
Survey paper on equilibrium and asymmetric information

Inga Baadshaug Eide
i.b.eide@cma.uio.no

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Preface

For more than three decades *asymmetric information* has been an important subject for economic research, and on at least two occasions the Bank of Sweden has awarded its "Prize in Economic Sciences in Memory of Alfred Nobel" (aka "The Nobel Prize in Economics") to researchers working in this field.¹ Parts of this research have been devoted to the study of asymmetric information in *financial markets*. Over the last 10-15 years, mathematicians working in the field of finance and stochastics have become increasingly interested in asymmetric information and have developed sophisticated mathematical tools to describe and model asymmetric information. Economists and mathematicians have had quite different approaches to the subject: Roughly speaking, we may say that economists have been more interested in how asymmetric information affects the market, while mathematicians have tended to study under which conditions given asset prices can be sustainable in a market where the agents have different information. Both "camps", however, strive to answer the fundamental questions:

Q1 Can informational asymmetries prevail in the market?

Q2 Is it possible to profit from an informational advantage?

This survey note aims at presenting some of the most important contributions from both mathematics and economics. I have chosen to treat the following papers in detail:

- R. Radner. Rational expectations equilibrium: generic existence and the information revealed by prices. *Econometrica*, 47:655–678, 1979
- A. S. Kyle. Continuous auctions and insider trading. *Econometrica*, 53:1315–1335, 1985
- J. Wang. A model of intertemporal asset prices under asymmetric information. *Review of Economic Studies*, 60:249–282, 1993
- B. Cornet and L. de Boisdeffre. Arbitrage and price revelation with asymmetric information and incomplete markets. *Journal of Mathematical Economics*, 38:393–410, 2002
- C. Hillairet. Existence of an equilibrium with discontinuous prices, asymmetric information and non-trivial initial σ -fields. *Mathematical Finance*, 15:99–117, 2005

¹ Mirrlees and Vickrey in 1996 and Akerlof, Spence and Stiglitz in 2001

Altogether, these papers give a good overview of the span of the research field. The reason for choosing these particular works can be summarised as follows: Radner was among the first to prove that under fairly general conditions rational expectations equilibrium prices are revealing. Kyle's work has become a cornerstone for the study of price formation under asymmetric information. Wang's article goes beyond the questions Q1 and Q2 above to explain the *effects* of asymmetric information on the stock prices themselves, their volatility and risk premium, and also on the agents' behaviour. Cornet and de Boisdeffre's paper represents an interesting alternative to the rational expectations models that have been dominating the field. Hillairet's paper illustrates the "mathematical finance" approach to asymmetric information. But unlike most other contributions in this field, her model is set in an equilibrium framework.

Before making an attempt to model asymmetric information in financial markets, there are certain issues that need to be clarified:

- What do we mean by "asymmetric information"?
 - Are some agent(s) *better* informed than the other agents or are their information just different?
 - Is the agents' private information available initially or revealed gradually?
- The time structure of the market - is it a single period, multiperiod or continuous time market?
- How do we model the sample space generated by the assets' payoffs: is it finite, or infinite?
- What are the characteristics of the agents?
 - Are they risk-neutral or risk-averse?
 - Do they trade to optimise their final wealth or to sustain an optimal consumption scheme?

The diversity of the selected papers in these respects is illustrated in Table 1. Another important issue is:

- to what extent do the agents extract information from the asset prices?

Radner, Kyle and Wang all assume that the agents are able to use asset prices and their knowledge about the market to infer something about the other agents' information. Cornet and de Boisdeffre assume a weaker form

		Radner	Kyle	Wang	Cornet and de Boisdeffre	Hillairet
time horizon	single period	x	x		x	
	multi-period, finite	o	x		o	
	continuous, finite		x			x
	continuous, infinite			x		
information structure	asymmetric	x			x	x
	informed vs. uninformed		x	x	!	
private information	available initially	x	x		x	x
	revealed gradually		o	x		
sample space	finite	x			x	
	infinite	!	x	x	!	x
agents' preferences	risk neutral		x		x	
	risk averse	x	o	x	x	x
objective	intermediate consumption	x		x	x	x
	final wealth	x	x		x	

Table 1: The modelling assumptions of the selected papers: x means "covered in the original paper", o means "covered in subsequent papers" and ! means "'spin-off' of this survey".

of rationality and let the agents extract information from prices only by analysing arbitrage opportunities. Hillairet assumes that no extra information can be extracted from the asset prices.

Not surprisingly, the papers provide different answers to the questions Q1 and Q2: Radner's paper shows that prices in a rational expectations equilibrium are "generically" revealing. Hence the answer to both questions is "generically, no". In Kyle's paper informational asymmetries can prevail, but are diminishing over time. A better informed agent can profit from his information, and the profit is calculated explicitly. In Wang's paper, informational asymmetries are prevailing over time. Though the profits are not explicitly calculated, it is clear that the informed agents can benefit from their additional information. Cornet and de Boisdeffre show that informational asymmetries can prevail if and only if the market is *incomplete*². In Hillairet's paper it is proved that the agents' private information is irrelevant, and hence the answer is "no" to both questions.

² Roughly speaking, a market is complete if any state-contingent claim to the considered assets is available on the market.

The paper is organised as follows: Part I introduces notation, stochastic processes, preferences and the classical Arrow-Debreu and Radner equilibria. Rational expectations models, including Radner's paper (Section 3) are treated in Part II. Part III concerns rational expectations models with *noise*, in particular Kyle's model (Section 6) and some of the numerous later contributions using his model. Wang's paper is also included in this part (Section 9). The concept of arbitrage under asymmetric information is treated in Part IV, using the ideas of Cornet and de Boisdeffre. Part V presents the standard approach from mathematical finance, in particular Hillairet's work (Section 11). In Part VI I summarise, draw parallels between the models and point out some directions for future research.

I have strived to present the papers in a uniform manner. The papers by Kyle, Wang and Hillairet are presented in a form close to the original, though I let out the multiperiod part of Kyle's paper and consider a simplified version of the asset price dynamics in Hillairet's paper. I present a simplified version of Radner's model, but generalised to the case of an infinite sample space. Hence the main result has to be reformulated to fit the new frame. The proof of the main result is inspired by, but different from the original. Cornet and de Boisdeffre's paper is presented in a framework quite different from the original, but the main results are the same. The main contributions of this survey are:

- A generalisation of Radner's main result to the case of an infinite discrete sample space.
- A generalisation of Cornet and de Boisdeffre's in two directions: From treating only a subset of the sample space to the whole space, and from a finite to an infinite sample space. I have also included a discussion of informed vs. uninformed agents not present in the original paper.
- An extensive, though not complete overview and discussion of subsequent works using Kyle's model.
- Some suggestions for future research projects that may be fruitful.

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I INTRODUCTION

1 Notation and preliminaries

The reader is assumed to have some background in probability and stochastics. Hence the following terms will *not* be explained

- σ -algebras
- (complete) probability spaces
- (right continuous) filtrations
- stopping times
- (local) martingales
- càdlàg and càglàd processes
- predictable and progressive stochastic processes
- Brownian motion

For explanations and definitions of these terms we refer to [48]. This does *not* mean that a reader has to be familiar with all these terms to be able to read this survey. To highlight the importance of the semimartingale property of asset prices, the definition of a semimartingale is provided in Appendix B.

1.1 Probability spaces and filtrations

All our analysis will take place in a *probability space* (Ω, \mathcal{F}, P) , which we assume to be *complete*. Our *time horizon* T may be either finite or infinite. Unless otherwise noted, expectations and "almost surely" (a.s.) and "almost all" statements are taken with respect to the measure P . The probability measure Q is said to be *absolutely continuous* with respect to P , denoted $Q \ll P$ if $Q(F) = 0$ whenever $P(F) = 0$, and *equivalent* to P , denoted $Q \sim P$ if $Q(F) = 0$ if and only if $P(F) = 0$.

We use the notation $\sigma\{\text{random variables/collection of sets}\}$ for the σ -algebra generated by some random variables or collection of sets. We will use calligraphic letters (\mathcal{F}, \mathcal{G} etc.) for σ -algebras and blackboard bold (\mathbb{F}, \mathbb{G} etc.) for filtrations. So that for given T

$$\mathbb{F} := \{\mathcal{F}_t; 0 \leq t \leq T\}.$$

When we say that (Ω, \mathcal{F}, P) is equipped with the filtration \mathbb{F} , we always assume that \mathcal{F}_0 contains all the P -null sets of \mathcal{F} and that $\mathcal{F}_T = \mathcal{F}$.

An \mathcal{F} -measurable random variable X belongs to $L_p(\mathcal{F}, P) := L_p(\Omega, \mathcal{F}, P)$ if

$$\int_{\Omega} |X|^p dP(\omega) < \infty.$$

When there is no ambiguity about the probability measure we say that $X \in L_p(\mathcal{F})$.

1.2 Preferences and utilities

Recall that a binary relation \succsim on some set \mathcal{X} is

- *complete* if for all $x, y \in \mathcal{X}$ we have $x \succsim y$ or $y \succsim x$ or both, and
- *transitive* if $x \succsim y$ and $y \succsim z$ implies that $x \succsim z$.

A complete and transitive binary relation is said to be *rational*. A *preference relation* is a rational binary relation. Note that from \succsim we can derive the

- *strict preference* \succ such that $x \succ y$ if $x \succsim y$ but $y \not\succsim x$ and
- *indifference* \sim such that $x \sim y$ if $x \succsim y$ and $y \succsim x$.

An element $x \in \mathcal{X}$ is said to be *maximal* for \succsim , if $x \succsim y$ for all $y \in \mathcal{X}$. We say that the preference relation is *non-satiabile* if in any neighbourhood of any point $x \in \mathcal{X}$, there is some y such that $y \succ x$.

A function $U : \mathcal{X} \rightarrow \mathbb{R}$ is a *utility function representing* \succsim if for all $x, y \in \mathcal{X}$

$$x \succsim y \iff U(x) \geq U(y).$$

A binary relation can be represented by a utility function only if it is rational. If \mathcal{X} is a set of \mathbb{R}^N -valued random variables on (Ω, \mathcal{F}) , a *von Neumann-Morgenstern* utility function can be expressed as

$$\int_{\Omega} U(X(\omega), \omega) dP(\omega), \quad X \in \mathcal{X}, \quad (1.1)$$

where $U : \mathbb{R}^N \times \Omega \rightarrow \mathbb{R}$ and P is a probability measure on (Ω, \mathcal{F}) . In this framework, we will refer to U as the utility function and assume that the agents seek to maximise their von Neumann-Morgenstern (or expected) utility (1.1). For more on preferences and utilities, we refer to [42, Chapter 3 and 6B].

2 Equilibrium under uncertainty

In this section we present two different notions of equilibrium under uncertainty: The *Arrow-Debreu equilibrium* is a generalisation of the Walrasian equilibrium (cf. e.g. [42]) where the traded goods are not the consumption goods themselves, but state-contingent *consumption plans* giving the timing of and conditions for the consumption of the goods³. In the *Radner equilibrium* model there is no market for consumption plans, but there is a *spot market* for all the goods at all times and states and a market for a certain number of *assets* with state-contingent prices. We shall see that the two equilibrium models are related, and under certain conditions equivalent. We will also discuss briefly these models under asymmetric information.

2.1 The Arrow-Debreu market model

Fix a time horizon T and a probability space (Ω, \mathcal{F}, P) equipped with the right continuous filtration \mathbb{F} . We assume that there are L goods available for consumption, and that the possible consumption plans constitute some subset C of \mathbb{F} -progressively measurable, \mathbb{R}_+^L -valued processes

$$c := \{c_t; 0 \leq t \leq T\}$$

giving the instantaneous rates of consumption of the different goods. The agents are indexed by $i = 1, \dots, I$. Agent i is characterised by

- a *consumption set* $C^{(i)} \subseteq C$,
- a *preference relation* $\succsim^{(i)}$ on $C^{(i)}$ and
- an \mathbb{F} -progressively measurable, L -dimensional *endowment* process,

$$e^{(i)} := \{e_t^{(i)}; 0 \leq t \leq T\} \in C^{(i)}.$$

The agents' consumption sets, preferences and endowments constitute a *pure exchange economy*⁴

$$\mathcal{E} := \{(C^{(i)}), (\succsim^{(i)}), (e^{(i)})\},$$

with the shorthand $(\cdot)^{(i)} := (\cdot)^{(1)}, \dots, \cdot^{(I)}$

³ This could be realised by a state contingent *forward market* (cf. e.g. [23]) for the consumption goods

⁴ as opposed to an economy with production

DEFINITION

An **Arrow-Debreu equilibrium** for the economy \mathcal{E} is an allocation of consumption plans $(c^{(i)})$ and a price $\Psi : C \rightarrow \mathbb{R}$ such that:

- For every i , $c^{(i)}$ is maximal for $\succsim^{(i)}$ in the *budget set*

$$\left\{ c \in C^{(i)}; \Psi(c) \leq \Psi(e^{(i)}) \right\}. \quad (2.1)$$

- The market for consumption plans clears, i.e.

$$\sum c^{(i)} = \sum e^{(i)} \text{ a.e. on } \Omega \times [0, T]. \quad (2.2)$$

Note that in this market all trades are settled at time 0. Establishing a consumption plan $c \in C$ means that an agent knows

$$c_t : \Omega \rightarrow \mathbb{R}^L$$

for any $t \in [0, T]$. Thinking of the ω 's as "states of the economy", he knows what he can consume at a certain time in a certain state, but he does *not* know in advance which states actually occur.

Under quite general conditions on preferences and consumption sets, one can show that an Arrow-Debreu equilibrium leads to an allocation of consumption goods that is optimal in the sense that one cannot make some agents better off without making others worse off. We refer to such allocations as *Pareto optimal (or - efficient)*:

DEFINITION

An allocation $(c^{(i)})$ is **feasible** for \mathcal{E} if $c^{(i)} \in C^{(i)}$ for all i and (2.2) holds. A feasible allocation is **Pareto optimal** for \mathcal{E} if there is no feasible allocation $(\hat{c}^{(i)})$ such that $\hat{c}^{(i)} \succsim^{(i)} c^{(i)}$ for all i and $\hat{c}^{(i)} \succ^{(i)} c^{(i)}$ for some i .

THE FIRST WELFARE THEOREM

An Arrow-Debreu equilibrium allocation is Pareto optimal.

For conditions and a proof of this theorem we refer to [23, Chapter 10].

The Arrow-Debreu model relies heavily on the assumption of *symmetric information*: Suppose that some agents are better informed than others in the sense that they can rule out certain states of the economy. Of course any consumption in these states must have zero cost, otherwise the better informed agents will profit risklessly from selling their endowments in these

states⁵. The zero cost, on the other hand, will lead the less informed agents to seek excess consumption in these states. In both cases the economy will be brought out of equilibrium. Hence, we can say that this model implicitly assumes that *all the agents have access to the same information*.

