STOCHASTIC MODELLING AND PRICING OF ENERGY AND WEATHER DERIVATIVES

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Introduction

Energy markets are primary commodity markets that deal with the trade and supply of energy for consumption. Initially, these markets were regulated in such a way that the energy delivery process including pricing was governed by a regulatory or government body and only the local utility was able to sell to consumers. The amount of energy and final price for it including transportation and distribution were fixed so that consumers had to accept these conditions. The deregulation of the energy market in the beginning of the 1990's resulted in the emergence of markets for trading in the underlying spot but also in derivative products in numerous countries and regions all over the world. The spot price is the current market price at which consumers and producers can respectively buy and sell the energy respectively for immediate payment and delivery. Energy derivatives are financial contracts traded in secondary markets whose prices are derived from the underlying energy spot price.

A well known problem in energy markets is to find efficient stochastic models for the underlying spot and derivative product prices. The models have to be analytically tractable and reproduce observable stylized facts in prices like stationarity, spikes, mean reversion, seasonality and volatility among others in order to make accurate future predictions. Lévy semistationary (LSS) processes have been recently proposed by Ole E. Barndorff-Nielsen *et al.* [6] as a new modelling framework for energy spot prices. An LSS process is defined as

$$Y(t) = \mu + \int_{-\infty}^{t} g(t-s)\sigma(s-)\, dL(s) + \int_{-\infty}^{t} q(t-s)a(s-)\, ds\,,$$

where μ is a constant, L is a Lévy process, g and q are nonnegative deterministic functions on $\mathbb{R}$, with $g(t) = q(t) = 0$ for $t < 0$, and σ and a are stationary right-continuous with left-limits processes. LSS processes encompass many classical models used in energy finance such as those based on the Schwartz one-factor mean-reversion model or the wider class of continuous-time autoregressive moving average (CARMA) models based on multivariate non-Gaussian Ornstein-Uhlenbeck processes. Furthermore, their structure allows one to reproduce these stylized facts in prices. We focus our interest on markets that are incomplete for the impossibility of hedging against adverse market conditions by trading on the underlying spot. The electricity market is an example of an incomplete market of this kind. Indeed, electricity is a commodity that cannot be stored directly and therefore, it has to be consumed after being produced. In the spot market, the price of the electricity is established for consumption according to supply and demand. For instance, The Nordic electricity exchange Nord Pool Spot is a deregulated electricity market which covers Denmark, Finland, Sweden, Norway, Estonia and Lithuania. Producers and suppliers which trade in electricity in this market are exposed to volume risk and price fluctuations. However, electricity can be stored indirectly in water reservoirs for hydro-production. Hydro pumped storage plants are another sophisticated mechanism that furthermore allows to use the surplus of energy coming from hydro-power to pump water from a lower to a higher reservoir. Both mechanisms make it possible to store electricity indirectly in the form of gravitational potential energy of that water. The high cost of disposing of water reservoirs together with the fact that the production of electricity is not immediate and the desire to continue offering competitive prices force producers and suppliers in the electricity spot market to look for risk management strategies by trading in the electricity derivatives market.

Weather conditions directly affect the production and consumption of electricity and consequently its final price. Producers and suppliers hedge against unfavourable weather conditions by trading in weather-based index products in weather markets. These markets, which emerged with the deregulation of energy markets, are also incomplete due to the impossibility of hedging with the underlying indexes based on weather variables for not having any value. We will pay special attention to the temperature derivatives traded at the Chicago Mercantile Exchange (CME).

The electricity and the weather market are then related markets that turn out to be incomplete. Throughout this dissertation we will see that these markets furthermore share similarities from a modelling point of view. For this purpose, we will price and analyse derivatives products like futures contracts on electricity and futures contracts on temperature indexes traded at the CME. Also, we will consider option contracts written on the temperature futures prices. To do so, first we accurately model the underlying spot price with an LSS process. We focus special attention on Lévy-driven CARMA processes which are the adaptation in continuous-time of the well-konwn *autoregressive moving average* (ARMA) processes used to model time series in discrete time. Empirical studies show the adequacy of Lévy-driven CARMA processes to model the dynamics of the electricity spot price and temperature. Garcia *et al.* [33] apply CARMA processes for modelling the dynamics of electricity spot prices and Benth *et al.* [16] consider a particular case of CARMA processes known as *continuous-time autoregressive* (CAR) processes for modelling the temperature dynamics. Lévy-driven CARMA processes are powerful mean-reverting and non-Markovian processes that allow for jumps. In this dissertation, we analyse arithmetic and geometric models where the noise term is a stationary Lévy-driven CARMA process. Benth *et al.* [16] provide the basic steps to fit a CAR model to historical data and in Benth and Šaltytė Benth [15] these are extended to a CARMA model. The classical asset pricing theory does not apply in this context due to the impossibility of trading in the electricity and weather indexes. The classical spot-forward relation does not hold, and consequently, a new pricing framework is required. The fact that the underlying is not tradeable gives rise to the existence of a huge class of pricing measures. We choose to work on the parametric class of pricing measures given by the Esscher transform where the parameter refers to the market price of risk. Barndorff-Nielsen *et al.* [6] and Benth and Šaltytė Benth [15] derive theoretically energy and weather futures prices respectively. Following these examples, we derive futures prices based on general LSS models with a special interest to CARMA models and we analyse the new spot-forward relationship.

1.1 Stochastic model for the spot price

In mathematical finance, the traditional models are based on stochastic processes driven by a Brownian motion $B = \{B(t), t \geq 0\}$. A stochastic process is said to be a geometric Brownian motion if it satisfies the following stochastic differential equation

$$dS(t) = \mu S(t)\, dt + \sigma S(t)\, dB(t)\,, \tag{1.1.1}$$

where μ and $\sigma > 0$ are constants denoting the drift and volatility, respectively. The solution of (1.1.1) is given by

$$S(t) = S(0)\mathrm{e}^{X(t)},$$

where $X(t) = (\mu - \frac{\sigma^2}{2})t + \sigma\, B(t)$ is a normally distributed with mean $(\mu - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$. Observe that the geometric Brownian motion is a non-negative variation of the Brownian motion which makes sense to use for pricing. The logarithmic returns

$$\ln S(t + \Delta t) - \ln S(t) = (\mu - \frac{\sigma^2}{2})\Delta t + \sigma(B(t + \Delta t) - B(t))$$

are normally distributed with mean $(\mu - \frac{\sigma^2}{2})\Delta t$, variance $\sigma^2 \Delta t$ and have independent and stationary increments. Eberlein and Keller [31] provided examples where the logarithmic returns of stock prices did not follow a normal distribution. They then suggested a model generalizing the geometric Brownian motion known as the exponential Lévy process which is defined as follows

$$S(t) = S(0)\mathrm{e}^{L(t)}\,.$$

The logarithmic returns of the exponential Lévy process are stationary and independent but not necessarily normally distributed allowing then for skewness and leptokurtic behavior. In the real world we observe spikes in prices, therefore it makes sense to consider Lévy processes. However, the jumps obtained with the Lévy process must be homogeneous in size and frequency so that the stationary condition, which is essential to make accurate forecasts, is preserved. Lévy processes describe the observed reality of financial markets in a more accurate way than models based on Brownian motion.

In an equilibrium setting when prices are relatively high, supply increases and demand reduces, producing a balancing effect. Conversely, when prices are relatively low, supply decreases and demand increases. Schwartz [43] considers this argument to introduce different mean reverting processes to model commodity prices. We emphasize the well-known one-dimension Ornstein-Uhlenbeck process defined as the solution of the following stochastic differential equation

$$dX(t) = \alpha(\mu - X(t))\, dt + \sigma\, dB(t)\,, \tag{1.1.2}$$

where $\alpha > 0$ measures the degree of mean reversion, μ is a constant representing the long run mean to which the process tends to revert and $\sigma > 0$ refers to the volatility in prices. If a Lévy process is considered in (1.1.2) rather than a Brownian motion, then spikes in prices tend to revert back to μ with a velocity given by the mean reverting coefficient α.

Consider the multidimensional stochastic differential equation

$$d\mathbf{X}(t) = A\mathbf{X}(t)\,dt + \sigma(t)\mathbf{e}_p\,dB(t) \tag{1.1.3}$$

where, $\mathbf{e}_p$ is the pth canonical basis vector in $\mathbb{R}^p$ for $p \in \mathbb{N} - \{0\}$. Furthermore, A is a real valued $p \times p$ matrix given by

$$A = \begin{pmatrix} \mathbf{0}_{p-1} & I_{p-1} \\ -\alpha_p .. & .. - \alpha_1 \end{pmatrix}, \tag{1.1.4}$$

where $\mathbf{0}_{p-1}$ is the $p-1$ dimensional vector of zeros, I_{p-1} is the identity matrix of dimension $p-1$ and the constants $\alpha_k, k = 1, \ldots, p$ satisfy $\alpha_k > 0$. The solution of (1.1.3) starting at $\mathbf{X}(s) = \mathbf{x} \in \mathbb{R}^p$ for a time $s \geq 0$ is given by

$$\mathbf{X}(t) = \exp(A(t-s))\mathbf{x} + \int_s^t \exp(A(t-u))\mathbf{e}_p\,dB(u), \tag{1.1.5}$$

for all $t \geq s \geq 0$. Let $q \in \mathbb{N}$ such that $0 \leq q < p$ and consider the following p-dimensional vector $\mathbf{b}$ defined as

$$\mathbf{b}^{\mathrm{T}} = \begin{pmatrix} b_0 & b_1 & \ldots & b_{q-1} & 1 & 0 & \ldots & 0 \end{pmatrix}, \tag{1.1.6}$$

with $b_q = 1$ and $b_j = 0$ for $q+1 \leq j < p$. A continuous-time autoregressive moving-average process of autoregressive order p and moving average order q, denoted CARMA(p, q), is defined then as

$$Y(t) = \mathbf{b}^{\mathrm{T}}\mathbf{X}(t). \tag{1.1.7}$$

CARMA(p, q)-processes can be understood as a linear combination of the $q+1$ coordinates of a particular case of multivariate Ornstein-Uhlenbeck process with the α's in (1.1.4) being the mean reverting coefficients. Thus, CARMA processes are powerful mean reverting processes. The subclass of CARMA(p, q)-processes with moving average order $q = 0$ for which the vector $\mathbf{b}$ in (1.1.6) reduces to $\mathbf{b} = \mathbf{e}_1$ is known as *continuous-time autoregressive* (CAR) processes of autoregressive order p, denoted CAR(p)-processes. CARMA processes as given in (1.1.7) are stationary if the the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A satisfy that all have negative real parts, *i.e.*

$$\mathrm{Re}(\lambda_i) < 0, \quad i = 1, \ldots, p.$$

Brockwell in [25] examines Lévy-driven CARMA processes with constant volatility, $\sigma = 1$, for financial applications. The stationary condition for these processes has been thoroughly studied by Brockwell and Linder [26].

Electricity markets are seasonally varying markets. We can observe seasonality in jump size and frequency. For instance, in the Nordpool market spikes are more frequent in the winter period. The stationarity feature is not effective in explaining seasonality in prices. Hence, the seasonally varying spot price for electricity is explained by an arithmetic or a geometric CARMA model defined respectively as

$$S(t) = \Lambda(t) + Y(t)$$
$$S(t) = \Lambda(t)\mathrm{e}^{Y(t)} ,$$

where Λ is a deterministic function modelling the seasonal feature for electricity prices and $Y(t)$ is the CARMA process defined as in (1.1.7).

1.2 Futures contracts

A future contract is an agreement between two parties to buy or sell an underlying commodity or asset in the future. The buyer and seller of such a contract agree on a price today for a product to be delivered or settled in cash at a future date or time period. Future contracts are risky financial products used primarily for hedging against the risk of fluctuations in prices but also to speculate by taking advantage of the price movements.

Throughout this section, we fix a probability space $(\Omega, \mathcal{F}, P)$ where $(\mathcal{F}_t)_{t\in\mathbb{R}}$ is a filtration with the market information up to time t and P is the market probability.

1.2.1 Pricing futures contracts

Suppose that our market model defined in $(\Omega, \mathcal{F}, P)$ consists of a risk-free asset A and a tradeable risky asset S. Denote $f(t, \tau)$ the futures price of a contract at time $t \geq 0$ delivering a unit of the underlying S at a future time τ, with $0 \leq t \leq \tau < \infty$. The payoff at time τ is given by $S(\tau) - f(t, \tau)$. The theory of mathematical finance, see Duffie [30], establishes that the price of any derivative is given as the conditional expected value under a risk-neutral probability Q of its discounted payoff with respect to the information of the market up to present. In the particular case of a futures contract, the price of entering into it is zero, *i.e.*

$$0 = \mathbb{E}_Q \left[\mathrm{e}^{-r(\tau-t)}(S(\tau) - f(t, \tau))|\mathcal{F}_t \right] ,$$

where $r > 0$ denotes the risk-free interest rate. We assume that $S(\tau) \in L^1(Q)$, the space of integrable random variables with respect to Q, and that $f(t, \tau)$ is $\mathcal{F}_t$-adapted as it is derived from the information on the market up to present. Hence, the futures price of a contract at time t delivering a unit of the underlying asset S at time $\tau \geq t$ reduces to

$$f(t, \tau) = \mathbb{E}_Q [S(\tau)|\mathcal{F}_t] . \tag{1.2.1}$$

A probability measure Q on $(\Omega, \mathcal{F})$ is called a *risk-neutral probability measure* if Q is equivalent to P, denote $Q \sim P$, and the discounted asset price $e^{-rt}S(t)$ is a (local)-martingale under Q. Risk-neutral probability measures are also known as *equivalent (local)-martingale measures*. The First Fundamental Theorem in Mathematical Finance, see e.g. Björk [21], presents the arbitrage-free pricing theory. The existence of risk-neutral probability measures is a necessary condition for applying this theory. The (local)-martingale property of the discounted asset S under Q reduces the futures price in (1.2.1) to

$$f(t,\tau) = \mathrm{e}^{r(\tau-t)} S(t)\,.$$

The strategy of trading the underlying spot is known as the *buy-and-hold strategy*. The traditional models in mathematical finance belong to the class of semimartingale processes. The reason for this is the existence of risk-neutral probability measures. In the temperature and electricity markets, the spot is not tradeable. Electricity cannot be kept in a portfolio over time and the temperature-based indexes do not have any value. Hence, the only necessary condition to price a futures contract in the electricity or weather market is to have a probability measure Q equivalent to the market probability P. In these markets then it is not necessary to model the underlying spot with a semimartingale process. LSS processes are not in general semimartingales, but Lévy-driven CARMA processes, for example, satisfy this condition. In the markets that we have in mind, any probability measure Q equivalent to the market probability P is considered a risk-neutral probability. As a consequence, we get a huge range of prices. We will consider the class of risk-neutral probabilities Q given through the Esscher transform.

At this point, we can define a new market model. To do so, we consider the same probability space as before with the risk-free asset A but we fix the risky asset being a futures contract f as defined in (1.2.1) instead. To trade in futures contracts without arbitrage opportunities the futures price must be a martingale under Q. We see next that this condition is directly satisfied just for pricing the futures contract under any probability $Q \sim P$. Indeed, as $S(\tau) = f(\tau,\tau)$ we get as follows the martingale requirement

$$f(t,\tau) = \mathbb{E}_Q[\mathbb{E}_Q[f(\tau,\tau)|\mathcal{F}_t]\,.$$

In the electricity market futures contracts are defined with delivery over a time period rather than a specific day. Denote $F(t,\tau_1,\tau_2)$ the futures price at time $t \geq 0$ of a contract delivering electricity over $[\tau_1,\tau_2]$. Entering into a futures contract at time $0 \leq t \leq \tau_1$ is costless, meaning that

$$0 = \mathbb{E}_Q[\mathrm{e}^{-r(\tau_1-t)}\big(\int_{\tau_1}^{\tau_2} (S(u) - F(t,\tau_1,\tau_2))\,du\big)\,|\,\mathcal{F}_t]\,.$$

Then,

$$F(t,\tau_1,\tau_2) = \mathbb{E}_Q[\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} S(t)\,dt\,|\,\mathcal{F}_t]\,.$$

In the derivatives market the delivery is settled financially, meaning that instead of receiving power over the time period $[\tau_1,\tau_2]$, at the end of the delivery period the buyer of such a futures contract receives the amount of money given by

$$\int_{\tau_1}^{\tau_2} S(t)\,dt$$

and the final payoff is then

$$\int_{\tau_1}^{\tau_2} S(t)\,dt - (\tau_2-\tau_1)F(t,\tau_1,\tau_2)\,.$$

1.2.2 The Esscher transform and the market price of risk

The Esscher transform for a Lévy process L introduces a parametric class of probability measures Q_θ in $(\Omega, \mathcal{F})$ equivalent to the market probability P defined by the following density function

$$Z(t) = \exp(\theta L(t) - \Psi_L(-i\theta)t)\,, \tag{1.2.2}$$

for $t \geq 0$, $\theta \in \mathbb{R}$ and $\Psi_L(-i\theta) = \ln \mathbb{E}(-i\theta L(1))$ being the logarithm of the moment generating function. To have the process Z well-defined, we assume that L has moments of exponential order, meaning that

$$\mathbb{E}(\exp(kL)) < +\infty, \quad k > 0\,.$$

The probability measure Q_θ is parametrized by θ which models the *market price of risk*, *i.e.* the trader's view exposing themselves to risk. Under Q_θ, L remains a Lévy process with characteristic exponent given by

$$\psi_{L,\theta}(x) = \psi_L(x - \mathrm{i}\theta) - \psi_L(-\mathrm{i}\theta)\,. \tag{1.2.3}$$

For the case $L = B$, the Esscher transform coincides with the Girsanov transform. Then, $\Psi_B(x) = -x^2/2$ and (1.2.2) reduces to

$$Z(t) = \exp(\theta B(t) - \frac{1}{2}\theta^2 t)\,. \tag{1.2.4}$$

Given a probability measure Q_θ defined by means of the density function Z in (1.2.4), Girsanov's Theorem establishes that the process W_θ defined by

$$W_\theta(t) = B(t) - \theta t$$

is a standard Brownian motion under the probability measure Q_θ.

1.3 The weather derivatives market

The amount of energy produced or demanded depends among other factors on the weather conditions. For example, during warm and cold time periods there is an increase in power consumption and huge rainfalls entail an increase in hydro-power production. The necessity of weather derivatives to hedge the risk of adverse price movements arose from the liberalization of the energy markets and it resulted in the creation of the weather market in 1996. The first participants in this market were energy companies which were exposed to weather risk after the deregulation.

Weather derivatives are financial instruments that can be used by organizations or individuals as part of a risk management strategy to reduce risk associated with adverse or unexpected weather conditions that occur with a high probability. Weather derivatives are calculated from an index based on a weather measure like temperature, wind speed, rainfall or snowfall, among others. The Chicago Mercantile Exchange (CME), a company that provides a financial derivatives marketplace, organized the first trade in weather derivatives between 1996 and 1997. We will focus our attention to the temperature market and work with temperature indexes traded in the CME.

1.3.1 Modelling of daily average temperatures

Temperature derivatives prices are calculated from an index based on the instantaneous temperature. The temperature variable is modeled in continuous-time by fitting a discrete time series of daily average temperatures (DATs) of a certain location. The DAT is computed as the average between the maximum and minimum temperature of a certain day. In Alexandris and Zapranis [2], an accurate description of different methods used up to the present day to price weather derivatives is presented together with their pros and cons. We propose to use the approach based on the daily modelling of temperatures and argue that the temperature model that fits the historical DATs can be used for all the available contracts on the market for the same location. A suitable model for temperature has to reproduce the yearly cycle with certain variations that make temperature predictable but not completely determined. Also, it has to consider the slight increase of temperatures over the years as a possible consequence of global warming and urbanization. The fact that temperatures move around a seasonal mean makes mean reverting processes interesting, but the known autoregressive behavior of the temperature encourages us to use a stationary CAR(p)-model with autoregressive order p determined by the historical analysis of the DATs. Letting $T(t)$ denote the instantaneous temperature of a location at time $t \geq 0$, we assume that

$$T(t) = \Lambda(t) + Y(t)\,,$$

where Λ is a deterministic function measuring the mean temperature at time t and the random term Y is assumed to be a CAR(p)-process obtained by taking $\mathbf{b} = \mathbf{e}_1$ in (1.1.7). The stationary condition for the temperatures is ensured by the drift matrix A in (1.1.4) having eigenvalues with strictly negative real part, see Benth and Šaltytė Benth [15]. To fit the continuous-time model for temperature to the historical DATs, first we let T_i be the temperature observed at day i which is defined as

$$T_i = \Lambda_i + y_i\,,$$

for $i = 0, 1, 2, \ldots$ with $i = 0$ being the starting day in the discrete time series of observed DATs. The seasonal term in discrete time Λ_i satisfies that $\Lambda_i = \Lambda(i)$ and y is assumed to be an AR(p)-process, i.e. an homologous process to a CAR(p)-process in discrete time. An AR(p)-process is defined as

$$y_{i+p} = \sum_{j=1}^{p} b_j y_{i+p-j} + \sigma\epsilon_i$$

where $b_j \in \mathbb{R}$ are the coefficients of the AR(p)-process, ϵ_i are independent, identically distributed random variables and the volatility σ is assumed to be constant for simplicity. Observe that the random term y_{i+p} is explained by means of the p previous random terms $y_i, \ldots, y_{i+p-1}$. The deterministic Λ function in continuous-time captures the increasing trend of temperatures over the years together with the seasonal mean. An analysis of the residuals $y_i = T_i - \Lambda_i$ by means of the *autocorrelation function* (ACF) and the *partial autocorrelation function* (PACF) determines the autoregressive order for the suitable AR-model. The next step is to determine the autoregressive coefficients which result from the Yule Walker equations, see Carmona [27]. Finally, there exists a connection established between CAR and AR processes which allows to determine the α coefficients for the CAR(p)-model. The link is obtained by means of a technique involving the

Euler approximation of the dynamic of the stochastic differential equation in (1.1.3), see Benth and Šaltytė Benth [15] for a detailed explanation.

1.3.2 Temperature-based indexes

The CME organizes trade in futures contracts written on three temperature indexes known as *heating-degree day* (HDD), *cooling-degree day* (CDD) and *cumulative average temperature* (CAT). Consider the indexes defined in continuous-time over a time period $[\tau_1, \tau_2],\ \tau_1 \geq 0$.

The HDD(τ_1, τ_2) and CDD(τ_1, τ_2) indexes are defined as

$$\text{HDD}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \max(c - T(t), 0)\, dt\,, \tag{1.3.1}$$

and

$$\text{CDD}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \max(T(t) - c, 0)\, dt\,, \tag{1.3.2}$$

where c is $65°$F (or $18°$C). In particular, HDD(τ_1, τ_2) aggregates the amount of $T(t)$ below c over the specific time period and CDD(τ_1, τ_2) the amount of $T(t)$ above c. The threshold c is a baseline selected by utility companies since furnaces and air conditioners are turned on above and below this benchmark. The HDD and CDD indexes are traded during the cold and warm season, respectively. For locations in the northern hemisphere, the cold season is considered to be from October to April and the warm season from April to October, respectively. For locations in the southern hemisphere the months corresponding to the cold and warm seasons are the opposite. April and October are border months where it is possible to trade both indexes.

The CAT(τ_1, τ_2) index is defined to be

$$\text{CAT}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} T(t)\, dt\,, \tag{1.3.3}$$

and it simply aggregates the temperature over the time period.

We distinguish five big areas in the world where there is trade in temperature index-based products: United States, Canada, Europe, the Pacific Rim and Australia. The following table compiles information about the indexes traded in these areas during the cold and warm seasons together with the time frames for which the contracts can be defined.

Temperature-based products

Locations	Index (cold season)	Index (warm season)	Time frames for the contracts $[\tau_1, \tau_2]$
United States	HDD	CDD	Weekly, monthly and [1]seasonal Strip
Canada	HDD	CAT, CDD	Monthly and seasonal Strip
Europe	HDD	CAT	Monthly and seasonal Strip
Pacific Rim	CAT	CAT	Monthly and seasonal Strip
Australia	HDD	CDD	Monthly and seasonal Strip

1.3.3 Pricing temperature derivatives

Weather and energy markets are similar from a modelling point of view for being incomplete markets in the sense of not being possible to hedge with the underlying. In the weather market the underlying indexes do not have any value. The CME offers trade in futures contracts on the temperature indexes HDD, CDD and CAT.

Denote $F_{Index}(t, \tau_1, \tau_2)$ the futures price of a contract at time $t \geq 0$ written on a temperature index defined over a time period $[\tau_1, \tau_2]$, where the variable *Index* refers to HDD, CDD or CAT. Entering into a futures contract at time $0 \leq t \leq \tau_2$ is costless, meaning that

$$0 = \mathbb{E}_Q[\mathrm{e}^{-r(\tau_2 - t)}(Index(\tau_1, \tau_2) - F_{Index}(t, \tau_1, \tau_2))|\mathcal{F}_t],$$

with $Index(\tau_1, \tau_2)$ defined respectively as in (1.3.1)-(1.3.3) and $F_{Index}(t, \tau_1, \tau_2)$ being $\mathcal{F}_t$-adapted. Therefore, the futures price reduces to

$$F_{Index}(t, \tau_1, \tau_2) = \mathbb{E}_Q[Index(\tau_1, \tau_2)|\mathcal{F}_t]. \tag{1.3.4}$$

Following the same approach as Alaton [1], it is interesting to estimate the market price of risk involved in the definition of Q from the market data.

The CME also offers trade in option contracts written on these temperature futures prices. The option holder pays a fee called a premium for the right - but not the obligation - to buy or sell a futures contract within a stated period of time at a predetermined price.

The arbitrage-free price at time $t \leq \tau$ of a call option written on a temperature futures as defined in (1.3.4), with strike price K at exercise time $\tau \leq \tau_1$, is given respectively as

$$C(t, \tau, \tau_1, \tau_2, K) = \mathrm{e}^{-r(\tau - t)}\mathbb{E}_Q[\max(F_{Index}(\tau, \tau_1, \tau_2) - K, 0)|\mathcal{F}_t], \tag{1.3.5}$$

where the constant $r > 0$ is the risk-free interest rate. The buyer of a call option pays the premium in (1.3.5) to the writer of the call option and has the right but not the obligation to buy a futures contract at exercise time τ at a strike price K. The final payoff for the buyer results in $F_{Index}(\tau, \tau_1, \tau_2) - K - C(t, \tau, \tau_1, \tau_2, K)$ if $F_{Index}(\tau, \tau_1, \tau_2) > K$ and, $-C(t, \tau, \tau_1, \tau_2, K)$ otherwise. Alternatively, the writer of a call option receives the premium in (1.3.5) and the buyer acquires the right but not the obligation to buy a futures contract at a exercise time τ at a strike price K. The final payoff for the writer is $K - F_{Index}(\tau, \tau_1, \tau_2) + C(t, \tau, \tau_1, \tau_2, K)$ if $F_{Index}(\tau, \tau_1, \tau_2) > K$ and, $C(t, \tau, \tau_1, \tau_2, K)$ otherwise.

1.4 Structure of the dissertation

This thesis is structured in four Chapters. The first two are devoted to price futures contracts written on arithmetic and geometric LSS processes. We focus especially on the particular case of LSS process known as Lévy-driven CARMA process. The last two Chapters pertain to the area

[1] A seasonal strip is considered to be a minimum of two, and a maximum of seven, consecutive calendar months within the period.

of weather derivatives. We price and analyse temperature futures and option contracts where the temperature dynamics is modelled with a CAR process.

Chapter 2 co-authored with F. E. Benth reprints from Sections 2.1 to 2.4 the paper published in the Proceedings of the International Workshop on Finance 2012, held at the University of Tokyo on October 30th and 31th, 2010. We consider a stationary Lévy-driven CAR(p)-process for the underlying spot and we compute the forward price by appealing to its definition as the conditional expected spot price at delivery of the contract, with the expectation taken under some pricing measure. We observe that given an arithmetic or a geometric model for the spot the forward price can be explicitly represented as a linear combination of specific term structures scaled by the spot and its derivatives up to order $p-1$, with p being the autoregressive order. To illustrate the theory, we consider a stationary CAR(3)-process fitted to temperature in Stockholm, Sweden, and we analyse the dynamics of the forward prices. Finally, given the path of the spot we proceed to recover its derivatives by means of backward finite differences with the aim of obtaining the forward price.

Chapter 3 co-authored with F. E. Benth reprints from Sections 3.1 to 3.4 the paper that will appear in the International Journal of Theoretical and Applied Finance. We consider an arithmetic and a geometric model for the spot given as follows by a stationary LSS process

$$Y(t) = \int_{-\infty}^{t} g(t-s)\sigma(s-)\,dL(s)\,.$$

The forward price derived by means of its definition, as done in Chapter 2, contains the term

$$Y(t,x) = \int_{-\infty}^{t} g(x+t-s)\sigma(s-)\,dL(s)\,,$$

which has a similar structure to the LSS process. Our main purpose is to establish a connection between the forward price dynamics and the observed spot path. To this end, we introduce a methodology involving the Laplace transform of a stochastic process that allows us to express the Laplace transform of $Y(t,x)$ in terms of the Laplace transform of $Y(t)$. By applying the inverse Laplace transform under certain conditions, $Y(t,x)$ gets expressed by means of $Y(t)$. We consider the Laplace transform approach for CAR(p)-processes, CARMA($p,1$)-processes and gamma-LSS processes. Section 2.5 contains an appenidx which is not included in the paper with the results obtained for CARMA($p,2$)-processes.

Chapter 4 co-authored with F. E. Benth reprints from Sections 4.1 to 4.5 the paper that will appear in the Journal of Energy Markets. We price HDD and CDD temperature futures contracts and call options written on these futures. Here, we propose an arithmetic model for the temperature given by a stationary CAR(p)-process. The pricing theory presented in Chapter 2 does not apply here as the HDD and CDD indexes are nonlinear functions of the temperature. However, the temperature futures prices can be derived directly from their definitions. The complexity of the futures prices distributions makes it impossible to derive a closed formula for the call option price and approximative methods like Monte Carlo are required. To tackle the problem, we propose approximative formulas for the HDD and CDD futures prices whose distribution is known. To discuss the validity of the approximative formulas, we perform an empirical study where we fit a CAR(3)-process to observed daily average temperatures in New York. Afterwards, we compare the approximated and the theoretical CDD futures prices in several numerical examples. We also include an empirical example where we estimate the market price of risk in the theoretical and approximative models that best fits real HDD futures prices. Finally, we derive a closed formula for the call option price based on the approximative formula for the CDD futures price.

Chapter 5 from Sections 5.1 to 5.6 contains a study of the local sensitivity of the (approximated) temperature HDD and CDD futures and call options prices derived in Chapter 4 with respect to perturbations in the deseasonalized temperature or in its derivatives up to a certain order determined by the CAR process which models the dynamics of the deseasonalized temperature. We also include an empirical analysis of the local sensitivity of these financial contracts for the New York temperatures which are modelled with a CAR(3)-process in Chapter 4. A posterior analysis of the results shows that these financial contracts are more sensitive to changes in the slope of the deseasonalized temperature.

Chapter 2

Forwards prices in markets driven by continuous-time autoregressive processes

Abstract
We analyse the forward price dynamics for contracts written on a spot following a continuous-time autoregressive dynamics. Prime examples of such spots could be power or freight rates, or weather variables like temperature and wind speed. It is shown that the forward price evolves according to template term structure functions, which are scaled by the deseasonalized spot and its derivatives. These template term structure functions can be expressed as a series of exponentially decaying functions with rates given by the (real parts) of the eigenvalues of the autoregressive dynamics. Moreover, the continuous-time autoregressive spot dynamics is differentiable up to an order less than the autoregressive order, and this is precisely the derivatives needed in the representation. The template term structures may produce humps in the forward curve. We consider several empirical examples for illustration based on a model relevant for the temperature market. A particular result of our analysis is that the paths of the forward price are non-differentiable, although the underlying spot is smooth. Our results offer insight into the dynamics of forward and futures prices for contracts in the markets for weather, shipping and power.

2.1 Introduction

Continuous-time autoregressive, and more generally, continuous-time autoregressive moving average processes have in recent years become popular in various financial applications. These non-Markovian processes allow for memory effects in the dynamics, and have been succesfully used in the modelling of different weather variables (see e.g. Benth and Šaltytė Benth [15]), interest rates (see Zakamouline *et al.* [44]) and commodity prices like power, freight rates or even oil (see e.g., Garcia *et al.* [33], Benth *et al.* [13] and Paschke and Prokopczuk [39]). The purpose of this paper is to investigate the forward price dynamics for contracts written on underlying spots following a continuous-time autoregressive dynamics. Such spots may be classical assets like a commodity, but also include temperature and wind, and even power or freight rates. Typically, all these examples are spots which cannot be traded in a portfolio sense.

If an underlying spot of a forward contract cannot be traded liquidly, like for example power, the classical spot-forward relationship based on the buy-and-hold hedging strategy breaks down. We resort to the no-arbitrage theory, and demand that the forward price is defined as the conditional expected spot price at delivery of the contracts, with the expectation taken under some pricing measure. This ensures that the forward price dynamics is a martingale under the pricing measure. Using the Esscher transform to construct a class of pricing measures (being simply a parametric class of equivalent probabilities), we can compute analytically the forward price dynamics. We show that this price dynamics is explicitly represented in terms of the spot and its derivatives up to order $p-1$, with p being the autoregressive order.

We consider general Lévy-driven continuous-time autoregressive processes of order $p \geq 1$ as defined by Brockwell [25]. The continuous-time autoregressive processes are semimartingales. In fact, they have differentiable paths of order up to $p-1$, where the dynamics of the derivatives can be explicitly stated. Of major importance in our analysis is the representation of the continuous-time autoregressive process as a p-dimensional Ornstein-Uhlenbeck process driven by the same (one-dimensional) Lévy process. The stationarity of the continuous-time autoregessive process is ensured by the drift matrix in this Ornstein-Uhlenbeck process having eigenvalues with negative real part. This matrix and its eigenvalues are crucial in understanding the forward price dynamics.

We compute the forward price dynamics, and show that it can be decomposed into a deterministic and stochastic part. The deterministic part includes possible seasonality structures and the market price of risk, where the latter comes from the parametric choice of pricing measures. In this paper, our concern is on the stochastic part of the forward dynamics, which is expressable as a sum of p functions in *time to maturity* x, denoted $f_i(x)$, $i = 1, \ldots, p$, where $f_i(x)$ is scaled by the $i-1$th derivative of this continuous-time autoregressive process. These derivatives are directly linked to derivatives of (some function) of the underlying spot dynamics. We call $f_i(x)$ template forward term structure functions, as they are the basic building blocks for the forward curve. These templates can make up forward curves in contango or backwardation, including humps of different sizes. The humps will occur when the slope (first derivative) and/or curvature (second derviative) of the spot is particularly big. We also include an analysis of forwards written on the average of the underlying spot, being relevant in weather and power markets as their contracts have a delivery *period* rather than a delivery *time*.

Due to stationarity of the spot model, forward prices far from maturity of the contract will be essentially non-stochastic (constant). However, close to maturity they will start to vary stochastically according to the size of the spot and its derivatives. Noteworthy is that the forward dynamics is much more erratic than the underlying spot, due to the fact that it depends on higher-order derivatives of the spot, which eventually have paths similar to the driving Lévy process. The spot, on the other hand, is smooth in the sense of being differentiable up to some order.

We illustrate our results by numerical examples. These examples are based on the empirical estimation of a continuous-time autoregressive model of order 3 on daily average temperature data observed over more than 40 years in Stockholm, Sweden. Apart from presenting different shapes of the forward curve and its dynamics, we also present how to apply our results to recover the derivatives from an observed path of spot prices (or, temperatures). Natural in our context is to resort to finite differencing of the continuous-time autoregressive process observed discretely, which turns out to give a reasonable approximation.

Our results are presented as follows: in the next Section we define continuous-time autoregressive processes and present some applications of these in different financial contexts. Sections 3 and 4 contain our main results, with the derivation of the forward prices and analysis of the forward curve as a function of time to delivery. Finally, we conclude and make an outlook.

2.2 Continuous-time autoregressive processes

In this Section we introduce the class of continuous-time autoregressive processes proposed by Doob [29] and later intensively studied by Brockwell [25].

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ satisfying the usual conditions (see e.g. Karatzas and Shreve [37]). We assume L is a real-valued two-sided square-integrable Lévy process, and choose to work with the RCLL version (right-continuous, with left-limits). Denote by $\mathbf{e}_k$ the kth canonical basis vector in $\mathbb{R}^p$ for $p \in \mathbb{N}$ and $k = 1, \ldots, p$. We define the *continuous-time autoregressive process* of order p (from now on, a CAR(p)-process) to be

$$Y(t) = \int_{-\infty}^{t} \mathbf{e}_1^{\mathrm{T}} \mathrm{e}^{A(t-s)} \mathbf{e}_p \, dL(s) \,, \tag{2.2.1}$$

for $t \geq 0$, whenever the stochastic integral makes sense. Here, $\mathbf{x}^{\mathrm{T}}$ denotes the transpose of a vector or matrix $\mathbf{x}$, and A is the $p \times p$ matrix

$$A = \begin{pmatrix} \mathbf{0}_{p-1\times 1} & I_{p-1} \\ -\alpha_p .. & .. - \alpha_1 \end{pmatrix} \tag{2.2.2}$$

for positive constants α_i, $i = 1, \ldots, p$. The expression $\exp(At)$ is interpreted as the matrix exponential for any time $t \geq 0$. In the next Lemma we derive the characteristic function of the CAR(p)-process $Y(t)$ and show that it is stationary:

Lemma 2.2.1. *Let A have eigenvalues with negative real part. For any $x \in \mathbb{R}$ and $t \geq 0$ it holds that*

$$\ln \mathbb{E}\left[\mathrm{e}^{\mathrm{i}xY(t)}\right] = \int_0^\infty \psi_L\left(x\mathbf{e}_1^{\mathrm{T}}\mathrm{e}^{As}\mathbf{e}_p\right) \, ds \,,$$

where $\psi_L(x) = \ln \mathbb{E}[\exp(\mathrm{i}xL(1))]$ is the characteristic exponent of L.

Proof. In the proof, we consider a process

$$\widetilde{Y}(t) = \int_0^t \mathbf{e}_1^{\mathrm{T}} \mathrm{e}^{A(t-s)} \mathbf{e}_p \, dL(s) \,,$$

and we show that this has a stationary limit as time t tends to infinity. Let $\{t_i\}_{i=1}^n$ be a sequence of nested partitions of the interval $[0, t]$. From the independent increment property of the Lévy process we find for $x \in \mathbb{R}$ and the defintion of its characteristic function,

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}x\widetilde{Y}(t)}\right] = \lim_{n\to\infty} \mathbb{E}\left[\prod_{j=1}^{n} \mathrm{e}^{\mathrm{i}[xg(t-t_j)]\Delta L(t_j)}\right]$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\sum_{j=1}^{n}\psi_L(xg(t-t_j))\Delta t_j \\
&= \exp(\int_0^t \psi_L(xg(t-s))\,ds) \\
&= \exp(\int_0^t \psi_L(xg(s))\,ds)\,.
\end{aligned}
$$

where $\Delta L(t_i) = L(t_{i+1}) - L(t_i)$, $\Delta t_i = t_{i+1} - t_i$, and where we have used the short-hand notation $g(u) = \mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{Au}\mathbf{e}_p$.

We can express $g(s) = \mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{As}\mathbf{e}_p$ as a sum of exponentials scaled by trigonometric functions. As the eigenvalues of A have negative real part, the exponentials decay at rates given by the negative real part of the eigenvalues. Hence, we can majorize $g(s)$ as

$$
|g(s)| \leq C\mathrm{e}^{\mathrm{Re}(\lambda_1)s}\,,
$$

where λ_1 is the eigenvalue with smallest absolute value of the real part. We apply Theorem 17.5 of Sato [42] to conclude that

$$
\lim_{t\to\infty}\int_0^t \psi_L(xg(s))\,ds = \int_0^\infty \psi_L(xg(s))\,ds\,.
$$

This concludes the proof. □

From now on we assume that A has eigenvalues with negative real part, which by the above Lemma makes $Y(t)$ in (2.2.1) well-defined and stationary.

A particular case of interest is when $L(t) = \sigma B(t)$, for $\sigma > 0$ a constant and B a Brownian motion. Then the characteristic function of L is $\psi_L(x) = -\sigma^2x^2/2$, and the distribution of Y has characteristic exponent

$$
\int_0^\infty \psi_L\left(x\mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{As}\mathbf{e}_p\right)\,ds = -\frac{1}{2}\sigma^2x^2\int_0^\infty \mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{As}\mathbf{e}_p\mathbf{e}_p^{\mathsf{T}}\mathrm{e}^{A^{\mathsf{T}}s}\mathbf{e}_1\,ds
$$

Hence, it becomes normally distributed, with zero mean and

$$
\sigma^2\int_0^\infty \mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{As}\mathbf{e}_p\mathbf{e}_p^{\mathsf{T}}\mathrm{e}^{A^{\mathsf{T}}s}\mathbf{e}_1\,ds
$$

being the variance.

