such that the test statistics for testing simultaneously the rational expectations, that is (4), and the linear restrictions on the adjustment parameters, that is $\alpha_3 = 0$, against $\mathcal{H}(1)$ is 5.66 with 6 degrees of freedom which corresponds to a p-value 0.46. The maximum likelihood estimate for δ is $\hat{\delta} = 0.66$. with a 95% confidence interval for δ is (0.48,0.81).

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7. CONCLUSION

The theme of this article has been to analyze cointegrated vector autoregressive models with restrictions on the adjustment parameters and in addition restrictions from exact rational expectations imposed. We considered estimation and testing in such models where the adjustment parameters satisfied the same restrictions, that is $\alpha = A\psi$, and also for some special cases the situation where some of the adjustment parameters were known.

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DATA AVAILABILITY STATEMENT

Data and R-code for the calculations in the application are available from the corresponding author on request. Data sharing not applicable to this article as no new datasets were generated or analyzed during the current study.

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APPENDIX A: PROOF OF PROPOSITION 1

Proof that (10) *implies* (11). Premultiplying $\alpha\beta' = A\psi\beta'$ by $(c, \overline{c}_\perp)'$ and postmultiplying by $(\overline{d}, \overline{d}_\perp)$ we find using $c'\alpha\beta' = \tau d'$ that

$$\begin{pmatrix} c' \\ \overline{c}'_\perp \end{pmatrix} \alpha\beta'(\overline{d}, \overline{d}_\perp) = \begin{pmatrix} \tau d'\overline{d} & \tau d'\overline{d}_\perp \\ \overline{c}'_\perp \alpha\beta'\overline{d} & \overline{c}'_\perp \alpha\beta'\overline{d}_\perp \end{pmatrix} = \begin{pmatrix} \tau & 0 \\ \overline{c}'_\perp \alpha\beta'\overline{d} & \overline{c}'_\perp \alpha\beta'\overline{d}_\perp \end{pmatrix}.$$

Next use $\overline{c}'_\perp A = vu'$ to simplify the entries $\overline{c}'_\perp \alpha\beta'\overline{d}$ and $\overline{c}'_\perp \alpha\beta'\overline{d}_\perp$. First $\overline{c}'_\perp \alpha\beta'\overline{d} = \overline{c}'_\perp A\psi\beta'\overline{d} = vu'\psi\beta'\overline{d} = v\theta$ for $\theta = u'\psi\beta'\overline{d}$. Furthermore, let $\nu = \psi\beta'\overline{d}_\perp$ such that

$$\overline{c}'_\perp \alpha\beta'\overline{d}_\perp = \overline{c}'_\perp A\psi\beta'\overline{d}_\perp = vu'\nu.$$

Then

$$\begin{pmatrix} c' \\ \overline{c}'_\perp \end{pmatrix} A\psi\beta'(\overline{d}, \overline{d}_\perp) = \begin{pmatrix} \tau & 0 \\ v\theta & vu'\nu \end{pmatrix}.$$

Since the matrix $\alpha\beta'$ has rank r and τ has rank n, the matrix $vu'\nu$ must have rank $r - n \geq 0$. Also $r - n = rank(vu'\nu) \leq rank(vu') = rank(\overline{c}'_\perp A) = o$.

We apply Sylvester's inequality, see Horn and Johnson (2013) p. 13, to the $(p-q)\times o$ matrix v and the $o\times(p-n)$ matrix $u'\nu$ and find

$$rank(v) + rank(u'\nu) - o \leq rank(vu'\nu) \leq min(rank(v), rank(u'\nu)).$$

Because $rank(v) = o$, this shows that $rank(u'\nu) \leq rank(vu'\nu)$. The reverse inequality is obvious so we find $rank(u'\nu) = rank(vu'\nu) = r - n$.

