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Homological Projective Duality

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Abstract

This thesis is concerned with the theory of homological projective duality of A. Kuznetsov. The varieties of interest are projective bundles. By considering the resolution $X = \operatorname{Hilb}^2 \mathbb{P}^2$ of the variety $\operatorname{Sym}^2 \mathbb{P}^2$ as a projective bundle, we show using results of Kuznetsov that the homological projective dual Y of X agrees with that of the smooth stack $[\mathbb{P}^2 \times \mathbb{P}^2 / S_2]$ as described by J. Rennemo. Further, we describe the homological projective dual of a family of projective bundles over the Grassmanian $G(n, n+1)$ to which X belongs. Lastly we study the duality of X and Y on linear sections and show an equivalence of derived categories of a pair of elliptic curves $X_L \subset X$ and $Y_L \subset Y$.

Contents

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Chapter 1

Introduction

1.1 Background

A classical duality relation in algebraic geometry is projective duality. It stems from the duality of points and hyperplanes of projective space $\mathbb{P}^n$. Hyperplanes can be considered as points of the dual projective space $(\mathbb{P}^n)^\vee$, and for a smooth projective variety X embedded into projective space, the set of points in $(\mathbb{P}^n)^\vee$ corresponding to the hyperplanes tangent to X describes a projective variety $X^\vee$ embedded into $(\mathbb{P}^n)^\vee$. We call this variety the projective dual of X. Homological projective duality of A. Kuznetsov first presented in [Kuz07], is a generalization of the projective duality in the sense that it carries the information of projective duality. However, the homological projective duality happens on the level of the derived categories of the varieties involved.

For a smooth projective variety X with a morphism $f : X \to \mathbb{P}(V)$ to projective space, denote by $D(X)$ its bounded derived category of coherent sheaves. For the duality to capture interesting information, we will need a particular semi-orthogonal decomposition of $D(X)$, called a Lefschetz decomposition with respect to the morphism f. We define the universal hyperplane section of X to be the variety $\mathscr{H}$ consisting of pairs $(x, H) \in X \times \mathbb{P}(V^\vee)$ with the incidence relation $x \in H$. Now given a Lefschetz decomposition of $D(X)$, the derived category of the universal hyperplane section $\mathscr{H}$ of X will inherit a semi-orthogonal decomposition

$$D(\mathscr{H}) = \langle \mathcal{C}_{\mathscr{H}}, \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle.$$

For the duality relation, the subcategory $\mathcal{C}_{\mathscr{H}}$ of $D(\mathscr{H})$ will be the category of interest. If $\mathcal{C}_{\mathscr{H}}$ can be identified with the derived category of some other smooth projective variety Y, we call this the homological projective dual of X. This identification is given by a Fourier-Mukai functor from the derived category of Y to the derived category of $\mathscr{H}$ that restricts to an equivalence of categories onto $\mathcal{C}_{\mathscr{H}}$. The variety Y comes with a morphism $g : Y \to \mathbb{P}(V^\vee)$ and a Lefschetz decomposition. Moreover, the duality is symmetric in the sense that if we now consider the derived category of the universal hyperplane section of Y and its decomposition with respect to g, then the subcategory appearing as the first term in the decomposition is equivalent to $D(X)$. Further, homological projective duality gives related semi-orthogonal decompositions of the derived category of linear sections of the two varieties involved.

A complication for finding homological projective dual pairs is the fact that finding a decomposition of the derived category of a variety X in general is a difficult task.

Question. *When does the derived category of a smooth projective variety X with a morphism f to projective space have a Lefschetz decomposition with respect to f?*

The approach to finding such a decomposition will often be to find a collection of exceptional objects and show that the collection is full. However, fullness of the exceptional collection is in general not easily verified either. Some types of varieties, such as Calabi-Yau varieties, have derived categories that do not admit semi-orthogonal decompositions. The varieties we are concerned with will in general not be Calabi-Yau. They will be projective bundles, and for such varieties, Lefschetz decompositions of their derived categories are know due to Orlov in [Orl92]. He showed that the derived category of a projective bundle X over a basescheme Y has a Lefschetz decomposition if the derived category of Y has a full exceptional collection. We give a proof of his result and write down the decompositions for the projective bundles we are interested in.

The main example we study in this thesis is the pair of varieties $\mathrm{Sym}^2 \mathbb{P}^2$ and $\mathbb{P}^2 \times \mathbb{P}^2$. Since $\mathrm{Sym}^2 \mathbb{P}^2$ is not smooth, the theory of homological projective duality does not apply to this variety. A natural choice is to replace $\mathrm{Sym}^2 \mathbb{P}^2$ with its resolution given by the Hilbert scheme $\mathrm{Hilb}^2 \mathbb{P}^2$. Alternatively, since the theory of A. Kuznetsov generalizes to smooth stacks, another approach would be to study the smooth stack $[\mathbb{P}^2 \times \mathbb{P}^2/S_2]$.

$$\begin{array}{ccc} [\mathbb{P}^2 \times \mathbb{P}^2/S_2] & & \mathrm{Hilb}^2 \mathbb{P}^2 \\ & \searrow \quad \swarrow & \\ & \mathrm{Sym}^2 \mathbb{P}^2 & \end{array}$$

By results of [BKR01] and [Hai01], there is an equivalence of derived categories

$$D(\mathrm{Hilb}^2 \mathbb{P}^2) \simeq D([\mathbb{P}^2 \times \mathbb{P}^2/S_n]).$$

Further, in [Ren20] J. Rennemo determined the HP-dual of the smooth stack $[\mathbb{P}(V) \times \mathbb{P}(V)/S_2]$ for a general vector space V by giving a Lefschetz decomposition of the derived category of the S_2-equivariant coherent sheaves on $\mathbb{P}(V) \times \mathbb{P}(V)$. He found that the HP-dual of $[\mathbb{P}(V) \times \mathbb{P}(V)/S_2]$ is a Clifford module category which in the case $\dim V = 3$ can be computed as the derived category of the variety $\mathbb{P}^2 \times \mathbb{P}^2$. With these two results in mind, we aim at answering the following question:

Question. *Does the homological projective dual of* $\mathrm{Hilb}^2 \mathbb{P}^2$ *agree with that of the smooth stack* $[\mathbb{P}^2 \times \mathbb{P}^2/S_2]$*?*

We find that even though the Lefschetz decomposition used to study $D(\mathrm{Hilb}^2 \mathbb{P}^2)$ differs from the one decomposing $D([\mathbb{P}^2 \times \mathbb{P}^2/S_2])$ given in [Ren20], the answer to the above question is yes nevertheless. Since HP-duality depends on the Lefschetz decompositions, this is not expected in general. We find that the method used for determining the dual of $\mathrm{Hilb}^2 \mathbb{P}^2$ generalizes to a family of projective bundles, and we determine the HP-dual of each variety in this family.

1.2 Outline

Chapter 2 is devoted to building the necessary foundations for studying the derived category. We give an introduction to the derived category at a level where notions of categories and functors are sufficient prerequisites. We introduce some essential notions of category theory and describe how the derived category of a general abelian category is constructed. We move on to study the derived category of coherent sheaves on a smooth projective variety more closely. We define the most essential derived functors used in algebraic geometry and illustrate some of their useful properties.

In Chapter 3 we study the geometry of projective bundles. We present the construction of relative projective spaces and present some results describing the sheaf cohomology of such schemes, making use of the derived functors from Chapter 2. We carry on to study closer the varieties of interest in this thesis for homological projective duality. To do so, we introduce quotient varieties and Hilbert schemes of points and describe the Picard groups of the varieties in question. Finally, we move on to study the varieties from a viewpoint of cohomology of their invertible sheaves. We prove the Künneth formula for smooth projective varieties and apply the results on cohomology of projective bundles to a few examples.

In Chapter 4 we study the decompositions of the derived categories necessary to apply the theory of homological projective duality. The decompositions are first introduced in the general setting of a triangulated category and later in the specific case of the derived category of a variety. We introduce the notion of an exceptional collection and describe the connection between full exceptional collections of the derived category and K-theory of coherent sheaves. Some further conditions on exceptional collections to be full are given and applied to some examples. Finally, we focus on the setting of projective bundles and present a way to decompose the derived category of a relative projective space in terms of the base scheme, as given in [Orl92].

In Chapter 5 we study homological projective duality. We state this duality in its original form given by Kuznetsov. We present the main theorem of A. Kuznetsov for studying pairs of homologically projectively dual varieties on linear sections. Lastly, we present a theorem of Kuznetsov that describes the homological projective dual of a general projective bundle which will also set the stage for the applications of homological projective duality.

Finally, in Chapter 6 we apply the theory and give a description of the homological projective dual of a family of projective bundles. This will also determine the dual Y of $X = \mathrm{Hilb}^2 \mathbb{P}^2$. We end this chapter by studying the duality of X and Y on linear sections and find a pair of closed subschemes $X_L \subset X$ and $Y_L \subset Y$ whose derived categories are equivalent.

1.3 Notations and Conventions

We let k denote an algebraically closed field of characteristic 0. Unless otherwise specified, we assume all schemes to be quasi-projective over k. We use the terms vector bundle and locally free sheaf, in particular line bundle and invertible sheaf, interchangeably. We assume familiarity with algebraic geometry at the level of [Har77] Chapters I-III, although some specific constructions will be recalled when necessary.

Chapter 2

The Derived Category

This chapter is devoted to studying the derived category. We construct it in a general setting but quickly turn our attention to the use of the derived category in algebraic geometry. The motivation for the derived category is to have a category for studying homological algebra. The main idea is to have as objects the complexes and work with them rather than their cohomology. We follow chapters 1-4 in [Huy06] by D. Huybrechts.

2.1 Preliminary Category Theory

We begin this section by recalling some basic notions from category theory. A category $\mathcal{C}$ is *additive* if for all objects $A, B \in \mathcal{C}$, the sets $\mathrm{Hom}_{\mathcal{C}}(A, B)$ are abelian groups and the compositions

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

are bilinear. We say that an additive category is *k-linear* if the groups $\mathrm{Hom}_{\mathcal{C}}(A, B)$ are k-vector spaces and the compositions k-bilinear. A k-linear functor $F : \mathcal{C} \to \mathcal{D}$ is a functor of k-linear categories such that the map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ is a homomorphism of k-vector spaces. An additive category is *abelian* if every morphism $f : A \to B$ admits a kernel and a cokernel fitting into an exact sequence

$$0 \to \ker f \to A \to B \to \operatorname{coker} f \to 0,$$

i.e. there are isomorphisms $\operatorname{coim} f \simeq \operatorname{im} f$. A functor of abelian categories is called *exact* if it preserves short exact sequences.

Definition 2.1.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that the functor $H : \mathcal{D} \to \mathcal{C}$ is *right adjoint* to F if there for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$ are isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(F(C), D) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(C, H(D))$$

which commute with the natural maps of Hom-spaces induced by morphisms in $\mathcal{C}$ and $\mathcal{D}$. If H is right adjoint to F, then F is left adjoint to H.

Definition 2.1.2. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *full* if for all $A, B \in \mathcal{C}$ the map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

is surjective. If it is injective we call it *faithful*, and if it is an isomorphism, *fully faithful*.

We say that two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exists a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G \simeq \mathrm{id}_{\mathcal{D}}$ and $G \circ F \simeq \mathrm{id}_{\mathcal{C}}$. Let $\mathcal{S} \subset \mathcal{C}$ be a subcollection of objects and morphisms of $\mathcal{C}$. Then $\mathcal{S}$ is a subcategory of $\mathcal{C}$ if the subcollection of morphisms is closed under composition, contains the identity $\mathrm{id}_A : A \to A$ for all $A \in \mathcal{S}$ and if $f : A \to B$ is in the collection of morphisms, then $A, B \in \mathcal{S}$. We denote by ι the inclusion functor $\iota : \mathcal{S} \to \mathcal{C}$.

Definition 2.1.3. A subcategory $\mathcal{S} \subset \mathcal{C}$ is *full* if

$$\mathrm{Hom}_{\mathcal{S}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$$

for all $A, B \in \mathcal{S}$.

A fully faithful functor $F : \mathcal{C} \to \mathcal{D}$ gives an equivalence of categories between $\mathcal{C}$ and the full subcategory of $\mathcal{D}$ of all objects $D \in \mathcal{D}$ isomorphic to $F(C)$ for some $C \in \mathcal{C}$. If every object $D \in \mathcal{D}$ is isomorphic to $F(C)$ for some object $C \in \mathcal{C}$ we call F *essentially surjective.*

Proposition 2.1.4. *[Huy06, Proposition 1.4] A fully faithful essentially surjective functor $F : \mathcal{C} \to \mathcal{D}$ defines an equivalence of categories $\mathcal{C} \cong \mathcal{D}$.*

Example 2.1.5. Let $X = \mathrm{Spec}(A)$ be an affine scheme. Then the category $\mathbf{QCoh}(X)$ of quasi-coherent sheaves on X and the category $\mathbf{Mod}_A$ of A-modules are equivalent. The functor

$$\sim : \mathbf{Mod}_A \to \mathbf{QCoh}(X)$$

defined by sending a module M to the sheaf $\widetilde{M}$ defined on the distinguished opens by $M(D(f)) = M_f$ and the global sections functor

$$\Gamma : \mathbf{QCoh}(X) \to \mathbf{Mod}_A$$

are fully faithful. Moreover, $\Gamma(X, \widetilde{M}) = M$ and $\widetilde{\Gamma(X, \mathcal{F})} = \mathcal{F}$. Hence the functors compose to the identity on $\mathbf{Mod}_A$ and $\mathbf{QCoh}(X)$.

Definition 2.1.6. [Huy06, p. 9] Let $\mathcal{C}$ be a k-linear category. Then a k-linear auto-equivalence $S : \mathcal{C} \to \mathcal{C}$ is called a *Serre functor* if for all $A, B \in \mathcal{C}$, there are isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(B, S(A))^{\vee}$$

of k-vector spaces functorial in A and B.

Definition 2.1.7. A subcategory $\mathcal{S} \subset \mathcal{C}$ is (left or right) *admissible* if the inclusion functor $\iota : \mathcal{S} \to \mathcal{C}$ admits a (left or right) adjoint.

Definition 2.1.8. Let $\mathcal{A}$ be an abelian category. Then

- $I \in \mathcal{A}$ is *injective* if $\mathrm{Hom}(-, I) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ is an exact functor. If for every $A \in \mathcal{A}$ there is an injection $0 \to A \to I$ with I injective, then we say that $\mathcal{A}$ has *enough injectives.*
- $P \in \mathcal{A}$ is *projective* if $\mathrm{Hom}(P, -) : \mathcal{A} \to \mathbf{Ab}$ is an exact functor. If for every $A \in \mathcal{A}$ there is a surjection $P \to A \to 0$ with P projective, then we say that $\mathcal{A}$ has *enough projectives.*

2.2 The Derived Category

For an abelian category $\mathcal{A}$ we can form the category of complexes of $\mathcal{A}$, denoted $\mathrm{Kom}(\mathcal{A})$. The objects of $\mathrm{Kom}(\mathcal{A})$ are complexes

$$\cdots \longrightarrow A^{i-1} \xrightarrow{d_A^{i-1}} A^i \xrightarrow{d_A^i} A^{i+1} \xrightarrow{d_A^{i+1}} \cdots$$

such that $A^i \in \mathcal{A}$ and $d_A^{i+1} \circ d_A^i = 0$ for all $i \in \mathbb{Z}$. The maps d_A^i are called *differentials* or *boundary maps*. If $A, B \in \mathrm{Kom}(\mathcal{A})$, then a morphism $f : A \to B$ of complexes is given by a morphism $f^i \in \mathrm{Hom}(A^i, B^i)$ for each $i \in \mathbb{Z}$ such that the f^i's commute with the differentials. The category of complexes is an abelian category; the complex consisting of only zeroes is the zero object and for each morphism f we can form the complex of the kernels $\ker f^i$ and the complex of the cokernels $\mathrm{coker}\, f^i$. For a complex $A \in \mathrm{Kom}(\mathcal{A})$, we define the cohomology objects by

$$H^i(A) = \frac{\ker d_A^i}{\mathrm{im}\, d_A^{i-1}}.$$

These are objects of $\mathcal{A}$ and they all vanish exactly when the complexes are exact sequences. A morphism $f : A \to B$ of complexes induces a morphism on cohomology objects $H(f) : H(A) \to H(B)$.

Definition 2.2.1. A complex $A \in \mathrm{Kom}(\mathcal{A})$ is *acyclic* if $H^i(A) = 0$ for all $i \in \mathbb{Z}$. We say that a functor $F : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{B})$ is acyclic with respect to a class of objects $\mathcal{S} \subset \mathcal{A}$ if it maps any acyclic complex of objects of $\mathcal{S}$ to an acyclic complex in $\mathrm{Kom}(\mathcal{B})$.

For any given complex A we can shift the entire complex to the left or right. We denote by $A[1]$ the complex A shifted one term to the left. More precisely, we let $A[1]$ be the complex such that $A[1]^i = A^{i+1}$ and $d_{A[1]}^i = -d_A^{i+1}$ for all i. For another complex B and a morphism $f \in \mathrm{Hom}(A, B)$ define $f[1]$ to be the morphism of complexes such that $f[1]^i = f^{i+1}$. This shifting of complexes gives rise to a functor.

Definition 2.2.2. Define the shift functor $T : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$ by $T(A) = A[1]$ and $T(f) = f[1] \in \mathrm{Hom}(A[1], B[1])$.

The shift functor is an auto-equivalence on $\mathrm{Kom}(\mathcal{A})$ and satisfies $T^i \circ T^j = T^{i+j}$ for all $i, j \in \mathbb{Z}$. For any $n \in \mathbb{Z}$ the complex $A[n]$ which is A shifted n terms is defined by $A[n] = T^n(A)$.

Definition 2.2.3. ([Huy06, p. 33]) Let $f : A \to B$ be a morphism of complexes. We define the cone of f, denoted $C(f)$ to be the complex defined by

$$C(f)^i = A^{i+1} \oplus B^i$$

with differentials

$$d_{C(f)}^i = \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}.$$

The cone construction gives a short exact sequence of complexes

$$0 \to B \to C(f) \to A[1] \to 0,$$

which gives a long exact sequence on cohomology

$$\cdots \to H^{i-1}(C(f)) \to H^i(A) \to H^i(B) \to H^i(C(f)) \to H^{i+1}(A) \to \cdots$$

We want to study complexes and morphisms of complexes up to cohomology, meaning that we would like to consider a morphism of complexes $f : A \to B$ as an isomorphism if the induced map on cohomology is an isomorphism.

Definition 2.2.4. (Chain Homotopy) Let A, B be complexes with boundary maps d_A and d_B, and let $f, g : A \to B$ be morphisms of complexes. We say that f and g are *homotopic* if there exists a morphism $s : A \to B[-1]$ such that

$$f - g = d_B \circ s + s \circ d_A.$$

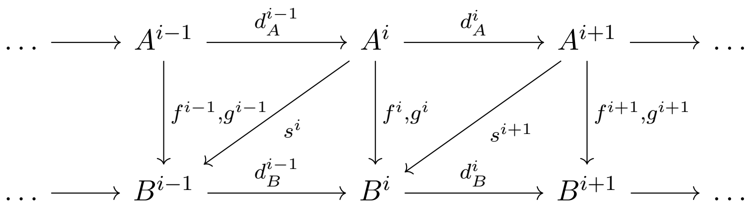


Figure 2.1: Chain Homotopy

Notice that if f and g are homotopic and we restrict to $\ker d_A$, then $s \circ d_A = 0$ and the difference of f and g can be factored through $\operatorname{im} d_B$. Hence, the induced morphisms on cohomology are equal. The homotopy equivalence is an equivalence relation on the morphisms of $\mathrm{Kom}(\mathcal{A})$ and gives rise to a new category called the homotopy category.

Definition 2.2.5. Let $\mathcal{A}$ be an abelian category. We define the homotopy category $\mathrm{K}(\mathcal{A})$ to be the category with the same objects as $\mathrm{Kom}(\mathcal{A})$ and with morphisms being chain homotopy equivalence classes of morphisms of $\mathrm{Kom}(\mathcal{A})$, i.e. for $A, B \in K(\mathcal{A})$ we let

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A, B) \coloneqq \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A, B)/\sim$$

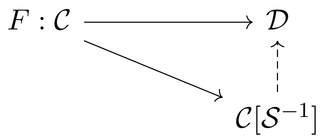
with $f \sim g$ if f and g are homotopic.

Some verification is needed to show that this is in fact a category, c.f. [Huy06, prop. 2.13]. The definition above defines a homotopy category for any additive category $\mathcal{A}$ which is not necessarily abelian [Huy06, p.32], however we will focus on the case where $\mathcal{A}$ is abelian.

Definition 2.2.6. Let $f : A \to B$ be a morphism of complexes. We say that f is a *quasi-isomorphism* if the induced maps $H^i(f) : H^i(A) \to H^i(B)$ are isomorphisms for all i.

If we would like to work in a category where the quasi-isomorphisms are actual isomorphisms, then the existence of inverses is necessary. However, there is no guarantee that a quasi-isomorphism has an inverse. The solution to this problem is a so-called 'localization' of the category $\mathrm{K}(\mathcal{A})$. The process is analogous to the localization of a ring by adding inverses of elements of a multiplicatively closed subset of the ring. Only now, we wish to add inverses of morphisms. The property we need is that the type of morphisms we want to invert, which in our case are the quasi-isomorphisms, form what is called a localizing class. For details, see [Ser13, p. 147].

Definition 2.2.7. Let $\mathcal{C}$ be an additive category and let $\mathcal{S}$ be a localizing class of $\mathcal{C}$. We define the category $\mathcal{C}[\mathcal{S}^{-1}]$ to be the category satisfying the universal property that any functor $F : \mathcal{C} \to \mathcal{D}$ factors uniquely through $\mathcal{C}[\mathcal{S}^{-1}]$ if and only if $F(\phi)$ is an isomorphism for all $\phi \in \mathcal{S}$.



This universal property determines $\mathcal{C}[\mathcal{S}^{-1}]$ uniquely up to equivalence of categories.

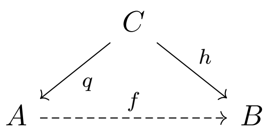
The class of quasi-isomorphisms form a localizing class and we define the derived category by the localization of the homotopy category by this class.

Definition 2.2.8. Let $\mathcal{A}$ be an abelian category and let $\mathcal{S}$ denote the class of quasi-isomorphisms of $\mathrm{K}(\mathcal{A})$. We define the derived category of $\mathcal{A}$, denoted $D(\mathcal{A})$, to be the category $\mathrm{K}(\mathcal{A})[\mathcal{S}^{-1}]$.

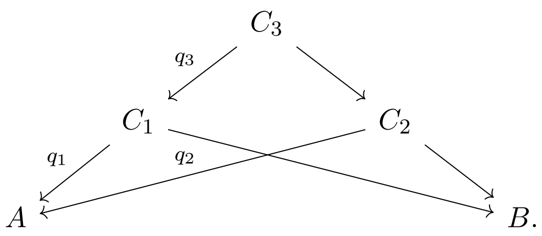
We denote by $Q_{\mathcal{A}} : \mathrm{K}(\mathcal{A}) \to D(\mathcal{A})$ the functor satisfying the universal property: it sends quasi-isomorphisms to isomorphisms, and any functor $F : \mathrm{K}(\mathcal{A}) \to \mathcal{D}$ with the same property factors uniquely through it. Notice that the objects in the derived category are the same as the objects of the category of complexes. We will often be interested in the complexes that are bounded in some way. We call a complex $A \in \mathrm{Kom}(\mathcal{A})$ *bounded above* if $A^i = 0$ for $i \gg 0$, *bounded below* if $A^i = 0$ for $i \ll 0$ and *bounded* if $A^i = 0$ for $|i| \gg 0$. Analogously we say that A has cohomology *bounded above* if $H^i(A) = 0$ for $i \gg 0$, *bounded below* if $H^i(A) = 0$ for $i \ll 0$ and *bounded* if $H^i(A) = 0$ for $|i| \gg 0$. If we want to restrict to complexes that are bounded above, bounded below or bounded, we write $D^-(\mathcal{A})$, $D^+(\mathcal{A})$ and $D^b(\mathcal{A})$, respectively. The categories $D^-(\mathcal{A})$, $D^+(\mathcal{A})$ and $D^b(\mathcal{A})$ are equivalent to the subcategories of $D(\mathcal{A})$ of complexes with cohomology bounded above, bounded below and bounded, respectively [Huy06, Prop 2.30]. While the objects of the derived category are just the complexes, the morphisms are more complicated to specify because of the localization involved.

2.2.1 Morphisms in the Derived Category

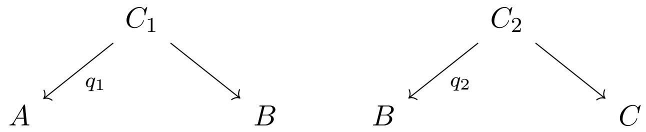
For complexes A and B, a morphism $f \in \mathrm{Hom}_{D(\mathcal{A})}(A, B)$ is given by a composition $f = h \circ q^{-1}$, where $h \in \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(C, B)$ and q^{-1} denotes the formal inverse of a quasi-isomorphism $q \in \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(C, A)$ for an object C. The morphism f can be represented as a roof diagram



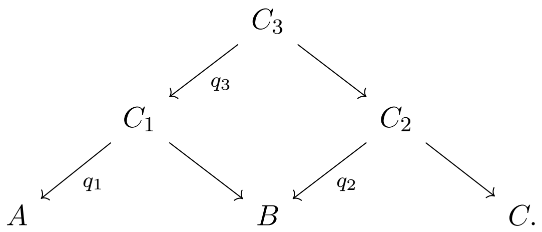
by adding an inverse to q. Two morphisms $f, g \in \mathrm{Hom}_{\mathrm{K}(\mathcal{A})[\mathcal{S}^{-1}]}(A, B)$ are equal if their roof diagrams are dominated, that is if there is an object C_3 and a quasi-isomorphism q_3 fitting into the commutative diagram



Given two morphisms represented by the roof diagrams



their composition is given by extending to a diagram



The existence of such an extension is ensured by the properties of a localizing class. Although the derived category and the homotopy category are not abelian categories, they do have the structure of *triangulated categories*. In the absence of exact sequences, what we do have are the distinguished triangles.

2.2.2 Triangulated Categories

Definition 2.2.9. A triangulated category is an additive category $\mathcal{T}$ equipped with an auto-equivalence $T^i : \mathcal{T} \to \mathcal{T}$ for each $i \in \mathbb{Z}$ and a collection of distinguished triangles subject to four axioms, see [Huy06].

Definition 2.2.10. [Stacks, Tag 05QK]. A distinguished triangle in an additive category $\mathcal{T}$ is a tuple (X, Y, Z, f, g, h) with $X, Y, Z \in \mathcal{T}$ giving a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X).$$

A morphism of triangles is given by morphisms α, β, γ such that the squares in the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X') \end{array}$$

commute.

The derived category and the homotopy category with the shift functor both form triangulated categories. The class of distinguished triangles is defined as follows

Definition 2.2.11. [Huy06, p. 36] A triangle

$$A \to B \to C \to A[1]$$

in the derived (or homotopy) category is *distinguished* if it is quasi-isomorphic to a triangle of the form

$$F \xrightarrow{f} G \to C(f) \to F[1]$$

where $C(f)$ denotes the cone of f.

A distinguished triangle induces a long exact sequence on the cohomology objects

$$\begin{aligned} \cdots \to H^{i-1}(A) \to H^{i-1}(B) \to & H^{i-1}(C) \to H^i(A) \to H^i(B) \\ & \to H^i(C) \to H^{i+1}(A) \to H^{i+1}(B) \to \cdots \end{aligned}$$

There is a natural inclusion functor $\iota : \mathcal{A} \to D(\mathcal{A})$ sending an object $A \in \mathcal{A}$ to the complex

$$\cdots \to 0 \to A \to 0 \to \cdots$$

with trivial differentials and A sits in degree 0. A morphism $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ is sent to the morphism

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow f & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

We will write just A instead of $\iota(A)$ whenever it is clear from context that we think of A as a complex in the derived category $D(\mathcal{A})$ centered in degree 0 rather than as an object of $\mathcal{A}$. The following proposition provides a way of calculating morphisms between complexes of this type.

