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Spectrahedra and their boundaries

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This master's thesis is submitted under the master's programme *Mathematics*, with programme option *Mathematics for applications*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 60 credits.

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

Spectrahedra are the solution set of linear matrix inequalities and is thus the set of feasible solutions in semidefinite optimization. This thesis is concerned with different representations of spectrahedra and the properties of spectrahedra given the representation. Combining theory from real algebraic geometry, matrix theory and linear algebra, we show several properties of spectrahedra, with focus on the boundary. Specifically, we introduce a new concept concerning the boundary of spectrahedra named *façades*, and relate this to more well known concepts in the theory of spectrahedra. In addition, we explore plane spectrahedra, which is a class of spectrahedra with a complete characterization.

Acknowledgements

First, I would like to express my sincere appreciation to my supervisor, Geir Dahl, for embracing my diverse academic background. He truly understood and valued the breadth of my mathematical studies, and skillfully guided me towards a thesis topic where I could effectively apply the knowledge gained throughout the years at UIO. My dad deserves a huge thank you, as my most loyal academic supporter, and for proofreading my thesis down to the smallest details. Also thanks to my mother and brother for all the support which does not involve mathematics. The biggest contribution is however from all my fellow students in the study halls 1101 and 1102. They have been extremely important for my academic achievements, mental health and latex abilities. And of course contributed to making all lunch and coffee breaks a bit too long by always creating a warm, uplifting and enthusiastic atmosphere.

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CHAPTER 1

Introduction

The concept of spectrahedra emerged in the late 1990's in the mathematical field of optimization. Since then, it has been discovered that spectrahedra reside at the intersection of convex geometry, linear algebra, and optimization, providing a powerful framework for modeling and solving a wide range of problems. This versatile concept finds applications in engineering, control theory, quantum information, and various other fields.

In this thesis we will investigate properties of spectrahedra already known, and present some new properties. A majority of our results will be concerning the boundary of spectrahedra. We will also take a closer look at some properties that are sometimes treated as they are obvious, but needs a bit of framework to be proved.

The main contribution in this thesis is the introduction of a new concept related to convex sets, which we have named *façades*. The name refers to the similarity to the faces of convex sets and moreover that they are on the "outside" of the spectrahedron, i.e., on the boundary. The idea to introduce the concept originated in a frustration of only being able to describe the "flat" parts of the boundary of a closed convex sets. To create a definition, the writer was inspired by the one of vertices of convex sets, and when it was apparent that it was possible to define the "curved faces", the idea was nurtured, and resulted in a thorough description of *façades*. Other than all results concerning façades, the writer has stated and proved all results in the thesis where it is not referred to any reference text.

The structure of the of the thesis is as follows: Chapter 2 introduces the basic mathematical concepts and results of interest. Chapter 3 presents the definitions of spectrahedra and present properties of general spectrahedra, moreover, this is the chapter where the concept of façades is introduced and explored. Chapter 4 concerns spectrahedra in the plane and explores properties which are based on a result that only apply for plane spectrahedra. Throughout the thesis, every result which is not cited or referred to any other text is stated and proved by the writer of the thesis.

On all matters of convexity, the references *Convex optimization* by Boyd and Vandenberghe [BV04] and *An introduction to convexity*[Dah10] by Geir Dahl are the main source. For spectrahedra and semidefinite optimization the

Semidefinite Optimization and Convex Algebraic Geometry[BPT12], especially the chapter by Parrilo and *Linear Matrix Inequality Representation of Sets*[HV03] by Helton and Vinnikov provide both important results, but also examples and figures this thesis. Most of the figures are created using GeoGebra, but some are also collected from other papers.

CHAPTER 2

Preliminaries

2.1 Euclidean spaces

In this thesis, we will focus on Euclidean spaces, which are finite-dimensional vector spaces defined over the real numbers or a translation of such. We will use the standard inner product for $\mathbb{R}^n$, defined as

$$\langle x, y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i.$$

This inner product gives rise to the Euclidean norm, $\|x\| = \sqrt{\langle x, x \rangle}$, and the metric $d(x, y) = \|x - y\|$. When $\langle x, y \rangle = 0$, we say that x and y are orthogonal.

For the space $\mathbb{R}^{m \times n}$, the inner product of $X, Y \in \mathbb{R}^{m \times n}$ is given by

$$\langle X, Y \rangle = \mathbf{tr}(X^T Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{i,j} Y_{i,j},$$

where **tr** denotes the trace of the given matrix, i.e. the sum of all its elements. The norm on $\mathbb{R}^{n \times n}$ is $\|X\| = \sqrt{\langle X, X \rangle}$.

If V is any finite dimensional real vector space, the **span** of $x_1, \ldots, x_k \in V$ are all elements of V that can be written as a linear combination $a_1 x_1 + a_2 x_2 + \ldots a_k x_k$. The elements $x_1, x_2, \ldots, x_k \in V$ are **linearly independent** if $a_1 x_1 + a_2 x_2 + \ldots a_k x_k = 0$ if and only if $a_1 = a_2 = \ldots = a_k = 0$. We say that a subset $B \subset V$ is a **basis** of V if all elements of B are linearly independent, and their span is equal to V. The **dimension** of V is then the number of elements in B. For any subset $U \subseteq V$ of a real vector space V, we can form the **orthogonal complement** of U, denoted $U^\perp$, by including all elements of V which are orthogonal to all elements of U. The following equality holds $\dim_{\mathbb{R}} U + \dim_{\mathbb{R}} U^\perp = \dim_{\mathbb{R}} V$.

If V, W are two real vector spaces, they are **isomorphic**, denoted $V \cong W$, if there is a bijection $T : V \to W$, such that for all $u, v \in V$ and all $\alpha \in \mathbb{R}$, $T(u + v) = T(u) + T(v)$ and $T(\alpha u) = \alpha T(u)$. Any two real vector spaces of the same (finite) dimension are isomorphic. This can be shown by letting the isomorphism T be a map taking basis elements to basis elements. Then T is bijective as the vector space have the same dimension, hence the same number of basis elements.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be any function. Then the gradient of f is given by $\nabla f(x_1, x_2, \ldots, x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$. The gradient is the direction and rate of fastest increase of the function at a given point. The set $L = \{x \in \mathbb{R}^n : f(x) = c\}$ for some $c \in \mathbb{R}$ is a **level set** of f, and for any $x \in L$, the gradient in x is orthogonal to L at x.

2.2 Affine geometry

Definition 2.2.1. A subset $A \subseteq \mathbb{R}^n$ is **affine** if the line through two distinct points in A lies in A, i.e. if for all $x, y \in A$ and all $\theta \in \mathbb{R}$, $(1-\theta)x + \theta y \in A$.

The combination of points from the definition can be generalized to an **affine combination** of some collection of points $\{x_i\}_{i=1,\ldots,k}$. That is a point on the form $\lambda_1 x_1 + \lambda_2 x_2 + \ldots \lambda_k x_k$, where $\sum_{i=1}^{k} \lambda_i = 1$ for some k. From induction on the definition of an affine set, it can be shown that an affine set contains all affine combinations of its points.[BV04].