Then, since $o \geq r - n$ there exists matrices κ and ζ of full rank and dimensions $o \times (r-n)$ and $(p-n)\times(r-n)$ respectively such that that $u'\nu = \kappa\zeta'$ and

$$A\psi\beta' = (\overline{c}, c_\perp) \begin{pmatrix} \tau & 0 \\ v\theta & v\kappa\zeta' \end{pmatrix} \begin{pmatrix} d' \\ d'_\perp \end{pmatrix} = \overline{c}\tau d' + c_\perp v\theta d' + c_\perp v\kappa\zeta' d'_\perp.$$

Proof that (11) *implies* (10). Sylvester's inequality applied to the $(p-q)\times o$ matrix v and the $o\times(r-n)$ matrix κ gives

$$rank(v) + rank(\kappa) - o \leq rank(v\kappa) \leq min(rank(v), rank(\kappa)),$$

or

$$o + (r-n) - o \leq rank(v\kappa) \leq min(o, r-n) = r - n$$

such that the equality holds and $rank(v\kappa) = r - n$. The last inequality follows from the assumption that $r - n \leq o$.

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The relation

$$\begin{pmatrix} c' \\ \bar{c}'_\perp \end{pmatrix} \alpha\beta'(\bar{d}, \bar{d}_\perp) = \begin{pmatrix} c' \\ \bar{c}'_\perp \end{pmatrix} (\bar{c}\tau d' + c_\perp v\theta d' + c_\perp v\kappa\zeta' d'_\perp)(\bar{d}, \bar{d}_\perp) = \begin{pmatrix} \tau & 0 \\ v\theta & v\kappa\zeta' \end{pmatrix}$$

shows that $\alpha\beta'$ has rank r because τ has rank n and $v\kappa\zeta'$ has rank $r-n$. This follows from an application of Sylvester's inequality since $rank(v\kappa) = r-n$ and

$$rank(v\kappa) + rank(\zeta') - (r-n) = (r-n) + (r-n) - (r-n) \leq rank(v\kappa\zeta') \leq (r-n).$$

Premultiplying the expression in (11) by c' and $A'_\perp$, implies that

$$c'\alpha\beta' = c'\bar{c}\tau d' = \tau d',$$
$$A'_\perp\alpha\beta' = A'_\perp(\bar{c}\tau d' + c_\perp v\theta d' + c_\perp v\kappa\zeta' d'_\perp).$$

But the assumption $sp(c) \subseteq sp(A)$ implies $A'_\perp\bar{c}\tau d' = 0$. The assumption also implies $sp(c_\perp v) \subseteq sp(A)$, see (9), such that $A'_\perp c_\perp v = 0$ and $A'_\perp\alpha\beta' = 0$. Therefore the space spanned by the columns of $\alpha\beta'$ is contained in $sp(A)$, that is $\alpha = A\psi$.

APPENDIX B: THE CASE $A'\bar{c}_\perp = 0$.

Proof that (12) *implies* (13). Assume $c'A\psi\beta' = \tau d'$. Then, using that $A'\bar{c}_\perp = 0$,

$$\begin{pmatrix} c' \\ \bar{c}'_\perp \end{pmatrix} A\psi\beta'(\bar{d}, \bar{d}_\perp) = \begin{pmatrix} c'A\psi\beta' \\ 0 \end{pmatrix} (\bar{d}, \bar{d}_\perp) = \begin{pmatrix} c'A\psi\beta'\bar{d} & c'A\psi\beta'\bar{d}_\perp \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \tau & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $rank(A\psi\beta') = n$.

Also,

$$A\psi\beta' = (\bar{c}, c_\perp) \begin{pmatrix} \tau & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d' \\ d'_\perp \end{pmatrix} = \bar{c}\tau d',$$

which is (13).

Proof that (13) *implies* (12). Assume $A\psi\beta' = \bar{c}\tau d'$. Multiplying with c', $c'A\psi\beta' = \tau d'$, which is (12). Also

$$\begin{pmatrix} c' \\ \bar{c}'_\perp \end{pmatrix} A\psi\beta'(\bar{d}, \bar{d}_\perp) = \begin{pmatrix} c' \\ \bar{c}'_\perp \end{pmatrix} \bar{c}\tau d'(\bar{d}, \bar{d}_\perp) = \begin{pmatrix} \tau & 0 \\ 0 & 0 \end{pmatrix},$$

and $rank(A\psi\beta')$ is therefore n.