Proposition 2.2.12. *[Huy06, p. 49] Let $\mathcal{A}$ be an abelian category with enough injectives. Then*

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B[n]) \simeq \mathrm{Ext}^n_{\mathcal{A}}(A, B) \simeq \mathrm{Hom}_{D(\mathcal{A})}(A[-n], B)$$

for all $A, B \in \mathcal{A}$.

Proposition 2.2.12 will prove useful later as we will be especially interested in objects of $\mathcal{A}$ considered as objects of $D(\mathcal{A})$ with no morphisms between them in the derived category. The proposition translates that problem into a question of homological algebra in the category $\mathcal{A}$.

Proposition 2.2.13. *The inclusion functor $\iota : \mathcal{A} \to D(\mathcal{A})$ is fully faithful and takes exact sequences to distinguished triangles.*

Proof. Under the assumption that $\mathcal{A}$ has enough injectives, Proposition 2.2.12 says that

$$\mathrm{Hom}_{\mathcal{A}}(A, B) = \mathrm{Ext}^0_{\mathcal{A}}(A, B) \simeq \mathrm{Hom}_{D(\mathcal{A})}(\iota(A), \iota(B)),$$

so ι is fully faithful. The functor is fully faithful for a general abelian category also, c.f. [Stacks, Tag 06XS]. For any exact sequence $0 \to A \to B \to C \to 0$ we get a quasi-isomorphism

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

The cone of $\iota(A \to B)$ is just the complex in the top row, hence $C(\iota(A \to B)) \cong \iota(C)$. The morphism $\iota(C \to A[1])$ is trivial since $\iota(A)$ is nonzero in degree 0 only. Thus

$$\iota(A) \to \iota(B) \to \iota(C) \to \iota(A)[1]$$

defines a distinguished triangle. ■

2.2.3 Truncation

In this section we briefly introduce the concept of truncation of complexes since it will be used througout. We follow the notation of [Stacks, Tag 0118]. Let $\mathcal{A}$ be an abelian category. Then for $\mathcal{E} \in D(\mathcal{A})$ we can define the morphisms of complexes given in the diagram

$$\begin{array}{lccccccccc}
\sigma_{\leq i-1}(\mathcal{E}^\bullet)[1] & \cdots & \longrightarrow & \mathcal{E}^{i-2} & \xrightarrow{-d_{\mathcal{E}}^{i-2}} & \mathcal{E}^{i-1} & \xrightarrow{0} & 0 & \longrightarrow & \cdots \\
\downarrow f & & & \downarrow 0 & & \downarrow d_{\mathcal{E}}^{i-1} & & \downarrow 0 & & \\
\sigma_{\geq i}(\mathcal{E}^\bullet): & \cdots & \longrightarrow & 0 & \xrightarrow{0} & \mathcal{E}^{i} & \xrightarrow{d_{\mathcal{E}}^{i}} & \mathcal{E}^{i+1} & \longrightarrow & \cdots \\
\downarrow & & & \downarrow 0 & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \\
\mathcal{E}^\bullet & \cdots & \longrightarrow & \mathcal{E}^{i-1} & \xrightarrow{d_{\mathcal{E}}^{i-1}} & \mathcal{E}^{i} & \xrightarrow{d_{\mathcal{E}}^{i}} & \mathcal{E}^{i+1} & \longrightarrow & \cdots
\end{array}$$

Then the cone of f is given by

$$C(f): \cdots \to \mathcal{E}^{i-2} \oplus 0 \xrightarrow{\begin{pmatrix} d_{\mathcal{E}}^{i-2} & 0 \\ 0 & 0 \end{pmatrix}} \mathcal{E}^{i-1} \oplus 0 \xrightarrow{\begin{pmatrix} 0 & 0 \\ d_{\mathcal{E}}^{i-1} & 0 \end{pmatrix}} 0 \oplus \mathcal{E}^{i} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & d_{\mathcal{E}}^{i} \end{pmatrix}} 0 \oplus \mathcal{E}^{i+1} \to \cdots$$

Hence $C(f) \simeq \mathcal{E}^\bullet$ and we get a distinguished triangle $\sigma_{\geq i-1}\mathcal{E}^\bullet[1] \to \sigma_{\leq i}\mathcal{E}^\bullet \to \mathcal{E}^\bullet$. The complex $\sigma_{\geq i}(\mathcal{E}^\bullet)$ and $\sigma_{\leq i}(\mathcal{E}^\bullet)$ are sometimes refer ed to as the *stupid truncations* of $\mathcal{E}^\bullet$. If we instead truncate the complex $\mathcal{E}^\bullet$ as

$$\begin{array}{lccccccccc}
\tau_{\leq i}(\mathcal{E}^\bullet): & \cdots & \longrightarrow & \mathcal{E}^{i-1} & \longrightarrow & \ker d_{\mathcal{E}}^{i} & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{E}^\bullet: & \cdots & \longrightarrow & \mathcal{E}^{i-1} & \longrightarrow & \mathcal{E}^{i} & \longrightarrow & \mathcal{E}^{i+1} & \longrightarrow & \cdots
\end{array}$$

then we obtain a complex with the property that

$$H^k(\tau_{\leq i}(\mathcal{E}^\bullet)) = \begin{cases} H^k(\mathcal{E}^\bullet) & \text{if } k \leq i \\ 0 & \text{if } k < i. \end{cases}$$

Similarly, the truncation

$$\begin{array}{lccccccccc}
\mathcal{E}^\bullet: & \cdots & \longrightarrow & \mathcal{E}^{i-1} & \longrightarrow & \mathcal{E}^{i} & \longrightarrow & \mathcal{E}^{i+1} & \longrightarrow & \cdots \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \\
\tau_{\geq i}(\mathcal{E}^\bullet): & \cdots & \longrightarrow & 0 & \longrightarrow & \operatorname{coker} d_{\mathcal{E}}^{i-1} & \longrightarrow & \mathcal{E}^{i+1} & \longrightarrow & \cdots
\end{array}$$

gives a complex with the property that

$$H^k(\tau_{\geq i}(\mathcal{E}^\bullet)) = \begin{cases} H^k(\mathcal{E}^\bullet) & \text{if } k \geq i \\ 0 & \text{if } k > i. \end{cases}$$

2.3 The Derived Category of Coherent Sheaves

Now that we have constructed the derived category for a general abelian category, we want to apply the framework to algebraic geometry. We would like to work with bounded complexes of coherent sheaves.

Definition 2.3.1. For a smooth projective variety X over $\operatorname{Spec} k$ we define the derived category of X as

$$D(X) = D^b(\mathbf{Coh}(X)).$$

In chapter 4 we will study exceptional collections of derived categories. We will then be interested in the homomorphisms in the derived category between locally free sheaves on X, specifically when the groups $\operatorname{Hom}_{D(X)}(\mathcal{E}, \mathcal{F}[n])$ are trivial for locally free sheaves $\mathcal{E}$ and $\mathcal{F}$. The following proposition will be useful for that problem.

Proposition 2.3.2. *Let X be a smooth projective variety and let $\mathcal{F}$ be coherent and $\mathcal{E}$ a locally free sheaf on X. Then*

$$\operatorname{Hom}_{\mathrm{D}(X)}(\mathcal{E}, \mathcal{F}[n]) \simeq H^n(X, \mathcal{F} \otimes \mathcal{E}^\vee).$$

Proof. We have $\operatorname{Ext}^n(\mathcal{E}, \mathcal{F}) \simeq \operatorname{Ext}^n(\mathcal{O}_X, \mathcal{F} \otimes \mathcal{E}^\vee)$ and the result follows from Proposition 2.2.12 and the fact that $\operatorname{Hom}(\mathcal{O}_X, -)$ is the global sections functor. ■

Proposition 2.3.3. *[Huy06, p. 63] The inclusion functor $\iota : D(X) \to D^+(\mathbf{QCoh}(X))$ defines an equivalence of $D(X)$ with the full subcategory of $D^+(\mathbf{QCoh}(X))$ of bounded complexes of quasi-coherent sheaves with coherent cohomology.*

Now that we have constructed the bounded derived category of coherent sheaves on a smooth projective variety, we would like a derived version of the usual functors - global sections, pushforward, pullback and tensor product. The naive approach is to just apply the functors term wise in each complex. This does preserve homotopy equivalence and gives a well defined functor between homotopy categories. However, it does not work in the derived category. Since the isomorphisms in the derived category are homotopy classes of quasi-isomorphisms of complexes, any bounded acyclic complex is trivial in the derived category - the zero morphism on each term in the complex to the trivial complex is a quasi-isomorphism. Hence, applying a functor term wise to an object in the derived category makes sense only when the functor is exact. What we will need to define the derived versions of the functors is a class of objects in the homotopy category for which each functor is acyclic, and the assertion that any complex is isomorphic in the derived category to a complex of objects from this class.

Definition 2.3.4. [Huy06, p. 48] For abelian categories $\mathcal{A}$ and $\mathcal{B}$ and a left exact functor $F : \mathcal{A} \to \mathcal{B}$, we say that a triangulated subcategory $\mathcal{K} \subset \mathrm{K}^+(\mathcal{A})$ is adapted to F if the following hold:

(i) F is acyclic with respect to the subcategory $\mathcal{K}$.

(ii) Any $B \in \mathrm{K}(\mathcal{A})$ is quasi-isomorphic to an object $A \in \mathcal{K}$.

We call $\mathcal{K}$ an adapted class.

Replace left exactness with right exactness and $\mathrm{K}^+(\mathcal{A})$ with $\mathrm{K}^-(\mathcal{A})$ for the definition of an adapted class to a right exact functor. We treat the case for left and right derived functors separately.

2.4 Right Derived Functors

For an abelian category $\mathcal{A}$ with enough injectives, any left exact functor $F : \mathcal{A} \to \mathcal{B}$ is acyclic with respect to the class of injective objects. The injective objects $\mathcal{I}$ form a full additive subcategory of $\mathcal{A}$ and one can define the homotopy category $\mathrm{K}^+(\mathcal{I})$. By the following proposition, the category $\mathrm{K}^+(\mathcal{I})$ form an adapted class to any left exact functor $F : \mathcal{A} \to \mathcal{B}$.

Proposition 2.4.1. *[Huy06, Proposition 2.35] If $\mathcal{A}$ is an abelian category with enough injectives, then any $A \in \mathrm{K}^+(\mathcal{A})$ has a quasi-isomorphism $A \to I$ to a complex $I \in \mathrm{K}^+(\mathcal{A})$ of injective objects of $\mathcal{A}$.*

To define a derived functor $\mathrm{R}F : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ we will need to go via the homotopy category $K^+(\mathcal{A})$.

Proposition 2.4.2. *[Huy06, Proposition 2.40] Let $\mathcal{I}$ denote the full additive subcategory of all injective objects of $\mathcal{A}$. Then there is an equivalence of categories*

$$Q_{\mathcal{I}} : \mathrm{K}^+(\mathcal{I}) \xrightarrow{\simeq} \mathrm{D}^+(\mathcal{A}).$$

For a left exact functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories with $\mathcal{A}$ having enough injectives, we define the right derived functor $\mathrm{R}F$ as the composition given in the diagram

$$\begin{array}{ccc} \mathrm{D}^+(\mathcal{A}) & \xrightarrow{\mathrm{R}F} & \mathrm{D}^+(\mathcal{B}) \\ {\scriptstyle Q_{\mathcal{I}}}\uparrow & & {\scriptstyle Q_{\mathcal{B}}}\uparrow \\ \mathrm{K}^+(\mathcal{I}) & \xrightarrow{F} & \mathrm{K}^+(\mathcal{B}), \end{array}$$

by choosing an inverse functor to the equivalence $Q_{\mathcal{I}}$. The bottom arrow is the functor that applies F to every term in the complex.

Now, let us confine our attention to the setting in algebraic geometry. While the category $\mathbf{Coh}(X)$ in general does not have enough injectives, the category $\mathbf{QCoh}(X)$ does [Huy06, p. 44]. Any $\mathcal{F} \in \mathbf{QCoh}(X)$ has a resolution

$$0 \to \mathcal{F} \to I^0 \to I^1 \to \cdots$$

of injective objects $I^i \in \mathbf{QCoh}(X)$. We will often write $0 \to \mathcal{F} \to I^\bullet$ where $I^\bullet \in \mathrm{D}(\mathbf{QCoh}(X))$ is the complex of injectives. Notice that

$$\begin{array}{cccccccccc} \mathcal{F}^\bullet : & & \cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 & \longrightarrow \cdots \\ \downarrow & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle 0} & \\ I^\bullet : & & \cdots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow \cdots \end{array}$$

is a quasi-isomorphism of complexes, and so every quasi-coherent sheaf is isomorphic in $\mathrm{D}(\mathbf{QCoh}(X))$ to its injective resolution. Moreover, Proposition 2.4.1 ensures that any complex $\mathcal{F}^\bullet$ of quasi-coherent sheaves is quasi-isomorphic to a complex of injectives.

2.4.1 Derived Global Sections

Let X be a k-scheme. The global sections functor

$$\Gamma : \mathbf{QCoh}(X) \to \mathbf{Vec}(k)$$

is left exact. Since $\mathbf{QCoh}(X)$ has enough injectives, any complex $\mathcal{F}^\bullet \in \mathrm{D}^+(\mathbf{QCoh}(X))$ is quasi-isomorphic to a complex of injectives $I^\bullet \in \mathrm{D}^+(\mathbf{QCoh}(X))$ and we define the *derived global sections* as the functor

$$\begin{aligned} \mathrm{R}\Gamma : \mathrm{D}^+(\mathbf{QCoh}(X)) &\longrightarrow \mathrm{D}^+(\mathbf{Vec}(k)) \\ \mathcal{F}^\bullet &\longmapsto \Gamma(X, I^\bullet) \end{aligned}$$

We denote by $\mathrm{R}^i\Gamma(X, \mathcal{F}^\bullet)$ the higher derived global sections defined as the cohomology objects of the complex $\Gamma(X, I^\bullet)$. If $\mathcal{F}$ is just a sheaf, then

$$\mathrm{R}^i\Gamma(X, \mathcal{F}) = H^i(X, \mathcal{F})$$

are the cohomology groups of X with respect to $\mathcal{F}$. If $\mathcal{F}$ is coherent, then these are finite dimensional and they vanish for $i > \dim(X)$. We can define the derived functor $\mathrm{R}\Gamma : \mathrm{D}(X) \to \mathrm{D}(\mathbf{Vec}(k))$ by the composition

$$\mathrm{D}(X) \to \mathrm{D}^b(\mathbf{QCoh}(X)) \to \mathrm{D}(\mathbf{Vec}(k)).$$

2.4.2 Derived Pushforward

For a morphism $f : X \to Y$ the pushforward functor

$$f_* : \mathbf{QCoh}(X) \to \mathbf{QCoh}(Y)$$

is left exact. We define the derived pushforward by

$$\begin{aligned} \mathrm{R}f_* : \mathrm{D}^+(\mathbf{QCoh}(X)) &\longrightarrow \mathrm{D}^+(\mathbf{QCoh}(X)) \\ \mathcal{F}^\bullet &\longmapsto f_*(I^\bullet) \end{aligned}$$

If f is projective or proper then f_* preserves coherence and the higher pushforwards $\mathrm{R}^i(\mathcal{F}^\bullet)$ are coherent as well. In this case we have a derived functor

$$\mathrm{R}f_* : \mathrm{D}(X) \to \mathrm{D}^b(\mathbf{QCoh}(X)) \to \mathrm{D}(Y)$$

by Proposition 2.3.3. If $g : Y \to Z$ is another morphism, then $g_* \circ f_* = (g \circ f)_*$. We would like

$$\mathrm{R}g_* \circ \mathrm{R}f_* = \mathrm{R}(g \circ f)_* \tag{2.1}$$

to be true also. However, the pushforward functor does not preserve injectives in general. The solution to this problem is the flasque sheaves. We say that a sheaf is *flasque* if all the restriction maps are surjective. For opens $V \subset U$ in Y, we have

$$\begin{array}{ccc} f_*\mathcal{F}(U) & \longrightarrow\!\!\!\!\rightarrow & f_*\mathcal{F}(V) \\ \| & & \| \\ \mathcal{F}(f^{-1}(U)) & \longrightarrow\!\!\!\!\rightarrow & \mathcal{F}(f^{-1}(V)), \end{array}$$

so flasque is preserved under pushforward. Any injective object of $\mathbf{QCoh}(X)$ is also flasque, and the flasque sheaves are adapted to the pushforward functor [Huy06, p. 74]. Thus if $I^\bullet$ is a complex of flasque sheaves quasi-isomorphic to $\mathcal{F}^\bullet$, then $\mathrm{R}f_*(\mathcal{F}^\bullet) = f_*(I^\bullet)$ and $f_*(I^\bullet)$ is again a complex of flasque sheaves.

Proposition 2.4.3. *[Ser13, p. 200] For abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors. Let $\mathcal{K}_\mathcal{A}$ and $\mathcal{K}_\mathcal{B}$ be adapted to F and G, respectively. Then the derived functors $\mathrm{R}F$, $\mathrm{R}G$ and $\mathrm{R}(G \circ F)$ exist, and if $F(\mathcal{K}_\mathcal{A}) \subset \mathcal{K}_\mathcal{B}$, then*

$$\mathrm{R}G \circ \mathrm{R}F = \mathrm{R}(G \circ F).$$

Since the flasque sheaves are also adapted to the global sections functor [Har77, p.208][Stacks, Tag 09SY], the above proposition says that there are well defined compositions $\mathrm{R}\Gamma(X, \mathrm{R}f_*(-)) = \Gamma(X, \mathrm{R}f_*(-))$ and $(\mathrm{R}f_* \circ \mathrm{R}g_*)(-) = \mathrm{R}(f \circ g)_*(-)$.

The derived pushforward and derived global sections have useful compatibilities. If $f : X \to \operatorname{Spec} k$, then $\mathrm{R}f_*(-) = \mathrm{R}\Gamma(X, -)$ since $f_*(-) = \Gamma(X, -)$. If Y is affine, then $f_*(\mathcal{F}) = \widetilde{\Gamma(X, \mathcal{F})}$ for a quasi-coherent sheaf $\mathcal{F}$. By taking an injective resolution $0 \to \mathcal{F} \to I^\bullet$, we get

$$\mathrm{R}f_*(\mathcal{F}) = f_*(I^\bullet) = \widetilde{\Gamma(X, I^\bullet)} = \mathrm{R}\widetilde{\Gamma(X, \mathcal{F})} \tag{2.2}$$

since the tilde-functor is exact. Taking cohomology of 2.2 gives the following proposition.

Proposition 2.4.4. *For a morphism $f : X \to \operatorname{Spec}(A)$ and a quasi-coherent sheaf on X we have*

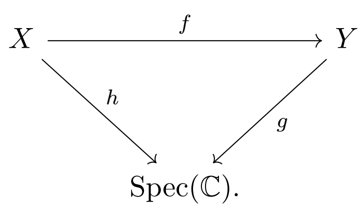
$$\mathrm{R}^i f_*(\mathcal{F}) \simeq \widetilde{H^i(X, \mathcal{F})}.$$

The derived pushforward and derived global sections are compatible in the following way.

Proposition 2.4.5. *Let $f : X \to Y$ be a morphism of schemes and let $\mathcal{F} \in \mathbf{QCoh}(X)$. Then*

$$\mathrm{R}\Gamma(X, \mathcal{F}) = \mathrm{R}\Gamma(Y, \mathrm{R}f_*(\mathcal{F})).$$

Proof. Consider the diagram



We have

$$\Gamma(X, \mathcal{F}) = h_*(\mathcal{F}) = (g \circ f)_*(\mathcal{F}) = g_*(f_*(\mathcal{F})) = \Gamma(Y, f_*\mathcal{F}).$$

Take a flasque resolution $0 \to \mathcal{F} \to I^\bullet$ of $\mathcal{F}$. Then

$$\mathrm{R}\Gamma(Y, \mathrm{R}f_*(\mathcal{F})) = \mathrm{R}\Gamma(Y, f_*I^\bullet) = \Gamma(Y, f_*I^\bullet) = \Gamma(X, I^\bullet) = \mathrm{R}\Gamma(X, \mathcal{F}).$$

■

2.5 Left Derived Functors

The left derived functors require less work. Since any coherent sheaf $\mathcal{F}$ on smooth projective scheme X of dimension n admits a resolution

$$0 \to \mathcal{E}^{n-1} \to \cdots \to \mathcal{E}^1 \to \mathcal{E}^0 \to \mathcal{F} \to 0$$

of locally free sheaves $\mathcal{E}^i$, any coherent sheaf $\mathcal{F}$ is isomorphic in $\mathrm{D}(X)$ to its locally free resolution $\mathcal{E}^\bullet$ by the quasi-isomorphism

$$\begin{array}{cccccccccc}
\mathcal{E}^\bullet : & \cdots & \longrightarrow & \mathcal{E}^1 & \longrightarrow & \mathcal{E}^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{F}^\bullet : & \cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

In fact, if X is regular then any $\mathcal{G}^\bullet \in D(X)$ is isomorphic to a complex $\mathcal{E}^\bullet \in D(X)$ of locally free sheaves [Huy06, p.77]. Thus for the right exact functor $\mathcal{F} \otimes (-) : \mathbf{Coh}(X) \to \mathbf{Coh}(X)$ we can define the derived tensor by

$$\begin{aligned}
\mathcal{F} \otimes^{\mathrm{L}} (-) : \mathrm{D}^-(X) &\longrightarrow \mathrm{D}^-(X) \\
\mathcal{G}^\bullet &\longmapsto \mathcal{F} \otimes \mathcal{E}^\bullet.
\end{aligned}$$

If $\mathcal{G}^\bullet$ is a complex of locally free sheaves then the derived tensor product and the usual tensor product coincide. For two honest complexes $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ with either $\mathcal{F}^\bullet$ or $\mathcal{G}^\bullet$ a complex of locally free sheaves, define the derived tensor product by

$$\begin{aligned}
\mathrm{L}(-) \otimes^{\mathrm{L}} (-) : \mathrm{D}^-(X) \times \mathrm{D}^-(X) &\longrightarrow \mathrm{D}^-(X) \\
(\mathcal{F}^\bullet, \mathcal{G}^\bullet) &\longmapsto \mathcal{F}^\bullet \otimes^{\mathrm{L}} \mathcal{G}^\bullet,
\end{aligned}$$

where $\mathcal{F}^\bullet \otimes^{\mathrm{L}} \mathcal{G}^\bullet$ is defined as the complex with entries $(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)^i = \bigoplus_{p+g=i} \mathcal{F}^p \otimes \mathcal{G}^q$ and differentials $d = d_{\mathcal{F}^\bullet} \otimes 1 + (-1)^i 1 \otimes \mathcal{G}^\bullet$ [Huy06, p. 79]. For a morphism $f : X \to Y$, let $\mathcal{E}^\bullet$ be a complex of locally free sheaves isomorphic to $\mathcal{G}^\bullet$ in $\mathrm{D}(Y)$. We define the derived pullback $\mathrm{L}f^*$ as the functor

$$\begin{aligned}
\mathrm{L}f^* : \mathrm{D}^-(\mathbf{QCoh}(Y)) &\longrightarrow \mathrm{D}^-(\mathbf{QCoh}(X)) \\
\mathcal{G}^\bullet &\longmapsto f^{-1}\mathcal{E}^\bullet \otimes^{\mathrm{L}}_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X.
\end{aligned}$$

The inverse image functor is exact and can be applied term wise in the complex. The tensor product is the derived tensor product. If $\mathcal{G}^\bullet$ itself is a complex of locally free sheaves or if f is flat, then the derived and usual pullbacks coincide.

2.6 Properties of Derived Functors

Proposition 2.6.1. *(The projection formula) Let $f : X \to Y$ be a morphism of ringed spaces, $\mathcal{F}$ an $\mathcal{O}_X$-module and $\mathcal{E}$ a locally free sheaf on Y of rank r. Then*

$$f_*(\mathcal{F} \otimes f^*\mathcal{E}) \simeq f_*(\mathcal{F}) \otimes \mathcal{E}.$$

Proof. Let $\{U_i\}$ be a cover of Y such that $\mathcal{E}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}$ for each i. We prove that an isomorphism exists for every U_i in the cover. So assume $X = f^{-1}(U_i)$ and $Y = U_i$. Then $\mathcal{E} \simeq \mathcal{O}_Y^{\oplus r}$ and we have

$$f_*(\mathcal{F} \otimes f^*\mathcal{E}) \simeq f_*(\mathcal{F} \otimes \mathcal{O}_X^{\oplus r}) \simeq f_*\mathcal{F}^{\oplus r} = (f_*\mathcal{F})^{\oplus r} \simeq f_*\mathcal{F} \otimes \mathcal{O}_Y^{\oplus r} \simeq f_*(\mathcal{F}) \otimes \mathcal{E}. \tag{2.3}$$

Since the isomorphism $f_*(\mathcal{F} \otimes f^*\mathcal{E}) \simeq f_*\mathcal{F} \otimes \mathcal{E}$ is independent of the choice of local isomorphism $\mathcal{E} \simeq \mathcal{O}_Y^{\oplus r}$, it is clear that the isomorphism commutes with the restriction maps. ■

Proposition 2.6.2. *[Har77, Excercise III 8.3] Let $f : X \to Y$ be a morphism of ringed spaces, $\mathcal{F}$ an $\mathcal{O}_X$-module and $\mathcal{E}$ a locally free sheaf on Y of rank r. Then*

$$\mathrm{R}f_*(\mathcal{F} \otimes f^*\mathcal{E}) \simeq \mathrm{R}f_*(\mathcal{F}) \otimes \mathcal{E}.$$

Proof. Let $I^\bullet$ be an injective resolution of $\mathcal{F}$. Then the derived pushforward of $\mathcal{F}$ is defined as $\mathrm{R}f_*(\mathcal{F}) = f_*(I^\bullet)$. Since $\mathcal{E}$ is a locally free sheaf, both $f^*(\mathcal{E})$ and $\mathcal{E} \otimes (-)$ are exact. Hence

$$\mathrm{R}f_*(\mathcal{F}) \otimes f^*\mathcal{E} \simeq f_*(I^\bullet) \otimes f^*\mathcal{E} \simeq f_*(I^\bullet \otimes f^*\mathcal{E}),$$

where we use the projection formula on each term in the complex $f_*(I^\bullet) \otimes \mathcal{E}$. To show that $I^\bullet \otimes f^*\mathcal{E}$ is an injective resolution of $\mathcal{F} \otimes f^*\mathcal{E}$, we can show that $\mathrm{Hom}(-, I^\bullet \otimes f^*\mathcal{E})$ is exact. We have

$$\mathrm{Hom}(-, I^\bullet \otimes f^*\mathcal{E}) = \mathrm{Hom}((-) \otimes (f^*\mathcal{E})^\vee, I^\bullet).$$

Since $\mathcal{E}$ is locally free, so is $(f^*\mathcal{E})^\vee$ and tensoring with locally free sheaves is exact. Since $I^\bullet$ is injective, $\mathrm{Hom}(-, I^\bullet)$ is also exact. Hence their composition is exact. It follows that

$$f_*(I^\bullet \otimes f^*\mathcal{E}) \simeq \mathrm{R}f_*(\mathcal{F} \otimes f^*\mathcal{E}),$$

which proves the proposition. ■

Lemma 2.6.3. *If X and Y are smooth projective, then for any $\mathcal{F}^\bullet \in D(X)$ and $\mathcal{G}^\bullet \in D(Y)$ we have*

$$\mathrm{R}f_*(\mathcal{F}^\bullet \otimes^{\mathrm{L}} \mathrm{R}f^*\mathcal{G}^\bullet) \simeq \mathrm{R}f_*\mathcal{F}^\bullet \otimes^{\mathrm{L}} \mathcal{G}^\bullet.$$

Proof. Replace $\mathcal{F}^\bullet$ with a complex $I^\bullet$ of inectives and $\mathcal{G}^\bullet$ with a complex $\mathcal{E}^\bullet$ of locally frees. Then

$$\mathrm{R}f_*\mathcal{F}^\bullet \otimes^{\mathrm{L}} \mathcal{G}^\bullet = f_*(\mathcal{I}) \otimes \mathcal{E}^\bullet = f_*(\mathcal{I} \otimes f^*\mathcal{E}) = \mathrm{R}f_*\mathcal{F}^\bullet \otimes^{\mathrm{L}} \mathcal{G}^\bullet.$$

■

Theorem 2.6.4 ([Huy06, p. 67]). *Let X be a smooth projective variety of dimension n over a field k, and denote by ω_X the cannonical sheaf of X. Then*

$$S_X : D(X) \longrightarrow D(X)$$

defined by $\mathcal{F}^\bullet \mapsto \omega_X \otimes (\mathcal{F}^\bullet)[n]$ is a Serre functor.