2.2 The Radner market model

At this market there is no market for consumption plans, but at any time t there is a *spot market*, i.e. a market for immediate delivery, for all the consumption goods at prices given by the \mathbb{F} -progressively measurable \mathbb{R}_+^L -valued spot price process

$$\psi := \{\psi_t; 0 \leq t \leq T\}.$$

There is also a market for N different assets whose (cum dividend) prices are given by the \mathbb{F} -semimartingale

$$S := \{S_t; 0 \leq t \leq T\}.$$

A *trading strategy* or *portfolio process* is an \mathbb{F} -predictable N -dimensional process

$$\theta := \{\theta_t; 0 \leq t \leq T\}$$

such that

$$\left\{ \int_0^t \theta_s^\top dS_s; 0 \leq t \leq T \right\}$$

is a finite-variance process. We say that the endowment process e and trading strategy θ *finance* the consumption plan c if

$$\theta_t^\top S_t \leq \theta_0^\top S_0 + \int_0^t \theta_s^\top dS_s + \int_0^t \psi_s^\top (e_s - c_s) ds, \quad t \in [0, T], \quad (2.3a)$$

and

$$\theta_T^\top S_T \geq 0. \quad (2.3b)$$

Noting that $\theta_t^\top S_t$ is the time t market value of the portfolio, (2.3a) states that this value cannot exceed its initial value plus the security trading gains and the values of the endowments net of consumption. The terminal constraint (2.3b) states that there cannot be any remaining obligations at time T . We can now define a *Radner equilibrium* for the extended economy

$$\tilde{\mathcal{E}} := \{\mathcal{E}, \text{assets}\}.$$

⁵ and, if allowed, even short-sell consumption goods

DEFINITION

The spot and asset prices ψ , S and the trading and consumption plans $((\theta^{(i)}, c^{(i)}))$ constitute a **Radner equilibrium** for $\tilde{\mathcal{E}}$ if:

- For every i ,
 - $e^{(i)}$ and $\theta^{(i)}$ finance $c^{(i)}$ and
 - $c^{(i)}$ is maximal for $\succsim^{(i)}$ among the consumption plans in $C^{(i)}$ that can be financed by $e^{(i)}$ and some trading strategy.
- The markets for assets and goods clear, i.e.

$$\sum \theta^{(i)} = 0, \text{ and } \sum c^{(i)} = \sum e^{(i)} \quad \text{a.e. on } \Omega \times [0, T]. \quad (2.4)$$

2.3 Connection between Arrow-Debreu and Radner

Consider for simplicity a single period market with L goods available for consumption. Suppose that C and all the $C^{(i)}$'s are the sets of \mathbb{R}_+^L -valued \mathcal{F} -measurable random variables whose components belong to $L_2(\mathcal{F})$ (cf. Section 1.1).

Suppose that there is no time 0 market for consumption plans but there is a *complete market* for the the good labeled 1, i.e. any claim to the amount $\theta \in L_2(\mathcal{F})$ of this good at time 1 is available at time 0 at a price given by some linear function $\phi : L_2(\mathcal{F}) \rightarrow \mathbb{R}$. At time 1, there is a spot market for all the goods, with spot prices given by the \mathbb{R}^L -valued \mathcal{F} -measurable random variable ψ .

As there are no initial endowments, each agent's time 0 budget constraint is

$$\phi(\theta) \leq 0 \quad (2.5a)$$

Having bought the claim θ , agent i 's time 1 budget constraint is

$$0 \leq \psi^\top (e^{(i)} - c) + \psi_1 \theta \quad \text{a.s.} \quad (2.5b)$$

Agent i thus seeks the $c^{(i)}$ that is maximal for $\succsim^{(i)}$ among the elements in C for which there exists some $\theta \in L_2(\mathcal{F})$ such that (2.5) hold. Thinking of the first good as "money" and claims to this good as "assets" whose time 0 price is q and time 1 price is the first component of ψ , we see that (2.5a) is (2.3a) evaluated at time 0, while (2.5b) is (2.3b) inserted into (2.3a) evaluated at time 1. Hence if (2.4) holds, $\{\psi, \phi, (\theta^{(i)}, c^{(i)})\}$ constitutes a Radner equilibrium.

In this setting we can prove the equivalence between the Arrow-Debreu and the Radner equilibria as follows: Suppose Ψ and $(c^{(i)})$ constitute an Arrow-Debreu equilibrium for $\{C, (\succsim^{(i)}), (e^{(i)})\}$ with a strictly positive price of any consumption of the first good. As $\Psi : C \rightarrow \mathbb{R}$ is a linear functional it has a *Riesz representation*, i.e. some $\psi \in C$ such that

$$\Psi(c) = E[\psi^\top c], \quad c \in C.$$

Noting that $\psi_1 > 0$ a.s., we define

$$\theta^{(i)} := \frac{\psi^\top (c^{(i)} - e^{(i)})}{\psi_1}, \quad i = 1, \dots, I$$

and ϕ such that

$$\phi(\theta) = E[\theta], \quad \theta \in L_2(\mathcal{F}).$$

It is easy to verify that given ϕ and ψ ,

- $\theta^{(i)}$ and $c^{(i)}$ satisfy the Radner budget condition (2.5) for every i ,
- the market clears, i.e. (2.4) holds and
- any $c \in C$ for which there exists some $\theta \in L_2(\mathcal{F})$ such that (2.5) holds is also in agent i 's budget set.

Hence $\{\psi, \phi, (\theta^{(i)}, c^{(i)})\}$ constitutes a Radner equilibrium.

Conversely, suppose $\{\psi, \phi, (\theta^{(i)}, c^{(i)})\}$ is a Radner equilibrium with $\psi_1 > 0$ a.s. and ϕ having the Riesz representation $\phi(\theta) = E[\varphi\theta]$. Then, with Ψ given by

$$\Psi(c) := E\left[\frac{\varphi\psi^\top c}{\psi_1}\right], \quad c \in C,$$

the $c^{(i)}$'s are in the Arrow-Debreu budget sets (2.1). Moreover if c is in agent i 's Arrow-Debreu budget set, it is easily verified that (2.5) holds with

$$\theta^{(i)} := \frac{\psi^\top (c^{(i)} - e^{(i)})}{\psi_1}.$$

Hence Ψ and the $(c^{(i)})$ form an Arrow-Debreu equilibrium.

The equivalence between Arrow-Debreu and Radner equilibria is treated in more general cases in [24] and [17]. The key idea behind the equivalence is the use of assets to transfer wealth between the different states of the economy. It is thus crucial that the asset market allows *complete* transfer of wealth between the states of the economy - a property referred to as (*asset*)

market completeness (cf. e.g. [45] for a definition). Reasoning as in Section 2.1 we see that such a market cannot be in equilibrium unless the agents are equally well informed. The Radner market can thus be viewed as a way to circumvent one of the main objections to the Arrow-Debreu market model: the unrealistic assumption of a state-contingent forward market for all the goods. The assumption of symmetric information is, however, crucial in a market that allows complete transfer of wealth across states of the economy. As the next example shows, informational asymmetries *may* prevail in markets with no transfer of wealth across states, but not if the agents are able to use equilibrium prices to extract information. This will be the subject of Part II.

EXAMPLE (CF. [42, EXAMPLE 19.H.2])

Consider the probability space (Ω, \mathcal{F}, P) where $\mathcal{F} := \{\emptyset, F, F^C, \Omega\}$ with $P(F) = \frac{1}{2}$. Suppose there are two consumption goods and two agents on the market with utility function

$$U(x, y, \omega) = (\chi_F(\omega) + 2\chi_{F^C}(\omega)) \ln x + y.$$

Each agent is supposed to receive the endowment 1 of the first good (as we are only considering relative spot prices we need not specify the endowment of the second good). Suppose the first agent has full information and the other agent has no information, i.e. $\mathcal{F}_1 = \mathcal{F}$ and $\mathcal{F}_2 = \{\emptyset, \Omega\}$. If the spot price of the first good in units of the second good is ψ the agents will demand the amounts

$$\frac{E[\chi_F + 2\chi_{F^C} | \mathcal{F}_i]}{\psi}, \quad i = 1, 2$$

to maximise their expected utilities. As $\mathcal{F}_1 = \mathcal{F}$ the first agent will demand

$$\frac{\chi_F + 2\chi_{F^C}}{\psi}$$

depending on the state, while the second agent will demand

$$\frac{E[\chi_F + 2\chi_{F^C}]}{\psi} = \frac{3}{2\psi}.$$

Adding the demands, the market clearing price is

$$\psi = \frac{7}{4} - \frac{1}{2}\chi_F.$$

In principle, it is possible to have such a state-dependent price function without each agent knowing the state. If, however, the second agent knows the first agent's endowment and utility function, and knows that his knowledge is superior, he might use the equilibrium price to infer the state of the economy.

In the sequel we will consider money as the only good in the market, or equivalently that there is only one consumption good and that the assets considered are real assets with prices and dividends in units of the consumption good. We assume that the endowments are given as money and shares of the risky assets.

II RATIONAL EXPECTATIONS

An equilibrium is only sustainable if the agents' probability assessments are not controverted by their observations of the market. As information in general is asymmetric, sophisticated agents will use their observations of the other agents' activities to update their own probability assessments. The theory of rational expectations is generally attributed to Muth ([44]) and Lucas ([41]). The idea of an equilibrium that is "time consistent" was already formulated by Hicks in the 30's (cf. [33] quoted in [50, Section 1]).

3 Revealing Rational Expectations Equilibria

In this section we present the results of [49] concerning the "generic" existence of a rational expectations equilibrium that reveals the agents' private information. This result is based on an *auxiliary proposition* which states that the set of probability that gives the same equilibrium price is "negligible", which allows us to conclude that "generically" two economies which differ only in the agents' probability assessments will have different equilibrium prices. In this section we present Radner's equilibrium concepts and the auxiliary proposition in a more general mathematical setting. We refer to [26] for a more detailed exposition and the proof of the auxiliary proposition.

Let (Ω, \mathcal{F}, P) be a complete probability space and assume that \mathcal{F} is separable. The J stocks are traded at time 0 and has the \mathcal{F} -measurable \mathbb{R}^J -valued time 1 payoff V . We let \mathcal{F}_V denote the σ -algebra generated by V and assume that $P(F) > 0$ for all non-empty $F \in \mathcal{F}_V$. There are I agents in the market. Agent i receives an initial endowment $\epsilon^{(i)} \in \mathbb{R}_+$ of cash and $e^{(i)} \in \mathbb{R}_+^J$ of stocks. The agents' utility functions are of the form

$$U_{0i}(\text{time 0 consumption}) + U_i(\text{time 1 wealth}).$$

The agents' time 0 decisions are based on their initial information given by the σ -algebra⁶ $\mathcal{G} \subseteq \mathcal{F}$. The meet $\mathcal{G} \wedge \mathcal{F}_V$ can be thought of as the agents' "payoff-relevant" initial information. Given an \mathbb{R}^J -valued \mathcal{G} -measurable asset price vector ϕ , agent i 's choice of initial consumption c and portfolio of stocks z must be \mathbb{R} - and \mathbb{R}^J -valued \mathcal{G} -measurable random variables satisfying the *budget constraint*

$$c + \phi^\top z \leq \epsilon^{(i)} + \phi^\top e^{(i)} \quad \text{a.s.} \quad (3.1)$$

⁶ Economists tend to prefer the term *signal* to describe an agent's information - in our setting \mathcal{G} can thus be thought of as the σ -algebra generated by some signal function.

Under fairly general assumptions the budget constraint hold with equality, so that $c = \epsilon^{(i)} + \phi^\top (e^{(i)} - z)$, and the solution to the optimisation problem

$$\max_{z \in \mathbb{R}^J} \left\{ E \left[U_{0i}(\epsilon^{(i)} + \phi^\top (e^{(i)} - z)) + U_i(V^\top z) \mid \mathcal{G} \right] \right\} \quad (3.2)$$

exists and is unique. Hence, given a \mathcal{G} -measurable ϕ , $z^{(i)}(\phi) : \Omega \rightarrow \mathbb{R}^J$ solving (3.2) is a \mathcal{G} -measurable \mathbb{R}^J -valued random variable. In equilibrium, the agents' demands for stocks must equal the total endowments of stocks i.e.

$$\sum z^{(i)} = \sum e^{(i)} \quad \text{a.s.}, \quad (3.3)$$

which implies that $\sum c^{(i)} = \sum \epsilon^{(i)}$ a.s. when the budget constraints (3.1) hold with equality.

DEFINITION

A **full communication equilibrium** is a collection $\{(z^{(i)}, \phi)\}$ of \mathcal{G} -measurable \mathbb{R}^J -valued random variables such that for any i , $z^{(i)}$ solves (3.2) and (3.3) holds. A full communications equilibrium is **revealing** if the payoff-relevant information revealed by the asset prices equal the agents initial payoff-relevant information, i.e.

$$\sigma\{\phi\} \wedge \mathcal{F}_V = \mathcal{G} \wedge \mathcal{F}_V.$$

Suppose the agents come to the market with *different* information $\mathcal{G}_1, \dots, \mathcal{G}_I$ and consider the *pooled information* given by the *join* $\underline{\mathcal{G}}$, defined as the smallest σ -algebra containing all the \mathcal{G}_i 's, i.e.

$$\underline{\mathcal{G}} := \bigvee \mathcal{G}_i.$$

We could proceed naively and define a "no communications equilibrium" as a collection $\{(z^{(i)}, \phi)\}$ of $\underline{\mathcal{G}}$ -measurable \mathbb{R}^J -valued random variables such that $z^{(i)}$ solves (3.2) *with \mathcal{G} replaced by \mathcal{G}_i* for each agent and (3.3) holds. But then we neglect the fact that *a sophisticated trader could use the equilibrium prices to extract information about the other agents' information*. This new information could in turn lead him to altering his demand for certain stocks. But if the total market demand changes significantly, the price vector is no more an equilibrium price vector. For a given \mathbb{R}^J -valued $\underline{\mathcal{G}}$ -measurable asset price vector ϕ , consider in stead the optimisation problem

$$\max_{z \in \mathbb{R}_+^J} \left\{ E \left[U_{0i}(\epsilon^{(i)} + \phi^\top (e^{(i)} - z)) + U_i(V^\top z) \mid \mathcal{G}_i \vee \sigma\{\phi\} \right] \right\}. \quad (3.4)$$

DEFINITION

A **rational expectations equilibrium** is a collection $\{(z^{(i)}, \phi)\}$ of \mathcal{G} -measurable \mathbb{R}^J -valued random variables such that for any i , $z^{(i)}$ solves (3.4) and (3.3) holds.

Clearly, a revealing full communications equilibrium is also a rational expectations equilibrium.

In the sequel, we shall deal with σ -algebras only indirectly via their conditional probabilities, or more precisely their conditional probabilities restricted to \mathcal{F}_V . Let $P(\cdot|\mathcal{G})$ denote a *regular version of the conditional probability* (cf. Appendix A). This measure is absolutely continuous with respect to P which implies that the restriction of $P(\cdot|\mathcal{G})$ to \mathcal{F}_V is absolutely continuous with respect to the restriction $P|_{\mathcal{F}_V}$ on (Ω, \mathcal{F}_V) . We let \mathcal{P} denote the set of probability measures on (Ω, \mathcal{F}_V) that are absolutely continuous with respect to $P|_{\mathcal{F}_V}$. Fix some $q \in \mathbb{R}^J$, $\mu \in \mathcal{P}$ and consider

$$\max_{\zeta \in \mathbb{R}^J} \left\{ U_{0i}(\epsilon^{(i)} + q^\top(e^{(i)} - \zeta)) + \int_{\Omega} U_i(V^\top \zeta) d\mu \right\}. \quad (3.5)$$

We say that $q \in \mathbb{R}^J$ is an *equilibrium price* for μ if the collection $(\zeta^{(i)})$ of solutions to (3.5) satisfies

$$\sum \zeta^{(i)} = \sum e^{(i)}.$$

The collection $\{(z^{(i)}, \phi)\}$ of \mathcal{G} -measurable \mathbb{R}^J -valued random variables is a full communication equilibrium if and only if for any i and almost all ω , $z^{(i)}(\omega)$ solves (3.5) with $q = \phi(\omega)$ and $\mu = P|_{\mathcal{F}_V}(\cdot|\mathcal{G})(\omega)$, and (3.3) holds (cf. [26, Lemma 3.1]).

