

FRÉCHET-HOEFFDING LOWER LIMIT COPULAS IN HIGHER DIMENSIONS

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ABSTRACT. Investigators have incorporated copula theories into their studies of multivariate dependency phenomena for many years. Copulas in general, which include the basic probability version as well as the Lévy and utility varieties, are enjoying a surge of popularity with applications to economics and finance. Ordinary copulas have a natural upper bound in all dimensions, the so-called Fréchet-Hoeffding limit, after the pioneering work of Wassily Hoeffding and, later, Maurice René Fréchet, working independently. Among the well-understood phenomena in the bivariate case is that a natural lower limit copula also exists. An extension of this copula, however, to the multidimensional case has not been forthcoming. This paper proposes such an extension of the lower limit distribution function and its copula, and examines some of their properties.

1. INTRODUCTION

Investigators have incorporated copula theories into their studies of multivariate dependency phenomena for many years. Copulas in general, which include the basic probability version as well as the Lévy and utility varieties, are enjoying a surge of popularity with applications to economics and finance. Ordinary copulas have a natural upper bound in all dimensions, the so-called Fréchet-Hoeffding limit, after the pioneering work of Wassily Hoeffding and, later, Maurice René Fréchet, working independently. Among the well-understood phenomena in the bivariate case is that a natural lower limit copula also exists.

In this two-variable case, on a domain of the unit square $[0, 1]^2$, the upper and lower limit copulas take this form.

$$(1.1) \quad C_{\uparrow}(u, v) := \min(u, v) \quad \text{for the upper}$$

$$(1.2) \quad C_{\downarrow}(u, v) := \max(u + v - 1, 0) \quad \text{for the lower}$$

The natural extension to a higher dimensions n for the upper limit copula is this.

$$(1.3) \quad C_{\uparrow}(u_1, u_2, \dots, u_n) := \min(u_1, u_2, \dots, u_n)$$

This function has all the necessary properties of a copula, justifying its name.

An extension of the bivariate lower limit copula, however, to the multidimensional case has not been forthcoming. One proposed extension takes the form

$$\tilde{C}_{\downarrow}(u_1, u_2, \dots, u_n) := \max(u_1 + u_2 + \dots + u_n - (n - 1), 0)$$

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This function certainly has the necessary range of $[0, 1]$, but is not a copula in dimensions $n > 2$ because it does not have the necessary n -increasing property. See (Nelsen 1998, Subsection 2.10 and Exercise 2.35) and (Fantazzini 2004, Subsection 2.1).

This paper proposes this extension of Fréchet–Hoeffding lower limit distribution function and its copula. Here are these functions, with the copula first expressed parametrically, then non-parametrically.

$$(1.4) \quad F_{\downarrow}(x_1, x_2, \dots, x_n) := \max \left[\left(\frac{1}{n-1} (x_1 + x_2 + \dots + x_n - 1) \right)^{n-1}, 0 \right],$$

and

$$(1.5) \quad C_{\downarrow} [1 - (1 - x_1)^{n-1}, 1 - (1 - x_2)^{n-1}, \dots, 1 - (1 - x_n)^{n-1}] := F_{\downarrow}(x_1, x_2, \dots, x_n),$$

so

$$(1.6) \quad C_{\downarrow}(u_1, u_2, \dots, u_n) = F_{\downarrow} \left[1 - (1 - u_1)^{\frac{1}{n-1}}, 1 - (1 - u_2)^{\frac{1}{n-1}}, \dots, 1 - (1 - u_n)^{\frac{1}{n-1}} \right]$$

The definition of Equation (1.6) forms the basis of the developing study in all dimensions $n \geq 2$, extending the definition of Equation (1.2) in dimension 2.

Look now to Figures 1 and 2. These show uniform probability measures in two dimensions concentrated, respectively, on the line segments for which the joint identity random variable (X, Y) has the relationships $X = Y$ and $X + Y = 1$. Shown also are sample domains of integration of the respective distribution functions. These measures provide the Fréchet–Hoeffding upper and lower limit copulas. Other relationships between random variables also produce these copulas. In particular the relationship $Y = -X$ produces the second.¹

Remark. Distribution functions in two dimensions such as these, which are defined on the unit square and have uniform margins, are their own copulas. If $F(x, y)$ is a distribution function with $F_1(x) = x$ and $F_2(y) = y$ its margins, and if $C(\cdot, \cdot)$ is its copula, then

$$C(F_1(x), F_2(y)) = F(x, y) = C(x, y)$$

2. DEVELOPMENT OF THE MULTIDIMENSIONAL LOWER LIMIT COPULA

Observe carefully that the domain of concentration in Figure 1 is referenced at the ‘diagonal,’ whereas the domain of concentration in Figure 2 is referenced at the ‘simplex.’ These distinctions are important, for in higher dimensions the corresponding domains continue to have the character of ‘diagonal’ and ‘simplex.’ The copulas of Equations (1.3) and (1.6) devolve from the following relationships (among others) of n random variables on unit hypercubes $[0, 1]^n$.