Any equivalence $F : D(X) \simeq D(Y)$ commutes with Serre functors S_X and S_Y, i.e. $F \circ S_X \simeq S_Y \circ F$. As the name might suggest, the Serre functor encodes the Serre duality in a derived category setting. By the property of a Serre functor, there are isomorphisms

$$\mathrm{Hom}(\mathcal{G}^\bullet, \mathcal{F}^\bullet) \simeq \mathrm{Hom}(\mathcal{F}^\bullet, \omega_X \otimes \mathcal{G}^\bullet[n])^\vee$$

for all $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D(X)$. If we set $\mathcal{G}^\bullet = \mathcal{O}_X[-i]$ and let $\mathcal{F}^\bullet$ be a coherent sheaf $\mathcal{F}$ on X both considered as complexes in $D(X)$, then

$$\mathrm{Hom}_{D(X)}(\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{F}) = H^i(X, \mathcal{F})$$

and

$$\mathrm{Hom}_{D(X)}(\mathcal{F}^\bullet, S_X(\mathcal{G}^\bullet)) = \mathrm{Ext}^{n-i}(\mathcal{F}, \omega_X).$$

This gives the more classical statement of Serre duality that for a coherent sheaf $\mathcal{F}$ on a smooth projective variety, there are isomorphisms

$$H^i(X, \mathcal{F})^\vee \simeq \mathrm{Ext}^{n-i}(\mathcal{F}, \omega_X)$$

for all i. If moreover $\mathcal{F}$ is locally free, then $\mathrm{Ext}^{n-i}(\mathcal{F}, \omega_X) \simeq \mathrm{Ext}^{n-i}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_x)$ and we get isomorphisms

$$H^i(X, \mathcal{F})^\vee \simeq H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)$$

for all i.

Example 2.6.5. Consider the Euler sequence on $\mathbb{P}^n$ given by

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n} \to 0,$$

where $\Omega_{\mathbb{P}^n}$ denotes the cotangent bundle on $\mathbb{P}^n$. Taking exterior powers gives an isomorphism

$$\wedge^{n+1}\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \simeq \wedge^n \Omega_{\mathbb{P}^n} \otimes \wedge^1 \mathcal{O}_{\mathbb{P}^n}.$$

Thus the canonical sheaf on $\mathbb{P}^n$ is given by

$$\omega_{\mathbb{P}^n} = \wedge^{n+1}\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} = \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes(n+1)} = \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

Then Serre duality says that

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))^\vee \simeq H^{n-i}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-m-n-1)). \tag{2.4}$$

We will refer to the isomorphism (2.4) later for cohomology computations.

Proposition 2.6.6. *[Har77, Proposition II 9.3] For schemes X, Y and S, let $f : X \to Y$ be a morphisms of finite type and $g : Y \to S$ be a flat morphism.*

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\quad p \quad} & X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\quad g \quad} & S \end{array}$$

Then for a quasi-coherent sheaf $\mathcal{F}$ on X, there are isomorphisms

$$g^* \mathrm{R}^i f_*(\mathcal{F}) \simeq \mathrm{R}^i q_*(p^* \mathcal{F})$$

for all $i \geq 0$.

Corollary 2.6.7. *[Stacks, Tag 0736] If $\mathcal{F} \in D(\mathbf{QCoh}(X))$ then*

$$g^* \mathrm{R} f_*(\mathcal{F}) \simeq \mathrm{R} q_*(p^* \mathcal{F}).$$

Definition 2.6.8. [Huy06, p. 114](Fourier-Mukai functor) Let X and Y be smooth projective varieties, and denote by $p : X \times Y \to X$ and $q : X \times Y \to Y$ the two projections. A Fourier-Mukai functor is a functor $\phi_{\mathcal{P}}$ defined by

$$\phi_{\mathcal{P}} : D(X) \longrightarrow D(Y)$$

$$\mathcal{F} \longmapsto \mathrm{R} q_*(\mathcal{P} \otimes^{\mathrm{L}} \mathrm{L} p^* \mathcal{F}).$$

We call $\mathcal{P} \in D(X \times Y)$ the *Fourier-Mukai kernel.*

We will not use Fourier-Mukai Functors explicitely, but quite remarkably, any equivalence of categories $D(X) \simeq D(Y)$ for X and Y smooth projective is given by a Fourier-Mukai functor [Orl97, Theorem 2.2]. One might ask what geometric information carries over to the derived category. The following theorem summarizes some of the important properties.

Theorem 2.6.9. *Let X and Y be smooth projective and assume $D(X) \simeq D(Y)$ is an exact equivalence of triangulated categories. Then*

1. *X and Y have the same dimension.*
2. *Their canonical bundles are of the same order.*
3. *If the (anti)-canonical bundle of X is ample, then $X \simeq Y$ and the (anti)-canonical bundle of Y is also ample.*

Proof. Proof of the first and second statement: [Huy06, Proposition 4.1]. Proof of the third statement: [Huy06, Proposition 4.11]. ■

Chapter 3

Geometry of Projective Bundles

The majority of the varieties in question in this thesis will be projective bundles. We call a scheme X a *projective bundle* if it has a morphism π to some base scheme Y such that for a cover of affine opens $U \subset Y$, the inverse image $\pi^{-1}(U)$ is isomorphic to $U \times \mathbb{P}^n$ for some n. In other words, for every U in the cover there exists a commutative diagram

$$\begin{array}{ccccc} U \times \mathbb{P}^n & \xrightarrow{\cong} & \pi^{-1}(U) & \hookrightarrow & X \\ & \searrow & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & U & \hookrightarrow & Y \end{array}$$

and the square is a fiber product. To any locally free sheaf $\mathcal{E}$ on Y one can associate a projective bundle $\mathbb{P}(\mathcal{E})$ by the relative Proj construction. Before giving the construction, we state some properties of symmetric and exterior powers of sheaves.

3.1 Projectivization of Vector Bundles

3.1.1 Properties of Symmetric and Exterior Powers

Recall the construction of the symmetric and exterior algebra $\operatorname{Sym}^n M$ and $\wedge^n M$ of a module M over a ring R. For the construction see for instance [EPS98, Appendix 2.A.3]. For a scheme X and a sheaf $\mathcal{F}$ on X we let $\operatorname{Sym}^n \mathcal{F}$ denote the sheafification of the presheaf defined by

$$U \mapsto \operatorname{Sym}^n \mathcal{F}(U)$$

for opens $U \subset X$. Similarly let $\wedge^n \mathcal{F}$ denote the sheafification of the preseheaf

$$U \mapsto \wedge^n \mathcal{F}(U).$$

Define $\operatorname{Sym} \mathcal{F} = \bigoplus_{n \geq 0} \operatorname{Sym}^n \mathcal{F}$ and $\wedge \mathcal{F} = \bigoplus_{n \geq 0} \wedge^n \mathcal{F}$. We will use several properties of the sheaves $\operatorname{Sym}^n(\mathcal{F})$ and $\wedge^n \mathcal{F}$ throughout the thesis, the most important of which are summarized in the proposition below.

Proposition 3.1.1. *Let $(X, \mathcal{O}_X)$ be a ringed space, let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{E}$ be locally free sheaves on X of finite rank and let $\mathcal{L}$ be an invertible sheaf on X. Assume there is an exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$. Then*

1. *There are isomorphisms $\operatorname{Sym}^n(\mathcal{F}^\vee) \simeq (\operatorname{Sym}^n \mathcal{F})^\vee$ for all $n \geq 0$.*
2. *There is an isomorphism $\wedge^{\operatorname{rk} \mathcal{F}} \mathcal{F} \simeq \wedge^{\operatorname{rk} \mathcal{G}} \mathcal{G} \otimes \wedge^{\operatorname{rk} \mathcal{E}} \mathcal{E}$.*

3. *There are isomorphisms* $\wedge^k \mathcal{L}^{\oplus n} \simeq (\mathcal{L}^{\otimes k})^{\oplus \binom{n}{k}}$ *and* $\wedge^n \mathcal{L}^{\oplus n} = \mathcal{L}^{\otimes n}$ *for all* $0 \leq k \leq n$*. If* $k > n$ *then* $\wedge^k \mathcal{L}^{\oplus n} = 0$*.*
4. *For all* $n > 0$ *there is an exact sequence*

$$0 \to \wedge^n \mathcal{E} \to \wedge^{n-1}\mathcal{E} \otimes \operatorname{Sym}^1 \mathcal{F} \to \cdots \to \wedge^2 \mathcal{E} \otimes \operatorname{Sym}^{n-2} \mathcal{F} \\ \to \mathcal{E} \otimes \operatorname{Sym}^{n-1} \mathcal{F} \to \operatorname{Sym}^n \mathcal{F} \to \operatorname{Sym}^n \mathcal{G} \to 0.$$

5. *Moreover, if* $\mathcal{E}$ *is invertible, then for all* $n > 0$ *there is a short exact sequence*

$$0 \to \mathcal{E} \otimes \operatorname{Sym}^{n-1} \mathcal{F} \to \operatorname{Sym}^n \mathcal{F} \to \operatorname{Sym}^n \mathcal{G} \to 0.$$

Proof. Properties 1.-3. are canonical isomorphisms that can be checked locally, c.f. [EPS98, Appendix 2.A]. Let E, M and N be free modules fitting into the exact sequence

$$0 \to E \xrightarrow{f} M \xrightarrow{g} N \to 0.$$

The maps of the sequence in property 4. are

$$\operatorname{Sym}^n M \to \operatorname{Sym}^n N$$

defined by $m_1 \cdots m_n \mapsto g(m_1) \cdots g(m_n)$ and

$$\wedge^k E \otimes \operatorname{Sym}^{n-k} M \to \wedge E^{k-1} \otimes \operatorname{Sym}^{n-k+1} M$$

defined by

$$e_1 \wedge \cdots \wedge e_k \otimes m_1 \cdots m_{n-k} \mapsto \sum_{i=1}^{k} (-1)^{i+1} e_1 \wedge \cdots \wedge \hat{e_i} \wedge \cdots \wedge e_k \otimes f(e_k) \cdot m_1 \cdots m_{n-k}.$$

One can check that the sequence in 4. with these maps is indeed exact. Property 5. is a consequence of 3. and 4. ■

Proposition 3.1.2. *[Wei94, p. 114] (The Koszul Resolution) Let* R *be a ring and* $x = (x_1, \ldots, x_n)$ *a regular sequence in* R*. Denote by* I *the ideal generated by* x*. Then there is an exact sequence*

$$0 \to \wedge^n(R^n) \to \cdots \to \wedge^2(R^n) \to R^n \xrightarrow{x} R \to R/I \to 0.$$

The maps $\wedge^k R^n \to \wedge^{k-1} R^m$ *are given by* $e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto \sum (-1)^{j+1} x_{i_j} e_{i_1} \wedge \cdots \wedge \hat{e_{i_j}} \wedge \cdots \wedge e_{i_k}$*.*

Example 3.1.3. Let $R = k[x_0, \ldots, x_n]$, let $x = (x_0, \ldots, x_n)$ be the regular sequence and let $I = (x)$. Then $R/I \simeq k$ and $\wedge^m(R^n) \simeq R^{\binom{n}{m}}$, so the sequence

$$0 \to R \to \cdots \to R^{\binom{n+1}{2}} \to R^{n+1} \to R \to k \to 0$$

is exact. The homomorphisms are all degree one maps, so

$$0 \to R \to \cdots \to R(n-1)^{\binom{n+1}{2}} \to R(n)^{n+1} \to R(n+1) \to k \to 0$$

is an exact sequence of graded R-modules. By applying the graded tilde functor, we get the exact sequence

$$0 \to \mathcal{O} \to \cdots \to \mathcal{O}(n-1)^{\binom{n+1}{2}} \to \mathcal{O}(n)^{n+1} \to \mathcal{O}(n+1) \to 0,$$

which gives a resolution of the $n+1$'st twist on $\mathbb{P}^n$ by the previous n twists.

3.1.2 Relative Projective Spaces

Let $\mathcal{F}$ be a quasi-coherent sheaf with the structure of a graded $\mathcal{O}_X$-algebra on a scheme X. That is, $\mathcal{F} \simeq \bigoplus_{i \geq 0} \mathcal{F}_i$ with $\mathcal{F}_0 \simeq \mathcal{O}_X$ and $\mathcal{F}_1$ coherent such that $\mathcal{F}_1$ locally generates $\mathcal{F}$. Then for a cover of X by affine opens $U = \operatorname{Spec}(A)$, each $\mathcal{F}(U)$ is a graded $\mathcal{O}_X(U)$-algebra, and so we get a morphism

$$\pi_U : \operatorname{Proj}(\mathcal{F}(U)) \to \operatorname{Spec}(A).$$

Gluing over the affine opens gives a scheme $\mathcal{P}roj(\mathcal{F})$ and a morphism

$$\pi : \mathcal{P}roj(\mathcal{F}) \to X.$$

The scheme $\mathcal{P}roj(\mathcal{F})$ comes with an invertible sheaf $\mathcal{O}_{\mathcal{P}roj(\mathcal{F})}(1)$ defined by the gluing of the invertible sheaves $\mathcal{O}(1)$ on each $\operatorname{Proj}(\mathcal{F}(U))$. If $\mathcal{F}$ is locally free, then $\operatorname{Sym} \mathcal{F}$ is a graded $\mathcal{O}_X$-algebra and we define the projectivization of $\mathcal{F}$ as the projective bundle given by

$$\mathbb{P}(\mathcal{F}) = \mathcal{P}roj(\operatorname{Sym}(\mathcal{F}^\vee)).$$

If the rank of $\mathcal{F}$ is $r+1$, then over an open affine $U = \operatorname{Spec}(A) \subset X$ we have $\pi^{-1}(U) = \operatorname{Proj}(\mathcal{F}(U)^\vee)$ and the graded $\mathcal{O}_X(U)$-algebra $\operatorname{Sym}(\mathcal{F}(U)^\vee)$ is isomorphic to the polynomial ring $A[x_0, \ldots, x_r]$ where the x_i form a basis for $\mathcal{F}(U)^\vee$. Thus we have commutative diagrams

$$\begin{array}{ccccc}
\mathbb{P}^r_{\operatorname{Spec} k(x)} & \hookrightarrow & \mathbb{P}^r_A & \hookrightarrow & \mathbb{P}(\mathcal{F}) \\
\downarrow & & \downarrow & & \downarrow \pi \\
\operatorname{Spec} k(x) & \hookrightarrow & \operatorname{Spec}(A) & \hookrightarrow & X
\end{array}$$

where each square is a fiber product. Notice that if X itself is affine, say $X = \operatorname{Spec}(A)$, then $\mathbb{P}(\mathcal{F}) = \mathbb{P}^r_A$. We define the projectivization $\mathbb{P}(V)$ of a vector space V analogously by

$$\mathbb{P}(V) = \operatorname{Proj}(\operatorname{Sym}(V^\vee)).$$

The projectivization of vector bundles behave nicely with respect to injections and surjections. For locally free sheaves $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$, if the morphism $\mathcal{E} \hookrightarrow \mathcal{F}$ is injective and $\mathcal{F} \twoheadrightarrow \mathcal{G}$ surjective, then $\mathbb{P}(\mathcal{E}) \hookrightarrow \mathbb{P}(\mathcal{F})$ is injective and $\mathbb{P}(\mathcal{F}) \twoheadrightarrow \mathbb{P}(\mathcal{G})$ surjective. Moreover, if $\mathcal{E}$ and $\mathcal{E}'$ are locally free sheaves, then $\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{E}')$ if and only if $\mathcal{E}' \simeq \mathcal{E} \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ [Har77, p. 170].

Proposition 3.1.4 ([Har77, p. 253]). *Let Y be a scheme and let $\mathcal{E}$ be a locally free sheaf on Y of rank $r+1$. If $X = \mathbb{P}(\mathcal{E})$ and $\pi : X \to Y$ is the projection morphism, then*

1. $\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(l) \simeq \operatorname{Sym}^l(\mathcal{E}^\vee)$ *for* $l \geq 0$
2. $\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(l) = 0$ *for* $l < 0$
3. $\mathrm{R}^i \pi_* (\mathcal{O}_{\mathbb{P}(\mathcal{E})}(l)) = 0$ *for* $0 < i < r$ *and all* $l \in \mathbb{Z}$
4. $\mathrm{R}^r \pi_* (\mathcal{O}_{\mathbb{P}(\mathcal{E})}(l) = 0$ *for* $l > -r-1$

Proof. To prove 1 and 2, we restrict to an open affine $U \subset Y$. Then $\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(l)|_{\pi^{-1}(U)} \simeq \mathcal{O}_Y(l)|_U = \mathrm{Sym}^l(\mathcal{O}_Y(1)|_U) \simeq \mathrm{Sym}^l(\mathcal{E}^\vee|_U)$. Hence

$$\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(l) \simeq \mathrm{Sym}^l(\mathcal{E}^\vee),$$

and if $l < 0$, then the right hand side is zero since there are only positive graded pieces, which proves 2. To prove 3 and 4, note that $R^i\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(l)$ is the sheafification of the presheaf defined by

$$U \mapsto H^i(p^{-1}(U), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(l)|_{p^{-1}(U)}),$$

c.f. [Har77, p. 251]. The right hand side is equal to $H^i(\mathbb{P}(\mathcal{E}(U)), \mathcal{O}_{\mathbb{P}(\mathcal{E}(U))}(l))$ which is just the cohomology of an invertible sheaf on a projective space of dimension $r-1$. We know that these are all zero for $0 < i < -r-1$ which proves 3. By Serre duality they are also zero for $i = r$ whenever $l > -r-1$, which proves 4. ∎

Proposition 3.1.5. *Let $f : X \to Y$ be a morphism of smooth projective varieties, $\mathcal{F}$ a coherent sheaf on X and $\mathcal{L}$ a locally free sheaf on Y. Then*

$$H^i(X, \mathcal{F} \otimes f^*\mathcal{L}) \simeq H^i(Y, \mathrm{R}f_*\mathcal{F} \otimes \mathcal{L}).$$

Proof. Take a flasque resolution $0 \to \mathcal{F} \otimes f^*\mathcal{L} \to I^\bullet$. We know that

$$\mathrm{R}\Gamma(X, \mathcal{F} \otimes f^*\mathcal{L}) = \mathrm{R}\Gamma(Y, \mathrm{R}f_*(\mathcal{F} \otimes f^*\mathcal{L}).$$

By taking cohomology on the left hand side we get

$$\mathrm{R}^i\Gamma(X, \mathcal{F} \otimes f^*\mathcal{L}) = H^i(\Gamma(X, I^\bullet)) = H^i(X, \mathcal{F} \otimes f^*\mathcal{L}).$$

Using the fact that pushforward preserves flasque and flasque is adapted to the global sections functor, we take cohomology on the right hand side to get

$$\mathrm{R}^i\Gamma(Y, \mathrm{R}f_*(\mathcal{F} \otimes f^*\mathcal{L}) = \mathrm{R}^i\Gamma(Y, f_*(I^\bullet)) = H^i(Y, \mathrm{R}f_*(\mathcal{F} \otimes f^*\mathcal{L})).$$

Lastly, the projection formula tells us that $\mathrm{R}f_*(\mathcal{F} \otimes f^*\mathcal{L}) \simeq \mathrm{R}f_*\mathcal{F} \otimes \mathcal{L}$. ∎

Corollary 3.1.6. *Let Y be smooth projective and $X = \mathbb{P}(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$ of rank $r+1$ on Y and let $\pi : X \to Y$ be the projection morphism. Then for a locally free sheaf $\mathcal{L}$ on Y we have*

$$H^i(X, \mathcal{O}_X(l) \otimes \pi^*\mathcal{L}) \simeq H^i(X, \mathrm{Sym}^l \mathcal{E}^\vee \otimes \mathcal{L})$$

for $l \geq 0$. The cohomology groups vanish for $0 < l < -r-1$.

Proof. From Proposition 3.1.4 we know that $\mathrm{R}\pi_*\mathcal{O}_X(l) = \mathrm{Sym}^l \mathcal{E}^\vee$ for $l \geq 0$ and $\mathrm{R}\pi_*\mathcal{O}_X(l) = 0$ for $0 < l < -r-1$. Then the corollary follows from Proposition 3.1.5. ∎

3.2 Grassmanians

Given a vector space V of dimension n, we can consider the set of all linear subspaces of a given dimension. Grassmannians are projective varieties that parameterize such subspaces. The points of the Grassmannian $G(k,n)$ can be described as the set

$$G(k,n) = \{[L] \ : \ L \subset V \text{ is a linear subspace of dimension } k\}.$$

The Grassmannian is a projective variety and can be embedded into projective space via the Plücker embedding [EH16, p. 90]. A point $[L] \in G(k,n)$ defines a linear subspace $L \subset V$. Then the orthogonal space of L is the space

$$L^{\perp} = \ker(V^{\vee} \to L^{\vee}).$$

Sending L to its orthogonal space $L^{\perp} \subset V^{\vee}$ defines an isomorphism

$$G(k,n) \xrightarrow{\simeq} G(n-k,n).$$

We can think of the elements of $G(k,n)$ either as k-dimensional linear subspaces of V or $k-1$-dimensional linear subspaces of $\mathbb{P}^{n-1}$. With our convention of the projectivization of vector spaces we have $\mathbb{P}(V) = G(1,n)$ and $\mathbb{P}(V^{\vee}) = G(n-1,n)$.

3.2.1 Sheaves Associated to Grassmannians

Let $\mathcal{O}_G$ denote the structure sheaf of $G(k,n)$. Then we can form a bundle $V \otimes \mathcal{O}_G \to G(k,n)$ called the tautological bundle. The fibers over points on $G(k,n)$ are isomorphic to V. The tautological bundle has a subbundle S of rank k whose fibers over points $[L] \in G(k,n)$ are isomorphic to the subspace $L \subset V$ [EH16, p. 95]. Since S is a subbundle of the tautological bundle, we get a short exact sequence

$$0 \to S \to V \otimes \mathcal{O}_G \to Q \to 0,$$

where Q is the quotient. We call S and Q the universal subbundle and universal quotient bundle, respectively.

3.3 Quotients and Hilbert Schemes of Points

3.3.1 Quotients

Let X be a variety and G a group. Denote by $q : X \to X/G$ the quotient map of X by G.

Definition 3.3.1. [Har95, p. 123] We call X/G *the quotient of* X *by* G if any regular map $f : X \to Y$ factors through q if and only if it is invariant under the action of G, that is if $f(x) = f(gx)$ for all $x \in X$ and all $g \in G$.

The quotient does not always exist as a variety. However, if we take the quotient of X by a finite group G it does always exist. The case of G finite is the one we are interested in. Consider the affine case where $X = \operatorname{Spec}(A)$ for a ring A and G is a finite group. We can form the G-invariant ring of A by

$$A^G = \{f \in A \;:\; gf = f \text{ for all } g \in G\}$$

and the quotient is given by $X/G = \operatorname{Spec}(A^G)$.

For a variety X we define the variety $\operatorname{Sym}^n X$ to be the quotient of X^n by the symmetric group S_n which acts on X^n by permutation of the coordinates. The points of $\operatorname{Sym}^n X$ are cycles of permutations of the points of X^n. They can be represented by formal sums $\sum_i n_i[p_i]$ such that $\sum n_i = n$ where $p_i \in X$ and the n_i are positive integers.

Example 3.3.2. Let $X = \operatorname{Spec} k[x]$. Then $X^n = \operatorname{Spec}(k[x]^{\otimes n}) = \operatorname{Spec} k[x_1, \ldots, x_n]$ and $\operatorname{Sym}^n X = \operatorname{Spec}(k[x_1, \ldots, x_n]^{S_n})$. The natural homomorphism

$$\phi : k[s_1, \ldots, s_n] \to k[x_1, \ldots, x_n]^{S_n}$$

where s_i denotes the symmetric functions in the x_i is an isomorphism and we conclude that

$$\operatorname{Sym}^n(\mathbb{A}^1) \simeq \mathbb{A}^n.$$

Similarly, one can show that $\operatorname{Sym}^n \mathbb{P}^1 = \mathbb{P}^n$.

Example 3.3.3. Let (x, y) be the coordinates on $\mathbb{P}^2 \times \mathbb{P}^2$, so $x = (x_0 : x_1 : x_2)$ and $y = (y_0 : y_1 : y_2)$. Consider the map

$$f : \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^5$$
$$(x, y) \longmapsto (x_i y_j + x_j y_i)$$

for $0 \leq i \leq j \leq 2$. Since f is given by homogeneous polynomials on the coordinates and is invariant under scaling, it defines a projective morphism of schemes. It is S_2-equivariant so it f factors through the universal quotient $\operatorname{Sym}^2 \mathbb{P}^2$ of $\mathbb{P}^2$. We get an injection $i : \operatorname{Sym}^2 \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, so the image of f is isomorphic to $\operatorname{Sym}^2 \mathbb{P}^2$ and is cut out by the zero locus of the homogeneous degree 3 polynomial defined by the determinant of the matrix

$$M = \begin{pmatrix} 2x_0y_0 & x_0y_1 + x_1y_0 & x_0y_2 + x_2y_0 \\ x_1y_0 + x_0y_1 & 2x_1y_1 & x_1y_2 + x_2y_1 \\ x_2y_0 + x_0y_2 & x_2y_1 + x_1y_2 & 2x_2y_2. \end{pmatrix}$$

Thus we can embed $\operatorname{Sym}^2 \mathbb{P}^2$ as a 4-dimensional closed subscheme of $\mathbb{P}^5$.

3.3.2 Hilbert Schemes of Points

Given a scheme X, we would like to study its closed subschemes. The closed and proper subschemes of X can be parametrized in a way that gives the parametrization itself the structure of a scheme. The resulting scheme is the Hilbert scheme of X. We give the definition of a general Hilbert scheme as presented in [Nak99, p. 5]. However we will quickly turn our attention to the Hilbert schemes of points, that is those that parametrize zero-dimensional closed subschemes of X.

Let X be a projective scheme with an ample line bundle $\mathcal{O}(1)$. For a sheaf $\mathcal{F}$ on X, define the *Hilbert polynomial* of $\mathcal{F}$ as

$$P_{\mathcal{F}}(m) = \chi(\mathcal{F} \otimes \mathcal{O}_X(m)) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F} \otimes \mathcal{O}_X(m)).$$

For a scheme T let $Z \overset{\iota}{\hookrightarrow} X \times T$ be a closed subscheme such that the projection $\pi : Z \to T$ is flat. For a point $t \in T$, denote by $Z_t = \pi^{-1}(\{t\})$ the fiber of π at t and define the Hilbert polynomial at t as

$$P_t(m) = \chi(\mathcal{O}_{Z_t} \otimes \mathcal{O}_X(m)).$$

Since π is flat, the Hilbert polynomial at t takes the same value for every $t \in T$ if T is connected. Define the contravariant functor

$$\mathcal{H}ilb_X^P : \mathbf{Sch} \to \mathbf{Set}$$

from the category of schemes to the category of sets by

$$\mathcal{H}ilb_X^P(T) = \{Z \subset X \times T \mid \pi : Z \to T \text{ is flat and } P_t(m) = P \text{ for all } t \in T\}.$$

This functor defines the Hilbert scheme $Hilb^P X$ by the following theorem.