Any affine set $A \subseteq \mathbb{R}^n$ can be expressed on the form

$$A = x_0 + L = \{x_0 + l : l \in L\}, \tag{2.1}$$

where $x_0 \in A$ and L is a linear subspace of $\mathbb{R}^n$. I.e., an affine set is a translated linear subspace [BV04]. The affine set A does not depend on the choice of $x_0 \in A$. To see this, consider the affine set A, and some point $x_0 \in A$, and define $L = A - x_0$. Then for $l_1, l_2 \in L$ and $\alpha, \beta \in \mathbb{R}$, we have $l_1 + x_0 \in A$ and $l_2 + x_0 \in A$. Thus

$$\alpha l_1 + \beta l_2 + x_0 = \alpha(l_1 + x_0) + \beta(l_2 - x_0) + (1 - \alpha - \beta)x_0 \in A$$

the last inclusion holds as it is an affine combination of elements in A. Now since $\alpha l_1 + \beta l_2 + x_0 \in A$ we can conclude that $\alpha l_1 + \beta l_2 \in L$, thus L is closed under sums and scalar multiplications, so it is a linear subspace of R^n. For a given affine space, the corresponding linear subspace is *unique*. Note that every linear space is also an affine set as x_0 can be chosen to be 0.

The **affine hull** of a set $S \in \mathbb{R}^n$, *aff*(S), is the set of all affine combinations of elements in S. The **dimension** of an affine set A is equal to the dimension of the associated linear space L, i.e. $dim\ A = dim\ L$. Moreover, we say that the **(affine) dimension** of an arbitrary set $S \in \mathbb{R}^n$, is the dimension of its affine hull. An affine set of dimension $n-1$ is called a **hyperplane**, and can be expressed as

$$H = \{x \in \mathbb{R}^n : \langle c, x \rangle = b\}.$$

A line is of dimension 1, and a point of dimension 0.

If the dimension of a subset $S \subseteq \mathbb{R}^n$ is strictly smaller than n, the interior of S will be empty. To describe the inner part of the set, we define the **relative interior**, $\mathrm{rint}(S)$ as the points in S for which there exist an open ball $B(x, \epsilon)$ such that $\mathrm{aff} S \cap B(x, \epsilon) \subseteq S$. We can say that this is the interior of an set with respect to its affine dimension. From the relative interior of a set, we define the **relative boundary** by $\mathrm{rb}(S) = \overline{S} \setminus \mathrm{rint}(S)$ where $\overline{S}$ is the closure of S. Note that if a set is full dimensional in the space it lies in, the relative boundary and relative interior coincide with the boundary and interior.

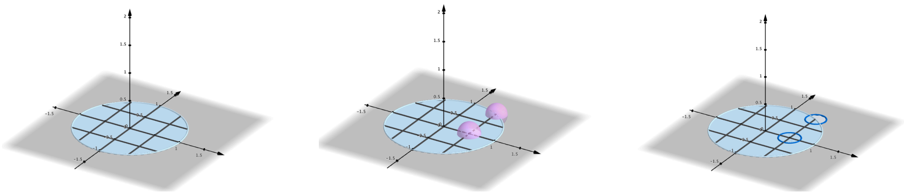

Figure 2.1: *Left.* The set $C = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 1, x_3 = 0\} \subseteq \mathbb{R}^3$. *Middle.* Two open balls in $\mathbb{R}^3$, intersecting both C and C^c. *Right.* The open balls intersected with the affine hull of C. Notice that one is included in C and is therefore in the relative interior.

Example 2.2.2. Consider the set $C = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 1, x_3 = 0\}$ pictured in Figure 2.1.

- *The affine hull of C* The origin is in C, so for all $x = (x_1, x_2, 0) \in C$, the affine combination $(1-\theta)\cdot 0 + \theta(x_1, x_2, 0) = \{x \in \mathbb{R}^3 : x_3 = 0\} \subseteq \mathit{aff}(C)$ for $\theta \in \mathbb{R}$. Furthermore if $x_3 \neq 0$, then $x \notin C$ hence $x \notin \mathit{aff}(C)$. So $\mathit{aff}(C) = \{x \in \mathbb{R}^3 : x_3 = 0\} \cong \mathbb{R}^2$.

- *The affine dimension of C* From above, we can easily see that the affine dimension of C is 2, as the affine hull is isomorphic to $\mathbb{R}^2$.

- *The interior and boundary of C* Each open ball in R^3 will be of dimension 3, and hence each ball intersecting C, will also intersect the compliment of C. Thus $C^o = \emptyset$. The same argument implies that the boundary of C equals C.

- *The relative interior and relative boundary of C* If we consider the set $\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$, the interior is the set where the inequality is strict. Thus the relative boundary is $\overline{C} \setminus \mathrm{rint}(C) = C \setminus \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1, x_3 = 0\} = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1, x_3 = 0\}$

2.3 Convexity

Convex sets

Definition 2.3.1. A subset $C \subseteq \mathbb{R}^n$ is a **convex set** if for each pair of points $x, y \in C$, $(1-\lambda)x + \lambda y \in C$ for $\lambda \in [0, 1]$.

If $x_1, \ldots x_k$ is a collection of points we denote a point on the form $\lambda_1 x_1 + \lambda_2 x_2 + \ldots \lambda_k x_k$, where $\lambda_i \geq 0$ for all i, and $\sum_{i=1}^k \lambda_i = 1$, a **convex combination** of the points $x_1, \ldots x_k$. Any convex sets include every convex combination of its points.

For any set $S \subseteq \mathbb{R}^n$ we can define the **convex hull** of a finite set of points $x_1, \ldots, x_k$ by

$$\mathrm{conv}\, S = \Big\{ \sum_{i=1}^{k} \lambda_i x_i : x_1, x_2, \ldots, x_k \in S,\ \lambda_1, \lambda_2, \ldots, \lambda_n \in [0,1],\ \sum_{i=1}^{k} \lambda_i = 1 \Big\},.$$

In other words, it is the set of all convex combinations of the elements of A. An alternative definition of the convex hull is that the convex hull of a set S

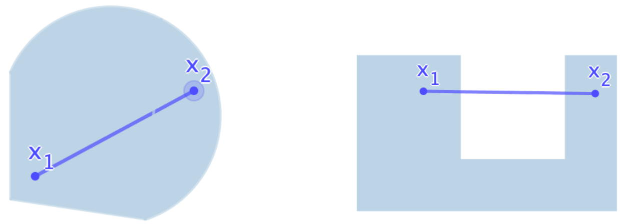

Figure 2.2: Two sets in $\mathbb{R}^2$. The set on the left is convex, and the one on the right is not.

is the intersection of all convex sets containing S, thus it is also the smallest convex set containing S. This definition allows us to form the convex hull of infinite sets, e.g., curves and surfaces. If S is a finite set of points, $\operatorname{conv} S$ is a **polytope**, which is an extensively studied convex set. A convex set is **line-free** if there are no linear space contained in the set.