Different probability measures with a common equilibrium price are said to be *confounding*. A full communications equilibrium is revealing if the conditional probability measures corresponding to different sets in $\mathcal{G} \wedge \mathcal{F}_V$ are non-confounding (cf. [26, Lemma 3.1]).

EXAMPLE

Suppose that $J = 2$, $\mathcal{F} = \sigma\{F_1, F_2, F_3\}$, $P(F_k) > 0$, $k = 1, \dots, 3$ and

$$V(\omega) = \begin{cases} \begin{bmatrix} 2 & 1 \end{bmatrix}^\top, & \omega \in F_1, \\ \begin{bmatrix} 1 & 2 \end{bmatrix}^\top, & \omega \in F_2, \\ \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \end{bmatrix}^\top, & \omega \in F_3. \end{cases}$$

Suppose that each agent's utility functions and endowments coincide, i.e.

$U_{0i} \equiv U_i \equiv U$ and $\epsilon^{(i)} \equiv \epsilon$, $e^{(i)} \equiv e$, given by

$$U(x) = 2\sqrt{x}, \quad x > 0,$$

$$\epsilon = \frac{4}{3}, \quad e = [1 \quad 1]^\top.$$

In this case $q = [1 \quad 1]^\top$ is a (no-trade) equilibrium price for any probability measure μ with $\mu(F_1) = \mu(F_2)$. Hence any couple of probability measures assigning the same probability to F_1 and F_2 is confounding.

As the example shows, one cannot in general rule out the occurrence of confounding probability measures. It is possible, however, to show that the set of confounding measures is negligible. In Radner's finite dimensional case, \mathcal{P} is the unit simplex of dimensionality determined by the number of "states". The *auxiliary proposition* states that the set of confounding couples is negligible in the sense that its closure has zero Lebesgue measure in $\mathcal{P}^{\otimes 2}$.

In a topological space a set is referred to as *meagre* if it can be expressed as a countable union of nowhere dense sets, i.e. sets for which the interior of the closure is empty. The complement of a meagre set is referred to as a *residual* set. A topological space is a *Baire space* if any residual set is dense. A property is said to hold *generically* in a Baire space if it holds on a residual subset. Any countable intersection of residual sets in a Baire space is in turn a residual set (cf. e.g. [43, Lemma 48.1]). Consequently, countable selections of generic properties hold simultaneously on a residual set and are thus generic. According to the *Baire category theorem* (cf. e.g. [43, Theorem 48.2]), any complete metric space, like \mathcal{P} equipped with the metric

$$d(\mu, \mu') := \sup\{|\mu(F) - \mu'(F)|; F \in \mathcal{F}_V\}$$

(cf. [26, Lemma 3.2]), is a Baire space. Moreover, as the set \mathcal{P}_+ of probability measures that are *equivalent* to P is a residual subset of \mathcal{P} (cf. [26, Lemma 3.3]), we may without loss of genericity restrict our analysis to this set.

We make the following assumptions regarding the agents' utility functions and endowments, the final payoffs and the possible equilibria:

ASSUMPTIONS

For every agent i ,

1. U_{0i}, U_i are twice continuously differentiable, strictly increasing and strictly concave, and $U'_{0i}(c) \rightarrow \infty$ and $U'_i(c) \rightarrow \infty$ as $c \rightarrow 0$
2. Denoting $\tilde{e}^{(i)} := [\epsilon^{(i)} \quad e^{(i)\top}]^\top$ we have that

- $\tilde{e}^{(i)} \in \mathbb{R}_+^J \setminus \{0\}$ for every i and
 - the sum has only strictly positive components, denoted $\sum \tilde{e}^{(i)} \in \mathbb{R}_{++}^J$.
3. V is bounded from above and away from zero below, in all components a.s.
 4. None of the assets are redundant, i.e. there is no non-zero $x \in \mathbb{R}^J$ such that

$$V^\top x = 0 \quad \text{a.s.}$$

5. In equilibrium there is no collection $(x^{(i)}) \in (\mathbb{R}^J)^I$ such that

$$\sum V^\top x^{(i)} U'_i(V^\top z^{(i)}) = 1 \quad \text{a.s.}$$

Assumption 5 may seem odd as an a priori assumption about the properties of an equilibrium. For a justification of this point, see [49, Appendix]. Radner also assumed that the market is incomplete. In the present setting this does not seem to be necessary - it is of course the case when \mathcal{F} is infinite.

These assumptions ensure that any equilibrium price must be strictly positive in all components and that each agent must exhaust his budget. Moreover, for any $\mu \in \mathcal{P}_+$ and $q \in \mathbb{R}_{++}^J$, there is a solution to (3.5), and this solution is a continuous function of μ and q in the vicinity of any equilibrium.

THE AUXILIARY PROPOSITION

Under the above assumptions, the set of confounding couples in $\mathcal{P}^{\otimes 2}$ is meagre.

The proposition is proved in three steps:

- For any $q \in \mathbb{R}_{++}^J$ the set $\mathcal{P}(q)$ of probability measures for which q is an equilibrium price is a meagre subset of \mathcal{P} (cf. [26, Lemma 4.1]).
- For any $\mu \in \mathcal{P}_+$, there is a countable number of equilibrium prices (cf. [26, Lemma 4.2]).
- These assertions imply the Auxiliary Proposition

In the first step we consider an increasing sequence of closed sets of probability measures whose union is \mathcal{P}_+ and prove that their intersections with $\mathcal{P}(q)$ are closed and have empty interior. Hence $\mathcal{P}(q)$ is contained in a countable union of meagre sets and is meagre itself. For the second step it is

sufficient to prove that the set of equilibrium prices is closed and that any perturbation of the asset prices will bring the economy out of equilibrium. Closedness follows from the continuity of the agents' demands as functions of q . Analysing the demand functions' sensitivity to price changes we see that equilibrium-preserving price perturbations would violate Assumption 5. The proposition is proved considering an increasing sequence of sets of confounding couples whose common equilibrium price is bounded, such that the union of the sets is the sets of confounding couples in $\mathcal{P}_+^{\otimes 2}$. The previous two steps and the continuity of the agent's demands near an equilibrium ensure that these sets are meagre. The set of confounding couples is thus contained in a countable union of meagre sets and meagre itself.

As the proof also entails *existence* of an equilibrium for a generic conditional probability, we state, with a slight abuse of terms⁷:

COROLLARY

Generically, full communications equilibria are revealing, and for a generic market with asymmetrically informed agents there exist a rational expectations equilibrium that reveals the agents' payoff relevant information.

4 Related results

Radner's paper has inspired several other authors treating the existence and indeterminacy of fully revealing, non-revealing and partially revealing rational expectations equilibria in both single- and multiperiod markets with nominal or real assets. For an overview see [21].

Grossman ([31]) refers to the full communications equilibrium based on the agents' pooled information as an *artificial fully-informed economy equilibrium*. Unlike Radner, Grossman studies a *complete market* (with production). Assuming that the agents are non-satiable and that the utility functions are twice continuously differentiable and strictly concave, he is able to prove that an artificial fully-informed economy equilibrium based on the agents' pooled information corresponds to a rational expectations equilibrium where each agent only uses his own information and the prices. Further he proves the existence of a rational expectations equilibrium that cannot be Pareto dominated by a central planner with access to the agents' pooled information.

These results suggest that the equilibrium prices reveal all the agents' private information. But if information is only available at a cost, it will not pay for any agent to acquire that information, because it is immediately perceived by the market. This paradox is treated by Grossman and Stiglitz in [32], where the authors also suggest that the presence of *noise* changes the situation - an idea we will pursue in the next part.

⁷ i.e. letting the market inherit the term "generic" from the conditional probabilities

THE EFFICIENT MARKET HYPOTHESIS

The results in this part support the strongest form of the *efficient market hypothesis*. This hypothesis, saying that stock prices reflect the relevant information available in the market was formulated by Fama (cf. e.g. [27]) about 40 years ago, and has been heavily discussed since (cf. e.g. [13] for an overview). We distinguish between three forms of market efficiency:

- A market is *weakly* efficient if the asset prices reflect all information contained in *stock data*, like historical prices and trading volumes.
- In a *semi-strongly* efficient market the asset prices reflect all *publicly available information* like general information about firms and industries, in addition to the stock data.
- In a *strongly* efficient market the asset prices reflect all *existing information* - both *public and private*.

Originally, Fama used the terms weak, semi-strong and strong to describe different types of *tests* for market efficiency. The various versions of the efficient market hypothesis rules out the possibility of earning an excessive expected profit by utilising the different types of information: stock data, publicly available information or private information. We have to stress that the efficient market hypothesis only rules out excessive *expected* profits: it is still possible to outperform the market, even year after year. As the calculations in [13] suggest, the efficient market hypothesis leaves plenty of room for a Soros, but under the same assumptions the reader can easily verify that a very rich Metsusalem is highly unlikely.

III RATIONAL EXPECTATIONS AND NOISE

As we have seen, the theory of rational expectations equilibria relies heavily on the agents' ability to extract information from asset prices. In this part we study how *noise* in the market can impair the information content of the asset prices.

5 Noise traders

In Kyle's seminal paper [40] the noise has the form of "uninformed *noise traders* who trade randomly" (p. 1315, emphasis added). He starts out with a simple single-period model where the *insider* knows the future value of a stock. The market, represented by a *market maker*, is aware of the presence of an insider, but cannot discern his trades from those of the noise traders. Trade takes place in two steps:

- The insider and noise traders place their orders (i.e. bids and offers) independently.
- The market maker observes the total order and determines the price of the stock, so that the market clears.

As the market maker is risk-neutral, the market-clearing price is the expected value of the stock, given the information revealed by the total order.

The insider faces a dilemma: If the stock is incorrectly priced, he wants to trade large volumes to profit more, on the other hand he knows that the more he trades, the more the stock price will move toward its "correct" value, which reduces his profit. Being aware of how his actions influence the stock price, the insider will place his bid or offer to maximise his profits. This *strategic* behaviour is what distinguishes Kyle's model from the models of rational expectations equilibria. Being aware of the insider's strategic behaviour, the market maker is able to interpret the total order as a noisy observation of the stock's true value, and revise his expectations. This combination of the insider trading optimally and the market maker responding rationally forms an *equilibrium*. An equilibrium where one agent acts optimally while the other is merely rational, may seem awkward - for an explanation see [40, comment following the definition on page 1318]. This idea is then extended to multiperiod and continuous time markets.

Apparently, the "noise traders" play the role of "useful idiots"⁸ for the informed agent, allowing him to "hide" his trades and earn a profit. In some sense they need not be irrational at all: they just assume that the stock prices are "correct" in the sense that they reflect the market information, i.e. they assume the efficient market hypothesis. For a more thorough treatment of noise traders, see [22] and the references therein. Glosten and Milgrom [30] introduced a similar class of agents who trade for liquidity reasons, thus noise traders are also referred to as *liquidity traders*. An early attempt at describing these agents and their role is found in [7].

6 The Kyle model

In the single-period Kyle model we consider a market with a zero-interest bank account and a stock with time 1 payoff $V \sim N(v_0, \sigma)$. At time 0, the insider, knowing V , places his order $X = X(V)$. Simultaneously and independently, the noise traders place their orders $Z \sim N(0, \sigma_z)$. The market maker observes the total orders $Y = X + Z$ and sets a price $P = P(Y)$.

DEFINITION

X is **optimal** for the insider if it maximises

$$E[(V - P)X|V] = E[(V - P(X + Z))X|V] = (V - E[P(X + Z)])X,$$

while P is **rational** if

$$P = E[V|Y].$$

A couple (X, P) such that X is optimal given P and P is rational given X is an **equilibrium**.

If we restrict ourselves to linear strategies, i.e.

$$X = \beta(V - v_0), \quad P = v_0 + \lambda Y,$$

where β and λ are constants, the unique equilibrium is

$$X = \frac{\sigma_z}{\sigma}(V - v_0), \quad P = v_0 + \frac{\sigma}{2\sigma_z}(Y). \quad (6.1)$$

The insider's expected profit in this case is

$$E[(V - v_0)X] = \frac{1}{2}\sigma\sigma_z \quad E[(V - v_0)X|V] = \frac{1}{2}\frac{\sigma_z}{\sigma}(V - v_0)^2. \quad (6.2)$$

To measure the extent to which the insider shares his knowledge with the market, we compare the pre-trade variance σ^2 to the post-trade variance

$$\text{var}[V|P] = \frac{1}{2}\sigma^2.$$

⁸ to our knowledge, neither Kyle nor Lenin used this term

Hence we may say that the insider "shares one half of his information". In [51] the insider is allowed to condition his order on Z , in this case

$$X = \frac{\sigma_z}{2\sigma}(V - v_0) - \frac{1}{2}Z, \quad P = v_0 + \frac{\sigma}{\sigma_z}Y$$

while the expected profit is as in (6.2). As $\text{var}[V|P]$ is still $\frac{1}{2}\sigma^2$, the "informativeness of prices" is as in Kyle's model

Kyle takes his ideas further to a *multi-period market* and obtains a difference equations scheme characterising the equilibrium. By letting the intertrade periods go to zero, he obtains a *continuous trade equilibrium* where

$$dX_t = \frac{\sigma_z}{\sigma} \frac{V - P_t}{1 - t} dt, \quad dP_t = \frac{V - P_t}{1 - t} dt + \frac{\sigma}{\sigma_z} dZ_t. \quad (6.3)$$

The asset price process thus forms a *Brownian bridge* from v_0 to V with respect to the insider's information and a Brownian motion with respect to the market information (cf. [39, Problem 5.6.11]). The total order process appears like a Brownian motion with respect to the market information.

Kyle's market model can be described as semi-strongly efficient because the equilibrium prices reflect both the market orders (the "stock data") and the probabilistic properties of the asset's payoff (public information), but it is clearly *not* strongly efficient, since the insider's information is not fully revealed.