CAR(p)-processes can be thought of as a subclass of the so-called *continuous-time autoregressive moving average* processes, denoted CARMA(p, q). Here, $q \in \mathbb{N}$ and $q < p$ is the order of the moving average part, and it is defined as

$$
Y(t) = \int_{-\infty}^t \mathbf{b}^{\mathsf{T}}\mathrm{e}^{A(t-s)}\mathbf{e}_p\,dL(s)\,,
$$

for a vector $\mathbf{b} \in \mathbb{R}^p$ with the property

$$\mathbf{b} = \begin{pmatrix} b_0 & b_1 & \dots & b_{q-1} & 1 & 0 & \dots & 0 \end{pmatrix}. \tag{2.2.3}$$

We obviously recover the CAR(p)-process by letting $\mathbf{b} = \mathbf{e}_1$. CAR($p$), or CARMA($p, q$), processes are members of the much more general class of Lévy semistationary (LSS) processes

$$X(t) = \int_{-\infty}^{t} g(t-s)\sigma(s)\, dL(s)\,, \tag{2.2.4}$$

where $g : \mathbb{R}_+ \mapsto \mathbb{R}$ is a deterministic function and $\sigma(t)$ a predictable stochastic process such that

$$\mathbb{E}\left[\int_{-\infty}^{t} g^2(t-s)\sigma^2(s)\, ds\right] < \infty\,, \tag{2.2.5}$$

for all $t \geq 0$. This makes the stochastic integral in the definition of the LSS-process well-defined in the sense of stochastic integration with respect to semimartingales, as defined in Protter [40]. The process $\sigma(t)$ is usually interpreted as a stochastic volatility or intermittency, see Barndorff-Nielsen *et al.* [5]. In the situation of a CAR(p)-process Y, we see that

$$g(t) = \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{At} \mathbf{e}_p\,,$$

for $t \geq 0$ and $\sigma(t) = 1$. It is of course not a problem to include a stochastic volatility in the definition (2.2.1) of the CAR(p)-process, however, we shall not do so here for the sake of simplicity. It is simple to see that for this particular g, the integrability condition (2.2.5) becomes

$$\int_0^{\infty} g^2(s)\, ds < \infty$$

which holds by estimating g using matrix norms.

We note that for $p = 1$, the matrix A collapses into $A = -\alpha_1$, and we find $g(t) = \exp(-\alpha_1 t)$. For this particular case we recognize $Y(t)$ as an Ornstein-Uhlenbeck process. In fact, general CAR(p)-processes can be defined via multivariate Ornstein-Uhlenbeck processes, as we show now: Introduce the $\mathbb{R}^p$-valued stochastic process $\mathbf{X}$ as the solution of the linear stochastic differential equation

$$d\mathbf{X}(t) = A\mathbf{X}(t)\, dt + \mathbf{e}_p\, dL(t)\,. \tag{2.2.6}$$

The stationary solution of this multivariate Ornstein-Uhlenbeck process is (using Itô's Formula for jump processes, see Ikeda and Watanabe [36]),

$$\mathbf{X}(t) = \int_{-\infty}^{t} \mathrm{e}^{A(t-s)} \mathbf{e}_p\, dL(s)\,. \tag{2.2.7}$$

We see that

$$Y(t) = \mathbf{e}_1^{\mathsf{T}} \mathbf{X}(t)\,. \tag{2.2.8}$$

Hence, the CAR(p)-process is the first coordinate process of the multivariate Ornstein-Uhlenbeck process $\mathbf{X}(t)$, which is a real-valued Ornstein-Uhlenbeck process for $p = 1$ as already observed.

The paths $t \mapsto Y(t)$ of the CAR(p)-process will be smooth. This is a consequence of the next Lemma:

Lemma 2.2.2. *Let $p > 1$. It holds that*

$$Y(t) = \int_0^t \int_{-\infty}^s \mathbf{e}_2^{\mathsf{T}} \mathrm{e}^{A(s-u)} \mathbf{e}_p \, dL(u) \, ds \,,$$

for $t \geq 0$.

Proof. From Basse and Pedersen [9] (see also Proposition 3.2 of Benth and Eyjolfsson [11] for the similar statement) we find that an LSS process $X(t)$ as in (2.2.4) with absolutely continuous kernel function g with a derivative satisfying $\int_0^\infty g'(s)^2 \, ds < \infty$ and $|g(0)| < \infty$ has the representation

$$dX(t) = \int_{-\infty}^t g'(t-s)\sigma(s) \, dL(s) \, dt + g(0)\sigma(t) \, dL(t) \,,$$

for $t \geq 0$. For the case of a CAR(p)-process, we observe that $g(t) = \mathbf{e}_1^{\mathsf{T}} \exp(At)\mathbf{e}_p$ is continuously differentiable with $g(0) = \mathbf{e}_1^{\mathsf{T}}\mathbf{e}_p = 0$ and

$$g'(t) = \mathbf{e}_1^{\mathsf{T}} A \mathrm{e}^{At} \mathbf{e}_p \,.$$

By the definition of A in (2.2.2), we see that $\mathbf{e}_1^{\mathsf{T}} A = \mathbf{e}_2^{\mathsf{T}}$. But as the eigenvalues of the matrix A have negative real part, we find that $\int_0^\infty g'(s)^2 \, ds < \infty$ as $|g'(s)|$ will be bounded by an exponentially decaying function. The result follows. □

This shows in particular that the process $Y(t)$ is of finite variation. Moreover, we see that the derivative of $Y(t)$ exists, and is equal to

$$Y'(t) = \int_{-\infty}^t \mathbf{e}_2^{\mathsf{T}} \mathrm{e}^{A(t-s)} \mathbf{e}_p \, dL(s) \,.$$

In fact, we can iterate the proof of Lemma 2.2.2 using the definition of A in (2.2.2) to show that the following smoothness result holds for $Y(t)$.

Proposition 2.2.3. *Let Y be a CAR(p)-process for $p > 1$. Then the paths $t \mapsto Y(t)$ for $t \geq 0$ are $p-1$ times differentiable, with ith derivative, $Y^{(i)}(t)$, given by*

$$Y^{(i)}(t) = \int_{-\infty}^t \mathbf{e}_{i+1}^{\mathsf{T}} \mathrm{e}^{A(t-s)} \mathbf{e}_p \, dL(s) \,,$$

for $i = 1, \ldots, p-1$.

Note that $Y(t)$ is not a Markovian process. However, due to the representation via $\mathbf{X}$ above, it can be viewed as a p-dimensional Markovian process. We remark that the smoothness property and the representation of the derivatives of $Y(t)$ could also be proven by resorting to the stochastic differential equation for $\mathbf{X}(t)$.

We next turn our attention to some of the applications of CAR(p)-processes. First, let us consider a model for the time dynamics of temperature in a specific location. In Figure 2.1 we see the daily average temperatures in Vilnius, Lithuania, a city located in north-east Europe. The figure shows the average value of the maximum and minimum temperature recorded on each day, ranging over a five year period. The temperatures are measured in degrees Celsius (°C).

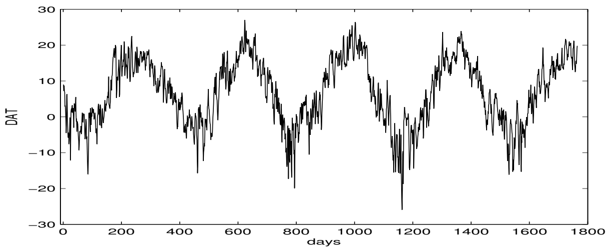

Figure 2.1: Five years of daily average temperatures measured in Vilnius.

In Benth and Šaltytė Benth [15] it is shown that an appropriate model for the time dynamics of these temperatures is given by

$$T(t) = \Lambda(t) + Y(t)\,, \tag{2.2.9}$$

where $T(t)$ is the temperature at time $t \geq 0$, $\Lambda(t)$ is some deterministic function modelling the seasonally varying mean level, and $Y(t)$ is a CAR(p)-process as above. For Vilnius, a seasonal mean function can be chosen as

$$\Lambda(t) = a_0 + a_1 t + a_2 \sin(2\pi(t - a_3)/365)\,,$$

i.e., a yearly cycle of amplitude a_2, shifted by a_3, and a linear trend a_0+a_1t. The level is a_0, and a_1 indicates an increasing average temperature over time that may be attributed to urbanization and climate change. Furthermore, a statistical analysis of the deseasonalized temperatures $T(t)-\Lambda(t)$ reveals an autoregressive structure of order $p = 3$, being stationary. Hence, a CAR(3)-process $Y(t)$ is appropriate. Moreover, the analysis in Benth and Šaltytė Benth [15] reveals that $L(t) = \sigma B(t)$, B being a Brownian motion. In fact, a time-dependent volatility $\sigma(t)$ is also proposed to capture the seasonal variance observed in the data.

Another weather variable that can be conveniently modelled by CAR(p)-processes is wind speed. In Figure 2.2 we have plotted the wind speed in meters per second (m/s) measured in Vilnius at the same location as the temperature measurements discussed above,

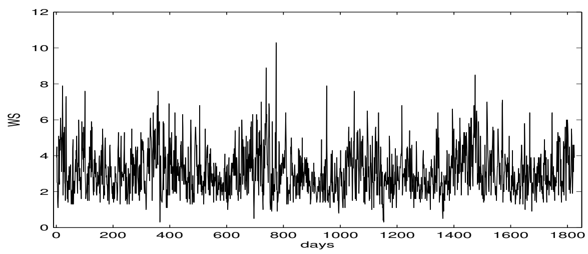

Figure 2.2: Five years of daily average wind speed measured in Vilnius.

Benth and Šaltytė Benth [15] propose an exponential model for the wind speed dynamics $W(t)$ given by

$$W(t) = \exp\left(\Lambda(t) + Y(t)\right) . \tag{2.2.10}$$

Hence, the logarithm of the wind speed follows a seasonal mean function $\Lambda(t)$ and a CAR(p)-process. In the case of wind speeds, the seasonal mean function $\Lambda(t)$ is more complex than the simple sine-function with trend chosen for temperatures. However, it is also for wind speeds in Vilnius appropriate to choose Y to be a CAR(3)-process driven by a Brownian motion. Also in this case one observes a seasonally varying variance $\sigma(t)$.

It is to be noted that many studies have confirmed the CAR(3)-structure of temperature dynamics, see e.g. Härdle and Lopez Cabrera [35] for analysis of German temperature data. Wind speeds at different locations vary between CAR(3) and CAR(4). For example, a study of wind speeds in New York by Benth and Šaltytė Benth [14] shows that CAR(4) is the best choice. In the empirical studies presented above, the stochastic part becomes stationary in the limit as the eigenvalues of the A matrix in the CAR(3)-process have negative real part. From a practical point of view, it is rather natural that wind and temperature are stationary phenomena around its mean level. These models for wind and temperature have been applied to weather derivatives pricing and hedging, in particular futures written on temperature and wind speed indexes, see Benth and Šaltytė Benth [15] and Benth *et al.* [16]. We will return to this, in the next Section.

Freight rates can be modelled using CAR(p)-processes. Benth *et al.* [13] perform an empirical study of the daily observed Baltic Capesize and Baltic Panamax Indexes, which are indexes created from assessments of 10, respectively 4, time charter rates of Capesize, respectively Panamax, vessels. Indeed, the dynamics of the logarithmic freight rates can in both cases be modelled by CAR(3)-processes, where the Lévy process is normal inverse Gaussian distributed. We have plotted the time evolution of the daily Baltic Capesize Index in Figure 2.3 over the period from March 1999 to November 2011.

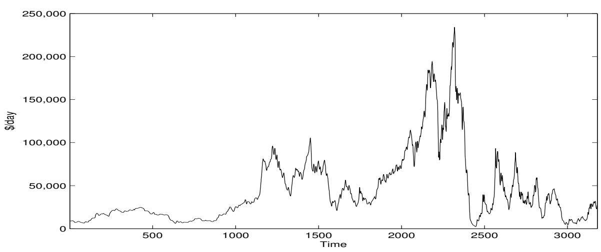

Figure 2.3: The Baltic Capesize Index over the period from March 1999 to November 2011

We note that the Baltic Exchange in London, UK, organizes a trade in futures contracts written on the average of these indexes over given time periods.

Zakamouline *et al.* [44] have shown that a Brownian-driven CARMA(2,1) model explains well the term structure of volatility for forward rates from UK treasury bonds. Another application area of CAR(p)-processes is commodity markets. Garcia *et al.* [33] demonstrated that deseasonalized electricity spot prices observed at the German power exchange EEX follow a

CARMA(2,1) process driven by an α-stable Lévy process. A more recent study by Benth *et al.* [12] takes electricity futures prices into account as well, and extends the spot model into a two-factor CAR(1) and CARMA(2,1) dynamics. At the EEX, and other power exchanges, futures and forward contracts are traded settled on the average electricity spot price over a specific time period. In [39] Paschke and Prokopzcuk have proposed a Brownian-driven CARMA(2,1)-model for the time dynamics of crude oil, and estimated this to crude oil futures with the aim of studying various term structures. We will analyse futures pricing based on general CARMA-processes in another context, but mention these areas of applications as they are close to CAR-processes.

2.3 Forward pricing

In this Section we consider the problem of deriving a forward price and how to represent this in terms of the underlying spot with a price dynamics given by $S(t)$. To this end, let $f(t,T)$ denote the forward price at time $t \geq 0$ of a contract delivering the underlying spot at time T. In classical financial theory, one resorts to the so-called buy-and-hold hedging strategy in the underlying asset to reach the spot-forward relationship

$$f(t,T) = S(t)\mathrm{e}^{r(T-t)}\,,$$

where $r > 0$ is the constant risk-free interest rate.

Considering forward contracts in markets for electricity, weather or freight, one cannot store the spot. There is no way one can buy and store temperature, nor wind, and by the very nature of power this is not storable either. The spot rates of freight are indexes, and hence cannot be used for hedging in a portfolio either, like the spot interest rate. Thus, in these markets we cannot resort to the hedging argument replicating a long forward position by holding a spot. This implies that the spot-forward relationship breaks down in the typical markets we have in mind.

From the arbitrage theory in mathematical finance (see e.g. Bingham and Kiesel [20]), we know that all tradeable assets in a market must have a martingale price dynamics under a risk neutral measure Q. In particular, forward prices must be martingales under some risk neutral probability Q. In a market where the spot cannot be traded (or stored), the probability Q does not need to be an equivalent martingale measure in the sense that the spot dynamics is a Q-martingale (after discounting). The only requirement is that Q is equivalent to P, and that we ensure the martingale property of the forward price. We refer to Q as a *pricing measure* rather than an equivalent martingale measure. As $f(T,T) = S(T)$, we get by such a martingale requirement that

$$f(t,T) = \mathbb{E}_Q\left[S(T)\,|\,\mathcal{F}_t\right]\,. \tag{2.3.1}$$

To have $f(t,T)$ well-defined, we suppose that $S(t) \in L^1(Q)$ for all $t \in [0,T]$. The pricing measure Q plays here much of the same role as risk loading does in insurance, as we may view the forward as an insurance contract locking in the spot at delivery. We refer to Benth *et al.* [16] and Benth and Šaltytė Benth [15] for more on the relationship between spot and forwards in markets where the spot cannot be stored.

In the remaining part of this Section we assume that the spot price dynamics is given by two possible models, arithmetic or geometric dynamics driven by a CAR(p)-process: In the arithmetic case, we assume

$$S(t) = \Lambda(t) + Y(t)\,, \tag{2.3.2}$$

with $\Lambda : \mathbb{R}_+ \mapsto \mathbb{R}$ being a deterministic measurable function assumed to be bounded on compacts. The process $Y(t)$ is the CAR(p) dynamics defined in (2.2.1). The geometric model is assumed to be

$$S(t) = \Lambda(t) \exp\left(Y(t)\right)\,. \tag{2.3.3}$$

We consider these two models in order to cover the models for wind, temperature and freight rates discussed above. We also have in mind applications to energy and commodity markets, as well as fixed-income theory, where also both arithmetic and geometric models are relevant. Note that we also include a deterministic function Λ to capture a possible seasonality. Although this will only play a technical role in the computations to follow, we include it for completeness.

It is worth mentioning that a forward contract on wind or temperature, or on power, delivers over a fixed period of time and not at a fixed point in time T. For example, a forward contract on power will typically deliver power continuously to the owner of the contract over an agreed period. On the EEX power exchange in Germany, such delivery periods can be specific weeks, months, quarters, and even years. Hence, buying a forward on power entitles you to the delivery of power over a period $[T_1, T_2]$, meaning that you receive

$$\int_{T_1}^{T_2} S(u)\,du\,,$$

with S being the power spot price at time u. In the market, the delivery is settled financially, meaning that the owner receives the money-equivalent of the above. The forward price of power is denoted per MWh, so by definition it becomes

$$F(t, T_1, T_2) = \mathbb{E}_Q\left[\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} S(u)\,du \,|\, \mathcal{F}_t\right]\,. \tag{2.3.4}$$

At the Chicago Mercantile Exchange (CME) in the US there is a market for forwards[1] written on temperature indexes measured in various cities world-wide. For example, forwards settled on the average temperature measured in Tokyo over given months can be traded. This will yield a temperature forward price as in (2.3.4), with $S(u)$ interpreted as the temperature in Tokyo at time u in the measurement period $[T_1, T_2]$. There are three other temperature indexes used for settlement of forwards in this weather market, called HDD, CDD and CAT. We refer to Benth and Šaltytė Benth [15] for a definition and analysis of these. Finally, in the freight market the forward is also settled on the average spot freight rates over a period in time. This yields then as well a forward price given by (2.3.4) using $S(u)$ as the spot freight rate.

[1]We will not distinguish between forwards and futures in this paper. The contracts on temperatures at CME are futures-style, whereas the power contracts mentioned earlier may be of forward and futures style.

Note that by the Fubini-Tonelli theorem (see e.g. Folland [32]), we find

$$F(t,T_1,T_2) = \frac{1}{T_2-T_1}\int_{T_1}^{T_2} \mathbb{E}_Q\left[S(u)\,|\,\mathcal{F}_t\right]\,du = \frac{1}{T_2-T_1}\int_{T_1}^{T_2} f(t,u)\,du\,. \tag{2.3.5}$$

Therefore, to price forwards in markets for weather, power and freight, it is sufficient to analyse forward prices with fixed time of delivery $f(t,T)$, as the forwards settling over a time period is simply an average of these.

We must fix a pricing measure Q, and we do so by introducing a parametric family of probabilities Q_θ for $\theta \in \mathbb{R}$ given by the Esscher transform. Introduce the process Z

$$Z(t) = \exp\left(\theta L(t) - \psi_L(-\mathrm{i}\theta)t\right)\,, \tag{2.3.6}$$

for $t \geq 0$. Observe that $\psi_L(-\mathrm{i}\theta)$ is the logarithm of the moment generating function of L. To have this process Z well-defined, we assume that L has moments of exponential order, meaning that

$$\mathbb{E}(\exp(kL)) < +\infty, \quad k > 0.$$

We observe that Z is a martingale with $Z(0) = 1$. Define a probability measure such that the Radon-Nikodym derivative has density process $Z(t)$, that is,

$$\left.\frac{dQ_\theta}{dP}\right|_{\mathcal{F}_t} = Z(t)\,. \tag{2.3.7}$$

Remark that we perform a measure change only for the part of the Lévy process living on the positive time $t \geq 0$. As L is two-sided, it is also defined for $t < 0$. However, this part is independent of L defined on $t \geq 0$. We do not make any change of measure for negative times. We can accommodate this by extending the definition of Z to be equal to one for all times $t < 0$.

From Benth and Šaltytė Benth [15], Proposition 8.3, one finds that the Lévy property of L is preserved under this change of measure, and the characteristic exponent of L with respect to Q_θ becomes

$$\psi_{L,\theta}(x) = \psi_L(x - \mathrm{i}\theta) - \psi_L(-\mathrm{i}\theta)\,. \tag{2.3.8}$$

It is easily seen that if the Lévy measure of L is $\ell(dz)$, then the Lévy measure under Q becomes $\exp(\theta z)\ell(dz)$, that is, the measure is exponentially tilted. One refers to the constant θ as the *market price of risk*. Remark that for the case $L = B$, we find that the Esscher transform coincides with the Girsanov transform. To see this, take into account that in this case $\psi_L(x) = -x^2/2$ and therefore

$$Z(t) = \exp\left(\theta B(t) - \frac{1}{2}\theta^2 t\right)\,,$$

which we recall from Girsanov's Theorem (see e.g. Karatzas and Shreve [37]) to be the density of a measure Q_θ such that the process W_θ defined by

$$dW_\theta(t) = -\theta\,dt + dB(t)\,,$$

is a Q_θ-Brownian motion. From (2.3.8) we see that the characteristic exponent of B with respect to Q_θ becomes $\theta x\mathrm{i} - x^2/2$, in line with the above.

We find the forward price for the arithmetic and geometric spot price models: these results can be found in Benth *et al.* [16] and Benth and Šaltytė Benth [15], but we recall them here with a slightly modified proof for the convenience of the reader.

Proposition 2.3.1. *Let $S(t)$ have the dynamics given by* (2.3.2)*. Then for a $\theta \in \mathbb{R}$ we have*

$$f(t,T) = \Lambda(T) + \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{A(T-t)} \mathbf{X}(t) - \mathrm{i}\psi_L'(-\mathrm{i}\theta) \mathbf{e}_1^{\mathsf{T}} A^{-1} \left(\mathrm{e}^{A(T-t)} - I \right) \mathbf{e}_p \,,$$

for $0 \le t \le T$ and with $\mathbf{X}(t)$ defined as in (2.2.7)*.*

Proof. Fix $\theta \in \mathbb{R}$. Then, by definition of the forward price and the spot dynamics we find

$$f(t,T) = \mathbb{E}_{Q_\theta}[S(T) \,|\, \mathcal{F}_t] = \Lambda(T) + \mathbb{E}_{Q_\theta}[\int_{-\infty}^{T} \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{A(T-s)} \mathbf{e}_p \, dL(s) \,|\, \mathcal{F}_t] \,,$$

But from splitting the integral into two pieces, one ranging from $-\infty$ to t, and the other from t to T, we find using $\mathcal{F}_t$-adaptedness on the former and independence of increments of Lévy processes of the latter, that

$$\mathbb{E}_{Q_\theta}\left[\int_{-\infty}^{T} \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{A(T-s)} \mathbf{e}_p \, dL(s) \,|\, \mathcal{F}_t \right] = \int_{-\infty}^{t} \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{A(T-s)} \mathbf{e}_p \, dL(s) + \mathbb{E}_{Q_\theta}\left[\int_{t}^{T} \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{A(T-s)} \mathbf{e}_p \, dL(s) \right] \tag{2.3.9}$$

For the expectation, we have that

$$\mathbb{E}_{Q_\theta}\left[\int_{t}^{T} \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{A(T-s)} \mathbf{e}_p \, dL(s) \right] = -\mathrm{i}\frac{d}{dx} \mathbb{E}_{Q_\theta}\left[\exp\left(\mathrm{i}x \int_{t}^{T} \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{A(T-s)} \mathbf{e}_p \, dL(s) \right) \right] \Big|_{x=0} \,.$$

Adapting the first part of the proof of Lemma 2.2.1 (working under Q_θ rather than the probability measure P), we find

$$\mathbb{E}_{Q_\theta}\left[\int_{t}^{T} \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{A(T-s)} \mathbf{e}_p \, dL(s) \right] = -\mathrm{i}\psi_{L,\theta}'(0) \int_{0}^{T-t} \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{As} \mathbf{e}_p \, ds$$

since $\psi_{L,\theta}(0) = 0$. But since $\psi_{L,\theta}'(0) = \psi_L'(-\mathrm{i}\theta)$, and integrating the matrix exponential, we get

$$\mathbb{E}_{Q_\theta}\left[\int_{t}^{T} \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{A(T-s)} \mathbf{e}_p \, dL(s) \right] = -\mathrm{i}\psi_L'(-\mathrm{i}\theta) \mathbf{e}_1^{\mathsf{T}} A^{-1} \left(\mathrm{e}^{A(T-t)} - I \right) \mathbf{e}_p \,.$$

This shows the last term of the forward price $f(t,T)$. Let us now consider the first term on the right-hand side of (2.3.9). From the representation (2.2.8) we have that $Y(t) = \mathbf{e}_1^{\mathsf{T}} \mathbf{X}(t)$ with $\mathbf{X}(t)$ given in (2.2.7). From the stochastic differential equation (2.2.6) and Itô's Formula for jump processes (see Ikeda and Watanabe [36]), we find

$$\mathbf{X}(T) = \mathrm{e}^{A(T-t)} \mathbf{X}(t) + \int_{t}^{T} \mathrm{e}^{A(T-s)} \mathbf{e}_p \, dL(s) \,.$$

But on the other hand we know that

$$\mathbf{X}(T) = \int_{-\infty}^{T} \mathrm{e}^{A(T-s)}\mathbf{e}_p \, dL(s) \,.$$

Hence, it follows that

$$\int_{-\infty}^{t} \mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{A(T-s)}\mathbf{e}_p \, dL(s) = \mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{A(T-t)}\mathbf{X}(t) \,.$$

This proves the Proposition. □

We observe that the second term in the forward price dynamics is closely related to the spot price $S(t)$ at time t. The third term is appearing as a result of the introduction of a pricing measure. The case $\theta = 0$ would lead to a similar term with $\psi_L'(0)$ rather than $\psi_L'(-\mathrm{i}\theta)$. Indeed, the θ measures the *risk premium* in the market since

$$f(t,T) - \mathbb{E}[S(T)\,|\,\mathcal{F}_t] = -\mathrm{i}(\psi_L'(-\mathrm{i}\theta) - \psi_L(0))\mathbf{e}_1^{\mathsf{T}}A^{-1}\left(\mathrm{e}^{A(T-t)} - I\right)\mathbf{e}_p \,.$$

By observing forward prices in the market in question, one can calibrate θ.

Let us state the forward price in case of a geometric spot price model:

Proposition 2.3.2. *Let $S(t)$ have the dynamics given by* (2.3.3)*. Then for a $\theta \in \mathbb{R}$ we have*

$$\ln f(t,T) = \Lambda(T) + \mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{A(T-t)}\mathbf{X}(t) + \int_0^{T-t} \psi_{L,\theta}\left(-\mathrm{i}\mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{As}\mathbf{e}_p\right)\, ds \,,$$

for $0 \le t \le T$ and with $\mathbf{X}(t)$ defined in (2.2.7)*.*

Proof. Arguing as in the proof of Proposition 2.3.1 by independence and measurability of the Lévy process, we find

$$f(t,T) = \exp\left(\Lambda(T) + \int_{-\infty}^{t} \mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{A(T-s)}\mathbf{e}_p \, dL(s)\right)\mathbb{E}_{Q_\theta}\left[\exp\left(\int_t^T \mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{A(T-s)}\mathbf{e}_p \, dL(s)\right)\right].$$

Adapting the proof of Lemma 2.2.1 we find

$$\mathbb{E}_{Q_\theta}\left[\exp\left(\int_t^T \mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{A(T-s)}\mathbf{e}_p \, dL(s)\right)\right] = \exp\left(\int_0^{T-t} \psi_{L,\theta}\left(-\mathrm{i}\mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{As}\mathbf{e}_p\right)\, ds\right).$$

The remaining part of the proof goes as for the proof of Proposition 2.3.1. □

If we are interested in the forward price for a contract which delivers the spot over a period, we recall the relationship (2.3.5) and see that for the arithmetic case one may actually obtain an analytic expression for $F(t, T_1, T_2)$. We calculate

$$(T_2 - T_1) \times F(t, T_1, T_2) = \int_{T_1}^{T_2} \Lambda(u)\, du + \mathbf{e}_1^{\mathsf{T}} \int_{T_1}^{T_2} \mathrm{e}^{A(u-t)}\, du\mathbf{X}(t) \qquad (2.3.10)$$

$$
\begin{aligned}
&\quad - \mathrm{i}\psi_L'(-\mathrm{i}\theta)\mathbf{e}_1^{\mathsf{T}}A^{-1}\int_{T_1}^{T_2}(\mathrm{e}^{A(u-t)}-I)\,du\mathbf{e}_p \\
&= \int_{T_1}^{T_2}\Lambda(u)\,du + \mathbf{e}_1^{\mathsf{T}}A^{-1}\left(\mathrm{e}^{A(T_2-t)}-\mathrm{e}^{A(T_1-t)}\right)\mathbf{X}(t) \\
&\quad - \mathrm{i}\psi_L'(-\mathrm{i}\theta)\mathbf{e}_1^{\mathsf{T}}A^{-1}\left(A^{-1}\mathrm{e}^{A(T_2-t)}-A^{-1}\mathrm{e}^{A(T_1-t)}-(T_2-T_1)I\right)\mathbf{e}_p\,.
\end{aligned}
$$

A similar analytic expression in the geometric case seems hard to obtain, if possible.

2.4 The spot-forward relationship

The main objective in this Section is to represent the forward price explicitly in terms of the spot price at time t. As we shall see, this will involve the derivatives of the spot price dynamics, and the forward price will become a linear combination of specific term structures scaled by derivatives of the spot.

To this end, denote by $X_i(t) = \mathbf{e}_i^{\mathsf{T}}\mathbf{X}(t)$, for $i = 1,\dots,p$. Obviously, $X_i(t)$ will be the ith coordinate of $\mathbf{X}(t)$, and in particular we have that $X_1(t) = \mathbf{e}_1^{\mathsf{T}}\mathbf{X}(t) = Y(t)$. Consider the term $\mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{A(T-t)}\mathbf{X}(t)$ in the forward price dynamics in Propositions 2.3.1 and 2.3.2. We find

$$
\mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{A(T-t)}\mathbf{X}(t) = \sum_{i=1}^{p} f_i(T-t)X_i(t)\,,
$$

where

$$
f_i(x) = \mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_i\,, \tag{2.4.1}
$$

for $i = 1,\dots,p$. Observe that $f_i(0) = \mathbf{e}_1^{\mathsf{T}}\mathbf{e}_i$ which is zero for $i > 1$ and one otherwise. Now, recall from Proposition 2.2.3 that the kth derivative of $Y(t)$ exits for $k = 1,\dots,p-1$, and that

$$
Y^{(k)}(t) = \int_{-\infty}^{t}\mathbf{e}_{k+1}^{\mathsf{T}}\mathrm{e}^{A(t-s)}\mathbf{e}_p\,dL(s) = \mathbf{e}_{k+1}^{\mathsf{T}}\mathbf{X}(t) = X_{k+1}(t)\,.
$$

Hence, we have shown the following Proposition:

Proposition 2.4.1. *Let $f_i(x)$ be defined as in* (2.4.1) *for $i = 1,\dots,p$. If S is an arithmetic spot price as in* (2.3.2)*, then*

$$
f(t,T) = \Lambda(T) + \sum_{i=1}^{p} f_i(T-t)Y^{(i-1)}(t) - \mathrm{i}\psi_L'(-\mathrm{i}\theta)\mathbf{e}_1^{\mathsf{T}}A^{-1}\left(\mathrm{e}^{A(T-t)}-I\right)\mathbf{e}_p\,,
$$

for $0 \le t \le T$. If the spot price is a geometric model as in (2.3.3)*, then*

$$
\ln f(t,T) = \Lambda(T) + \sum_{i=1}^{p} f_i(T-t)Y^{(i-1)}(t) + \int_0^{T-t}\psi_{L,\theta}\left(-\mathrm{i}\mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{As}\mathbf{e}_p\right)\,ds\,,
$$

for $0 \le t \le T$.

Consider the arithmetic spot model case: We see that the forward price will depend explicitly on the deseasonalized spot price and its derivatives up to order $p-1$. If we have that the seasonal function Λ is sufficiently differentiable, we can rewrite this into a dependency on the spot price and its derivatives up to order $p-1$. This result is very different from the classical spot-forward relationship, where the forward price is simply proportional to the current spot price only. In our setting we find that also the rate of growth, the curvature etc. of the spot price matter in the forward price. This is a result of our choice of spot price model having a CAR(p)-dynamics, along with the non-tradeability assumption of the spot.

We next investigate the term structure shapes defined by the functions $f_i(x)$ in (2.4.1). As a case study, we take parameters from the fitting of the CAR(3)-model to daily average temperature data collected over more than 40 years in Stockholm, Sweden. The statistical estimation procedure along with the estimates are all reported in Benth *et al.* [16], and of particular interest here is the parameters in the CAR(3)-model. It was found that the α-parameters of the A matrix become

$$\alpha_1 = 2.043, \, , \quad \alpha_2 = 1.339\,, \quad \alpha_3 = 0.177\,.$$

These values give the eigenvalues $\lambda_1 = -0.175$ and $\lambda_{2,3} = -0.934 \pm 0.374\mathrm{i}$ for A, yielding a stationary CAR(3)-model. In Figure 2.4 we have plotted the resulting f_1, f_2 and f_3 defined in (2.4.1) as a function of time to maturity x.

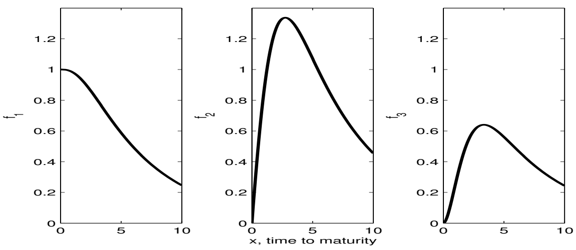

Figure 2.4: f_i for $i = 1, 2, 3$ for Stockholm, Sweden.

All three functions f_1, f_2 and f_3 tend to zero as time to maturity goes to infinity, so in the long end of the forward market the contribution from these functions will become negligible. In the short end, that is, for small times to maturity x, the main contribution comes from f_1, as the two others start at zero for $x = 0$. We also clearly see that f_2 is bigger than f_1 and f_3 around its peak at $x \approx 3$.

We may view the functions f_i as *template* forward curves, which give the shape scaled by the corresponding values of $\mathbf{X}(t)$, that is, the value and its derivatives of the deseasonalized temperature. Thinking in terms of principal component analysis, we have that f_1 gives the *level* of the forward curve, corresponding to a shape decreasing from one towards zero in an exponential fashion. The template curve f_2 is scaled by the derivative of the deseasonalized temperature, and hence one can interpret f_2 as the *slope* in this context. The curve f_2 is increasing towards a maximum value, after which it decreases to zero in a seemingly exponential way. It contributes with a hump in the overall forward curve. If the deseasonalized temperature is increasing (has a positive) slope, there will be an upward pointing hump in the curve, while a deseasonalized temperature experiencing a decline at time t will yield a downward pointing hump. This hump will be most significant at around 3 days to maturity, and obviously the size is determined by how strong the slope of the deseasonalized temperature is at the time in question. The final template f_3 is scaled by the double-derivative of the deseasonalized temperature, and thus we relate this to the *curvature*. The curve f_3 will also contribute with a hump, which will point upward in the case of a convex deseasonalized temperature, and downward pointing if the deseasonalized temperature is concave. We see that the hump is smaller than for f_2, and f_2 is the template that contributes most among the three when x is around 3. This means that the slope of the deseasonalized temperature is more important than level and curvature for times to maturity at around 3. Also, we see that the shape of f_3 is different in the very short end, with a significantly smaller increase than f_2. In fact, f_2 is concave for small x, while f_3 in convex. Note that an increasing but concave deseasonalized temperature at time t (positive derivative, but negative double-derivative) will dampen the hump in the overall forward curve, while an increasing deseasonalized temperature being convex yields possibly a large hump in the short end of the curve. In Figure 2.5 we have plotted these two cases for some illustrative values of the vector $\mathbf{X}(t)$. Note that in this plot we have ignored the contribution from the seasonality function and other terms, and only focused on the part given by the templates f_1, f_2 and f_3.

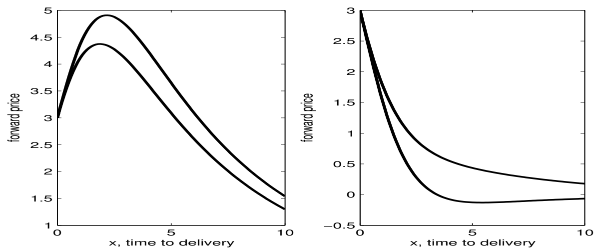

Figure 2.5: Forward curves as a function of time to delivery for various combinations of $Y'(t)$ and $Y''(t)$ with $Y(t) = 3$. To the left, we have $Y'(t) = \pm 1.5$ and $Y''(t) = 0.5$, whereas to the right we have $Y'(t) = \pm 1.5$ and $Y''(t) = -0.5$. The highest value of $Y'(t)$ gives the largest values of the forward price.

We have chosen $Y(t) = 3$, that is, the current temperature is three degrees above its mean. The slope of the temperature is ± 1.5, meaning that the temperature is rapidly increasing or decreasing. Finally, we have used $Y''(t) = \pm 0.5$. We observe that for a negative curvature (concave temperature), we have decreasing forward curves, with the one having negative slope being smallest. If the temperature is convex (positive curvature), we get a hump in the forward curve, again the smallest curve stemming from a negative slope in the temperature.

To gain further insight into the shape of $f_i(x)$, $i = 1, \ldots, p$ defined in (2.4.1), we apply the spectral representation of A to re-express it into a sum of exponentials. To do this, let us first assume that A has p distinct eigenvalues $\lambda_1, \ldots, \lambda_p$, with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

One easily verifies that

$$\mathbf{v}_j = (1, \lambda_j, \lambda_j^2, \ldots, \lambda_j^{p-1})^{\mathrm{T}} ,$$

for $j = 1, \ldots, p$. Letting the matrix C consist of columns being the eigenvectors, we find that $\mathbf{e}_i = \sum_{j=1}^p a_j^i \mathbf{v}_j$ where the vector $\mathbf{a}_i = (a_1^i, \ldots, a_p^i)^{\mathrm{T}}$ is given as $\mathbf{a}_i = C^{-1}\mathbf{e}_i$. Therefore,

$$\mathbf{e}_i = \sum_{j=1}^{p} (\mathbf{e}_j^{\mathrm{T}} C^{-1} \mathbf{e}_i) \mathbf{v}_j \,.$$

Hence, we obtain

$$f_i(x) = \mathbf{e}_1^{\mathrm{T}} \mathrm{e}^{Ax} \mathbf{e}_i = \sum_{j=1}^{p} (\mathbf{e}_j^{\mathrm{T}} C^{-1} \mathbf{e}_i) \mathrm{e}^{\lambda_j x}$$

for a given $i = 1, \ldots, p$. We see that the shapes of all $f_i(x)$ can be represented as a weighted sum of exponentials. Due to stationarity, these exponentials are decaying as functions of time to maturity, at the speed determined by the real part of the eigenvalues. Each exponential term is scaled by factors $\mathbf{e}_j^{\mathrm{T}} C^{-1} \mathbf{e}_i$, or, the jith element of the inverse of the eigenvectors matrix C.

Let us consider forwards with delivery period, like forwards on the average temperature or electricity spot price over a given period. Letting S be defined as the arithmetic spot price model defined in (2.3.2), we recall the expression for the forward price $F(t, T_1, T_2)$ in (2.3.10), where the averaging takes place in the time interval $[T_1, T_2]$. We find that

$$\begin{aligned} F(t, T_1, T_2) = & \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \Lambda(u)\, du + \sum_{i=1}^{p} F_i(T_1 - t, T_2 - T_1) Y^{(i-1)}(t) \\ & + \Psi(T_1 - t, T_2 - T_1) \,, \end{aligned} \tag{2.4.2}$$

where

$$\Psi(x, y) = -\mathrm{i}\psi_L'(-\mathrm{i}\theta)(\mathbf{e}_1^{\mathrm{T}} A^{-2} \frac{1}{y} \left(\mathrm{e}^{Ay} - I\right) \mathrm{e}^{Ax} \mathbf{e}_p - \mathbf{e}_1^{\mathrm{T}} A^{-1} \mathbf{e}_p) \,, \tag{2.4.3}$$

and

$$F_i(x, y) = \mathbf{e}_1^{\mathrm{T}} A^{-1} \frac{1}{y} \left(\mathrm{e}^{Ay} - I\right) \mathrm{e}^{Ax} \mathbf{e}_i \,, \tag{2.4.4}$$

for $i = 1, \ldots, p$. Here, $x = T_1 - t$ denotes *time to delivery period starts* and $y = T_2 - T_1$ *length of delivery period*. Note that $F_i(0, y) \neq 0$ for all $i = 1, \ldots, p$. This is a reflection that $F(t, T_1, T_2)$ is not converging to the underlying spot as $t \to T_1$, due to the delivery period. Such a behaviour is particular in forward markets with delivery period, contrary to "classical" commodity markets where the forward is equal to the spot when time to delivery is zero. In Figure 2.6 we have plotted the templates F_1, F_2 and F_3 for the case of monthly "delivery" period, that is, a forward on the average over a month on the underlying. This would correspond to a monthly temperature forward as the A matrix is also here borrowed from Stockholm, and we see that all three curves are decaying. Interestingly, the dominating factor will be F_2, which is scaled by the derivative of the deseasonalized temperature. Hence, the forward curve is most sensitive to the *change* in temperature, and not the level or the curvature. We also see no humps.