For any family $\{C_i\}_{i\in I}$ of convex sets, the intersection $\bigcap_{i\in I} C_i$ is convex. The Minkowski sum of convex sets C and D, defined by $C + D = \{x + y : x \in C \textit{ and } y \in D\}$ is convex, moreover this extends to any family of convex sets, i.e., the set $\{\sum_{i\in I} x_i : x_i \in C_i\}$ is convex.

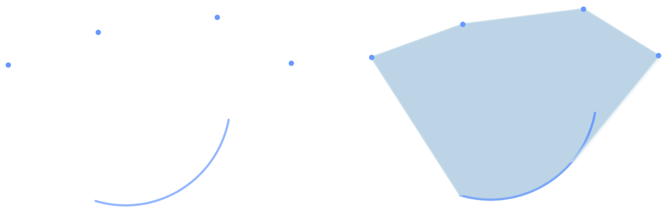

Figure 2.3: The first figure show the set $A \in \mathbb{R}^2$, and the second show the convex hull of A, $conv(A)$.

Hyperplanes and halfspaces

As earlier stated a hyperplane H in a real inner product space V is denoted

$$H = \{x \in V : \langle c, x\rangle = b\}$$

$c \in V$ orthogonal to the hyperplane, usually referred to as the normal vector of H, and $b \in \mathbb{R}$. Corresponding to each hyperplane in V, there are two **halfspaces**, $H^+ = \{x \in V : \langle c, x\rangle \geq b\}$ and $H^- = \{x \in V : \langle c, x\rangle \leq b\}$. If $S \subseteq V$ is contained in either H^+ or H^- and $S \cap H \neq \emptyset$, we say that H is a **supporting hyperplane** of S, more precisely we say that for each $x \in S \cap H$, H supports S at x. We also sometimes refer to the halfspace containing S as the **supporting halfspace** of S. When the set of concern is of a smaller dimension than the space it lies in, there are hyperplanes that include the whole set. We say they are **trivial supporting hyperplanes**. For example, the xy-plane is a supporting hyperplane of C in Figure 2.1, but it also fully contains C. The trivial supporting hyperplanes are not of much interest, so we will only refer to H as a supporting hyperplane of S, if S is not fully contained in H. For convex sets, we have the following useful connection to supporting hyperplanes.

Theorem 2.3.2. *[Dah10] Let $C \subseteq V$, be a nonempty convex subset of an euclidean space, and let $x \in \mathbf{rb}(C)$. Then there is a hyperplane supporting C in x.*

Corollary 2.3.3. *[Dah10] Let $C \subseteq V$ be a nonempty closed convex subset of an euclidean space. Then C is the intersection of all its supporting halfspaces.*

Convex cones

Definition 2.3.4. A subset K in a real vector space V is a **convex cone** if for each pair of points $x, y \in V$, and for all $\alpha, \beta \in \mathbb{R}_+$ we have that $\alpha x + \beta y \in K$, i.e., if K is closed under non-negative linear combinations, or **conical combination**. Similar to affine and convex hull, we can define the **conical hull** cone A, by including all conical combinations of the points in A.

A cone $K \in V$ is **pointed** if $K \cap \{-K\} = \{0\}$. If a cone is closed, convex, pointed and full-dimensional, we say that it is a **proper cone**. For every convex cone, $K \subset V$, there is a **dual cone**;

$$K^* = \big\{y \in V : \langle x, y\rangle \geq 0 \text{ for all } x \in K\big\}. \tag{2.2}$$

If a cone is proper, then the dual cone is proper. For several of the cones we will study in this thesis, the cones are **self-dual**, which is when the dual cone equals the cone, i.e., $K^* = K$. In general, the pair (K, K^*) is of great importance in optimization, as *duality* is an important tool for finding and confirming optimal solutions.

Another convex cone that will be useful is the **normal cone** of a convex set, also sometimes referred to as the *polar cone*. For a convex set $C \subseteq \mathbb{R}^m$, and a point $x \in C$, the normal cone is

$$N_C(x) = \{z \in \mathbb{R}^m : \langle z, x\rangle \geq \langle z, y\rangle \text{ for all } y \in C\}. \tag{2.3}$$

The gradient of a real valued function is orthogonal to any level set of the functions, thus for a convex where the boundary is defined by a level set of a function, we have that $\nabla f(x) \in N_C(x)$ for all x in the level set.

Lemma 2.3.5. *Let $A = x_0 + L \subseteq \mathbb{R}^m$ be an affine set as in Equation (2.1). Then for any $a \in A$, $N_A(a) = L^\perp$*

Proof. The affine set does not depend on the choice of $x_0 \in A$, thus we choose $x_0 = a$ and express $A = a + L$, where L is the corresponding linear subspace of

$\mathbb{R}^m$. The normal cone of A in a is

$$\begin{aligned} N_A(a) =& \{z \in \mathbb{R}^m : \langle z, a \rangle \geq \langle z, a + l \rangle \ \forall l \in L\} \\ =& \{z \in \mathbb{R}^m : \langle z, a \rangle \geq \langle z, a \rangle + \langle z, l \rangle \ \forall l \in L\} \\ =& \{z \in \mathbb{R}^m : 0 \geq \langle z, l \rangle \ \forall l \in L\} \\ =& \{z \in \mathbb{R}^m : 0 = \langle z, l \rangle \ \forall l \in L\} \\ =& L^\perp \end{aligned}$$

■

Faces of convex sets

Definition 2.3.6. [Dah10] Let C be a convex set in $\mathbb{R}^m$. A convex subset $F \subseteq C$ is a **face** of C if the following holds: if $x_1, x_2 \in C$ is such that $(1-\lambda)x_1 + \lambda x_2 \in F$ for some $0 < \lambda < 1$, then $x_1, x_2 \in F$.

We include $\emptyset$ and C in the faces of C, and refer to them as the **trivial faces** of C. The dimension of a face is its affine dimension. Faces of dimension 0 are called **extreme points**, and dimension one less than the convex set are called **facets**.

Definition 2.3.7. An extreme point $x \in C$ is a **vertex** if $\dim N_C(x) = \dim C$.

Example 2.3.8. Consider the polytope in Figure 2.4. For every line segment in the polytope, where the relative interior of the line segment is included in one of the edges of the polytope, the endpoints of the line segment are also on that edge. Hence the edges are faces of the polytope. An example is depicted on the right in Figure 2.4. The other faces of the depicted polytope are the "corners". The corners are faces of dimension 0, or extreme points, because if there are two points, x_1, x_2 in the polytope such that $(1-\lambda)x_1 + \lambda x_2$ intersects the corner point for any $\lambda \in (0, 1)$, then both x_1 and x_2 must be the corner point. Note moreover that for a polytope, the extreme points are also vertices.