7 Continuous-time trading

In [5], Back elaborates Kyle's continuous time model in a more mathematically stringent manner. In Back's model, the final asset value V is not necessarily normally distributed, but has the distribution function F and support I_V . Before the trade starts, the insider gets to know V . The price of the asset at time t , P_t is based on the total orders received up to time t according to a *pricing rule* H ,

$$P_t = H(t, Y_t).$$

It is assumed that $H \in C^{1,2}((0, 1), \mathbb{R})$ is continuous in t on $[0, 1]$, and strictly monotone in y on $[0, 1]$, and

$$E[H^2(1, Z_1)] < \infty, \quad E\left[\int_0^1 H^2(t, Z_t) dt\right] < \infty.$$

Denote by $\mathbb{F}^{(\cdot)}$ (the completion of) the filtrations generated by the processes Y, P, Z . Due to the monotonicity of H , observing P is equivalent to observing

Y , and hence $\mathbb{F}^Y = \mathbb{F}^P$. We denote the informed agent's information by \mathbb{F}^I , for him observing Y is equivalent to observing Z , hence

$$\mathcal{F}_t^I = \mathcal{F}_t^P \vee \sigma\{V\} = \mathcal{F}_t^Y \vee \sigma\{V\} = \mathcal{F}_t^Z \vee \sigma\{V\}, \quad 0 \leq t \leq 1.$$

The market maker's information is

$$\mathbb{F}^M = \mathbb{F}^Y.$$

An *admissible trading strategy* for the insider is an \mathbb{F}^I -semimartingale, such that $X_0 = 0$ and for any pricing rule H ,

$$E \left[\int_0^T H^2(t, X_t + Z_t) dt \right] < \infty.$$

DEFINITION

Given a trading strategy X , a pricing rule H is **rational** if

$$H(t, X_t + Z_t) = E[V | \mathcal{F}_t^M] \quad 0 \leq t \leq 1.$$

A trading strategy is **optimal** if it maximises the expected *post announcement final wealth* of the insider,

$$E \left[\int_{[0,1]} (V - P_{t-}) dX_t - [P, X]_1 | \mathcal{F}_0^I \right],$$

where the subscript on \int is used to explicitly include the endpoints in the integration and the bracket process is as defined in e.g. [48]. An **equilibrium** is a couple (H, X) such that X is optimal given H and H is rational given X .

Denoting the insider's expected profit on the investments made in $(t, T]$ provided he follows an optimal trading strategy by $J(t, Y_t, V)$ and his time t rate of investment by α (i.e. $dX_t = \alpha dt$) and using Itô calculus, we obtain the *Hamilton-Jacobi-Bellman* equation

$$\max_{\alpha \in \mathbb{R}} \left\{ J_t(t, y, v) + \alpha J_y(t, y, v) + \frac{1}{2} \sigma_z^2 J_{yy}(t, y, v) + \alpha(v - H(t, y)) \right\} = 0$$

on $(0, 1) \times \mathbb{R} \times I_V$ (7.1a)

Observing that the insider can make a profit on investments made at time 1 if and only if $H(1, Y_1) \neq V$, the boundary condition is

$$J(1, y, v) > J(1, H^{-1}(1, v), v) = 0 \quad \forall v \in I_V, \forall y \neq H^{-1}(1, v), \quad (7.1b)$$

(inverse with respect to the second variable). A solution to (7.1) is *sufficiently smooth* if $J \in C^{1,2,2}$ on $(0, T) \times \mathbb{R} \times I_V$ and J, J_y are continuous from the left at the endpoint 1.

If a sufficiently smooth solution to (7.1) exists, then

- the insider's expected profit cannot exceed $J(V, 0, 0)$,
- this expected profit is attained if and only if the trading strategy has continuous paths, zero martingale part and ensures that $H(1, Y_1) = V$ a.s. and
- any equilibrium trading strategy must be *inconspicuous*, i.e. such that Y is a Brownian motion with instantaneous variance σ_z^2 with respect to its own filtration.

Let $h = F^{-1} \circ N_c$ where N_c is the cumulative $N(0, \sigma_z)$ -distribution. Then (H, X) given by

$$H(t, y) = E[h(y + Z_1 - Z_t)] \quad (7.2a)$$

$$X_t = (1 - t) \int_0^t \frac{h^{-1}(V) - Z_s}{(1 - s)^2} ds = \int_0^t \frac{h^{-1}(V) - Y_s}{1 - s} ds \quad (7.2b)$$

is an equilibrium. The pricing rule (7.2a) is the only one for which there exists a sufficiently smooth solution to (7.1).

Suppose (H, X) is an equilibrium and H is such that there exists a sufficiently smooth solution to (7.1), then

$$dP_t = H_y(t, Y_t) dY_t. \quad (7.3)$$

Y is a \mathbb{F}^M -Brownian motion with instantaneous variance σ_z and

$$\{H(t, Z_t); 0 \leq t \leq 1\}$$

is an \mathbb{F}^I -martingale. Further, if F has a density function and $E[H_y(1, Z_1)] < \infty$, then

$$\{H_y(t, Z_t); 0 \leq t \leq 1\}$$

is an \mathbb{F}^I -martingale and

$$\{H_y(t, Y_t); 0 \leq t \leq 1\}$$

is an \mathbb{F}^M -martingale.

[5, LEMMA 2 AND 5, THEOREM 1, 2 AND 3]

Back considers the two special cases of normally and log-normally dis-

tributed random variables. Not surprisingly, if $V \sim N(v_0, \sigma^2)$, $H(t, y) = v_0 + \frac{\sigma}{\sigma_z}y$, and (7.2) and (7.3) correspond to (6.3). The total amount of insider trade in this case is

$$\begin{aligned}
X_t &= (1-t) \int_0^t \frac{\frac{\sigma_z}{\sigma}(V-v_0) - Z_s}{(1-s)^2} ds \\
&= \frac{\sigma_z}{\sigma}(V-v_0)t - (1-t) \int_0^t \frac{\int_0^s dZ_u}{(1-s)^2} ds \\
&= \frac{\sigma_z}{\sigma}(V-v_0)t - (1-t) \int_0^t \int_u^t \frac{ds}{(1-s)^2} dZ_u \\
&= \frac{\sigma_z}{\sigma}(V-v_0)t - Z_t + (1-t) \int_0^t \frac{dZ_u}{1-u} \\
&\longrightarrow \frac{\sigma_z}{\sigma}(V-v_0) - Z_1, \quad t \rightarrow 1,
\end{aligned} \tag{7.4}$$

cf. [45, Exercise 5.11]. If, on the other hand $\ln V \sim N(\alpha, \sigma^2)$,

$$H(t, y) = e^{\alpha + \frac{\sigma}{\sigma_z}y + \frac{1}{2}(1-t)\sigma^2}$$

and

$$dX_t = \left(\frac{\sigma_z}{\sigma(1-t)} \ln \frac{V}{P_t} + \frac{1}{2} \sigma \sigma_z \right) dt, \quad dP_t = \frac{\sigma}{\sigma_z} P_t dY_t.$$

The insider's expected profits in the two cases are $\sigma \sigma_z$ and $\sigma_z \sigma e^{\alpha + \frac{1}{2}\sigma^2}$.

8 Further developments

8.1 Insiders observing the value process

8.1.1 Multiperiod case

Suppose that the values V_0, \dots, V_N are announced publicly at times $\{t_0, \dots, t_N\}$ with $0 = t_0 < t_1 < \dots < t_N = 1$ and that the insider gets to know V_n at time t_{n-1} . For simplicity we only consider the case where $V_n \sim N(V_{n-1}, \sigma^2 \Delta t_n)$ where $\Delta t_n := t_n - t_{n-1}$. In this case, for any $t \in (t_{n-1}, t_n]$,

$$\begin{aligned}
H(t, y) &= E[h(y + Z_{t_n} - Z_t)], \\
X_t &= X_{t_{n-1}} + (t_n - t) \int_{t_{n-1}}^t \frac{\frac{\sigma_z}{\sigma} - Z_s}{(t_n - s)^2} ds.
\end{aligned}$$

Proceeding as in (7.4) we see that the amount of insider trading in each period is

$$\Delta X_n := X_{t_n} - X_{t_{n-1}} = \frac{\sigma_z}{\sigma}(V_n - V_{n-1}) - (Z_{t_n} - Z_{t_{n-1}}).$$

The insider's expected total profit is

$$\sum \sigma \sigma_z \Delta t_n = \sigma \sigma_z.$$

As this profit is independent of the number and sizes of time intervals, it is tempting to conclude that it also holds when the Δt_n 's approach zero. But as the "limiting" trading strategy

$$dX_t = \frac{\sigma_z}{\sigma} dV_t - dZ_t \tag{8.1}$$

is a martingale, this is *not* the case. In fact the expected profit following the strategy (8.1) is 0.

8.1.2 Observing a Gaussian martingale

In his PhD thesis [53], Wu studied the Kyle/Back model in the case where the insider observes the value process, or equivalently observes some signal process

$$S := \{S_t; 0 \leq t \leq 1\},$$

with $V = h(S_1)$. S is assumed to be a continuous, square-integrable centered Gaussian martingale independent of the noise trade, which is a standard Brownian motion (i.e. $\sigma_z = 1$). It is further assumed that

$$\text{var}(S_1) = 1, \tag{8.2a}$$

$$\frac{1}{\text{var}(S_t) - t} \in L^1_{\text{loc}}([0, 1)) \tag{8.2b}$$

and

$$\int_0^t \frac{u}{(\text{var}(S_u) - u)^2} du < \infty, \quad 0 \leq t < 1. \tag{8.2c}$$

PROPOSITION

Under the above assumptions, (H, X) with H given by (7.2a) and

$$X_t = \int_0^t \frac{S_s - Y_s}{\text{var}(S_s) - s} ds, \quad 0 \leq t \leq 1$$

is an equilibrium.

[53, PROPOSITION 4.2]

Note that a standard Brownian motion does not satisfy (8.2). It is proved ([53, Proposition 2.3] and [29, Proposition 5.1]) that in this case an inconspicuous insider trading strategy with zero martingale part cannot have the

property that $H(1, Y_1) = V$ a.s. Relaxing the latter condition does not help (cf. [53, Example 4.3]).

Another approach could be to relax the condition that an equilibrium trading strategy has to be such that the total demand is a Brownian motion with instantaneous variance σ_z with respect to the market maker's information. We are then, however, not in a situation where there is a solution to (7.1). To justify that there might be equilibria where this is not the case, note that in the derivation of (7.1) it is assumed that the insider's strategy at time t is of the form $dX_t = \alpha dt$. Attempts at establishing and solving the Hamilton-Jacobi-Bellman equation under more general conditions have *thus far* not been successful. From an economical point of view, the requirement that the insider's strategy is inconspicuous seem unnecessary strict: Even if the total order process differs from the "pure" noise trade, it may still be impossible to detect the insider's order process. In this line of reasoning it is interesting to note that in the single period equilibrium (6.1), $Y \sim N(0, 2\sigma_z)$ which means that Y and Z do *not* agree in law.

8.1.3 Time varying noise trade

In [6] Back and Pedersen study the above problem in a market with time-varying noise trade,

$$dZ_t := \beta_t dW_t^{(z)},$$

where $W^{(z)}$ is a Brownian motion and β a deterministic, strictly positive and continuous function. The signal process is given by $S_0 \sim N(0, \sigma_0^2)$ and

$$dS_t := \sigma_t dW_t^{(s)},$$

where $W^{(s)}$ is a Brownian motion independent of $W^{(z)}$ and σ a deterministic and continuous function. It is assumed that there exist some $\epsilon > 0$ such that for any $t \in (0, 1)$,

$$\frac{\int_t^1 \beta_u^2 du}{\int_0^1 \beta_u^2 du} > (1 + \epsilon) \frac{\int_t^1 \sigma_u^2 du}{\sigma_0^2 + \int_0^1 \sigma_u^2 du}. \quad (8.3)$$

For notational convenience we assume that

$$\int_0^1 \beta_u^2 du = \sigma_0^2 + \int_0^1 \sigma_u^2 du.$$

THEOREM
Define

$$\begin{aligned}\Sigma_t &:= \int_t^1 (\beta_u^2 - \sigma_u^2) du, \\ \theta_t &:= \beta_t \frac{S_t - Y_t}{\Sigma_t} \\ dX_t &:= \theta_t dt \\ H(t, y) &:= \int_{-\infty}^{\infty} h(x) \pi(t, y, x) dx \\ \lambda(t, y) &:= \frac{\partial}{\partial y} H(t, y)\end{aligned}$$

where $\pi(t, y, \cdot)$ is the density function corresponding to $N(y, \int_t^1 \beta_u^2 du)$. Then (H, X) is an equilibrium. The equilibrium price P evolves as

$$dP_t = \lambda(t, Y_t) dY_t.$$

If h is continuously differentiable and satisfies $E[h'(S_1)] < \infty$ then

$$\{\lambda(t, Y_t); 0 \leq t \leq 1\}$$

is a martingale with respect to the market makers' information.

CF. [6, THEOREM]

8.2 Risk averse insiders

In Cho's paper ([12]) the insider is allowed to be risk-averse with a utility function of the form

$$u(x) = \gamma e^{\gamma x} \tag{8.4}$$

with $\gamma < 0$. The setting is quite similar to that in [5], but the prices are allowed to depend on the *whole* trading history in the sense that $P_t = H(t, \xi_t)$, where

$$\xi_t := \int_0^t \lambda_s dY_s$$

and λ is a smooth, deterministic and strictly positive function of time. The regularity conditions on H are as in Section 7. The insider's trading strategy is assumed to be of the form $dX_t = \alpha_t dt$ where α is \mathbb{F}^I -adapted. An equilibrium in this economy is a triplet $((H^*, \lambda^*), \alpha^*)$ such that the price given by (H^*, λ^*) is rational and α^* maximises

$$E \left[u \left(\int_0^1 (V - H^*(t, \xi_t^{\alpha^*})) \alpha_t^* dt \right) \right],$$

where

$$\xi_t^{*\alpha} := \int_0^t \lambda_s^* (dZ_s + \alpha_s ds).$$

If the insider is risk neutral λ^* is proved to be constant, so that the results coincide with those of [5]. If the insider is risk-averse with utility function of the form (8.4) an equilibrium can only exist if V is normally distributed (cf. [12, Proposition 3]).

PROPOSITION

Suppose $V \sim N(v_0, \sigma_v^2)$. Define

$$\begin{aligned} H^*(t, \xi) &:= v_0 + \xi, \\ \lambda_1^* &:= \sqrt{\frac{\sigma_v^2}{\sigma_z^2} + \left(\frac{\gamma\sigma_v^2}{2}\right)^2} - \frac{\gamma\sigma_v^2}{2}, \\ \lambda_t^* &:= \frac{\lambda_1^*}{\gamma\sigma_z^2\lambda_1^*(1-t) + 1}, \\ \alpha_t^* &:= \frac{V - P_t}{\lambda_1^*(1-t)}. \end{aligned}$$

Then $((H^*, \lambda^*), \alpha^*)$ is an equilibrium.

[12, PROPOSITION 4]

The reader can easily verify that this equilibrium converges to the risk neutral equilibrium when $\gamma \rightarrow 0$.

8.3 Insiders knowing the default time

In [11] Campi and Ceti study a defaultable zero-coupon bond with maturity one and face value one. The insider knows in advance the default time τ of the bond, the other agents in the market only observe the default when it happens. The insider's optimal trading strategy is such that $1 +$ the total order process is a *Bessel bridge* from 1 to 0 whose length equals the default time of the bond. In this setting, the presence of an insider makes the default time *predictable* with respect to the market information.

8.4 A forward calculus approach

In the papers [5], [12] and [11] special measures are taken to ensure that the price process is a semimartingale: The insider's trading strategy is a semimartingale with respect to both his own and the market's information, hence the total order process is also a semimartingale. The regularity of the pricing rule ensures that the price is also a semimartingale. As discussed in Appendix B, the semimartingale property is crucial for using stochastic calculus to evaluate the insider's profit. In [1], Aase, Bjuland and Øksendal

are able to do without the semimartingale assumption, by using techniques of forward integration. Without any other a priori assumptions on the price process than rationality and time continuity, they derive the results of [5] in the case where V is normally distributed. Their results are actually slightly more general, because they also allow time varying noise trade.