$$(2.1) \quad \begin{aligned} X_1 &= X_2 = \dots = X_n \\ X_1 + X_2 + \dots + X_n &= 1 \end{aligned}$$

The proposed definition of the multidimensional lower limit copula as in Equation (1.6) is in fact the copula of the distribution function exhibited by Equation (1.4) for the multivariate identity random variable (X_1, X_2, \dots, X_n) with probability measure uniformly concentrated

¹Perhaps this relationship caused the initial misunderstanding that the Fréchet–Hoeffding lower limit copula could not be extended beyond two dimensions. The false reasoning would go in the direction that multiple variables could not all have inverse binary relationships without all being zero, a.e.

on the simplex of Equation (2.1). This is the unique possibility for a distribution function specified in this manner, which must increase by order $(n - 1)$ as the sum $x_1 + x_2 + \dots + x_n$ increases linearly, passing from 0 when the sum is 1, to 1 when the sum is n .

Remark. To visualize this polynomial growth — quadratic in the 3-dimensional case, cubic in the 4-dimensional case, etc. — simply imagine an orthant-shaped domain of integration for the measure encroaching into the simplex, and therefore into the support of the measure. Figure 2 shows this encroachment in the 2-dimensional case. The intersection of this orthant with the simplex, itself a simplex, grows in the described manner.

Definition 2.1. The random variable terms of the simplex of Equation (2.1) shall be said to exhibit *complementary dependence*. The 2-dimensional relationship (with a 1-dimensional simplex) shall be said equivalently to exhibit *inverse dependence*.

Necessary to developing the copula is the specification the marginal distributions. In the two dimensional case these were uniform on the unit interval, but in higher dimensions the corresponding measures concentrate increasingly toward the origin. This fact made it necessary first to calculate the multivariate distribution function, and now its margins.

To specify the marginal distributions one need only look to the iterated projections to lower dimensions of the uniformly concentrated measure of the $(n - 1)$ -simplex to observe that the n -dimensional marginal measures are

$$\mu_n(x_i) := (n - 1)(1 - x_i)^{n-2}, \text{ for } i = 1, 2, \dots, n$$

These integrate to

$$F_{\downarrow i} := 1 - (1 - x_i)^{n-1}$$

for the marginal distribution functions. Hence one has the arguments of Equation 1.5, the $\{F_{\downarrow i}\}$, for the parametric form of the extended Fréchet–Hoeffding lower bound copula.

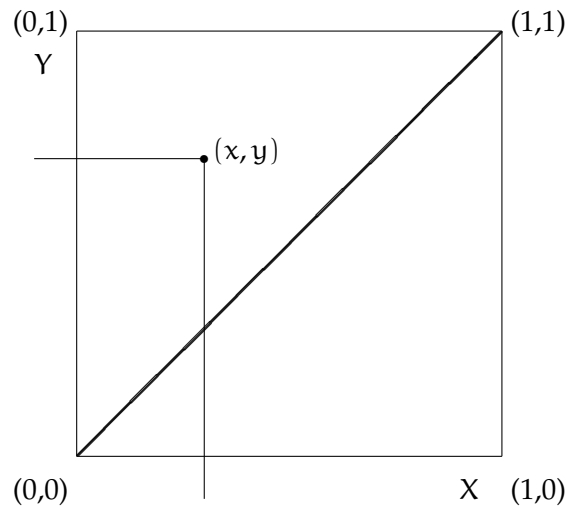


FIGURE 1. Mass concentration on diagonal $X = Y$, with sample domain of integration for the Fréchet – Hoeffding upper limit joint distribution function and copula $F_{\uparrow}(x, y) = C_{\uparrow}(x, y) = \min(x, y)$

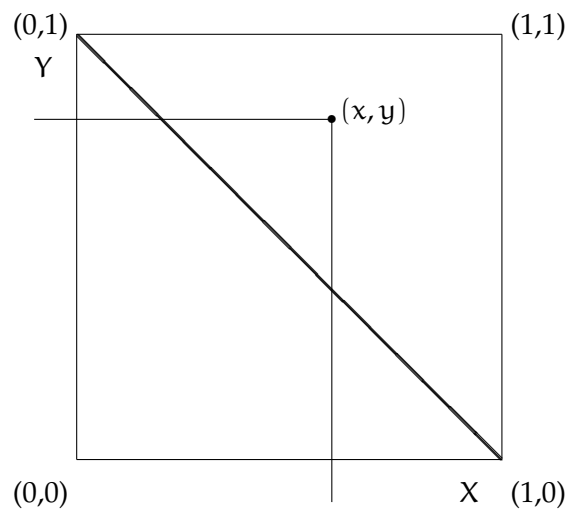


FIGURE 2. Mass concentration on simplex $X + Y = 1$, with sample domain of integration for Fréchet – Hoeffding lower limit joint distribution function and copula $F_{\downarrow}(x, y) = C_{\downarrow}(x, y) = \max(x + y - 1, 0)$

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