Figure 2.4: "Line segments" with relative interior contained in what turns out to be the faces of a polytope in $\mathbb{R}^2$

Definition 2.3.9. [Dah10] Let C be a convex set in $\mathbb{R}^m$ and H a supporting hyperplane of C.Then the intersection $C \cap H$ is called an **exposed face** of C.

Example 2.3.10. Consider the closed unit disk, $B \subseteq \mathbb{R}^2$. For each point y on the boundary, the set $H = \{x \in \mathbb{R}^2 : \langle x, y\rangle = 1\}$ is a supporting hyperplane of H, and $y \in B \cap H$. Moreover for any other point $z \in B$, $z = \lambda x$ for some $\lambda < 1$ or z and x are linearly independent. Thus $z \notin H$ as $\langle x, z\rangle < \|x\|\|z\| = 1$ by the Cauchy-Schwarz inequality. This shows that each boundary point of the unit circle is an exposed face, in particular an extreme point.

Proposition 2.3.11. *[Dah10] Let $C \in \mathbb{R}^m$ be a convex set. Then every exposed face of C is also a face of C.*

The contrary is not necessarily true, though we will see that for our main object of interest, spectrahedra, the two concepts coincide. First we will see an example of a face that is not exposed.

Example 2.3.12. Consider the set $C = B + ([0,1] \times 0) \in \mathbb{R}^2$, which is the Minkowski sum of the closed unit disc B and the line segment from $[0,1]$ on the x-axis. See Figure 2.5. In Example 2.3.10 we saw that for every point which is only on the "curved" part of ∂C (blue in the figure) is an exposed face, hence also a face. The line segments $[0,1] \times \{-1\}$ and $[0,1] \times \{1\}$ (black in the figure) are also exposed faces. The hyperplanes $y = 1$, and $y = -1$ intersects C exactly in these line segments. Now consider the point $a = (0,1)$. The supporting hyperplane of C in $a = (0,1)$ is $y = 1$. Then a can not be an exposed face, since the intersection between C and the hyperplane includes more than just the point a. On the other hand, if we let $a_1, a_2 \in C$ be points such that $a = \frac{1}{2}a_1 + \frac{1}{2}a_2$, this implies that $a_1 = a_2 = a$, hence a is a face. A similar argument hold for the points $(0,-1), (1,-1), (1,1)$.

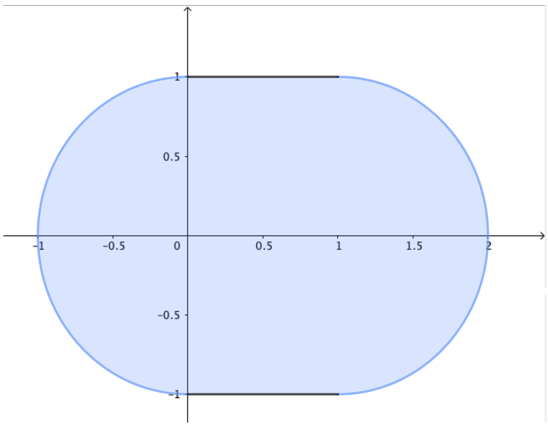

Figure 2.5: The set $C = B + ([0,1] \times 0) \in \mathbb{R}^2$.

Proposition 2.3.13. *[NPS09] Let C be a closed convex set with non-empty interior in an euclidean space, and let $\{F_i\}_{i \in I}$ be the set of faces of C. Then the following holds.*

(i) *For every face $F_i \subsetneq C$, there exist a supporting hyperplane H of C such that $F_i \subseteq H$.*

(ii) *F_i is closed for all $i \in I$.*

(iii) *If* $F_1 \subsetneq F_2$*, then* $\dim F_1 < \dim F_2$.

(iv) *If* G *is a face of any element in* $\{F_i\}_{i\in I}$*, then* $G \in \{F_i\}_{i\in I}$*. I.e. any face of a face of* C *is also a face of* C.

2.4 Functions and maps

Convex functions

Definition 2.4.1. A function **convex function** is a function $f : C \to \mathbb{R}$, where for any convex set $C \subseteq \mathbb{R}^m$, we have

$$f((\mathbf{1}-\lambda)x + \lambda y) \leq (\mathbf{1}-\lambda)f(x) + \lambda f(y)$$

for all $x, y \in C$ and for all $0 \leq \lambda \leq 1$.

Definition 2.4.2. The **epigraph** of a function $f : \mathbb{R}^m \to \mathbb{R}$ is the set

$$\text{epi}(f) = \{(x, y) \in \mathbb{R}^m \times \mathbb{R} : y \geq f(x)\}$$

Theorem 2.4.3. *[Dah10] Let* $f : C \to \mathbb{R}$ *where* $C \subseteq R^m$ *is convex. Then* f *is a convex function if and only if* $\text{epi}(f)$ *is a convex set.*

Affine maps

Definition 2.4.4. Let A, B be affine sets with corresponding linear spaces L_A, L_B. A map $\alpha : A \to B$ is an **affine map** if there exist a linear map $\beta : L_A \mapsto L_B$ such that $\beta(x - y) = \alpha(x) - \alpha(y)$ for all $x, y \in A$.

Proposition 2.4.5. *Affine maps preserve the following*

(i) *Colinearity, three or more points that lie on the same line, will still be on the same line under the map*

(ii) *Paralellism, the image of paralell lines will be paralell*

(iii) *Convexity, the image of a convex set is convex*

(iv) *Extreme points, the image of extreme points of a set will be extreme points of the image of the set*

Projections

Definition 2.4.6. Let $C \subseteq \mathbb{R}^m$ be closed subset, then the **projection** of $x \in \mathbb{R}^m \setminus C$ onto the set $C$is defined by $\pi_C(x) = \arg\min_{y\in C} \|x - y\|$.

A necessary condition for the existence of a projection is that C is closed. For an arbitrary set in $\mathbb{R}^m$, the projection is not unique. Consider for example the projection of $(0, 0)$ onto the set $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}$. Then every boundary point will be a projection of origo onto the set.

If however $C \subseteq \mathbb{R}^m$ is a closed convex set, then for every $x \in \mathbb{R}^n \setminus C$ the projection $\pi_C(x)$ is a *unique* point in C, more specific in ∂S. Furthermore, for each boundary point of a closed convex set, $y \in \partial C$, there exist a point $x \in R^n \setminus C$ such that $\pi_C(x) = y$. [LV16]

2.5 Matrix theory

Let $A = (a_{ij})_{1 \le i,j \le n}$ be a real valued with $n \times n$ matrix. The **rank** of a matrix is the dimension of the vector space spanned by the columns of the matrix, i.e. the number of linearly independent columns. The **null space** of a matrix A consist of all the vectors $x \neq 0$ such that $Ax = 0$. The dimension of the null space is called the **nullity**, denoted $\operatorname{nl} A$.