9 Dividends and underlying information

9.1 The market

Wang ([52]) studies an economy consisting of a riskless bank account with constant rate $r > 0$ and a dividend-paying stock. Dividends are paid at the rate D , given by the process

$$dD_t = (\Pi_t - kD_t)dt + \sigma_D dB_t^{(D)},$$

where $k \geq 0$ and σ_D are constants and $B^{(D)}$ a standard Brownian motion. The state variable Π follows a mean-reverting Ornstein-Uhlenbeck process

$$d\Pi_t = a_\Pi(\bar{\Pi} - \Pi_t)dt + \sigma_\Pi dB_t^{(\Pi)},$$

where $a_\Pi > 0$, $\bar{\Pi}$ and σ_Π are constants and $B^{(\Pi)}$ is a standard Brownian motion, independent of $B^{(D)}$.

The theoretical possibility of negative dividends is a drawback of this model: we may either think of the stock as an "unlimited liability" asset as in [52] or accept that "the model breaks down as prices and dividends approach zero" as in the related paper [10]. Another approach is to replace the infinite time horizon in the current model by a stochastic default time,

$$\tau := \inf\{t > 0; D_t \leq 0\}.$$

Hopefully, a planned follow-up paper of [11] treating the case where information regarding the default time is revealed gradually to the insider, will provide valuable insights for this approach.

A fraction γ of the agents in the market are *uninformed agents* who only observe D , whereas the remaining *informed agents* also observe Π . In addition all the agents observe the asset price process S , i.e. we have

$$\begin{aligned}\mathcal{F}_t^{(i)} &= \sigma\{D_s, \Pi_s, S_s; 0 \leq s \leq t\}, \\ \mathcal{F}_t^{(u)} &= \sigma\{D_s, S_s; 0 \leq s \leq t\}.\end{aligned}$$

All the agents have the same utility function

$$u(x, t) = -e^{-\rho t - x}$$

where $\rho \geq 0$ is a constant.

The total amount of risky equity (i.e. the number of stocks on the market) is $1 + \Theta$ where Θ is given by the Ornstein-Uhlenbeck process

$$d\Theta_t = -a_\Theta \Theta_t dt + \sigma_\Theta dB_t^{(\Theta)}$$

where $a_\Theta > 0$ and σ_Θ are constants and $B^{(\Theta)}$ is a standard Brownian motion, independent of $B^{(D)}$ and $B^{(\Pi)}$. To see the link to the Kyle/Back model, one can think of $-\Theta$ as the noise trade.

Given the stock price S the instantaneous *excess return* on a stock is

$$d\bar{S}_t = (D_t - rS_t)dt + dS_t.$$

If an agent's portfolio of stocks is X and his consumption rate is c , his total wealth W satisfies

$$dW_t = (rW_t - c_t)dt + X_t d\bar{S}_t.$$

The agents are assumed to maximise their expected infinite lifetime utility

$$E \left[\int_t^\infty -e^{-\rho s - c_s} ds \mid \mathcal{F}_t^{(\cdot)} \right]. \quad (9.1)$$

The price S is an *equilibrium price* if the agents' optimal portfolios sum up to $1 + \Theta$.

9.2 The asset price

THEOREM

In the case where *all* the agents are informed (i.e. $\gamma = 0$) the price, S^* , is a function of the state variables D, Π, Θ ,

$$S^* = \Phi + p_0^* + p_\Theta^* \Theta,$$

where

$$\begin{aligned} \Phi_t &:= E \left[\int_t^\infty e^{-rs} D_s ds \mid D_t, \Pi_t \right] = \phi + p_D^* D_t + p_\Pi^* \Pi_t \\ \phi &:= \frac{a_\Pi p_\Pi^* \bar{\Pi}}{r}, \quad p_D^* := \frac{1}{r+k}, \quad p_\Pi^* := \frac{p_D^*}{r+a_\Pi}, \\ p_0^* &:= -((p_D^* \sigma_D)^2 + (p_\Pi^* \sigma_\Pi)^2), \quad p_\Theta^* < 0. \end{aligned}$$

[52, THEOREM 3.1]

Under asymmetric information the stock price will clearly depend on the state variables D, Π, Θ and the uninformed traders' estimates of the latter,

denoted by $\hat{\cdot} = E[\cdot | \mathcal{F}_t^{(u)}]$. Because the stock price is assumed to reveal some linear combination of Π and Θ to the uninformed agents, we only need to take one of these estimates, say $\hat{\Pi}$ into account (cf. [52, Lemma 4.1]). Denoting the estimation error $\Delta := \hat{\Pi} - \Pi$, the proposed form of the equilibrium price is

$$\begin{aligned} S &= \Phi + p_0 + p_\Theta \Theta + p_\Delta \Delta \\ &= \phi + p_0 + p_D^* D + p_\Pi \Pi + p_\Theta \Theta + p_\Delta \hat{\Pi}, \end{aligned}$$

where $p_\Pi = p_\Pi^* - p_\Delta$, and p_0, p_Θ, p_Δ are constants to be determined. Denoting $\Lambda := p_\Pi \Pi + p_\Theta \Theta$ and noting that the informed agents can deduce Θ from S , we have

$$\begin{aligned} \mathcal{F}_t^{(i)} &= \sigma\{D_s, \Pi_s, \Theta_s; 0 \leq s \leq t\}, \\ \mathcal{F}_t^{(u)} &= \sigma\{D_s, \Lambda_s; 0 \leq s \leq t\}. \end{aligned}$$

The "rational expectations" problem of the uninformed agents amounts to estimating Π and Θ from observations of D and Λ . This is a standard filtering problem for the process

$$\begin{bmatrix} d\Pi_t \\ d\Theta_t \end{bmatrix} = \begin{bmatrix} a_\Pi \bar{\Pi} \\ 0 \end{bmatrix} dt + \begin{bmatrix} -a_\Pi & 0 \\ 0 & -a_\Theta \end{bmatrix} \begin{bmatrix} \Pi_t \\ \Theta_t \end{bmatrix} dt + \begin{bmatrix} 0 & \sigma_\Pi & 0 \\ 0 & 0 & \sigma_\Theta \end{bmatrix} dB_t,$$

where $B := [B^{(D)} \ B^{(\Pi)} \ B^{(\Theta)}]^\top$, with the observation process

$$\begin{aligned} \begin{bmatrix} dD_t \\ d\Lambda_t \end{bmatrix} &= \begin{bmatrix} 0 \\ p_\Pi a_\Pi \bar{\Pi} \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ -p_\Pi a_\Pi & -p_\Theta a_\Theta \end{bmatrix} \begin{bmatrix} \Pi_t \\ \Theta_t \end{bmatrix} dt + \begin{bmatrix} -k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D_t \\ \Lambda_t \end{bmatrix} dt + \\ &\quad \begin{bmatrix} \sigma_D & 0 & 0 \\ 0 & p_\Pi \sigma_\Pi & p_\Theta \sigma_\Theta \end{bmatrix} dB_t. \end{aligned}$$

From standard filtering theory (cf. [37] and [52, Appendix A]⁹)

$$\begin{aligned} \begin{bmatrix} d\hat{\Pi}_t \\ d\hat{\Theta}_t \end{bmatrix} &= \begin{bmatrix} a_\Pi (\bar{\Pi} - \hat{\Pi}_t) \\ -a_\Theta \hat{\Theta}_t \end{bmatrix} dt + \\ &\quad \begin{bmatrix} h_{\Pi D} & h_{\Pi \Lambda} \\ h_{\Theta D} & h_{\Theta \Lambda} \end{bmatrix} \begin{bmatrix} dD_t - (\hat{\Pi}_t - kD_t)dt \\ d\Lambda_t - p_\Pi a_\Pi (\bar{\Pi} - \hat{\Pi}_t)dt + p_\Theta a_\Theta \hat{\Theta}_t dt \end{bmatrix}, \quad (9.2) \end{aligned}$$

where

$$\begin{bmatrix} h_{\Pi D} & h_{\Pi \Lambda} \\ h_{\Theta D} & h_{\Theta \Lambda} \end{bmatrix} := \begin{bmatrix} \frac{x}{\sigma_D^2} & \frac{p_\Pi}{\sigma_\Pi^2} (\sigma_\Pi^2 - x(a_\Pi - a_\Theta)) \\ -\frac{x}{\sigma_D^2} \frac{p_\Pi}{p_\Theta} & \frac{1}{\sigma_\Lambda^2 p_\Theta} (p_\Pi^2 (a_\Pi - a_\Theta)x + (p_\Theta \sigma_\Theta)^2) \end{bmatrix}$$

with

$$\sigma_\Lambda^2 := (p_\Theta \sigma_\Theta)^2 + (p_\Pi \sigma_\Pi)^2$$

⁹ beware of a few misprints

and

$$x := \frac{\sigma_D^2}{\sigma_\Lambda^2 + (a_\Pi - a_\Theta)^2 p_\Pi^2 \sigma_D^2} \left(-a_\Pi (p_\Theta \sigma_\Theta)^2 - a_\Theta (p_\Pi \sigma_\Pi)^2 + \sigma_\Lambda \sqrt{(a_\Pi p_\Theta \sigma_\Theta)^2 + (a_\Theta p_\Pi \sigma_\Pi)^2 + \left(\frac{p_\Theta \sigma_\Pi \sigma_\Theta}{\sigma_D}\right)^2} \right).$$

The estimation error Δ is an Ornstein-Uhlenbeck process

$$d\Delta_t = -a_\Delta \Delta_t dt + b_\Delta dB_t,$$

where $a_\Delta = a_\Pi + h_{\Pi D} + (a_\Theta - a_\Pi)h_{\Pi\Lambda}$ and b_Δ is a 3-dimensional row vector that can be calculated based on (9.2) and the dynamics of D , Π , Θ and Λ . For the excess return on a stock we have

$$d\bar{S}_t = (e_0 + e_\Theta \Theta + e_\Delta \Delta) dt + b_S dB_t,$$

where $e_0 := -rp_0$, $e_\Theta := -(r + a_\Theta)p_\Theta$, $e_\Delta := -(r + a_\Delta)p_\Delta$ and b_S is a 3-dimensional row vector that can be calculated based on (9.2) and the dynamics of D , Π , Θ and Λ .

9.3 The agents' optimisation problems

Both the informed and the uninformed agents now face the optimisation problem of choosing the amount of stocks X and consumption rate c that maximise (9.1). Because the agents' expectations of the excess return differ:

$$E[d\bar{S}_t | \mathcal{F}_t^{(i)}] = (e_0 + e_\Theta \Theta + e_\Delta \Delta) dt$$

while

$$E[d\bar{S}_t | \mathcal{F}_t^{(u)}] = (e_0 + e_\Theta \hat{\Theta}) dt,$$

the agents will have different perceptions of the wealth dynamics. Hence, the informed and uninformed agents' optimisation problems must be solved separately. Denoting the optimal value of (9.1) by J we have

$$J^{(i)} = J^{(i)}(W, \Theta, \Delta, t) \text{ and } J^{(u)} = J^{(u)}(W, \hat{\Theta}, t).$$

Solving the optimisation problems for the informed and uninformed agents (cf. [52, Appendix B]) and denoting $\Psi^{(i)} = [1 \ \Theta \ \Delta]^\top$ and $\Psi^{(u)} = [1 \ \hat{\Theta}]^\top$ we get

$$c^{(\cdot)} = rW - \frac{1}{2} \Psi^{(\cdot)\top} v^{(\cdot)} \Psi^{(\cdot)} - \ln r,$$

$$X^{(\cdot)} = f^{(\cdot)} \Psi^{(\cdot)}$$

where $v^{(i)}$ and $v^{(u)}$ are 3×3 - and 2×2 -dimensional symmetric matrices and $f^{(i)}$ and $f^{(u)}$ are 3- and 2-dimensional row vectors (cf. [52, Appendix B]).

The market-clearing condition

$$(1 - \gamma)X^{(i)} + \gamma X^{(u)} = 1 + \Theta$$

i.e.

$$(1 - \gamma)(f_1^{(i)} + f_2^{(i)}\Theta + f_3^{(i)}\Delta) + \gamma(f_1^{(u)} + f_2^{(u)}\hat{\Theta}) = 1 + \Theta$$

is then used to find the values of the coefficients p_0, p_Θ, p_Δ . No analytical solutions are given, but in general we have that

$$p_0 < 0, \quad p_\Theta \leq 0, \quad 0 \leq p_\Delta \leq p_\Pi^*.$$

9.4 The effect of asymmetric information

As Wang points out, the factor γ measures both the degree of uncertainty in the market and the degree of asymmetric information. When a market parameter depends on γ it can be related to the degree of uncertainty in the market, the degree of asymmetry or both. To "isolate" the effect of asymmetric information he studies what happens near the extrema $\gamma = 0$ (all the agents are informed) and $\gamma = 1$ (all the agents are uninformed). The price in the case $\gamma = 1$ is

$$S^{**} = (\phi + p_0^{**}) + p_D^* D + p_\Pi^* \hat{\Pi} + p_\Theta^{**} \Theta$$

where p_0^{**} and p_Θ^{**} are determined by the market-clearing condition in the case $\gamma = 1$.

According to "conventional wisdom" rational and better-informed agents should stabilise prices. Denoting the instantaneous variance of the stock price by σ_S (i.e. $\sigma_S := \|b_S\|$), it is proved that

$$\sigma_{S^{**}} > \sigma_{S^*}.$$

However, σ_S is *not* a monotonically increasing function of γ . To see this, note that with $\gamma = 1$, *the uninformed agents are able to observe* Θ . Introducing a small fraction of informed investors in this market, will provide the uninformed agents with more information about Π and hence reduce the uncertainty about future cash flows and accordingly the instantaneous price variability (measured by σ_S). On the other hand, an increase in the asset price caused by a decrease in Θ will (rationally) be interpreted as partially caused by an increase in Π , which will cause a further increase in the asset price, meaning an *increased* instantaneous price variability. The net effect

of introducing informed agents in the market is the off-set of these forces. It is proved that for a high level of noise trade (i.e. high σ_Θ) the latter force can dominate, as illustrated in [52, Figure 2].

The unconditional expected excess return on the stock is $e_0 = -rp_0$. From the theorem on page 28 we have that this entity is independent of σ_Θ when $\gamma = 0$, this is also proved to be the case when $\gamma = 1$. Hence under symmetric information, the excess return on the stock demanded by the investors is independent of the level of noise trade. Under asymmetric information, an increased level of noise trade increases the uninformed agents' uncertainty about future cash flows and accordingly increases the demanded excess return on the stock. It is proved that $p_0^{**} < p_0^*$, which means that $e_0^{**} > e_0^*$.

Wang also shows how asymmetric information can lead to negative serial correlation of asset returns. Further, it is proved that for certain parameters, the uninformed agents will (rationally) behave like trend chasers. Both these results agree with empirical studies.

IV NO ARBITRAGE UNDER ASYMMETRIC INFORMATION

As an alternative to the theory of rational expectations, Cornet and de Boisdeffre ([14]) assume that the agents extract information from asset prices only by analysing *arbitrage opportunities*. Contrary to rational expectations models the agents need no a priori knowledge of the other agents' preferences or behaviour.