APPENDIX C: PROOF OF PROPOSITION 3

In matrix notation the transformed equations can be written

$$KG\Delta X_t = KG\alpha\beta' X_{t-1} + KG\Gamma\Delta X_{t-1} + KG\mu + KG\varepsilon_t.$$

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We now introduce the restrictions. Under (14), $\alpha\beta' = \bar{c}\tau d' + c_\perp v(\theta d' + \kappa\zeta' d_\perp')$. We find

$$\begin{pmatrix} \bar{v}'\bar{c}_\perp' \\ v_\perp'\bar{c}_\perp' \\ c' \end{pmatrix} \alpha\beta' = \begin{pmatrix} \bar{v}'\bar{c}_\perp' \\ v_\perp'\bar{c}_\perp' \\ c' \end{pmatrix} [(\bar{c}\tau + c_\perp v\theta)d' + c_\perp v\kappa\zeta' d_\perp'] = \begin{pmatrix} \theta \\ 0 \\ \tau \end{pmatrix} d' + \begin{pmatrix} \kappa\zeta' \\ 0 \\ 0 \end{pmatrix} d_\perp'.$$

After premultiplication by the matrix KG defined in (16) and (18)

$$KG\alpha\beta' = \begin{pmatrix} \alpha_1^{**} \\ \alpha_2^{**} \\ \alpha_3^{*} \end{pmatrix} \beta' = \begin{pmatrix} \theta - \omega_{13.2}\tau \\ -\omega_{2.3}\tau \\ \tau \end{pmatrix} d' + \begin{pmatrix} \kappa\zeta' \\ 0 \\ 0 \end{pmatrix} d_\perp'.$$

Similarly, since $c'\Gamma = \tau_1 d_1'$,

$$KG\Gamma = \begin{pmatrix} \Gamma_1^{**} \\ \Gamma_2^{**} \\ \Gamma_3^{*} \end{pmatrix} = K \begin{pmatrix} \bar{v}'\bar{c}_\perp' \\ v_\perp'\bar{c}_\perp' \\ c' \end{pmatrix} \Gamma = \begin{pmatrix} (\bar{v}' - \omega_{12.3}v_\perp')\bar{c}_\perp'\Gamma - \omega_{13.2}\tau_1 d_1' \\ v_\perp'\bar{c}_\perp'\Gamma - \omega_{2\cdot 3}\tau_1 d_1' \\ \tau_1 d_1' \end{pmatrix}.$$

The conditional equations, when the restrictions are taken into account, are therefore

$$\begin{aligned} \Delta X_{1t}^* &= \mu_1^* - \omega_{12.3}\mu_2^* - \omega_{13.2}d_\mu + \omega_{12.3}\Delta X_{2t}^* + \omega_{13.2}\Delta X_{3t}^* + (\theta - \omega_{13.2}\tau)d'X_{t-1} + \kappa\zeta' d_\perp' X_{t-1} \\ &\quad + ((\bar{v}' - \omega_{12.3}v_\perp')\bar{c}_\perp'\Gamma - \omega_{13.2}\tau_1 d_1')\Delta X_{t-1} + \varepsilon_{1t}^{**}, \\ \Delta X_{2t}^* &= \mu_2^* - \omega_{2.3}d_\mu + \omega_{2.3}\Delta X_{3t}^* - \omega_{2.3}\tau d'X_{t-1} \\ &\quad + (v_\perp'\bar{c}_\perp'\Gamma - \omega_{2.3}\tau_1 d_1')\Delta X_{t-1} + \varepsilon_{2t}^{**}, \\ \Delta X_{3t}^* &= d_\mu + \tau d'X_{t-1} + \tau_1 d_1'\Delta X_{t-1} + \varepsilon_{3t}^{**}. \end{aligned}$$

The first equation describes the conditional distribution of X_{1t}^* given X_{2t}^*, X_{3t}^* and the past, the second equation describes the distribution of X_{2t}^* given X_{3t}^* and the past and the last equation is the marginal one for the variable $X_{3t}^* = c'X_t$.