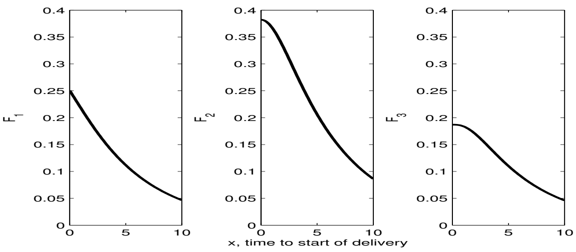

Figure 2.6: F_i for $i = 1, 2, 3$ for Stockholm, Sweden with y being set to one month.

In Figure 2.7 we choose y to be one week, and see that there are humps coming into the forward curve stemming from the *slope* and *curvature*. Again the slope is significantly more important in contributing to the forward curve than the other two values.

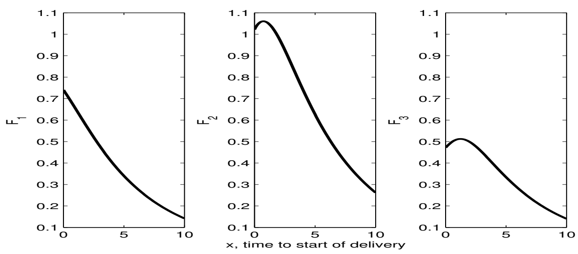

Figure 2.7: F_i for $i = 1, 2, 3$ for Stockholm, Sweden with y being set to one week.

Note that at the Chicago Mercantile Exchange, there is trade in temperature forwards settled over one week.

Next, let us consider how the forward price dynamics is evolving as a function of the spot price empirically. We consider a numerical example, simulating the spot price path by the path of $\mathbf{X}(t)$, which can be done exact as this is an Ornstein-Uhlenbeck process. Hence, we can simulate the path of a forward price $t \mapsto f(t, T)$ for a given delivery time T. To this end, from (2.2.6) we find, for $\Delta > 0$,

$$\mathbf{X}(t+\Delta) = \mathrm{e}^{A\Delta}\mathbf{X}(t) + \int_t^{t+\Delta} \mathrm{e}^{(t+\Delta-s)}\mathbf{e}_p \, dL(s) \,. \tag{2.4.5}$$

Hence, we can simulate $\mathbf{X}(t+\Delta)$ from $\mathbf{X}(t)$ and an independent noise given by the stochastic integral $\int_t^{t+\Delta} \exp(A(t+\Delta-s))\mathbf{e}_p \, dL(s)$. In the case $L = \sigma B$, a Brownian motion with volatility σ, this stochastic integral is a p-dimensional Gaussian random variable with mean zero and variance given by the Itô isometry as

$$\begin{aligned}\mathrm{Var}\left(\int_t^{t+\Delta} \sigma \exp(A(t+\Delta-s))\mathbf{e}_p \, dB(s)\right) &= \sigma^2 \int_t^{t+\Delta} \mathrm{e}^{A(t+\Delta-s)}\mathbf{e}_p\mathbf{e}_p^{\mathsf{T}}\mathrm{e}^{A^{\mathsf{T}}(t+\Delta-s)} \, ds \\ &= \sigma^2 \int_0^{\Delta} \mathrm{e}^{As}\mathbf{e}_p\mathbf{e}_p^{\mathsf{T}}\mathrm{e}^{A^{\mathsf{T}}s} \, ds \,.\end{aligned}$$

Hence, the noises are independent and identically distributed, being only a function of the time step Δ.

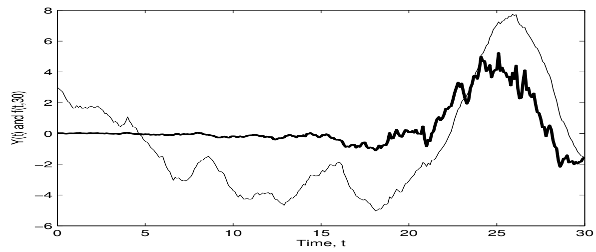

Figure 2.8: The path of $Y(t)$ along with $f(t,T)$ where $T = 30$. Seasonality and market price of risk are assumed to be zero. The spot path $Y(t)$ is the thin curve.

We simulated the dynamics of $\mathbf{X}(t)$ for $p = 3$ with the matrix A as before and with $L = \sigma B$ where $\sigma = 1$. The time step was chosen to be $\Delta = 0.1$, measured in days, and in Figure 2.8 we have plotted the path of $Y(t)$ over the time interval 0 to 30 days, along with the corresponding forward price for a contract with delivery $T = 30$. We have assumed the seasonality Λ being identically equal to zero and supposed zero market price of risk $\theta = 0$. As it is evident from the plot, far from maturity there is essentially no variation in the forward price, a result of the stationarity of Y (or, more precisely, of $\mathbf{X}$). Closer to maturity, the variations in the forward price become bigger, and we see how they follow the slope and level of the spot price $Y(t)$. It is harder to see the effect of the curvature directly. But interestingly, it seems that the forward price path is much rougher than that of the spot. This can be attributed to the fact that the spot is twice differentiable, whereas the forward is explicitly depending on all the coordinates of $\mathbf{X}(t)$, in particular $X_3(t) = \mathbf{e}_3^{\mathrm{T}}\mathbf{X}(t)$ which is not differentiable.

A more realistic situation is that when we only observe the path of the spot, and we must recover its derivatives in order to compute the forward price. As the dynamics of the spot $Y(t)$ is on a state-space form, we could use a Kalman filter for this purpose (see Benth and Šaltytė Benth [15]). However, we can also use numerical differentiation of the past and present spot observations. Backward finite differencing yields the approximations

$$Y'(t) \approx \frac{Y(t) - Y(t-\Delta)}{\Delta} \tag{2.4.6}$$

$$Y''(t) \approx \frac{Y(t) - 2Y(t-\Delta) + Y(t-2\Delta)}{\Delta^2}\,. \tag{2.4.7}$$

We applied this routine on a simulated example to check its performance. In Figure 2.9 we have simulated the path of $Y(t)$ for the same set of parameters as above and applied the finite differences to recover the first and second derivative (depicted as dotted lines in the figure). To be more in line with applications, we assume that we have daily observations of Y, and simulated a path over 100 days. In the figure, we have included the actual paths of $X_2(t) = \mathbf{e}_2^{\mathrm{T}}\mathbf{X}(t)$ and $X_3(t) = \mathbf{e}_3^{\mathrm{T}}\mathbf{X}(t)$ realized from the simulation (depicted as complete lines on the figure). We see that the finite difference approximations of the path of $Y(t)$ are recovering the actual derivatives very well, motivating that this procedure would make sense in practical applications.

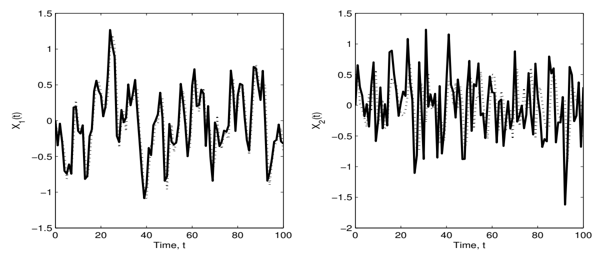

Figure 2.9: Estimation of $Y'(t)$ (left) and $Y''(t)$ (right) based on finite differencing of the path of $Y(t)$ (dotted lines), versus the exact first and second derivatives (complete lines).

2.5 Conclusions and outlook

In this paper we have consider the forward price dynamics as a function of the spot price modelled as a continuous-time autoregressive dynamics. In the case of autoregression of order $p > 1$, the forward curve is driven by the derivatives of the spot price up to order $p-1$, where each component gives a contribution to the structure of the forward curve. The components may be viewed as templates for the forward curve, as they are the basis functions for the possible shapes that can be achieved by this model. The forward curve can vary between contango and backwardation, as well as incorporating humps.

The different forward term structures appear depending on the state of the spot and its derivatives. In fact, we may explain humps in the forward curve as appearing stochastically as a result of the underlying spot having a positive slope (that is, the spot price is currently increasing). In many ”physical markets” like weather and energy/commodities, one can identify trends in spot prices (by technical analysis, say), and this may be applied to identify and explain occurence of humps. It would be interesting to see whether this holds true in an empirical setting. After all, stochastic volatility may explain humps in the forward curve as well (see Benth [10]), and this raises the question whether a hump occurs because of the state of the volatility, or due to some structural propetrties of the spot price path (or both).

We also would like to refer to a study of Diebold and Li [28] which uses the popular Nelson-Siegel yield curve in the context of forward rates and drives the components of this curve by autoregressive time series of order 1. Our study gives results on the forward process in the same spirit as this study, however, in a specific framework that ensures an arbitrage-free forward price dynamics.

In a future study we will extend our analysis to general Lévy semistationary dynamics of the spot, which would allow for a moving average structure in the continuous-time autoregressive model. It is expected that in this case the forward dynamics will depend on the history of the spot, and not only on the current value and its derivatives.

Chapter 3

Forward prices as functionals of the spot path in commodity markets modeled by Lévy semistationary processes

Abstract
We show that the forward price can be represented as a functional of the spot price path in the case of Lévy semistationary models for the spot dynamics. The functional is a weighted average of the historical spot price in general, and is derived by means of the Laplace transform. For the special cases of continuous-time autoregressive moving average and gamma-Lévy semistationary processes for the spot dynamics, we are able to produce an analytical weight function. Both classes of processes are of interest in markets for power, weather and shipping, and we provide a discussion of the results based on numerical examples.

3.1 Introduction

The classical relationship between spot and forward prices is based on the so-called buy-and-hold hedging strategy. By purchasing the underlying commodity on the spot market and holding it until delivery, one perfectly replicates the forward contract. The forward price thus becomes the cost of financing the spot purchase. Extensions of this no-arbitrage pricing of forwards include the cost of storage and convenience yield (see e.g. Geman [34] for a discussion of these notions in various commodity markets).

Power markets constitute an important case where one cannot trade the spot in a financial sense. By the very nature of power spot, it cannot be stored, and hence the buy-an-hold strategy breaks down. For the same reason the notion of convenience yield is not relevant either (see Geman [34]). Hence, the classical spot-forward relationship is no longer valid. Since power forwards are financially tradeable assets, we know from the theory of arbitrage pricing that the forward price dynamics has to be a (local) martingale under some pricing measure. On the other hand, the discounted power spot dynamics does not to possess the (local) martingale property under the same pricing probability due to its non-storable feature (see Benth *et al.* [16]). We define the forward price dynamics under the pricing measure as the predicted spot at delivery conditioned on the current market information. Weather and shipping are two other examples of markets where this reasoning is relevant, as weather cannot be traded and shipping spot rates are only indexes defined by trader's opinions (see Benth and Šaltytė Benth [15] for weather markets, and Alizadeh and Nomikos [3] for freight rates). These are the markets on which we are focused in this paper.

An important question is of course which pricing measure one should use, as there are infinitely many equivalent probabilities to choose from. We will not investigate this question here, but refer the reader to Benth *et al.* [16] and Benth and Šaltytė Benth [15], where the selection of such pricing measures is discussed with many references to relevant work on this. Typically, one is choosing a parametric class of pricing measures which preserves certain stochastic properties of the dynamics. The parameter is interpreted as the market price of risk, providing us with a model of the risk premium in the market. Hence, one is aiming for a pricing measure which can explain the observed risk premium, defined as the difference between the observed forward price and the predicted spot at delivery (the so-called ex-ante risk premium). In Benth and Šaltytė Benth [15] the relationship to utility-based pricing approaches, involving pricing kernels/densities, is discussed. In many situations, the stochastic properties of the weather factors and spot prices are preserved under the pricing measure, with the only difference that parameters are changed. Hence, in this paper we will model the stochastic dynamics of the underlying directly under the pricing measure, a usual approach in interest-rate theory.

Empirical studies of power, weather and freight rates have proven the relevance of applying continuous-time autoregressive moving average processes (so-called CARMA processes), and more generally Lévy semistationary (LSS) processes in modelling the spot dynamics. We refer to Benth and Šaltytė Benth [15] and Härdle and Lopez Cabrera [35] for weather studies, Garcia *et al.* [33] and Barndorff-Nielsen *et al.* [6] for analysis of power, and Benth *et al.* [13] for freight markets. CARMA and LSS processes have a memory effect in their dynamics, and in the case of LSS processes, they may fail to be semimartingales. We also would like to mention the work by Paschke and Prokopczuk [39] on oil prices using CARMA processes, as well as the application of such in fixed-income markets Andresen *et al.* [4].

In Barndorff-Nielsen *et al.* [6] and Benth and Šaltytė Benth [15], forward prices have been derived theoretically for various contract specifications in the markets that we have in mind. Interestingly, the prices will not depend explicitly on the current spot price, but some stochastic process related to the spot dynamics. In this paper we show how this process may be represented as a functional of the *historical path* of spot prices, that is, the forward price will not only depend on the current spot price, but also in its past values. This is a completely new relationship between forward and spot price, and it is possible due to the stochastic properties of the spot price fluctuations along its non-tradeability.

More specifically, we apply the Laplace transform to show that if the spot price dynamics follow an LSS process, then the forward price dynamics can be represented as a weighted average of all spot prices up to current time. In the special case of a CARMA process, this reduces to an exponential weighting factor in the averaging of historical prices, as well as dependency on the present spot price and its derivatives up to a certain order. Hence, not only the past spot prices, but also the current trend and curvature of the spot will impact the forward price.

Our results significantly generalize the analysis in Benth and Solanilla Blanco [17], where continuous-time autoregressive (CAR) processes where studied by other tools than here. In the present paper we also include stochastic volatility in our general spot model, and present detailed results in some special cases being highly relevant for power markets, namely CARMA(2,1) and Gamma-LSS processes which have been applied in several empirical studies (see e.g. Garcia *et al.* [33] and Barndorff-Nielsen *et al.* [6]). Moreover, we present some numerical case studies which demonstrate how the weight functions and typical forward curves look like for these models. Importantly, the forward curves will have a hump shape.

Our results are presented as follows. In the next section we introduce Lévy semistationary processes, and present the forward prices. We then proceed in Section 3.3 with applying the Laplace transform to represent the forward price as a functional on the spot price and volatility paths. The particular cases of CARMA and Gamma-LSS processes are treated in Section 3.4, which also contains several numerical examples.

3.2 Spot and forward pricing based on Lévy semistationary processes

Let $(\Omega, \mathcal{F}, Q)$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{-\infty<t<\infty}$ satisfying the usual hypotheses, and define L to be a two-sided square integrable Lévy process having paths which are right-continuous with left-limits (RCLL). We consider the commodity spot price $S(t)$ as presented in Barndorff-Nielsen *et al.* [6] by means of a geometric model

$$S(t) = \Lambda(t) \exp(Y(t)) . \tag{3.2.1}$$

Here, $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded and measurable deterministic seasonal function and Y a Lévy semistationary process (from now on called an *LSS process*) defined as

$$Y(t) = \int_{-\infty}^{t} g(t-s)\sigma(s-)\, dL(s) , \tag{3.2.2}$$

for a deterministic function $g : \mathbb{R}_+ \to \mathbb{R}_+$. The *volatility* $\sigma(t)$ is supposed to be a square integrable stationary RCLL process which is independent of the Lévy process L. As σ is square-integrable and stationary, we find from the independence between L and the volatility that the condition

$$\int_0^\infty g^2(s)\, ds < \infty , \tag{3.2.3}$$

guarantees the existence of the stochastic integral in (3.2.2) (see Protter [41]).

Although $S(t)$ is well-defined for all $t \in \mathbb{R}$, we shall focus on its dynamics for non-negative times exclusively. The Lévy process L is two-sided in order to have a stationary dynamics of Y. We refer to Barndorff-Nielsen *et al.* [6] for an extensive analysis of this spot price model applied to power markets.

We have chosen to model the spot price dynamics directly under the pricing measure Q, explaining the unusual notation for the given probability space. More commonly, the dynamics of the spot price is modeled under the market probability, here denoted P. Next, one introduces an equivalent probability Q under which pricing of derivatives (for example forwards) is done. If the commodity can be liquidly traded in the spot market, the probability Q is a martingale measure in the sense that the spot price becomes a Q-(local) martingale after discounting with the risk-free interest rate. As discussed in the Introduction, in many commodity markets one can not trade liquidly in the spot, that is, one cannot form portfolios of the spot. This is for example the case of power markets, as the physical nature of power demands immediate consumption (or production). In other markets, like that for weather derivatives, we face a similar situation as the underlying (temperature or rain, say) is obviously not a tradeable asset class. In these commodity markets, the spot price dynamics does not need to be a (local) martingale with respect to Q, and we refer to Q as the *pricing measure*. In an insurance context, the pricing measure can be thought of as modelling the premium incurred by the insurer for taking on a non-hedgeable risk.

As we see from the definition of S in (3.2.1), the spot price is in general not a Q-martingale after discounting. In the context of energy and weather markets, one typically chooses a class of pricing measures that preserves the Lévy process property of the driving noise L in the spot dynamics (see Benth *et al.* [16] for energy and Benth *et al.* [16] for weather). We choose to work directly under the pricing measure Q to save some notation and technicalities in the exposition. Although power and weather are our prime examples, also markets like gas and coal, and freight may be included in our analysis (see Geman [34] and Benth *et al.* [16]).

We want to point out that the spot model in (3.2.1) encompasses many of the classical models that have been applied in commodity markets, such as the one-factor Schwartz model (see Schwartz [43]) and continuous-time autoregressive moving average processes, also known as CARMA-processes (see Garcia *et al.* [33] for an application of these processes to power spot), among others. We will return to these special cases in Section 3.4.

Let $f(t,T)$ denote the forward price at time $t \geq 0$ of a contract delivering electricity at time $T \geq t$. We use the definition of a forward price in incomplete markets from Duffie [30] (see also Benth *et al.* [16]),

$$f(t,T) = \mathbb{E}_Q[S(T)|\mathcal{F}_t]\,. \tag{3.2.4}$$

As long as $S(T) \in L^1(Q)$, the space of integrable random variables, this yields a martingale dynamics of the forward price process $t \to f(t,T), 0 \leq t \leq T$, ensuring an arbitrage-free model.

We have defined S as the exponential of an LSS process Y. In order to have $S(T) \in L^1(Q)$, we impose that $L(1)$ has finite exponential moments, and introduce the notation $\phi_L(x)$ for the logarithm of the moment generating function of $L(1)$, defined as

$$\phi_L(x) = \ln \mathbb{E}\left[\mathrm{e}^{xL(1)}\right]\,. \tag{3.2.5}$$

Obviously, exponential integrability implies square integrability of L. Furthermore, we assume that the following condition holds for the driving noise L and the stochastic volatility σ: for all $0 \leq t \leq \tau < \infty$, with τ being some terminal time horizon for the financial market in question, it holds

$$\mathbb{E}\left[\exp\left(\int_{-\infty}^{t} \phi_L(g(t-s)\sigma(s))\,ds\right)\right] < \infty\,. \tag{3.2.6}$$

This condition ensures that $S(T) \in L^1(Q)$.

In the following Proposition we derive a general expression for the forward price based on (3.2.4) and (3.2.1):

Proposition 3.2.1. *The forward price* $t \mapsto f(t,T)$ *for* $0 \le t \le T$ *is*

$$f(t,T) = \Lambda(T) \exp\left(\int_{-\infty}^{t} g(T-s)\sigma(s-)\, dL(s)\right) \mathbb{E}\left[\exp\left(\int_t^T \phi_L\left(g(T-s)\sigma(s)\right)\, ds\right) \mid \mathcal{F}_t\right].$$

Proof. By adaptedness, we find

$$f(t,T) = \Lambda(T) \exp\left(\int_{-\infty}^{t} g(T-s)\sigma(s-)\, dL(s)\right) \mathbb{E}\left[\exp\left(\int_t^T g(T-s)\sigma(s-)\, dL(s)\right) \mid \mathcal{F}_t\right].$$

Define the filtration $\mathcal{G}_t = \sigma\{\sigma(s)\,, 0 \le s \le T\} \vee \mathcal{F}_t$. As $\sigma(s)$ is independent of L, we find by double conditioning that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\int_t^T g(T-s)\sigma(s-)\, dL(s)\right) \mid \mathcal{F}_t\right] \\
&\qquad = \mathbb{E}\left[\mathbb{E}\left[\exp\left(\int_t^T g(T-s)\sigma(s-)\, dL(s)\right) \mid \mathcal{G}_t\right] \mid \mathcal{F}_t\right] \\
&\qquad = \mathbb{E}\left[\exp\left(\int_t^T \phi_L\left(g(T-s)\sigma(s)\right)\, ds\right) \mid \mathcal{F}_t\right].
\end{aligned}$$

In the last equality we made use of the independent increment property of the Lévy process L. We note that all steps above are verified under the moment assumption (3.2.6). Hence, the proof is complete. □

Let us consider the special case when $\sigma(s) \equiv \sigma$, a positive constant. Since $\sigma L(t)$ again is a Lévy process, we can without any loss of generality assume that $\sigma = 1$. Inserting this choice of $\sigma(s)$ into the moment condition (3.2.6) yields that $S(T) \in L^1(Q)$ whenever

$$\int_0^\infty |\phi_L(g(s))|\, ds < \infty\,.$$

We find the following corollary:

Corollary 3.2.2. *Let* $\sigma(s) \equiv 1$. *Then the forward price* $t \mapsto f(t,T)$ *for* $0 \le t \le T$ *is*

$$f(t,T) = \Lambda(T) \exp\left(\int_{-\infty}^{t} g(T-s)\, dL(s) + \int_0^{T-t} \phi_L\left(g(s)\right)\, ds\right).$$

Proof. This follows immediately from Proposition 3.2.1. □

Another case of interest is when $L = B$, a Brownian motion, and $\sigma^2(t)$ being again an LSS process of the simple form

$$\sigma^2(t) = \int_{-\infty}^{t} h(t-s)\, dU(s)\,, \tag{3.2.7}$$

where U is a two-sided subordinator process being independent of B, and $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ being square-integrable. We suppose that U has exponential moments and denote by ϕ_U the logarithm of the moment generating function. In the following Lemma we state a sufficient condition for $S(T) \in L^1(Q)$.

Lemma 3.2.3. *Condition* (3.2.6) *holds as long as*

$$\int_0^\infty \phi_U\left(\frac{1}{2}\int_0^s g^2(s-v)h(v)\, dv\right) ds < \infty\,.$$

Proof. When $L = B$, we have that $\phi_L(x) = \phi_B(x) = \frac{1}{2}x^2$. Hence, condition (3.2.6) reads

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_{-\infty}^{t} g^2(t-s)\sigma^2(s)\, ds\right)\right] < \infty\,,$$

for all $0 \le t \le \tau < \infty$. From (3.2.7), we find after appealing to the stochastic Fubini's theorem (see Protter [41])

$$\begin{aligned}\int_{-\infty}^{t} g^2(t-s)\int_{-\infty}^{s} h(s-u)\, dU(u)\, ds &= \int_{-\infty}^{t}\int_{u}^{t} g^2(t-s)h(s-u)\, ds\, dU(u)\\ &= \int_{-\infty}^{t}\int_{0}^{t-u} g^2(t-u-v)h(v)\, dv\, dU(u)\,.\end{aligned}$$

From the independent increment property of U we find the conclusion of the Lemma. □

From now on, we suppose that the condition in Lemma 3.2.3 holds whenever we analyse this model. We have the following corollary.

Corollary 3.2.4. *Let $\sigma(t)$ have dynamics given in* (3.2.7) *and $L = B$. Then the forward price $t \mapsto f(t,T)$ for $0 \le t \le T$ is*

$$\begin{aligned} f(t,T) = \Lambda(T)\exp&\left(\int_{-\infty}^{t} g(T-s)\sigma(s-)\, dB(s) + \frac{1}{2}\int_{-\infty}^{t}\int_{t}^{T} g^2(T-s)h(s-u)\, ds\, dU(u)\right)\\ &\times\exp\left(\int_{t}^{T}\phi_U\left(\frac{1}{2}\int_{u}^{T} g^2(T-s)h(s-u)\, ds\right) du\right).\end{aligned}$$

Proof. When $L = B$, the logarithm of the moment generating function becomes $\phi_L(x) = x^2/2$. Hence, from Proposition 3.2.1 we must compute the conditional expectation

$$\mathbb{E}\left[\exp\left(\int_t^T \phi_L\left(g(T-s)\sigma(s)\right)\, ds\right) \mid \mathcal{F}_t\right] = \mathbb{E}\left[\exp\left(\frac{1}{2}\int_t^T g^2(T-s)\sigma^2(s)\, ds\right) \mid \mathcal{F}_t\right].$$

By inserting the dynamics of $\sigma^2(s)$, it follows from the stochastic Fubini's theorem (see Protter [41]) that

$$
\begin{aligned}
&\int_t^T g^2(T-s)\sigma^2(s)\,ds \\
&\qquad = \int_t^T g^2(T-s)\int_{-\infty}^s h(s-u)\,dU(u)\,ds \\
&\qquad = \int_t^T g^2(T-s)\int_{-\infty}^t h(s-u)\,dU(u)\,ds + \int_t^T g^2(T-s)\int_t^s h(s-u)\,dU(u)\,ds \\
&\qquad = \int_{-\infty}^t \int_t^T g^2(T-s)h(s-u)\,ds\,dU(u) + \int_t^T \int_u^T g^2(T-s)h(s-u)\,ds\,dU(u)\,.
\end{aligned}
$$

But then the result follows since the first term is $\mathcal{F}_t$-adapted and the second is independent of $\mathcal{F}_t$. □

In the rest of this paper we shall mostly focus on the two situations in Corollaries 3.2.2 and 3.2.4. The main emphasis will be put on the simple case of constant volatility, but we will also discuss the stochastic volatility case in some detail.

For completeness, we also briefly discuss forward prices for so-called *arithmetic* LSS processes. To this end, consider the spot price dynamics to be

$$
S(t) = \Lambda(t) + Y(t)\,, \tag{3.2.8}
$$

for $t \geq 0$. Here Λ and Y are defined as for the geometric model of S in (3.2.1). We find the following forward price:

Proposition 3.2.5. *If the spot price is defined by* (3.2.8)*, then for* $0 \leq t \leq T$

$$
f(t,T) = \Lambda(T) + \mathbb{E}[L(1)]\int_t^T g(T-s)\mathbb{E}[\sigma(s)\,|\,\mathcal{F}_t]\,ds + \int_{-\infty}^t g(T-s)\sigma(s-)\,dL(s)\,.
$$

Proof. By $\mathcal{F}_t$-measurability, we find

$$
\begin{aligned}
\mathbb{E}\left[Y(T)\,|\,\mathcal{F}_t\right] &= \mathbb{E}\left[\int_{-\infty}^t g(T-s)\sigma(s-)\,dL(s)\,|\,\mathcal{F}_t\right] + \mathbb{E}\left[\int_t^T g(T-s)\sigma(s-)\,dL(s)\,|\,\mathcal{F}_t\right] \\
&= \int_{-\infty}^t g(T-s)\sigma(s-)\,dL(s) + \mathbb{E}\left[\int_t^T g(T-s)\sigma(s-)\,dL(s)\,|\,\mathcal{F}_t\right]\,.
\end{aligned}
$$

By double conditioning, using that σ is independent of L, the second term above becomes

$$
\mathbb{E}\left[\int_t^T g(T-s)\sigma(s-)\,dL(s)\,|\,\mathcal{F}_t\right] = \mathbb{E}[L(1)]\mathbb{E}\left[\int_t^T g(T-s)\sigma(s)\,ds\,|\,\mathcal{F}_t\right]\,.
$$

By the square integrability of σ and g, all steps above are valid without any additional conditions. The proof is complete. □

It is worth noticing that the term $\int_{-\infty}^t g(T-s)\sigma(s-)\,dL(s)$ appears explicitly in the forward price for the two cases of arithmetic and geometric spot price dynamics (recall Proposition 3.2.1 for the geometric case). The dependency on the stochastic volatility is apparently of a different nature in the arithmetic case compared to the geometric. We see in Proposition 3.2.5 that the forward price depends on the conditional expectation of $\sigma(s)$, while in the geometric case we must compute the conditional expectation of the exponential of some functional of $\sigma(s)$, involving the moment generating function of L (recall Proposition 3.2.1). In case L has zero expectation, the forward price is not depending on $\sigma(s)$ for the arithmetic spot model, while a constant σ gives a simple deterministic integral term.

Let us consider the case $\sigma(t)$ being defined as the LSS process in (3.2.7). We must compute the conditional expectation of the *square-root* of an LSS process, which seems rather difficult. However, by slightly extending the analysis in Benth and Taib [19], we can represent this conditional expectation in terms of a Fourier integral and the characteristic exponent of $\sigma^2(t)$, which is known. This is the content of the next result:

Proposition 3.2.6. *Suppose $\sigma^2(t)$ follows the dynamics in* (3.2.7). *Then for $s \geq t \geq 0$ it holds that*

$$\mathbb{E}[\sigma(s)\,|\,\mathcal{F}_t] =$$
$$\frac{\Gamma\left(\frac{3}{2}\right)}{2\pi}\int_{\mathbb{R}}(\gamma+\mathrm{i}y)^{-3/2}\exp\left(\int_0^{s-t}\psi_U\left((\gamma+\mathrm{i}y)h(v)\right)\,dv+(\gamma+\mathrm{i}y)\int_{-\infty}^t h(s-v)\,dU(v)\right)\,dy$$

for an arbitrary constant $\gamma > 0$ and ψ_U being the logarithm of the characteristic function of $U(1)$. Γ denotes the Gamma-function.

Proof. The argument follows the lines of the proof of Lemma 2.2 in Benth and Taib [19]. Define, for $x \geq 0$ the function $g_\gamma(x) = \sqrt{x}\exp(-\gamma x)$ for a constant $\gamma > 0$. Let $g_\gamma(x) = 0$ for $x < 0$, by Lemma 2.1 in Benth and Taib [19] we find that its Fourier transform becomes

$$\widehat{g}_\gamma(y) = \int_{\mathbb{R}} g_\gamma(x)\mathrm{e}^{-\mathrm{i}xy}\,dx = \Gamma\left(\frac{3}{2}\right)(\gamma+\mathrm{i}y)^{-3/2}\,.$$

As this function is integrable on $\mathbb{R}$, we can apply the Fourier inversion formula to have

$$\sqrt{x} = \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{g}_\gamma(y)\mathrm{e}^{(\gamma+\mathrm{i}y)x}\,dy\,.$$

Thus, using this representation together with Fubini-Tonelli's Theorem, we obtain

$$\begin{aligned}\mathbb{E}[\sigma(s)\,|\,\mathcal{F}_t] &= \mathbb{E}[\sqrt{\sigma^2(s)}\,|\,\mathcal{F}_t]\\ &= \frac{1}{2\pi}\mathbb{E}\left[\int_{\mathbb{R}}\widehat{g}_\gamma(y)\exp\left((\gamma+\mathrm{i}y)\sigma^2(s)\right)\,dy\,|\,\mathcal{F}_t\right]\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{g}_\gamma(y)\mathbb{E}\left[\exp\left((\gamma+\mathrm{i}y)\sigma^2(s)\right)\,|\,\mathcal{F}_t\right]\,dy\,.\end{aligned}$$

We compute the conditional characteristic function of $\sigma^2(s)$ by appealing to independence of increments of the Lévy process U along with its $\mathcal{F}_t$-measurability:

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left((\gamma+\mathrm{i}y)\sigma^2(s)\right) \mid \mathcal{F}_t\right] \\
&\quad= \exp\left((\gamma+\mathrm{i}y)\int_{-\infty}^{t} h(s-v)\,dU(v)\right)\mathbb{E}\left[\exp\left(\int_t^s (\gamma+\mathrm{i}y)h(s-v)\,dU(v)\right) \mid \mathcal{F}_t\right] \\
&\quad= \exp\left((\gamma+\mathrm{i}y)\int_{-\infty}^{t} h(s-v)\,dU(v)\right)\mathbb{E}\left[\exp\left(\int_t^s (\gamma+\mathrm{i}y)h(s-v)\,dU(v)\right)\right] \\
&\quad= \exp\left((\gamma+\mathrm{i}y)\int_{-\infty}^{t} h(s-v)\,dU(v)\right)\exp\left(\int_t^s \psi_U((\gamma+\mathrm{i}y)h(s-v))\,dv\right).
\end{aligned}
$$

This concludes the proof. □

Arithmetic spot price models have been applied in several studies of power prices, see e.g. Lucia and Schwartz [38], Garcia *et al.* [33], Barndorff-Nielsen *et al.* [6] and Benth *et al.* [12].

3.3 Forward prices as functionals of the spot path

In this section we will express the foward price in terms of the path of the spot. By means of the Laplace transform, we establish a connection between the forward price dynamics and the path of the spot. This is in sharp contrast to the usual spot-forward relationship in classical liquid financial markets, where the current discounted forward price coincides with the current spot price.

Observe that in the general forward price in Proposition 3.2.1, we have dependency on the stochastic integral

$$
Y(t,x) := \int_{-\infty}^{t} g(x+t-s)\sigma(s-)\,dL(s)\,, \tag{3.3.1}
$$

with $x = T - t \geq 0$. The deseasonalised logarithmic spot price is

$$
\ln S(t) - \ln \Lambda(t) = Y(t)\,,
$$

where $Y(t)$ defined as the LSS process in (3.2.2). As we immediately see, $Y(t,x)$ is different than $Y(t)$ for all $x > 0$, and only coincides for $x = 0$. However, both $Y(t,x)$ and $Y(t)$ are generated from the same path of the Lévy process L and the volatility process σ. The mathematical aim for this Section is to express $Y(t,x)$ as some functional of the path of $Y(t)$.

Introducing a stochastic volatility σ as in (3.2.7) with $L = B$, we see from Corollary 3.2.4 that the forward price depends on the factor

$$
\Sigma(t,x) := \int_{-\infty}^{t}\int_0^x g^2(x-v)h(t+v-s)\,dv\,dU(s)\,, \tag{3.3.2}
$$

for $x = T - t \geq 0$. Hence, the forward price depends on the path of U up to time t, which is the same path generating the state of the stochastic volatility $\sigma(t)$. Note that $\Sigma(t,0) = 0$ and therefore different than $\sigma^2(t)$, which is contrary to the case of $Y(t)$ and $Y(t,x)$. In this Section we aim at understanding the relationship between $\Sigma(t,x)$ and $\sigma^2(t)$.

As a simple first example, consider Y being an Ornstein-Uhlenbeck process, that is $g(t) = \exp(-\alpha t)$ for $\alpha > 0$ a constant. The parameter α is sometimes referred to as the speed of mean reversion of the process, as it measures how fast Y reverts back to its mean. It is simple to see that

$$Y(t,x) = \mathrm{e}^{-\alpha x} Y(t)\,,$$

and therefore we find from Proposition 3.2.1

$$\begin{aligned} f(t,T) = {} & \Lambda(T) \exp\left(\mathrm{e}^{-\alpha(T-t)} \left(\ln S(t) - \ln \Lambda(t)\right)\right) \\ & \times \mathbb{E}\left[\exp\Big(\int_t^T \phi_L\left(\exp(-\alpha(T-s))\sigma(s)\right)\, ds\Big) \mid \mathcal{F}_t\right]. \end{aligned}$$

Thus, we see that the forward price is explicitly a function of the current spot price. If we further let $h(t) = \exp(-\lambda t)$, that is, the stochastic volatility process is generated by an Ornstein-Uhlenbeck dynamics as well, we find after appealing to Corollary 3.2.4 for the case $L = B$

$$\begin{aligned} f(t,T) = {} & \Lambda(T) \exp\left(\mathrm{e}^{-\alpha(T-t)} Y(t) + \frac{1}{2(2\alpha - \lambda)}\left(\mathrm{e}^{-\lambda(T-t)} - \mathrm{e}^{-2\alpha(T-t)}\right)\sigma^2(t)\right) \\ & \times \exp\left(\int_t^T \phi_U\left(\frac{1}{2}\int_u^T g^2(T-s)h(s-u)\, ds\right) du\right). \end{aligned}$$

Of course, the integral in the argument of ϕ_U can be computed analytically. For this specification, the forward price will depend on the current level of the stochastic volatility. This model was proposed and analysed for UK gas spot prices by Benth [10].

As we shall see, going beyond Ornstein-Uhlenbeck process will lead to much more involved relationships between forward and spot prices, where in fact the whole path of the spot may be taken into account in the forward price. Let us start by recalling the definition of the Laplace transform:

Definition 3.3.1. *Let Z be a stochastic process and suppose that*

$$\int_0^\infty \mathrm{e}^{-\theta t} |Z(t)|\, dt < \infty,\ a.s..$$

Then, the Laplace transform of Z is defined as

$$\mathcal{L}Z(\theta) = \int_0^\infty \mathrm{e}^{-\theta t} Z(t)\, dt\,,$$

for $\theta > 0$ constant.

In the next Lemma we show that the Laplace transform can be applied to the process $t \mapsto Y(t,x)$ for any $x \geq 0$.

Lemma 3.3.2. *The Laplace transform of the process* $t \mapsto Y(t,x), t \geq 0$ *defined in* (3.3.1) *exists for all* $x \geq 0$.

Proof. Indeed, by the stochastic Fubini's theorem (see Protter [41]) and Cauchy-Schwartz' inequality

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\theta t}|Y(t,x)|\,dt\right] = \int_0^\infty \mathrm{e}^{-\theta t}\mathbb{E}\left[|Y(t,x)|\right]\,dt \leq \int_0^\infty \mathrm{e}^{-\theta t}\mathbb{E}\left[Y^2(t,x)\right]^{1/2}\,dt\,.$$

But, by double conditioning using the independence between σ and L along with the stationarity of σ, we find

$$\begin{aligned}\mathbb{E}\left[\left(\int_{-\infty}^t g(t+x-s)\sigma(s-)\,dL(s)\right)^2\right] &= \mathbb{E}[L^2(1)]\mathbb{E}[\sigma^2(s)]\int_{-\infty}^t g^2(t+x-s)\,ds \\ &= \mathbb{E}[L^2(1)]\mathbb{E}[\sigma^2(s)]\int_x^\infty g^2(u)\,du \\ &\leq \mathbb{E}[L^2(1)]\mathbb{E}[\sigma^2(s)]\int_0^\infty g^2(u)\,du < \infty\,.\end{aligned}$$

Hence, the Lemma follows. □

We compute the Laplace transform of $Y(t,x)$:

Proposition 3.3.3. *Denote by* $g_x(t) := g(x+t)$*. Then for* $Y(t,x)$ *defined in* (3.3.1) *it holds*

$$\int_0^\infty \mathrm{e}^{-\theta t}Y(t,x)\,dt = \int_{-\infty}^0 \mathcal{L}g_{x-s}(\theta)\sigma(s-)\,dL(s) + \mathcal{L}g_x(\theta)\int_0^\infty \mathrm{e}^{-\theta s}\sigma(s-)\,dL(s)\,.$$

Proof. By Lemma 3.3.2 we know that the Laplace transform of $Y(t,x)$ exists *a.s.* By the stochastic Fubini's theorem (see Protter [41]) we find,

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\theta t}Y(t,x)\,dt &= \int_0^\infty \mathrm{e}^{-\theta t}\int_{-\infty}^t g_x(t-s)\sigma(s-)\,dL(s)\,dt \\ &= \int_0^\infty \mathrm{e}^{-\theta t}\int_{-\infty}^\infty g_x(t-s)1(s\leq t)\sigma(s-)\,dL(s)\,dt \\ &= \int_{-\infty}^\infty\int_0^\infty \mathrm{e}^{-\theta t}g_x(t-s)1(s\leq t)\,dt\sigma(s-)\,dL(s) \\ &= \int_{-\infty}^\infty\int_{\max(0,s)}^\infty \mathrm{e}^{-\theta t}g_x(t-s)\,dt\,\sigma(s-)\,dL(s)\,.\end{aligned}$$

Splitting the integral and applying the definition of the Laplace transform yield,

$$\int_0^\infty \mathrm{e}^{-\theta t}Y(t,x)\,dt$$

$$= \int_{-\infty}^{0} \int_{0}^{\infty} \mathrm{e}^{-\theta t} g_x(t-s)\, dt \sigma(s-)\, dL(s) + \int_{0}^{\infty} \int_{s}^{\infty} \mathrm{e}^{-\theta t} g_x(t-s)\, dt \sigma(s-)\, dL(s)$$

$$= \int_{-\infty}^{0} \mathcal{L} g_{x-s}(\theta) \sigma(s-)\, dL(s) + \mathcal{L} g_x(\theta) \int_{0}^{\infty} \mathrm{e}^{-\theta s} \sigma(s-)\, dL(s)\,.$$

Thus, the Proposition follows. □

Consider the factor

$$\int_{0}^{\infty} \mathrm{e}^{-\theta s} \sigma(s-)\, dL(s)\,. \tag{3.3.3}$$

Note that by letting $x = 0$ in Proposition 3.3.3 the Laplace transform of $Y(t, x)$ reduces to

$$\int_{0}^{\infty} \mathrm{e}^{-\theta t} Y(t)\, dt = \int_{-\infty}^{0} \mathcal{L} g_{-s}(\theta) \sigma(s-)\, dL(s) + \mathcal{L} g(\theta) \int_{0}^{\infty} \mathrm{e}^{-\theta s} \sigma(s-)\, dL(s)\,.$$

Hence, the Laplace transforms of $Y(t)$ and $Y(t, x)$ share the factor in (3.3.3). We can express the Laplace transform of $Y(t, x)$ in terms of that of $Y(t)$ by,

$$\int_{0}^{\infty} \mathrm{e}^{-\theta t} Y(t, x)\, dt \tag{3.3.4}$$

$$= \int_{-\infty}^{0} \mathcal{L} g_{x-s}(\theta) \sigma(s-)\, dL(s) + \frac{\mathcal{L} g_x(\theta)}{\mathcal{L} g(\theta)} \left(\int_{0}^{\infty} \mathrm{e}^{-\theta t} Y(t)\, dt - \int_{-\infty}^{0} \mathcal{L} g_{-s}(\theta) \sigma(s-)\, dL(s) \right)$$

$$= \int_{-\infty}^{0} \left(\mathcal{L} g_{x-s}(\theta) - \frac{\mathcal{L} g_x(\theta)}{\mathcal{L} g(\theta)} \mathcal{L} g_{-s}(\theta) \right) \sigma(s-)\, dL(s) + \frac{\mathcal{L} g_x(\theta)}{\mathcal{L} g(\theta)} \int_{0}^{\infty} \mathrm{e}^{-\theta t} Y(t)\, dt\,.$$

We observe the dependency on the ratio between the Laplace transforms of g_x and g in the above expression. As g is a positive-valued function, the Laplace transforms will be strictly bigger than zero and the ratio is well-defined. Under certain conditions, we can find the inverse Laplace transform of this ratio.