The **eigenvalues** of A are all λ such that $Ax = \lambda x$ for some non-zero vector $x \in \mathbb{R}^n$, in particular x is an **eigenvector** corresponding to the eigenvalue. The eigenvalues can be found by computing the roots of the **characteristic polynomial** $p_A(\lambda) = det(\lambda I - A)$. If $\{\lambda_i\}_{i=1,\ldots,n}$ are the eigenvalues of A, $p_A(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i)$. From this it can be deduced that $\det A = (-1)^n \det(0 \cdot I - A) = (-1)^n \prod_{i=1}^n (0 - \lambda_i) = \prod_{i=1}^n \lambda_i$
A square matrix $D \in \mathbb{R}^{n \times n}$ **diagonal** if all elements off the diagonal are zero. A square matrix $A \in \mathbb{R}^{n \times n}$ is **diagonalizable** if there exist an $n \times n$ invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. A matrix $A = (a_{ij})_{1 \le i,j \le n}$ is **symmetric** if $a_{ij} = a_{ji}$ for all $i, j = 1, 2, \ldots, n$ or equivalently if $A^T = A$. The set of real symmetric matrices of dimension n will be denoted $\mathcal{S}^n$. All eigenvalues of a real symmetric matrix are real.

For any matrix A we can form a **submatrix** by including a subset of rows and columns of A, its determinant is called a **minor** of A. If A is quadratic matrix, i. e $m = n$, then a **principal submatrix** is formed by including the rows and columns of the same index, the determinant of this submatrix is called a **principal minor** of A. Moreover, we say that the principal submatrix and principal minor are **leading** if they are formed by a submatrix in the upper left corner of the matrix.

Theorem 2.5.1. *[LV16] Any real symmetric matrix $A \in \mathcal{S}^n$ can be decomposed as*

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^T,$$

where $\{\lambda_i\}$ are the eigenvalues of A, and $\{u_i\}$ are the corresponding eigenvectors which form an orthonormal basis for $\mathbb{R}^n$. Moreover, the decomposition can be expressed on matrix form

$$A = UDU^T$$

where U is an real orthogonal (columns are orthonormal) $n \times n$- matrix, and D is a diagonal matrix with the eigenvalues of A on the diagonal.

A consequence of the spectral theorem of symmetric matrices is that all symmetric matrices are diagonalizable.

Positive semidefinite matrices

A *positive semidefinite matrix* is a symmetric matrix A such that $x^T Ax \ge 0$ for all $x \in \mathbb{R}^n$. We denote this by $A \succeq 0$. If the inequality is strict for all $x \in \mathbb{R}^n \setminus \{0\}$ we say that the matrix is *positive definite*, or $A \succ 0$.

Theorem 2.5.2 ([BPT12, p. 448])**.** *Let $A \in \mathcal{S}^n$. Then the following are equivalent*

(i) *A is positive semidefinite ($A \succeq 0$).*

(ii) *All eigenvalues of A are non-negative.*

(iii) *All principal minors of A are non-negative.*

(iv) *$A = U^T U$ for some matrix $U \in \mathbb{R}^{n \times r}$, where r is the rank of A.*

(v) *The coefficients of $p_A(\lambda)$ weakly alternates in signs.*

A similar set of equivalences holds for positive definite matrices.

Theorem 2.5.3 ([BPT12, p. 448])**.** *Let $A \in \mathcal{S}^n$. Then the following are equivalent.*

(i) *A is positive definite ($A \succ 0$).*

(ii) *All eigenvalues of A are strictly positive*

(iii) *All n leading principal minors are strictly positive.*

(iv) *The coefficients of $p_A(\lambda)$ alternates in signs.*

(v) *$A = B^T B$ for some invertible matrix $B \in \mathbb{R}^{n \times n}$.*

There are several ways to show that a matrix is positive semidefinite using the equivalent definitions. We will take a look at a few results which can be easily deduced from Theorem 2.5.2.

Proposition 2.5.4. *If $X \in \mathcal{S}^n$ and $B \in \mathbb{R}^{n \times n}$ is an invertible matrix. Then $X \succeq 0$ if and only if $BXB^T \succeq 0$, we say that this is a congruence transformation of X.*

Given two symmetric matrices $A \in \mathcal{S}^n$ and $B \in \mathcal{S}^m$, we can define the following block diagonal matrix.

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{S}^{n+m}$$

As a direct consequence of (iii) in Theorem 2.5.2 we get the following proposition

Proposition 2.5.5. *Let $A, B \in \mathcal{S}^n$, then $A \oplus B \succeq 0$ if and only if both $A \succeq 0$ and $B \succeq 0$.*

Proposition 2.5.6. *Let X be a real symmetric matrix, such that the Schur complement exist, then $X \succeq 0$ if and only if its Schur complement is positive semidefinite.*

A matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is **diagonally dominated** if $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$ for all i.

Proposition 2.5.7. *Let A be areal valued symmetric matrix with non-negative diagonal elements which is diagonally dominated. Then A is positive semidefinite.*

2.6 Positive semidefinite cone

If we now consider the set of all symmetric $n \times n$-matrices $\mathcal{S}^n$, they form a real vector space under the usual matrix addition, and scalar multiplication. As $\mathcal{S}^n$ is a real vector space, it is also an inner product space, with the trace inner product. The standard basis of this vector space, is the set of matrices that are zero everywhere except either one diagonal element equal 1, or except that the elements (i,j) and (j,i) are $\frac{1}{\sqrt{2}}$. It can be checked that this is an orthonormal basis. As the non-zero elements for this basis can be placed in $\frac{n^2+n}{2}$ different ways, we have that $\mathcal{S}^n \cong \mathbb{R}^{\frac{n^2+n}{2}}$.

For $A, B \succeq 0$, and $\alpha, \beta \geq 0$,

$$x^T(\alpha A + \beta B)x = \alpha x^T Ax + \beta x^T Bx \geq 0,$$

so $\alpha A + \beta B \succeq 0$, which shows that the set of all positive semidefinite matrices form a convex cone $\mathcal{S}^n_+ \subseteq \mathcal{S}^n$. In fact this cone is generated by the symmetric matrices of rank 1, i.e., $\mathcal{S}^n_+ = cone\{xx^T : x \in \mathbb{R}^n\}$. This can easily be seen by observing that the conical combinations of rank 1 matrices, is exactly the symmetric matrices described in the spectral decomposition theorem (Theorem 2.5.1) with non-negative eigenvalues, thus it is the positive semidefinite symmetric matrices.