The scope of [14] is firstly to extend the concept of arbitrage to the case of asymmetric information and secondly to study how no-arbitrage prices reveal information. In the follow-up paper [15], the authors study how agents can extract information by successively ruling out "arbitrage states" i.e. states in which an arbitrage opportunity would give a strictly positive payoff. The existence of a no-arbitrage equilibrium in a market with asymmetric information is dealt with in [18]. Extensions to the multiperiod case are studied in [3]. All these papers are limited to a finite dimensional state space, and the agents' information are represented by subsets of the state space (sub-trees in the multiperiod case).

This part provides a summary of the results in [25] where the ideas from [14] are used in a more mathematically profound analysis of a financial market. For proofs of the results, we refer to [25].

10 Information and arbitrage

Consider the complete probability space (Ω, \mathcal{F}, P) where \mathcal{F} . The J assets in the economy are traded at time 0 and gives the \mathcal{F} -measurable \mathbb{R}^J -valued payoff V at some later time $T > 0$. We assume that \mathcal{F} is separable and generated by V . A *portfolio* is a (possibly random) J -dimensional vector whose components denote the holdings of the assets. The payoff of the portfolio z is the random variable $V^\top z$. A *price function* is an \mathcal{F} -measurable \mathbb{R}^J -valued random variable.

ASSUMPTION

V is bounded and there exist some $z^* \in \mathbb{R}^J$ such that $V^\top z^* > 0$ a.s. and $\frac{V}{V^\top z^*}$ is integrable.

The portfolio z^* has the interpretation of a *riskless* portfolio, and will be used as a *numéraire* in our version of the Fundamental Theorem of Asset Pricing (page 35).

An agent's information will be represented by a σ -algebra $\mathcal{G} \subseteq \mathcal{F}$. We assume all σ -algebras to be completed. For any σ -algebra $\mathcal{G} \subseteq \mathcal{F}$ we let $P(\cdot|\mathcal{G})$ denote a *regular version of the conditional probability* (cf. Appendix A).

An *information structure* is a collection $(\mathcal{H}_i) := (\mathcal{H}_1, \dots, \mathcal{H}_I)$ of σ -algebras representing the agents' information. The agents' *pooled* information is given by the *join*, $\underline{\mathcal{H}}$, defined as the smallest σ -algebra containing all the \mathcal{H}_i 's, i.e.

$$\underline{\mathcal{H}} := \bigvee \mathcal{H}_i,$$

while their *common information* is given by the *meet*, $\overline{\mathcal{H}}$, defined as the largest σ -algebra contained in all the \mathcal{H}_i 's, i.e.

$$\overline{\mathcal{H}} := \bigwedge \mathcal{H}_i.$$

The information structure (\mathcal{H}_i) is *symmetric* if all the \mathcal{H}_i 's coincide. The information structure (\mathcal{G}_i) is a *refinement* of (\mathcal{H}_i) if $\mathcal{H}_i \subseteq \mathcal{G}_i$ for all i , we also say that (\mathcal{H}_i) is *coarser* than (\mathcal{G}_i) . Clearly for any refinement (\mathcal{G}_i) , $\underline{\mathcal{H}} \subseteq \underline{\mathcal{G}}$. The refinement is *self-attainable* if $\underline{\mathcal{H}} = \underline{\mathcal{G}}$.

10.1 Arbitrage

DEFINITION

Given the price function ϕ , a vector $z \in \mathbb{R}^J$ is a ϕ -**arbitrage** for the σ -algebra $\mathcal{G} \subseteq \mathcal{F}$ at ω if

$$\phi(\omega)^\top z \leq 0, \quad V^\top z \geq 0 P(\cdot|\mathcal{G})(\omega)\text{-a.s. and } P(V^\top z > 0|\mathcal{G})(\omega) > 0. \quad (10.1)$$

The price function ϕ is a **no-arbitrage price function** for \mathcal{G} and \mathcal{G} is ϕ -**arbitrage-free** if at almost all ω there are no ϕ -arbitrages for \mathcal{G} . The set of no-arbitrage price functions for \mathcal{G} is denoted $\Phi(\mathcal{G})$.

Note that we do *not* assume that the price function is \mathcal{G} -measurable. This may seem odd when thinking of \mathcal{G} as the agent's information: clearly the agent will observe the asset prices. But as the asset prices can depend on information that is not available to the agent we cannot assume that the asset price as a *mapping* $\phi : \Omega \rightarrow \mathbb{R}^J$ is \mathcal{G} -measurable. If, however, the price function is \mathcal{G} -measurable, the above property of no-arbitrage coincides with the standard definition, namely that there is no \mathcal{G} -measurable \mathbb{R}^J -valued random variable ξ such that

$$\phi^\top \xi \leq 0, \quad V^\top \xi \geq 0 P\text{-a.s. and } P(V^\top \xi > 0) > 0.$$

For a proof, see [16, Lemma 2.3].

The following theorem is a version of the *Fundamental Theorem of Asset Pricing* (cf. e.g. [28, Theorem 1.6]):

THE FUNDAMENTAL THEOREM OF ASSET PRICING FOR A GENERAL PRICE FUNCTION

The price function ϕ is a no-arbitrage price function for \mathcal{G} if and only if for almost all ω there exist some probability measure $P^{(0)} \sim P$ on (Ω, \mathcal{F}) such that

$$\frac{\phi(\omega)}{\phi(\omega)^\top z^*} = E^{(0)} \left[\frac{V}{V^\top z^*} \mid \mathcal{G} \right] (\omega). \quad (10.2)$$

[25, THEOREM 2.1]

An immediate consequence is that for every σ -algebra \mathcal{G} , there exist some \mathcal{G} -measurable no-arbitrage price function, with $E[V|\mathcal{G}]$ as a trivial example. Absence of arbitrage can also be defined for information structures:

DEFINITION

The price function ϕ is a **common no-arbitrage price function** for (\mathcal{H}_i) and (\mathcal{H}_i) is **ϕ -arbitrage-free** if all the \mathcal{H}_i 's are ϕ -arbitrage-free. The set of common no-arbitrage price functions for (\mathcal{H}_i) is denoted

$$\Phi_c((\mathcal{H}_i)) := \bigcap \Phi(\mathcal{H}_i). \quad (10.3)$$

(\mathcal{H}_i) is **arbitrage-free** if there exist some common no-arbitrage price function, i.e. $\Phi_c((\mathcal{H}_i)) \neq \emptyset$.

Clearly the Fundamental Theorem of Asset Pricing ensures that the sets on the right hand side in (10.3) are non-empty, but their intersection can be empty. Hence a symmetric information structure is arbitrage-free, whereas an asymmetric information structure may fail to be so.

AN "ASYMMETRIC INFORMATION VERSION" OF THE FUNDAMENTAL THEOREM OF ASSET PRICING

The price function ϕ is a common no-arbitrage price function for (\mathcal{H}_i) if and only if for almost all ω , there exist some collection of measures $P^{(1)}, \dots, P^{(I)} \sim P$ such that

$$\frac{\phi(\omega)}{\phi(\omega)^\top z^*} = E^{(i)} \left[\frac{V}{V^\top z^*} \mid \mathcal{H}_i \right] (\omega), \quad i = 1, \dots, I.$$

[25, COROLLARY 2.1.3]

It would be natural to assume that the asset prices are based on the agents' pooled information only, i.e. ϕ is \mathcal{H} -measurable. As the following result shows, this does not affect the existence of common no-arbitrage price functions.

PROPOSITION

If there exists some common no-arbitrage price function for (\mathcal{H}_i) , there exists some \mathcal{H} -measurable common no-arbitrage price function for (\mathcal{H}_i) . [25, PROPOSITION 2.1]

EXAMPLE

Suppose that $I = 2$, $J = 3$, \mathcal{F} is generated by the partition F_1, \dots, F_4 of Ω ,

$$V(\omega) := \begin{cases} \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^\top, & \omega \in F_1, \\ \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top, & \omega \in F_2, \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top, & \omega \in F_3, \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\top, & \omega \in F_4, \end{cases}$$

and the information structure is $\mathcal{H}_1 := \sigma\{F_4\}$ and $\mathcal{H}_2 := \sigma\{F_3\}$. Now, if $\omega \in F_4^C$, $G(\omega, \mathcal{H}_1) = F_4^C$, which means that in the first agent's view

- the payoff of the first asset can be -1, 0 or 1, all with positive probability,
- the second asset's payoff is almost surely non-negative and almost surely higher than or equal to the payoff of the first asset, and moreover there is a positive probability that the payoff is strictly positive and strictly higher than the payoff of the first asset, and
- the third asset's payoff is almost surely non-negative and strictly positive with positive probability.

The reader can now easily verify that any no-arbitrage price for \mathcal{H}_1 must be such that the price of the second asset is strictly positive and strictly higher than the price of the first asset, and that the price of the third asset is strictly positive on F_4^C . Clearly, if $\omega \in F_4$, the first agent knows that the payoff of the second asset is 1 and of the other assets 0, which means that any no-arbitrage price for \mathcal{H}_1 must be such that the price of the second asset is strictly positive and the prices of the other assets are 0 on F_4 . Hence

$$\Phi(\mathcal{H}_1) = \left\{ \begin{aligned} & [p_1 \ p_2 \ p_3]^\top \chi_{F_4^C} + [0 \ q_2 \ 0]^\top \chi_{F_4}; \\ & p_1 < p_2, p_2 > 0, p_3 > 0, q_2 > 0 \end{aligned} \right\}.$$

The same reasoning for the second agent yields

$$\Phi(\mathcal{H}_2) = \left\{ [p_1 \ p_2 \ 0]^\top \chi_{F_3^C} + [0 \ 0 \ q_3]^\top \chi_{F_3}; p_1 < p_2, p_2 > 0, q_3 > 0 \right\}.$$

Hence

$$\Phi_c((\mathcal{H}_1, \mathcal{H}_2)) = \emptyset.$$

We now equip every agent i with a strictly increasing utility function $U_i : \mathbb{R} \rightarrow \mathbb{R}$ and consider the economy

$$\mathcal{E} := \{V, (\mathcal{H}_i), (U_i)\}.$$

This is a simplification of the economy considered in [14, Section 2.4] which includes consumption goods, spot prices and endowments.

DEFINITION

A collection $\{(z^{(i)}, \phi)\}$, where $z^{(i)} : \Omega \rightarrow \mathbb{R}^J$ is \mathcal{F} -measurable for all i and ϕ is a price function, constitutes a **no-arbitrage equilibrium** for the economy \mathcal{E} if:

- For all i and almost all ω , $z^{(i)}(\omega)$ solves

$$\max_{z \in \mathbb{R}^J} E[U_i(V^\top z) | \mathcal{H}_i](\omega) \text{ subject to } \phi(\omega)^\top z \leq 0. \quad (10.4)$$

- $\sum z^{(i)} = 0$.

Observing that if $\phi \notin \Phi_c((\mathcal{H}_i))$, then for at least one $F \in \mathcal{F}$ and one agent i , there exists some arbitrage opportunity such that (10.4) has no solution, it is easy to prove:

PROPOSITION

If there is a solution to (10.4) for every agent, then $\phi \in \Phi_c((\mathcal{H}_i))$.

An alternative approach to arbitrage, not dealing explicitly with asset prices, is the following:

DEFINITION

An allocation $(z^{(i)}) \in (\mathbb{R}^J)^I$ is a **future arbitrage opportunity** for (\mathcal{G}_i) at ω if

$$\sum z^{(i)} = 0, \quad (10.5a)$$

$$V^\top z^{(i)} \geq 0 \quad P(\cdot | \mathcal{H}_i)(\omega)\text{-a.s.} \quad (10.5b)$$

for all $i = 1, \dots, I$ and

$$P(V^\top z^{(j)} > 0 | \mathcal{H}_j)(\omega) > 0 \quad (10.5c)$$

for some $j \in \{1, \dots, I\}$.

A practical interpretation of a future arbitrage is a *redistribution* of assets (10.5a) such that each agent would almost surely not loose on his extra

assets (10.5b) and at least one agent perceives a potential profit on his extra assets (10.5c). In this respect a future arbitrage can be regarded as a form of Pareto improvement (cf. page 4). Recalling that the First Welfare Theorem rules out the possibility of Pareto improvements in equilibrium the following result is not surprising.

PROPOSITION

The information structure (\mathcal{H}_i) is arbitrage-free if and only if at almost all ω there are no future arbitrage opportunities for (\mathcal{H}_i) .

[25, PROPOSITION 2.2]

EXAMPLE (CONTINUED FROM PAGE 36)

The allocation

$$z^{(1)} := [0 \ 0 \ 1] =: -z^{(2)}$$

is a future arbitrage opportunity on $F_1 \cup F_2$. While

$$z^{(1)} := [-1 \ 1 \ 0] =: -z^{(2)}$$

is a future arbitrage opportunity on F_3 .

10.2 Arbitrage-free refinements

As symmetric information structures are arbitrage-free, the agents can find a no-arbitrage price by sharing their information. We say that given the information structure (\mathcal{H}_i) , the *pooled refinement* where $\mathcal{G}_i = \underline{\mathcal{H}}$ for all i is an *arbitrage-free and self-attainable refinement* of (\mathcal{H}_i) . But as the following example shows, the agents do not necessarily have to share *all* their information to reach an arbitrage-free refinement.

EXAMPLE (CONTINUED FROM PAGE 36)

The information structure $(\mathcal{G}_1, \mathcal{G}_2)$ given by

$$\mathcal{G}_1 := \mathcal{H}_1 \vee \mathcal{H}_2 = \sigma\{F_3, F_4\},$$

$$\mathcal{G}_2 := \mathcal{H}_2 = \sigma\{F_3\},$$

is clearly a self-attainable refinement of $(\mathcal{H}_1, \mathcal{H}_2)$. Proceeding as before, we find that

$$\Phi(\mathcal{G}_1) = \left\{ [p_1 \ p_2 \ 0]^\top \chi_{F_1 \cup F_2} + [0 \ 0 \ q_3]^\top \chi_{F_3} + [0 \ r_2 \ 0]^\top \chi_{F_4}; \right. \\ \left. p_1 < p_2, p_2 > 0, q_3 > 0, r_2 > 0 \right\}.$$

As

$$\Phi(\mathcal{G}_2) \equiv \Phi(\mathcal{H}_2) \supset \Phi(\mathcal{G}_1),$$

we have that

$$\Phi_c((\mathcal{G}_1, \mathcal{G}_2)) = \Phi(\mathcal{G}_1).$$

PROPOSITION

For any information structure there exists a unique coarsest refinement that is arbitrage-free. Moreover, this refinement is self-attainable. [25, PROPOSITION 3.1]

If the agents have to share all their information to reach an arbitrage-free information structure, i.e. the pooled refinement is the coarsest arbitrage-free refinement, we say that the information structure is *fully revealing*. As we shall see, the revealing properties of an information structure is linked to *market completeness*.

DEFINITION

A **contingent \mathcal{F} -claim** is a nonnegative and finite-valued random variable X on (Ω, \mathcal{F}, P) . Such a claim is **attainable for the σ -algebra $\mathcal{G} \subseteq \mathcal{F}$** if there exists some \mathcal{G} -measurable portfolio z such that

$$V^\top z = X \quad \text{a.s.} \quad (10.6)$$

We say that the market $\{(\Omega, \mathcal{F}, P), V\}$ is **complete** for \mathcal{G} if every contingent \mathcal{F} -claim is attainable for \mathcal{G} .