Theorem 3.3.4. *[Gro95] The functor $\mathcal{H}ilb_X^P$ is representable by a projective scheme $\operatorname{Hilb}^P X$.*

The degree of the Hilbert polynomial is the maximal dimension of the subschemes it parametrizes. Thus for a constant Hilbert polynomial, the above functor will parametrize zero-dimensional subschemes. The Hilbert scheme of points is defined to be exactly this scheme.

Definition 3.3.5. For a projective scheme X we define the Hilbert scheme of n points of X denoted by $\operatorname{Hilb}^n X$ as the projective scheme representing the functor $\mathcal{H}ilb_X^n$.

If T is just a point then the set $\mathcal{H}ilb_X^n(T)$ describes the points of $\operatorname{Hilb}^n X$. Indeed, if $Z \overset{\iota}{\hookrightarrow} X \times T$ is a closed subscheme and $\pi : Z \to T$ is the flat projection, then $\mathcal{O}_{Z_T} = \mathcal{O}_Z$ and $n = \chi(\mathcal{O}_Z \otimes \mathcal{O}_X(m))$. Since Z is zero-dimensional it is supported in finitely many points and we define the length of Z as

$$len(Z) = \sum_p \dim_k H^0(Z, \mathcal{O}_{Z,p}) = \dim_k H^0(Z, \mathcal{O}_Z),$$

where the sum is taken over all $p \in supp(Z)$. If $Z \in \operatorname{Hilb}^n X$ then $len(Z) = n$. If we let $Z \subset X$ be the closed subscheme supported in n distinct points $x_1, \ldots, x_n \in X$, then $\mathcal{O}_Z = \bigoplus_{i=1}^n k(x_i)$, where $k(x_i)$ denotes the skyscraper sheaf at x_i and we see that $len(Z) = n$. The Hilbert scheme of points has a map

$$\begin{aligned} \phi : \operatorname{Hilb}^n X &\longrightarrow \operatorname{Sym}^n X \\ Z &\longmapsto \textstyle\sum_p len(\mathcal{O}_{Z,p})[p]. \end{aligned}$$

This map defines a morphism of schemes, called the Hilbert-Chow morphism [Fan+05].

Proposition 3.3.6. *[Fan+05, Theorem 7.3.4] If X is a nonsingular quasiprojective surface, then the Hilbert-Chow morphism $\operatorname{Hilb}^n(X) \to \operatorname{Sym}^n(X)$ is a resolution of the singularities of $\operatorname{Sym}^n(X)$.*

Remark 3.3.7. If X is smooth of dimension one then $\operatorname{Sym}^n X$ is smooth and the Hilbert-Chow morphism is an isomorphism.

Example 3.3.8. The variety $\operatorname{Sym}^2 \mathbb{P}^2$ is singular along the image of the diagonal $\Delta \subset \mathbb{P}^2 \times \mathbb{P}^2$ by the quotient map. The Hilbert scheme $\operatorname{Hilb}^2 \mathbb{P}^2$ is a resolution of the singularities of $\operatorname{Sym}^2 \mathbb{P}^2$ and $Bl_\Delta \operatorname{Sym}^2 \mathbb{P}^2 \simeq \operatorname{Hilb}^2 \mathbb{P}^2$.

$\operatorname{Hilb}^2 \mathbb{P}^2$ as a Projective Bundle

The variety $\operatorname{Hilb}^2 \mathbb{P}^2$ has a morphism $\phi : \operatorname{Hilb}^2 \mathbb{P}^2 \to G(2,3)$ given by sending a closed zero-dimensional subscheme Z of length 2 to the unique projective line containing Z [BOR20]. This projective line defines a point in $(\mathbb{P}^2)^\vee = G(2,3)$. If Z is supported in two different points p_1 and p_2, then Z is mapped to the projective line $\mathrm{H} \in G(2,3)$ spanned by p_1 and p_2. If p_1 and p_2 coincide, then they specify a tangent direction to the point $p_1 = p_2$ which again defines a unique projective line $\mathrm{H} \in G(2,3)$ [Fan+05, p. 169]. The fibers of ϕ are given by the set of pairs of points on a given projective line. So for a $\mathrm{H} \in G(2,3)$ the fiber is given by

$$\phi^{-1}(\mathrm{H}) = \mathbb{P}(\mathrm{Sym}^2 \mathrm{H}) = \mathrm{Sym}^2 \mathbb{P}(H) \simeq \mathbb{P}^2.$$

Thus $\mathrm{Hilb}^2 \mathbb{P}^2$ has the structure of a $\mathbb{P}^2$-fibration over $G(2,3)$. We want to describe $\mathrm{Hilb}^2 \mathbb{P}^2$ as a projective bundle $\mathbb{P}(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$ on $G(2,3)$ of rank 3. Specifying a point of $\mathbb{P}(\mathcal{E})$ should be equivalent to specifying a projective line $\mathbb{P}(\mathrm{H})$ and a pair of points on the line. So we can describe the points of $\mathbb{P}(\mathcal{E})$ as the set

$$Y = \{(p_1 + p_2, \mathrm{H}) \in \mathrm{Sym}^2 \mathbb{P}^2 \times G(2,3) \ : \ p_i \in \mathbb{P}(\mathrm{H})\}.$$

Now, Y has a projection to $G(2,3)$, and the fiber of a point H is all pairs of points on $\mathbb{P}(\mathrm{H})$, which are in bijection with the points of $\mathrm{Sym}^2 \mathbb{P}(\mathrm{H})$. On $G(2,3)$ the universal subbundle S parameterizes the 2-dimensional linear subspaces, which are 1-dimensional projective spaces. So the fibers $S|_{\mathrm{H}}$ are exactly the projective lines. We see that

$$\mathrm{Sym}^2 S|_{\mathrm{H}} = \mathrm{Sym}^2 \mathbb{P}^1 \simeq \mathbb{P}^2,$$

so by setting $\mathcal{E} = \mathrm{Sym}^2 S$ we obtain the desired projective bundle $\mathbb{P}(\mathcal{E})$, and one can show that $\mathbb{P}(\mathcal{E}) \simeq \mathrm{Hilb}^2 \mathbb{P}^2$ is in fact an isomorphism of schemes.

3.4 The Picard Group

Invertible sheaves will play a central role when finding decompositions of the derived category of a variety. In this section we briefly describe the invertible sheaves of projective bundles.

Definition 3.4.1. On a ringed space X we define the Picard group of X denoted $\mathrm{Pic}(X)$ to be the abelian group of isomorphism classes of locally free shaves under the tensor operation.

If $f : X \to Y$ is a morphism of ringed spaces then the pullback $f^*\mathcal{L}$ of an invertible sheaf $\mathcal{L}$ on Y is again invertible and it defines a group homomorphism

$$f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X).$$

Example 3.4.2. [Har77, p. 145] The invertible sheaves on $\mathbb{P}^n$ are, up to isomorphism, given by $\mathcal{O}_{\mathbb{P}^n}(m)$ for some $m \in \mathbb{Z}$. The assignment $\mathcal{O}_{\mathbb{P}^n}(m) \mapsto m$ gives an isomorphism

$$\mathrm{Pic}(\mathbb{P}^n) \xrightarrow{\simeq} \mathbb{Z}.$$

Proposition 3.4.3. *[Har77, p. 170] Let Y by a scheme and $X = \mathbb{P}(\mathcal{E})$ a projective bundle over Y. Then*

$$\mathrm{Pic}(X) \simeq \mathbb{Z} \oplus \mathrm{Pic}(Y)$$

and the invertible sheaves on X are of the form $\mathcal{O}_X(i) \otimes \pi^*\mathcal{O}(j)$, where π is the projection morphism.

Example 3.4.4. ($\mathbb{P}^n \times \mathbb{P}^m$) By Proposition 3.4.3 we have $\mathrm{Pic}(\mathbb{P}^n \times \mathbb{P}^m) \simeq \mathbb{Z} \oplus \mathbb{Z}$. All the invertible sheaves on $\mathbb{P}^n \times \mathbb{P}^m$ are of the form $p^*\mathcal{O}_{\mathbb{P}^n}(i) \otimes q^*\mathcal{O}_{\mathbb{P}^m}(j)$ for $i, j \in \mathbb{Z}$ where p and q are the two projections.

Example 3.4.5. By the identification of $\mathrm{Hilb}^2 \mathbb{P}^2$ with the projective bundle $\mathbb{P}(\mathcal{E})$ over $G(2,3) = \mathbb{P}^2$, the Proposition 3.4.3 says that $\mathrm{Pic}(\mathrm{Hilb}^2 \mathbb{P}^2) = \mathbb{Z} \oplus \mathbb{Z}$, and the invertible sheaves are of the form $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(i) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(i)$ for $i, j \in \mathbb{Z}$.

3.5 Cohomology of Invertible Sheaves on Projective Bundles

3.5.1 A Künneth Formula

In this section, we prove the Künneth formula for the fiber product of two projective schemes over an algebraically closed field. Let X and Y be two such schemes, and consider the diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & \operatorname{Spec} k. \end{array}$$

Given a pair of coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X and Y respectively, we can form the sheaf $\mathcal{F} \boxtimes \mathcal{G} = p^*\mathcal{F} \otimes q^*\mathcal{G}$ on $X \times Y$. The Künneth formula gives a way to calculate the cohomology groups $H^i(X \times Y, \mathcal{F} \boxtimes \mathcal{G})$ by knowing the cohomology groups of X and Y with respect to this pair of sheaves.

Theorem 3.5.1. *(Künneth Theorem) Let X and Y be smooth projective schemes over a field k, and let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on X and Y, respectively. Let p and q be as in the diagram above. Then*

$$H^n(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) \simeq \bigoplus_{i+j=n} H^i(X, \mathcal{F}) \otimes H^j(Y, \mathcal{G}).$$

Proof. Since pushforward to a point defines the same functor as taking global sections, we have that $\mathrm{R}(f \circ p)_*(-) = \mathrm{R}\Gamma(X \times Y, -)$. Since f and g are flat, separated morphisms of finite type, we know by Corollary 2.6.7 that cohomology commutes with flat base change. In other words, we have

$$f^*\mathrm{R}g_*(\mathcal{G}) \simeq \mathrm{R}p_*(q^*\mathcal{G}).$$

By the symmetry of the diagram, we also have

$$g^*\mathrm{R}f_*(\mathcal{F}) \simeq \mathrm{R}q_*(p^*\mathcal{F}).$$

Using this and the derived projection formula (Lemma 2.6.3), we have

$$\begin{aligned} \mathrm{R}\Gamma(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) &= \mathrm{R}(f \circ p)_*(p^*\mathcal{F} \otimes q^*\mathcal{G}) \\ &\simeq \mathrm{R}f_*(\mathcal{F} \otimes \mathrm{R}p_*(q^*\mathcal{G})) \\ &\simeq \mathrm{R}f_*(\mathcal{F} \otimes f^*(\mathrm{R}g_*\mathcal{G})) \\ &\simeq \mathrm{R}f_*\mathcal{F} \otimes \mathrm{R}g_*\mathcal{G} \\ &= \mathrm{R}\Gamma(X, \mathcal{F}) \otimes \mathrm{R}\Gamma(Y, \mathcal{G}). \end{aligned}$$

Now, $\mathrm{R}\Gamma(X \times Y, \mathcal{F} \boxtimes \mathcal{G})$ is a complex of k-vector spaces, and since every complex of vector spaces splits, we have a quasi-isomorphism

$$\mathrm{R}\Gamma(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) \simeq \bigoplus_n H^n(X \times Y, \mathcal{F} \boxtimes \mathcal{G})[-n], \tag{3.1}$$

where the complex on the right has trivial differentials. For $A, B \in D(\operatorname{Spec} k)$ we define $A \otimes^L B$ as the total complex of the double complex

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A^{i-1} \otimes B^{i+1} & \xrightarrow{d_A^{i-1}\otimes 1} & A^{i} \otimes B^{i+1} & \xrightarrow{d_A^{i}\otimes 1} & A^{i+1} \otimes B^{i+1} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle (-1)^{i-1}\otimes d_B^{i}} & & \uparrow{\scriptstyle (-1)^{i}\otimes d_B^{i}} & & \uparrow{\scriptstyle (-1)^{i+1}\otimes d_B^{i}} & & \\
\cdots & \longrightarrow & A^{i-1} \otimes B^{i} & \xrightarrow{d_A^{i-1}\otimes 1} & A^{i} \otimes B^{i} & \xrightarrow{d_A^{i}\otimes 1} & A^{i+1} \otimes B^{i} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle (-1)^{i-1}\otimes d_B^{i-1}} & & \uparrow{\scriptstyle (-1)^{i}\otimes d_B^{i-1}} & & \uparrow{\scriptstyle (-1)^{i+1}\otimes d_B^{i-1}} & & \\
\cdots & \longrightarrow & A^{i-1} \otimes B^{i-1} & \xrightarrow{d_A^{i-1}\otimes 1} & A^{i} \otimes B^{i-1} & \xrightarrow{d_A^{i}\otimes 1} & A^{i+1} \otimes B^{i-1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where $(A \otimes B)^i = \bigoplus_{p+q=i} A^p \otimes B^q$. Replacing A and B with $\bigoplus_i H^i(A)[-i]$ and $\bigoplus_i H^i(B)[-i]$ respectively, the double complex will have trivial differentials and entries $H^i(A) \otimes H^j(B)$. Hence, the total complex will have trivial differentials, and each term will have the form

$$\bigoplus_{i+j=n} H^i(A) \otimes H^j(B).$$

It follows that what sits in degree n of the complex $\bigoplus_i H^i(X \times Y, \mathcal{F} \boxtimes \mathcal{G})[-i]$ is isomorphic to

$$\bigoplus_{i+j=n} H^i(X, \mathcal{F}) \otimes H^j(Y, \mathcal{G}).$$

■

Example 3.5.2. (Cohomology of invertible sheaves on $\mathbb{P}^2 \times \mathbb{P}^2$) Let $X = \mathbb{P}^2 \times \mathbb{P}^2$, $Y = \mathbb{P}^2$ and let p and q denote the first and second projections, respectively. We have seen that $\operatorname{Pic}(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$ and that the invertible sheaves on X are given by $\mathcal{O}_X(i,j) = \mathcal{O}_Y(i) \boxtimes \mathcal{O}_Y(j)$ for $i, j \in \mathbb{Z}$. From the Künneth formula, we have

$$H^n(X, \mathcal{O}_X(i,j)) = \bigoplus_{l+k=n} H^l(Y, \mathcal{O}_Y(i) \otimes H^k(Y, \mathcal{O}_Y(j)).$$

so we can recover the cohomology groups of X by knowing the cohomology groups of Y. For projective space, we have the perfect pairing

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \times H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-m-n-1)) \longrightarrow \mathbb{C} \tag{3.2}$$

by the isomorphism (2.4) in (Example 2.6.5). Since all the intermediate cohomology of projective space vanish, we only have to consider the cases where l and k are equal to either 0 or 2. Consequently, $\mathbb{P}^2 \times \mathbb{P}^2$ can have cohomology only in even degree. By the perfect pairing (3.2), we know that if $i \geq -2$ then $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(i)) = 0$. We have

$$H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(i,j)) = \begin{cases} 0 & \text{if either } i < 0 \text{ or } j < 0, \\ \mathbb{C}^{\binom{i+2}{2}} \otimes \mathbb{C}^{\binom{j+2}{2}} & \text{otherwise .} \end{cases}$$

In degree 2, we have

$$H^2(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(i,j)) = H^0(\mathbb{P}^2, \mathcal{O}(i)) \otimes H^2(\mathbb{P}^2, \mathcal{O}(j)) \tag{3.3}$$

$$\oplus H^2(\mathbb{P}^2, \mathcal{O}(i)) \otimes H^0(\mathbb{P}^2, \mathcal{O}(j)). \tag{3.4}$$

We know that $H^k(\mathbb{P}^2, \mathcal{O}(i))$ vanishes for $k = 0$ when $i < 0$, while for $k = 2$ it vanishes for $i \geq -2$. So in any case, either the first or second term in (3.3) will vanish. If $0 > i \geq -2$ then both terms vanish. We get

$$H^2(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(i,j)) = \begin{cases} 0 & \text{if } 0 > i \geq -2 \text{ or } 0 > j \geq -2, \\ \mathbb{C}^{\binom{i+2}{2}} \otimes \mathbb{C}^{\binom{-j-1}{2}} & \text{if } i > 0 \text{ and } j < -2, \\ \mathbb{C}^{\binom{-i-1}{2}} \otimes \mathbb{C}^{\binom{j+2}{2}} & \text{if } i < -2 \text{ and } j > 0. \end{cases}$$

Lastly, in degree 4 we get

$$H^4(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(i,j)) = \begin{cases} 0 & \text{if } i \geq -2 \text{ or } j \geq -2, \\ \mathbb{C}^{\binom{-i-1}{2}} \otimes \mathbb{C}^{\binom{-j-1}{2}} & \text{otherwise }. \end{cases}$$

Example 3.5.3 (Cohomology of invertible sheaves on $\operatorname{Sym}^2 \mathbb{P}^2$). We have seen that $X = \operatorname{Sym}^2 \mathbb{P}^2$ can be embedded in $Y = \mathbb{P}^5$ as the image of an S_2-equivariant morphism from $\mathbb{P}^2 \times \mathbb{P}^2$, and that the image of this morphism is cut out by a degree 3 homogenous polynomial F. Let $\mathbb{P}^5 = \operatorname{Proj}(R)$ for the graded ring $R = \mathbb{C}[x_0, \ldots, x_5]$. If $I = (F)$ is the ideal generated by F, then the sequence

$$0 \to R(-3) \xrightarrow{\cdot F} R \to R/I \to 0$$

is a short exact sequence of graded rings. Applying the graded tilde functor, we get a short exact sequence

$$0 \to \mathcal{O}_Y(-3) \to \mathcal{O}_Y \to \mathcal{O}_X \to 0$$

where $\mathcal{O}_X$ denotes the pushforward by the inclusion of the structure sheaf on X. We can tensor this sequence by an invertible sheaf $\mathcal{O}_Y(k)$ to get the sequence

$$0 \to \mathcal{O}_Y(k-3) \to \mathcal{O}_Y(k) \to \mathcal{O}_X(k) \to 0.$$

This gives a long exact sequence on cohomology

$$\begin{aligned} \cdots \to H^i(Y, \mathcal{O}_Y(k-3)) \to H^i(Y, \mathcal{O}_Y(k)) \to H^i(X, \mathcal{O}_X(k)) \\ \to H^{i+1}(Y, \mathcal{O}_Y(k-3) \to H^{i+1}(Y, \mathcal{O}_Y(k)) \to \cdots . \end{aligned}$$

Since $H^i(Y, \mathcal{O}_Y(l)) = 0$ for all $0 < i < 5$ and all l, and $H^5(X, \mathcal{O}_X(k)) = 0$ since X has dimension 4, we have just two short exact sequences to consider, namely

$$0 \to H^0(Y, \mathcal{O}_Y(k-3)) \to H^0(Y, \mathcal{O}_Y(k)) \to H^0(X, \mathcal{O}_X(k)) \to 0 \tag{3.5}$$

and

$$0 \to H^4(X, \mathcal{O}_X(k)) \to H^5(Y, \mathcal{O}_Y(k-3)) \to H^5(Y, \mathcal{O}_Y(k)) \to 0. \tag{3.6}$$

Lets consider the global sections first. If $k < 0$, everything vanishes. If $0 < k < 3$, the first term vanishes and we get an isomorphism $H^0(X, \mathcal{O}_X(k)) \simeq H^0(Y, \mathcal{O}_Y(k))$. If $k \geq 3$ then (3.5) is isomorphic to

$$0 \to \mathbb{C}^{\binom{k+2}{5}} \to \mathbb{C}^{\binom{k+5}{5}} \to H^0(X, \mathcal{O}_X(k)) \to 0.$$

Hence, $H^0(X, \mathcal{O}_X(k)) \simeq \mathbb{C}^{\binom{2-k}{5}-\binom{-1-k}{5}}$ since it is a short exact sequence of finite dimensional vector spaces over $\mathbb{C}$. As for (3.6), we can do a similar analysis. By the perfect pairing (3.2), the sequence is isomorphic to

$$0 \to H^4(X, \mathcal{O}_X(k)) \to H^0(Y, \mathcal{O}_Y(-k-3))^\vee \to H^0(Y, \mathcal{O}_Y(-k-6))^\vee \to 0.$$

If $k \geq -2$ everything vanishes. If $-2 > k > -6$ the last term vanishes so we have an isomorphism $H^4(X, \mathcal{O}_X(k)) \simeq H^0(Y, \mathcal{O}_Y(-k-3))^\vee$. If $k \leq -6$, then (3.6) is isomorphic to

$$0 \to H^4(X, \mathcal{O}_X(k)) \to \mathbb{C}^{\binom{2-k}{5}} \to \mathbb{C}^{\binom{-1-k}{5}} \to 0$$

so $H^4(X, \mathcal{O}_X(k)) \simeq \mathbb{C}^{\binom{2-k}{5}-\binom{-1-k}{5}}$. To summarize, we have

$$H^0(X, \mathcal{O}_X(k)) = \begin{cases} 0 & \text{if } k < 0, \\ \mathbb{C}^{\binom{k+5}{5}} & \text{if } 0 < k < 3, \\ \mathbb{C}^{\binom{k+5}{5}-\binom{k+2}{5}} & \text{if } 3 \leq k. \end{cases}$$

and

$$H^4(X, \mathcal{O}_X(k)) = \begin{cases} 0 & \text{if } k \geq -2, \\ \mathbb{C}^{\binom{2-k}{5}} & \text{if } -2 > k > -6, \\ \mathbb{C}^{\binom{2-k}{5}-\binom{-1-k}{5}} & \text{if } k \leq -6. \end{cases}$$

Example 3.5.4. (Cohomology of invertible sheaves on $\mathbb{P}(\operatorname{Sym}^2 S)$) Let $X = \mathbb{P}(\operatorname{Sym}^2 S)$, $Y = G(2,3)$ and let $\pi : X \to Y$ be the projection morphism. The invertible sheaves on X are of the form

$$\mathcal{O}_X(k) \otimes \pi^*\mathcal{O}_Y(j)$$

for $i, j \in \mathbb{Z}$. From 3.1.6 we know that for $k \geq 0$ we have isomorphisms

$$H^i(X, \mathcal{O}_X(k) \otimes \pi^*\mathcal{O}_Y(j)) \simeq H^i(Y, \operatorname{Sym}^k(\operatorname{Sym}^2 S^\vee) \otimes \mathcal{O}_Y(j)).$$

If $k = 0$ then these are just the cohomology groups of the twists $\mathcal{O}(j)$ on $\mathbb{P}^2$. To treat the cases $k > 0$, consider the exact sequence

$$0 \to S \to \mathcal{O}_Y^{\oplus 3} \to Q \to 0.$$

We dualize and get

$$0 \to Q^\vee \to \mathcal{O}_Y^{\oplus 3} \to S^\vee \to 0.$$

Since $Q = \mathcal{O}_Y(1)$ has rank one, we can apply Proposition 3.1.1 to get an exact sequence

$$0 \to Q^\vee \otimes \mathcal{O}_Y^{\oplus 3} \to \operatorname{Sym}^2 \mathcal{O}_Y^{\oplus 3} \to \operatorname{Sym}^2 S^\vee \to 0.$$

We have $\operatorname{Sym}^2 \mathcal{O}_Y^{\oplus 3} \simeq \mathcal{O}_Y^{\oplus 6}$ and $Q^\vee \otimes \mathcal{O}_Y^{\oplus 3} \simeq \mathcal{O}_Y(-1)^{\oplus 3}$. So If $k = 1$, then

$$H^i(X, \mathcal{O}_X(1)) \simeq H^i(Y, \operatorname{Sym}^2 S^\vee) \simeq H^i(Y, \mathcal{O}_X^{\oplus 6})$$

which are zero for $i > 0$ and $k^{\oplus 6}$ for $i = 0$. Now assume $k > 2$. Again by Proposition 3.1.1 we get a long exact sequence

$$\begin{aligned} 0 \to (Q^\vee)^{\otimes 3} \to \wedge^2(Q^\vee)^{\oplus 3} \otimes \operatorname{Sym}^{k-2}\mathcal{O}_Y^{\oplus 6} \to (Q^\vee)^{\oplus 3} \otimes \operatorname{Sym}^{k-1} \mathcal{O}_Y^{\oplus 6} \\ \to \operatorname{Sym}^k \mathcal{O}_Y^{\oplus 6} \to \operatorname{Sym}^k(\operatorname{Sym}^2 S^\vee) \to 0. \end{aligned}$$

For simplicity, rewrite the sequence as

$$0 \to A \xrightarrow{i} B \xrightarrow{h} C \xrightarrow{g} D \xrightarrow{f} E \to 0.$$

The cohomology groups of A, B, C and D are not hard to compute. Since

$$\operatorname{Sym}^k \mathcal{O}_Y^{\oplus 6} \simeq \mathcal{O}_Y^{\oplus \binom{k+5}{5}}$$

we get

$$H^i(Y, D) = H^i(Y, \mathcal{O}_Y)^{\oplus \binom{k+5}{5}}$$

which only has cohomology in degree zero. We have $H^i(Y, A) \simeq H^i(Y, \mathcal{O}_Y(-3))$ which has cohomology only in degree 2. Since

$$C = (Q^\vee)^{\oplus 3} \otimes \operatorname{Sym}^{k-1} \mathcal{O}_Y^{\oplus 6} \simeq \mathcal{O}_Y(-1)^{\oplus 3} \otimes \mathcal{O}_Y(-1)^{\oplus \binom{4+k}{5}},$$

we have

$$H^i(Y, C) = H^i(Y, \mathcal{O}_Y(-2))^{\oplus 3 \cdot \binom{4+k}{5}}$$

which is zero for all i. By the identification $\wedge^2 \mathcal{O}_Y(-1)^{\oplus 3} \simeq \mathcal{O}_Y(-2)^{\oplus 3}$ it is clear that $H^i(Y, B) = 0$ for all i also. The cohomology is summarized in table 3.1.

Table 3.1: Cohomology Groups

i	0	1	2
$H^i(Y, A)$	0	0	$\mathbb{C}$
$H^i(Y, B)$	0	0	0
$H^i(Y, C)$	0	0	0
$H^i(Y, D)$	$\mathbb{C}^{\oplus \binom{k+5}{5}}$	0	0

Now, we split the sequence into short exact sequences

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & B & \longrightarrow & \operatorname{im} h & \longrightarrow & 0 & & & & & & \\
 & & & & & & \| & & & & & & & & \\
 & & & & 0 & \longrightarrow & \ker g & \longrightarrow & C & \longrightarrow & \operatorname{im} g & \longrightarrow & 0 & & \\
 & & & & & & & & & & \| & & & & \\
 & & & & & & & & 0 & \longrightarrow & \ker f & \longrightarrow & D & \xrightarrow{f} & E \longrightarrow 0.
\end{array}$$

From the long exact sequences on cohomology, we get

$$H^0(Y, \ker f) = H^0(Y, \operatorname{im} g) \simeq H^1(Y, \ker g) = H^1(Y, \operatorname{im} h) \simeq H^2(Y, A) \simeq \mathbb{C}$$

and

$$H^1(Y, \ker f) = H^1(Y, \operatorname{im} g) \simeq H^2(Y, \ker g) = H^2(Y, \operatorname{im} h) \simeq H^3(Y, A) = 0.$$

So we get a short exact sequence of $\mathbb{C}$-vector spaces

$$0 \to H^0(Y, \ker f) \to H^0(Y, D) \to H^0(Y, E) \to 0,$$

thus

$$H^0(Y, E) \simeq \mathbb{C}^{\oplus \binom{5+k}{5} - 1}.$$

Further, we have

$$H^1(Y, E) \simeq H^2(Y, \ker f) = H^2(Y, \operatorname{im} g) \simeq H^3(Y, \ker g) = 0$$

and

$$H^2(Y, E) \simeq H^3(Y, \ker f) = 0.$$

However, introducing twists $\mathcal{O}_Y(j)$ in the long exact sequence quickly complicates things. The isomorphisms above rely on C having no cohomology in any degree and B and D having no cohomology in degree 1 and 2, which will not be the case when tensoring with $\mathcal{O}_Y(j)$ for $j \neq 0$.