Proposition 3.3.4. *Suppose for $x \geq 0$ there exists a function $t \mapsto \xi_x(t)\,, t \geq 0$ such that its Laplace transform is well-defined and*

$$\mathcal{L} \xi_x(\theta) = \frac{\mathcal{L} g_x(\theta)}{\mathcal{L} g(\theta)}\,. \tag{3.3.5}$$

If furthermore the Laplace transform of the function

$$t \mapsto \int_{-\infty}^{0} \left(\int_{0}^{t} \xi_x(t-u) g(u-s)\, du \right)^2 ds$$

exists, then

$$Y(t, x) = \int_{-\infty}^{0} \left(g(x+t-s) - \int_{0}^{t} \xi_x(t-u) g(u-s)\, du \right) \sigma(s-)\, dL(s) + \int_{0}^{t} \xi_x(t-s) Y(s)\, ds\,.$$

Proof. Consider the second term in (3.3.4), and recall that the Laplace transform of the convolution product is equal to the product of Laplace transforms. Hence,

$$\mathcal{L}\xi_x(\theta)\int_0^\infty \mathrm{e}^{-\theta t}Y(t)\,dt = \mathcal{L}(\int_0^\cdot \xi_x(\cdot - s)Y(s)\,ds)(\theta)\,.$$

This proves the second term in the representation of $Y(t,x)$.

As for the first term, note that by applying Cauchy-Schwartz' inequality

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-\theta t}\mathbb{E}\left[|\int_{-\infty}^0 g(x+t-s)\sigma(s-)\,dL(s)|\right]\,dt \\
&\quad\leq \int_0^\infty \mathrm{e}^{-\theta t}\mathbb{E}\left[(\int_{-\infty}^0 g(x+t-s)\sigma(s-)\,dL(s))^2\right]^{1/2}\,dt \\
&\quad= \mathbb{E}[L^2(1)]^{1/2}\mathbb{E}[\sigma^2(s)]^{1/2}\int_0^\infty \mathrm{e}^{-\theta t}(\int_{-\infty}^0 g^2(x+t-s)\,ds)^{1/2}\,dt\,.
\end{aligned}$$

But

$$\int_{-\infty}^0 g^2(x+t-s)\,ds = \int_{t+x}^\infty g^2(u)\,du \leq \int_0^\infty g^2(u)\,du < \infty\,,$$

which shows that the Laplace transform of the stochastic process $t\mapsto\int_{-\infty}^0 g(x+t-s)\sigma(s-)\,dL(s)$ exists, and by the stochastic Fubini's theorem (see Protter [41]),

$$\mathcal{L}(\int_{-\infty}^0 g_{x-s}(\cdot)\sigma(s-)\,dL(s))(\theta) = \int_{-\infty}^0 \mathcal{L}g_{x-s}(\theta)\sigma(s-)\,dL(s)\,.$$

A similar argument shows that, after appealing to the Cauchy-Schwartz' inequality

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-\theta t}\mathbb{E}\left[|\int_{-\infty}^0\int_0^t \xi_x(t-u)g(u-s)\,du\sigma(s-)\,dL(s)|\right]\,dt \\
&\quad\leq \mathbb{E}[L^2(1)]^{1/2}\mathbb{E}[\sigma^2(s)]^{1/2}\int_0^\infty \mathrm{e}^{-\theta t}(\int_{-\infty}^0(\int_0^t \xi_x(t-u)g(u-s)\,du)^2\,ds)^{1/2}\,dt \\
&\quad\leq \mathbb{E}[L^2(1)]^{1/2}\mathbb{E}[\sigma^2(s)]^{1/2}(\int_0^\infty \mathrm{e}^{-\theta t}\,dt)^{1/2}(\int_0^\infty \mathrm{e}^{-\theta t}\int_{-\infty}^0(\int_0^t \xi_x(t-u)g(u-s)\,du)^2\,ds\,dt)^{1/2}\,.
\end{aligned}$$

But by assumption this is finite, and we have

$$\mathcal{L}(\int_{-\infty}^0\int_0^\cdot \xi_x(\cdot-u)g(u-s)\,du\sigma(s-)\,dL(s))(\theta) = \int_{-\infty}^0 \mathcal{L}\xi_x(\theta)\mathcal{L}g_{-s}(\theta)\sigma(s-)\,dL(s)\,.$$

The Proposition follows. □

From the above consideration we reach the result that $Y(t,x)$ can be represented in terms of convolution between the path of Y and some function ξ_x, as well as a stochastic integral ranging over the negative time axis. In passing, we remark that the integrability condition on the convolution product $\int_0^t \xi_x(t-u)g(u-s)\,du$ in the Proposition is only sufficient, and may be sharpened. Recalling that $Y(t)$ is the logarithmic deseasonalized spot price, and $Y(t,x)$ in (3.3.1) is connected to the forward price as in Proposition 3.2.1, we find that the forward price $f(t,T)$ is a functional of the path of the spot price up to current time. In fact, we find the following Corollary:

Corollary 3.3.5. *Under the assumptions of Proposition 3.3.4, it holds that*

$$\begin{aligned}\ln f(t,T) = \ln \Lambda(T) - \int_0^t \xi_{T-t}(t-s)\ln\Lambda(s)\,ds + \int_0^t \xi_{T-t}(t-s)\ln S(s)\,ds \\ + \ln \mathbb{E}\left[\exp\left(\int_t^T \phi_L(g(T-s)\sigma(s))\,ds\right) |\mathcal{F}_t\right] \\ + \int_{-\infty}^0 \left(g(T-s) - \int_0^t \xi_{T-t}(t-u)g(u-s)\,du\right)\sigma(s-)\,dL(s)\,.\end{aligned}$$

Proof. The result follows by combining Propositions 3.2.1 and 3.3.4. □

The convolution can be regarded as a weighted average of the historical spot price up to the current time t, with ξ_x as the *weight function*. We see that the weight function will depend on x, time to maturity of the forward. If we restrict our attention to the constant volatility case $\sigma(t)=1$, then from Corollary 3.2.2 we obtain the simple relationship

$$\begin{aligned}\ln f(t,T) = \ln\Lambda(T) - \int_0^t \xi_{T-t}(t-s)\ln\Lambda(s)\,ds + \int_0^t \xi_{T-t}(t-s)\ln S(s)\,ds + \int_0^{T-t}\phi_L(g(s))\,ds \\ + \int_{-\infty}^0 \left(g(T-s) - \int_0^t \xi_{T-t}(t-u)g(u-s)\,du\right)dL(s)\,.\end{aligned}$$

In the next Section, we shall analyse some particular cases of g which allow for an explicit expression for the function $\xi_x(t)$.

We may separate the definition of $Y(t,x)$ into two parts,

$$Y(t,x) = Y_0(t,x) + Y_+(t,x)\,, \tag{3.3.6}$$

where

$$Y_0(t,x) = \int_{-\infty}^0 g(t+x-s)\sigma(s-)\,dL(s)\,, \tag{3.3.7}$$

and

$$Y_+(t,x) = \int_0^t g(t+x-s)\sigma(s-)\,dL(s)\,. \tag{3.3.8}$$

The variance of $Y_0(t,x)$ tends to zero with an increasing t since

$$\text{Var}(Y_0(t,x)) = \text{Var}(\sigma(s))\text{Var}(L(1)) \int_t^\infty g^2(x+s)\, ds \,.$$

We refer to Benth and Eyjolfsson [11] for more on this limiting behavior. Hence, $Y(t,x) \approx Y_+(t,x)$, and from the considerations above we can approximately have the representation

$$Y(t,x) \approx \int_0^t \xi_x(t-s)Y(s)\, ds \tag{3.3.9}$$

This simplifies the relationship between the forward and spot, as we get rid of the stochastic integral term over the negative times in Proposition 3.3.4

$$\int_{-\infty}^0 \left(g(T-s) - \int_0^t \xi_{T-t}(t-u)g(u-s)\, du \right) \sigma(s-)\, dL(s) \,.$$

We also see that this factor is $\mathcal{F}_0$-measurable, and therefore observable at time zero. Hence, starting our price dynamics at time zero, we "know" this term by observation of the history of the spot up to time zero.

Next, we move on to analyse the case of a stochastic volatility with dynamics (3.2.7) and $L = B$, that is, the relationship between the spot and forward price when the latter is given as in Corollary 3.2.4.

To this end, introduce the function

$$G_x(t) = \int_0^x g^2(x-v)h(t+v)\, dv \,, \tag{3.3.10}$$

for $t, x \geq 0$, and note

$$\Sigma(t,x) = \int_{-\infty}^t G_x(t-s)\, dU(s) \,.$$

As U is a subordinator and G_x obviously is non-negative, $\Sigma(t,x) \geq 0$ for all $t, x \geq 0$. Following the same line of arguments as in the proof of Proposition 3.3.3, we find

$$\int_0^\infty \mathrm{e}^{-\theta t}\Sigma(t,x)\, dt = \int_{-\infty}^0 \mathcal{L}G_{x,-s}(\theta)\, dU(s) + \mathcal{L}G_x(\theta) \int_0^\infty \mathrm{e}^{-\theta s}\, dU(s) \,. \tag{3.3.11}$$

Here, $G_{x,-s}(t) = G_x(t-s)$. We recall that U has finite exponential moments, and therefore in particular it holds that

$$\mathbb{E}\left[(\int_0^\infty \mathrm{e}^{-\theta s}\, dU(s))^2 \right] = \mathbb{E}[U^2(1)] \int_0^\infty \mathrm{e}^{-2\theta s}\, ds = \frac{1}{2\theta}\mathbb{E}[U^2(1)] < \infty \,.$$

Hence, $\int_0^\infty \exp(-\theta s)\, dU(s)$ exists $a.s.$, and the Laplace transform of $\Sigma(t,x)$ is well-defined. Similarly, it holds that

$$\int_0^\infty \mathrm{e}^{-\theta t}\sigma^2(t)\, dt = \int_{-\infty}^0 \mathcal{L}h_{-s}(\theta)\, dU(s) + \mathcal{L}h(\theta) \int_0^\infty \mathrm{e}^{-\theta s}\, dU(s) \,. \tag{3.3.12}$$

By combining the two expressions (3.3.11) and (3.3.12), we conclude that the Laplace transform of $\Sigma(t,x)$ can be expressed in terms of the Laplace transform of $\sigma^2(t)$ by

$$\int_0^\infty \mathrm{e}^{-\theta t}\Sigma(t,x)\,dt \tag{3.3.13}$$

$$= \int_{-\infty}^0 \left(\mathcal{L}G_{x,-s}(\theta) - \frac{\mathcal{L}G_x(\theta)}{\mathcal{L}h(\theta)}\mathcal{L}h_{-s}(\theta)\right)\,dU(s) + \frac{\mathcal{L}G_x(\theta)}{\mathcal{L}h(\theta)}\int_0^\infty \mathrm{e}^{-\theta t}\sigma^2(t)\,dt\,. \tag{3.3.14}$$

The relationship between $\Sigma(t,x)$ and $\sigma^2(t)$ has the same form as that for $Y(t,x)$ and $Y(t)$, although the ratio $\mathcal{L}G_x(\theta)/\mathcal{L}h(\theta)$ is more complex.

Let us investigate this ratio in more detail. A direct computation using Tonelli's theorem reveals that

$$\begin{aligned}\mathcal{L}G_x(\theta) &= \int_0^\infty \mathrm{e}^{-\theta t}\int_0^x g^2(x-v)h(t+v)\,dv\,dt\\ &= \int_0^x g^2(x-v)\int_0^\infty \mathrm{e}^{-\theta t}h(t+v)\,dt\,dv\\ &= \int_0^x g^2(x-v)\mathcal{L}h_v(\theta)\,dv\,.\end{aligned}$$

But then,

$$\frac{\mathcal{L}G_x(\theta)}{\mathcal{L}h(\theta)} = \int_0^x g^2(x-v)\frac{\mathcal{L}h_v(\theta)}{\mathcal{L}h(\theta)}\,dv\,.$$

From this we find the analogous result to Proposition 3.3.4 for the stochastic volatility case:

Proposition 3.3.6. *Suppose for $v \geq 0$ there exists a function $t \mapsto \eta_v(t)\,, t \geq 0$ such that its Laplace transform is well-defined and*

$$\mathcal{L}\eta_v(\theta) = \frac{\mathcal{L}h_v(\theta)}{\mathcal{L}h(\theta)}\,.$$

If furthermore the Laplace transform of the functions

$$t \mapsto \int_{-\infty}^0 \left(\int_0^t\int_0^x g^2(x-v)\eta_v(t-u)\,dvh(u-s)\,du\right)^2 ds$$

and

$$t \mapsto \int_{-\infty}^0 G_x^2(t-s)\,ds$$

exist, then

$$\begin{aligned}\Sigma(t,x) = &\int_{-\infty}^0 \left(G_x(t-s) - \int_0^t\int_0^x g^2(x-v)\eta_v(t-u)\,dvh(u-s)\,du\right)\,dU(s)\\ &+ \int_0^t\int_0^x g^2(x-v)\eta_v(t-s)\,dv\sigma^2(s)\,ds\,.\end{aligned}$$

Proof. It holds that

$$\frac{\mathcal{L}G_x(\theta)}{\mathcal{L}h(\theta)} = \int_0^x g^2(x-v)\mathcal{L}\eta_v(\theta)\,dv = \int_0^\infty \mathrm{e}^{-\theta t}\left\{\int_0^x g^2(x-v)\eta_v(t)\,dv\right\}\,dt\,.$$

Hence, by following the same line of arguments as in the proof of Proposition 3.3.4 we reach the desired result. □

The *weight* function η_v is associated to the kernel function h in the LSS dynamics of the stochastic volatility as ξ_x to the kernel function g of the LSS process Y. In the next Section we investigate some concrete specifications of the kernel function g, but notice that the analysis could be done for h in a completely analogous manner.

From Corollary 3.2.4 we can now express the forward price in terms of the weighted average spot and volatility. Applying Propositions 3.3.4 and 3.3.6, we derive the relationship,

$$\begin{aligned}\ln f(t,T) &= \ln\Lambda(T) - \int_0^t \xi_{T-t}(t-s)\ln\Lambda(s)\,ds + \int_0^t \xi_{T-t}(t-s)\ln S(s)\,ds \\ &+ \frac{1}{2}\int_0^t\int_0^{T-t} g^2(T-t-v)\eta_v(t-s)\,dv\,\sigma^2(s)\,ds \\ &+ \int_t^T \phi_U\left(\frac{1}{2}\int_u^T g^2(T-s)h(s-u)\,ds\right)\,du \\ &+ \int_{-\infty}^0\left(g(T-s) - \int_0^t \xi_{T-t}(t-u)g(u-s)\,du\right)\,\sigma(s-)\,dB(s) \\ &+ \frac{1}{2}\int_{-\infty}^0\left(G_{T-t}(t-s) - \int_0^t\int_0^{T-t} g^2(T-t-v)\eta_v(t-u)\,dv h(u-s)\,du\right)\,dU(s)\,.\end{aligned}$$

The last two terms are $\mathcal{F}_0$-measurable. If we argue as above and approximate them by zero when t is large, then the forward price is given as weighted averages of the spot and the stochastic volatility, as well as some seasonal adjustment factors.

Before closing this Section, let us consider the so-called Barndorff-Nielsen and Shephard (BNS) stochastic volatility model (see Barndorff-Nielsen and Shephard [7]). In this case, $\sigma^2(t)$ is an Ornstein-Uhlenbeck process, defined as

$$\sigma^2(t) = \int_{-\infty}^t \mathrm{e}^{-\lambda(t-s)}\,dU(s)\,, \tag{3.3.15}$$

for a constant $\lambda > 0$. From the definition of $\Sigma(t,x)$, we find

$$\begin{aligned}\Sigma(t,x) &= \int_{-\infty}^t\int_0^x g^2(x-v)\mathrm{e}^{-\lambda(t-s+v)}\,dv\,dU(s) \\ &= \int_{-\infty}^t \mathrm{e}^{-\lambda(t-s)}\int_0^x g^2(x-v)\mathrm{e}^{-\lambda v}\,dv\,dU(s)\end{aligned}$$

$$= \int_0^x g^2(x-v)\mathrm{e}^{-\lambda v}\,dv\,\sigma^2(t)\,.$$

Hence, it is not necessary to use the Laplace transform in this situation. We notice that the factor in front of $\sigma^2(t)$ is the convolution between the square of the kernel function of the LSS process Y and the kernel function of $\sigma^2(t)$, evaluated at x, the time of maturity.

3.4 Particular cases of LSS processes

In this Section we study some concrete specifications of g that lead to the existence of a weight function $\xi_x(t)$ defined in Proposition 3.3.4. We look at the cases of g coming from a CARMA or a Gamma-LSS process, which both have been applied to energy price modelling (see Garcia *et al.* [33] and Barndorff-Nielsen *et al.* [6] for an application of the respective models).

3.4.1 The CARMA case

For $Y(t)$ in (3.2.2), let us consider the kernel function g given as

$$g(t) = \mathbf{b}^{\mathsf{T}}\mathrm{e}^{At}\mathbf{e}_p\,, \tag{3.4.1}$$

for $t \in \mathbb{R}_+$. Here, $\mathbf{e}_p$ is the pth canonical basis vector in $\mathbb{R}^p$ for p being a natural number. Furthermore, A is a $p \times p$ matrix defined by

$$A = \begin{pmatrix} \mathbf{0}_{p-1} & I_{p-1} \\ -\alpha_p .. & .. - \alpha_1 \end{pmatrix} \tag{3.4.2}$$

with $\mathbf{0}_{p-1}$ being the $p-1$-dimensional vector of zeros and I_{p-1} is the identity matrix of dimension $p-1$. The constants $\alpha_1, \ldots, \alpha_p$ are supposed to be positive. Moreover, we assume that the eigenvalues of A all have negative real part. Finally, for $q \in \mathbb{N}_0$, where $q < p$ and $\mathbb{N}_0$ are the natural numbers including zero, we assume $\mathbf{b}$ is the p-dimensional vector

$$\mathbf{b}^{\mathsf{T}} = \begin{pmatrix} b_0 & b_1 & \ldots & b_{q-1} & 1 & 0 & \ldots & 0 \end{pmatrix}. \tag{3.4.3}$$

We use the notation $\mathbf{x}^{\mathsf{T}}$ for transposition of a matrix or vector $\mathbf{x}$.

With the above choice, we say that

$$Y(t) = \int_{-\infty}^t \mathbf{b}^{\mathsf{T}}\mathrm{e}^{A(t-s)}\mathbf{e}_p\sigma(s-)\,dL(s)$$

is a CARMA(p, q)-process. We note from the assumption on the eigenvalues of A that g becomes a bounded continuous function decaying exponentially as time tends to infinity, and therefore the process $Y(t)$ is well-defined. We refer to Brockwell [25] for a detailed study of CARMA(p, q)-processes.

As it turns out, the relationship between $Y(t)$ and $Y(t, x)$ in the case of a CARMA(p, q)-process involves taking derivatives of the former. For this to be valid, we of course need to have certain differentiability properties of the paths of Y to hold.

Proposition 3.4.1. *Let Y be a CARMA(p,q)-process with $p \geq 2$ and $p-q-1 \geq 1$. Then $t \mapsto Y(t)$ is $p-q-1$ times differentiable, with the n'th derivative given by*

$$Y^{(n)}(t) = \int_{-\infty}^{t} \mathbf{b}^{\mathrm{T}} A^n e^{A(t-s)} \mathbf{e}_p \sigma(s-)\, dL(s)\,,$$

for $n = 0, \ldots, p-q-1$.

Proof. From Proposition 3.2 of Benth and Eyjolfsson [11] we find that an LSS process Y as defined in (3.2.2) with absolutely continuous kernel function g with a derivative g' almost everywhere satisfying $\int_0^\infty |g'(s)|^2\, ds < \infty$ and $|g(0)| < \infty$ has the representation

$$dY(t) = \int_{-\infty}^{t} g'(t-s)\sigma(s-)\, dL(s)\, dt + g(0)\sigma(t-)\, dL(t)\,, \tag{3.4.4}$$

for $t \geq 0$. For the case of a CARMA(p,q)-process, we observe that $g(t) = \mathbf{b}^{\mathrm{T}}\exp(At)\mathbf{e}_p$ is continuously differentiable with

$$g'(t) = \mathbf{b}^{\mathrm{T}} A e^{At} \mathbf{e}_p. \tag{3.4.5}$$

The fact that the eigenvalues of A have negative real part allows us to bound (3.4.5) as

$$|g'(t)| \leq C e^{\mathrm{Re}(\lambda_1)t}\,, \tag{3.4.6}$$

where C is a constant and λ_1 is the eigenvalue with the smallest absolute value of the real part. This bound for $g'(t)$ ensures that $\int_0^\infty g'(s)^2\, ds < \infty$. Hence, since $g(0) = \mathbf{b}^{\mathrm{T}}\mathbf{e}_p = 0$ due to the fact that the last coordinate of $\mathbf{b}$ is zero, we have that $Y(t)$ is differentiable, with

$$Y'(t) = \int_{-\infty}^{t} \mathbf{b}^{\mathrm{T}} A e^{A(t-s)} \mathbf{e}_p \sigma(s-)\, dL(s).$$

The Proposition holds for $n = 1$.

We iterate inductively on the expression (3.4.4) to find the n'th derivative of Y. Note that the n'th derivative of g is

$$g^{(n)}(t) = \mathbf{b}^{\mathrm{T}} A^n e^{At} \mathbf{e}_p\,,$$

for which we have the bound

$$|g^{(n)}(t)| \leq C_n e^{\mathrm{Re}(\lambda_1)t},$$

ensuring integrability. Here, C_n is a constant which depends on the order of derivative of g. To determine the order of differentiability of Y, we must check for which values of n we have $g^{(n)}(0) = 0$.

Obviously,

$$g^{(n)}(0) = \mathbf{b}^{\mathrm{T}} A^n \mathbf{e}_p\,.$$

The vector $\mathbf{b}^{\mathrm{T}}$ has the last $p-q-1$ coordinates being zero, and be repeated application of the matrix A we see that for $n = 0, \ldots, p-q-2$, the row vector $\mathbf{b}^{\mathrm{T}} A^n$ becomes

$$\mathbf{b}^{\mathrm{T}} A^n = \begin{pmatrix} \overbrace{0 \quad \cdots \quad 0}^{n} & b_0 & \cdots & b_{q-1} & 1 & \overbrace{0 \quad \cdots \quad 0}^{p-q-1-n} \end{pmatrix}.$$

Hence, $\mathbf{b}^{\mathsf{T}}A^n$ is n forward shifts of the non-zero elements of $\mathbf{b}^{\mathsf{T}}$. Moreover, this yields that $g^{(n)}(0) = \mathbf{b}^{\mathsf{T}}A^n\mathbf{e}_p = 0$ for $n \leq p-q-2$, but $g^{(p-q-1)}(0) \neq 0$. Thus, the Proposition follows. □

With this smoothness result at hand, we now move on with an analysis of the weight function $\xi_x(t)$ defined in Proposition 3.3.4, or, in other words, an analysis of the ratio $\mathcal{L}g_x(\theta)/\mathcal{L}g(\theta)$.

A direct calculation reveals the following result:

Proposition 3.4.2. *Let g be as in* (3.4.1). *Then,*

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)} = \frac{\sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_m\theta^{m-1}}{\sum_{m=1}^{p} b_{m-1}\theta^{m-1}}\,.$$

Proof. We have that

$$\begin{aligned}
\mathcal{L}g_x(\theta) &= \int_0^\infty g(u+x)\mathrm{e}^{-\theta u}\,du \\
&= \int_0^\infty \mathbf{b}^{\mathsf{T}}\mathrm{e}^{A(x+u)}\mathbf{e}_p e^{-\theta u}\,du \\
&= \mathbf{b}^{\mathsf{T}}\mathrm{e}^{Ax}\int_0^\infty \mathrm{e}^{(A-\theta I_p)u}\,du\mathbf{e}_p \\
&= \mathbf{b}^{\mathsf{T}}\mathrm{e}^{Ax}(\theta I_p - A)^{-1}\mathbf{e}_p\,.
\end{aligned}$$

Letting $x = 0$, we get $\mathcal{L}g(\theta) = \mathbf{b}^{\mathsf{T}}(\theta I - A)^{-1}\mathbf{e}_p$. Note that $(\theta I_p - A)^{-1}\mathbf{e}_p$ is equal to the pth column of $(\theta I_p - A)^{-1}$, and thus, we do not need to know the whole inverse of $(\theta I_p - A)^{-1}$ but only the last column of it. Let C denote the inverse of $(\theta I_p - A)$, then we can get the pth column of C if we solve the linear system

$$(\theta I - A)\mathbf{c}_p = \mathbf{e}_p\,,$$

where $\mathbf{c}_p^{\mathsf{T}} = (c_{1p}, c_{2p}, \ldots, c_{pp})$ is the pth column vector of C. By the structure of $\theta I_p - A$ we find

$$\begin{aligned}
c_{1p} &= (\theta^p + \sum_{k=1}^{p}\alpha_k\theta^{p-k})^{-1}\,, \\
c_{kp} &= c_{1p}\theta^{k-1}, \quad k = 2, \ldots, p\,,
\end{aligned}$$

and we can rewrite the pth column of $(\theta I_p - A)^{-1}$ as follows:

$$\begin{aligned}
(\theta I_p - A)^{-1}\mathbf{e}_p &= (\theta^p + \sum_{k=1}^{p}\alpha_k\theta^{p-k})^{-1}\mathbf{e}_1 + \sum_{m=2}^{p}\theta^{m-1}(\theta^p + \sum_{k=1}^{p}\alpha_k\theta^{p-k})^{-1}\mathbf{e}_m \\
&= (\theta^p + \sum_{k=1}^{p}\alpha_k\theta^{p-k})^{-1}(\mathbf{e}_1 + \sum_{m=2}^{p}\theta^{m-1}\mathbf{e}_m) \\
&= (\theta^p + \sum_{k=1}^{p}\alpha_k\theta^{p-k})^{-1}\sum_{m=1}^{p}\theta^{m-1}\mathbf{e}_m\,.
\end{aligned}$$

Finally, since $\mathbf{b}^{\mathsf{T}}\mathbf{e}_m = b_{m-1}$ for all $m = 1, \ldots, p$, we obtain

$$\begin{aligned}\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)} &= \frac{(\theta^p + \sum_{k=1}^p \alpha_k \theta^{p-k})^{-1} \sum_{m=1}^p \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \theta^{m-1}}{(\theta^p + \sum_{k=1}^p \alpha_k \theta^{p-k})^{-1} \sum_{m=1}^p \mathbf{b}^{\mathsf{T}} \mathbf{e}_m \theta^{m-1}} \\ &= \frac{\sum_{m=1}^p \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \theta^{m-1}}{\sum_{m=1}^p b_{m-1} \theta^{m-1}} .\end{aligned}$$

This concludes the proof. □

As a first specific case, let us consider the situation when $q = 0$ which means that $\mathbf{b} = \mathbf{e}_1$ in the definition of g in (3.4.1). For this choice of $\mathbf{b}$, we say that Y is a CAR(p)-process (continuous-time autoregressive process of order p). From Proposition 3.4.2, we find

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)} = \sum_{m=1}^p \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \theta^{m-1}.$$

Hence, from (3.3.4) and the Laplace transform of higher-order derivatives of a function,

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-\theta t} Y(t,x)\,dt \\ &\quad= \sum_{m=1}^p \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \theta^{m-1} \int_0^\infty \mathrm{e}^{-\theta t} Y(t)\,dt \\ &\qquad+ \int_{-\infty}^0 \left(\mathcal{L}g_{x-s}(\theta) - \sum_{m=1}^p \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_p \theta^{m-1} \mathcal{L}g_{-s}(\theta) \right) \sigma(s-)\,dL(s) \\ &\quad= \sum_{m=1}^p \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \left\{ \mathcal{L}Y^{(m-1)}(\theta) + \sum_{k=0}^{m-2} \theta^k Y^{(m-2-k)}(0) \right\} \\ &\qquad+ \int_{-\infty}^0 \left(\mathcal{L}g_{x-s}(\theta) - \sum_{m=1}^p \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \left\{ \mathcal{L}g_{-s}^{(m-1)}(\theta) + \sum_{k=0}^{m-2} \theta^k g_{-s}^{(m-2-k)}(0) \right\} \right) \sigma(s-)\,dL(s) .\end{aligned}$$

Hence we use the convention that an empty sum is equal to zero. We recall from Proposition 3.4.1 that the CAR(p)-process Y is $p-1$ times differentiable for $p \geq 2$, exactly the regularity we need in the above derivation. From the same Proposition it holds furthermore that

$$Y^{(i)}(0) = \int_{-\infty}^0 g^{(i)}(-s)\sigma(s-)\,dL(s) = \int_{-\infty}^0 g_{-s}^{(i)}(0)\sigma(s-)\,dL(s) ,$$

for $i = 0, \ldots, p-1$. Hence,

$$\int_0^\infty \mathrm{e}^{-\theta t} Y(t,x)\,dt = \sum_{m=1}^p \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \mathcal{L}Y^{(m-1)}(\theta)$$

$$+\int_{-\infty}^{0}\left(\mathcal{L}g_{x-s}(\theta)-\sum_{m=1}^{p}\mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_m\mathcal{L}g_{-s}^{(m-1)}(\theta)\right)\sigma(s-)\,dL(s)\,.$$

The above stochastic integral vanishes as $g_{x-s}(t)=\sum_{m=1}^{p}\mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_m g_{-s}^{(m-1)}(t)$. To see this, note that we can express $\mathbf{e}_1^{\mathsf{T}}\exp(Ax)$ in the basis vectors of $\mathbb{R}^p$ as

$$\mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{Ax}=\sum_{m=1}^{p}(\mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_m)\mathbf{e}_m^{\mathsf{T}}\,,$$

to find

$$g_{x-s}(t)=g(t-s+x)=\mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{Ax}\mathrm{e}^{A(t-s)}\mathbf{e}_p=\sum_{m=1}^{p}(\mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_m)(\mathbf{e}_m^{\mathsf{T}}\mathrm{e}^{A(t-s)}\mathbf{e}_p)\,.$$

On the other hand, by an induction argument it holds that

$$g^{(k)}(t)=\mathbf{e}_{k+1}^{\mathsf{T}}\mathrm{e}^{At}\mathbf{e}_p$$

for $k=1,\ldots,p-1$ since $\mathbf{e}_k^{\mathsf{T}}A=\mathbf{e}_{k+1}^{\mathsf{T}}$. Hence, we conclude that for the case of a CAR(p)-process the relationship between $Y(t,x)$ and $Y(t)$ in (3.3.1) and (3.2.2), respectively, reduces to

$$Y(t,x)=\sum_{m=1}^{p}\mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_m\,Y^{(m-1)}(t)\,. \tag{3.4.7}$$

The forward price will depend on the current spot and its derivatives up to an order one less than the autoregression order. This result was derived by other methods in Benth and Solanilla Blanco [17].

So far, we have not made explicit the form of the weight function $\xi_x(t)$ for the CAR(p) case. If $\delta_0(t)$ is the Dirac-δ function at zero, it holds that $\mathcal{L}\delta_0(\theta)=1$. Moreover, by the concept of weak derivatives,

$$\mathcal{L}\delta_0^{(n)}(\theta)=\theta^n\,.$$

Hence, we can represent $\xi_x(t)$ as the "function"

$$\xi_x(t)=\sum_{m=1}^{p}\mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_m\,\delta_0^{(m-1)}(t)$$

Of course, the Dirac-δ function does not make sense as a classical function, so this representation must be interpreted as a sum of generalized functions (that is, linear functionals on a space of functions).

As our next case we consider a CARMA(p, 1)-process $Y(t)$ for $p\geq 2$. The vector $\mathbf{b}\in\mathbb{R}^p$ in the definition of the kernel function g is then $\mathbf{b}^{\mathsf{T}}=(b_0\quad 1\quad 0\quad \ldots\quad 0)$. Garcia *et al.* [33] proposed a linear CARMA(2,1) model for power prices observed at the Singapore New Electricity Market driven by a stable Lévy process. Later, Benth *et al.* [12] applied a two-factor model for

EEX power prices, where one of the factors was assumed to be a CARMA(2,1)-process, while Barndorff-Nielsen *et al.* [6] also found evidence for a CARMA(2,1)-kernel in an LSS model of EEX spot prices.

From Proposition 3.4.2 we have

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)} = \sum_{m=1}^{p} \frac{\mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \theta^{m-1}}{b_0 + \theta}\,. \tag{3.4.8}$$

Again like in the situation of a CAR(p)-process, we find the Laplace transform of the weight function being a polynomial of θ of order $p-1$. However, the moving average part contributes with a factor $1/(b_0+\theta)$. If $b_0 > 0$, then it is simple to see that the Laplace transform of the function $\exp(-b_0 t)$ is $1/(b_0+\theta)$ for all $\theta > 0$. On the other hand, if $b_0 < 0$, then the Laplace transform of $\exp(-b_0 t)$ is only defined for $\theta > -b_0$. At the first glance, one could be tempted to compute as in the CAR(p)-case above, taking into account that the first factor $1/(b_0+\theta)$ comes in as a convolution. However, such a computation would require $p-1$-differentiability of Y, which does not hold in the situation of a CARMA($p,1$)-process. From Proposition 3.4.1 we know that Y is only $p-2$ times differentiable.

Introduce the notation $f(t) = \exp(-b_0 t)$. Then,

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)} = \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \theta^{m-1} \mathcal{L}f(\theta)\,.$$

Again we apply the general expression in (3.3.4) in order to analyse the Laplace transform of $Y(t,x)$ in terms of $Y(t)$. First, observe that

$$\begin{aligned}\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)}\mathcal{L}Y(\theta) &= \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \theta^{m-1} \mathcal{L}(f \star Y)(\theta) \\ &= \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \mathcal{L}((f \star Y)^{(m-1)})(\theta) \\ &\quad + \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \sum_{k=1}^{m-1} \theta^{k-1} (f \star Y)^{(m-1-k)}(0)\,.\end{aligned}$$

In the second equality we have applied the rules of Laplace transform and differentiation, while $\star$ means the convolution product defined as

$$(f \star Y)(t) = \int_0^t f(t-s) Y(s)\, ds\,. \tag{3.4.9}$$

By repeated differentiation, it holds for $m \in \mathbb{N}$ with $m \leq p$ that

$$(f \star Y)^{(m-1)}(t) = (f^{(m-1)} \star Y)(t) + \sum_{i=0}^{m-2} f^{(m-2-i)}(0) Y^{(i)}(t)\,. \tag{3.4.10}$$

Here, we will only differentiate Y at the most $p-2$ times. Thus, convoluting with f is decreasing the order of differentiation of Y from $p-1$ in the CAR(p) case, to $p-2$ in the CARMA($p,1$) case. Since $(f^{(m-1)} \star Y)(0) = 0$, for $m = 1, \ldots, p$ we get

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)}\mathcal{L}Y(\theta) = \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \mathcal{L}((f \star Y)^{(m-1)})(\theta) + \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \sum_{k=1}^{m-1} \theta^{k-1} \sum_{i=0}^{m-k-2} f^{(m-k-2-i)}(0) Y^{(i)}(0) .$$

Performing the same line of arguments for the second term in (3.3.4), we find in particular that

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)}\mathcal{L}g_{-s}(\theta) = \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \mathcal{L}((f \star g_{-s})^{(m-1)})(\theta) + \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \sum_{k=1}^{m-1} \theta^{k-1} \sum_{i=0}^{m-k-2} f^{(m-k-2-i)}(0) g_{-s}^{(i)}(0) .$$

From Proposition 3.4.1

$$Y^{(i)}(0) = \int_{-\infty}^{0} g^{(i)}(-s)\sigma(s-)\, dL(s) = \int_{-\infty}^{0} g_{-s}^{(i)}(0)\sigma(s-)\, dL(s) .$$

After simple algebra in (3.3.4), where all terms involving θ cancel, we end up with the expression

$$\int_0^\infty \mathrm{e}^{-\theta t} Y(t,x)\, dt = \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \mathcal{L}((f \star Y)^{(m-1)})(\theta) + \int_{-\infty}^{0} \left(\mathcal{L}g_{x-s}(\theta) - \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \mathcal{L}((f \star g_{-s})^{(m-1)})(\theta) \right) \sigma(s-)\, dL(s) .$$

After Laplace inversion, we conclude the following:

Proposition 3.4.3. *With g being a CARMA($p,1$) kernel for $p \geq 2$, it holds that*

$$Y(t,x) = \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m (f \star Y)^{(m-1)}(t) + \int_{-\infty}^{0} \left(g(t-s+x) - \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m (f \star g_{-s})^{(m-1)}(t) \right) \sigma(s-)\, dL(s) ,$$

where $f(t) = \exp(-b_0 t)$, b_0 being the first coordinate of the vector $\mathbf{b}$.

Let us inspect the integrand in the last term of $Y(t,x)$ in Proposition 3.4.3. By the definition of f, we find that $f^{(n)}(t) = (-b_0)^n f(t)$ for any $n \in \mathbb{N}$. Therefore we find from the same relationship as (3.4.10) with g_{-s} instead of Y that

$$(f \star g_{-s})^{(m-1)}(t) = (-b_0)^{m-1}(f \star g_{-s})(t) + \sum_{i=0}^{m-2} (-b_0)^{m-2-i} g_{-s}^{(i)}(t) .$$

A direct differentiation of g reveals that

$$g^{(i)}_{-s}(t) = b_0 \mathbf{e}_1^{\mathsf{T}} A^i \mathrm{e}^{A(t-s)} \mathbf{e}_p + \mathbf{e}_2^{\mathsf{T}} A^i \mathrm{e}^{A(t-s)} \mathbf{e}_p \,,$$

for $i = 0, \ldots, p-2$. By repeated application of the matrix A, we see that $\mathbf{e}_1^{\mathsf{T}} A^i = \mathbf{e}_{i+1}^{\mathsf{T}}$, and similarly $\mathbf{e}_2^{\mathsf{T}} A^i = \mathbf{e}_{i+2}^{\mathsf{T}}$. The convolution product $(f \star g_{-s})(t)$ can be computed explicitly as follows:

$$\begin{aligned}
(f \star g_{-s})(t) &= \int_0^t \mathrm{e}^{-b_0(t-u)} g_{-s}(u)\, du \\
&= \mathrm{e}^{-b_0 t} \int_0^t \mathrm{e}^{b_0 u} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{A(u-s)} \mathbf{e}_p \, du \\
&= \mathrm{e}^{-b_0(t-s)} \mathbf{b}^{\mathsf{T}} \int_{-s}^{t-s} \mathrm{e}^{(b_0 I + A)v}\, dv \mathbf{e}_p \\
&= \mathbf{b}^{\mathsf{T}} (b_0 I + A)^{-1} \left(\mathrm{e}^{At} - \mathrm{e}^{-b_0 I t} \right) \mathrm{e}^{A(-s)} \mathbf{e}_p \,.
\end{aligned}$$

In the third equality we used the fact that $\exp(b_0 t) I = \exp(b_0 I t)$ where I is the $p \times p$ identity matrix. Since,

$$\begin{aligned}
g(t-s+x) &= \mathbf{b}^{\mathsf{T}} \mathrm{e}^{A(t-s+x)} \mathbf{e}_p \\
&= (\mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax}) \mathrm{e}^{A(t-s)} \mathbf{e}_p \\
&= \sum_{m=1}^{p} (\mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m) \mathbf{e}_m^{\mathsf{T}} \mathrm{e}^{A(t-s)} \mathbf{e}_p \,,
\end{aligned}$$

we reach after some manipulations the expression

$$\begin{aligned}
g(t-s+x) - &\sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m (f \star g_{-s})^{(m-1)}(t) \\
&= \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \left\{ \mathbf{e}_m^{\mathsf{T}} \mathrm{e}^{A(t-s)} \mathbf{e}_p - \sum_{i=0}^{m-2} (-b_0)^{m-2-i} \left(b_0 \mathbf{e}_{i+1}^{\mathsf{T}} + \mathbf{e}_{i+2}^{\mathsf{T}} \right) \mathrm{e}^{A(t-s)} \mathbf{e}_p \right\} \\
&\quad - \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m (-b_0)^{m-1} (f \star g_{-s})(t) \,.
\end{aligned}$$

Unfortunately, this term is not canceling to zero like the CAR(p) case. For example, supposing $p = 2$, we find

$$\begin{aligned}
g(t-s+x) - &\sum_{m=1}^{2} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m (f \star g_{-s})^{(m-1)}(t) \\
&= \left(\mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_1 - b_0 \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_2 \right) \left(\mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{A(t-s)} \mathbf{e}_2 - (f \star g_{-s})(t) \right) ,
\end{aligned}$$

which is in general non-zero. Indeed, if $t = 0$ we find $(f \star g_{-s})(0) = 0$ and thus

$$\mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{A(-s)} \mathbf{e}_2 - (f \star g_{-s})(0) = \mathbf{e}_1^{\mathsf{T}} \mathrm{e}^{A(-s)} \mathbf{e}_2 \,,$$

which is non-zero. Furthermore, if $x = 0$ we have that the term in the first parenthesis becomes

$$\mathbf{b}^{\mathsf{T}}\mathbf{e}_1 - b_0\mathbf{b}^{\mathsf{T}}\mathbf{e}_2 = 0\,,$$

while its derivative at zero is

$$\mathbf{b}^{\mathsf{T}}A\mathrm{e}^{Ax}\mathbf{e}_1 - b_0\mathbf{b}^{\mathsf{T}}A\mathrm{e}^{Ax}\mathbf{e}_2|_{x=0} = -\alpha_2 - b_0\alpha_1 - b_0^2 \neq 0\,.$$

Therefore we find that for $x > 0$, the term in the first parenthesis is non-zero as well.