Proposition 2.6.1. *[BPT12] For the positive semidefinite cone $\mathcal{S}^n_+$ the following properties hold*

(i) *$\mathcal{S}^n_+$ is self dual, i.e., $(\mathcal{S}^n_+)^* = \mathcal{S}^n_+$.*

(ii) *$\mathcal{S}^n_+$ is a proper cone.*

(iii) *The interior of $\mathcal{S}^n_+$ is $\{X \in \mathcal{S}^n : X \succ 0\}$.*

2.7 Convex optimization

Convex optimization is the task of minimizing a convex functions over convex sets. A convex optimization problem is on the form[BV04]

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0 \;\; i = 1, \ldots, m \\ & a_j^T x = b_j \;\; j = 1, \ldots, p \end{array}$$

Where $f_0, f_1, \ldots, f_m : V \to \mathbb{R}$ are convex functions, for some finite dimensional real vector space. We refer to the function to be optimized, $f_0(x)$ as the **objective function**, and the inequalities and equations that must hold for the variable we are optimizing over as the **constraints**. We say that any x such that the constraints hold is a feasible solution of the optimization problem.

There are no general analytic ways to solve a convex optimization problem, but for various classes of problems there are effective methods for solving them.

For other classes the known methods for solving them might be either "slow" or with low accuracy. For practical use, the methods known today are sufficient, but usually the challenge is to formulate the real world problems as convex optimizations problems, both the objective function and the constraints. The types of optimization that are most widely used in practical applications are all various versions of *conic optimization*. Convex sets are in general hard to express, the solution sets in conic optimization are somewhat "nicer". In conic optimization, the feasible solutions are given by a proper convex cone intersected with an affine space.

Let V be a real vector space, for example $\mathbb{R}^m$, $c \in V$, $a_1, \ldots, a_k \in V$, $b_1, \ldots b_k \in \mathbb{R}$, and K a proper cone in V. Then we define a primal conic optimization problem by

$$\inf \Big\{ \langle c, x \rangle : x \in K,\ \langle a_i, x \rangle = b_i,\ i = 1, 2, \ldots, k \Big\} \tag{2.4}$$

If $x \in K^o$, we say that x is **strictly feasible**.

Conic optimization holds another powerful property, the existence of a dual problem that generates lower (or upper, in the case of maximization problems) bounds on the original problem. The duality property makes is possible to find better approximations to the optimal solution, and also more ways of assuring optimality. If the primal of a conic optimization problem is given by Equation (2.4), then the dual problem is

$$\sup \Big\{ \sum_{i=j}^{k} y_i b_i : y = (y_1, \ldots y_k) \in V,\ \sum_{i=1}^{k} y_i a_i - c \in K^* \Big\} \tag{2.5}$$

If x is a feasible solution to the primal problem, and y is a feasible solution to the dual problem, we sometimes say that (x, y) is a feasible solution to the primal-dual problem. For all feasible solutions of a primal-dual pair of a conic program weak duality holds, i.e., the value of the function we minimize in the primal problem, is greater than or equal to the value of the function we maximize in the dual problem. We will return to this property in two special cases of conic optimization; linear and semidefinite.

Linear Optimization

To get an overview of the main concepts of conic optimization, we take a look at the most famous, but also most straight forward type. Linear optimization is also known as linear programming (LP). Linear programming is a cornerstone in mathematical optimization, and was one of the first types where efficient algorithm for solving problems was developed. As a consequence, linear optimization have been a vital tool in several practical applications, as logistics, economics, engineering, and resource allocation.

A linear optimization is the problem of minimizing (or maximizing) a linear function, given linear constraints and has a standard form given by

$$\begin{aligned} (P) \quad & \underset{x \in \mathbb{R}^n}{\text{minimize}} && \langle c, x \rangle \\ & \text{subject to} && Ax \leq b_i, \quad i = 1, \ldots, m \\ & && x_1, x_2, \ldots, x_n \geq 0 \end{aligned} \tag{2.6}$$

The parameters of the problem is $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$ and $b_1, \ldots, b_m \in \mathbb{R}$, and $x \in \mathbb{R}^n$ is the decision variable.

The inequality $x \geq 0$ is what makes linear programming a conic optimization problem. It defines the non-negative orthant which is the set $\{x_i \geq 0, \ i = 1, 2, \ldots, n\}$. The non-negative orthant $\mathbb{R}^n_+$ is a convex cone, as for each pair $x, y = (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n_+$, every non-negative combination is also in $\mathbb{R}^n_+$. Furthermore, for any $x \geq 0$, $\langle y, x \rangle \geq 0$ if and only if $y \geq 0$. It follows that the non-negative orthant is a self-dual cone. Hence the dual of a linear program (Equation (2.6)) is

$$\begin{aligned} (D) \quad & \underset{y \in \mathbb{R}^m}{\text{maximize}} && \langle b, y \rangle \\ & \text{subject to} && A^T y \geq c_j, \quad j = 1, \ldots, n \\ & && y_1, y_2, \ldots, y_m \geq 0 \end{aligned} \tag{2.7}$$

Theorem 2.7.1. *[BPT12] For a primal-dual linear optimization problem the following holds.*

(i) ***Weak duality:*** *For any feasible solution (x, y) for (P,D), we have $\langle b, y \rangle \leq \langle c, x \rangle$.*

(ii) ***Strong duality:*** *If both (P) and (D) are feasible, there exist optimal solutions (x^*, y^*) and they (P) and (D) have the same optimal value, i.e., $\langle c, x^* \rangle = \langle b, y^* \rangle$.*

(iii) ***Complementary slackness:*** *If (x^*, y^*) are optimal solutions, then $x_i^*(c - A^T y^*)_i = 0$ for $i = 1, 2, \ldots, n$.*

We will return to some of these terms in Chapter 3, but then in the context of semidefinite optimization, which is another type of conic optimization.

Polyhedra

The set of feasible solutions of a linear program, both primal and dual, is a convex object known as *polyhedra.*

Definition 2.7.2. A **polyhedron** is a set in $\mathbb{R}^n$ defined by finitely many inequalities or equations, i.e., a set which can be written on the form

$$\{x \in \mathbb{R}^n : Ax \leq b, \ x \geq 0\}.$$

Earlier, we defined a polytope as the convex hull of a finite set of points, but a result from convexity theory show that a set is a polytope if and only if it is a bounded polyhedron. Also unbounded polyhedra are related to polytopes in the theorem we now present, called *The main theorem for polyhedra.*

Theorem 2.7.3. *[Dah10] Each polyhedron $P \subseteq \mathbb{R}^n$ may be written as*

$$P = \operatorname{conv} V + \operatorname{cone} W$$

where $V, W \subseteq \mathbb{R}^n$ are finite sets. In particular, if P is pointed, we may here let V be the set of vertices, and W consist of a direction vectors of each extreme halfline of P. Conversely, if V, W are finite sets in $\mathbb{R}^n$, then the set $P = \operatorname{conv} V + \operatorname{cone} W$ is a polyhedron, i.e.,

$$P = \operatorname{conv} V + \operatorname{cone} W = \{x \in \mathbb{R}^n : Ax \leq b\}$$

for some matrix $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ for some m.