Chapter 4

Decompositions of Derived Categories

Before we can study homological projective duality, we will need a certain type of decomposition of the derived categories of the varieties involved, called a Lefschetz decomposition. This chapter is devoted to studying decompositions, both in the general case of triangulated categories as well as in the specific case of the derived category of a smooth projective variety. From here on out we will not distinguish notation between a functor and its derived functor. It should be clear from context whether it is derived or not.

4.1 Semi-orthogonal Decompositions

Recall that a subcategory $\mathcal{S}$ of a category $\mathcal{C}$ is called *full* if $\mathrm{Hom}_{\mathcal{S}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$ for all objects $A, B \in \mathcal{S}$.

Definition 4.1.1. [Kuz07, Definition 2.1](Semi-orthogonal decomposition) Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be a collection of full triangulated subcategories such that for any $A_i \in \mathcal{A}_i$ and any $A_j \in \mathcal{A}_j$, we have $\mathrm{Hom}_{\mathcal{T}}(A_i, A_j) = 0$ for all $0 < j < i \leq n$. Then $\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ is a *semi-orthogonal collection* of $\mathcal{T}$. If for any $T \in \mathcal{T}$ there exist a chain of morphisms

$$0 = T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 = T$$

such that the cone of $T_{k-1} \to T_k$ is contained in $\mathcal{A}_k$ then $\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ is a *semi-orthogonal decomposition* of $\mathcal{T}$.

If $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a two term semi-orthogonal decomposition, then for any T there is a chain of morphisms

$$0 \to T_1 \to T.$$

The cone of the morphism $0 \to T_1$ which of course is quasi-isomorphic to T_1 is an object of $\mathcal{B}$, and the cone of the morphism $T_1 \to T$ is an object of $\mathcal{A}$. In other words, any $T \in \mathcal{T}$ fits into a distinguished triangle

$$B \to T \to A \to B[1]$$

with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Given a triangulated category $\mathcal{T}$ and a full triangulated subcategory $\mathcal{A} \subset \mathcal{T}$, we can form the left orthogonal category

$$\mathcal{A}^{\perp} := \{F \in \mathcal{T} \ : \ \mathrm{Hom}_{\mathcal{T}}(A[n], F) = 0 \text{ for all } A \in \mathcal{A} \text{ and } n \in \mathbb{Z}\}$$

of $\mathcal{A}$. Similarly, the right orthogonal category of $\mathcal{A}$ is given by

$$^{\perp}\mathcal{A} := \{F \in \mathcal{T} \;:\; \mathrm{Hom}_{\mathcal{T}}(F, A[n]) = 0 \text{ for all } A \in \mathcal{A} \text{ and } n \in \mathbb{Z}\}.$$

If $\mathcal{A}$ is admissible, i.e. if the inclusion functor has a left and right adjoint, then $\mathcal{T} = \langle \mathcal{A}^{\perp}, \mathcal{A}\rangle$ and $\mathcal{T} = \langle \mathcal{A}, ^{\perp}\mathcal{A}\rangle$ are semi-orthogonal decompositions of $\mathcal{T}$. The semi-orthogonal decompositions we will be concerned with when studying homological projective duality are those given by an exceptional collection.

4.2 Exceptional Collections

Definition 4.2.1. [Huy06, Definition 1.57](Exceptional Collection) Let $\mathcal{T}$ be a k-linear triangulated category. We say that $E \in \mathcal{T}$ is an exceptional object if $\mathrm{Hom}_{\mathcal{T}}(E, E[n]) = k$ for $n = 0$ and 0 otherwise. A sequence $E_1 \ldots, E_m$ of exceptional objects is called an *exceptional collection* if $\mathrm{Hom}_{\mathcal{T}}(E_i, E_j[n]) = 0$ for all $j < i$ and all $n \in \mathbb{Z}$.

Given a category $\mathcal{T}$ and a collection of exceptional objects $E_1, \ldots, E_n$ we can generate a full triangulated subcategory $\mathcal{C} \subset \mathcal{T}$ by the exceptional objects. We write $\mathcal{C} = \langle E_1, \ldots, E_n\rangle$ for this category, and it is generated by the exceptional objects under the following operations.

1 Sums: For any two objects $A, B \in \mathcal{C}$ let $A \oplus B \in \mathcal{C}$.

2 Shifts: For any object $A \in \mathcal{C}$ let $T(A) \in \mathcal{C}$, where T denotes the shift functor.

3 Cones: For any two objects $A, B \in \mathcal{C}$ and any $f \in \mathrm{Hom}_{\mathcal{T}}(A, B)$ let $C(f) \in \mathcal{C}$, where $C(f)$ denotes the cone of f.

We say that the exceptional collection generates all of $\mathcal{T}$ if the smallest full triangulated subcategory containing the exceptional objects which is closed under the above operations is equivalent to $\mathcal{T}$, in which case we write $\mathcal{T} = \langle E_1, \ldots, E_n\rangle$, c.f. [Căl05, p. 10].

Definition 4.2.2. Let $\mathcal{T}$ be a k-linear triangulated category, and let $E = \{E_1 \ldots, E_m\}$ be an exceptional collection of objects of $\mathcal{T}$. We say that E is full if $\langle E\rangle^{\perp} = 0$, in which case $\mathcal{T} = \langle E\rangle$.

Remark 4.2.3. Note that if $\mathcal{T}$ is the derived category of some abelian category $\mathcal{A}$, $\mathcal{T}$ can have a full exceptional collection of objects of $\mathcal{A}$ considered as objects of $\mathcal{T}$ by the fully faithful inclusion functor. In fact, this will be the case for the varieties we will consider.

The simplest examples of full exceptional collections in the algebraic geometry setting are those that decompose the derived category of projective space.

Example 4.2.4. (Beilinson's collection on $\mathbb{P}^n$) Consider the derived category $D(\mathbb{P}^n)$. As it turns out, this category can be generated by $n+1$ invertible sheaves on $\mathbb{P}^n$, namely the sheaves $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)$. In other words, the sheaves form a full exceptional collection for $D(\mathbb{P}^n)$, hence

$$D(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)\rangle.$$

One easily checks that these are exceptional objects. For $0 \le i < j \le n$ and all $k \in \mathbb{Z}$, we have

$$\mathrm{Hom}_{D(\mathbb{P}^n)}(\mathcal{O}(i), \mathcal{O}(i)) \simeq \mathrm{Ext}^0(\mathcal{O}(i), \mathcal{O}(i)) \simeq \mathrm{Hom}(\mathcal{O}, \mathcal{O}) = \Gamma(\mathbb{P}^n, \mathcal{O}) = k.$$

The exceptional objects also form an exceptional collection since

$$\begin{aligned}\mathrm{Hom}_{D(\mathbb{P}^n)}(\mathcal{O}(i),\mathcal{O}(j)[k]) &= \mathrm{Ext}^k(\mathcal{O}(i),\mathcal{O}(j))\\ &= \mathrm{Ext}^k(\mathcal{O},\mathcal{O}(i)\otimes\mathcal{O}(j)^\vee)\\ &= H^k(\mathbb{P}^n,\mathcal{O}(i)\otimes\mathcal{O}(j)^\vee)=0\end{aligned}$$

whenever $0<i-j<-n-1$. What remains to show is that $\mathcal{C}=\langle\mathcal{O},\mathcal{O}(1),\dots,\mathcal{O}(n)\rangle$ is full, i.e. that $\mathcal{C}^\perp$ is trivial. We will later see that it suffices to show that the exceptional collection generates all line bundles on X. So we solve the easier problem of showing that the objects of $\mathcal{C}$ generates all line bundles of X. We confine to the case of $\mathbb{P}^1$ and show that any line bundle is generated as an object of $D(\mathbb{P}^1)$ by the collection $\{\mathcal{O},\mathcal{O}(1)\}$. Let $\mathbb{P}^1=\mathrm{Proj}(R)$ with $R=\mathbb{C}[x,y]$. Then the sequence

$$0\to R\xrightarrow{(-y,x)}R(1)\oplus R(1)\xrightarrow{\binom{x}{y}}R(2)\to R(2)/(x,y)\to 0$$

is an exact sequence of graded rings. It is just the Koszul resolution from Example 3.1.3 on $\mathbb{P}^1$. Applying the graded tilde functor gives an exact sequence

$$0\to\mathcal{O}\to\mathcal{O}(1)\oplus\mathcal{O}(1)\to\mathcal{O}(2)\to 0. \tag{4.1}$$

Since a short exact sequence gives a distinguished triangle

$$\mathcal{O}\to\mathcal{O}(1)\oplus\mathcal{O}(1)\to\mathcal{O}(2)\to\mathcal{O}[1]$$

in the derived category by the inclusion functor, it follows that

$$C(\mathcal{O}\to\mathcal{O}(1)\oplus\mathcal{O}(1))\simeq\mathcal{O}(2)$$

is a quasi-isomorphism, hence $\mathcal{O}(2)$ is the cone of a morphism of objects in the collection. More generally we have $C(\mathcal{O}(n-1)\to\mathcal{O}(n)\oplus\mathcal{O}(n))\simeq\mathcal{O}(n+1)$, so all the positive twists are generated inductively by the collection. As for the negative twists, denote by $A^\bullet$ the exact complex (4.1) and tensor by $\mathcal{O}(-1)$. Then truncating gives a quasi-isomorphism

$$\begin{array}{ccccccccccc}\tau_{\leq 0}(A^\bullet): & \dots & \longrightarrow & 0 & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow\dots\\ \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \sigma_{\geq 0}(A^\bullet): & \dots & \longrightarrow & 0 & \longrightarrow & \mathcal{O}\oplus\mathcal{O} & \longrightarrow & \mathcal{O}(1) & \longrightarrow & 0 & \longrightarrow\dots\end{array}$$

Inductively we obtain all the negative twists, hence Beilinson's collection on $\mathbb{P}^1$ generates all the line bundles. The argument for $\mathbb{P}^1$ generalizes to $\mathbb{P}^n$ by the Koszul sequence from Example 3.1.3 given by

$$0\to\mathcal{O}\to\mathcal{O}(1)^{\oplus n+1}\to\mathcal{O}(2)^{\oplus\binom{n+1}{2}}\to\dots\to\mathcal{O}(n)^{\oplus n+1}\to\mathcal{O}(n+1)\to 0.$$

Hence the twisting sheaf $\mathcal{O}(n+1)$ on $\mathbb{P}^n$ fits into a resolution by sums of the first n twists. If $A^\bullet$ denotes the corresponding complex, then the morphism $\sigma_{\leq n}(A^\bullet)\to\sigma_{\geq n+1}(A^\bullet)[-1]$ pictured in the diagram

$$\begin{array}{ccccccccc}\sigma_{\leq n}(A^\bullet): & \dots & \longrightarrow & \mathcal{O}(n-1)^{\oplus\binom{n+1}{2}} & \longrightarrow & \mathcal{O}(n)^{\oplus n} & \longrightarrow & 0 & \longrightarrow\dots\\ \downarrow & & & \downarrow & & \downarrow & & \downarrow & \\ \sigma_{\geq n+1}(A^\bullet)[-1]: & \dots & \longrightarrow & 0 & \longrightarrow & \mathcal{O}(n+1) & \longrightarrow & 0 & \longrightarrow\dots\end{array}$$

is a quasi-isomorphism. Thus $\mathcal{O}(n+1)$ is generated by the sequence. Analogously to the case of $\mathbb{P}^1$ we obtain all the positive twists inductively. The line bundle $\mathcal{O}(-1)$ is generated by the quasi-isomorphism $\tau_{\leq 0}(A^\bullet \otimes \mathcal{O}(-1)) \to \sigma_{\geq 0}(A^\bullet \otimes \mathcal{O}(-1))$. Similarly, all the negative twists are generated inductively by twisting down the Koszul sequence. The same argument will show that

$$D(\mathbb{P}^n) = \langle \mathcal{O}(d), \mathcal{O}(d+1), \ldots, \mathcal{O}(d+n) \rangle$$

is a semi-orthogonal decomposition of $D(\mathbb{P}^n)$ for all $d \in \mathbb{Z}$.

Lemma 4.2.5. *A fully faithful functor $F : \mathcal{C} \to \mathcal{D}$ takes exceptional collections to exceptional collections.*

Proof. Let $\langle E_1, \ldots, E_n \rangle$ be a collection of exceptional objects E_i of $\mathcal{C}$. Then

$$\mathrm{Hom}_{\mathcal{D}}(F(E_i), F(E_j)[k]) = \mathrm{Hom}_{\mathcal{C}}(E_i, E_j[k])$$

for all i, j and all $k \in \mathbb{Z}$. Hence $F(E_1), \ldots, F(E_n)$ is an exceptional collection. ∎

Lemma 4.2.6. *Let $X = \mathbb{P}(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$ on a base scheme Y and denote by $\pi : X \to Y$ the projection. Then the functor*

$$\pi^* : D(Y) \to D(X)$$

is fully faithful.

Proof. Let $\mathcal{G}, \mathcal{F} \in D(Y)$. Then

$$\begin{aligned} \mathrm{Hom}(\pi^*\mathcal{G}, \pi^*\mathcal{F}) &= \mathrm{Hom}(\mathcal{G}, \pi_*\pi^*\mathcal{F}) \\ &= \mathrm{Hom}(\mathcal{G}, \pi_*(\pi^*\mathcal{F} \otimes \mathcal{O}_X)) \\ &= \mathrm{Hom}(\mathcal{G}, \mathcal{F} \otimes \pi_*\mathcal{O}_X) \\ &= \mathrm{Hom}(\mathcal{G}, \mathcal{F} \otimes \mathcal{O}_Y) = \mathrm{Hom}(\mathcal{G}, \mathcal{F}). \end{aligned}$$

Here we have used adjunction of pushforward and pullback, the projection formula from Lemma 2.6.3 and the fact that $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ by Proposition 3.1.4. ∎

The question of whether an exceptional collection is full is difficult to answer in general. We can however give criteria on the length of a full exceptional collection in the cases where we know the K_0-theory of the variety.

4.3 K-theory

Definition 4.3.1. For a smooth projective variety X we define the *K_0-theory of X* denoted $K_0(X)$ as the quotient of the free abelian group generated by isomorphism classes of coherent sheaves on X by the relation that $[\mathcal{E}] = [\mathcal{F}] + [\mathcal{G}]$ if there is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

of coherent sheaves $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$. We define $K_0(D(X))$ analogously by taking quasi-isomorphism classes of complexes with the relation that $[\mathcal{E}^\bullet] = [\mathcal{F}^\bullet] + [\mathcal{G}^\bullet]$ if

$$\mathcal{F}^\bullet \to \mathcal{E}^\bullet \to \mathcal{G}^\bullet \to \mathcal{F}^\bullet[1]$$

is a distinguished triangle.

An exact functor $F : \mathcal{C} \to \mathcal{D}$ gives rise to a homomorphism $K_0F : K_0(\mathcal{C}) \to K_0(\mathcal{D})$ defined by $K_0([\mathcal{F}]) = [F(\mathcal{F})]$. If $D(X)$ admits a two term semi-orthogonal decomposition $\langle \mathcal{A}, \mathcal{B} \rangle$, then any object $\mathcal{F} \in D(X)$ fits uniquely into a distinguished triangle $\mathcal{F}_\mathcal{B} \to \mathcal{F} \to \mathcal{F}_\mathcal{A}$ for objects $\mathcal{F}_\mathcal{A} \in \mathcal{A}$ and $\mathcal{F}_\mathcal{B} \in \mathcal{B}$. Consider the exact functors

$$i_\mathcal{A} : \mathcal{A} \to D(X)$$

$$i_\mathcal{B} : \mathcal{B} \to D(X)$$

given by the inclusions and denote by $K_0 i_\mathcal{A}$ and $K_0 i_\mathcal{B}$ the homomorphisms on K_0-theory arising from these functors. Since any $\mathcal{F} \in D(X)$ fits into a distinguished triangle $\mathcal{F}_\mathcal{B} \to \mathcal{F} \to \mathcal{F}_\mathcal{A}$ for unique $\mathcal{F}_\mathcal{A}$ and $\mathcal{F}_\mathcal{B}$, we also get projection functors

$$p_\mathcal{A} : D(X) \to \mathcal{A}$$

$$p_\mathcal{B} : D(X) \to \mathcal{B}$$

defined by $p_\mathcal{A}(\mathcal{F}) = \mathcal{F}_\mathcal{A}$ and $p_\mathcal{B}(\mathcal{F}) = \mathcal{F}_\mathcal{B}$. Similarly, denote by $K_0 p_\mathcal{A}$ and $K_0 p_\mathcal{B}$ the homomorphisms induced by these projection functors.

Proposition 4.3.2. *The homomorphism* $\phi := K_0 i_\mathcal{A} \oplus K_0 i_\mathcal{B}$ *given by*

$$\begin{aligned} \phi : K_0(\mathcal{A}) \oplus K_0(\mathcal{B}) &\longrightarrow \mathrm{K}_0(D(X)) \\ ([\mathcal{F}_\mathcal{A}], [\mathcal{F}_\mathcal{B}]) &\longmapsto [\mathcal{F}_\mathcal{A}] + [\mathcal{F}_\mathcal{B}] \end{aligned}$$

is an isomorphism.

Proof. We have

$$K_0 p_\mathcal{A} \circ K_0 i_\mathcal{A} = \mathrm{id}_\mathcal{A}$$

$$K_0 p_\mathcal{B} \circ K_0 i_\mathcal{B} = \mathrm{id}_\mathcal{B}$$

and

$$K_0 p_\mathcal{A} \circ K_0 i_\mathcal{B} = K_0 p_\mathcal{B} \circ K_0 i_\mathcal{A} = 0.$$

Let $(\alpha, \beta) \in K_0(\mathcal{A}) \oplus K_0(\mathcal{B})$ and assume $\phi(\alpha, \beta) = 0$. Then

$$K_0 p_\mathcal{A}(\phi(\alpha, \beta)) = (K_0 p_\mathcal{A} \circ K_0 i_\mathcal{A})(\alpha) + (K_0 p_\mathcal{A} \circ K_0 i_\mathcal{B})(\beta) = \alpha + 0$$

implies that $\alpha = 0$ since $K_0 p_\mathcal{A}$ is a group homomorphism. Similarly,

$$K_0 p_\mathcal{B}(\phi(\alpha, \beta)) = (K_0 p_\mathcal{B} \circ K_0 i_\mathcal{A})(\alpha) + (K_0 p_\mathcal{B} \circ K_0 i_\mathcal{B})(\beta) = 0 + \beta$$

implies that $\beta = 0$. Hence ϕ is injective. We have

$$\begin{aligned} (K_0 i_\mathcal{A} \circ K_0 p_\mathcal{A} + K_0 i_\mathcal{B} \circ K_0 p_\mathcal{B})([\mathcal{F}]) &= K_0 p_\mathcal{A}([\mathcal{F}_\mathcal{A}]) + K_0 i_\mathcal{B}([\mathcal{F}_\mathcal{B}]) \\ &= [\mathcal{F}_\mathcal{A}] + [\mathcal{F}_\mathcal{B}] = [\mathcal{F}]. \end{aligned}$$

So composing with the projections is the identity on $K_0(D(X))$ and it follows that ϕ is also surjective. Hence ϕ is an isomomorphism. ■

Proposition 4.3.3. *For a smooth projective variety X we have*

$$K_0(D(X)) = K_0(X).$$

Proof. Consider $\phi : K_0(X) \to K_0(D(X))$ defined by $[\mathcal{E}] \mapsto [\mathcal{E}^\bullet]$ where

$$\mathcal{E}^\bullet \;:\; \cdots \to 0 \to \mathcal{E} \to 0 \to \ldots$$

and $\mathcal{E}$ sits in degree 0. Let $\psi : K_0(D(X)) \to K_0(X)$ be defined by $[\mathcal{F}^\bullet] \mapsto \sum_i(-1)^i[H^i(\mathcal{F}^\bullet[-i])]$. Then ϕ and ψ are inverses and respect the respective group operations of $K_0(X)$ and $K_0(D(X))$ [Stacks, Tag 0FCP]. ■

Theorem 4.3.4. *Let X be a smooth projective variety and assume that $E = \langle E_1, \ldots, E_n\rangle$ is a full exceptional collection of $D(X)$. Then*

$$K_0(D(X)) \simeq K_0(\langle E_1\rangle) \oplus \cdots \oplus K_0(\langle E_n\rangle) \simeq \mathbb{Z}^{\oplus n}.$$

Proof. We know from Proposition 4.3.2 that if $D(X)$ admits a two term semi-orthogonal decomposition $\langle \mathcal{A}, \mathcal{B}\rangle$, then

$$K_0(D(X)) \simeq K_0(\mathcal{A}) \oplus K_0(\mathcal{B}).$$

Now, if $\langle E_1, \ldots, E_n\rangle$ is a full exceptional collection of $D(X)$, then $\langle E_1, \langle E_2, \ldots, E_n\rangle\rangle$ is a semi-orthogonal decomposition. If we let $\mathcal{B}_1 = \langle E_2, \ldots, E_n\rangle$, then

$$K_0(D(X)) \simeq K_0(\langle E_1\rangle) \oplus K_0(\mathcal{B}_1).$$

So if $\mathcal{B}_i = \langle E_{i+1}, \ldots, E_n\rangle$, then $K_0(\mathcal{B}_i) \simeq K_0(\langle E_{i+1}\rangle) \oplus K_0(\mathcal{B}_{i+1})$. We then have

$$\begin{aligned} K_0(D(X)) &\simeq K_0(\langle E_1\rangle) \oplus K_0(\mathcal{B}_1) \\ &\simeq K_0(\langle E_1\rangle) \oplus \cdots \oplus K_0(\langle E_i\rangle) \oplus K_0(\mathcal{B}_i) \\ &\simeq K_0(\langle E_1\rangle) \oplus \cdots \oplus K_0(\langle E_n\rangle). \end{aligned}$$

It remains to show that $K_0(\langle E_i\rangle) \simeq \mathbb{Z}$. It is clear that $K_0(\mathrm{Spec}(k)) = \mathbb{Z}$ since any two k-vector spaces of the same rank are isomorphic, and

$$0 \to k^l \to k^m \to k^n \to 0$$

is exact only if $m = l + n$. Moreover, the exact equivalence

$$\begin{aligned} \Phi : \langle E\rangle &\longrightarrow D(\mathrm{Spec}(k)) \\ \mathcal{F} &\longmapsto \textstyle\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(E, \mathcal{F}[n])[-n] \end{aligned}$$

induces an isomorphism $K_0\Phi : K_0(\langle E\rangle) \xrightarrow{\simeq} K_0(D(\mathrm{Spec}(k)))$, so

$$K_0(D(X)) \simeq \bigoplus_{i=1}^n K_0(\langle E_i\rangle) \simeq \bigoplus_{i=1}^n K_0(D(\mathrm{Spec}(k))) \simeq \bigoplus_{i=1}^n K_0(\mathrm{Spec}(k)) \simeq \mathbb{Z}^{\oplus n}.$$

■

As an immediate consequence we have the following condition on full exceptional collections.

Corollary 4.3.5. *Any two full exceptional collections of $D(X)$ have the same length.*

Proof. If $E_1, \ldots, E_n$ and $G_1, \ldots, G_m$ are full exceptional collections then

$$\mathbb{Z}^{\oplus n} \simeq K_0(D(X)) \simeq \mathbb{Z}^{\oplus m},$$

which implies that $n = m$. ■

In other words, if $D(X)$ has a full exceptional collection, then $K_0(X) = \mathbb{Z}^{\oplus n}$ and n is the length of the collection. So knowing the K_0-theory of X is sufficient to say what the length of a full exceptional collection of $D(X)$ should be. Since we are concerned with projective bundles, the following result will be useful for our applications.

Proposition 4.3.6. *[Qui75, Thm. 2.1] Let $\mathcal{E}$ be a locally free sheaf on $\mathbb{P}^n$ of rank r. Then*

$$K_0(\mathbb{P}(\mathcal{E})) = K_0(\mathbb{P}^n)^{\oplus r} = \mathbb{Z}^{\oplus(n+1)r}.$$

4.4 Lefschetz Decompositions

To use the machinery of homological projective duality, we will need a decomposition which has the form of a Lefschetz decomposition. This decomposition is defined specifically for the derived category of a projective variety.

Definition 4.4.1 ([Kuz14, p. 6]). Let X be a smooth projective variety and let $\mathcal{L}$ be an invertible sheaf on X. We call a semi-orthogonal decomposition of $D^b(X)$ a *right Lefschetz decomposition* with respect to $\mathcal{L}$ if it has the form

$$D(X) = \langle \mathcal{A}_0, \mathcal{A}_1 \otimes \mathcal{L}, \dots, \mathcal{A}_n \otimes \mathcal{L}^{\otimes n} \rangle,$$

where the $\mathcal{A}_i$ are full subcategories of $D^b(X)$ such that $\mathcal{A}_n \subset \mathcal{A}_{n-1} \subset \cdots \subset \mathcal{A}_0$. If it has the form

$$D(X) = \langle \mathcal{B}_n \otimes \mathcal{L}^{\otimes -n}, \mathcal{B}_{n-1} \otimes \mathcal{L}^{\otimes -n+1}, \dots, \mathcal{B}_0 \rangle,$$

with the $\mathcal{B}_i$ full subcategories such that $\mathcal{B}_n \subset \mathcal{B}_{n-1} \subset \cdots \subset \mathcal{B}_0$, we call it a *left Lefschetz decomposition* with respect to $\mathcal{L}$. If the subcategories $\mathcal{A}_i$ (resp. $\mathcal{B}_i$) are all equal, the Lefschetz decomposition is called *rectangular*.

Example 4.4.2. Beilinson's collection for $\mathbb{P}^n$ from example 4.2.4 can be written as a right Lefschetz decomposition with respect to the invertible sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ by letting $\mathcal{A}_0 = \mathcal{A}_1 = \cdots = \mathcal{A}_n = \langle \mathcal{O}_{\mathbb{P}^n} \rangle$. We see that this is also rectangular.

4.4.1 Spanning Classes

In this section we introduce two classes of objects which are so called spanning classes. They are useful for proving fullness of exceptional collections.

Definition 4.4.3. (Spanning Class) [Huy06, p. 20] A class of objects $\mathcal{C} \subset D(X)$ form a *spanning class* of $D(X)$ if $\mathrm{Hom}(A, B[i]) = \mathrm{Hom}(B[i], A) = 0$ for all $A \in \mathcal{C}$, $B \in D(X)$ and all $i \in \mathbb{Z}$ implies that $B \simeq 0$.

Remark 4.4.4. Importantly, if an exceptional collection of $D(X)$ generates all objects of a spanning class of $D(X)$, then it is full. By definition of a spanning class, the objects with no morphisms to or from the shifted objects in the spanning class are trivial. So if the exceptional collection generates all objects in the spanning class, any object in the orthogonal must be trivial and the collection is therefore full.

For the derived category of a smooth projective variety, we will make use of two spanning classes to show fullness of exceptional collections. These are line bundles and the skyscraper sheaves.