EXAMPLE

Suppose that $I = 2$, $J = 2$, \mathcal{F} is generated by the partition $\{F_1, F_2, F_3\}$ of Ω and

$$V(\omega) := \begin{cases} \begin{bmatrix} 1 & 0 \end{bmatrix}^\top, & \omega \in F_1, \\ \begin{bmatrix} 0 & 1 \end{bmatrix}^\top, & \omega \in F_2, \\ \begin{bmatrix} 1 & 1 \end{bmatrix}^\top, & \omega \in F_3. \end{cases} \quad (10.7)$$

Suppose $\mathcal{G} := \sigma\{F_1\}$, then as

$$V^\top \begin{bmatrix} 1 \\ 0 \end{bmatrix} \chi_{F_1} = \chi_{F_1}, \quad V^\top \begin{bmatrix} -1 \\ 1 \end{bmatrix} \chi_{F_1^c} = \chi_{F_2} \quad \text{and} \quad V^\top \begin{bmatrix} 1 \\ 0 \end{bmatrix} \chi_{F_1^c} = \chi_{F_3}$$

the market is complete for \mathcal{G} . Proceeding similarly, the reader can easily verify that this market is complete for any σ -algebra generated by any one of the sets.

PROPOSITION

The following are equivalent

- The market $\{(\Omega, \mathcal{F}, P), V\}$ is complete for \mathcal{G} .
- Any arbitrage-free information structure (\mathcal{G}_i) with $\mathcal{G}_i \supseteq \mathcal{G}$ for all i is symmetric.

Consequently, an information structure (\mathcal{H}_i) is fully revealing if the market is complete for the agents' common information.

[25, PROPOSITION 3.2, COROLLARY 3.2.1]

The following example shows that an information structure need not be fully revealing even if the market is complete for every agent.

EXAMPLE

Suppose $I = 2$, $J = 2$, \mathcal{F} is generated by the partition $\{F_1, \dots, F_4\}$ of Ω ,

$$V(\omega) := \begin{cases} \begin{bmatrix} 1 & 0 \end{bmatrix}^\top, & \omega \in F_1, \\ \begin{bmatrix} 2 & 0 \end{bmatrix}^\top, & \omega \in F_2, \\ \begin{bmatrix} 0 & 1 \end{bmatrix}^\top, & \omega \in F_3, \\ \begin{bmatrix} 1 & 1 \end{bmatrix}^\top, & \omega \in F_4, \end{cases}$$

and that the information structure is given by $\mathcal{H}_1 := \sigma\{F_1 \cup F_3\}$ and $\mathcal{H}_2 := \sigma\{F_2 \cup F_3\}$. Proceeding as in the example on page 39, the reader can easily verify that the market is complete for each agent's information. But the common information is the trivial σ -algebra for which the market is not complete. Any price vector ϕ with $\phi_1 > \phi_2 > 0$ a.s. belongs to $\Phi_c(\mathcal{H}_1, \mathcal{H}_2)$. Hence the information structure is *not* fully revealing.

10.3 Information revealed by prices

As previously indicated one way the agents can share their information is via prices:

PROPOSITION

Suppose that $\phi \in \Phi(\mathcal{F}')$ for some $\mathcal{F}' \subseteq \mathcal{F}$, then for any σ -algebra $\mathcal{H} \subseteq \mathcal{F}'$ there is a *unique coarsest* σ -algebra $\mathcal{G} \supseteq \mathcal{H}$ such that $\phi \in \Phi(\mathcal{G})$. This σ -algebra is referred to as the **σ -algebra revealed by ϕ** and denoted by $\mathcal{S}(\phi, \mathcal{H})$.

[25, PROPOSITION 4.1]

The observation that a price function that is not a common no-arbitrage price function for an information structure might still be a common no-arbitrage price function for some refinement of the information structure motivates the following:

DEFINITION

The $\underline{\mathcal{H}}$ -measurable function $\phi : \Omega \rightarrow \mathbb{R}^J$ is a **no-arbitrage price function** for (\mathcal{H}_i) , denoted $\phi \in \Phi_0((\mathcal{H}_i))$ if ϕ is a common no-arbitrage price function for some self-attainable refinement of (\mathcal{H}_i) .

Clearly, $\Phi_0((\mathcal{H}_i)) \neq \emptyset$. We also have that

$$\phi \in \Phi(\underline{\mathcal{H}}) \implies \phi \in \Phi_0((\mathcal{H}_i)).$$

The reverse implication holds if the market is complete for the common information. But it does not hold in general. As pointed out in the previous example, any price ϕ with $\phi_1 > \phi_2 > 0$ a.s. is a common no-arbitrage price for $(\mathcal{H}_1, \mathcal{H}_2)$ but is not necessarily a no-arbitrage price for the pooled information $\underline{\mathcal{H}} = \mathcal{F}$.

PROPOSITION

Given some information structure (\mathcal{H}_i) and price function ϕ the following are equivalent

- $\phi \in \Phi_0((\mathcal{H}_i))$
- $\mathcal{S}(\phi, \mathcal{H}_i)$ exists and $\mathcal{S}(\phi, \mathcal{H}_i) \subseteq \underline{\mathcal{H}}$ for all $i = 1, \dots, I$.
- $(\mathcal{S}(\phi, \mathcal{H}_i))$ is the coarsest self-attainable refinement of (\mathcal{H}_i) that is ϕ -arbitrage-free.

[25, PROPOSITION 4.2]

These observations motivate the following:

DEFINITION

The refinement $(\mathcal{S}(\phi, \mathcal{H}_i))$ is referred to as the **refinement revealed by ϕ** . A self-attainable arbitrage-free refinement (\mathcal{G}_i) of (\mathcal{H}_i) is **price-revealable** if there is some price function $\phi \in \Phi_0((\mathcal{H}_i))$ such that for every i , $\mathcal{G}_i = \mathcal{S}(\phi, \mathcal{H}_i)$.

As the following example shows, not all self-attainable arbitrage-free refinements are price-revealable, but the coarsest arbitrage-free refinement is (cf. [25, Proposition 4.3]).

EXAMPLE

Suppose $I = 2$, $J = 2$ and \mathcal{F} and V are as in the example on page 40, and consider the information structure

$$(\sigma\{F_1\}, \sigma\{F_1 \cup F_2\}).$$

The coarsest arbitrage free refinement of this information structure is

$$(\sigma\{F_1, F_2\}, \sigma\{F_1 \cup F_2\}).$$

The pooled refinement is not price revealable, because there is no price that will enable agent 2 to distinguish between F_1 and F_2 .

In [25, Section 5] we study *how* the refinements are revealed to the agents as they successively "arbitrage sets", i.e. elements in \mathcal{F} where a perceived arbitrage opportunity would give a strictly positive payoff. Similarly the coarsest arbitrage-free refinement (cf. Proposition on page 39) can be obtained as the agents analyse sets where a future arbitrage opportunity would give a strictly positive payoff.

V ENLARGEMENT OF FILTRATIONS

As indicated in the preface, the standard approach of mathematical finance to problems of asymmetric information is very different from the approaches discussed in the previous parts. Mathematicians have normally taken the price process as given and modelled the agents as *price takers* following some optimal strategy given their information and preferences. One of the earliest attempts at such models is found in [46]. In a standard problem of insider trading, the public information is given by some filtration \mathbb{F} satisfying the usual hypotheses while the insider is assumed to have access to some larger filtration \mathbb{G} . The price process of the assets is given as a vector-valued semimartingale with respect to \mathbb{F} . As pointed out in Appendix B, the semimartingale property of the asset prices is crucial for the stochastic integral (and hence the value process of a trading strategy) to be well-defined in the framework of Itô calculus. A process that is a semimartingale with respect to a certain filtration is not necessarily a semimartingale with respect to a larger filtration. The main developments in the field result from using one of the following approaches to the semimartingale issue:

- Assuming that the enlarged filtrations satisfy certain conditions (e.g. Jacod's Hypothesis H' [36]) which ensure that the semimartingale property of the price processes is preserved. Without possibility or aim to be complete we refer to the seminal paper by [46] and the more recent [35]. This is also the approach followed in Hillairet's work ([34]) that will be treated in Section 11.
- Applying another framework of stochastic calculus to give meaning to the stochastic integral. One example is *forward calculus*, cf. e.g. [8], [20] and [1] (treated in Section 8.4).
- Observing that certain requirements on a "reasonable" price process, like local boundedness and a "no free lunch with vanishing risk"-condition ([19, Definition 2.8]) for trading strategies that are \mathbb{G} -predictable and satisfies certain other regularity requirements (cf. [19] or [4]), imply that the price process is a \mathbb{G} -semimartingale. For more on this approach, see [4].

The price process obtained via forward calculus in [1] (see Section 8.4) is indeed a semimartingale. In [8], the authors calculate the expected (logarithmic) utility increment of an insider using forward calculus and show

that if the expected utility is finite, the price process is a semimartingale. In this respect we can say (cf. [4, Section 1]) that the authors use an *extrinsic* approach to prove that the price process is a semimartingale, whereas the third approach above is to prove that the semimartingale property is *intrinsic* for any "reasonable" price process.

11 Existence of an equilibrium

Following up the idea in [47], Hillairet [34] uses enlargement of filtration techniques to study equilibrium prices in a market with asymmetric information. The article describes a market with a multiple of risky assets whose price dynamics are given by jump diffusions, but for the sake of notational simplicity we restrict ourselves to the case of a single risky asset whose price follows a geometric Brownian motion.

11.1 The information structure

Fix some time horizon T and a complete probability space (Ω, \mathcal{F}, P) equipped with the filtration \mathbb{F} , which we assume to be the completion of the filtration generated by the Brownian motion B . The I agents are characterised by their information given by the filtrations $\mathbb{G}^{(1)}, \dots, \mathbb{G}^{(I)}$. We assume that agent i 's information is of the form

$$\mathcal{G}_t^{(i)} = \mathcal{F}_t \vee \sigma\{L_i\}, \quad t \in [0, T],$$

where L_i is an \mathcal{F} -measurable random variable¹⁰. The agents' *common* information is denoted by \mathbb{G} , i.e.

$$\mathcal{G}_t = \bigwedge \mathcal{G}_t^{(i)}, \quad t \in [0, T].$$

The following assumption is made to ensure that any \mathbb{F} -semimartingale is also a $\mathbb{G}^{(i)}$ -semimartingale (cf [2, Corollary 2.6 (1)]).

DECOUPLING ASSUMPTION

For all i , there exist some probability measure $Q^{L_i} \sim P$ under which $\sigma\{L_i\}$ and \mathcal{F}_t are independent for all $t \in [0, T]$. Further, it is assumed that Q^{L_i} is identical to P on \mathcal{F} .

[34, ASSUMPTION 2.2, REMARK 2.3]

For a justification of the term "decoupling", see [2, Lemma 2.4]. The assumption is sufficient for the following theorem to hold.

¹⁰a slight simplification of Hillairet's approach

A MARTINGALE REPRESENTATION THEOREM

For any local $(\mathbb{G}^{(i)}, Q^{L_i})$ -martingale M there exists some $\mathbb{G}^{(i)}$ -predictable ψ such that $\int_0^t \psi_s dB_s \in L_1(\mathcal{G}_t^{(i)}, Q^{L_i})$ for all $t \in [0, T]$ and

$$M_t = M_0 + \int_0^t \psi_s dB_s, \quad t \in [0, T].$$

[34, THEOREM 2.4]

The following assumption is crucial to establish both the existence of an equilibrium and its revealing properties (cf. Lemma on page 48).

CONDITIONAL INDEPENDENCE ASSUMPTION

The σ -algebras $\mathcal{G}_t^{(1)}, \dots, \mathcal{G}_t^{(I)}$ are conditionally independent given \mathcal{G}_t for all $t \in [0, T]$.

[34, ASSUMPTION 5.2]

11.2 The economy

At the market there is one riskless asset (bank account) and a risky asset (stock). The *interest rate* of the bank account is given by an \mathbb{F} -predictable and bounded stochastic process r . The dynamics of the stock price is given by

$$dS_t = S_t(b_t dt + \sigma_t dB_t), \quad t \in [0, T],$$

where b and σ are \mathbb{F} -predictable stochastic processes that are sufficiently regular for a solution to exist.

Agent i is assumed to receive *endowments* at the \mathbb{F} -predictable¹¹, nonnegative and uniformly bounded rate

$$e^{(i)} := \{e_t^{(i)}; 0 \leq t \leq T\}.$$

The sum e of all the agents' endowment rates is given by

$$e_t = e_0 + \int_0^t \mu_s ds + \int_0^t \eta_s dB_s, \quad t \in [0, T],$$

where e_0 is a constant and μ and η are \mathbb{F} -progressive processes such that e is bounded away from zero below. In addition, each agent is endowed with the *initial wealth* $W_0^{(i)} \in L_1(\mathcal{G}_0^{(i)})$.

¹¹It seems unnecessary to require *all* the endowments to be \mathbb{F} -predictable. To solve the agents' optimisation problems in Section 11.4 it suffices that $e^{(i)}$ is $\mathbb{G}^{(i)}$ -predictable for every i . Clearly the argument preceding the Lemma on page 48 requires the *sum* of the endowments to be \mathbb{F} -predictable for an equilibrium to exist.