The coefficients of the regressors in the three equations are

$$\begin{aligned} &1\{\mu_1^* - \omega_{12.3}\mu_2^* - \omega_{13.2}d_\mu, \omega_{12.3}, \omega_{13.2}, (\theta - \omega_{13.2}\tau), \kappa\zeta', (\bar{v}' - \omega_{12.3}v_\perp')\bar{c}_\perp'\Gamma - \omega_{13.2}\tau_1 d_1'\}, \\ &\{\mu_2^* - \omega_{2.3}d_\mu, \omega_{2.3}, -\omega_{2.3}\tau, v_\perp'\bar{c}_\perp'\Gamma - \omega_{2.3}\tau_1 d_1'\}, \\ &\{d_\mu, \tau, \tau_1\}. \end{aligned}$$

If $c_i(R)$ denotes the coefficient of regressor R in equation i, $i = 1, 2, 3$, then we see that τ appears in the three coefficients

$$c_1(d'X_{t-1}) = \theta - \omega_{13.2}\tau, c_2(d'X_{t-1}) = -\omega_{2.3}\tau, \text{ and } c_3(d'X_{t-1}) = \tau.$$

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We can reparameterize the first equation, introducing $\theta^* = \theta - \omega_{13.2}\tau$, but there is a constraint among the regression coefficients in equations for ΔX^*_{2t} and ΔX^*_{3t}, given by

$$c_2(d'X_{t-1}) = -\omega_{2.3}\tau = -c_2(\Delta X^*_{3t})c_3(d'X_{t-1}).$$

This constraint is not satisfied by the estimates of the unconstrained regression coefficients, so the regression coefficients are not variation free, but will have to be estimated by a constrained regression.

Another solution is as follows: We choose a value for τ, and define the regressor $R_{t-1} = \tau d'X_{t-1}$, then the equations for ΔX^*_{2t} and ΔX^*_{3t} become

$$\Delta X^*_{2t} = \mu^*_2 - \omega_{2.3}d_\mu + \omega_{2.3}(\Delta X^*_{3t} - R_{t-1}) + (v'_\perp \overline{c}'_\perp \Gamma - \omega_{2.3}\tau_1 d'_1)\Delta X_{t-1} + \varepsilon^{**}_{2t},$$
$$\Delta X^*_{3t} - R_{t-1} = d_\mu + \tau_1 d'_1 \Delta X_{t-1} + \varepsilon^{**}_{3t}.$$

In these equations, the parameters $\mu^*_2 - \omega_{2.3}d_\mu$, $\omega_{2.3}$, and $v'_\perp \overline{c}'_\perp \Gamma - \omega_{2.3}\tau_1 d'_1$ can be estimated by unconstrained regression in the equation for ΔX^*_{2t} and d_μ and τ_1 in the equation for $\Delta X^*_{3t} - R_{t-1}$.

The value of the likelihood can now be computed as a function of τ. If τ is unknown an estimator can be found using a general maximization procedure.

APPENDIX D: DETAILS OF $rank(A'\overline{c}_\perp) < r - n$ AND $sp(c) \subseteq sp(A)$ OF SECTION 3.2

The model multipying (1) by $(\overline{c}_\perp, c)'$ and using the restrictions $\alpha\beta' = A\psi\beta' = vu'\psi\beta'$ and $c'\alpha\beta' = c'A\psi\beta' = \tau d_1$, can be written

$$\overline{c}'_\perp \Delta X_t = vu'\psi\beta' X_{t-1} + \sum_{i=1}^{k} \overline{c}'_\perp \Gamma_i \Delta X_{t-i} + \overline{c}'_\perp \mu + \overline{c}'_\perp \varepsilon_t,$$

$$c'\Delta X_t = \tau d' X_{t-1} + \sum_{i=1}^{\ell} \tau_i d'_i \Delta X_{t-i} + c'\mu + c'\varepsilon_t.$$