Proposition 4.4.5. *Let X be a smooth projective variety with an exceptional collection $\mathscr{C} = \langle E_1, \dots, E_n \rangle$ that generates all the line bundles of X. Then $\mathscr{C}^{\perp} = 0$, i.e. the collection is full.*

Proof. We prove that for any $\mathcal{E}$ there is a line bundle $\mathcal{L}$ and some integer $n \in \mathbb{Z}$ such that $\operatorname{Hom}(\mathcal{L}[n], \mathcal{E}) \neq 0$. This will show that the line bundles is a spanning class, and in particular that the orthogonal $\mathscr{C}^\perp$ is zero. For any $\mathcal{E} \in D(X)$ there is a minimal $l \in \mathbb{Z}$ such that $H^i(\mathcal{E}) \neq 0$. Now, replace $\mathcal{E}$ with $\mathcal{E}[l]$, i.e. shift $\mathcal{E}$ by $[l]$ to get a complex with no cohomology in negative degree. Then $H^0(\mathcal{E}) = \ker d^0_{\mathcal{E}}$ and we get a morphism $f : \tau_{\leq 0}(\mathcal{E}) \to \mathcal{E}$ given by the morphism of complexes

$$\begin{array}{ccccccccc} \tau_{\leq 0}(\mathcal{E}): & \dots \longrightarrow & 0 & \longrightarrow & H^0(\mathcal{E}) & \longrightarrow & 0 & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{E}: & \dots \longrightarrow & 0 & \longrightarrow & \mathcal{E}^0 & \longrightarrow & \mathcal{E}^1 & \longrightarrow & \dots \end{array}$$

This gives a distinguished triangle

$$\tau_{\leq 0}(\mathcal{E}) \to \mathcal{E} \to C(f) \to \tau_{\leq 0}(\mathcal{E})[1] \tag{4.2}$$

where $C(f)$ denotes the cone of f. It induces a long exact sequence on cohomology

$$\dots \to H^i(\tau_{\leq 0}(\mathcal{E})) \to H^i(\mathcal{E}) \to H^i(C(f)) \to H^{i+1}(\tau_{\leq 0}(\mathcal{E})) \to \dots$$

and since the complex $\tau_{\leq 0}(\mathcal{E})$ has no cohomology in degrees $i \neq 0$, the long exact sequence gives an exact sequence

$$0 \to H^{-1}(\mathcal{E}) \xrightarrow{\alpha} H^{-1}(C(f)) \xrightarrow{\beta} H^0(\tau_{\leq 0}(\mathcal{E})) \xrightarrow{\gamma} H^0(\mathcal{E}) \xrightarrow{\delta} H^0(C(f)) \to 0.$$

By construction of $\tau_{\leq 0}(\mathcal{E})$, γ is an isomorphism which implies that $\beta = \delta = 0$. Since we have shifted $\mathcal{E}$ to have no cohomology in negative degree, $H^{-1}(\mathcal{E}) = 0$, implying that $\alpha = 0$. We get

$$H^i(C(f)) = 0 \text{ for } i \leq 0,$$
$$H^i(C(f)) = H^i(\mathcal{E}) \text{ for } i \geq 1.$$

We will need the following lemma to complete the proof:

Lemma 4.4.6. *For any nonzero coherent sheaf $\mathcal{F}$ on a smooth projective scheme X, there exists a line bundle $\mathcal{L} \in Pic(X)$ such that* $\operatorname{Hom}(\mathcal{L}, \mathcal{F}) \neq 0$.

Proof. Since X is projective, it has an ample line bundle $\mathcal{L}$. Then by definition, for any coherent sheaf $\mathcal{F}$ there is an integer n_0 so that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for $n \geq n_0$. Then $\operatorname{Hom}(\mathcal{O}_X, \mathcal{L}^{\otimes n} \otimes \mathcal{F}) \neq 0$, so $\operatorname{Hom}((\mathcal{L}^{\otimes n})^\vee, \mathcal{F}) \neq 0$. ■

Assume $\mathcal{L}$ is a line bundle such that $g : \mathcal{L} \to H^0(\mathcal{E})$ is a nonzero morphism. Such an $\mathcal{L}$ exists by Lemma 4.4.6. Then applying $\operatorname{Hom}(\mathcal{L}, -)$ to the triangle (4.2) gives a long exact sequence

$$\begin{aligned} \dots \to \operatorname{Ext}^{-1}(\mathcal{L}, C(f)) \to & \operatorname{Hom}(\mathcal{L}, \tau_{\leq 0}(\mathcal{E})) \to \operatorname{Hom}(\mathcal{L}, \mathcal{E}) \\ & \to \operatorname{Hom}(\mathcal{L}, C(f)) \to \operatorname{Ext}^1(\mathcal{L}, \tau_{\leq 0}(\mathcal{E})) \to \operatorname{Ext}^1(\mathcal{L}, \mathcal{E}) \to \dots \end{aligned}$$

We claim that $\operatorname{Ext}^i(\mathcal{L}, C(f)) = \operatorname{Hom}(\mathcal{L}, C(f)[i]) = 0$ for $i \leq 0$. This follows from the general fact that for any abelian category $\mathcal{C}$ and any two complexes $A, B \in D(\mathcal{C})$ such that $H^i(A) = 0$ for $i \geq N$ and $H^i(B) = 0$ for $N < 0$ for some $N \in \mathbb{Z}$, we have $\operatorname{Hom}_{D(\mathcal{C})}(A, B) = 0$ [Huy06, p. 70]. In other words, if A has only cohomology up to a certain degree and B has only cohomology in higher degree, then there are no morphism between them. In our case, we know that $H^i(C(f)) = 0$ for $i \leq 0$ and $H^i(C(f)[-1]) = H^{i+1}(C(f)) = 0$ for $i \leq 1$. Hence, $\mathrm{Hom}(\mathcal{L}, C(f)) = 0$ and $\mathrm{Ext}^i(L, C(f)) = \mathrm{Hom}(L, C(f)[i]) = 0$ for $i \leq 0$. So the long exact sequence induces isomorphisms

$$\mathrm{Ext}^i(\mathcal{L}, \tau_{\leq 0}(\mathcal{E})) \simeq \mathrm{Ext}^i(\mathcal{L}, \mathcal{E})$$

for $i \leq 0$. In particular, $\mathrm{Hom}(\mathcal{L}, \tau_{\leq 0}(\mathcal{E})) \simeq \mathrm{Hom}(\mathcal{L}, \mathcal{E}) \neq 0$. Replacing $\mathcal{E}$ with $\mathcal{E}[-l]$, i.e. shifting back $\mathcal{E}$ to the complex we started with, we get

$$\mathrm{Hom}(\mathcal{L}[l], \mathcal{E}) \neq 0$$

which proves the proposition. ■

We showed earlier that Beilinson's collection generates all line bundles on $\mathbb{P}^n$. The proposition above completes the proof that the collection is indeed full. Another usefull spanning class that we will make use of later is the class of skyscraper sheaves.

Proposition 4.4.7. *[Huy06, p. 69] Let X be a smooth projective variety over* $\mathrm{Spec}(k)$. *Then the skyscraper sheaves $k(x)$ form a spanning class of $D(X)$.*

4.4.2 Calabi-Yau Varieties have no Semi-orthogonal Decomposition

An interesting example of smooth projective varieteies with no semi-orthogonal decomposition of their derived categories are the Calabi-Yau varieties. Recall that a variety X is Calabi-Yau if the canonical sheaf ω_X is isomorphic to the structure sheaf.

Example 4.4.8. A non-singular degree 3 curve $C \overset{i}{\hookrightarrow} \mathbb{P}^2$ is Calabi-Yau. We saw in Example 2.6.5 that $\omega_{\mathbb{P}^2} = \mathcal{O}_{P^2}(-3)$, and it follows from the adjunction formula that

$$\omega_C = i^*\omega_{\mathbb{P}^2} \otimes \mathcal{O}_C(3) \simeq \mathcal{O}_C.$$

More generally, if $Z = V(f_1, \dots, f_r) \overset{i}{\hookrightarrow} \mathbb{P}^n$ is a complete intersection with $d_i = \deg f_i$, then Z is Calabi-Yau if $\sum d_i = n + 1$. By the projection formula we have

$$\omega_Z = i^*\omega_{\mathbb{P}^n} \otimes \mathcal{O}_C(d_1 + \cdots + d_r) \simeq \mathcal{O}_C(-n - 1 + d_1 \cdots + d_r)$$

which is isomorphic to $\mathcal{O}_C$ when $\sum d_i = n + 1$.

The fact that the derived category of a Calabi-Yau variety has no semi-orthogonal decomposition is a consequence of Serre duality. Assume for contradiction that X is Calabi-Yau and that $D(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semi-orthogonal decomposition. Let $\mathcal{F} \in \mathcal{A}$ and $\mathcal{G} \in \mathcal{B}$. Then semi-orthogonality implies that

$$\mathrm{Hom}(\mathcal{G}, \mathcal{F}[i]) = 0$$

for all $i \in \mathbb{Z}$. By Serre duality there are isomorphisms

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}[i])^\vee \simeq \mathrm{Hom}(\mathcal{G}, \mathcal{F}[\dim X - i])$$

for all $i \in \mathbb{Z}$ since the canonical sheaf ω_X is trivial. But since the right hand side is zero, we have

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}[i]) = 0$$

for all $i \in \mathbb{Z}$. This implies that the derived category actually splits as $D(X) = \mathcal{A} \oplus \mathcal{B}$. In other words, Calabi-Yau implies that semi-orthogonality is actual orthogonality. This is a contradiction whenever X is connected. Assume for contradiction that $D(X) = \mathcal{A} \oplus \mathcal{B}$. Then for $x \in X$ the complex $k(x)$ splits as $k(x) = A \oplus B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Since $H^i(k(x)) = k(x)$ for $i = 0$ and zero otherwise we have $H^i(A) = H^i(B) = 0$ for $i \neq 0$ so $k(x) = H^0(A) \oplus H^0(B)$ but $k(x)$ does not split as a k-vector space. Since x was chosen arbitrarily we have for all $x \in X$ that $k(x) \in \mathcal{A}$ or $k(x) \in \mathcal{B}$. Now, consider the structure sheaf $\mathcal{O}_X$ as a complex in $D(X)$. Assuming $\mathcal{O}_X = A' \oplus B'$ for $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$ also implies that $\mathcal{O}_X = H^0(A') \oplus H^0(B')$ and if X is connected, then $H^0(A') = 0$ or $H^0(B') = 0$ which in turn implies that $\mathcal{O}_X \in \mathcal{A}$ or $\mathcal{O}_X \in \mathcal{B}$. Since $\mathrm{Hom}(\mathcal{O}_X, k(x)) \neq 0$ for all $x \in X$, the skyscraper sheaves $k(x)$ all belong to the same subcategory as $\mathcal{O}_X$, say $\mathcal{A}$. Since the skyscraper sheaves form a spanning class of $D(X)$, we have $\mathcal{A} = D(X)$ and $\mathcal{B} = 0$.

4.5 Decompositions for Projective Bundles

We start this section by considering an exceptional collection for $\mathbb{P}^2 \times \mathbb{P}^2$ which generalizes to $\mathbb{P}^n \times \mathbb{P}^m$. We move on to giving exceptional collections of projective bundles over $\mathbb{P}^n$ before discussing the more general setting of a projective bundle over a general base scheme Y.

Example 4.5.1. ($\mathbb{P}^2 \times \mathbb{P}^2$) Consider the diagram

$$\begin{array}{ccc} \mathbb{P}^2 \times \mathbb{P}^2 & \xrightarrow{\quad p \quad} & \mathbb{P}^2 \\ {\scriptstyle q}\downarrow & & \downarrow \\ \mathbb{P}^2 & \longrightarrow & \mathrm{Spec}(\mathbb{C}) \end{array}$$

and define $\mathcal{O}(i,j) = \mathcal{O}(i) \boxtimes \mathcal{O}(j)$. We know that all the line bundles on $\mathbb{P}^2 \times \mathbb{P}^2$ are of this form. From Proposition 2.3.2 we have

$$\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)[n]) = \mathrm{Ext}^n(\mathcal{O}, \mathcal{O}(j-i)) = H^n(\mathbb{P}^2, \mathcal{O}(j-i)).$$

It follows from our cohomology computations of $\mathbb{P}^2 \times \mathbb{P}^2$ in Example 3.5.2 that

$$\mathrm{Hom}_{D(\mathbb{P}^2 \times \mathbb{P}^2)}(\mathcal{O}(i,j), \mathcal{O}(k,l)[n]) = H^n(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(k-i, l-j)) = 0$$

for all $n \in \mathbb{Z}$ and all $i,j,k,l \in \mathbb{Z}$ such that $0 > k-i > -3$ or $0 > l-j > -3$. So we get an exceptional collection

$$\langle \mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,0), \mathcal{O}(2,0), \mathcal{O}(0,2), \mathcal{O}(1,1), \mathcal{O}(1,2), \mathcal{O}(2,1), \mathcal{O}(2,2) \rangle \tag{4.3}$$

Note that $K_0(\mathbb{P}^2 \times \mathbb{P}^2) \simeq \mathbb{Z}^{\oplus 9}$ so the collection is of expected length. To show that it it is full, we exploit the fact that Beilinson's collection on $\mathbb{P}^2$ generates all line bundles. The exact sequence on $\mathbb{P}^2$ given by

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus 3} \to \mathcal{O}(2)^{\oplus 3} \to \mathcal{O}(3) \to 0$$

pulls back by p to the exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1,0)^{\oplus 3} \to \mathcal{O}(2,0)^{\oplus 3} \to \mathcal{O}(3,0) \to 0,$$

which gives a resolution of $\mathcal{O}(3,0)$ by the exceptional objects. Tensoring by $\mathcal{O}(0,j)$ for $j = 1,2$ gives a resolution of $\mathcal{O}(3,1)$ and $\mathcal{O}(3,2)$ by objects in the sequence. Symmetrically, we obtain $\mathcal{O}(3,0)$, $\mathcal{O}(3,1)$ and $\mathcal{O}(3,2)$ by pulling back by q and tensor with $\mathcal{O}(i,0)$ for $i=0,1,2$. An induction argument shows that all line bundles can be generated from pullbacks of the Koszul resolution on $\mathbb{P}^2$. Since the line bundles on a variety span the entire derived category of X, (4.3) gives a semi-orthogonal decomposition of $\mathbb{P}^2 \times \mathbb{P}^2$. The decomposition can be made Lefschetz with respect to the line bundle $\mathcal{L} = \mathcal{O}(1,1)$. Firstly, let

$$\begin{aligned}
\mathcal{A}_0 &= \langle \mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,0), \mathcal{O}(0,2), \mathcal{O}(2,0) \rangle, \\
\mathcal{A}_1 &= \langle \mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,0) \rangle, \\
\mathcal{A}_2 &= \langle \mathcal{O} \rangle.
\end{aligned}$$

Then $\mathcal{A}_2 \subset \mathcal{A}_1 \subset \mathcal{A}_0$ is satisfied, $\mathcal{A}_1 \otimes \mathcal{L} = \langle \mathcal{O}(1,1), \mathcal{O}(1,2), \mathcal{O}(2,1) \rangle$ and $\mathcal{A}_2 \otimes \mathcal{L}^{\otimes 2} = \langle \mathcal{O}(2,2) \rangle$ Hence

$$\mathrm{D}(\mathbb{P}^2 \times \mathbb{P}^2) = \langle \mathcal{A}_0, \mathcal{A}_1 \otimes \mathcal{L}, \mathcal{A}_2 \otimes \mathcal{L}^{\otimes 2} \rangle.$$

is a Lefschetz decomposition of $D(\mathbb{P}^2 \times \mathbb{P}^2)$ with respect to $\mathcal{L}$.

Proposition 4.5.2. *Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and denote by p and q the projection to the first and second factor. Define $\mathcal{O}_X(i,j) = p^*\mathcal{O}(i) \otimes q^*\mathcal{O}(j)$. Let $\mathcal{L}_1 = \mathcal{O}(1,0)$ and $\mathcal{L}_2 = \mathcal{O}(0,1)$. Define the full triangulated subcategories $\mathcal{A} = \langle \mathcal{O}(i,0) \rangle_{0 \le i \le n}$ and $\mathcal{B} = \langle \mathcal{O}(0,j) \rangle_{0 \le j \le m}$. Then $D(X)$ has semi-orthogonal decompositions of the form*

$$\begin{aligned}
D(X) &= \langle \langle \mathcal{O}(i,j)_{0 \le j \le m} \rangle_{0 \le i \le n} \rangle = \langle \mathcal{B} \otimes \mathcal{L}_1, \mathcal{B} \otimes \mathcal{L}_1^{\otimes 2}, \dots, \mathcal{B} \otimes \mathcal{L}_1^{\otimes n} \rangle, \\
D(X) &= \langle \langle \mathcal{O}(i,j)_{0 \le i \le n} \rangle_{0 \le j \le m} \rangle = \langle \mathcal{A} \otimes \mathcal{L}_2, \mathcal{A} \otimes \mathcal{L}_2^{\otimes 2}, \dots, \mathcal{A} \otimes \mathcal{L}_2^{\otimes m} \rangle.
\end{aligned}$$

If $m = n$ and $\mathcal{L} = \mathcal{O}(1,1)$ then there is a semi-orthogonal decomposition

$$D(X) = \langle \mathcal{C}_0, \mathcal{C}_1 \otimes \mathcal{L}, \dots, \mathcal{C}_n \otimes \mathcal{L}^{\otimes n} \rangle$$

where $\mathcal{C}_k = \langle \mathcal{O}(k.k), \mathcal{O}(k+1,k), \dots \mathcal{O}(n,k), \mathcal{O}(k,k+1), \dots \mathcal{O}(k,n) \rangle$.

Proof. The proof is analogous to showing that the exceptional collection for $\mathbb{P}^2 \times \mathbb{P}^2$ is full. For any $\mathcal{O}(i,j)$ we have

$$\mathrm{Hom}(\mathcal{O}(i,j), \mathcal{O}(i,j)[r]) = H^r(X, \mathcal{O}) = \mathbb{C}$$

for $r = 0$ and 0 otherwise, so the objects are exceptional. For any $\mathcal{O}(k,l)$ appearing later in the sequence, we have

$$\mathrm{Hom}(\mathcal{O}(i,j), \mathcal{O}(k,l)[r]) = H^r(X, \mathcal{O}(k-i, l-j) = 0$$

since either $0 < k-i < -n-1$ or $0 < l-j < -m-1$. So the objects form an exceptional collection. To show that it is full, we show that all line bundles on X are generated by the collection and argue that it is full Proposition 4.4.5.

All the line bundles on X are of the form $\mathcal{O}(i,j)$, so we can think of a line bundle on X as a point in the $\mathbb{Z} \times \mathbb{Z}$-plane. If we denote by $[0,N] \times [0,M]$ the $N \times M$ rectangle in the first quadrant of the $\mathbb{Z} \times \mathbb{Z}$-plane with a vertex at the origin, then the collections above can be represented by the rectangle $[0,n] \times [0,m]$. We show that we can expand the rectangle with one row in all directions. The Koszul resolution on $\mathbb{P}^n$ pulls back by p to an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1,0)^{\oplus n+1} \to \mathcal{O}(2,0)^{\oplus \binom{n+1}{2}} \to \cdots \to \mathcal{O}(n,0)^{\oplus n+1} \to \mathcal{O}(n+1,0) \to 0. \quad (4.4)$$

Tensoring by $\mathcal{O}(0,j)$ for $j = 0, 1, \ldots, m$ we obtain all the line bundles that extends the rectangle to $[0, n+1] \times [0, m]$. Symmetrically, by pulling back the Koszul resolution from $\mathbb{P}^m$ by q and tensoring with $\mathcal{O}(i, 0)$ for $i = 1, 2, \ldots, n$ extends the original rectangle to the rectangle $[0, n] \times [0, m+1]$. Now that we have obtained $\mathcal{O}(0, m+1)$ we obtain $\mathcal{O}(n+1, m+1)$ by the resolution

$$0 \to \mathcal{O}(0, m+1) \to \mathcal{O}(1, m+1)^{\oplus n+1} \to \mathcal{O}(2, m+1)^{\oplus \binom{n+1}{2}} \to \\ \cdots \to \mathcal{O}(n, m+1)^{\oplus n+1} \to \mathcal{O}(n+1, m+1) \to 0.$$

which is just (4.4) tensored by $\mathcal{O}(0, m+1)$. Then the original rectangle extendeds to $[0, n+1] \times [0, m+1]$. The extension in the negative directions is analogous. Simply twist the Koszul resolution by $\mathcal{O}(-1)$ and take the pullback by p and q. This proves the base case in an induction argument that the rectangle extends to the entire $\mathbb{Z} \times \mathbb{Z}$-plane. The inductive step is similar. Assume the collection generates all line bundles $\mathcal{O}(i,j)$ for $i \leq N$ and $j \leq M$ for arbitrary N and M, i.e. it extends to the rectangle $[0, N] \times [0, M]$. Now resolve $\mathcal{O}(N+1, j)$ and $\mathcal{O}(i, M+1)$ by the previous n and m twists, respectively. The line bundle $\mathcal{O}(N+1, M+1)$ has a resolution

$$0 \to \mathcal{O}(N-n, M+1) \to \mathcal{O}(N-n{+}1, M+1)^{\oplus n+1} \to \mathcal{O}(N-n+2, M+1)^{\oplus \binom{n+1}{2}} \to \\ \cdots \to \mathcal{O}(N, M+1)^{\oplus n+1} \to \mathcal{O}(M+1, N+1) \to 0.$$

This proves by induction that the rectangle can be extended to $[0, N+1] \times [0, M+1]$ and it follows by induction that it extends to the entire first quadrant. In other words, the collection generates all the positive twists. To show that it extends to all combinations of negative and positive twists follows the same method by twisting accordingly. ■

The variety $\mathbb{P}^n \times \mathbb{P}^m$ is the trivial projective bundle over $\mathbb{P}^m$. It is the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^m}^{\oplus n+1})$. We wish to consider a more general case. So let $\mathcal{E}$ be a locally free sheaf of rank $r+1$ on $\mathbb{P}^n$, and consider its projectivization $X = \mathbb{P}(\mathcal{E})$. We know that the Picard group of X is $\mathbb{Z}^{\oplus 2}$, and if we let $\pi : X \to \mathbb{P}^n$ be the projection morphism then the sheaves $\mathcal{O}_X(1)$ and $\pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ generate $\mathrm{Pic}(X)$. So let

$$\mathcal{H}(i,j) = \mathcal{O}_X(i) \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(j).$$

We seek an exceptional collection of $D(\mathbb{P}(\mathcal{E}))$ given by the line bundles $\mathcal{H}(i,j)$. Since $(\mathcal{O}_X(i) \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(j))^\vee = \mathcal{O}_X(-i) \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(-j)$, we have

$$\begin{aligned}\mathrm{Hom}_{D(\mathbb{P}(\mathcal{E}))}(\mathcal{H}(i,j), \mathcal{H}(k,l)[m]) &= \mathrm{Ext}^n(\mathcal{O}_{\mathbb{P}(\mathcal{E})}, \mathcal{H}(k-i, l-j)) \\ &= H^n(\mathbb{P}(\mathcal{E}), \mathcal{H}(k-i, l-j)),\end{aligned}$$

which is zero for $0 < k-i < -r-1$. From Corollary 3.1.6 we know that if $k - i \geq 0$ then

$$H^n(\mathbb{P}(\mathcal{E}), \mathcal{O}_X(k-i) \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(l-j)) \simeq H^n(\mathbb{P}^n, \mathrm{Sym}^{k-i}(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^n}(l-j)).$$

If $k = i$ then the right hand side is just the cohomology of a line bundle on $\mathbb{P}^n$, which of course is zero whenever $0 < l-j < -n-1$. If $-r-1 < k-i < 0$ then the left hand side vanishes. We have

$$\mathrm{Hom}_{D(\mathbb{P}(\mathcal{E}))}(\mathcal{H}(i,j), \mathcal{H}(i,j)[m]) = H^n(\mathbb{P}^n, \mathrm{Sym}^0(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^n}) = H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$$

which is $\mathbb{C}$ for $n = 0$ and zero otherwise. So the sheaves $\mathcal{H}(i, j)$ are indeed exceptional objects. If we assume $k = i > -r - 1$ then

$$\mathrm{Hom}_{D(\mathbb{P}(\mathcal{E}))}(\mathcal{H}(i, j), \mathcal{H}(k, l)[m]) = H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l - j))$$

which is zero whenever $0 < l - j < -n - 1$. Hence the sequence of sheaves

$$\langle \mathcal{H}(i, 0), \mathcal{H}(i, 1), \ldots, \mathcal{H}(i, n)\rangle$$

satisfies the conditions for being a collection of exceptional objects. If we assume $j = l$ then

$$\mathrm{Hom}_{D(\mathbb{P}(\mathcal{E}))}(\mathcal{H}(i, j), \mathcal{H}(k, l)[n]) = H^n(\mathbb{P}^n, \mathrm{Sym}^{k-i}(\mathcal{E}))$$

which is zero whenever $0 < k - i < -r - 1$. Hence the sequence of sheaves

$$\langle \mathcal{H}(0, 0), \mathcal{H}(1, 0), \ldots, \mathcal{H}(r, 0)\rangle$$

satisfies the same condition. Combining the two, we get a collection

$$\begin{aligned}\mathscr{H} = \langle &\mathcal{H}(0, 0), \mathcal{H}(0, 1), \ldots, \mathcal{H}(0, n),\\ &\mathcal{H}(1, 0), \mathcal{H}(1, 1), \ldots, \mathcal{H}(1, n),\\ &\ldots\\ &\mathcal{H}(r, 0), \mathcal{H}(r, 1), \ldots, \mathcal{H}(r, n)\rangle\end{aligned}$$

which is an exceptional collection for $D(\mathbb{P}(\mathcal{E}))$. It is also of expected length by Proposition 4.3.6. Our base scheme is $\mathbb{P}^n$ and the bundle $\mathcal{E}$ has rank $r + 1$, so a full exceptional collection for $D(\mathbb{P}(\mathcal{E}))$ must have length $(n+1)\cdot(r+1)$. Showing that the collection is full is more difficult. We will make use of the spanning class of skyscraper sheaves.

Proposition 4.5.3. *Let $\mathcal{E}$ be locally free on $\mathbb{P}^n$ of rank $r + 1$, let $X = \mathbb{P}(\mathcal{E})$ and denote by $\pi : X \to \mathbb{P}^n$ the projection morphism. Then $\mathscr{H}$ is a full exceptional collection of $D(X)$ and it gives a rectangular Lefschetz decomposition*

$$D(X) = \langle \pi^* D(\mathbb{P}^n) \otimes \mathcal{O}_X, \pi^* D(\mathbb{P}^n) \otimes \mathcal{O}_X(1), \ldots, \pi^* D(\mathbb{P}^n) \otimes \mathcal{O}_X(r)\rangle \qquad (4.5)$$

where $D(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots \mathcal{O}_{\mathbb{P}^n}(n)\rangle$.