CHAPTER 3

Spectrahedra

"Spectrahedron" is a term introduced by Ramana and Goldman in their paper from 1995 called "Some geometric results in Semidefinite programming"[RG95]. After they define the spectrahedron they write the following to present it;
"Spectrahedra are nothing but the feasible regions of Semidefinite Programs (SDP). The name Spectrahedron can perhaps be justified as follows: the definition of this class of sets involves the spectrum, and they bear a resemblance to polyhedra. Indeed, spectrahedra may be considered "next natural successors" to polyhedra, as one moves beyond linear constraints in optimization theory."

Since then, spectrahedra has indeed become more than just the feasible region of SDP problems, and is a familiar term in branches of mathematics from algebraic geometry to control theory and optimization.

3.1 Defining a spectrahedron

There are two formal definitions of a spectrahedron, which are not "equivalent", but the sets they describe have a clear connection and the same properties in their respective spaces. The first and most commonly used, especially in the study of spectrahedra as geometric objects, is the following;

Definition 3.1.1. A set $S \subseteq \mathbb{R}^m$ is a **spectrahedron** if it has the form

$$S = \left\{ (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : A(x) = A_0 + \sum_{i=1}^{m} A_i x_i \succeq 0 \right\} \tag{3.1}$$

where $A_0, A_1, \ldots, A_m \in \mathcal{S}^n$.

We say that $A(x)$ is a **linear pencil** and that $A(x) \succeq 0$ is a **linear matrix inequality(LMI)**, both of **size** n. If $A_0 = I$ the linear pencil and the LMI are **monic**. For convenience the linear pencil is usually expressed as one single matrix with linear polynomials in all elements.

Example 3.1.2. Many well known convex sets in $\mathbb{R}^n$ are in fact spectrahedra, for example the unit ball. Let

$$S = \{\mathbf{x} \in \mathbb{R}^3 : I_4 + x_1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \succeq 0\}$$

By taking the Schur complement of the linear pencil above and using Proposition 2.5.6, we get

$$1-\begin{pmatrix}x_1 & x_2 & x_3\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=1-(x_1^2+x_2^2+x_3^2)$$

Our linear pencil is PSD if and only if its Schur complement is PSD, and for a 1×1-matrix, this is the same as being non-negative. Thus the linear pencil in S is PSD if and only if $x_1^2+x_2^2+x_3^2\leq 1$ which holds for the unit ball in $\mathbb{R}^3$. A similar argument can be used for the unit ball in any dimension.

Example 3.1.3. Another example of a spectrahedron is the *elliptope*, $\mathcal{E}_n$. It can be expressed by

$$\mathcal{E}_3=\{(x_1,x_2,x_3)\in\mathbb{R}^3:\begin{pmatrix}1&x_1&x_2\\x_1&1&x_3\\x_2&x_3&1\end{pmatrix}\succeq 0\}$$

The elliptope is also the set of all correlation matrices of size n. The elliptope was introduced as a *semidefinite relaxation* of the famous maximum cut problem, i.e., it is a spectrahedron containing all the solutions of the maximum cut problem. To learn more about the elliptope and maximum cut problem, see [LP95].

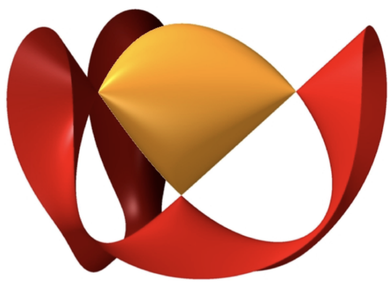

Figure 3.1: The zero set of the determinant of the linear pencil of $\mathcal{E}_3$. The elliptope is restricted to the set yellow part by the other principal minors of the linear pencil.

When the spectrahedron is the set of feasible solution in semidefinite optimization, the usual choice of definition is the following

Definition 3.1.4. A set $S'\subseteq\mathcal{S}^n$ is a **h-spectrahedron** if it has the form

$$S'=\left\{X\in\mathcal{S}^n:\langle D_j,X\rangle=d_j\ for\ j=1\ldots,k,\ X\succeq 0\right\}\tag{3.2}$$

where $D_j\in\mathcal{S}^n$ and $d_j\in\mathbb{R}$ for $j=1,\ldots,k$. Note that $S'\subseteq\mathcal{S}^n_+\subseteq\mathcal{S}^n$

The notation "h-spectrahedron" is a term used in this thesis to differ the usual definitions of spectrahedra. The h is a reference to the hyperplanes in $\mathcal{S}^n$ that defines the spectrahedron in Equation (3.2).

Example 3.1.5. An example of an h-spectrahedron is a subset $S' \subseteq \mathcal{S}^4_+$ such that the following equations holds for all $X \in S'$.

$$\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, X\rangle = 1, \qquad \langle \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, X\rangle = 1$$

$$\langle \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, X\rangle = 1, \qquad \langle \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, X\rangle = 1$$

$$\langle \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, X\rangle = 0, \qquad \langle \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, X\rangle = 0$$

$$\langle \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, X\rangle = 0,$$

In this example each equality put a constraint on either one element on the diagonal, or two equal elements off the diagonal. More often we see that the constraints create more complex relations between the elements of the variable matrix. For example, in eigenvalue optimization we find the constraint $Tr(X) = 1$ which equals $\langle J, X\rangle = 1$. This constraint includes all the elements of the variable matrix.

Connecting the definitions

At first glance these two definitions seem completely different, and actually they are not equivalent as we usually expect definitions of the "same" object to be. Nevertheless, [BPT12] claims they are *affinely equivalent* for linearly independent matrices in both definitions. In the reference text, the claim stands alone, so in this thesis we choose topresent a framework and provide a proof to support it.

Definition 3.1.6. Let E and F be two euclidean spaces. Then $P \subseteq E$ and $Q \subseteq F$ are **affinely equivalent** if there exist an affine map $\alpha : E \mapsto F$ such that α is bijective on its image and $\alpha|_P = Q$.

Lemma 3.1.7. *The set*

$$P = \{A_0 + \sum_{i=1}^{m} A_i x_i : (x_1, x_2 \ldots, x_m) \in \mathbb{R}^m\} \tag{3.3}$$

is an affine set of $\mathcal{S}^n$.

Proof. Let $X, Y \in P$, and $t \in \mathbb{R}$, then

$$\begin{aligned} tX + (1-t)Y &= t \cdot (A_0 + \sum_{i=1}^{m} A_i x_i) + (1-t)(A_0 + \sum_{i=1}^{m} A_i y_i) \\ &= tA_0 + (1-t)A_0 + \sum_{i=1}^{m} A_i(x_i + y_i) \\ &= A_0 + \sum_{i=1}^{m} A_i z_i \in P. \end{aligned}$$

By Definition 2.2.1, P is an affine set of $\mathcal{S}^n$. ■

Lemma 3.1.8. *Let P be as in Equation* (3.3) *and assume that $\{A_i\}_{i=0,1,\ldots,m}$ are linearly independent matrices. Then $\mathbb{R}^m$ is affinely equivalent to the set $P \subseteq \mathcal{S}^n$. Moreover,* $\dim_{\mathbb{R}} P = m$.