Consider next the first term in the expression for $Y(t,x)$ in Proposition 3.4.3. Using the explicit form of f, we get from (3.4.10),

$$(f \star Y)^{(m-1)}(t) = (-b_0)^{m-1}(f \star Y)(t) + \sum_{i=0}^{m-2}(-b_0)^{m-2-i}Y^{(i)}(t)\,.$$

Hence,

$$\begin{aligned}\sum_{m=1}^{p}\mathbf{b}^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_m(f \star Y)^{(m-1)}(t) &= \sum_{m=1}^{p}\mathbf{b}^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_m\sum_{i=0}^{m-2}(-b_0)^{m-2-i}Y^{(i)}(t)\\ &\quad + (f \star Y)(t)\sum_{m=1}^{p}(-b_0)^{m-1}\mathbf{b}^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_m\,.\end{aligned}$$

Therefore, $Y(t,x)$ depends explicitly on $Y(t), Y'(t), \ldots, Y^{(p-2)}(t)$, as well as on the exponentially weighted average $(f \star Y)(t)$ of $Y(s)$ for $0 \leq s \leq t$, *i.e.*,

$$(f \star Y)(t) = \int_0^t \mathrm{e}^{-b_0(t-s)}Y(s)\,ds\,.$$

The exponential decay rate is equal to the moving average component b_0, which will determine the influence of the past observations of $Y(s), s \leq t$, on the value of $Y(t,x)$.

We end with the complete expression for the case CARMA(2,1), which is the most interesting one in energy markets. Using the expression for $Y(t,x)$ in Proposition 3.4.3 for $p = 2$, and nesting up the different derivations above, we find

$$\begin{aligned}Y(t,x) &= (\mathbf{b}^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_1 - b_0\mathbf{b}^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_2)\int_0^t \mathrm{e}^{-b_0(t-s)}Y(s)\,ds + \mathbf{b}^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_2 Y(t)\\ &\quad + \int_{-\infty}^0 \left(\mathbf{b}^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_1 - b_0\mathbf{b}^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_2\right)\left(\mathbf{e}_1^{\mathsf{T}}\mathrm{e}^{A(t-s)}\mathbf{e}_2 - (f \star g_{-s})(t)\right)\sigma(s-)\,dL(s)\,.\end{aligned}$$

We now focus on the two *shape functions* $f_1(x)$ and $f_2(x)$ defined as

$$f_1(x) = \mathbf{b}^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_1 - b_0 f_2(x)\,, \tag{3.4.11}$$

$$f_2(x) = \mathbf{b}^{\mathsf{T}}\mathrm{e}^{Ax}\mathbf{e}_2\,. \tag{3.4.12}$$

We see that the $Y(t,x)$ depends on $Y(t)$ through the scaling of $f_2(x)$ in the x-direction, while the exponential average is scaled by $f_1(x)$. In terms of forward prices, $f_2(x)$ and $f_1(x)$ will give two *forward curve templates*, which are scaled by the random behavior of the spot and its exponential average, respectivy. We now discuss an empirical example in order to gain more understanding on how these two forward curve template functions look like.

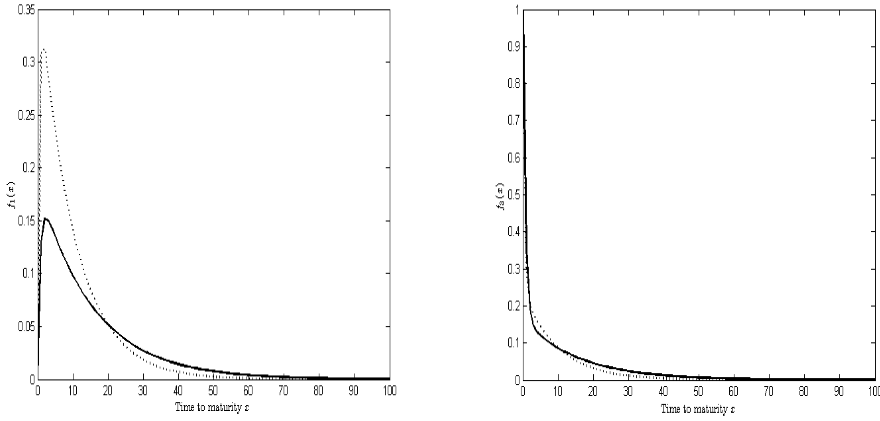

Figure 3.1: The forward curve template function f_1 (left) and f_2 (right) in the case of a CARMA(2,1). Parameters collected from estimation of peak and base load spot prices at EEX performed in Benth *et al.* [12]. Peak template curves are depicted in dotted lines, while base curves are complete lines.

In Benth *et al.* [12] a two-factor spot model is fitted to spot price data observed at the German power exchange EEX. One of these two factors is assumed to be a CARMA(2,1)-process, and we apply the estimated parameters in our empirical study here in order to have accessible reasonable numbers. Admittedly, the study in Benth *et al.* [12] made use of an arithmetic model, so the estimated numbers are not directly linked to a geometric-type specification as we consider. However, as our analysis is just meant for illustration, we ignore this fact. Both peak and base load prices are considered, and Benth *et al.* [12] find the estimates $\alpha_1 = 1.4854(2.3335)$, $\alpha_2 = 0.0911(0.2263)$ and $b_0 = 0.2861(0.6127)$ (peak estimates in paranthesis). In Figure 3.1 we have plotted the two forward curve templates, with the base case as a complete line and the peak case as a dotted line. We see that $f_2(x)$ decays rapidly towards zero for both peak and base cases, starting at 1. This means that we will have a decaying forward curve in terms of $Y(t)$ if the spot price is above its average, while it will be rapidly increasing to zero if it is below its average. We get forward curve shapes which are either contango or backwardation, being the classical situation in forward commodity markets.

However, in addition we get a contribution form the exponential average of the deseasonalized spot price. The forward curve template $f_1(x)$ produces a hump, with a maximum value in the short end of the curve. If the exponential average is large and positive, we get a hump in the forward curve in the short end. With changing signs of the exponential average and the current value of Y we may get a negative hump as well. The hump is much more pronounced for the peak case, having a maximum more than twice as big as the base case. This means a stronger influence from the exponential averaging factor.

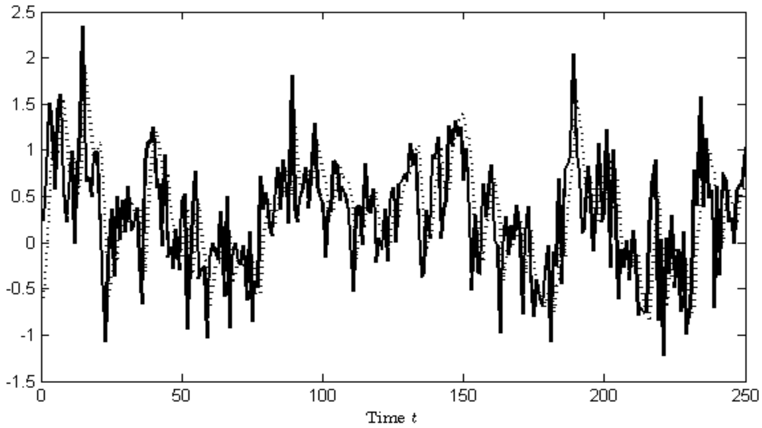

Figure 3.2: Simulation of $Y(t)$ (complete line) and its exponential average (dotted line) for peak parameters.

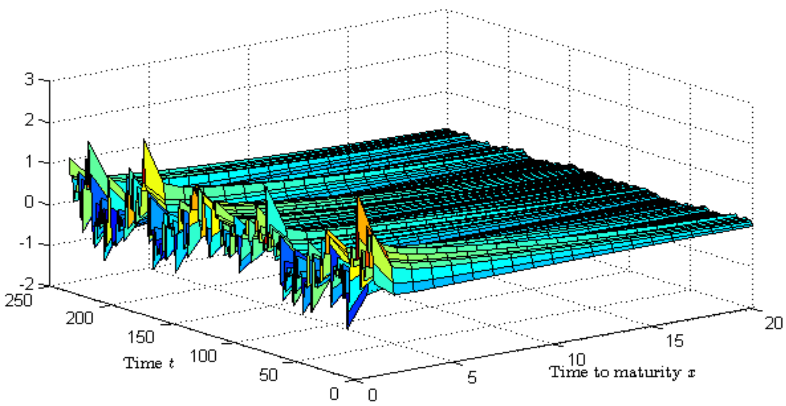

Figure 3.3: The field $Y(t, x)$ for the CARMA(2,1)-process.

We next simulated a realization of the field $Y(t, x)$ based on a simulated path of $Y(t)$. We assumed a Brownian motion for the driving noise $L = B$, and volatility equal to one. We simulated a path of the process $Y(t)$ based on an Euler discretization, using the peak-estimated parameters. For the exponential averaging, we used a rolling window of 100 days. Simulating over a financial year of 250, we obtained the two paths shown in Figure 3.2. Since the moving average parameter of the peak data is rather big, we see that the difference between the values of

Y and its average is not very big. However, the averaging process is smoothing off the biggest peaks in the paths of Y. The resulting field $Y(t,x)$ is shown in Figure 3.3. Notably is the fast reversion to zero for $Y(t,x)$ as a function of x. This is a reflection of the stationarity, and tells us that the forward prices in the long end will only move according to deterministic factors. For small values of x, the fluctuations in the field $Y(t,x)$ are close to the spot dynamics $Y(t)$.

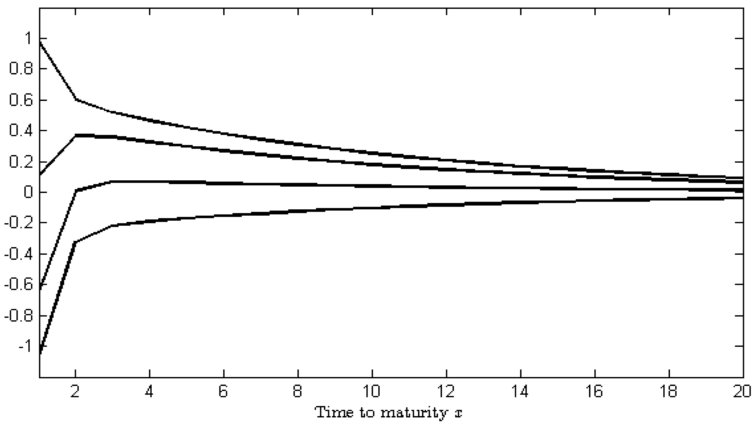

Figure 3.4: Four shapes of forward curves.

In Figure 3.4 we have extracted four of the curves of $x \mapsto Y(t,x)$ at the time instances $t = 20, \ldots, 23$ to gain more insight into the flexibility of the shapes. At these times we start with a declining curve at time 20, next followed by two hump shapes (for times 21 and 22), before we get an increasing shape (at time 23). When we observe the hump shapes, the values of Y are positive while the average is negative. For the decreasing curve, both Y and its average are positive, while for the increasing last curve both are negative. Indeed, this shows the broad range of different forward curves that this model can accommodate.

3.4.2 The gamma-LSS process

In Barndorff-Nielsen *et al.* [6] it was found that an LSS process $Y(t)$ with a gamma-kernel g explained very well the logarithmic deseasonalized spot price dynamics at the German power exchange EEX. A gamma kernel is defined as

$$g(t) = t^{\nu-1}\mathrm{e}^{-\lambda t} \tag{3.4.13}$$

with $\lambda > 0$ and $\nu > 1/2$. The latter condition is needed to satisfy the integrability condition (3.2.3). We note in passing that this type of kernel function has also proved to be useful in turbulence modelling (see Barndorff-Nielsen and Schmiegel [8]).

We compute the ratio $\mathcal{L}g_x(\theta)/\mathcal{L}g(\theta)$ for this case in the next Proposition:

Proposition 3.4.4. *It holds that*

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)} = \mathrm{e}^{\theta x}\frac{\Gamma(\nu,(\lambda+\theta)x)}{\Gamma(\nu)},$$

where $\Gamma(a,b) = \int_b^\infty s^{a-1}\mathrm{e}^{-s}\,ds$ *is the upper incomplete* Γ*-function and* $\Gamma(a) = \Gamma(a,0)$ *is the* Γ*-function.*

Proof. A direct computation gives

$$\mathcal{L}g_x(\theta) = \int_0^\infty (t+x)^{\nu-1}\mathrm{e}^{-\lambda(t+x)}e^{-\theta t}\,dt = \frac{\mathrm{e}^{\theta x}}{(\lambda+\theta)^\nu}\Gamma(\nu,(\lambda+\theta)x).$$

Since $g_0 = g$, the result follows. □

With this result at hand, we can next find explicitly the weight function $\xi_x(t)$ for this kernel g. This is done in the following Proposition:

Proposition 3.4.5. *Suppose that* $\nu \notin \mathbb{N}$*. Then, for any* $x > 0$*, it holds that*

$$\xi_x(t) = \frac{1}{\Gamma(\nu)\Gamma(1-\nu)}\frac{x^\nu}{t^\nu(x+t)}\mathrm{e}^{-\lambda(t+x)},$$

where $\Gamma(a)$ *is the* Γ*-function.*

Proof. Let $x > 0$, and compute the Laplace transform of the function $f(t) = t^{-\nu}\exp(-\lambda t)x/(t+x)$ using Tonelli's theorem:

$$\begin{aligned}\mathcal{L}f(\theta) &= \int_0^\infty t^{-\nu}\mathrm{e}^{-(\lambda+\theta)t}\frac{x}{x+t}\,dt \\ &= \int_0^\infty t^{-\nu}\mathrm{e}^{-(\lambda+\theta)t}\int_0^\infty \mathrm{e}^{-u(1+t/x)}\,du\,dt \\ &= \int_0^\infty\int_0^\infty t^{-\nu}\mathrm{e}^{-(\theta+\lambda+u/x)t}\,dt\mathrm{e}^{-u}\,du.\end{aligned}$$

Changing variables in the inner integral yields

$$\begin{aligned}\mathcal{L}f(\theta) &= \int_0^\infty \left(\theta+\lambda+\frac{u}{x}\right)^{\nu-1}\int_0^\infty s^{-\nu}\mathrm{e}^{-s}\,ds\mathrm{e}^{-u}\,du \\ &= \Gamma(1-\nu)\int_0^\infty \left(\theta+\lambda+\frac{u}{x}\right)^{\nu-1}\mathrm{e}^{-u}\,du \\ &= \Gamma(1-\nu)x^{1-\nu}\mathrm{e}^{x(\lambda+\theta)}\Gamma(\nu,x(\lambda+\theta)).\end{aligned}$$

Note that since $\nu \notin \mathbb{N}$, $\Gamma(1-\nu)$ is well-defined. Since

$$\mathcal{L}\xi_x(\theta) = \frac{1}{\Gamma(\nu)\Gamma(1-\nu)}x^{\nu-1}\mathrm{e}^{-\lambda x}\mathcal{L}f(\theta) = \mathrm{e}^{\theta x}\frac{\Gamma(\nu,x(\lambda+\theta))}{\Gamma(\nu)},$$

the result follows. □

The above result does not hold for $\nu \in \mathbb{N}$, since then $\Gamma(1-\nu)$ is not defined. The case $\nu = 1$ gives the trivial $g(t) = \exp(-\lambda t)$, which is the kernel of an Ornstein-Uhlenbeck process $Y(t)$. For $\nu \in \mathbb{N}$ with $\nu \geq 2$ we find by integration by parts (here, $b \geq 0$)

$$\begin{aligned}\Gamma(\nu, b) &= \int_b^\infty u^{\nu-1} \mathrm{e}^{-u}\, du \\ &= -\left[u^{\nu-1}\mathrm{e}^{-u}\right]_{u=b}^{\infty} + (\nu-1)\int_b^\infty u^{\nu-2}\mathrm{e}^{-u}\, du \\ &= b^{\nu-1}\mathrm{e}^{-b} + (\nu-1)\Gamma(\nu-1, b)\,.\end{aligned}$$

Thus, iterating this until $\Gamma(1, b) = \exp(-b)$, we reach

$$\Gamma(\nu, x(\lambda+\theta)) = \mathrm{e}^{-x(\lambda+\theta)} \sum_{k=0}^{\nu-1} a_\nu(k) x^k (\lambda+\theta)^k\,,$$

where $a_\nu(0) = 1$ and $a_\nu(k) = (\nu-1)(\nu-2)\cdots(\nu-k)$ for $k \geq 1$. Therefore, we get that

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)} = \mathrm{e}^{-\lambda x}\Gamma^{-1}(\nu) \sum_{k=0}^{\nu-1} a_\nu(k) x^k (\lambda+\theta)^k\,.$$

In the case of $\nu \in \mathbb{N}$ and $\nu \geq 2$ we conclude that the ratio $\mathcal{L}g_x(\theta)/\mathcal{L}g(\theta)$ is a polynomial in θ of order $\nu - 1$. Therefore, the weight function $\xi_x(t)$ will be analogous to the CAR(p) case as studied in the previous Subsection, being a sum of derivatives of Dirac-δ functions. This implies that $Y(t, x)$ in (3.3.1) will become a function of $Y^{(k)}(t)$ for $k = 0, \ldots, \nu - 1$. Of course, for this to be valid we need that $Y(t)$ is differentiable up to the order $\nu - 1$. This is proven in the next Lemma:

Lemma 3.4.6. *Suppose $n \geq \nu > n-1$ for some $n \in \mathbb{N}$ with $n \geq 2$. Then Y is $n-1$ times differentiable.*

Proof. Note that

$$g'(t) = (\nu-1)t^{\nu-2}\mathrm{e}^{-\lambda t} - \lambda t^{\nu-1}\mathrm{e}^{-\lambda t}\,,$$

which is square integrable on $\mathbb{R}_+$ and is zero in zero. Hence, from Proposition 3.2. of Benth and Eyjolfsson [11] we have that Y is differentiable with

$$Y'(t) = \int_{-\infty}^t g'(t-s)\sigma(s-)\, dL(s)\,.$$

Iterating this argument $n-1$ times eventually leads to the existence of the $n-1$-derivative being

$$Y^{(n-1)}(t) = \int_{-\infty}^t g^{(n-1)}(t-s)\sigma(s-)\, dL(s)\,.$$

Since $g^{(n-1)}(0) \neq 0$ (the function may explode at zero), higher-order derivatives do not exist. □

Remark that when $1 > \nu > 1/2$, $g(0)$ does not exist (in fact, $\lim_{t\downarrow 0} g(t) = \infty$), and the process is not differentiable.

Let us return back to the case when $\nu \notin \mathbb{N}$ as in Proposition 3.4.5. In Figure 3.5 we have plotted $\xi_x(t)$ as a function of x and t for the parameters $\nu = 0.55$ and $\lambda = 0.02$.

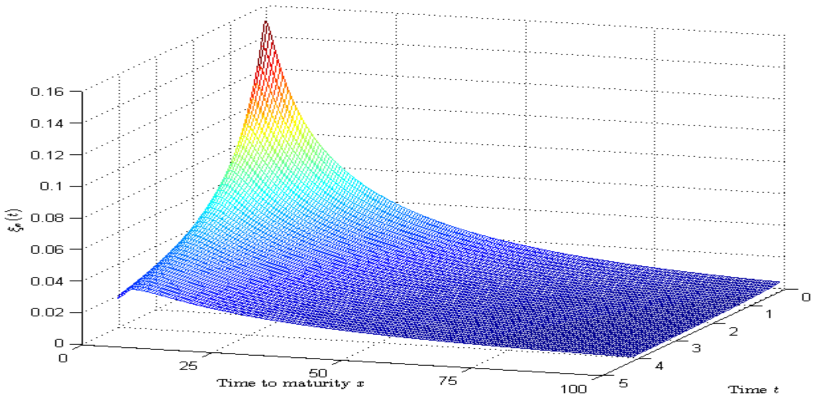

Figure 3.5: The weight function for the Gamma-LSS case.

We have plotted $\xi_x(t)$ for times to maturity ranging up to 100 days, while time is going up to 5 days. Obviously, the values of $\xi_x(t)$ explode when both t and x are close to zero. We notice the fast decay to zero as both variables increase, which means that the emphasis is put on the most current spot prices. This can be seen by recalling (3.3.4) with Proposition 3.4.5, yielding

$$\int_0^t \xi_x(t-s)Y(s)\,ds = \frac{x^\nu}{\Gamma(\nu)\Gamma(1-\nu)}\mathrm{e}^{-\lambda x}\int_0^t (t-s)^{-\nu}(x+t-s))^{-1}\mathrm{e}^{-\lambda(t-s)}Y(s)\,ds\,.$$

As $\xi_x(t-s)$ will be largest for s close to t, the values of Y for times closest to t will matter most in the convolution. Notable in Figure 3.5 is also the slight increasing in the function $x \mapsto \xi_x(t)$ for small values of x, before it decays again. This can be viewed as a hump-shape in the forward curve.

3.5 Appendix: The CARMA(p, 2) case

Appendix A contains the forward price expressed as a functional of the spot price path $Y(t)$ being a CARMA(p, 2)-process for $p \geq 2$.

Proposition 3.5.1. *Let* $\mathbf{b}^{\mathrm{T}} \in \mathbb{R}^p$, $\mathbf{b}^{\mathrm{T}} = (b_0 \quad b_1 \quad 1 \quad 0 \quad \ldots \quad 0)$ *and* $\theta^2 + b_1\theta + b_0 = (\theta - \theta_1)(\theta - \theta_2)$. *If* $\theta_1, \theta_2 \in \mathbb{R}$ *and* $\theta_1 \neq \theta_2$, *let* $\theta > 0$ *such that* $\theta_1 + \theta > 0$ *and* $\theta_2 + \theta > 0$ *then*

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)} = \sum_{m=1}^{p} \frac{\mathbf{b}^{\mathrm{T}}\mathrm{e}^{Ax}\mathbf{e}_m}{\theta_1 - \theta_2}\Big(\mathcal{L}f^{(m-1)}(\theta) + \sum_{k=1}^{m-1}\mathcal{L}\delta^{(k-1)}(\theta)f^{(m-1-k)}(0)\Big)$$

and

$$Y(t,x) = \sum_{m=1}^{p} \frac{\mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m}{\theta_1 - \theta_2} \Big(\int_0^t f^{(m-1)}(t-\tau) Y(\tau)\, d\tau + \sum_{k=1}^{m-1} f^{(m-1-k)}(0) Y^{(k-1)}(t) \Big)$$

where $f(t) = \mathrm{e}^{\theta_1 t} - e^{\theta_2 t}$. *Otherwise if* $\theta_1 = \theta_2$ *then*

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)} = \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \Big(\mathcal{L}f^{(m-1)}(\theta) + \sum_{k=1}^{m-1} \mathcal{L}\delta^{(k-1)}(\theta) f^{(m-1-k)}(0) \Big)$$

and

$$Y(t,x) = \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \Big(\int_0^t f^{(m-1)}(t-\tau) Y(\tau)\, d\tau + \sum_{k=1}^{m-1} f^{(m-1-k)}(0) Y^{(k-1)}(t) \Big)$$

where $f(t) = t\mathrm{e}^{\theta_1 t}$.

Proof. From Proposition 3.4.2 and the fact that $b_2 = 1$ and $b_m = 0$ for all $m = 3, \dots, p-1$ we have

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)} = \sum_{m=1}^{p} \frac{\mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \theta^{m-1}}{b_0 + b_1\theta + \theta^2}\,. \tag{3.5.1}$$

If $\theta^2 + b_1\theta + b_0$ has two different real roots θ_1 and θ_2, we can consider the following decomposition using partial fractions

$$\frac{1}{\theta^2 + b_1\theta + b_0} = \frac{1}{\theta_1 - \theta_2} \Big(\frac{1}{\theta - \theta_1} - \frac{1}{\theta - \theta_2} \Big)\,. \tag{3.5.2}$$

Define $f(t) = e^{\theta_1 t} - e^{\theta_2 t}$, then (3.5.2) can be rewritten by means of the Laplace transform as

$$\frac{1}{\theta^2 + b_1\theta + b_0} = \frac{1}{\theta_1 - \theta_2} \mathcal{L}\{e^{\theta_1 t} - e^{\theta_2 t}\}(\theta)\,. \tag{3.5.3}$$

Hence, substituting (3.5.3) into (3.5.1) and considering the Laplace transform of higher-order derivatives of $f(t)$, then (3.5.1) finally reduces to

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)} = \sum_{m=1}^{p} \frac{\mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m}{\theta_1 - \theta_2} \Big(\mathcal{L}f^{(m-1)}(\theta) + \sum_{k=1}^{m-1} \mathcal{L}\delta^{(k-1)}(\theta) f^{(m-1-k)}(0) \Big)\,. \tag{3.5.4}$$

Taking inverse Laplace transform in (3.5.4) we get,

$$\xi_x(t) = \sum_{m=1}^{p} \frac{\mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m}{\theta_1 - \theta_2} \Big(f^{(m-1)}(t) + \sum_{k=1}^{m-1} \delta^{(k-1)}(t) f^{(m-1-k)}(0) \Big)\,.$$

Therefore

$$Y(t,x) = \int_0^t \xi_x(t-\tau) Y(\tau)\, d\tau$$

$$= \sum_{m=1}^{p} \frac{\mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m}{\theta_1 - \theta_2} \Big(\int_0^t f^{(m-1)}(t-s) Y(s)\, ds + \sum_{k=1}^{m-1} f^{(m-1-k)}(0) Y^{(k-1)}(t) \Big) .$$

If $\theta^2 + b_1\theta + b_0$ has a double real root θ_1, then we can consider the following decomposition using partial fractions

$$\frac{1}{\theta^2 + b_1\theta + b_0} = \frac{1}{(\theta - \theta_1)^2} . \tag{3.5.5}$$

Define $f(t) = te^{\theta_1 t}$, then (3.5.5) can be rewritten by means of the Laplace transform as

$$\frac{1}{\theta^2 + b_1\theta + b_0} = \mathcal{L}\{te^{\theta_1 t}\}(\theta) . \tag{3.5.6}$$

Hence, substituting (3.5.6) into (3.5.1) and considering the Laplace transform of higher-order derivatives of $f(t)$, then (3.5.1) finally reduces to

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)} = \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \Big(\mathcal{L}f^{(m-1)}(\theta) + \sum_{k=1}^{m-1} \mathcal{L}\delta^{(k-1)}(\theta) f^{(m-1-k)}(0) \Big) . \tag{3.5.7}$$

Taking inverse Laplace transform in (3.5.7) we get,

$$\xi_x(t) = \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \Big(f^{(m-1)}(t) + \sum_{k=1}^{m-1} \delta^{(k-1)}(t) f^{(m-1-k)}(0) \Big) .$$

Therefore

$$\begin{aligned} Y(t,x) &= \int_0^t \xi_x(t-s) Y(s)\, ds \\ &= \sum_{m=1}^{p} \mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \Big(\int_0^t f^{(m-1)}(t-s) Y(s)\, ds + \sum_{k=1}^{m-1} f^{(m-1-k)}(0) Y^{(k-1)}(t) \Big) . \end{aligned}$$

□

In both cases $f^{(0)}(0)Y^{(p-2)}(t) = 0$ so that we have the same degree of differentiability for the CARMA$(p,2)$-process as established in Proposition 3.4.1.

Proposition 3.5.2. *Let* $\mathbf{b}^{\mathsf{T}} \in \mathbb{R}^p$, $\mathbf{b}^{\mathsf{T}} = (b_0 \quad b_1 \quad 1 \quad 0 \quad \cdots \quad 0)$ *and* $\theta^2 + b_1\theta + b_0 = (\theta - \theta_1)(\theta - \overline{\theta}_1)$ *with* $\theta_1, \overline{\theta}_1 \in \mathbb{C}$ *such that* $\overline{\theta}_1$ *is the conjugate of* θ_1 *then*

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)} = \sum_{m=1}^{p} \frac{\mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m}{\theta_1 - \overline{\theta}_1} \Big(\mathcal{L}f^{(m-1)}(\theta) + \sum_{k=1}^{m-1} \mathcal{L}\delta^{(k-1)}(\theta) f^{(m-1-k)}(0) \Big)$$

and

$$Y(t,x) = \sum_{m=1}^{p} \frac{\mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m}{\theta_1 - \overline{\theta}_1} \Big(\int_0^t f^{(m-1)}(t-\tau) Y(\tau)\, d\tau + \sum_{k=1}^{m-1} f^{(m-1-k)}(0) Y^{(k-1)}(t) \Big) .$$

Proof. From Proposition 3.4.2 and the fact that $b_2 = 1$ and $b_m = 0$ for all $m = 3, \dots, p-1$ we have

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)} = \sum_{m=1}^{p} \frac{\mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m \theta^{m-1}}{b_0 + b_1\theta + \theta^2} \,. \tag{3.5.8}$$

If $\theta^2 + b_1\theta + b_0$ has two complex roots $\theta_1, \overline{\theta}_1$, we can consider the following decomposition using partial fractions

$$\frac{1}{\theta^2 + b_1\theta + b_0} = \frac{1}{\theta_1 - \overline{\theta}_1}\Big(\frac{1}{\theta - \theta_1} - \frac{1}{\theta - \overline{\theta}_1}\Big) \,. \tag{3.5.9}$$

Define $f(t) = e^{\theta_1 t} - e^{\overline{\theta}_1 t}$, then (3.5.9) can be rewritten by means of the Laplace transform as

$$\frac{1}{\theta^2 + b_1\theta + b_0} = \frac{1}{\theta_1 - \overline{\theta}_1}\mathcal{L}\{e^{\theta_1 t} - e^{\overline{\theta}_1 t}\}(\theta) \,. \tag{3.5.10}$$

Hence, substituting (3.5.10) into (3.5.8) and considering the Laplace transform of higher-order derivatives of $f(t)$, then (3.5.8) finally reduces to

$$\frac{\mathcal{L}g_x(\theta)}{\mathcal{L}g(\theta)} = \sum_{m=1}^{p} \frac{\mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m}{\theta_1 - \overline{\theta}_1}\Big(\mathcal{L}f^{(m-1)}(\theta) + \sum_{k=1}^{m-1} \mathcal{L}\delta^{(k-1)}(\theta) f^{(m-1-k)}(0)\Big) \,. \tag{3.5.11}$$

Taking inverse Laplace transform in (3.5.11) we get,

$$\xi_x(t) = \sum_{m=1}^{p} \frac{\mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m}{\theta_1 - \overline{\theta}_1}\Big(f^{(m-1)}(t) + \sum_{k=1}^{m-1} \delta^{(k-1)}(t) f^{(m-1-k)}(0)\Big) \,.$$

Therefore

$$\begin{aligned} Y(t,x) &= \int_0^t \xi_x(t-s) Y(s)\, ds \\ &= \sum_{m=1}^{p} \frac{\mathbf{b}^{\mathsf{T}} \mathrm{e}^{Ax} \mathbf{e}_m}{\theta_1 - \overline{\theta}_1}\Big(\int_0^t f^{(m-1)}(t-s) Y(s)\, ds + \sum_{k=1}^{m-1} f^{(m-1-k)}(0) Y^{(k-1)}(t)\Big) \,. \end{aligned}$$

□

Observe that $f^{(0)}(0) Y^{(p-2)}(t) = 0$ so that we have the same degree of differentiability for the CARMA($p, 2$)-process as established in Proposition 3.4.1.

Chapter 4

Approximation of the HDD and CDD temperature futures prices dynamics

Abstract
We propose an approximation which makes the HDD and CDD temperature futures price dynamics linearily dependent on the underlying temperature. The approximation is analysed both theoretically and empirically. We base our analysis on a continuous-time autoregressive stochastic dynamics for the time evolution of temperature in a given location. The model is fitted to temperature data collected in New York over a long time period. We apply our results to derive a simple version of the Black-76 formula for pricing a call option on CDD and HDD futures.

4.1 Introduction

The Chicago Mercantile Exchange (CME) organizes trade in futures contracts written on temperature indexes measured in several cities world-wide. Typical indexes are the so-called heating-degree day (HDD) and cooling-degree day (CDD), which are used for cities in the US. The CDD index accumulates the difference between the observed daily average temperature and a threshold of $65°$F over a time period. In the CME market place, the period where the CDD index is measured is typically a month in the summer season. The index provides a measure of the demand for air-conditioning cooling, and futures written on this can be applied for hedging such demand by power producer or retailers. On the other hand, the HDD index measures the demand for heating in the winter season.

The CDD and HDD indexes are nonlinear functions of the underlying temperature. In fact, the two indexes can be viewed as a strip of call (respectively put) payoff structures on the underlying temperature. As a result, the futures price will have nonlinear dependency on the temperature as well (see Benth and Šaltytė Benth [15]). The CME organizes a market for plain vanilla call and put options on temperature futures, and the pricing of these derivatives will in fact become very complex due to the nonlinear nature of the underlying futures. The objective of the current paper is to propose some methods to linearize the futures price dynamics, which will pave the way for simple option pricing formulas. In fact, we can price call and put options using a variant of the Black-76 formula, avoiding highly inefficient Monte Carlo-based pricing (see Benth and Šaltytė Benth [15]). The suggested approximative price dynamics will also significantly simplify the statistical analysis of futures prices.

Our approximation is based on a linearization of a function appearing in the theoretical futures price formula based on arbitrage-free pricing. The function in question is closely connected to the cumulative distribution function, and possesses nice analytical properties which enables us to perform an error analysis of the approximative futures price. We discuss the approximation in several numerical examples based on temperature observations from New York. As it turns out, our proposed approximation works well in several cases, however, there are also examples where it does not perform good and care must be taken.

Our analysis of CDD and HDD futures prices is based on a continuous-time autoregressive (CAR) temperature dynamics. Several empirical studies of temperature data have shown that CAR models explain very well the statistical properties of the dynamics (see Härdle and Lopez Cabrera [35], Benth and Šaltytė Benth [15] and the references therein). We fit a long series of daily average temperature data observed in New York to a CAR process, which will be the reference point in several of our numerical and empirical examples. CAR models have been applied in several markets, like power (see Garcia *et al.* [33]), oil (see Paschke and Prokopczuk [39]) and fixed income (Andresen *et al.* [4]). General class of CARMA (continuous-time autoregressive moving average) processes have been extensively studied by Brockwell and co-authors (see Brockwell [24] for an overview).

We present our results in the following way: in the next section the stochastic model for the temperature dynamics is presented, including an empirical analysis of New York data. We also present the theoretical arbitrage-free prices of HDD and CDD futures here. Section 3 presents and analyses the linearization of the HDD and CDD futures prices, with several empirical examples given illustrating the theory. We continue in Section 4 with an analytical formula for the price of a call written on a temperature futures using its linearized dynamics. Finally, we conclude in Section 5.

4.2 Temperature modelling and futures pricing

In this Section we introduce a stochastic dynamics for the time evolution of temperatures measured in a location. Introduce a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. Let $T(t)$ denote the temperature in the location at time $t \geq 0$, and assume that

$$T(t) = \Lambda(t) + Y(t)\,. \tag{4.2.1}$$

Here, Λ is a deterministic function measuring the mean temperature at time t, and $Y(t)$ some stochastic process modelling the random variations around this mean level. We assume Λ to be a smooth function, with at least linear growth. We frequently refer to Y as the *deseasonalized temperature*.

As a model for Y, we introduce the class of CAR(p)-processes for $p \in \mathbb{N}$. To this end, let $\mathbf{X}(t)$ be a vector-valued stochastic process with values in $\mathbb{R}^p$, having the dynamics

$$d\mathbf{X}(t) = A\mathbf{X}(t)\,dt + \sigma(t)\mathbf{e}_p\,dB(t)\,, \tag{4.2.2}$$

for a one-dimensional Brownian motion $B(t)$. Here, $\mathbf{e}_i$, $i = 1, \dots, p$ are the canonical unit vectors in $\mathbb{R}^p$, $\sigma(t)$ is a bounded, real-valued deterministic function and A is the $p \times p$-matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_p & -\alpha_{p-1} & -\alpha_{p-2} & \dots & -\alpha_1 \end{bmatrix}. \tag{4.2.3}$$

The constants $\alpha_i, i = 1, \dots, p$ are all assumed to be strictly positive. Moreover, we assume that the eigenvalues of A have negative real part, which implies that the process $\mathbf{X}(t)$ has a stationary distribution (see Benth and Šaltytė Benth [15]).

A CAR(p)-process is defined as

$$Y(t) = \mathbf{e}_1'\mathbf{X}(t)\,, \tag{4.2.4}$$

that is, the first coordinate of the vector $\mathbf{X}(t)$. We use the notation $\mathbf{x}'$ to denote the transpose of a vector (or matrix) $\mathbf{x}$. We remark in passing that one may generalize to so-called continuous-time autoregressive *moving average* processes by mixing of the $q + 1 \leq p$, q being a natural number or zero, first coordinates of $\mathbf{X}(t)$ (we refer to Benth and Šaltytė Benth [15] for more on this).

From the dynamics of $\mathbf{X}(t)$, we see that it is a vector-valued Ornstein-Uhlenbeck process, with a particular "mean-reversion" matrix A and the stochastic evolution driven by a one-dimensional Brownian motion. Hence, it has simple analytical properties that allow for reasonably explicit expressions for the dynamics of various temperature futures prices. We briefly recall some results from Benth and Šaltytė Benth [15].

Consider a temperature futures settled on the CDD index measured in a location over the period $[\tau_1, \tau_2]$, $\tau_1 < \tau_2$. Typically, this measurement period is a month in the warm season of the year, ranging from April to October for US cities. The index is defined as

$$\text{CDD}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \max(T(t) - c, 0)\, dt\,, \tag{4.2.5}$$

where c is 65°F (or 18°C). As we see, the CDD index is aggregating the temperatures above the threshold c over the measurement period, and as such, provides a measure for the demand for air-conditioning cooling, say, in the period $[\tau_1, \tau_2]$. In the actual market, the index is measured as the sum over the average daily temperature, where the average is calculated as the average over the recorded minimum and maximum temperature. We apply the definition for notational convenience. The HDD index is analogously defined as

$$\text{HDD}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \max(c - T(t), 0)\, dt\,, \tag{4.2.6}$$

and measures the demand for heating in the cold season, ranging from October to April in the US market.

A CDD futures contract with measurement period $[\tau_1, \tau_2]$ is financially settled on the CDD index $\text{CDD}(\tau_1, \tau_2)$. The settlement takes place at time τ_2, the end of the measurement period, where one gets a payment equal to USD20 per index point. To save on notation, we assume that one simply gets the index value in Dollars. The arbitrage-free futures price $F_{\text{CDD}}(t, \tau_1, \tau_2)$ at time $t \leq \tau_2$ is defined as

$$F_{\text{CDD}}(t, \tau_1, \tau_2) = \mathbb{E}_Q \left[\text{CDD}(\tau_1, \tau_2) \,|\, \mathcal{F}_t\right] \,, \tag{4.2.7}$$

where $\mathbb{E}_Q[\cdot \,|\, \mathcal{F}_t]$ is the conditional expectation operator with respect to some probability $Q \sim P$. The probability Q is some pricing measure modelling the risk premium in the market charged by the actors for the inability to hedge the underlying index. We have the obvious analogous expression for $F_{\text{HDD}}(t, \tau_1, \tau_2)$.