Proof. It is clear that (4.5) is just a rearrangement of the collection $\mathscr{H}$. We have

$$\begin{aligned}\pi^* D(\mathbb{P}^n) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(i) &= \langle \pi^*\mathcal{O}_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(i), \ldots, \pi^*\mathcal{O}_{\mathbb{P}^n}(n) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(i)\rangle\\ &= \langle \mathcal{H}(i, 0), \ldots, \mathcal{H}(i, n)\rangle\end{aligned}$$

and by letting i range from 0 to r we obtain the collection $\mathscr{H}$. To prove that $\mathscr{H}$ is full, we show that the collection generates all the skyscraper sheaves $k(x)$ for $x \in X$ and use the fact that these form a spanning class of $D(X)$. On the base scheme Y, the Koszul resolution gives an exact sequence

$$0 \to \mathcal{O}_Y(-n) \to \cdots \to \mathcal{O}_Y(-2)^{\oplus\binom{n}{2}} \to \mathcal{O}_Y(-1)^{\oplus n} \to \mathcal{O}_Y \to k(y) \to 0. \qquad (4.6)$$

For instance, if $R = \mathbb{C}[x_0, \ldots, x_n]$ and $y = (1 : 0 : \cdots : 0)$, then the Koszul resolution with respect to the regular sequence $I = (x_1, \ldots, x_n)$ reads

$$0 \to R(-n) \to \cdots \to R(-2)^{\oplus\binom{n}{2}} \to R(-1)^{\oplus n} \to R \to \mathbb{C}[x_0] \to 0 \qquad (4.7)$$

and (4.6) is obtained by applying the graded tilde functor to (4.7). Since $k(y) \otimes \mathcal{O}_Y(n) = k(y)$, we can twist (4.6) to get the sequence

$$0 \to \mathcal{O}_Y \to \cdots \to \mathcal{O}_Y(n-2)^{\oplus \binom{n}{2}} \to \mathcal{O}_Y(n-1)^{\oplus n} \to \mathcal{O}_Y(n) \to k(y) \to 0. \tag{4.8}$$

The projection morphism π is flat, so this resolution pulls back to an exact sequence on X. For every $x \in X$ we can choose a fiber $\mathbb{P}(\mathcal{E}|y)$ containing x. Let $i : \mathbb{P}(\mathcal{E}|_y) \hookrightarrow \mathbb{P}(\mathcal{E})$ be the inclusion. Since $\mathbb{P}(\mathcal{E}|_y) \simeq \mathbb{P}^r_{k(y)}$ there is a resolution

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E}|_y)} \to \cdots \to \mathcal{O}_{\mathbb{P}(\mathcal{E}|_y)}(r-2)^{\oplus \binom{r}{2}} \to \mathcal{O}_{\mathbb{P}(\mathcal{E}|_y)}(r-1)^{\oplus r} \to \mathcal{O}_{\mathbb{P}(\mathcal{E}|_y)}(r) \to k(x) \to 0. \tag{4.9}$$

Further $i_* k(x) = k(x)$ and i is a closed immersion, so pushing forward (4.9) by i gives a resolution of $k(x)$ by the sheaves $i_*\mathcal{O}_{\mathbb{P}(\mathcal{E}|_y)}(j)$ for $0 \leq j \leq r$. The sheaves $i_*\mathcal{O}_{\mathbb{P}(\mathcal{E}|_y)}(j)$ we obtain inductively by pulling back the sequence (4.8) by π and then twist by $\mathcal{O}_X(j)$. So for each $x \in X$ we get a resolution of $k(x)$ by sheaves in the collection by choosing a fiber containing x. So every $k(x)$ is isomorphic in $D(X)$ to its resolution, hence the collection generates all skyscraper sheaves $k(x)$ and is therefore full. ■

We have shown that any projective bundle over $\mathbb{P}^n$ has a semi-orthogonal decomposition given by the exceptional collection $\mathscr{H}$. We can actually stretch the result a bit further and consider any projective bundle over a general base scheme Y with a full exceptional collection. The result is by D. Orlov [Orl92, Cor. 2.7].

Proposition 4.5.4 ([Orl92, Cor. 2.7]). *Let $X = \mathbb{P}(\mathcal{E})$ be the projectivization of a locally free sheaf $\mathcal{E}$ of rank $r+1$ on a smooth projective variety Y and let $p : X \to Y$ be the projection morphism. If $D(Y)$ has a full exceptional collection $\langle E_1, \ldots, E_n \rangle$ then*

$$D(X) = \langle p^*D(Y), p^*D(Y) \otimes \mathcal{O}_X(1), \ldots, p^*D(Y) \otimes \mathcal{O}_X(r) \rangle$$

is a semi-orthogonal decomposition of $D(X)$.

Proof. We first show that the objects indeed form an exceptional collection. From Lemma 4.2.6 we know that π_* is fully faithful, so

$$\mathrm{Hom}(\pi^* E_j \otimes \mathcal{O}_X(i), \pi^* E_k \otimes \mathcal{O}_X(i)[m]) = \mathrm{Hom}(E_j, E_k[m]) = 0$$

if $j > k$. Also by adjunction and projection we have

$$\begin{aligned} \mathrm{Hom}(\pi^* E_j \otimes \mathcal{O}_X(i), \pi^* E_k \otimes \mathcal{O}_X(l)[m]) &= \mathrm{Hom}(E_j, \pi_*(\pi^* E_k[m] \otimes O_X(l-i))) \\ &= \mathrm{Hom}(E_j, E_k[m] \otimes \pi_*\mathcal{O}_X(l-i)) = 0 \end{aligned}$$

if $0 < l-i < -r-1$ since $\pi_*\mathcal{O}_X(l-i) = 0$ in that case. So the objects form an exceptional collection. It remains to show that the collection is full. We will make use of the following lemma.

Lemma 4.5.5. *Let $\mathcal{C} = \langle \pi^* E_1, \ldots \pi^* E_n \rangle$. If $\mathcal{F} \in D(Y)$, then $\pi^*\mathcal{F} \in \mathcal{C}$.*

Proof. Any $\mathcal{F} \in \langle E_1, \ldots, E_n \rangle$ is an iteration of sums, shifts and cones of morphisms $E_i \to E_j$. But π^* distributes over sums and shifts. Moreover π^* is an exact functor of triangulated categories, so for any $A, B \in D(Y)$ and an exact triangle

$$A \xrightarrow{f} B \to C(f)$$

we get an exact triangle

$$\pi^*A \to \pi^*B \to \pi^*C(f).$$

If we take $A = E_i$ and $B = E_j$ then this implies that $\pi^*C(f) \in \mathcal{C}$. It follows that any iteration of cones of morphisms $E_i \to E_j$ pulls back by π to an object in $\mathcal{C}$. Thus $\pi^*\mathcal{F} \in \mathcal{C}$. ■

Now for any $x \in X$ choose a point $y \in Y$ so that $x \in \mathbb{P}(\mathcal{E}|_y) \simeq \mathbb{P}^r_{k(y)}$. Then there is a resolution of $k(x)$ given by

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E}|_y)} \to \cdots \to \mathcal{O}_{\mathbb{P}(\mathcal{E}|_y)}(r-2)^{\oplus\binom{r}{2}} \to \mathcal{O}_{\mathbb{P}(\mathcal{E}|_y)}(r-1)^{\oplus r} \to \mathcal{O}_{\mathbb{P}(\mathcal{E}|_y)}(r) \to k(x) \to 0.$$

Pushing forward by the inclusion $i : \mathbb{P}(\mathcal{E}|_y) \hookrightarrow X$ gives a resolution of $k(x)$ by the sheaves $i_*\mathcal{O}_{\mathbb{P}(\mathcal{E}|_y)}(k)$ for $0 \leq k \leq r$. By Lemma 4.5.5 the object $\pi^*k(y)$ is generated by the exceptional collection, and $\pi^*k(y) \otimes \mathcal{O}_X(k) = i_*\mathcal{O}_{\mathbb{P}(\mathcal{E}|_y)}(k)$. So the sheaves $i_*\mathcal{O}_{\mathbb{P}(\mathcal{E}|_y)}(k)$ are generated by the collection and gives a resolution of $k(x)$, i.e. the collection generates $k(x)$. So for any $x \in X$ we can choose a fiber containing it and find a resolution of locally free sheaves generated by the collection. Thus the collection generates all skyscraper sheaves $k(x)$ and is therefore full by Proposition 4.4.7. ■

Example 4.5.6. ($\mathrm{Hilb}^2 \mathbb{P}^2$) From the identification of $\mathrm{Hilb}^2 \mathbb{P}^2$ with the projectivization of the rank three locally free sheaf $\mathrm{Sym}^2 S$ on $G(2,3)$, an exceptional collection for $\mathrm{Hilb}^2 \mathbb{P}^2$ becomes a special case of the general case discussed above. So let $X = \mathbb{P}(\mathrm{Sym}^2 S) \simeq \mathrm{Hilb}^2 \mathbb{P}^2$ and $\mathbb{P}^2 = G(2,3)$ and define $\mathcal{H}(i,j) = \mathcal{O}_X(i) \otimes \mathcal{O}_{\mathbb{P}^2}(j)$. Then we have a full exceptional collection

$$\begin{aligned} D(X) = \langle &\mathcal{H}(0,0), \mathcal{H}(0,1), \mathcal{H}(0,2),\\ &\mathcal{H}(1,0), \mathcal{H}(1,1), \mathcal{H}(1,2),\\ &\mathcal{H}(2,0), \mathcal{H}(2,1), \mathcal{H}(2,2)\rangle. \end{aligned}$$

Consider the morphism

$$f : \mathbb{P}(\mathrm{Sym}^2 S) \hookrightarrow \mathbb{P}(\mathrm{Sym}^2 \mathcal{O}_{G(2,3)}^{\oplus 3}) \simeq \mathbb{P}^5 \times G(2,3) \to \mathbb{P}^5$$

given by the inclusion $\mathrm{Sym}^2 S \hookrightarrow \mathrm{Sym}^2 \mathcal{O}_{G(2,3)}^{\oplus 3}$ and the projection $\mathbb{P}^5 \times G(2,3) \to \mathbb{P}^5$. By this composition it is clear that the sheaf $\mathcal{O}(1)$ pulls back to $\mathcal{O}_X(1)$ via f. If we let

$$\mathcal{A} = \langle \mathcal{H}(0,0), \mathcal{H}(0,1), \mathcal{H}(0,2)\rangle$$

then

$$D(X) = \langle \mathcal{A}, \mathcal{A} \otimes \mathcal{O}_X(1), \mathcal{A} \otimes \mathcal{O}_X(1)^{\otimes 2}\rangle$$

is a rectangular Lefschetz decomposition with respect to $\mathcal{O}_X(1)$ and of the form of Proposition 4.5.4.

Chapter 5

Homological Projective Duality

In this chapter we finally state the definition of homological projective duality. Throughout the chapter, the schemes in question will always be smooth projective varieties over $\mathrm{Spec}(\mathbb{C})$, and for such a scheme X we denote by $D(X)$ the bounded derived category of coherent sheaves on X. We follow the two articles [Kuz14] and [Kuz07] by Alexander Kuznetsov, the latter of which is the original article stating the homological projective duality relation, as well as the notes on homological projective duality by Richard Thomas [Tho18].

5.1 The Universal Hyperplane Section

For a scheme X with a globally generated line bundle $\mathcal{O}_X(1)$, let $V = H^0(X, \mathcal{O}_X(1))^\vee$ and assume given a morphism $f : X \to \mathbb{P}(V)$. Denote by $Q \subset \mathbb{P}(V) \times \mathbb{P}(V^\vee)$ the incidence quadric defined by

$$Q = \{(x, H) \in \mathbb{P}(V) \times \mathbb{P}(V^\vee) \ : \ x \in H\}.$$

Here we think of the points H in the dual space $\mathbb{P}(V^\vee)$ as their corresponding hyperplanes in $\mathbb{P}(V)$. Equivalently, we can think of a point of $\mathbb{P}(V^\vee)$ as the section $s \in H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))$ up to scaling whose zero locus defines a hyperplane of $\mathbb{P}(V)$. Then we can describe the points of Q as pairs (x, s) such that $s(x) = 0$. The universal hyperplane section $\mathscr{H} \subset X \times \mathbb{P}(V^\vee)$ of X is defined as the fiber product $X \times_{\mathbb{P}(V)} Q$:

$$\begin{array}{ccc} \mathscr{H} & \longrightarrow & Q \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{\ f\ } & \mathbb{P}(V). \end{array}$$

The points of $\mathscr{H}$ can also be described by an incidence relation as

$$\mathscr{H} = \{(x, s) \in X \times \mathbb{P}(V^\vee) \ : \ s(x) = 0\}.$$

Remark 5.1.1. If f is an embedding, which will be the case when $\mathcal{O}_X(1)$ is very ample, $\mathscr{H}$ carries the information of the usual projective dual $X^\vee$ of X. Under the projection down to $\mathbb{P}(V^\vee)$, the discriminant locus of $\mathscr{H}$, which is the set of points $(x, H) \in \mathscr{H}$ such that H is tangent to x (i.e. $H \subset \mathrm{T}_x X$), is the projective dual $X^\vee$ of X.

5.2 The Homological Projective Dual

For a variety X with a morphism $f : X \to \mathbb{P}(V)$ with $\dim V = N$, let $\mathcal{O}_X(1) \coloneqq f^*\mathcal{O}_{\mathbb{P}(V)}(1)$. Assume given a Lefschetz decomposition

$$D(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_n(n) \rangle \tag{5.1}$$

where $\mathcal{A}_i(i) \coloneqq \mathcal{A}_i \otimes \mathcal{O}_X(i)$. Consider the base change diagram

$$\begin{array}{ccc} X \times \mathbb{P}(V^\vee) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathbb{P}(V^\vee) & \xrightarrow{\ \phi\ } & \operatorname{Spec}(\mathbb{C}). \end{array}$$

Proposition 5.2.1. *[Kuz14] The flat base change morphism* $\mathbb{P}(V^\vee) \to \operatorname{Spec}(\mathbb{C})$ *gives a semi-orthogonal decomposition*

$$D(X \times \mathbb{P}(V^\vee)) = \langle \mathcal{A}_0 \boxtimes D(\mathbb{P}(V^\vee)), \mathcal{A}_1(1) \boxtimes D(\mathbb{P}(V^\vee)), \ldots, \mathcal{A}_n(n) \boxtimes D(\mathbb{P}(V^\vee)) \rangle \tag{5.2}$$

from the pullback by the two projections.

Notice that

$$\mathcal{A}_k(k) \boxtimes D(\mathbb{P}(V^\vee)) = \langle \mathcal{A}_k(k) \boxtimes \mathcal{O}_{\mathbb{P}(V^\vee)}, \mathcal{A}_k(k) \boxtimes \mathcal{O}_{\mathbb{P}(V^\vee)}(1), \ldots, \mathcal{A}_k(k) \boxtimes \mathcal{O}_{\mathbb{P}(V^\vee)}(N) \rangle,$$

so in the case where $X = \mathbb{P}^n$ we recognize this as the decomposition of $\mathbb{P}^n \times \mathbb{P}^{N-1}$ from Proposition 4.5.2. Let $\mathscr{H}$ denote the universal hyperplane section of X, and let $\alpha : \mathscr{H} \hookrightarrow X \times \mathbb{P}(V^\vee)$ be the embedding.

Proposition 5.2.2. *[Kuz07, Lemma 5.3] For any* $1 \leq k \leq n$, *the functor*

$$\mathcal{A}_k(k) \boxtimes D(\mathbb{P}(V^\vee)) \subset D(X \times \mathbb{P}(V^\vee)) \xrightarrow{\alpha^*} D(\mathscr{H})$$

is fully faithful and gives a semi-orthogonal collection

$$\langle \mathcal{A}_1(1) \boxtimes D(\mathbb{P}(V^\vee), \ldots, \mathcal{A}_n(n) \boxtimes D(\mathbb{P}(V^\vee)) \rangle \subset D(\mathscr{H}). \tag{5.3}$$

Define $\mathcal{C}_{\mathscr{H}}$ *to be the left orthogonal of (5.3). Then* $D(\mathscr{H})$ *admits a semi-orthogonal decomposition*

$$D(\mathscr{H}) = \langle \mathcal{C}_{\mathscr{H}}, \mathcal{A}_1(1) \boxtimes D(\mathbb{P}(V^\vee), \ldots, \mathcal{A}_n(n) \boxtimes D(\mathbb{P}(V^\vee)) \rangle. \tag{5.4}$$

Definition 5.2.3. For a smooth projective variety X with morphism $f : X \to \mathbb{P}(V)$ and a given Lefschetz decomposition (5.1), we define the homological projective dual category of $D(X)$ to be the category $\mathcal{C}_{\mathscr{H}}$. If moreover the category $\mathcal{C}_{\mathscr{H}}$ is geometric, that is equivalent to the derived category $D(Y)$ of some smooth projective variety Y with a morphism $g : Y \to \mathbb{P}(V^\vee)$, we call (Y, g) *the homological projective dual* of (X, f).

Let us make precise what we mean by $\mathcal{C}_{\mathscr{H}}$ being geometric. Since any equivalence of derived categories is given by a Fourier-Mukai functor, there should exist an object $\mathcal{E} \in D(\mathscr{H} \times_{\mathbb{P}(V^\vee)} Y)$ so that the Fourier-Mukai functor $\phi_{\mathcal{E}} : D(Y) \to D(\mathscr{H})$ is fully faithful and defines an equivalence of categories onto $\mathcal{C}_{\mathscr{H}}$ [Kuz07, Def. 6.1].

Example 5.2.4. Let $X = \mathbb{P}^n$, $V = H^0(X, \mathcal{O}_X(1))^\vee$ and let $f : X \to \mathbb{P}(V)$ be the identity. We seek the homological projective dual of (X, f) with Beilinson's collection

$$D(X) = \langle \mathcal{A}_0, \mathcal{A}(1), \dots, \mathcal{A}(n) \rangle.$$

So $\mathcal{A}(i) = \mathcal{A} \otimes \mathcal{O}(i)$ and $\mathcal{A} = \langle \mathcal{O} \rangle$. Let $\mathscr{H} \subset \mathbb{P}(V) \times \mathbb{P}(V^\vee)$ be the universal hyperplane section. Over every point $x \in \mathbb{P}(V)$, $\mathscr{H}$ parametrizes all hyperplanes in $\mathbb{P}(V^\vee)$ incident to x. So $\mathscr{H}$ is a projective bundle with $\mathbb{P}^{n-1}$-fibers over $\mathbb{P}^n$, i.e. the projectivization of a rank n locally free sheaf on $\mathbb{P}^n$. One can show that $\mathscr{H}$ is in fact the projectivization $\mathbb{P}(\Omega_{\mathbb{P}(V)})$ of the cotangent bundle. Now, the flat base-change $\mathbb{P}(V^\vee) \to \operatorname{Spec} k$ gives a semi-orthogonal decomposition

$$D(X \times \mathbb{P}(V^\vee) = \langle \mathcal{A}_0 \boxtimes D(\mathbb{P}(V^\vee)), \mathcal{A}_1(1) \boxtimes D(\mathbb{P}(V^\vee)), \dots, \mathcal{A}_n(n) \boxtimes D(\mathbb{P}(V^\vee)) \rangle.$$

Here

$$A_k(k) \boxtimes D(\mathbb{P}(V^\vee)) = \langle \mathcal{O}(k) \boxtimes \mathcal{O}, \mathcal{O}(k) \boxtimes \mathcal{O}(1), \dots, \mathcal{O}(k) \boxtimes \mathcal{O}(n) \rangle. \tag{5.5}$$

Pulling back to $\mathscr{H}$ by the inclusion gives a decomposition

$$D(\mathscr{H}) = \langle \mathcal{C}_{\mathscr{H}}, \mathcal{A}_1(1) \boxtimes D(\mathbb{P}(V^\vee), \dots, \mathcal{A}_n(n) \boxtimes D(\mathbb{P}(V^\vee)) \rangle \tag{5.6}$$

which is fully faithful onto the last $n-1$ terms. The subcategory $\mathcal{C}_{\mathscr{H}}$ is the orthogonal of $\langle \mathcal{A}_1(1) \boxtimes D(\mathbb{P}(V^\vee), \dots, \mathcal{A}_n(n) \boxtimes D(\mathbb{P}(V^\vee)) \rangle$. Now, the $\mathcal{A}_i$ are all exceptional objects and they pull back to exceptional objects on $D(\mathscr{H})$. But since $\mathscr{H}$ is a projective bundle, we know from K_0-theory (Proposition 4.3.6) that a full exceptional collection of $D(\mathscr{H})$ must have length $n(n+1)$. Writing (5.6) out in terms of exceptional objects, that is writing each of the last $n-1$ terms of (5.6) as in (5.5), it is clear that

$${}^\perp\mathcal{C}_{\mathscr{H}} = \langle \mathcal{A}_1(1) \boxtimes D(\mathbb{P}(V^\vee), \dots, \mathcal{A}_n(n) \boxtimes D(\mathbb{P}(V^\vee)) \rangle$$

can be written as a collection of length $n(n+1)$. Since the pullback of each term by the inclusion is fully faithful (Proposition 5.2.2), the objects are all exceptional. Thus $\mathcal{C}_{\mathscr{H}} = 0$, and the homological projective dual of (X, f) is the empty variety.

From the above example, the duality relation can seem somewhat odd. The interesting part of the duality is what happens on linear sections. The following section is dedicated to studying the duality relation on linear sections.

5.3 The Main Theorem of Homological Projective Duality

The setting of the main theorem of HP duality is the following. We let $X \to \mathbb{P}(V)$ and $Y \to \mathbb{P}(V^\vee)$ be smooth projective, V denotes the dual of the vector space $H^0(X, O, X(1))$ of global sections of a globally generated line bundle $\mathcal{O}_X(1)$ on X. We assume given a right Lefschetz decomposition

$$D(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_n(n) \rangle \tag{5.7}$$

of $D(X)$ with respect to $f^*\mathcal{O}_{\mathbb{P}(V)}(1)$. Here $\mathcal{A}_i(i) = \mathcal{A}_i \otimes f^*\mathcal{O}_{\mathbb{P}(V)}(1)$. For a linear subspace $L \subset V^\vee$ consider the fiber products X_L and Y_L given by the diagrams

$$\begin{array}{ccc} X_L & \hookrightarrow & X \\ \downarrow & & \downarrow f \\ \mathbb{P}(L^\perp) & \hookrightarrow & \mathbb{P}(V) \end{array} \qquad\qquad \begin{array}{ccc} Y_L & \hookrightarrow & Y \\ \downarrow & & \downarrow g \\ \mathbb{P}(L) & \hookrightarrow & \mathbb{P}(V^\vee), \end{array}$$

where $L^\perp$ is the orthogonal of L defined as the kernel $\ker(V \to L^\vee)$.

Definition 5.3.1. (Expected dimension) Assume L has dimension r and V has dimension N. Then $L^\perp$ has dimension $N-r$ and we say that X_L and Y_L have *expected dimension* if

$$\dim X_L = \dim X - r \quad \text{and} \quad \dim Y_L = \dim Y - (N-r).$$

The duality statement of X and Y is summarized in the following theorem.

Theorem 5.3.2. *[Kuz14, Thm. 2.5] Let (Y,g) be the HP dual of (X,f) with respect to the Lefschetz decomposition (5.7). If V has dimension N and $L \subset V^\vee$ is an r-dimensional linear subspace such that X_L and Y_L have expected dimension, then*

1. *$D(Y)$ has an admissible subcategory $\mathcal{B}_0$ equivalent to $\mathcal{A}_0$ extending to a left Lefschetz decomposition*
$$D(Y) = \langle \mathcal{B}_n(-n), \mathcal{B}_{n-1}(1-n), \dots, \mathcal{B}_0 \rangle \tag{5.8}$$
2. *(X,f) is HP dual to (Y,g) with respect to the Lefschetz decomposition (5.8)*
3. *There are semi-orthogonal decompositions*
$$\begin{aligned} D(X_L) &= \langle \mathcal{C}_L, \mathcal{A}_r(r), \dots, \mathcal{A}_n(n) \rangle, \\ D(Y_L) &= \langle \mathcal{B}_n(-n), \dots, \mathcal{B}_{N-r}(r-N), \mathcal{C}_L \rangle, \end{aligned}$$
where
$$\mathcal{C}_L = \langle \mathcal{A}_r(r), \dots, \mathcal{A}_n(n) \rangle^\perp = {}^\perp \langle \mathcal{B}_n(-n), \dots, \mathcal{B}_{N-r}(r-N) \rangle.$$

If L has dimension 1, then $\mathbb{P}(L)$ defines a point in $\mathbb{P}(V^\vee)$ and $\mathbb{P}(L^\perp)$ defines a hyperplane in $\mathbb{P}(V)$. Denote by $H = \mathbb{P}(L^\perp)$ the hyperplane in $\mathbb{P}(V)$ and assume L is such that X_L and Y_L have expected dimension. Then $X_L = f^{-1}(H)$ and $Y_L = g^{-1}(H)$, which is the fiber of g over the point $H \in \mathbb{P}(V^\vee)$.

$$\begin{array}{ccc} f^{-1}(H) & \hookrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ H & \hookrightarrow & \mathbb{P}(V) \end{array} \qquad\qquad \begin{array}{ccc} g^{-1}(H) & \hookrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ H & \hookrightarrow & \mathbb{P}(V^\vee) \end{array}$$

By Theorem 5.3.2 we have semiorthogonal decompositions

$$\begin{aligned} &D(X_L) = \langle \mathcal{C}_L, \mathcal{A}_1(1), \dots \mathcal{A}_n(n) \rangle \\ &D(Y_L) = \mathcal{C}_L. \end{aligned}$$

So intersecting X with a hyperplane corresponds to taking a fiber Y_H on the other side of the duality, and $D(Y_L)$ consist of only the category $\mathcal{C}_L$.

Remark 5.3.3. As mentioned, the usual projective dual $X^\vee$ is the discriminant locus of the morphism $\mathscr{H} \to \mathbb{P}(V^\vee)$. But this discrimant locus consist of exactly those hyperplanes $H \in \mathbb{P}(V^\vee)$ for which the fiber $g^{-1}(H)$ is singular. In other words $X^\vee$ is also the discriminant locus of the morphism $Y \to \mathbb{P}(V^\vee)$, c.f. [Tho18, p. 3].

Theorem 5.3.4. *[Kuz07, Thm. 7.9] If $g : Y \to \mathbb{P}(V^\vee)$ is homologically projectively dual to $f : X \to \mathbb{P}(V)$, then the discriminant locus of g is the projective dual of X.*

Example 5.3.5. Let $X = \mathbb{P}^1$ and consider the morphism $v : X \to \mathbb{P}^2$ given by the veronese embedding $(x_0 : x_1) \mapsto (x_0^2 : x_0x_1 : x_2^2)$. Then $v^*\mathcal{O}_{P^2}(1) = \mathcal{O}_X(2)$, and we seek a decomposition of $D(X)$ which is Lefsceßhtz with respect to $\mathcal{O}_X(2)$. Since no nontrivial Lefschetz decomposition of $D(\mathbb{P}^1)$ with respect to $\mathcal{O}_X(2)$ exists, we take the trivial decomposition $D(X) = \langle \mathcal{A} \rangle$. Then

$$D(X \times \mathbb{P}(V^\vee)) = \langle \mathcal{A} \boxtimes D(\mathbb{P}(V^\vee)) \rangle,$$

and we see that

$$D(\mathscr{H}) = \langle \mathcal{C}_{\mathscr{H}} \rangle.$$

So the homological projective dual is the universal hyperplane section $\mathscr{H}$ of X itself with the projection morphism $\mathscr{H} \to \mathbb{P}(V^\vee)$.

The example above illustrates what happens in general if we take a trivial decomposition of $D(X)$ for some $X \to \mathbb{P}(V)$. Then the homological projective dual is just the universal hyperplane section with the projection $\mathscr{H} \to \mathbb{P}(V^\vee)$. We end this section by stating a theorem of A. Kuznetsov which will set the stage for the next chapter. We have broken the theorem into two parts. The first is the statement of Proposition 4.5.4 which gives a way to obtain a Lefschetz decomposition of a projective bundle over a basescheme with a full exceptional collection. The second part gives a description of the homological projective dual of the projective bundle.

Theorem 5.3.6 ([Kuz07, Cor. 8.4]). *Let $X = \mathbb{P}(E)$ be the projectivization of a locally free sheaf E of rank $r + 1$ on a smooth projective variety Y with a full exceptional collection and let $p : X \to Y$ be the projection morphism. If $V = H^0(X, \mathcal{O}_X(1))^\vee$ and E is globally generated, then the homological projective dual of X with the morphism $X \to \mathbb{P}(V)$ and the decomposition from Proposition 4.5.4 is given by $Y = \mathbb{P}(E^\perp)$, where*

$$E^\perp = Ker(V^\vee \otimes_{\mathbb{C}} \mathcal{O}_Y \to E^\vee).$$

Note: $H^0(X, \mathcal{O}_X(1))^\vee = H^0(Y, E^\vee)^\vee$ by Corollary 3.1.6.