The conditional equation, conditioning on $c'\Delta X_t$ and the past, of $\overline{c}'_\perp \Delta X_t$ and the marginal equation of $c'\Delta X_t$ are

$$\overline{c}'_\perp \Delta X_t = vu'\psi\beta' X_{t-1} + \sum_{i=1}^{\ell} (\overline{c}'_\perp \Gamma_i - \omega_{\overline{c}_\perp \cdot c}\tau_i d'_i)\Delta X_{t-i} + \sum_{i=\ell+1}^{k} \overline{c}'_\perp \Gamma_i \Delta X_{t-i} \tag{D1}$$
$$- \omega_{\overline{c}_\perp \cdot c}(c'\Delta X_t - \tau d' X_{t-1} - c'\mu) + \overline{c}'_\perp \mu + \overline{c}'_\perp \varepsilon_t - \omega_{\overline{c}_\perp \cdot c} c'\varepsilon_t,$$

$$c'\Delta X_t = \tau d' X_{t-1} + \sum_{i=1}^{\ell} \tau_i d'_i \Delta X_{t-i} + c'\mu + c'\varepsilon_t \tag{D2}$$

where $\omega_{\overline{c}_\perp \cdot c} = E(\overline{c}'_\perp \varepsilon_t \varepsilon'_t c)[E(c'\varepsilon_t \varepsilon'_t c)]^{-1}$ and $\omega_{cc} = E(c'\varepsilon_t\varepsilon'_t c)$.

The coefficient matrix $vu'\psi\beta'$ in (D1) has dimension $(p-q) \times p$ and rank o. Since $sp(c) \subseteq sp(A)$ such that $o = s - q \le p - q$, the matrix $vu'\psi\beta'$ has reduced rank. But the coefficient $-\omega_{\overline{c}_\perp .c}\tau d'$ represents a cross equation restriction as a product of $\omega_{\overline{c}_\perp .c}$ from the equation for $\overline{c}'_\perp \Delta X_t$, and τ from the equation for $c'\Delta X_t$. Hence by keeping τ fixed (D1) can be estimated by a combination of reduced rank and ordinary OLS regressions. The parameters in (D2) can be estimated by OLS regressions. One can then optimize over τ as explained in Proposition 3.

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APPENDIX E: CASE 2 OF SECTION 4.1

When $0 < m = r$, $\alpha = a$ all adjustment parameters are known. The transformed model with the restrictions incorporated is

$$\overline{c}_{\perp}'\Delta X_t = \overline{c}_{\perp}' a\beta' X_{t-1} + \sum_{i=1}^{k}\overline{c}_{\perp}'\Gamma_i\Delta X_{t-i} + \overline{c}_{\perp}'\mu + \overline{c}_{\perp}'\varepsilon_t,$$
$$c'\Delta X_t = \tau d' X_{t-1} + \sum_{i=1}^{\ell}\tau_i d_i'\Delta X_{t-i} + c'\mu + c'\varepsilon_t.$$

The only parameters of the matrix Π are the elements of β. When $r > rank(\overline{c}_{\perp}'a)$ the model is not identified since the elements of β cannot be distinguished. The possibility $r < rank(\overline{c}_{\perp}'a)$ is impossible since $rank(\overline{c}_{\perp}'a) \leq rank(a) = m = r$. Therefore, for identified models $v_2 = \overline{c}_{\perp}'a$ has dimensions $(p-q)\times r$ and full rank r. Note also that $r = m = rank(\overline{c}_{\perp}'a) \leq \min(rank(\overline{c}_{\perp}'), rank(a)) = \min(p-q, r) \leq p-q$.