We let Q be given by a Girsanov transform such that the process

$$dW(t) = -\frac{\theta(t)}{\sigma(t)}\, dt + dB(t)\,, \tag{4.2.8}$$

is a Q-Brownian motion. Here, θ is a bounded deterministic function, and in order to have this Girsanov transform validated, we must assume that $\sigma(t)$ is bounded below by a constant strictly bigger than zero. The parameter function θ is referred to as the *market price of risk*, and is an implicit parametrization of the risk premium in the market. We note that the Q-dynamics of $\mathbf{X}(t)$ becomes

$$d\mathbf{X}(t) = (\theta(t)\mathbf{e}_p + A\mathbf{X}(t))\; dt + \sigma(t)\mathbf{e}_p\, dW(t)\,.$$

We find the following result, which is a slight extension of Proposition 5.4, page 121, in Benth and Šaltytė Benth [15]:

Proposition 4.2.1. *It holds for $t \leq \tau_2$ that*

$$F_{CDD}(t, \tau_1, \tau_2) = \int_{\tau_1}^{\max(t,\tau_1)} \max(T(s)-c, 0)\, ds + \int_{\max(t,\tau_1)}^{\tau_2} \Sigma(t,s)\Psi\left(\frac{m_\theta(t,s,\mathbf{X}(t)) - c}{\Sigma(t,s)}\right) ds\,,$$

where, for a vector $\mathbf{x} \in \mathbb{R}^p$,

$$m_\theta(t,s,\mathbf{x}) = \Lambda(s) + \mathbf{e}_1' \exp(A(s-t))\mathbf{x} + \int_t^s \mathbf{e}_1' \exp(A(s-u))\mathbf{e}_p\theta(u)\, du$$

$$\Sigma^2(t,s) = \int_t^s (\mathbf{e}_1' \exp(A(s-u))\mathbf{e}_p)^2\sigma^2(u)\, du\,.$$

Furthermore, $\Psi(x) = x\Phi(x) + \Phi'(x)$, with Φ being the cumulative standard normal distribution function.

Proof. First, if $t \leq \tau_1$, the result follows from Proposition 5.4 in Benth and Šaltytė Benth [15]. Let now $\tau_1 \leq t \leq \tau_2$. Then, as

$$\int_{\tau_1}^t \max(T(s) - c, 0)\, ds$$

is $\mathcal{F}_t$-measurable, we have

$$F_{\text{CDD}}(t, \tau_1, \tau_2) = \int_{\tau_1}^t \max(T(s) - c, 0)\, ds + \mathbb{E}_Q\left[\text{CDD}(t, \tau_2) \,|\, \mathcal{F}_t\right]\,.$$

The second term is the price of a CDD futures at time t with measurement period $[t, \tau_2]$, which we again find from Proposition 5.4 in Benth and Šaltytė Benth [15]. Hence, the proof is complete. $\square$

We remark that $m_\theta(t, s, \mathbf{X}(t))$ is the mean of $T(s)$ given $\mathbf{X}(t)$, $s \geq t$, while $\Sigma^2(t, s)$ is its conditional variance. Note in Proposition 4.2.1 that $\exp(At)$, for $t \geq 0$ is the matrix exponential defined as $\exp(At) = \sum_{n=0}^{\infty} A^n t^n / n!$. Furthermore, we recall $\Phi'(x)$ to be $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$, the standard normal density function.

There is a similar expression for the HDD futures price, which may be conveniently derived from the so-called CDD-HDD parity. From Corollary 5.1, page 119 in Benth and Šaltytė Benth [15] we find

Corollary 4.2.2. *It holds that*

$$F_{CDD}(t, \tau_1, \tau_2) - F_{HDD}(t, \tau_1, \tau_2) = \mathbb{E}_Q \left[\int_{\tau_1}^{\tau_2} T(s)\, ds \,|\, \mathcal{F}_t \right] - c(\tau_2 - \tau_1)\,.$$

The expectation on the right-hand side is in fact the price of a so-called CAT futures at time t (CAT being the acronym for cumulative average temperature). From a slight generalization of Proposition 5.6, page 123, in Benth and Šaltytė Benth [15] we have

Proposition 4.2.3. *It holds for $t \leq \tau_2$ that*

$$F_{HDD}(t, \tau_1, \tau_2) = \int_{\tau_1}^{\max(t,\tau_1)} \max(c - T(s), 0)\, ds + \int_{\max(t,\tau_1)}^{\tau_2} \Sigma(t, s) \Psi \left(\frac{c - m_\theta(t, s, \mathbf{X}(t))}{\Sigma(t, s)} \right) ds\,,$$

where Ψ, m_θ and Σ are defined in Proposition 4.2.1.

Observe that both the HDD and CDD futures price depend nonlinearily on the state of the vector $\mathbf{X}(t)$ in the temperature dynamics. The nonlinearity stems from the function $\Psi(x)$ defined in Proposition 4.2.1 combined with an integration. For example, this will make parameter estimation a complex matter, as the dynamics of the futures prices will have a state-dependent volatility which is nonlinear. Also, it seems impossible to derive analytic formulas for call and put option prices on the CDD and HDD futures, products that are traded at the CME. Accurate pricing must resort on Monte Carlo simulation, which is slow, or on numerical solution of certain partial differential equations. The latter approach is complicated by the fact that the associated partial differential equation will be defined on a p-dimensional domain with diffusion in only one direction, while having (strong) gradients in all directions. Furthermore, the integration over the measurement period gives complex boundary conditions.

We propose in the next Section ways to linearize the dynamics of the HDD and CDD futures based on approximating the function Ψ. Such a linarization will yield analytically tractable dynamics for the HDD and CDD futures. As we shall demonstrate empirically and theoretically, the approximation is an many cases performing very well, and therefore is an attractive alternative to the exact pricing formulas.

We end this Section with an empirical case study of the CAR-dynamics for daily average temperatures recorded in New York. The study will demonstrate the validity of our proposed dynamics for the stochastic time evolution of temperatures, as well as providing us with a basis for later examples illustrating the results in this paper.

We have available a time series of daily average temperatures (DATs) observed in New York from January 1st, 1960 to April 20th, 2013. The DAT is calculated as the average of the minimum and maximum temperature recorded over the day. The data are measured in Farenheit degrees. The DATs from February 20th in each leap year is deleted from the data set in order to equalise the length of all years. In total, we have a time series of 19.455 observations.

In the following, we estimate the parameters in our stochatsic model for the temperature dynamics. We let the observed DATs be measurements of the dynamics $T(t)$ defined in (4.2.1). We let T_i be the temperature observed at day i for $i = 0, 1, 2, \ldots$, with $i = 0$ being January 1st, 1960, and assume

$$T_i = \Lambda_i + y_i \,.$$

The seasonal function $\Lambda(t)$ is assumed to have the form

$$\Lambda(t) = a_0 + a_1 t + a_2 \cos\Big(2\pi(t - a_3)/365\Big)\,,$$

where $a_t + a_1 t$ is a linear trend capturing the possible influence of global warming and urbanization, say, and the cosine-term models the yearly seasonal cycle of the DATs. The parameters $a_0, \ldots, a_3$ are all constants. Fitting $\Lambda(t)$ to the observed DATs by least-squares resulted in the estimates $\widehat{a}_0 = 54.19$, $\widehat{a}_1 = 0.0001$, $\widehat{a}_2 = 22.15$ and $\widehat{a}_3 = 204.81$. In Fig 4.1, a snapshot of the observed DATs from the first 10 years of the data set is depicted together with the fitted seasonality function $\Lambda(t)$.

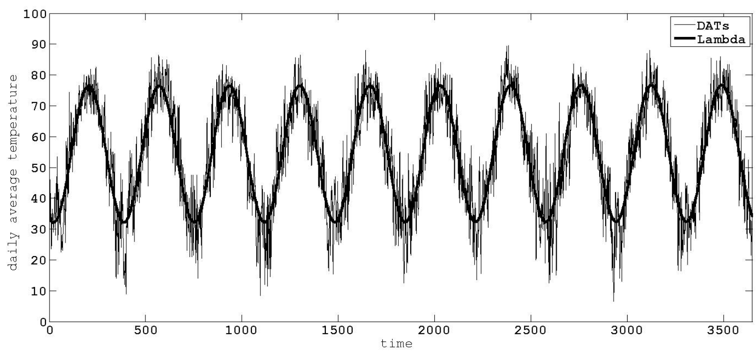

Figure 4.1: Observed DATs in New York together with the fitted seasonal function $\Lambda(t)$. A snapshot of ten years starting from January 1st, 1960.

We next move to the deseasonalized temperatures $y_i = T_i - \Lambda(i)$. In Figure 4.2(left) we have plotted the autocorrelation function (ACF) of the time series, which decays in a seemingly exponential manner to zero. This is a clear sign of a stationary autoregressive dynamics. This is further emphasized by looking at the partial ACF (PACF) shown in Figure 4.2(right), where it is suggested that the data follows an autoregressive time series dynamics of order 3, denoted AR(3).

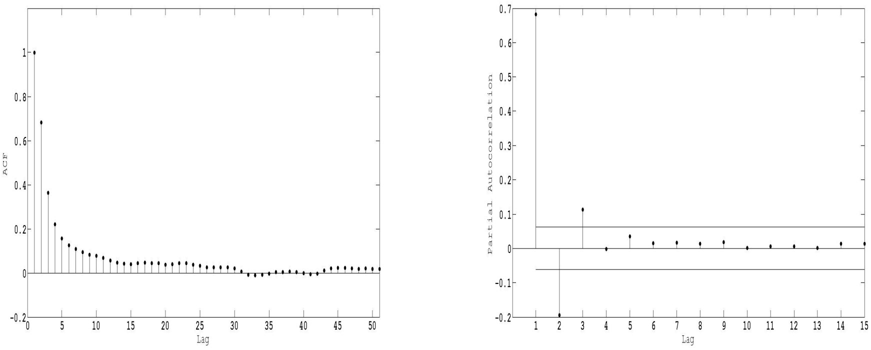

Figure 4.2: The ACF (left) and PACF (right) of the deseasonalized temperature series $T_i - \Lambda(i)$.

We suppose that y_i follows an AR(3) dynamics, which means that for $i = 0, 1, 2, \ldots,$

$$y_{i+3} = \sum_{j=1}^{3} b_j y_{i+3-j} + \sigma\epsilon_i \, ,$$

for constant autoregression parameters b_j, $j = 1, 2, 3$, σ being the the constant volatility function and $\{\epsilon_i\}_{i=0,1,\ldots}$ are independent standard normally distributed random variables. By using standard techniques for estimating the parameters of AR(p)-processes, we find $\widehat{b}_1 = 0.8382$, $\widehat{b}_2 = -0.2869$ and $\widehat{b}_3 = 0.1123$, and $\widehat{\sigma} = 5.25$.

In Figure 4.3 we have plotted the empirical density of the normalized residuals of the estimated AR(3) times series along with the standard normal distribution. Although not perfect, we obtain a reasonable fit to the standard normal distribution. To obtain a better fit, one might introduce a seasonally varying volatility $\sigma(t)$ as advocated in Benth and Šaltytė Benth [15] and empirially observed in many cities. As our purpose is here to study an approximation of futures prices, we refrain from further generality of the temperature model as this is not likely to influence the approximation itself.

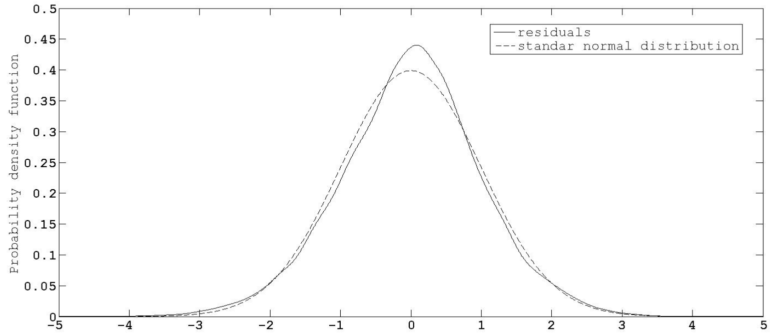

Figure 4.3: Empirical density of the normalized residuals from the estimated AR(3) time series together with the standard normal density (broken line).

The link between the coefficients of an AR(p)-process and a CAR(p)-process $Y(t)$ defined in (4.2.4) is established in Lemma 10.2 in Benth and Šaltytė Benth [15]. Based on the estimated b_j's, we find the coefficients in the matrix A of the process $\mathbf{X}(t)$ in (4.2.2) defining the CAR(3)-process to be $\widehat{\alpha}_1 = 2.1618$, $\widehat{\alpha}_2 = 1.6105$ and $\widehat{\alpha}_3 = 0.3364$. The eigenvalues of the matrix A become $\lambda_1 = -0.34, \lambda_{2,3} = -0.91 \pm 0.40\mathrm{i}$. Hence, Y is a stationary CAR(3) process as the eigenvalues have negative real part. We apply this model as the basic case study in an empirical investigation of the approximation of HDD and CDD futures prices that we introduce next.

4.3 Approximation of the HDD and CDD futures price dynamics

In this Section we propose a linearization of the HDD and CDD futures prices. This is done on analysing the function $\Psi(x)$ defined in Proposition 4.2.1. Let us start with some initial considerations on this function.

As the derivative of $\Psi(x)$ is

$$\Psi'(x) = \Phi(x) \in (0, 1)\,,$$

for Φ being the cumulative standard normal distribution function, we find that Ψ is monotonely increasing. Also, we find that $\Psi(x)$ tends to infinity as $x \to \infty$, and to zero as $x \to -\infty$. In fact, since

$$\lim_{x\to\infty} \frac{\Psi(x)}{x} = 1$$

we have that $\Psi(x) \sim x$ for large x. This is of course a reflection of the fact that $\max(x - c, 0) = x - c$ when x is larger than c. In the next Lemma we quantify the convergence rate:

Lemma 4.3.1. *For $x > 0$ it holds that*

$$|\Psi(x) - x| \leq \left(\frac{x}{2} + \frac{1}{\sqrt{2\pi}}\right) \mathrm{e}^{-x^2/2}\,.$$

Proof. By the triangle inequality we find

$$|\Psi(x) - x| \leq x(1 - \Phi(x)) + \Phi'(x)\,.$$

But, after a change of variables,

$$1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \mathrm{e}^{-y^2/2}\, dy = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-(z+x)^2/2}\, dz = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-z^2/2} \mathrm{e}^{-zx}\, dz \mathrm{e}^{-x^2/2}\,.$$

After noting that $\exp(-zx) \leq 1$ for $z, x > 0$, the Lemma follows. □

From the Lemma, we see that $\Psi(x)$ is converging to x at a rate $x \exp(-x^2/2)$ as x tends to infinity. This is a very rapid convergence, indicating that x does not need to be big before we have $\Psi(x) \approx x$.

One could also consider a Taylor approximation of Ψ. Doing a Taylor expansion of Ψ of order 1 with remainder around zero entails in

$$\begin{aligned} \Psi(x) &= \Psi(0) + \Psi'(0)x + \frac{1}{2}\Psi''(z)x^2 \\ &= \frac{1}{\sqrt{2\pi}} + \frac{1}{2}x + \frac{1}{2}\phi(z)x^2, \end{aligned}$$

since $\Psi'(x) = \Phi(x)$, and hence $\Psi''(x) = \phi(x)$ where we recall that $\phi(x)$ is the density function of the standard normal distribution. Note that $|z| \leq |x|$. Thus, a reasonable approximation of $\Psi(x)$ around zero is

$$\Psi(x) \approx \frac{1}{\sqrt{2\pi}} + \frac{1}{2}x\,.$$

Since $\phi(z) \leq (2\pi)^{-1/2}$, the error in this approximation will be

$$|\Psi(x) - \Big(\frac{1}{\sqrt{2\pi}} + \frac{1}{2}x\Big)| \leq \frac{1}{\sqrt{2\pi}}x^2.$$

We see that the error is of order x^2 close to origo.

In Figure 4.4 we have plotted the function Ψ together with the two linear approximations discussed above.

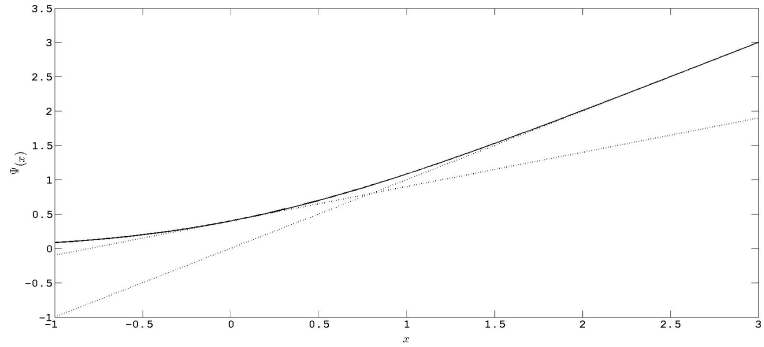

Figure 4.4: The function $\Psi(x)$ defined in Proposition 4.2.1 together with its Taylor approximation and the function x (dotted lines).

We see that x is a good approximation of $\Psi(x)$ for values of x above 1, and that the Taylor approximation works well around zero. In Figure 4.5 we have plotted the relative error in percent of the two approximations.

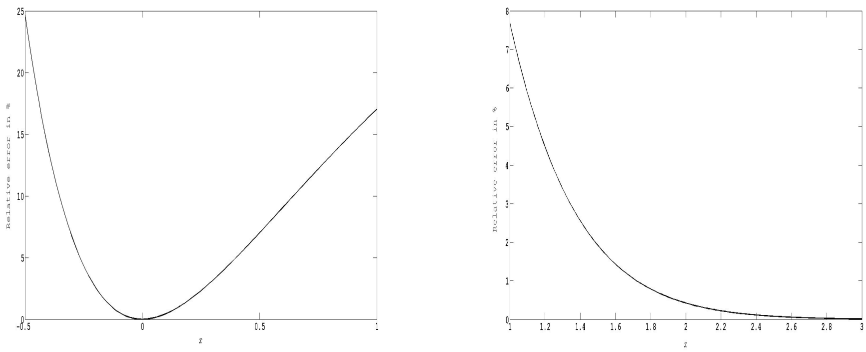

Figure 4.5: The relative error in percent between Ψ and its Taylor approximation (left), and x (right).

For about $|x| \leq 0.25$ the relative error of the Taylor approximation is below 5%. 5% relative error is achieved for the other linear approximation for about $x > 1.2$.

We compute the approximative CDD futures price using $\Psi(x) \approx x$. Inserting x for $\Psi(x)$ in the formula in Proposition 4.2.1 yields

$$
\begin{aligned}
F_{\text{CDD}}(t, \tau_1, \tau_2) &= \int_{\tau_1}^{\max(t,\tau_1)} \max(T(s) - c, 0)\, ds + \int_{\max(t,\tau_1)}^{\tau_2} \Sigma(t, s)\Psi\left(\frac{m_\theta(t, s, \mathbf{X}(t)) - c}{\Sigma(t, s)}\right) ds \\
&\approx \int_{\tau_1}^{\max(t,\tau_1)} \max(T(s) - c, 0)\, ds + \int_{\max(t,\tau_1)}^{\tau_2} \Sigma(t, s)\frac{m_\theta(t, s, \mathbf{X}(t)) - c}{\Sigma(t, s)}\, ds
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\tau_1}^{\max(t,\tau_1)} \max(T(s)-c,0)\,ds + \int_{\max(t,\tau_1)}^{\tau_2} m_\theta(t,s,\mathbf{X}(t)) - c\,ds \\
&= \int_{\tau_1}^{\max(t,\tau_1)} \max(T(s)-c,0)\,ds + \int_{\max(t,\tau_1)}^{\tau_2} \Lambda(s) - c\,ds \\
&\quad + \mathbf{e}_1' A^{-1}\left(\exp(A(\tau_2-t)) - \exp(A(\max(t,\tau_1)-t))\right)\mathbf{X}(t) \\
&\quad + \int_{\max(t,\tau_1)}^{\tau_2}\int_t^s \mathbf{e}_1' \exp(A(s-u))\mathbf{e}_p\theta(u)\,du\,ds\,.
\end{aligned}
$$

In conclusion, we find that

$$
F_{\text{CDD}}(t,\tau_1,\tau_2) \approx \int_{\tau_1}^{\max(t,\tau_1)} \max(T(s)-c,0)\,ds + \Theta_x(t,\tau_1,\tau_2) + \mathbf{a}_x(t,\tau_1,\tau_2)\mathbf{X}(t)\,, \quad (4.3.1)
$$

for

$$
\mathbf{a}_x(t,\tau_1,\tau_2) = \mathbf{e}_1' A^{-1}\left(\exp(A(\tau_2-t)) - \exp(A(\max(t,\tau_1)-t))\right) \quad (4.3.2)
$$

and

$$
\Theta_x(t,\tau_1,\tau_2) = \int_{\max(t,\tau_1)}^{\tau_2} \Lambda(s) - c\,ds + \int_{\max(t,\tau_1)}^{\tau_2}\int_t^s \mathbf{e}_1' \exp(A(s-u))\mathbf{e}_p\theta(u)\,du\,ds\,. \quad (4.3.3)
$$

Hence, the approximative CDD futures price can be represented as a linear function in the coordinates of $\mathbf{X}(t)$.

Recall the seasonal function estimated on the New York temperature data. From this, we can obtain mean temperatures for New York over a given month. For example, in the summer period the mean temperature for June is 72.41°F, for July 76.79°F and finally for August 75.27°F. The estimated monthly summer means are all significantly higher than the threshold value of $c = 65$°F in the CDD futures, indicating that the approximation $\Psi(x) \approx x$ should work well.

In Figure 4.6 we have plotted the exact CDD price from the expression in Proposition 4.2.1 together with the approximated CDD price in (4.3.1). The measurement month of the CDD index is chosen to be August, and we compute the prices based on observations of the New York temperatures. Note that the vector $\mathbf{X}(t) \in \mathbb{R}^3$ in the CAR(3) dynamics will have the first coordinate being the deseasonalized temperature at time t, the second being the derivative and the third being the second derivative (see Benth and Solanilla Blanco [17]). To find the first and second derivative, we apply numerical differentiation of the daily observations of deseasonalized temperatures. The CDD prices are computed from March 3 until July 31, 2011 with a market price of risk being equal to zero. As we see from Figure 4.6, the approximative prices (complete line) are very close to the exact. In fact, the maximal relative error is less than 0.5%.

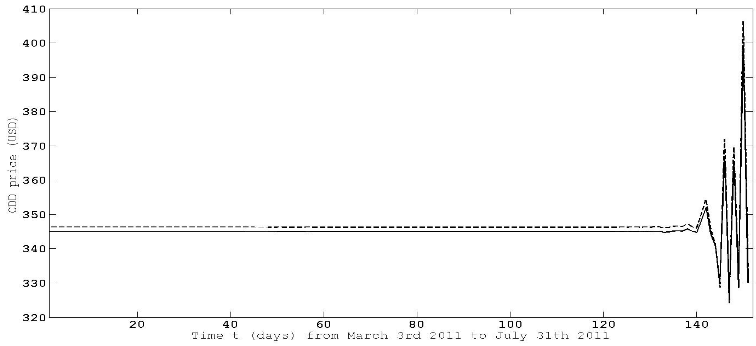

Figure 4.6: Forward prices for CDD contracts from March 3rd to July 31th, 2011 with measurement period August 2011. Theoretical (broken line) versus approximated (complete line).

Let us investigate the approximation a bit closer. In the derivation of the approximative futures price we use that $\Psi(x) \approx x$ for an argument x being

$$x = \frac{m_\theta(t, s, \mathbf{X}(t)) - c}{\Sigma(t, s)}.$$

We let σ be constant (as in the New York data example), which yields

$$\Sigma^2(t, s) := \Sigma^2(s - t) = \sigma^2 \int_0^{s-t} (\mathbf{e}_1' \mathrm{e}^{Au} \mathbf{e}_p)^2 \, du.$$

Supposing that the random variable $\mathbf{X}(t)$ is stationary, we have

$$\mathbf{X}(t) = \sigma \int_{-\infty}^{t} \mathrm{e}^{A(t-s)} \mathbf{e}_p \, dW(s)$$

as the stationary representation. Hence, $\mathbf{X}(t)$ becomes a p-variate Gaussian variable with zero mean and variance-covariance matrix

$$\mathrm{Cov}(\mathbf{X}(t)) = \sigma^2 \int_0^{\infty} \mathrm{e}^{Au} \mathbf{e}_p \mathbf{e}_p' \mathrm{e}^{A'u} \, du.$$

Hence, for $s > t$ we have that $(m_\theta(t, s, \mathbf{X}(t)) - c)/\Sigma(s - t)$ is a normally distributed random variable with mean

$$\mathbf{E}_Q \left[\frac{m_\theta(t, s, \mathbf{X}(t)) - c}{\Sigma(s - t)} \right] = \frac{\Lambda(s) - c + \int_t^s \mathbf{e}_1' \mathrm{e}^{A(s-u)} \mathbf{e}_p \theta(u) \, du}{\Sigma(s - t)} \tag{4.3.4}$$

and variance

$$\mathrm{Var}_Q \left(\frac{m_\theta(t, s, \mathbf{X}(t)) - c}{\Sigma(s - t)} \right) = \sigma^2 \frac{\mathbf{e}_1' \mathrm{e}^{A(s-t)} \int_0^\infty \mathrm{e}^{Au} \mathbf{e}_p \mathbf{e}_p' \mathrm{e}^{A'u} \, du \mathrm{e}^{A'(s-t)} \mathbf{e}_1}{\Sigma^2(s - t)}. \tag{4.3.5}$$

If $s - t \to \infty$, we have that $\Sigma^2(s-t)$ tends to a constant since A is assumed to have eigenvalues with negative real part. By the same argument it therefore follows that the variance of $(m_\theta(t, s, \mathbf{X}(t)) - c)/\Sigma(s-t)$ converges to zero as $s - t \to \infty$. From this we conclude that when s is significantly bigger than t,

$$\frac{m_\theta(t, s, \mathbf{X}(t)) - c}{\Sigma(s-t)} \approx \frac{\Lambda(s) - c + \int_t^s \mathbf{e}_1' \mathrm{e}^{A(s-u)} \mathbf{e}_p \theta(u)\, du}{\Sigma(s-t)}.$$

On the other hand, if $s - t \downarrow 0$, then the variance tends to ∞ and expected value to $\pm\infty$.

In Figure 4.7 we have plotted the expected value in (4.3.4) for the month of June, based on the parameters estimated from the New York data and a market price of risk $\theta = 0$. We have chosen $t = \tau_1$, the start of the measurement month, being June 1. The mean (bold complete line) is starting around 7.5, decaying to around 1.25 before it increases towards a value slightly below 2.5. There is a lot of uncertainty for the first 1-5 days within the measurement period, before the uncertainty becomes basically zero. This is due to the stationarity of the model, of course. In the beginning of the measurement period, the approximation $\Psi(x) \approx x$ will work very good by looking at the mean only, however, the large variations may induce a large error. From around 13 days when the mean is above 1.5 and we have basically no error, the approximation will be good. We are integrating over all the measurement period to obtain the CDD price, and large errors in the beginning may not be as influential as this is a smaller part of the total price.

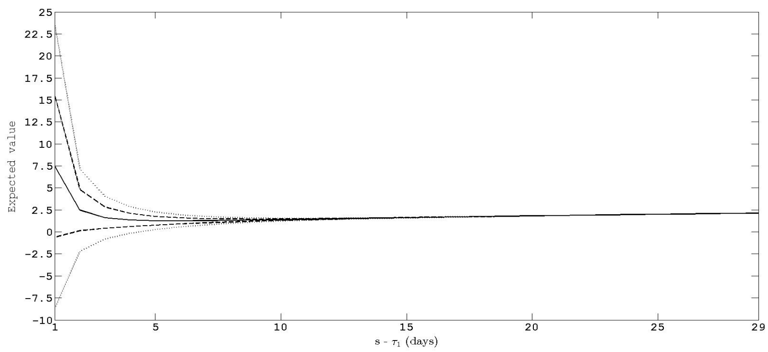

Figure 4.7: Expected value function (complete line) in (4.3.4) for the month of June 2011, as a function of $s - \tau_1$, where we have chosen $t = \tau_1 =$ June 1st, 2011. In addition, we have inserted the bounds for ± 1 std (slashed line) and ± 2 std (dotted line).

Taking $t < \tau_1$ will make the uncertainty of the expected value smaller, and in fact if $t << \tau_1$, we are basically considering the case with no uncertainty. We remark that the expected value in (4.3.4) is varying with s mostly because of the volatility function $\Sigma(t-s)$ scaling the seasonality function less the threshold c.

The similar CDD prices for the months of July and August indicate a much better situation for the validity of the approximation $\Psi(x) \approx x$. In Figure 4.8 we observe the expected value in (4.3.4) for a July contract to be slightly below 2.5 for the larger part of the measurement period. For CDD measured in August, the expected value decays (see Figure 4.9), but is above 2.3 throughout the measurement period. Both months start out with expected values much higher than 2.5 for the first 1-4 days, say.

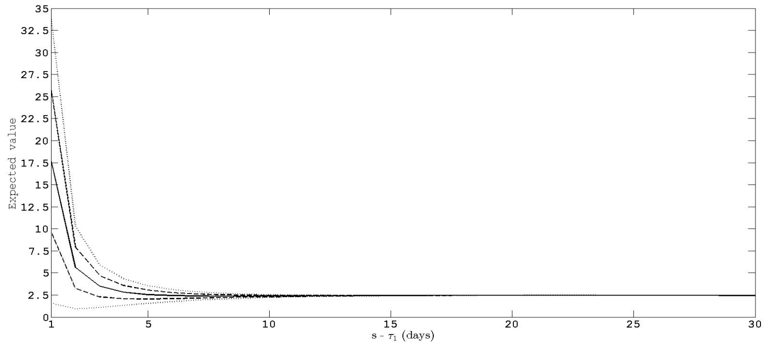

Figure 4.8: Expected value function (complete line) in (4.3.4) for the month of July 2011, as a function of $s - \tau_1$, where we have chosen $t = \tau_1 =$ July 1st, 2011. In addition, we have inserted the bounds for ± 1 std (slashed line) and ± 2 std (dotted line).

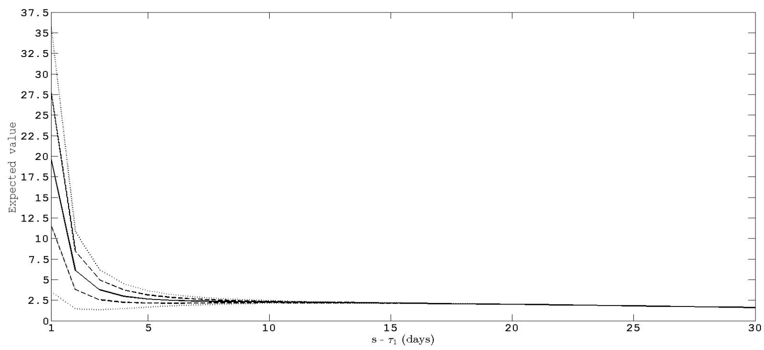

Figure 4.9: Expected value function (complete line) in (4.3.4) for the month of August 2011, as a function of $s - \tau_1$, where we have chosen $t = \tau_1 =$ August 1st, 2011. In addition, we have inserted the bounds for ± 1 std (slashed line) and ± 2 std (dotted line).

Let us turn our attention to the HDD futures contracts. For New York, these are traded in the months of October to April. Based on the estimated seasonal function $\Lambda(t)$ for New York, we find the monthly mean temperatures to be $46.51°$F in November, $37.59°$F in December, $33.34°$F in January, $34.89°$F in February and $41.62°$F in March. For the "border months" we have $57.76°$F for October and $52.05°$F for April. As we see, all the relevant months for HDD futures have mean temperatures significantly less that the threshold $c = 65°$F, and it is highly reasonable to apply the approximation $\Psi(x) \approx x$. The approximative HDD futures price based on this becomes

$$F_{\text{HDD}}(t,\tau_1,\tau_2) \approx \int_{\tau_1}^{\max(t,\tau_1)} \max(c - T(s), 0)\,ds + \Theta_x(t,\tau_1,\tau_2) - \mathbf{a}_x(t,\tau_1,\tau_2)\mathbf{X}(t)\,, \quad (4.3.6)$$

for

$$\mathbf{a}_x(t,\tau_1,\tau_2) = \mathbf{e}_1' A^{-1}\left(\exp(A(\tau_2 - t)) - \exp(A(\max(t,\tau_1) - t))\right) \quad (4.3.7)$$

and

$$\Theta_x(t,\tau_1,\tau_2) = \int_{\max(t,\tau_1)}^{\tau_2} c - \Lambda(s)\,ds - \int_{\max(t,\tau_1)}^{\tau_2}\int_t^s \mathbf{e}_1' \exp(A(s-u))\mathbf{e}_p\theta(u)\,du\,ds\,. \quad (4.3.8)$$

CDD futures are also traded for the months of October and April, where we have seen that the mean temperature is "far" below the threshold $c = 65°$. Hence, it is not reasonable to expect the approximation $\Psi(x) \approx x$ to work very well for CDD futures measured in these two months. On the other hand, the HDD futures can be approximated rather well, and by resorting to the CDD-HDD parity in Corollary 4.2.2, we can work out an alternative approximation of the CDD futures also for these two months.

There are months where the first order Taylor expansion of $\Psi(x)$ may provide a useful approximation of the futures prices. In the months May and September the average daily temperatures are 63.34°F and 68.30°F, respectively. For these months the daily average temperatures will evolve in the close vicinity of the threshold $c = 65$°F. As May and September are months where CDD futures are traded, it is reasonable to look at an approximation of the CDD futures price based on the Taylor expansion of $\Psi(x)$ around zero. The same approximation analysis as above except using $\Psi(x) \approx 1/\sqrt{2\pi} + 0.5x$, yields,

$$F_{\text{CDD}}(t,\tau_1,\tau_2) \approx \int_{\tau_1}^{\max(t,\tau_1)} \max(T(s) - c, 0)\,ds + \Theta_{\text{taylor}}(t,\tau_1,\tau_2) + \mathbf{a}_{\text{taylor}}(t,\tau_1,\tau_2)\mathbf{X}(t)\,, \quad (4.3.9)$$

for

$$\mathbf{a}_{\text{Taylor}}(t,\tau_1,\tau_2) = \frac{1}{2}\mathbf{e}_1' A^{-1}\left(\exp(A(\tau_2 - t)) - \exp(A(\max(t,\tau_1) - t))\right) \quad (4.3.10)$$

and

$$\Theta_{\text{Taylor}}(t,\tau_1,\tau_2) = \int_{\max(t,\tau_1)}^{\tau_2} \frac{1}{2}(\Lambda(s) - c) + \frac{1}{\sqrt{2\pi}}\Sigma(t,s)\,ds + \frac{1}{2}\int_{\max(t,\tau_1)}^{\tau_2}\int_t^s \mathbf{e}_1' \exp(A(s-u))\mathbf{e}_p\theta(u)\,du\,ds\,. \quad (4.3.11)$$

In Figure 4.10 we have plotted the exact CDD price from the expression in Proposition 4.2.1 together with the approximated CDD price in (4.3.9). The measurement month of the CDD index is chosen to be September, and we compute the prices based on observations of the New York temperatures. The plot shows that both the approximated and the exact move similarily, however being quite far from each other. The approximation in not working satisfactory in this case. Possibly one could move to a second-order Tayolor expansion of $\Psi(x)$ to obtain a better approximation.

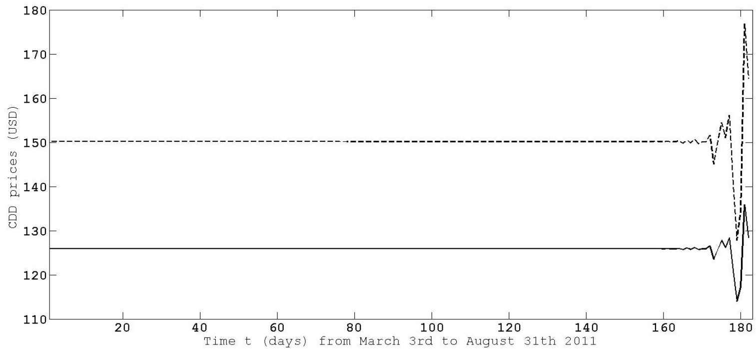

Figure 4.10: Forward prices for CDD contracts from March 3rd 2011 to August 31th, 2011 with measurement period September 2011. Theoretical (broken line) versus approximated (complete line).

We remark that our analysis of the goodness of the approximations are based on the temperature evolution in New York, and the results may be different for other cities. Obviously, the mean temperatures are different for different locations. As the CDD and HDD futures are traded for locations over all of US (and in fact also other places in the world), the linearizatioins that we have proposed may fail, or work, for different months or periods over the year.

Finally, we include an empirical example where we have estimated the market price of risk in the theoretical and approximative model that best fits real forward prices of HDD contracts based on observations of the New York temperatures, see Figure 4.11. For the purpose, we have used non-linear fitting and we have got $\theta = -0.102$ for the theoretical model and $\theta = -0.095$ for the approximative model. Not unexpectedly, the calibrated market price of risk for the approximative model is reasonably close to the corresponding value for the theoretical. On the other hand, both the approximative and the theoretical prices (seen as dotted lines in Figure 4.11) are not explaining the observed forward prices well as we are close to beginning of the settlement period. Far away, we have a good match between the model and market prices, but seems that the market prices are much more sensitive to the variations in the underlying temperature than the model. Also, one may expect weather forecasts to play a significant role, which are not accounted for in our theoretical model. Hence, one may argue for much more sophisticated pricing measure Q than what we have introduced in this paper. Another aspect is liquidity, which is rather low for these weather derivatives compared to other finanacial assets like stocks and commodities.

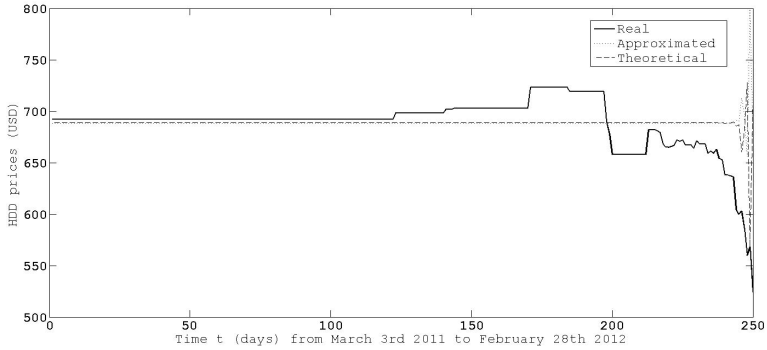

Figure 4.11: Forward prices for HDD contracts from March 3rd 2011 to February 28th 2012 with measurement period March 2012 for an estimated market price of risk.

4.4 Application to pricing of plain vanilla options

As an application of our approximation of CDD and HDD futures, we include a discussion on pricing of call options on these futures. Such options are offered for trade at the CME, and hence it is highly relevant to have available efficient pricing methods.