A few things need to be specified in the above theorem. First of all by the slightly abusive notation $V^\vee \otimes_{\mathbb{C}} \mathcal{O}_Y$ we mean the sheaf obtained by letting the vector space $V^\vee$ be the constant sheaf of global sections of $E^\vee$. So for any $U \subset X$ the sections of $V^\vee$ on U are the elements of the vector space $V^\vee$. The morphism

$$V^\vee \otimes \mathcal{O}_Y \to E^\vee$$

is defined over an open $U \subset X$ as the $\Gamma(U, \mathcal{O}_G)$-module homomorphism

$$\Gamma(G, \operatorname{Sym}^k S) \otimes \Gamma(U, \mathcal{O}_Y) \to \Gamma(U, \operatorname{Sym}^k S)$$

given by $s \otimes p \mapsto s|_U \cdot p$. Secondly, HP dual varieties X and Y come with a pair of morphisms $f : X \to \mathbb{P}(V)$ and $g : Y \to \mathbb{P}(V^\vee)$ which need to be specified. The surjection $V^\vee \otimes \mathcal{O}_Y \to E^\vee$ dualize to an inclusion $E \hookrightarrow H^0(Y, E^\vee) \otimes \mathcal{O}_Y$. Taking projectivizations gives an inclusion

$$\mathbb{P}(E) \hookrightarrow \mathbb{P}(H^0(Y, E^\vee) \otimes \mathcal{O}_Y) \simeq \mathbb{P}(V) \times Y.$$

Finally, composing with the projection to the first factor describes the morphism $f : \mathbb{P}(E) \to \mathbb{P}(V)$.

$$\begin{array}{ccccc} \mathbb{P}(E) & \hookrightarrow & \mathbb{P}(V \otimes \mathcal{O}_Y) & \cong & \mathbb{P}(V) \times Y \\ & \searrow^{f} & & & \downarrow \\ & & & & \mathbb{P}(V). \end{array}$$

Similarly, projectivization of the inclusion $E^{\perp} \hookrightarrow V^{\vee} \otimes \mathcal{O}_Y$ gives a morphism

$$\mathbb{P}(E^{\perp}) \hookrightarrow \mathbb{P}(V^{\vee} \otimes \mathcal{O}_Y) \simeq \mathbb{P}(V^{\vee}) \times Y,$$

and by composing with the projection to the first factor we obtain the morphism $g : \mathbb{P}(E^{\perp}) \to \mathbb{P}(V^{\vee})$.

$$\begin{array}{ccccc} \mathbb{P}(E^{\perp}) & \hookrightarrow & \mathbb{P}(V^{\vee} \otimes \mathcal{O}_Y) & \cong & \mathbb{P}(V^{\vee}) \times Y \\ & \searrow^{g} & & & \downarrow \\ & & & & \mathbb{P}(V^{\vee}). \end{array}$$

It is clear that $f_* \mathcal{O}_{\mathbb{P}(V)}(1) = \mathcal{O}_X(1)$ and the Lefschetz decomposition of $D(X)$ is the one given in Corollary 3.1.6.

Chapter 6

Applications

6.1 The Homological Projective Dual of a Family of Projetive Bundles

Now that we have the machinery of HP duality at hand, we can calculate the HP dual of $\mathrm{Hilb}^2 \mathbb{P}^2$. We have identified $\mathrm{Hilb}^2 \mathbb{P}^2$ with the projective bundle $\mathbb{P}(E)$ where $E = \mathrm{Sym}^2 S$ and S is the universal subbundle on $G(2,3)$. This identification and Kuznetsovs result on the HP dual of projective bundles (Theorem 5.3.6) gives us enough information to calculate the HP dual of $\mathrm{Hilb}^2 \mathbb{P}^2$ with the morphism

$$f : \mathrm{Hilb}^2 \mathbb{P}^2 \to \mathbb{P}(H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))^\vee).$$

It turns out that the method for this calculation generalizes to a whole family of projective bundles. So we give the proof of the more general case, and the HP dual of $\mathrm{Hilb}^2 \mathbb{P}^2$ will follow.

Consider the Grassmanian $G = G(n, n+1)$ and let S denote the universal subbundle of rank n. Let

$$V = H^0(\mathbb{P}(\mathrm{Sym}^k S, \mathcal{O}(1))^\vee) = H^0(G, \mathrm{Sym}^k S^\vee)^\vee.$$

We aim to prove the following theorem.

Theorem 6.1.1. *Let $X = \mathbb{P}(\mathrm{Sym}^k S)$. The homological projective dual $(Y \to \mathbb{P}(V^\vee))$ of $(X \to \mathbb{P}(V))$ is the variety*

$$Y = \mathbb{P}^n \times \mathbb{P}^{M-1}.$$

where $M = \binom{n+k-1}{k-1}$.

Proof. Let $E = \mathrm{Sym}^k S$. We wish to apply c5.3.6 stating that the homological projective dual of X is given by $\mathbb{P}(E^\perp)$ where $E^\perp$ is the kernel of the map

$$f : V^\vee \otimes \mathcal{O}_G \to E^\vee.$$

To determine $E^\perp$ we aim to obtain a different description of f. So consider the exact sequence

$$0 \to S \to \mathcal{O}_G^{\oplus n+1} \to Q \to 0$$

where S is the universal subbundle of rank n and Q is the universal quotien tbundle of rank one. More specifically, on $G = \mathbb{P}^n$, the universal quotient bundle Q is the invertible sheaf $\mathcal{O}_G(-1)$. We dualize the sequence to get

$$0 \to Q^\vee \to \mathcal{O}_G^{\oplus n+1} \to S^\vee \to 0.$$

Applying Proposition 3.1.1 we get the exact sequence

$$0 \to Q^\vee \otimes \operatorname{Sym}^{k-1} \mathcal{O}_G^{\oplus n+1} \to \operatorname{Sym}^k \mathcal{O}_G^{\oplus n+1} \to (\operatorname{Sym}^k S)^\vee \to 0. \tag{6.1}$$

Again, by Proposition 3.1.1 we have have an isomorphism $\operatorname{Sym}^k(S^\vee) \simeq (\operatorname{Sym}^k S)^\vee$. Set

$$\mathcal{F} = Q^\vee \otimes \operatorname{Sym}^{k-1} \mathcal{O}_G^{\oplus n+1} \simeq \mathcal{O}_G(-1)^{\oplus M}$$

and write the sequence (6.1) as

$$0 \to \mathcal{F} \to \mathcal{O}_G^{\oplus N} \to E^\vee \to 0 \tag{6.2}$$

where $N = \binom{n+k}{k}$. This gives a long exact sequence on cohomology, and since $H^i(G(2,3), \mathcal{F}) = 0$ for all i, there are isomorphisms

$$H^i(G, \mathcal{O}_G^{\oplus N}) \xrightarrow{\simeq} H^i(G, \operatorname{Sym}^k S^\vee)$$

for all i. Thus

$$V^\vee = H^0(G, \operatorname{Sym}^k S^\vee) \simeq \mathbb{C}^{\oplus N}$$

and so there is a natural identification

$$V^\vee \otimes \mathcal{O}_G \simeq \mathcal{O}_G^{\oplus N}.$$

We see that $E^\vee$ is globally generated and f is surjective. Thus Theorem 5.3.6 applies, and We wish to identify $\mathcal{F}$ with the kernel $E^\perp$. If we can find an isomorphism β so that the diagram

$$\begin{array}{ccccc} E^\perp & \hookrightarrow & V^\vee \otimes \mathcal{O}_G & \xrightarrow{f} & \operatorname{Sym}^k S^\vee \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle=} \\ \mathcal{F} & \hookrightarrow & \operatorname{Sym}^k \mathcal{O}_G^{\oplus n+1} & \xrightarrow{g} & \operatorname{Sym}^k S^\vee \end{array}$$

with exact rows commutes, then there is a morphism α of the kernels. The Five Lemma [Wei94, p. 13] ensures that α too is an isomorphism. On global sections there is a diagram

$$\begin{array}{ccc} H^0(G, V^\vee \otimes \mathcal{O}_G) & \xrightarrow{\cong} & H^0(G, \operatorname{Sym}^k S^\vee) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle=} \\ H^0(G, \operatorname{Sym}^k \mathcal{O}_G^{\oplus n+1}) & \xrightarrow{\cong} & H^0(G, \operatorname{Sym}^k S^\vee) \end{array}$$

where the isomophism $H^0(G, V^\vee \otimes \mathcal{O}_G) \xrightarrow{\simeq} H^0(G, \operatorname{Sym}^k \mathcal{O}_G^{\oplus n+1})$ is just the composition of the other three. We ask the question whether this isomorphism on global sections induces a unique isomorphism β of sheaves fitting into the diagram above. To answer the question, we will prove two lemmas from which the result will follow.

Lemma 6.1.2. *Let X be a smooth projective variety over $\operatorname{Spec}(\mathbb{C})$ and let $\mathcal{E}$ be a sheaf on X. Then*

$$H^0 : \operatorname{Hom}(\mathcal{O}_X^N, \mathcal{E}) \to \operatorname{Hom}_{\mathbb{C}}(H^0(X, O_X^N), H^0(X, \mathcal{E}))$$

is an isomorphism.

Proof. We have

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}_X^N,\mathcal{E}) &= \mathrm{Hom}(\mathcal{O}_X,\mathcal{E})^{\oplus N} = H^0(X,\mathcal{E})^{\oplus N} = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C},H^0(X,\mathcal{E}))^{\oplus N}\\ &= \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\oplus N},H^0(X,\mathcal{E})) = \mathrm{Hom}_{\mathbb{C}}(H^0(X,\mathcal{O}_X^N),H^0(X,\mathcal{E})).\end{aligned}$$

■

Lemma 6.1.3. *Let X be as above and assume given two surjections $f,g:\mathcal{O}_X^{\oplus N}\to\mathcal{E}$ to a locally free sheaf $\mathcal{E}$. Assume further that f and g induce isomorphisms on global sections. Then there exists a unique isomorphism $h:\mathcal{O}_X^{\oplus N}\to\mathcal{O}_X^{\oplus N}$ making the following diagram commute:*

$$\begin{array}{ccc}\mathcal{O}_X^{\oplus N} & \xrightarrow{\;f\;} & \mathcal{E}\\ \big\downarrow{\scriptstyle \exists! h} & \nearrow{\scriptstyle g} & \\ \mathcal{O}_X^{\oplus N} & & \end{array}$$

Proof. Let

$$H^0(f),H^0(g):H^0(X,\mathcal{O}_X^{\oplus N})\xrightarrow{\simeq}H^0(X,\mathcal{E})$$

denote the two isomorphisms on global sections induced by f and g. Define $H^0(h)=(H^0(g))^{-1}\circ H^0f$. We wish to show that $H^0(h)$ is induced by a unique isomorphism $h:\mathcal{O}_X^{\oplus N}\to\mathcal{O}_X^{\oplus N}$. By Lemma 6.1.2 the inverse $H^0(h)^{-1}$ is the image of a unique morphism $h':\mathcal{O}_X^{\oplus N}\to\mathcal{O}_X^{\oplus N}$, and by functoriality of H^0, we have

$$id = H^0(h)\circ H^0(h)^{-1} = H^0(h)\circ H^0(h') = H^0(h\circ h').$$

Hence $h\circ h' = h'\circ h = id$ and we conclude that h is an isomorphism. Uniqueness of h follows from Lemma 6.1.2. Again, by functoriality of H^0 we have

$$H^0(f) = H^0(g)\circ H^0(h) = H(g\circ h)$$

so $f=g\circ h$ and the diagram commutes. ■

Now let $\phi:\mathrm{Sym}^k\,\mathcal{O}_G^{\oplus n+1}\to\mathcal{O}_G^{\oplus N}$ and $\psi:V^\vee\otimes\mathcal{O}_G\to\mathcal{O}_G^N$ be the two isomorphisms and define the two surjections by the compositions

$$f\circ\psi^{-1}, g\circ\phi^{-1}:\mathcal{O}_G^{\oplus N}\to\mathrm{Sym}^k S^\vee.$$

Then by Lemma 6.1.3 the isomorphism $H^0(G,\mathcal{O}_G^{\oplus N})\xrightarrow{\simeq}H^0(G,\mathcal{O}_G^{\oplus N})$ induces a unique isomorphism $\mathcal{O}_G^{\oplus N}\to\mathcal{O}_G^{\oplus N}$ making the diagram in Figure 6.1 commute.

Thus α gives an isomorphism $E^\perp\simeq\mathcal{F}=\mathcal{O}_G(-1)^{\oplus M}$. Then

$$\mathbb{P}(E^\perp)\simeq\mathbb{P}(\mathcal{O}_G(-1)^{\oplus M})\simeq\mathbb{P}(\mathcal{O}_G(-1)^{\oplus M}\otimes\mathcal{O}(1))\simeq\mathbb{P}(\mathcal{O}_G^{\oplus M})\simeq\mathbb{P}^{M-1}\times G$$

since $\mathbb{P}(E^\perp)\simeq\mathbb{P}(E^\perp\otimes\mathcal{L})$ for a line bundle $\mathcal{L}$. So

$$Y=\mathbb{P}(E^\perp)\simeq\mathbb{P}^{M-1}\times\mathbb{P}^2$$

is the homological projective dual of X.

■

By the identification of $\mathrm{Hilb}^2\,\mathbb{P}^2$ with the projective bundle $\mathbb{P}(\mathrm{Sym}^2 S)$ over $G(2,3)$, we also obtain the homological projective dual of $\mathrm{Hilb}^2\,\mathbb{P}^2$.

Corollary 6.1.4. *Let $X=\mathrm{Hilb}^2\,\mathbb{P}^2$ and let $f:X\to\mathbb{P}(V)$ where V is the vector space $H^0(\mathbb{P}(\mathrm{Sym}^2 S),\mathcal{O}(1))^\vee = H^0(G(2,3),\mathrm{Sym}^2 S^\vee)^\vee$ and S is the universal subbundle on $G(2,3)$. Then the homological projective dual Y of X is*

$$Y=\mathbb{P}^2\times\mathbb{P}^2\to\mathbb{P}(V^\vee).$$

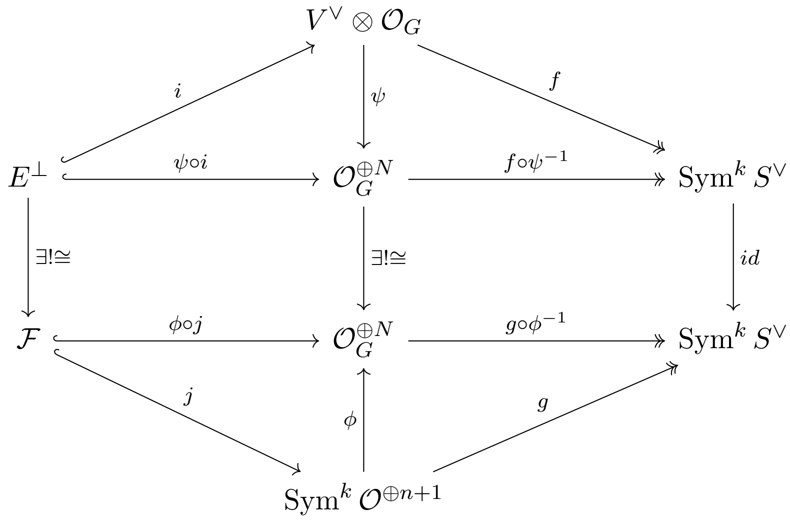


Figure 6.1: Commutative Diagram

6.2 Duality of $\operatorname{Hilb}^2 \mathbb{P}^2$ and $\mathbb{P}^2 \times \mathbb{P}^2$

Let $X = \operatorname{Hilb}^2 \mathbb{P}^2$ and $Y = \mathbb{P}^2 \times \mathbb{P}^2$, and denote by $f : X \to \mathbb{P}(V)$ and $g : Y \to \mathbb{P}(V^\vee)$ the two morphism so that (X, f) and (Y, g) are homologically projectively dual by Corollary 6.1.4. So $X = \mathbb{P}(E)$ for $E = \operatorname{Sym}^2 S$ and $V = H^0(G, E^\vee)^\vee$. Denote by G the Grassmanian $G(2,3)$ over the vector space $W \simeq \mathbb{C}^3$, so $G = \mathbb{P}(W^\vee)$. Since we are interested in the HP-dual of X with the morphism to $\mathbb{P}^5$ factoring through the Hilbert-Chow morphism, we need to show that this morphism actually agrees with the morphism f. Recall that f is given by the composition

$$f : \mathbb{P}(E) \hookrightarrow \mathbb{P}(V \otimes \mathcal{O}_G) \simeq \mathbb{P}(V) \times G \to \mathbb{P}(V).$$

From the proof of Theorem 6.1.1 we can identify the exact sequence

$$0 \to E^\perp \to V^\vee \otimes \mathcal{O}_G \to E^\vee \to 0$$

with the exact sequence

$$0 \to Q^\vee \otimes W^\vee \to \operatorname{Sym}^2(W \otimes \mathcal{O}_G)^\vee \to E^\vee \to 0$$

inducing an isomorphism $V^\vee \simeq H^0(G, \operatorname{Sym}^2(W \otimes \mathcal{O}_G)^\vee)$. By identifying the morphisms $V^\vee \otimes \mathcal{O}_G \to E^\vee$ and $\operatorname{Sym}^2(W \otimes \mathcal{O}_G)^\vee \to E^\vee$, dualizing and taking projectivizations we get the morphisms $\mathbb{P}(E) \hookrightarrow \mathbb{P}(V \otimes \mathcal{O}_G)$ and $\mathbb{P}(E) \hookrightarrow \mathbb{P}(\operatorname{Sym}^2 W \otimes \mathcal{O}_G)$. By composing the latter with the projection to $\mathbb{P}(\operatorname{Sym}^2 W)$ gives a description of f as the composition

$$f : \mathbb{P}(E) \hookrightarrow \mathbb{P}(\operatorname{Sym}^2 W \otimes \mathcal{O}_G) \to \mathbb{P}(\operatorname{Sym}^2 W).$$

We now wish to show that there is a commutative diagram

$$\begin{array}{ccc}
\mathbb{P}(\operatorname{Sym}^2 S) & \xhookrightarrow{\;i\;} & \mathbb{P}(\operatorname{Sym}^2 W \otimes \mathcal{O}_G) \\
\Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle p} \\
\operatorname{Sym}^2 \mathbb{P}(W) & \xhookrightarrow{\;\iota\;} & \mathbb{P}(\operatorname{Sym}^2 W)
\end{array}$$

where $f = p \circ i$ and the composition $\iota \circ \phi$ is the morphism factoring through the Hilbert-Chow morphisms. Since we are dealing with varieties, it suffices to show that f and $\iota \circ \phi$ agree on every closed point. A point of $\mathbb{P}(E)$ can be specified by the data

$$\{(H, [w]) \in G(2,3) \times \mathbb{P}(\mathrm{Sym}^2 H) \ : \ w \in \mathrm{Sym}^2 H\}$$

where $[w]$ denotes the class of the element $w \in \mathrm{Sym}^2 H$ when projectivizing and $H \subset W$ is a two-dimensional linear subspace. Write $w = [v_1 \otimes v_2 + v_2 \otimes v_1]$ for elements $v_i \in H$ and denote by $[v_1], [v_2] \in \mathbb{P}(H) \subset \mathbb{P}(W)$ the classes of v_1 and v_2. Then the morphism $\iota \circ \phi$ is given by

$$\begin{aligned} \iota \circ \phi : \mathbb{P}(E) &\longrightarrow \mathrm{Sym}^2 \mathbb{P}(W) \longrightarrow \mathbb{P}(\mathrm{Sym}^2 W) \\ (H, [w]) &\longmapsto \{[v_1] + [v_2]\} \longmapsto [v_1 \otimes v_2 + v_2 \otimes v_1]. \end{aligned}$$

Since ι is the inclusion and p the projection to $\mathbb{P}(\mathrm{Sym}^2 W)$, we have $\iota : (H, [w]) \mapsto (H, [w])$ and $p : (H, [w]) \mapsto [w]$. Hence the diagram commutes, so f factors through the Hilbert-Chow morphism. Now the morphism g is given by the composition

$$g : \mathbb{P}(E^\perp) \hookrightarrow \mathbb{P}(\mathcal{O}_G \otimes V^\vee) \simeq \mathbb{P}(W^\vee) \times \mathbb{P}(V^\vee) \to \mathbb{P}(V^\vee).$$

We have

$$\mathbb{P}(E^\perp) = \mathbb{P}(\mathcal{O}_G(-1) \otimes W^\vee) \simeq \mathbb{P}(W^\vee) \times \mathbb{P}(W^\vee)$$

and identify $\mathbb{P}(E^\perp) \hookrightarrow \mathbb{P}(O_G \otimes V^\vee)$ with $\mathbb{P}(E^\perp) \hookrightarrow \mathbb{P}(\mathrm{Sym}^2 W^\vee \otimes \mathcal{O}_G)$. Let

$$\begin{aligned} \tau \circ \psi : \mathbb{P}(W^\vee) \times \mathbb{P}(W^\vee) &\longrightarrow \mathrm{Sym}^2(W^\vee) \hookrightarrow \mathbb{P}(\mathrm{Sym}^2 W^\vee) \\ ([v_1], [v_2]) &\longmapsto \{[v_1] + [v_2]\} \longmapsto [v_1 \otimes v_2 + v_2 \otimes v_1] \end{aligned}$$

be the composition of the quotient map and the inclusion from Example 3.3.3. In order to show that g factors through $\mathrm{Sym}^2 \mathbb{P}(W^\vee)$, it remains the check that the following diagram commutes

$$\begin{array}{ccc} \mathbb{P}(Q^\vee \otimes W^\vee) & \xhookrightarrow{\ j\ } & \mathbb{P}(\mathrm{Sym}^2 W^\vee \otimes \mathcal{O}_G)) \\ \downarrow \simeq & & \downarrow q \\ \mathbb{P}(W^\vee) \times \mathbb{P}(W^\vee) & \xrightarrow{\ \tau \circ \psi\ } & \mathbb{P}(\mathrm{Sym}^2 W^\vee) \end{array}$$

A point of $\mathbb{P}(Q^\vee \otimes W^\vee)$ is specified by the data $p = (H, [w \otimes \rho])$ where $H \in G(2,3)$, w is in the fiber $Q^\vee|_H$ and $\rho \in W^\vee|_H$. Then the point p is included via j by $(H, [w \otimes \rho + \rho \otimes w])$ and projecting down to $\mathbb{P}(\mathrm{Sym}^2 W^\vee)$ just forgets the point H in the base. The fiber $Q^\vee|_H$ is one-dimensional so $[w] = H$. The isomorphism in the diagram maps p to $(H, [\rho]) = ([w], [\rho])$, and $\psi \circ \tau([w], [\rho]) = [w \otimes \rho + \rho \otimes w]$. So the diagram commutes showing that g factors through $\mathrm{Sym}^2 \mathbb{P}(W^\vee)$.

Now that we know that f factors through $\mathrm{Sym}^2 \mathbb{P}(W)$ and g factors through $\mathrm{Sym}^2 \mathbb{P}(W^\vee)$ we can study the duality closer on linear sections. Recall that the image of the morphism $\mathrm{Sym}^2 \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ is cut out by the zero locus of a homogeneous degree 3 polynomial with a two dimensional singular locus. The Hilbert-Chow morphism

$\mathrm{Hilb}^2\,\mathbb{P}^2 \to \mathrm{Sym}^2\,\mathbb{P}^2$ is one-to-one away from the singular locus of $\mathrm{Sym}^2\,\mathbb{P}(W)$. On the other hand g factors through $\mathrm{Sym}^2\,\mathbb{P}(W^\vee)$ and is two-to-one away from the singular locus. For a linear subspace $[L] \in G(3,6)$ of $V^\vee$ let $\mathbb{P}(L)$ denote its projectivization. Consider the subset $U \subset G(3,6)$ defined by

$$U = \{[L] \in G(3,6) \;:\; Y \times_{\mathbb{P}(V^\vee)} \mathbb{P}(L) \text{ is smooth}\}.$$

Denote by $L^\perp$ the orthogonal of L and let $U' \subset G(3,6)$ be the subset defined by

$$U' = \{[L] \in G(3,6) \;:\; X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp) \text{ is smooth}\}.$$

One can show that U and U' are both Zariski-open subsets of the variety $G(3,6)$, hence they are dense and have non-empty intersection. Choose an $[L] \in U \cap U'$. Let

$$X_L = X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp) \quad \text{and} \quad Y_L = Y \times_{\mathbb{P}(V^\vee)} \mathbb{P}(L).$$

Since the Hilbert-Chow morphism $\mathrm{Hilb}^2\,\mathbb{P}(W) \to \mathrm{Sym}^2\,\mathbb{P}(W)$ is one-to-one away from the singular locus of $\mathrm{Sym}^2\,\mathbb{P}(W)$ and $f(X)$ avoids this singular locus, we see that $X_L = f^{-1}(\mathbb{P}(L))$ embeds into $\mathbb{P}(L^\perp)$ via f as a smooth one-dimensional subvariety defined by a homogeneous degree 3 polynomial. That is X_L is an elliptic curve.

On the other hand, g is two-to-one away from the singular locus, so Y_L does not embed into $\mathbb{P}(V^\vee)$ via g. However, we can try to understand $Y_L = g^{-1}(\mathbb{P}(L))$ as a closed subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$. Take three hyperplanes $H_1, H_2, H_3 \subset \mathbb{P}^5$ so that $\mathbb{P}(L) = H_1 \cap H_2 \cap H_3$. Denote by $s_1, s_2, s_3 \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1))$ the sections defining H_1, H_2 and H_3. Then

$$Y_L = g^{-1}(H_1 \cap H_2 \cap H_3) = g^{-1}(H_1) \cap g^{-1}(H_2) \cap g^{-1}(H_3).$$

Since $g^*\mathcal{O}_{\mathbb{P}^5}(1) = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1)$, the s_i pull back to sections

$$t_i = g^* s_i \in H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1)).$$

Let $D_i = V(t_i)$ denote the corresponding effective divisors of $\mathbb{P}^2 \times \mathbb{P}^2$. Then $\mathcal{O}(D_i) = \mathcal{O}(1,1)$ and by the adjunction formula we have

$$\omega_{D_1} =\simeq \mathcal{O}(-3,-3)|_{D_1} \otimes \mathcal{O}(1,1) = \mathcal{O}(-2,-2).$$

Then

$$\omega_{D_1\cap D_2} \simeq \mathcal{O}(-2,-2)|_{D_1\cap D_2} \otimes \mathcal{O}(1,1) = \mathcal{O}(-1,-1),$$

and finally we find that

$$\omega_{Y_L} \simeq \mathcal{O}(-1,-1)|_{Y_L} \otimes \mathcal{O}(1,1) = \mathcal{O}_{Y_L}.$$

In other words, the closed subscheme Y_L of $\mathbb{P}^2 \times \mathbb{P}^2$ is Calabi-Yau. It is smooth of dimension one, so it is also an elliptic curve.

Since both X_L and Y_L have expected dimension, the main theorem of homological projective duality applies.

$$\begin{array}{ccc} X_L & \hookrightarrow & \mathrm{Hilb}^2\,\mathbb{P}^2 \\ \downarrow & & \downarrow f \\ \mathbb{P}^2 & \hookrightarrow & \mathbb{P}^5 \end{array} \qquad \begin{array}{ccc} Y_L & \hookrightarrow & \mathbb{P}^2 \times \mathbb{P}^2 \\ \downarrow & & \downarrow g \\ \mathbb{P}^2 & \hookrightarrow & \mathbb{P}^5 \end{array}$$

By the 3rd property of Theorem 5.3.2 and the fact that the derived category of smooth Calabi-Yau varieties does not decompose, we deduce that there is an equivalence

$$D(X_L) \simeq D(Y_L)$$

of derived categories of the elliptic curves X_L and Y_L.

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