The restricted model, by multiplying the equation for $\overline{c}_{\perp}'\Delta X_t$ with $(\overline{v}_2, v_{2\perp})'$, can be decomposed into three parts

$$\overline{v}_2'\overline{c}_{\perp}'\Delta X_t = \beta' X_{t-1} + \sum_{i=1}^{k}\overline{v}_2'\overline{c}_{\perp}'\Gamma_i\Delta X_{t-i} + \overline{v}_2'\overline{c}_{\perp}'\mu + \overline{v}_2'\overline{c}_{\perp}'\varepsilon_t,$$
$$v_{2\perp}'\overline{c}_{\perp}'\Delta X_t = \sum_{i=1}^{k}v_{2\perp}'\overline{c}_{\perp}'\Gamma_i\Delta X_{t-i} + v_{2\perp}'\overline{c}_{\perp}'\mu + v_{2\perp}'\overline{c}_{\perp}'\varepsilon_t,$$
$$c'\Delta X_t = \tau d' X_{t-1} + \sum_{i=1}^{\ell}\tau_i d_i'\Delta X_{t-i} + c'\mu + c'\varepsilon_t.$$

Define the parameters as in Section 3.2 and premultiply $(\beta', 0, d\tau')'$ with the matrix K, see (18), to get

$$K\begin{pmatrix}\beta\\ 0\\ \tau d'\end{pmatrix} = \begin{pmatrix}\beta' - \omega_{13.2}\tau d'\\ -\omega_{2.3}\tau d'\\ \tau d'\end{pmatrix}.$$

The conditional equations of $(\Delta X_t'\overline{c}_{\perp}\overline{v}_2, \Delta X_t'\overline{c}_{\perp}v_{2\perp}, \Delta X_t'c)'$ are therefore after also taking the restrictions on $\Gamma_1, \dots, \Gamma_\ell$ into account given by

$$\begin{aligned}\overline{v}_2'\overline{c}_{\perp}'\Delta X_t &= \mu_1^* + \omega_{12.3}(\Delta X_{2t}^* - \mu_2^*) + \omega_{13.2}(\Delta X_{3t}^* - \mu_3^*) + (\beta' - \omega_{13.2}\tau)d' X_{t-1}\\ &\quad + \sum_{i=1}^{k}(\overline{v_2}' - \omega_{12.3}v_{2\perp}')\overline{c}_{\perp}'\Gamma_i\Delta X_{t-i} - \sum_{i=1}^{\ell}\omega_{13.2}\tau_i d_i\Delta X_{t-i} + \varepsilon_{1t}^{**},\\ v_{2\perp}'\overline{c}_{\perp}'\Delta X_t &= \mu_2^* + \omega_{2.3}(\Delta X_{3t}^* - \mu_3^* - \tau d' X_{t-1}\\ &\quad - \sum_{i=1}^{\ell}\tau_i d_i'\Delta X_{t-i}) + \sum_{i=1}^{k}v_{2\perp}'\overline{c}_{\perp}'\Gamma\Delta X_{t-i} + \varepsilon_{2t}^{**},\\ c'\Delta X_t &= \mu_3^* + \tau d' X_{t-1} + \sum_{i=1}^{\ell}\tau_i d_i'\Delta X_{t-i} + \varepsilon_{3t}^{**}.\end{aligned}$$

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where $\mu^* = (\overline{c}_\perp \overline{v}_2, \overline{c}_\perp v_{2\perp}, c)'\mu$. The parameters in the equation for $\overline{v}_2' \overline{c}_\perp' \Delta X_t$, using $\beta' - \omega_{13.2}\tau d'$ as coefficient of X_{t-1}, are variation independent of the parameters in the equations for $v_{2\perp}' \overline{c}_\perp' \Delta X_t$ and $c'\Delta X_t$. But the coefficient $-\omega_{2.3}\tau d'$ represents cross equation restrictions among the regression coefficients in the equations for $v_{2\perp}' \overline{c}_\perp' \Delta X_t$ and $c'\Delta X_t$ as a product of $\omega_{2.3}$ from the equation for $v_{2\perp}' \overline{c}_\perp' \Delta X_t$, and τ from the equation for $c'\Delta X_t$. Arguing as in Section 3.2 one can first assume that τ are known, introduce the variable $c'\Delta X_t - \tau d' X_{t-1}$, estimate by ordinary least squares and finally optimize over τ.

An alternative procedure is also in this case to estimate τ using the SUR procedure of Remark 7.

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