To this end, we consider a call option with strike K at exercise time τ, written on a CDD futures with measurement period $[\tau_1, \tau_2]$. We suppose that exercise takes place prior to measurement period, so that $\tau \leq \tau_1$, which is the typical situation for the options traded at the CME. The arbitrage-free price at time $t \leq \tau$ will become

$$C(t, \tau, \tau_1, \tau_2, K) = \mathrm{e}^{-r(\tau - t)} \mathbb{E}_Q \left[\max \left(F_{\text{CDD}}(\tau, \tau_1, \tau_2) - K, 0 \right) \mid \mathcal{F}_t \right] , \tag{4.4.1}$$

where the constant $r > 0$ is the risk-free interest rate. We recall from (4.3.1) and (4.3.9) that the CDD futures price dynamics can be approximated by a dynamics of the form

$$\widetilde{F}_{\text{CDD}}(t, \tau_1, \tau_2) = \Theta(t, \tau_1, \tau_2) + \mathbf{a}(t, \tau_1, \tau_2) \mathbf{X}(t) , \tag{4.4.2}$$

for $t \leq \tau_1$. Here, Θ and $\mathbf{a}$ are generic notations referring to Θ_x, Θ_{Taylor} and $\mathbf{a}_x$, $\mathbf{a}_{\text{Taylor}}$, respectively. We find the following:

Proposition 4.4.1. *The price of a call option at time t with strike K and exercise $\tau \geq t$ written on a CDD futures with measurement $[\tau_1, \tau_2]$, $\tau \leq \tau_1$, having approximative dynamics $F_{CDD}(t, \tau_1, \tau_2) \approx \widetilde{F}_{CDD}(t, \tau_1, \tau_2)$ defined in* (4.4.2) *is $\widetilde{C}(t, \tau, \tau_1, \tau_2, K, \widetilde{F}_{CDD}(t, \tau_1, \tau_2))$ where*

$$\widetilde{C}(t, \tau, \tau_1, \tau_2, K, x) = \mathrm{e}^{-r(\tau - t)} S(t, \tau, \tau_1, \tau_2) \Psi\left(\frac{d(t, \tau, \tau_1, \tau_2, x) - K}{S(t, \tau, \tau_1, \tau_2)} \right)$$

with

$$d(t, \tau, \tau_1, \tau_2, K, x) = x + \Theta(\tau, \tau_1, \tau_2) - \Theta(t, \tau_1, \tau_2) + \int_t^\tau \theta(s) \mathbf{a}(s, \tau_1, \tau_2) \mathbf{e}_p \, ds$$

and

$$S^2(t,\tau,\tau_1,\tau_2) = \int_t^\tau \sigma^2(s)(\mathbf{a}(s,\tau_1,\tau_2)\mathbf{e}_p)^2\,ds\,.$$

We recall $\Psi(x) = x\Phi(x) + \Phi'(x)$, *with* Φ *being the cumulative standard normal distribution function.*

Proof. First, we note that

$$\mathbf{X}(\tau) = \mathrm{e}^{A(\tau-t)}\mathbf{X}(t) + \int_t^\tau \theta(s)\mathrm{e}^{A(\tau-s)}\mathbf{e}_p\,ds + \int_t^\tau \sigma(s)\mathrm{e}^{A(\tau-s)}\mathbf{e}_p\,dW(s)\,,$$

for $t \le \tau$. Thus,

$$\begin{aligned}
\widetilde{F}_{\mathrm{CDD}}(\tau,\tau_1,\tau_2) &= \Theta(\tau,\tau_1,\tau_2) + \mathbf{a}(\tau,\tau_1,\tau_2)\mathbf{X}(\tau)\\
&= \Theta(\tau,\tau_1,\tau_2) + \mathbf{a}(\tau,\tau_1,\tau_2)\mathrm{e}^{A(\tau-t)}\mathbf{X}(t) + \int_t^\tau \theta(s)\mathbf{a}(\tau,\tau_1,\tau_2)\mathrm{e}^{A(\tau-s)}\mathbf{e}_p\,ds\\
&\quad + \int_t^\tau \sigma(s)\mathbf{a}(\tau,\tau_1,\tau_2)\mathrm{e}^{A(\tau-s)}\mathbf{e}_p\,dW(s)\\
&= \Theta(\tau,\tau_1,\tau_2) + \mathbf{a}(t,\tau_1,\tau_2)\mathbf{X}(t) + \int_t^\tau \theta(s)\mathbf{a}(s,\tau_1,\tau_2)\mathbf{e}_p\,ds\\
&\quad + \int_t^\tau \sigma(s)\mathbf{a}(s,\tau_1,\tau_2)\mathbf{e}_p\,dW(s)\\
&= \widetilde{F}_{\mathrm{CDD}}(t,\tau_1,\tau_2) + \Theta(\tau,\tau_1,\tau_2) - \Theta(t,\tau_1,\tau_2) + \int_t^\tau \theta(s)\mathbf{a}(s,\tau_1,\tau_2)\mathbf{e}_p\,ds\\
&\quad + \int_t^\tau \sigma(s)\mathbf{a}(s,\tau_1,\tau_2)\mathbf{e}_p\,dW(s)\,,
\end{aligned}$$

since $\mathbf{a}(\tau,\tau_1,\tau_2)\exp(A(\tau-s)) = a(s,\tau_1,\tau_2)$ for $t \le s \le \tau$. Since the Itô integral in the last term above is independent of $\mathcal{F}_t$, we find by the $\mathcal{F}_t$-measurability of $\widetilde{F}_{\mathrm{CDD}}(t,\tau_1,\tau_2)$ that

$$\begin{aligned}
&\mathbb{E}_Q\left[\max\left(\widetilde{F}_{\mathrm{CDD}}(\tau,\tau_1,\tau_2) - K, 0\right) \mid \mathcal{F}_t\right]\\
&\quad= \mathbb{E}_Q\left[\max\left(d(t,\tau,\tau_1,\tau_2,\widetilde{F}_{\mathrm{CDD}}(t,\tau_1,\tau_2)) + \int_t^\tau \sigma(s)\mathbf{a}(s,\tau_1,\tau_2)\mathbf{e}_p\,dW(s) - K\right) \mid \mathcal{F}_t\right]\\
&\quad= \mathbb{E}\left[\max\left(d(t,\tau,\tau_1,\tau_2,x) - K + S(t,\tau,\tau_1,\tau_2)Z, 0\right)\right]\big|_{x=\widetilde{F}_{\mathrm{CDD}}(t,\tau_1,\tau_2)}
\end{aligned}$$

for a standard normally distributed random variable Z. Here we use that $\int_t^\tau \sigma(s)\mathbf{a}(s,\tau_1,\tau_2)\mathbf{e}_p\,dW(s)$ is normally distributed with variance given by $S^2(t,\tau,\tau_1,\tau_2)$. A straightforward calculation using properties of the normal distribution completes the proof. □

We observe that the option price becomes explicitly dependent on the current (approximative) CDD futures price. The pricing formula will be a version of the famous Black-76 formula (see Black [22]) for the price of a call option on a futures when the underlying dynamics is a linear

Brownian model rather than a geometric Brownian motion. Furthermore, we also allow for time-dependent volatility, leading to the term $S(t, \tau, \tau_1, \tau_2)$ in the pricing formula. Due to the very complex nature of the CDD futures price $F_{\text{CDD}}(\tau, \tau_1, \tau_2)$, it is hard to derive an analytical pricing formula for call options, and one must resort to numerical procedures to find a price. Hence, the approximative formula that we have derived in the above Proposition provides an attractive alternative for efficient pricing.

If we consider the approximative case $\Psi(x) \approx x$, we have that $\Theta(t, \tau_1, \tau_2) = \Theta_x(t, \tau_1, \tau_2)$. A straightforward computation of the involved integrals shows that

$$d(t, \tau, \tau_1, \tau_2, x) = x + \Theta_x(\tau, \tau_1, \tau_2) - \Theta_x(t, \tau_1, \tau_2) + \int_t^\tau \mathbf{a}_x(s, \tau_1, \tau_2)\mathbf{e}_p\theta(s)\, ds\,.$$

The Taylor case becomes slighty more involved, and we leave the derivation of the corresponding expression to the interested reader.

4.5 Conclusions and outlook

We have analysed a linear approximation of the HDD and CDD temperature futures price, and demonstrated both theoretically and empirically that such approximations work well in several cases. Our investigations are based on a continuous-time autoregressive model with seasonal mean estimated to temperature data observed in New York. For this city, we find a satisfactory fit of the approximative CDD prices for the summer monts, while for autumn months there might be a larger error. Our results tell that one may price call and put options in many cases by resorting to the approximative linear price dynamics, for which we compute a "Black-76-like" pricing formula. Hence, we avoid numerical pricing, and the option's Greeks are easily available.

When comparing the theoretical and approximative prices to real data, we observe large difference which may be explained by liquidity issues, price sensitivity to weather forecasts and more complex market price of risk structures. We believe that liquidity is a serious issue in this market, arguing that our analysis has validity for benchmarking purposes.

In future studies we would like to study more general pricing measures Q that will be more sensitive to variations in the underlying temperature variations (see the general change of measure in Benth and Šaltytė Benth [15]). We believe that this will result in a better calibration to the actual observed prices, while still preserving the validity of the approxitmation studies made in this paper. By introducing weather forecasts, like for example being strongly correlated stochastic processes with the actual temperature, we can obtain an even better fit with the actual futures prices.

Chapter 5

Local sensitivity analysis of CDD and HDD derivatives prices

Abstract
We study the local sensitivity of CDD and HDD temperature futures prices and option prices written on these futures with respect to perturbations in the deseasonalized temperature or in one of its derivatives up to a certain order determined by the continuous-time autoregressive process modelling the deseasonalized temperature in the HDD and CDD indexes. We also consider an empirical case where a CAR process of autoregressive order 3 is fitted to New York temperatures and we perform a study of the local sensitivity of these financial contracts and a posterior analysis of the results.

5.1 Introduction

Weather related risks can be hedged by trading in weather derivatives. The Chicago Mercantile Exchange (CME) organizes trade in futures contracts written in weather indexes in several cities around the world. We focus on the temperature indexes HDD (heating-degree day) and CDD (cooling-degree day) which measure the aggregation of temperature below and above a threshold of $65°$F over a time period, respectively. The daily modelling of temperature is an approach that can be used to get the non-arbitrage price of temperature derivatives. A continuous-time function which consists of a deterministic term modelling the seasonal cycle of temperatures and a noise term modelling uncertainty is fitted to historical time series of daily average temperatures (DATs). Several empirical studies of temperature data have shown that continuous time autoregressive (CAR) models explain very well the statistical properties of the deseasonalized temperature dynamics (see Härdle and Lopez Cabrera [35], Benth and Šaltytė Benth [15] and the references therein). Although this approach requires a model for the instantaneous temperature contained in the indexes, it has the advantage that the model can be used for all available contracts on the market on the same location. In Benth and Solanilla Blanco [18], HDD and CDD futures prices based on CAR temperature dynamics and option prices on these futures are derived theoretically. An approximative model for the HDD and CDD futures is suggested in order to derive a closed formula for the call option price. The (approximative) formulas for HDD and CDD futures and option prices depend on the deseasonalized temperature and its derivatives up to $p-1$, where $p \in \mathbb{N} - \{0\}$ refers to the autoregressive order of the CAR process which models the temperature dynamics of these indexes.

The objective of this paper is to study the local sensitivity of the (approximative) HDD and CDD futures and option prices derived in Benth and Solanilla Blanco [18] with respect to perturbations in the deseasonalized temperature or in one of its derivatives up to order $p-1$. To do so, we consider the partial derivatives of such financial contracts with respect to these variables evaluated at a fixed point. Local sensitivity measures parameter importance by considering infinitesimal variations in a specific variable.

Sensitivity analysis is widely used in mathematical modelling to determine the influence of parameter values on response variables. The local sensitivity analysis with partial derivatives is a first step to study the response of a model to changes in their inputs variables. In mathematical finance there is extensive literature of Greeks, which are quantities representing the sensitivity of derivatives prices to a change in underlying parameters. The sensitivity analysis focused on HDD and CDD futures and option prices where the dynamics follows a CAR process has not been considered yet so our contribution is a first analysis of the local sensitivities of such financial contracts with respect to a perturbation in the deseasonalized temperature or in one of its $p-1$ derivatives.

The paper is structured as follows: in the next section we review results concerning the arbitrage-free pricing of temperature HDD and CDD futures with measurement over a period and call option written on these. We also derive some results to study the sensitivity of these financial contracts to perturbations in the deseasonalized temperature or in one of its derivatives up to order $p-1$, with p being the order of the CAR process used to model the deseasonalized temperature in these indexes. In Section 3 we adapt all the results in Section 2 for the case where the measurement time is a day instead. In Section 4 we consider a previous empirical study of New York temperatures where the deseasonalized temperature dynamics follows a CAR(3)-process and we study local sensitivity of HDD and CDD futures and option prices with a measurement day. In Section 5 we include an empirical analysis of the sensitivity of the HDD and CDD futures prices with measurement over a period. Finally, in Section 6 we present a conclusion of the results.

5.2 Local sensitivity of CDD and HDD derivatives prices with measurement over a period

Introduce a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. The CDD and HDD indexes over a time period $[\tau_1, \tau_2]$, $\tau_1 < \tau_2$, are defined respectively as

$$\mathrm{CDD}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \max(T(t) - c, 0)\, dt \tag{5.2.1}$$

and

$$\text{HDD}(\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} \max(c - T(t), 0)\, dt\,, \tag{5.2.2}$$

where $T(t)$ is the temperature of the location at time $t \geq 0$ and the threshold c is 65°F (or 18°C). The temperature is modelled as $T(t) = \Lambda(t) + Y(t)$ by means of a seasonal function Λ and a CAR(p)-process $Y(t) = \mathbf{e}_1'\mathbf{X}(t)$ defined as the first component of a multivariate Ornstein-Uhlenbeck process $\mathbf{X}(t)$ with dynamics

$$d\mathbf{X}(t) = A\mathbf{X}(t)\, dt + \sigma(t)\mathbf{e}_p\, dB(t)\,. \tag{5.2.3}$$

Here $\mathbf{z}'$ denotes the transpose of a vector (or matrix) $\mathbf{z}$ and A is a $p \times p$ matrix given by

$$A = \begin{pmatrix} \mathbf{0}_{p-1\times 1} & I_{p-1} \\ -\alpha_p .. & .. - \alpha_1 \end{pmatrix},$$

where $\alpha_i > 0$ for $i = 1 \cdots p$ and σ is the time-dependent volatility of the process. We assume that A has eigenvalues with strictly negative real part in order to get a stationary CAR-model. The arbitrage-free futures price written on a CDD index as in (5.2.1) at time $t \leq \tau_2$ is given by

$$F_{\text{CDD}}(t,\tau_1,\tau_2) := \mathbb{E}_Q\left[\text{CDD}(\tau_1,\tau_2) \mid \mathcal{F}_t\right]\,, \tag{5.2.4}$$

where the conditional expectation is taken under some probability $Q \sim P$. We work with Q given by a Girsanov transform such that the process

$$dW(t) = -\frac{\theta(t)}{\sigma(t)}\, dt + dB(t)\,,$$

is a Q-Brownian motion. The time-dependent function θ refers to the market price of risk. Analogously, we have the expression for $F_{\text{HDD}}(t,\tau_1,\tau_2)$. We refer to Benth and Solanilla Blanco [18] for a better understanding of this setting. We have extracted from this reference the following summary for pricing HDD and CDD futures contracts and call options written on these.

We recall the CDD and HDD futures prices formulas provided in Proposition 2.1 and Proposition 2.3 respectively. For our convenience in this setting we restrict $t \leq \tau_1$, so that

$$F_{\text{CDD}}(t,\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} \Sigma(t,s)\Psi\left(\frac{m_\theta(t,s,\mathbf{X}(t)) - c}{\Sigma(t,s)}\right) ds \tag{5.2.5}$$

and

$$F_{\text{HDD}}(t,\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} \Sigma(t,s)\Psi\left(\frac{c - m_\theta(t,s,\mathbf{X}(t))}{\Sigma(t,s)}\right) ds\,. \tag{5.2.6}$$

Note that $\Psi(x) = x\Phi(x) + \Phi'(x)$, with Φ being the cumulative standard normal distribution function and for $\mathbf{x} \in \mathbb{R}^p$

$$m_\theta(t,s,\mathbf{x}) = \Lambda(s) + \mathbf{e}_1' e^{A(s-t)}\mathbf{x} + \int_t^s \mathbf{e}_1' e^{A(s-u)}\mathbf{e}_p\theta(u)\, du$$

$$\Sigma^2(t,s) = \int_t^s (\mathbf{e}_1' e^{A(s-u)}\mathbf{e}_p)^2 \sigma^2(u)\, du\,.$$

If we consider the initial condition $\mathbf{X}(0) = \mathbf{x} \in \mathbb{R}^p$, then $F_{\text{CDD}}(t, \tau_1, \tau_2)$ as defined in (5.2.4) is a random variable for $t > 0$ with all the stochasticity contained in the term $\mathbf{X}(t)$. In our setting we loose this condition when for $t \geq 0$ we fix $\mathbf{X}(t) = \mathbf{x} \in \mathbb{R}^p$ with $\mathbf{x}' = (\mathrm{x}_1, \ldots, \mathrm{x}_p)$. In such a case the CDD futures price becomes a deterministic function. Denote by $X_i(t) = \mathbf{e}_i'\mathbf{X}(t)$, for $i = 1, \ldots, p$, the ith component of $\mathbf{X}(t)$. Note that $Y(t) = \mathbf{e}_1'\mathbf{X}(t) = X_1(t)$. We find that the term $\mathbf{e}_1' e^{A(s-t)}\mathbf{X}(t)$ in m_θ can be rewritten as follows as a linear combination of the components of $\mathbf{X}(t)$

$$\mathbf{e}_1' e^{A(s-t)}\mathbf{X}(t) = \sum_{i=1}^{p} f_i(s-t) X_i(t)\,, \tag{5.2.7}$$

where $f_i(s-t) = \mathbf{e}_1' \exp(A(s-t))\mathbf{e}_i$ for $i = 1, \ldots, p$. From now and to the end of this paper, we focus on CDD futures contracts and call options written on these. Similar results can be obtained considering the HDD index. The new notation in (5.2.7) let us to rewrite the CDD futures price formula as follows:

$$F_{\text{CDD}}(t, \tau_1, \tau_2, \mathrm{x}_1, \ldots, \mathrm{x}_p)\Big|_{\mathbf{x}=\mathbf{X}(t)} = \int_{\tau_1}^{\tau_2} \Sigma(t,s)\Psi\left(\frac{m_\theta(t,s,\mathbf{X}(t)) - c}{\Sigma(t,s)}\right) ds\,. \tag{5.2.8}$$

To answer the question to what extent an infinitesimal change in a component of $\mathbf{X}(t)$ is affecting the behaviour of the CDD futures price, we need to consider the partial derivatives of this with respect to the components of $\mathbf{X}(t)$, say x_i for $i = 1, \ldots, p$, to avoid misunderstandings in the notation. In the next proposition we consider the partial derivatives of the CDD futures price with respect to the components of $\mathbf{x}$.

Proposition 5.2.1. *Let $t \leq \tau_1$, then for $i = 1, \ldots, p$ it holds that*

$$\Big(\frac{\partial}{\partial \mathrm{x}_i} F_{CDD}(t, \tau_1, \tau_2, \mathrm{x}_1, \ldots, \mathrm{x}_p)\Big)\Big|_{\mathbf{x}=\mathbf{X}(t)} = \int_{\tau_1}^{\tau_2} \Phi\left(\frac{m_\theta(t,s,\mathbf{X}(t)) - c}{\Sigma(t,s)}\right) \mathbf{e}_1' \exp(A(s-t))\mathbf{e}_i\, ds\,.$$

Proof. The proof follows by exchanging the derivative and the integral and afterwards applying the chain-rule on the integrand. In this last step consider that $\Psi'(x) = \Phi(x)$ and the fact that m_θ can be rewritten as follows a linear combination of the components of $\mathbf{x}$

$$m_\theta(t,s,\mathbf{x}) = \Lambda(s) + \sum_{i=1}^{p} \mathbf{e}_1' e^{A(s-t)}\mathbf{e}_i \mathrm{x}_i + \int_t^s \mathbf{e}_1' e^{A(s-u)}\mathbf{e}_p \theta(u)\, du\,,$$

in such a way that

$$\frac{\partial}{\partial \mathrm{x}_i} m_\theta(t,s,\mathbf{x}) = \mathbf{e}_1' e^{A(s-t)}\mathbf{e}_i\,.$$

□

Observe that CDD futures prices depend nonlinearly on the vector $\mathbf{X}(t)$ which is included in the function Ψ. This fact makes difficult to derive analytic formulas for plain vanilla options (call options). To this aim, we recall some useful linearized formulas that approximate CDD futures prices and make possible then to derive call option prices. Let $t \leq \tau_1$, by setting $\Psi(x) \approx x$ in (5.2.8), we get an approximation of the CDD futures prices which is linear on $\mathbf{X}(t)$ given by

$$F_{\text{CDD}}(t,\tau_1,\tau_2,\mathrm{x}_1,\ldots,\mathrm{x}_p)\Big|_{\mathbf{x}=\mathbf{X}(t)} \approx \Theta_x(t,\tau_1,\tau_2) + \mathbf{a}_x(t,\tau_1,\tau_2)\mathbf{X}(t)\,, \tag{5.2.9}$$

where

$$\begin{aligned}
\mathbf{a}_x(t,\tau_1,\tau_2) &= \int_{\tau_1}^{\tau_2} \mathbf{e}_1' \exp(A(s-t))\,ds\,,\\
\Theta_x(t,\tau_1,\tau_2) &= \int_{\tau_1}^{\tau_2} c-\Lambda(s)\,ds + \int_{\tau_1}^{\tau_2}\int_t^s \mathbf{e}_1'\exp(A(s-u))\mathbf{e}_p\theta(u)\,du\,ds\,.
\end{aligned}$$

By setting the first order Taylor approximation $\Psi(x) \approx \frac{1}{\sqrt{2\pi}} + \frac{1}{2}x$ in (5.2.8) instead, CDD futures prices reduce to

$$F_{\text{CDD}}(t,\tau_1,\tau_2,\mathrm{x}_1,\ldots,\mathrm{x}_p)\Big|_{\mathbf{x}=\mathbf{X}(t)} \approx \Theta_{\text{Taylor}}(t,\tau_1,\tau_2) + \mathbf{a}_{\text{Taylor}}(t,\tau_1,\tau_2)\mathbf{X}(t)\,, \tag{5.2.10}$$

where

$$\begin{aligned}
\mathbf{a}_{\text{Taylor}}(t,\tau_1,\tau_2) &= \frac{1}{2}\int_{\tau_1}^{\tau_2} \mathbf{e}_1'\exp(A(s-t))\,ds\,,\\
\Theta_{\text{Taylor}}(t,\tau_1,\tau_2) &= \int_{\tau_1}^{\tau_2} \frac{1}{2}(\Lambda(s)-c) + \frac{1}{\sqrt{2\pi}}\Sigma(t,s)\,ds\\
&\quad + \frac{1}{2}\int_{\tau_1}^{\tau_2}\int_t^s \mathbf{e}_1'\exp(A(s-u))\mathbf{e}_p\theta(u)\,du\,ds\,.
\end{aligned}$$

We introduce a new notation that encompasses both approximated formulas in (5.2.9) and (5.2.10). To this end let $t \leq \tau_1$, then

$$\widetilde{F}_{\text{CDD}}(t,\tau_1,\tau_2,\mathrm{x}_1,\ldots,\mathrm{x}_p)\Big|_{\mathbf{x}=\mathbf{X}(t)} = \Theta(t,\tau_1,\tau_2) + \mathbf{a}(t,\tau_1,\tau_2)\mathbf{X}(t)\,, \tag{5.2.11}$$

where Θ and $\mathbf{a}$ are generic notations for Θ_x and $\mathbf{a}_x$ or Θ_{Taylor} and $\mathbf{a}_{\text{Taylor}}$. The next proposition provides the partial derivatives of the approximated CDD futures prices with respect to the components of $\mathbf{x}$.

Proposition 5.2.2. *Let $t \leq \tau_1$, then it holds that*

$$\frac{\partial}{\partial \mathrm{x}_i}\widetilde{F}_{CDD}(t,\tau_1,\tau_2,x_1,\ldots,x_p) = \mathbf{a}(t,\tau_1,\tau_2)\mathbf{e}_i\,, \tag{5.2.12}$$

for $i = 1,\ldots,p$ where $\mathbf{a}$ is the generic notation for $\mathbf{a}_x$ and $\mathbf{a}_{Taylor}$.

Proof. Rewrite (5.2.11) in terms of the components of $\mathbf{X}(t)$ as

$$\widetilde{F}_{\text{CDD}}(t, \tau_1, \tau_2, \mathrm{x}_1, \ldots, \mathrm{x}_p)\Big|_{\mathbf{x}=\mathbf{X}(t)} = \Theta(t, \tau_1, \tau_2) + \sum_{i=1}^{p} \mathbf{a}(t, \tau_1, \tau_2)\mathbf{e}_i X_i(t)\,.$$

Afterwards, differentiate $\widetilde{F}_{\text{CDD}}(t, \tau_1, \tau_2, \mathrm{x}_1, \ldots, \mathrm{x}_p)$ with respect to x_i for $i = 1, \ldots, p$. □

Observe that (5.2.12) does not depend on $\mathbf{X}(t)$.

We take now CDD and HDD futures prices as the underlying to price call options. The arbitrage-free price for a call option with strike K at exercise time τ, written on a CDD futures with measurement over a period $[\tau_1, \tau_2]$, with $\tau \leq \tau_1$ and for times $t \leq \tau_1$ is defined as

$$C(t, \tau, \tau_1, \tau_2, K) := \mathrm{e}^{-r(\tau-t)}\mathbb{E}_Q\left[\max\left(F_{\text{CDD}}(\tau, \tau_1, \tau_2) - K, 0\right) \mid \mathcal{F}_t\right]\,, \tag{5.2.13}$$

where $r > 0$ is the risk-free interest rate. All the stochasticity in the call option price is in the term $\mathbf{e}_1'\mathrm{e}^{A(s-\tau)}\mathbf{X}(\tau)$ which, in turn is contained in the CDD futures price at exercise time and more specifically in m_θ. For $\tau \geq t$,

$$\mathbf{X}(\tau) = \mathrm{e}^{A(\tau-t)}\mathbf{X}(t) + \int_t^\tau \theta(s)\mathrm{e}^{A(\tau-s)}\mathbf{e}_p\,ds + \int_t^\tau \sigma(s)\mathrm{e}^{A(\tau-s)}\mathbf{e}_p\,dB(s)$$

is a solution of the stochastic differential equation in (5.2.3), then $\mathbf{e}_1'\mathrm{e}^{A(s-\tau)}\mathbf{X}(\tau)$ reduces to

$$\begin{aligned} &\mathbf{e}_1'\mathrm{e}^{A(s-\tau)}\mathbf{X}(\tau) \\ &\quad= \mathbf{e}_1'\mathrm{e}^{A(s-t)}\mathbf{X}(t) + \int_t^\tau \theta(u)\mathbf{e}_1'\mathrm{e}^{A(s-u)}\mathbf{e}_p\,du + \int_t^\tau \sigma(u)\mathbf{e}_1'\mathrm{e}^{A(s-u)}\mathbf{e}_p\,dB(u)\,. \end{aligned} \tag{5.2.14}$$

As $F_{\text{CDD}}(\tau, \tau_1, \tau_2)$ depends explicitely on $\mathbf{X}(t)$ we can rewritte the call option price in (5.2.13) as follows

$$C(t, \tau, \tau_1, \tau_2, K, \mathrm{x}_1, \ldots, \mathrm{x}_p)\Big|_{\mathbf{x}=\mathbf{X}(t)} = \mathrm{e}^{-r(\tau-t)}\mathbb{E}_Q\left[\max\left(F_{\text{CDD}}(\tau, \tau_1, \tau_2) - K, 0\right) \mid \mathcal{F}_t\right]. \tag{5.2.15}$$

To study the sensitivity of the call option price with respect to infinitesimal changes in the components of $\mathbf{X}(t)$ we have to consider partial derivatives which is not an easy task, if possible. To avoid differentiating the payoff $\max(F_{\text{CDD}}(\tau, \tau_1, \tau_2, \mathrm{x}_1, \ldots, \mathrm{x}_p) - K, 0)$, we apply the density approach which moves the dependency of $\mathbf{X}(t)$ from the payoff to the required density function to compute the conditional expectation, (see Broadie and Glasserman [23]). Nevertheless, this method does not work here because the payoff function is path-dependent on $\mathbf{X}(t)$ from τ_1 to τ_2. Indeed, $\max\left(F_{\text{CDD}}(\tau, \tau_1, \tau_2) - K, 0\right)$ contains the term $\mathbf{e}_1'\mathrm{e}^{A(s-\tau)}\mathbf{X}(\tau)$ in m_θ which depends on $\mathbf{X}(t)$ as we have seen in (5.2.14).

The linearized CDD futures price as defined in (5.2.11) makes it possible to get an approximate call option price formula which is analytically treatable in the sense that approximation methods like Monte Carlo are not required. This problem is thoroughly tackled by setting in

(5.2.15) the linearized CDD futures prices defined in (5.2.11), see (Benth and Solanilla Blanco [18]) for a detailed explanation. The approximate formula for the call option price then reduces to

$$\widetilde{C}(t,\tau,\tau_1,\tau_2,K,\widetilde{F}_{\text{CDD}}(t,\tau_1,\tau_2,\text{x}_1,\ldots,\text{x}_p))\Big|_{\mathbf{x}=\mathbf{X}(t)} = \tag{5.2.16}$$
$$\mathrm{e}^{-r(\tau-t)}S(t,\tau,\tau_1,\tau_2)\Psi\Big(\frac{d(t,\tau,\tau_1,\tau_2,\widetilde{F}_{\text{CDD}}(t,\tau_1,\tau_2))-K}{S(t,\tau,\tau_1,\tau_2)}\Big)$$

with

$$d(t,\tau,\tau_1,\tau_2,K,x) = x + \Theta(\tau,\tau_1,\tau_2) - \Theta(t,\tau_1,\tau_2) + \int_t^\tau \theta(s)\mathbf{a}(s,\tau_1,\tau_2)\mathbf{e}_p\,ds$$

and

$$S^2(t,\tau,\tau_1,\tau_2) = \int_t^\tau \sigma^2(s)(\mathbf{a}(s,\tau_1,\tau_2)\mathbf{e}_p)^2\,ds\,.$$

Observe that the approximate call option becomes explicitly dependent on the approximative futures price. The next proposition provides the partial derivatives of the approximated call option price with respect to the components of $\mathbf{x}$.

Proposition 5.2.3. *Let $t \leq \tau_1$, then it holds that*

$$\Big(\frac{\partial}{\partial x_i}\widetilde{C}(t,\tau,s,K,\widetilde{F}_{CDD}(t,\tau_1,\tau_2,,x_1,\ldots,x_p))\Big)\Big|_{\mathbf{x}=\mathbf{X}(t)}$$
$$= e^{-r(\tau-t)}\Phi(\frac{d(t,\tau,s,\widetilde{F}_{CDD}(t,\tau_1,\tau_2))-K}{S(t,\tau,\tau_1,\tau_2)})\mathbf{a}(t,\tau_1,\tau_2)\mathbf{e}_i\,,$$

for $i = 1,\ldots,p$.

Proof. The proof follows by taking partial derivatives in (5.2.16). Take into account that $\Psi'(x) = \Phi(x)$ and that the only component in the function d dependent on the components of $\mathbf{X}(t)$ is the approximated futures price whose partial derivatives are provided in Proposition 5.2.2. □

In the next section we simplify our setting and consider futures prices with a measurement day and call options written on these.

5.3 Local sensitivity of CDD and HDD derivatives prices with measurement over a day

In this Section we perform a complete study of the sensitivity of CDD and HDD futures prices with a measurement day and call option prices derived from these futures to infinitesimal changes on the components of $\mathbf{X}(t)$.

The Fubini-Tonelli theorem (see e.g. Folland [32]) connects futures prices setting over a time period and futures prices with a delivery day running over a time period as follows

$$F_{\text{CDD}}(t,\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} \mathbb{E}_Q[\max(T(s)-c,0)|\mathcal{F}_t]\,ds = \int_{\tau_1}^{\tau_2} F(t,s)\,ds\,,$$

for $t \leq s$. We see that $F_{\text{CDD}}(t,\tau_1,\tau_2)$ is expressed as the CDD futures price at time t with a measurement day s running over the time period $[\tau_1,\tau_2]$. Consequently, we deduce from (5.2.5) and (5.2.6) respectively that

$$F_{\text{CDD}}(t,s) = \Sigma(t,s)\Psi\left(\frac{m_\theta(t,s,\mathbf{X}(t))-c}{\Sigma(t,s)}\right)$$

and

$$F_{\text{HDD}}(t,s) = \Sigma(t,s)\Psi\left(\frac{c-m_\theta(t,s,\mathbf{X}(t))}{\Sigma(t,s)}\right)\,,$$

where, for $\mathbf{x} \in \mathbb{R}^p$

$$m_\theta(t,s,\mathbf{x}) = \Lambda(s) + \mathbf{e}_1' e^{A(s-t)}\mathbf{x} + \int_t^s \mathbf{e}_1' e^{A(s-u)}\mathbf{e}_p\theta(u)\,du$$

$$\Sigma^2(t,s) = \int_t^s (\mathbf{e}_1' e^{A(s-u)}\mathbf{e}_p)^2\sigma^2(u)\,du\,.$$

Recall that $\Psi(x) = x\Phi(x) + \Phi'(x)$, with Φ being the cumulative standard normal distribution function. The same notation introduced in (5.2.8) can be used in this context, then

$$F_{\text{CDD}}(t,s,\mathrm{x}_1,\ldots,\mathrm{x}_p)\Big|_{\mathbf{x}=\mathbf{X}(t)} = \Sigma(t,s)\Psi\left(\frac{m_\theta(t,s,\mathbf{X}(t))-c}{\Sigma(t,s)}\right).$$

The term $\mathbf{e}_1'\exp(A(s-t))\mathbf{X}(t)$ included in m_θ and rewritten as in (5.2.7) provides information about the evolution of the futures price. Note that when the time to maturity $s-t \to \infty$, the function $f_i(s-t)$ tends to zero for $i = 1,\cdots,p$ since we asume that the real parts of all the eigenvalues of the matrix A are strictly negative in order to have a stationary model. Hence, at the long end the behaviour of the futures prices is not affected by this term. But, if $s-t$ is approaching to zero, the term $\mathbf{e}_1'\exp(A(s-t))\mathbf{X}(t)$ is influenced for all the components of the $\mathbf{X}(t)$. Finally, for $x=0$, only the first component of $\mathbf{X}(t)$ takes part on the evolution of the futures price. These arguments determine the evolution of futures prices for times $t>0$ when time to delivery is a day $s, s \geq t$.

In the next proposition we consider the partial derivatives of the CDD futures price with respect to the components of $\mathbf{x}$.

Proposition 5.3.1. *Let $t \leq s$, it holds that*

$$\left(\frac{\partial}{\partial \mathrm{x}_i}F_{CDD}(t,s,\mathrm{x}_1,\ldots,\mathrm{x}_p)\right)\Big|_{\mathbf{x}=\mathbf{X}(t)} = \Phi\left(\frac{m_\theta(t,s,\mathbf{X}(t))-c}{\Sigma(t,s)}\right)\mathbf{e}_1'\exp(A(s-t))\mathbf{e}_i\,,$$

for $i = 1,\ldots,p$.

Proof. The proof follows by applying the chain-rule. Consider that $\Psi'(x) = \Phi(x)$ and also take into account that m_θ can be written as a linear combination of the components of $\mathbf{x}$ as follows

$$m_\theta(t, s, \mathbf{x}) = \Lambda(s) + \sum_{i=1}^{p} \mathbf{e}_1' e^{A(s-t)} \mathbf{e}_i \mathrm{x}_i + \int_t^s \mathbf{e}_1' e^{A(s-u)} \mathbf{e}_p \theta(u)\, du\,,$$

so that

$$\frac{\partial}{\partial \mathrm{x}_i} m_\theta(t, s, \mathbf{x}) = \mathbf{e}_1' e^{A(s-t)} \mathbf{e}_i\,.$$

□

Next, we also adapt the linearized formulas for CDD futures prices presented in Section 5.2 to our setting. To this end let $t \leq s$, formulas (5.2.9) and (5.2.10) reduce respectively to

$$F_{\text{CDD}}(t, s, \mathrm{x}_1, \ldots, \mathrm{x}_p) \approx \Theta_x(t, s) + \mathbf{a}_x(t, s)\mathbf{x}\,,$$

where

$$\begin{aligned} \mathbf{a}_x(t, s) &= \mathbf{e}_1' \exp(A(s-t)) \\ \Theta_x(t, s) &= \Lambda(s) - c + \int_t^s \mathbf{e}_1' \exp(A(s-u)) \mathbf{e}_p \theta(u)\, du\,, \end{aligned}$$

and

$$F_{\text{CDD}}(t, s, \mathrm{x}_1, \ldots, \mathrm{x}_p) \approx \Theta_{\text{Taylor}}(t, s) + \mathbf{a}_{\text{Taylor}}(t, s)\mathbf{x}\,,$$

where

$$\begin{aligned} \mathbf{a}_{\text{Taylor}}(t, s) &= \frac{1}{2} \mathbf{e}_1' \exp(A(s-t)) \\ \Theta_{\text{Taylor}}(t, s) &= \frac{1}{2}(\Lambda(s) - c) + \frac{1}{\sqrt{2\pi}} \Sigma(t, s) + \frac{1}{2} \int_t^s \mathbf{e}_1' \exp(A(s-u)) \mathbf{e}_p \theta(u)\, du\,. \end{aligned}$$

We provide also the new notation that encompasses both approximated CDD futures prices formulas. To this end let $t \leq s$, then

$$\widetilde{F}_{\text{CDD}}(t, s, \mathrm{x}_1, \ldots, \mathrm{x}_p)\Big|_{\mathbf{x}=\mathbf{X}(t)} = \Theta(t, s) + \mathbf{a}(t, s)\mathbf{X}(t) \tag{5.3.1}$$

where Θ and $\mathbf{a}$ are generic notations for Θ_x and $\mathbf{a}_x$ or Θ_{Taylor} and $\mathbf{a}_{\text{Taylor}}$. The next proposition provides the partial derivatives of the approximated CDD futures prices with respect to the components of $\mathbf{x}$.

Proposition 5.3.2. *Let $t \leq s$, then it holds that*

$$\frac{\partial}{\partial x_i} \widetilde{F}_{CDD}(t, s, x_1, \ldots, x_p) = \mathbf{a}(t, s)\mathbf{e}_i$$

for $i = 1, \ldots, p$, where $\mathbf{a}$ is the generic notation for $\mathbf{a}_x$ and $\mathbf{a}_{Taylor}$.

Proof. Rewrite (5.3.1) in terms of the components of $\mathbf{x}$ as

$$\widetilde{F}_{\text{CDD}}(t,s,\mathrm{x}_1,\ldots,\mathrm{x}_p)\Big|_{\mathbf{x}=\mathbf{X}(t)} = \Theta(t,s) + \sum_{i=1}^{p} \mathbf{a}(t,s)\mathbf{e}_i X_i(t)\,,$$

and differentiate $\widetilde{F}_{\text{CDD}}(t,s,\mathrm{x}_1,\ldots,\mathrm{x}_p)$ with respect to the components x_i for $i=1,\ldots,p$. □

Observe that unlike the result in Proposition 5.3.1, here we loose the dependency on $\mathbf{X}(t)$.

The arbitrage-free price for a call option with strike K at exercise time τ, written on a CDD futures with a measurement day s, for a time $t>0$ with $t\leq\tau\leq s$ is defined as

$$C(t,\tau,s,K) := \mathrm{e}^{-r(\tau-t)}\mathbb{E}_Q\left[\max\left(F_{\text{CDD}}(\tau,s)-K,0\right) \mid \mathcal{F}_t\right]\,, \tag{5.3.2}$$

where $r>0$ is the risk-free interest rate. For our purpose and making use of the same argument as in Section 5.2 we can rewritte (5.3.2) as

$$C(t,\tau,s,K,\mathrm{x}_1,\ldots,\mathrm{x}_p)\Big|_{\mathbf{x}=\mathbf{X}(t)} := \mathrm{e}^{-r(\tau-t)}\mathbb{E}_Q\left[\max\left(F_{\text{CDD}}(\tau,s)-K,0\right) \mid \mathcal{F}_t\right]\,.$$

Next, we see that the density approach here works well as the payoff function of the call option price is not path-dependent on $\mathbf{X}(t)$ over a time period. In the next Proposition, we present the partial derivatives of the call option price with respect to the components of $\mathbf{x}$.

Proposition 5.3.3. *Let $t\leq s$, then it holds that*

$$\Big(\frac{\partial}{\partial x_i}C(t,s,\tau,K,x_1,\ldots,x_p)\Big)\Big|_{\mathbf{x}=\mathbf{X}(t)} = e^{-r(\tau-t)}\mathbb{E}_Q\left[g(Z,t,s,\tau,\mathbf{X}(t))\right]$$

for $i=1,\ldots,p$, where for $\mathbf{x}\in\mathbb{R}^p$

$$g(Z,t,s,\tau,\mathbf{x}) = \max(F_{CDD,Z}(\tau,s)-K,0)\left(\frac{Z-\widetilde{m}_\theta(t,s,\tau,\mathbf{x})}{\widetilde{\Sigma}^2(t,s,\tau)}\right)\mathbf{e}_1'\exp(A(s-t))\mathbf{e}_i\,.$$

$Z=\mathbf{e}_1'\exp(A(s-\tau))\mathbf{X}(\tau)$ is a normally distributed random variable and

$$\widetilde{m}_\theta(t,s,\tau,\mathbf{x}) = \mathbf{e}_1'\exp(A(s-t))\mathbf{x} + \int_t^\tau \mathbf{e}_1'\exp(A(s-u))\mathbf{e}_p\theta(u)\,du$$
$$\widetilde{\Sigma}^2(t,s,\tau) = \int_t^\tau (\mathbf{e}_1'\exp(A(s-u))\mathbf{e}_p)^2\sigma^2(u)\,du$$

are the mean and the variance of Z conditioned on $\mathbf{X}(t)$, respectively.