Each agent has some utility function $U^{(i)} : [0, T] \times (0, \infty) \rightarrow \mathbb{R}$ such that the partial derivatives $U_t^{(i)}, U_{tc}^{(i)}, U_{cc}^{(i)}, U_{ccc}^{(i)}$ exist and are continuous on $[0, T] \times (0, \infty)$. Further for every $t \in [0, T]$, $U^{(i)}(t, \cdot)$ is strictly increasing and strictly concave, and

$$U_c^{(i)}(t, c) \xrightarrow{c \rightarrow \infty} 0 \text{ and } U_c^{(i)}(t, c) \xrightarrow{c \searrow 0} \infty.$$

For any $t \in [0, T]$ $I^{(i)}(t, \cdot)$ denotes the inverse of $U_c^{(i)}(t, \cdot)$

11.3 Finding an equivalent martingale measure

There are no arbitrage opportunities for agent i if there exists some probability measure $P^{(i)} \sim P$ such that the discounted asset price $\hat{S} := S e^{-\int r_s ds}$ is a $(\mathbb{G}^{(i)}, P^{(i)})$ -local martingale. Under the Decoupling Assumption, the density process $Z^{(i)}$ defined by

$$Z_t^{(i)} := E_{Q^{L_i}} \left[\frac{dP}{dQ^{L_i}} \Big| \mathcal{G}_t^{(i)} \right]$$

is a positive $(\mathbb{G}^{(i)}, Q^{L_i})$ -local martingale. Hence, by Theorem 2.4 there exist some $\mathbb{G}^{(i)}$ -predictable process $\rho^{(i)}$ such that

$$dZ_t^{(i)} = Z_t^{(i)} \rho_t^{(i)} dB_t.$$

Denote $\theta := \frac{b-r}{\sigma}$, and define $Y^{(i)}$ by the Doléans-Dade exponential (see e.g. [48])

$$Y_t^{(i)} := \mathcal{E} \left(- \int_0^t (\theta_s + \rho_s^{(i)}) (dB_s - \rho_s^{(i)} ds) \right).$$

We can now define the deflator $\zeta^{(i)} := Y^{(i)} e^{-\int r_s ds}$. Assuming that $Y^{(i)}$ is a $(\mathbb{G}^{(i)}, P)$ -martingale (cf. [34, Assumption 3.3]¹²), the discounted asset price \hat{S} is a local martingale with respect to $\mathbb{G}^{(i)}$ and the probability measure $P^{(i)}$ defined by

$$P^{(i)}(A) := E[Y_T^{(i)} \chi_A], \quad A \in \mathcal{G}_T^{(i)}.$$

11.4 The agents' trading and consumption strategies

If

$$\pi := \{ \pi_t \in \mathbb{R}^J; 0 \leq t \leq T \}$$

denotes the amount agent i has invested in the stock and

$$c := \{ c_t; 0 \leq t \leq T \}$$

¹²In our simplified version some sufficient conditions for this to hold true are the Kazamaki and Novikov conditions given in [45, Remark following Exercise 4.4].

his consumption rate, his *wealth* is governed by

$$dW_t^{(i)} = r_t W_t^{(i)} dt + \pi_t (b_t - r_t) dt + \sigma_t dB_t + (e_t^{(i)} - c_t) dt.$$

A portfolio/consumption pair (π, c) is *admissible* for agent i if c and $\frac{\pi}{S}$ are $\mathbb{G}^{(i)}$ -predictable,

$$\int_0^T (c_t + (\pi_t \sigma_t)^2) dt < \infty \quad P\text{-a.s.}$$

and the corresponding wealth process is bounded from below and satisfies

$$W_T^{(i)} \geq 0 \quad P\text{-a.s.}$$

Agent i chooses a portfolio/consumption pair among the admissible ones to maximise

$$E \left[\int_0^T U^{(i)}(t, c_t) dt \middle| \mathcal{G}_0^{(i)} \right].$$

The following theorem is the "non-trivial initial information"-version of [38, Theorem 9.4]:

THEOREM

Suppose that there exist some $\mathcal{G}_0^{(i)}$ -measurable random variable λ such that

$$E \left[\int_0^T \zeta_t^{(i)} (I^{(i)}(t, \lambda \zeta_t^{(i)}) - e_t^{(i)}) dt \middle| \mathcal{G}_0^{(i)} \right] = W_0^{(i)}. \quad (11.1)$$

In that case, there exist a unique solution to agent i 's optimisation problem, where the optimal consumption rate is

$$c_t^{(i)} = I^{(i)}(t, \lambda \zeta_t^{(i)}), \quad t \in [0, T].$$

[34, THEOREM 4.4]

The existence and uniqueness of λ is ensured by the following (cf. [34, Proposition 4.5]) for any $x \in \mathbb{R}_+$

$$E \left[\int_0^T \zeta_t^{(i)} (I^{(i)}(t, x \zeta_t^{(i)}) - e_t^{(i)}) dt \middle| \mathcal{G}_0^{(i)} \right] < \infty, \quad P\text{-a.s.}$$

and

$$E \left[\int_0^T \zeta_t^{(i)} e_t^{(i)} dt \middle| \mathcal{G}_0^{(i)} \right] + W_0^{(i)} > 0, \quad P\text{-a.s.}$$

11.5 Equilibrium

Clearly, in equilibrium¹³ we must have that

$$\sum c_t^{(i)} = \sum e_t^{(i)} = e_t, \quad t \in [0, T], \quad (11.2)$$

i.e., fixing some i

$$\sum_{j \neq i} c_t^{(j)} = e_t - c_t^{(i)}, \quad t \in [0, T]. \quad (11.3)$$

Note that (11.2) implies that the corresponding optimal portfolio processes satisfy

$$\sum \pi_t^{(i)} = 0, \quad t \in [0, T],$$

(cf. [34, proof of Lemma 5.11] or [38, Theorem 9.2]).

The left-hand side of (11.3) is $\mathcal{G}_t^{(-i)} := \bigvee_{j \neq i} \mathcal{G}_t^{(j)}$ -measurable, i.e.

$$\mathcal{F}_L := \sigma \left\{ \sum_{j \neq i} c_s^{(j)}; 0 \leq s \leq t \right\} \subseteq \mathcal{G}_t^{(-i)},$$

while the right-hand side is $\mathcal{G}_t^{(i)}$ -measurable, i.e.

$$\mathcal{F}_R := \sigma \left\{ e(s) - c_s^{(i)}; 0 \leq s \leq t \right\} \subseteq \mathcal{G}_t^{(i)}.$$

By the conditional independence between $\mathcal{G}_t^{(i)}$ and $\mathcal{G}_t^{(-i)}$ given \mathcal{G}_t we have that for any $A \in \mathcal{F}_L = \mathcal{F}_R$:

$$E[\chi_A | \mathcal{G}_t] = E[(\chi_A)^2 | \mathcal{G}_t] = E[E[\chi_A | \mathcal{G}_t^{(-i)}] E[\chi_A | \mathcal{G}_t^{(i)}] | \mathcal{G}_t] = E[\chi_A | \mathcal{G}_t] E[\chi_A | \mathcal{G}_t].$$

Hence, $P(A | \mathcal{G}_t)$ is either 0 or 1. As \mathcal{G}_t contains all P -null sets, we have that $A \in \mathcal{G}_t$. Thus, both sides of (11.3) are \mathcal{G}_t -measurable:

LEMMA

Under the assumption of conditional independence, in order to achieve an equilibrium each agent's consumption process must be adapted to the filtration \mathbb{G} .

[34, LEMMA 5.4]

If we also assume that each agent's initial wealth is \mathcal{G}_0 -measurable, it is (cf. [34, Proposition 4.2]) possible to find a portfolio process such that $(\pi^{(i)}, c^{(i)})$ would be admissible for agent i if he only had access to the information \mathbb{G} . Hence, each agent's optimal consumption is also \mathbb{G} -optimal.

¹³Hillaiet refers to this notion of equilibrium as an Arrow-Debreu equilibrium, this is not consistent with our previous definition of an Arrow-Debreu equilibrium.

Hillairet also proves the existence of such a "revealing equilibrium". It is assumed that the common information is of the form

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma\{L\}, \quad t \in [0, T],$$

where L is some \mathcal{F} -measurable random variable that satisfies the decoupling assumption and that \mathcal{G}_T and $\mathcal{G}_t^{(i)}$ are conditionally independent given \mathcal{G}_t for all i and $t \in [0, T]$. One can then proceed as in Section 11.3 to find an equivalent martingale measure for the common information. It is also assumed that (11.1) has a \mathcal{G}_0 -measurable solution for all i . Introducing a *representative agent* as in [38], it is proved that the existence of an equilibrium is equivalent to the existence of a solution to this agent's optimisation problem ([34, Proposition 5.9]). Finally, under some extra assumptions regarding the relation between the representative agent's utility function and the parameters r and b , the existence of an equilibrium is established ([34, Proposition 5.14]).

VI CONCLUSIONS

We have seen that both economical rational expectations models and models from mathematical finance support the idea of strong market efficiency.

Critics of the rational expectations approach often say that it "demands too much" concerning the agents' ability to observe market activity and draw the right conclusions. The approaches by Kyle and Wang on one hand and Cornet and de Boisdeffre on the other, take this criticism into account. Adding noise to the model limits the agents' ability to observe the market activity. But when there is no noise in the market, both Kyle's and Wang's models result in fully revealing equilibria. Analysing and ruling out arbitrage opportunities entails a much weaker form of rationality than assumed in the theory of rational expectations. In this sense one may think of Cornet and de Boisdeffres approach as one of "not irrational" expectations.

As seen in Kyle's paper and the subsequent ones and in Wang's paper: Agents with different information have *different perceptions* of the price dynamics. Apart from the semimartingale property, it is not clear how the agents in Hillairet's model perceive the asset price dynamics. The price taking assumption however, rules out arbitrage opportunities like the Brownian bridge price process in Kyle's paper.

In principle, the core difference between the approach of Cornet and de Boisdeffre and the traditional no-arbitrage models in mathematical finance is that the latter models only consider the case where the price is adapted to the information of the least informed agents. An extension of Cornet and de Boisdeffre's work to the continuous time case, could probably benefit from the results from the mathematical finance approaches.

APPENDIX

A Conditional probability

From [9, Section 33] we present some facts about conditional probability. Given the probability space (Ω, \mathcal{F}, P) . Let \mathcal{G} be a σ -algebra $\mathcal{G} \subseteq \mathcal{F}$. For any $F \in \mathcal{F}$, the random variable $P(F|\mathcal{G})$ is a version of the conditional probability of F if it is \mathcal{G} -measurable, integrable and satisfies

$$\int_G P(F|\mathcal{G})(\omega) dP(\omega) = P(F \cap G), \quad G \in \mathcal{G}.$$

There will in general be many such random variables, but any two of them are equal with probability 1.

THEOREM

For any version of $P(\cdot|\mathcal{G})$,

$$\begin{aligned} P(\emptyset|\mathcal{G}) &= 0, & P(\Omega|\mathcal{G}) &= 1, & \text{a.s.}, \\ P(F|\mathcal{G}) &\in [0, 1], & F \in \mathcal{F}, & & \text{a.s.}, \end{aligned}$$

and if F_1, F_2, \dots is a countable sequence of disjoint sets in \mathcal{F} , then

$$P\left(\bigcup_n F_n|\mathcal{G}\right) = \sum_n P(F_n|\mathcal{G}), \quad \text{a.s.}$$

[9, THEOREM 33.2]

If \mathcal{G} is generated by a finite partition G_1, \dots, G_N one can always define conditional probability as follows: for any $F \in \mathcal{F}$,

$$P(F|\mathcal{G})(\omega) := \begin{cases} P(F), & \omega \in G_n, P(G_n) = 0, \\ \frac{P(F \cap G_n)}{P(G_n)} & \omega \in G_n, P(G_n) > 0. \end{cases} \quad (\text{A.1})$$

In this case, $P(\cdot|\mathcal{G})(\omega)$ is a probability measure on (Ω, \mathcal{F}) for any ω . For more general σ -algebras this is not always the case. But the above theorem ensures that for any ω_0 such that $P(G) > 0$ for any $G \in \mathcal{G}$ containing ω_0 , $P(\cdot|\mathcal{G})(\omega_0)$ is a probability measure on (Ω, \mathcal{F}) . Moreover, the proof of the theorem also entails that $P(\cdot|\mathcal{G})(\omega_0)$ is absolutely continuous with respect to P . Further, we have that for any $F \in \mathcal{F}$,

$$P(F|\mathcal{G}) = E[\chi_F|\mathcal{G}] \quad \text{a.s.}$$

B Semimartingales

This presentation of semimartingales is based entirely upon [48, Parts II and III]. Suppose that the probability space (Ω, \mathcal{F}, P) is complete and equipped with the right continuous filtration \mathbb{F} . A *simple predictable process* is a process with representation

$$H_t = H_0 \chi_{t=0} + \sum_{i=1}^n H_i \chi_{t \in (T_i, T_{i+1}]}, \quad (\text{B.1})$$

where $0 = T_1 \leq \dots \leq T_{n+1} < \infty$ is a finite sequence of \mathbb{F} -stopping times and $H_i \in \mathcal{F}_{T_i}$, $i = 1, \dots, n$. We denote the set of such processes by \mathbf{S} , this set topologised by uniform convergence in (t, ω) by \mathbf{S}_u and the set of finite-valued random variables topologised by convergence in probability by \mathbf{L}^0 . For any stochastic process X we may define the linear mapping $I_X : \mathbf{S}_u \rightarrow \mathbf{L}^0$

$$I_X(H) := H_0 X_0 + \sum_{i=1}^n H_i (X_{T_{i+1}} - X_{T_i}).$$

Denote by $X^{(\tau)}$ the process X stopped at the stopping time τ .

DEFINITION

X is an **\mathbb{F} -semimartingale** if it is adapted to \mathbb{F} and continuous from the right with left-hand limits, and the mapping $I_{X^{(t)}} : \mathbf{S}_u \rightarrow \mathbf{L}^0$ is continuous for any $t \in \mathbb{R}_+$.

Denote the space of càdlàg¹⁴ \mathbb{F} -adapted processes by \mathbb{D} , and the space of càglàd \mathbb{F} -adapted processes by \mathbb{L} and \mathbf{S} topologised by uniform convergence on compacts in probability by \mathbf{S}_{ucp} . The definition of a semimartingale and the fact that \mathbf{S} is dense in \mathbb{L} under the topology of uniform convergence on compacts in probability justifies the following:

DEFINITION

For $H \in \mathbf{S}$ with the representation (B.1) the **stochastic integral of H with respect to the semimartingale X** is given by the continuous mapping $J_X : \mathbf{S}_{ucp} \rightarrow \mathbb{D}_{ucp}$

$$J_X(H) := H_0 X_0 + \sum_{i=1}^n H_i (X^{(T_{i+1})} - X^{(T_i)}).$$

The extension of J_X as a mapping $\mathbb{L}_{ucp} \rightarrow \mathbb{D}_{ucp}$ is called the **stochastic integral** and written

$$J_X(H) = \int H dX_s.$$

¹⁴ Recall that càdlàg (càglàd) processes are continuous from the right (left) and has left- (right-) hand limits

Roughly speaking one may say that a semimartingale is a stochastic process that can serve as an integrator for stochastic processes in \mathbb{L} . A process of the form (B.1) can be interpreted as a trading strategy starting with the portfolio H_0 and buying H_i assets at each time T_i and keeping them until time T_{i+1} . Strategies of this form are referred to as *buy and hold strategies*. Given the price process X the strategy given by (B.1) is *self-financing* if

$$H_{i-1}^\top X_{T_i} = H_i^\top X_{T_i}, \quad i = 1, \dots, n.$$

The *wealth* generated by the strategy (B.1) is

$$\begin{aligned} W_t &:= H_t^\top X_t = W_0 + \sum_{T_i \leq t} (H_i^\top X_{T_i} - H_{i-1}^\top X_{T_{i-1}}) \\ &= W_0 + \sum_{T_i \leq t} (H_{i-1}^\top X_{T_i} - H_{i-1}^\top X_{T_{i-1}}) \\ &= W_0 + \sum_{T_i \leq t} H_{i-1}^\top (X_{T_i} - X_{T_{i-1}}). \end{aligned}$$

Now, if X is a semimartingale we may extend this idea to trading strategies $Y \in \mathbb{L}$. The self-financing condition is now

$$dW_t = Y_t^\top dX_t$$

(with the Itô interpretation) and the wealth generated by Y is

$$W_t = W_0 + \int_0^t Y_s dX_s.$$

Hence, the semimartingale property of prices enables us to give a mathematically tractable expression of the wealth generated by a fairly wide range of trading strategies. In the case of symmetric information it is proved that a price process that is "reasonable" i.e. locally bounded and satisfying the "no free lunch with vanishing risk"-condition ([19, Definition 2.8]) is a semimartingale ([19, Theorem 7.2]). Further, it is proved that if there exists some strictly concave and strictly increasing utility function U such that

- $U(x) \rightarrow \infty$ as $x \rightarrow \infty$ and
- the supremum of the expected utility of the final value of a self-financing trading strategy whose wealth process is bounded from below, is finite,

then the price process is a semimartingale (cf. [4, Corollary 1.8] or [8] for a similar result).

Semimartingality is a property that is preserved by stochastic integration, i.e. $J_x(H)$ is itself a semimartingale (cf. e.g. [48, Part II, Theorem 19]). The economical interpretation of this property is that we may regard the set of self-financing trading strategies in \mathbb{L}_{ucp} as investment opportunities or assets themselves whose prices are given by the wealth they generate.

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