Proof. The random variable $\mathbf{e}_1'\exp(A(s-\tau))\mathbf{X}(\tau) = Z$ included in $m_\theta(\tau,s,\mathbf{x})$ is normally distributed and

$$\tilde{m}_\theta(t,s,\tau,\mathbf{x}) = \mathbf{e}_1'\mathrm{e}^{A(s-t)}\mathbf{x} + \int_t^\tau \theta(u)\mathbf{e}_1'\mathrm{e}^{A(s-u)}\mathbf{e}_p\,du$$

and

$$\tilde{\Sigma}^2(t,s,\tau) = \int_t^\tau \sigma^2(u)(\mathbf{e}_1' \exp(A(s-u))\mathbf{e}_p)^2\,du$$

are the mean and the variance of Z, respectively, conditioned on $\mathbf{X}(t)$. The probability density function of Z is then

$$p_Z(z,t,s,\tau,\mathbf{x}) = \frac{1}{\sqrt{2\pi}\tilde{\Sigma}(t,s,\tau)} \exp\Big(-\frac{1}{2}\Big(\frac{z-\tilde{m}(t,s,\tau,\mathbf{x})}{\tilde{\Sigma}(t,s,\tau)}\Big)^2\Big).$$

We see that we have moved the dependency of $\mathbf{X}(t)$ contained in Z from the payoff function to the required density function in order to compute the the conditional expectation as follows

$$\begin{aligned}
&\big(\frac{\partial}{\partial \mathrm{x}_i} C(t,s,\tau,K,\mathrm{x}_1,\ldots,\mathrm{x}_p)\big)\Big|_{\mathbf{x}=\mathbf{X}(t)} \\
&\quad = e^{-r(\tau-t)} \int_{\mathbb{R}} \max(F_{\mathrm{CDD},z}(\tau,s)-K,0)\big(\frac{\partial}{\partial \mathrm{x}_i} P_Z(z,t,s,\tau,\mathbf{x})\big)\,dz\Big|_{\mathbf{x}=\mathbf{X}(t)} \\
&\quad = e^{-r(\tau-t)} \int_{\mathbb{R}} \max(F_{\mathrm{CDD},z}(\tau,s)-K,0)\big(\frac{z-\widetilde{m}_\theta(t,s,\tau,\mathbf{x})}{\widetilde{\Sigma}^2(t,s,\tau)}\big) \\
&\qquad \times \mathbf{e}_1' \mathrm{e}^{A(s-t)} \mathbf{e}_i P_Z(z,t,s,\tau,\mathbf{x})\,dz\Big|_{\mathbf{x}=\mathbf{X}(t)} \\
&\quad = \mathbb{E}_Q\left[\max(F_{\mathrm{CDD},Z}(\tau,s)-K,0)\big(\frac{Z-\widetilde{m}_\theta(t,s,\tau,\mathbf{X}(t))}{\widetilde{\Sigma}^2(t,s,\tau)}\big)\mathbf{e}_1'\mathrm{e}^{A(s-t)}\mathbf{e}_i \mid \mathcal{F}_t\right].
\end{aligned}$$

□

Next, we adapt to our setting the approximated call option price formula in (5.2.16) which reduces to

$$\begin{aligned}
&\widetilde{C}(t,\tau,s,K,\widetilde{F}_{\mathrm{CDD}}(t,s,\mathrm{x}_1,\ldots,\mathrm{x}_p))\Big|_{\mathbf{x}=\mathbf{X}(t)} \\
&\quad = \mathrm{e}^{-r(\tau-t)} S(t,\tau,s)\Psi\Big(\frac{d(t,\tau,s,\widetilde{F}_{\mathrm{CDD}}(t,s))-K}{S(t,\tau,s)}\Big)
\end{aligned} \tag{5.3.3}$$

with

$$d(t,\tau,s,K,x) = x + \Theta(\tau,s) - \Theta(t,s) + \int_t^\tau \theta(u)\mathbf{a}(u,s)\mathbf{e}_p\,du$$

and

$$S^2(t,\tau,s) = \int_t^\tau \sigma^2(u)(\mathbf{a}(u,s)\mathbf{e}_p)^2\,du\,.$$

We end up this Section with a result for the partial derivatives of the approximated call option price with respect to the components of $\mathbf{x}$.

Proposition 5.3.4. *Let $t \leq s$, then it holds that*

$$\left(\frac{\partial}{\partial x_i}\widetilde{C}(t,\tau,s,K,\widetilde{F}_{CDD}(t,s,x_1,\dots,x_p))\right)\Big|_{\mathbf{x}=\mathbf{X}(t)} = e^{-r(\tau-t)}\Phi\left(\frac{d(t,\tau,s,\widetilde{F}_{CDD}(t,s))-K}{S(t,\tau,s)}\right)\mathbf{a}(t,s)\mathbf{e}_i\,.$$

for $i = 1,\dots,p$.

Proof. The proof follows by taking partial derivatives in (5.3.3). Take into account that $\Psi'(x) = \Phi(x)$ and that the only component in function d dependent on the components of $\mathbf{X}(t)$ is the approximated futures price whose partial derivatives are provided in Proposition 5.3.2. □

5.4 Empirical study of the local sensitivity of CDD futures and option prices with a measurement day

Consider the stationary CAR(3)-process obtained in Benth and Solanilla Blanco [18] reprinted in Chapter 4 to model the temperature data from New York which is defined by the following mean reverting matrix A,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.3364 & -1.6105 & -2.1618 \end{pmatrix}$$

and a constant volatility, $\sigma = 5.25$. The function Σ^2 which defines the CDD and HDD futures prices now reduces to

$$\Sigma^2(t,s) := \Sigma^2(s-t) = \sigma^2 \int_0^{s-t} (\mathbf{e}_1' \mathrm{e}^{Au}\mathbf{e}_p)^2\,du\,.$$

Furthermore, we choose to work with $\theta = 0$ and fix the measurement day s to be August 1st, 2011. We focus our empirical study on CDD futures contracts and call options written on these. Further, we will do some empirics on CDD futures prices with measurement over August 2011. We choose to work with delivery in August 2011, whether it is a particular day or the whole month, as in Benth and Solanilla Blanco [18] it is proved that the linearized formula for the CDD futures price works well.

We start analysing the sensitivity of CDD futures prices with a measurement day. To do so, we consider the random variable $\Phi(m_\theta(t,s,\mathbf{X}(t)) - c)/\Sigma(s-t))$ which makes the difference between the results provided in Propositions 5.3.1 and 5.3.2. To simplify the notation, we write $\Phi(Z(t,s))$ where

$$Z(t,s) := (m_\theta(t,s,\mathbf{X}(t)) - c)/\Sigma(s-t)\,, \tag{5.4.1}$$

and $s-t$ is the time to maturity. In Benth and Solanilla Blanco [18], a more general study where θ is a time dependent function concludes that when $s-t \downarrow 0$ the expected value of $Z(t,s)$ tends to $\pm\infty$ and the variance to ∞. Such a case indicates too much dispersion. On the other hand, when

$s-t \to \infty$, the variance of $Z(t,s)$ tends to zero. As a consequence $Z(t,s)$ can be approximated as $Z(t,s) \approx \mathbb{E}_Q(Z(t,s))$. Figure 5.1 shows the tendency $Z(t,s)$ when the measurement day s is August 1st, 2011.

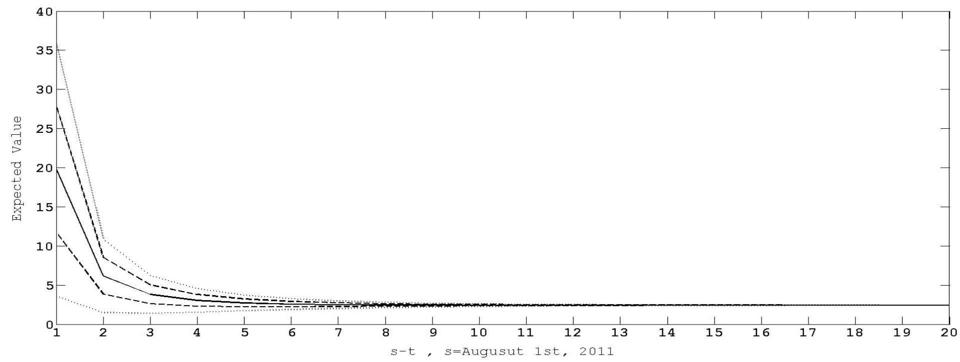

Figure 5.1: Expected value of $Z(t,s)$ (complete line) as a function of $s-t$ with measurement day s being August 1st, 2011. In addition, we have inserted the bounds for ± 1 std (slashed line) and ± 2 std (dotted line).

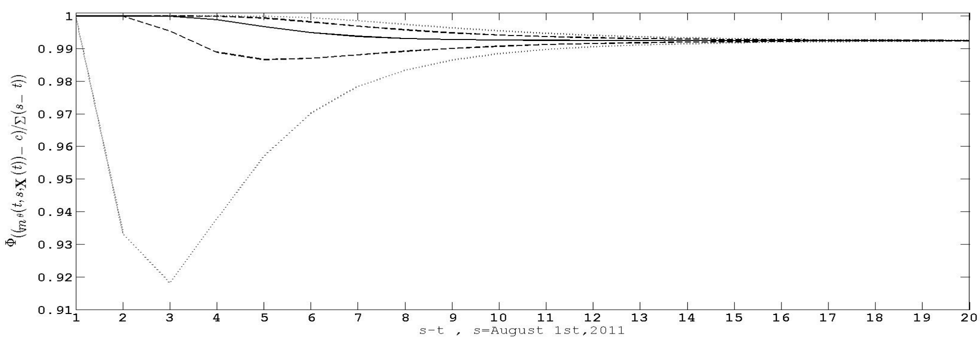

Figure 5.2: $\Phi(\mathbb{E}_Q[Z(t,s)])$ (complete line) as a function of $s-t$, where s is August 1st, 2011. In addition, we have inserted $\Phi(\mathbb{E}_Q[Z(t,s)] \pm i\,\text{std})$ for $i=1$ (slashed line) and $i=2$ (dotted line).

Figure 5.2 results from applying the function Φ to Figure 5.1. We see that $\Phi(Z(t,s)) = 1$ for $s-t = 1,2$. From $s-t=3$ to $s-t=11$ there is a small decay and finally $\Phi(Z(t,s))$ stabilizes at 0.9924 for $s-t \geq 12$. The values between the dashed lines are more probable than the ones in between the dotted lines. First of all, we deduce that the partial derivatives of the CDD futures price in Proposition 5.3.1 and the partial derivatives of the approximated CDD futures price in Proposition 5.3.2 are close and satisfy that

$$\Big(\frac{\partial}{\partial \mathrm{x}_i} F_{\text{CDD}}(t,s,\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3)\Big)_{|\mathbf{x}=\mathbf{X}(t)} \leq \frac{\partial}{\partial \mathrm{x}_i} \widetilde{F}_{CDD}(t,s,\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3),$$

for $i = 1, \ldots, 3$. As a consequence, the approximated CDD futures prices are more sensitive to changes in the components of $\mathbf{X}(t)$ than the CDD futures prices. Recall from (Benth and Solanilla Blanco [17]) that the vector $\mathbf{X}(t)$ contains the deseasonalized temperature $Y(t)$ and its derivatives up to order $p-1$. In our setting $\mathbf{X}(t)$ reduces to

$$\mathbf{X}(t) = \begin{pmatrix} Y(t) \\ Y'(t) \\ Y''(t) \end{pmatrix},$$

where $Y(t)$ is the deseasonalized temperature and $Y'(t)$ and $Y''(t)$ are respectively the slope and the curvature of $Y(t)$.

Next, we show the plot with the partial derivatives of the approximated CDD futures price with respect the coordinates of $\mathbf{x}$ derived in Proposition 5.3.2.

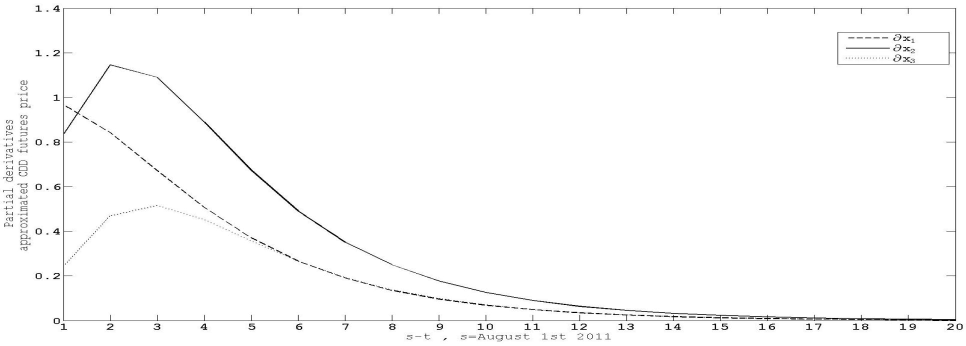

Figure 5.3: $\partial \widetilde{F}_{\mathrm{CDD}}(t, s, \mathrm{x}_1, \ldots, \mathrm{x}_p)/\partial \mathrm{x}_i$ for $i = 1, 2, 3$ as a function of $s-t$ with measurement day s being August 1st, 2011.

The x-axis considers the time to maturity, $s-t$, where the measurement day s is August 1st, 2011. The y-axis shows the different partial derivatives. Firstly, we observe that the partial derivatives of the approximated CDD futures price are positive. We see that at time to maturity $s-t=1$ any perturbation in the component x_1 affects the tendency of the approximated CDD futures price more than in the components x_2 and x_3. However, when time to maturity increases the contribution of x_1 decreases gradually. From $s-t=1$ to $s-t=2$ the contribution of x_2 increases to the extent that at time to maturity $s-t=2$ perturbations in x_2 dominate the evolution of the approximated CDD futures price. From $s-t>2$ the contribution of x_2 decreases gradually but it remains always above x_1. The contribution of x_3 increases from $s-t=1$ to $s-t=3$. For bigger times to maturity it decreases gradually. We point out that variations in x_3 always contribute less than variations in x_1 or x_2. At the long end, small variations in any component hardly affect the tendency of the approximated CDD futures price. This fact makes sense as the term $\mathbf{e}_1 e^{A(s-t)}\mathbf{x}$, which is dependent on the coordinates of $\mathbf{x}$, tends to zero at the long end.

The partial derivatives of the CDD futures price with respect the coordinates of $\mathbf{x}$, as presented in Proposition 5.3.1, depend on $\mathbf{X}(t)$. The first component $Y(t) = T(t) - \Lambda(t)$ corresponds to the deseasonalized temperature. We approximate the derivatives of $Y(t)$ with backward finite differences. Hence $Y'(t) \approx Y(t) - Y(t-1)$ is approximated by the difference between the deseasonalized temperature at times t and $t-1$. $Y''(t) \approx Y(t) - 2Y(t-1) + Y(t-2)$ is approximated by a linear combination of the deseasonalized temperatures at times t and the two prior times $t-1$ and $t-2$. Finally, we get the following relation between the temperature and the seasonal function:

$$\begin{pmatrix} T(t) \\ T(t-1) \\ T(t-2) \end{pmatrix} \approx \begin{pmatrix} \Lambda(t) + Y(t) \\ \Lambda(t-1) + Y(t) - Y'(t) \\ \Lambda(t-2) + Y''(t) - 2Y'(t-1) + Y(t) \end{pmatrix}$$

$$= \begin{pmatrix} \Lambda(t) + \mathrm{x}_1 \\ \Lambda(t-1) + \mathrm{x}_1 - \mathrm{x}_2 \\ \Lambda(t-2) + \mathrm{x}_3 - 2\mathrm{x}_2 + \mathrm{x}_1 \end{pmatrix}\Bigg|_{\mathbf{X(t)=x}} . \tag{5.4.2}$$

Observe that given a fixed $\mathbf{X}(t)$ with $0 \leq t \leq s$, the temperature at time t is approximately x_1 degrees above the seasonal mean function and one and two days prior to t, it is approximately $\mathrm{x}_1 - \mathrm{x}_2$ and $\mathrm{x}_3 - 2\mathrm{x}_2 + \mathrm{x}_1$ degrees above the seasonal mean function, respectively. Consider $\mathbf{X}(t) = \mathbf{0}$ where $\mathbf{0}$ is the null vector in $\mathbb{R}^3$. The partial derivatives of the CDD futures price are completely deterministic. By the relation between the temperature and the seasonal function established in (5.4.2), for this particular case the temperature for the time t and the two prior times $t-1$ and $t-2$ is approximately the seasonal mean function. Next, we show the plot with the partial derivatives of the CDD futures price with respect the coordinates of $\mathbf{x}$ derived in Proposition 5.3.1.

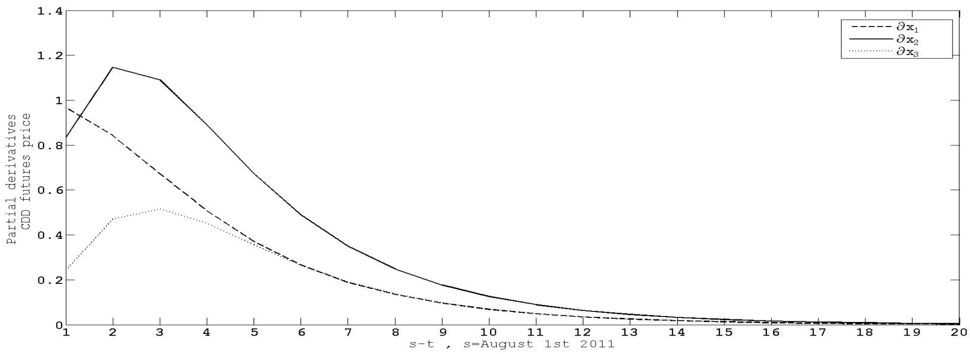

Figure 5.4: $(\partial F_{\text{CDD}}(t, s, \mathrm{x}_1, \ldots, \mathrm{x}_p)/\partial \mathrm{x}_i)_{|\mathbf{x=0}}$ for $i = 1, 2, 3$ as a function of $s - t$ with measurement day s being August 1st, 2011.

The x-axis considers the time to maturity, $s-t$, where the measurement day s is August 1st, 2011. The y-axis shows the different partial derivatives. Observe that at first sight Figures 5.3 and 5.4 seem to coincide. Indeed, Figure 5.5 below shows that the relative error between them is less than 1% entirely.

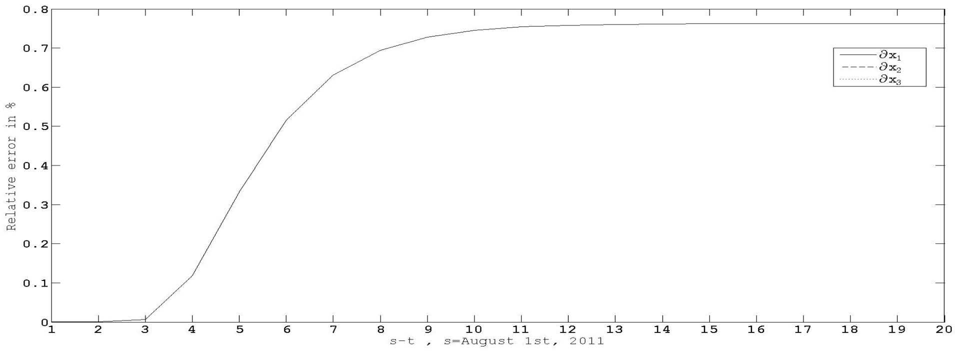

Figure 5.5: The relative error in percent between $\partial \widetilde{F}_{\mathrm{CDD}}(t, s, \mathrm{x}_1, \ldots, \mathrm{x}_p)/\partial \mathrm{x}_i$ and $(\partial F_{\mathrm{CDD}}(t, s, \mathrm{x}_1, \ldots, \mathrm{x}_p)/\partial \mathrm{x}_i)_{|\mathbf{x}=\mathbf{0}}$ for $i = 1, 2, 3$ as a function of $s - t$ with measurement day s being August 1st, 2011.

We consider also the case where $\mathbf{X}(t) = \mathbf{e}_k$ is the kth canonical basis vector in $\mathbb{R}^3$ for $k = 1, \ldots, 3$. For $\mathbf{X}(t) = \mathbf{e}_1$ the temperature at the present time t and the two consecutive prior times to t, say $t-1$ and $t-2$, is approximately one degree above the seasonal mean function. For $\mathbf{X}(t) = \mathbf{e}_2$, the temperature is close to the seasonal mean at present time t and it is approximately one and two degrees below the seasonal mean at the times $t - 1$ and $t - 2$, respectively. Finally, for $\mathbf{X}(t) = \mathbf{e}_3$ the temperature is close to the seasonal mean at present time t and one prior time, but two days prior to t it is nearly one degree above its seasonal mean. The partial derivatives of the CDD futures price evaluated at the canonical basis vectors in $\mathbb{R}^3$ also behave in a similar way to the partial derivatives on the approximated CDD futures price. In Figure 5.6 we show for the case $\mathbf{X}(t) = \mathbf{e}_1$ the relative error between the partial derivatives of the approximated CDD futures price and the CDD futures price is also less than 1% entirely.

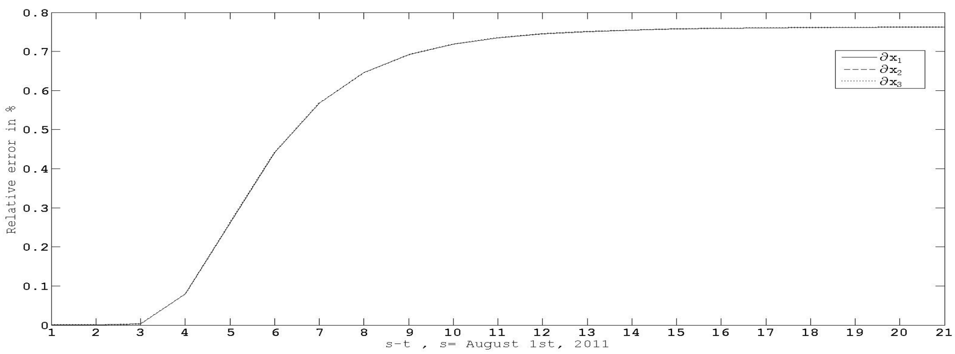

Figure 5.6: The relative error in percent between $\partial \widetilde{F}_{\mathrm{CDD}}(t, s, \mathrm{x}_1, \ldots, \mathrm{x}_p)/\partial \mathrm{x}_i$ and $(\partial F_{\mathrm{CDD}}(t, s, \mathrm{x}_1, \ldots, \mathrm{x}_p)/\partial \mathrm{x}_i)_{|\mathbf{x}=\mathbf{e}_1}$ for $i = 1, 2, 3$ as a function of $s - t$ with measurement day s being August 1st, 2011.

We proceed now with the analysis of call option prices written on CDD futures prices and call option prices written on the approximated CDD futures prices. To do so, we consider the results in Propositons 5.3.3 and 5.3.4. Observe that in both results there is dependency on $\mathbf{X}(t)$. We focus on an at-the-money call opiton prices. In view of Figure 5.7 we fix the strike price K being $K = 13$.

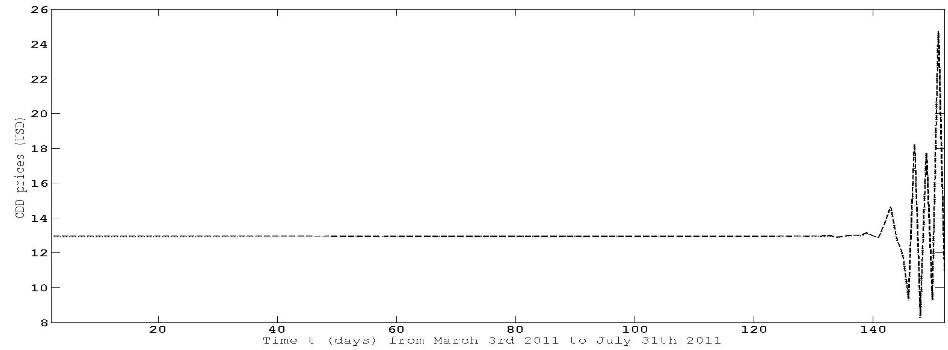

Figure 5.7: Forward prices and approximated forward prices for CDD contracts from March 3rd, 2011 to July 31th, 2011 with measurement day August 2nd, 2011.

For the study of the sensitivity of call option prices we restrict to the same cases of $\mathbf{X}(t)$ as considered for futures prices. Nevertheless, for the sake of simplicity we show the results for $\mathbf{X}(t) = \mathbf{0}$ and $\mathbf{X}(t) = \mathbf{e}_1$. Next, we show the plot with the partial derivatives of the call option prices with respect to the coordinates of $\mathbf{x}$ derived in Proposition 5.3.3.

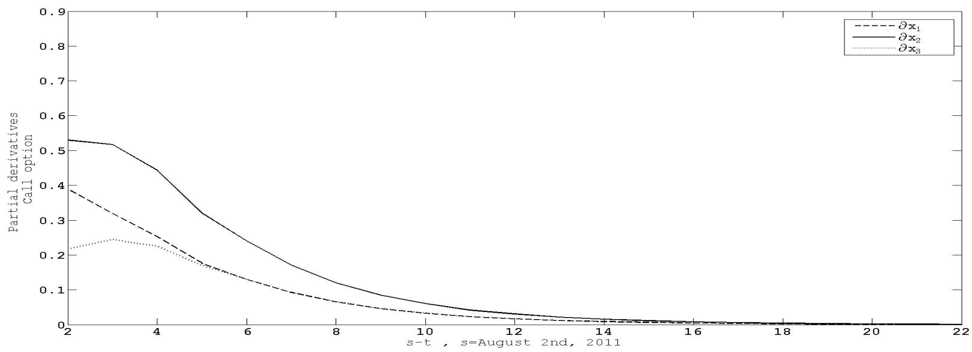

Figure 5.8: $(\partial C(t, \tau, s, \mathrm{x}_1, \ldots, \mathrm{x}_p)/\partial \mathrm{x}_i)_{|\mathbf{x}=\mathbf{0}}$ for $i = 1, 2, 3$ with exercise time τ being August 1st, 2011 and measurement day s August 2nd, 2011.

The x-axis considers the time to maturity, $s - t$, where s is August 2nd, 2011. We have fixed the exercise time τ being August 1st, 2011. The y-axis shows the different partial derivatives of the call option price when $\mathbf{X}(t) = \mathbf{0}$. Firstly, we observe that all the partial derivatives of the approximated call option are positive. We see that for all times to maturity the call option price is more sensitive to any infinitesimal change in the component x_2, followed by x_1 and x_3 respectively. This tendency follows as time to maturity increases but on the other hand the approximated call option prices become less affected to any pertubation in a component of $\mathbf{x}$. At the long end, small variations in any component hardly affect the tendency of the call option prices.

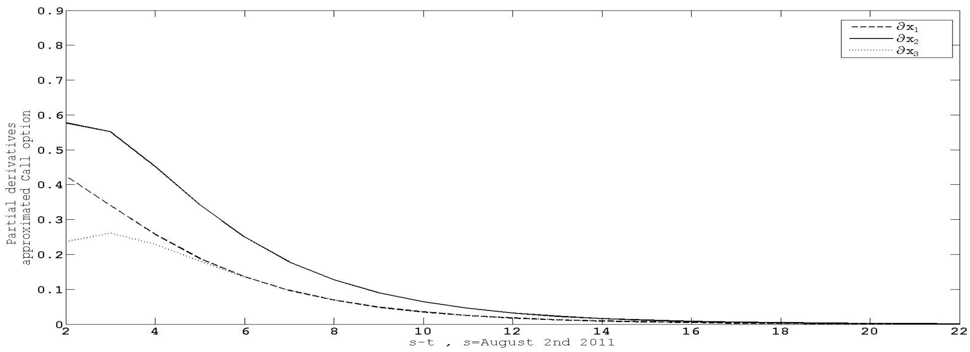

Figure 5.9: $(\partial\widetilde{C}(t,\tau,s,\widetilde{F}_{\mathrm{CDD}}(t,s,\mathrm{x}_1,\ldots,\mathrm{x}_p))/\partial\mathrm{x}_i)_{|\mathbf{x}=\mathbf{0}}$ for $i=1,2,3$ with exercise time τ being August 1st, 2011 and measurement day s August 2nd, 2011.

Figure 5.9 shows the partial derivatives of the approximated call option prices with respect to the coordinates of $\mathbf{x}$ derived in Proposition 5.3.4 when $\mathbf{X}(t)=\mathbf{0}$. We observe that the partial derivatives in Figures 5.8 and 5.9 show a close habavior. We also see that for small times to maturity the partial derivatives of the approximated call option price are bigger than the partial derivatives of the call option price. Next, we show the plots with the partial derivatives of the approximated call option prices with respect to the coordinates of $\mathbf{x}$ derived in Proposition 5.3.4.

We end the analysis of sensitivity showing the results for the case $\mathbf{X}(t)=\mathbf{e}_1$. Figure 5.10 contains the partial derivatives for the call options prices and Figure 5.11 the partial derivatives for the approximated call option prices. We see also here that both prices are more sensitive to changes in the second component of $\mathbf{x}$ in all the domain. Furthermore, the sensitivity to this component decreases as time to maturity increases.

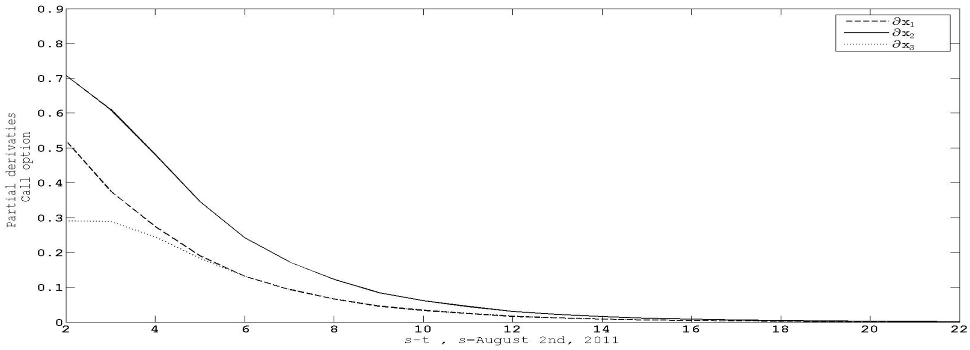

Figure 5.10: $(\partial C(t,\tau,s,\mathrm{x}_1,\ldots,\mathrm{x}_p))/\partial \mathrm{x}_i)_{|\mathbf{x}=\mathbf{e_1}}$ for $i=1,2,3$ with exercise time τ being August 1st, 2011 and measurement day s August 2nd, 2011.

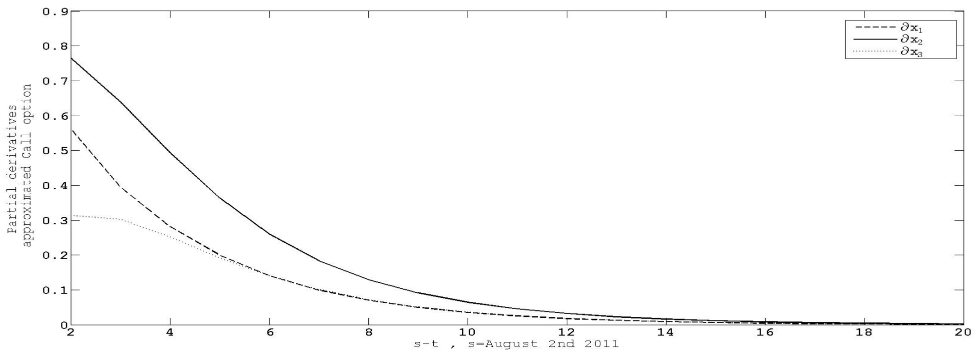

Figure 5.11: $(\partial \widetilde{C}(t,\tau,s,\widetilde{F}_{\mathrm{CDD}}(t,s,\mathrm{x}_1,\ldots,\mathrm{x}_p))/\partial \mathrm{x}_i)_{|\mathbf{x}=\mathbf{e_1}}$ for $i=1,2,3$ with exercise time τ being August 1st, 2011 and measurement day s August 2nd, 2011.

5.5 Empirical study of the local sensitivity of CDD futures prices with measurement over a period

The sensitivity analysis of CDD futures prices with measurement over a period $[\tau_1,\tau_2]$ with respect to infinitesimal changes in the components of $\mathbf{X}(t)$ can be performed by means of the partial derivatives provided in Propositions 5.2.1 and 5.2.2. We proceed analogously as in Section 5.4 with the CDD futures prices with a measurement day.

The random variable $\Phi(Z(t,s))$ with $Z(t,s)$ in (5.4.1) makes here also the difference between the results provided in these two Propositions. The same reasoning followed for CDD futures prices with a measurement day is valid to conclude that

$$\Big(\frac{\partial}{\partial \mathrm{x}_i}F_{\mathrm{CDD}}(t,\tau_1,\tau_2,\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3)\Big)_{|\mathbf{x}=\mathbf{X}(t)} \leq \frac{\partial}{\partial \mathrm{x}_i}\widetilde{F}_{CDD}(t,\tau_1,\tau_2,\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3)\,. \qquad (5.5.1)$$

Hence, the approximated CDD futures price becomes more sensitive to any infinitesimal change in the coordinates of $\mathbf{X}(t)$ than the CDD futures price. Next, we show the plot with the partial derivatives of the approximated CDD futures price with respect the coordinates of $\mathbf{x}$ derived in Proposition 5.2.2.

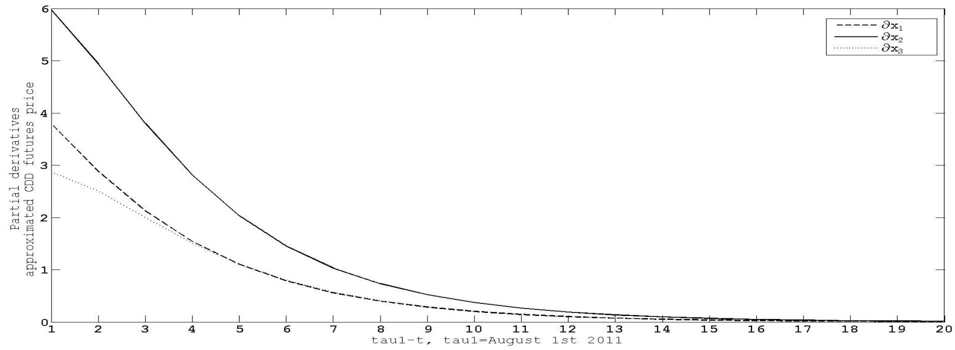

Figure 5.12: $\partial \widetilde{F}_{\mathrm{CDD}}(t, \tau_1, \tau_2, \mathrm{x}_1, \ldots, \mathrm{x}_p)/\partial \mathrm{x}_i$ for $i = 1, 2, 3$ as a function of $\tau_1 - t$ with measurement over a period $[\tau_1, \tau_2]$ being August, 2011.

The x-axis considers the time to maturity, $\tau_1 - t$, where τ_1 is August 1st, 2011 and the measurement period $[\tau_1, \tau_2]$ is August 2011. The y-axis shows the different partial derivatives of the approximated CDD futures price. Firstly, we observe that the values here of the partial derivatives are greater than those corresponding to the partial derivatives of the approximated CDD futures price with a measurement day, with more emphasis on times t which are close to τ_1. Indeed, the partial derivatives of the approximated CDD futures prices can be understood as the partial derivatives of the approximated CDD futures prices with measurement day s which runs over $[\tau_1, \tau_2]$ as shown as follows

$$\frac{\partial}{\partial \mathrm{x}_i} \widetilde{F}_{\mathrm{CDD}}(t, \tau_1, \tau_2, \mathrm{x}_1, \ldots, \mathrm{x}_p) = \int_{\tau_1}^{\tau_2} \frac{\partial}{\partial \mathrm{x}_i} \widetilde{F}_{\mathrm{CDD}}(t, s, \mathrm{x}_1, \ldots, \mathrm{x}_p)\, ds\,.$$

Recall that Figure 5.3 shows that the derivatives of the approximated CDD futures price with a measurement day are positive and when time to maturity increases tend to zero. This fact together with the relation in (5.5.1) let us to justify why the derivatives of the approximated CDD futures prices with measurement over a period behave in this way. We also observe that any infinitesimal change in x_2 dominate more the behaviour of the approximated CDD futures price, followed than any change in x_1 and x_3. This was exactly the same tendency followed by the partial derivatives of the approximated CDD futures price with a measurement day, see Figure 5.3, for times to maturity greater than 2.

For the study of the sensitivity of the partial derivatives of the CDD futures price with measurement over a period derived in Proposition 5.2.1 we consider the cases $\mathbf{X}(t) = \mathbf{0}$ and $\mathbf{X}(t) = \mathbf{e}_1$.

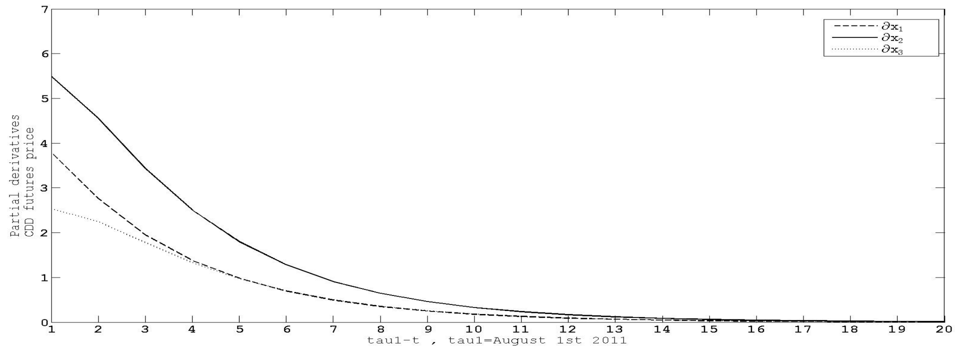

Figure 5.13: $(\partial F_{\text{CDD}}(t, \tau_1, \tau_2, \mathrm{x}_1, \ldots, \mathrm{x}_p)/\partial \mathrm{x}_i)_{|\mathbf{x}=\mathbf{0}}$ for $i = 1, 2, 3$ as a function of $\tau_1 - t$ with measurement over a period $[\tau_1, \tau_2]$ being August, 2011.

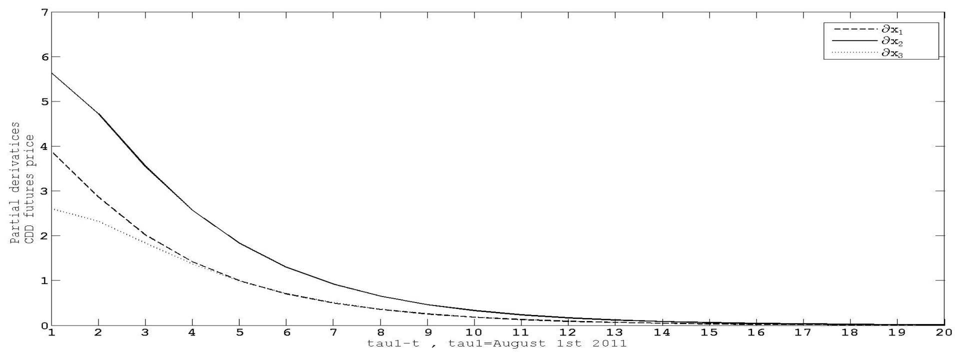

Figure 5.14: $(\partial F_{\text{CDD}}(t, \tau_1, \tau_2, \mathrm{x}_1, \ldots, \mathrm{x}_p)/\partial \mathrm{x}_i)_{|\mathbf{x}=\mathbf{e}_1}$ for $i = 1, 2, 3$ as a function of $\tau_1 - t$ with measurement over a period $[\tau_1, \tau_2]$ being August, 2011.

Figures 5.13 and 5.14 show also that perturbations in x_2 affect also more in these cases the CDD futures price.

5.6 Conclusions and outlook

In this paper we have studied the local sensitivity of the (approximated) CDD and HDD futures and options prices with respect to a perturbation in the deseasonalized temperature or in one of its derivatives up to a certain order determined by the CAR process modelling the deseasonalized temperature. To do so, we have considered the partial derivatives of these financial contracts with respect to these variables (deseasonalized temperature and its derivatives). The HDD and CDD futures and call option prices and their approximative formulas were derived in Benth and Solanilla Blanco [18] where we also checked that the approximative formulas worked well. We have considered and empirical analysis where we have taken the same CAR(3)-process fitted to the time series of New York temperatures in Benth and Solanilla Blanco [18]. The sensitivity study of these financial contracts with a fixed measurement day shows first that the approximated futures prices are more sensitive to any perturbation in one of these variables than the theoretical futures prices. Nevertheless, the relative error between both partial derivatives is rather small. We also observe that one time prior to the considered measurement day the behaviour of the (approximated) futures prices remains more affected by a perturbation in the deseasonalized temperature but when time to maturity increases then a perturbation in the slope of the deseasonalized temperature dominates the behaviour of the (approximated) futures prices. At the long end any perturbation of these variables hardly affect the behaviour of the (approximated) futures prices. For the call option prices we also observe that the approximated call option prices are more sensitive to any pertubation in one of the variables than the call option prices. We emphasize that unlike (approximated) futures prices any perturbation in the slope of the deseasonalized temperature dominates the behaviour of the (approximated) call option prices at all times. We have also extended the analysis of sensitivity to (approximated) futures prices with measurement over a fixed month. We have seen that in this case the slope of the deseasonalized temperature dominates the bahaviour of the (approximated) futures prices at all the times.

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