Proof. Let $\alpha : \mathbb{R}^m \to \mathcal{S}^n, x \mapsto A_0 + \sum_{i=1}^{m} A_i x_i$, and $\beta : \mathbb{R}^m \to \mathcal{S}^n, x \mapsto \sum_{i=1}^{m} A_i x_i$. Then β is a linear map because

$$\begin{aligned} \beta(\mu_1 x + \mu_2 y) &= \sum_{i=1}^{m} A_i(\mu_1 x_i + \mu_2 y_i) \\ &= \mu_1 \sum_{i=1}^{m} A_i x_i + \mu_2 \sum_{i=1}^{m} A_i y_i \\ &= \mu_1 \beta(x) + \mu_2 \beta(y). \end{aligned}$$

Furthermore $\beta(x-y) = \sum_{i=1}^{m} A_i x_i - \sum_{i=1}^{m} A_i y_i + A_0 - A_0 = \alpha(x) - \alpha(y)$, thus α is an affine map. Since the A_i's are linearly independent, α is injective, and hence bijective on its image. This shows that $\mathbb{R}^m$ is affinely equivalent to its image under α, which is P. The image of the map α can at most have the dimension of its domain, which is m. Moreover $\dim_{\mathbb{R}} \alpha(\mathbb{R}^m) + \dim_{\mathbb{R}} \ker \alpha = m$, so $dim_{\mathbb{R}} P = m$ if and only if α is bijective, which we have already proven. ■

It follows from Lemma 3.1.8 that the spectrahedra S from Definition 3.1.1 is affinely equivalent to the set $\{A_0 + \sum_{i=1}^{m} A_i x_i : x \in \mathbb{R}^m\} \cap \mathcal{S}_+^n \subset \mathcal{S}^n$.

Lemma 3.1.9. *The set*

$$Q = \{X \in \mathcal{S}_+^n : \langle B_j, X\rangle = b_j \ for \ j = 1 \ldots, k\} \tag{3.4}$$

is an affine set of $\mathcal{S}^n$.

Proof. Let $X, Y \in Q$, then

$$\begin{aligned} \langle B_j, \theta X + (1-\theta)Y\rangle &= \langle B_j, \theta X\rangle + \langle B_j, (1-\theta)Y\rangle \\ &= \theta\langle B_j, X\rangle + (1-\theta)\langle B_j, Y\rangle \\ &= \theta b_j + (1-\theta)b_j = b_j \end{aligned}$$

so $\theta X + (1-\theta)Y \in Q$, and Q is an affine set. ■

Lemma 3.1.10. *The sets P in Equation (3.3) and Q in Equation (3.4) are equal for a suitable choice of $\{A_i\}_{i=0,1,\ldots,m}$, $\{B_j\}_{j=1,\ldots,k}$ and $\{b_j\}_{j=1,\ldots,k}$.*

Proof. Let $P = \{A_0 + \sum_{i=1}^m A_i x_i : (x_1, x_2 \ldots, x_m) \in \mathbb{R}^m\}$, choose $\{B_j\}_{j=1,\ldots k}$ to be a basis of the orthogonal complement of $span\{A_1, \ldots, A_m\}$. Then $\langle B_j, A_i\rangle = 0$ for all j, i. Also let $b_j = \langle B_j, A_0\rangle$ for all j. Then for all $X \in P$ and for all $j = 1, \ldots, k$, $\langle B_j, X\rangle = \langle B_j, A_0\rangle + \sum_{i=0}^m x_i \langle B_j, A_i\rangle = b_j + \sum_{i=0}^m x_i \cdot 0 = b_j$, which shows that $X \in Q$.

Now let $X \in Q$. Then $\langle B_j, X\rangle = b_j$. As we have chosen $b_j = \langle B_j, A_0\rangle$, we get $\langle B_j, X\rangle = \langle B_j, A_0\rangle$, and $\langle B_j, X - A_0\rangle = 0$, and $X - A_0$ is the orthogonal to B_j for all j. Since $\{B_j\}_{j=1,\ldots k}$ is a basis of the orthogonal complement of $span\{A_1, \ldots, A_m\}$, $X - A_0 \in ((span\{A_1, \ldots, A_m\})^\perp)^\perp = span\{A_1, \ldots, A_m\}$. Thus $X - A_0 = \sum_{i=1}^m A_i x_i$, and $X \in P$. ■

The main theorem of this section connecting the definitions of spectrahedra and h-spectrahedra follows from the above lemmas.

Theorem 3.1.11. *A spectrahedra $S \subseteq \mathbb{R}^m$ on the form Equation (3.1) is affinely equivalent to an h-spectrahedra $S' \subseteq \mathcal{S}^n$ on the form Equation (3.2) where $span\{\{B_j\}_{j=1,\ldots,k}\}$ is the orthogonal complement of $span\{A_1, \ldots A_m\}$ and $b_j = \langle B_j, A_0\rangle$ for all j.*

Corollary 3.1.12. *If $S \subseteq \mathbb{R}^m$ is a spectrahedron defined by linearly independent matrices $A_1, \ldots, A_m$, then an affinely equivalent h-spectrahedron $S' \subset \mathcal{S}^n$ is defined by at least $\frac{n^2+n}{2} - m$ hyperplanes in $\mathcal{S}^n$.*

Proof. By Lemma 3.1.8, the dimension of a spectrahedron defined by linearly independent matrices $A_1, \ldots A_m$ has dimension m. The space $\mathcal{S}^n$ has $\dim_{\mathbb{R}} \mathcal{S}^n = \frac{n^2+n}{2}$. Thus the orthogonal complement of $span\{A_1, \ldots, A_m\}$ has dimension $\frac{n^2+n}{2} - m$, which shows that k must be at least $\frac{n^2+n}{2} - m$ for $span\{B_1, \ldots, B_k\}$ to form a basis of a subspace of the correct dimension. ■

With this theorem in mind, lets another look at Example 3.1.2 and Example 3.1.5 which we now refer to as S and S'. Any $X \in S'$ is on the form

$$X = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{1,2} & 1 & 0 & 0 \\ x_{1,3} & 0 & 1 & 0 \\ x_{1,4} & 0 & 0 & 1 \end{pmatrix}$$

which is exactly the linear pencil described in S. Furthermore, notice that the dimension of the affine space in S' is $\frac{4^2+4}{2} - 7 = 3$ as we are in the space $\mathcal{S}^4$ and there are 7 linearly independent matrices creating the equations. Also $S \subseteq \mathbb{R}^3$, so as it should, the dimension matches in these two spectrahedral representations of the unit ball.

3.2 Properties of spectrahedra

In this section, it is natural to start with a proposition describing the two fundamental properties of spectrahedra. Since these properties are commonly assumed without rigorous verification, we provide a formal proof in this thesis .

Proposition 3.2.1. *Let $S \subseteq \mathbb{R}^m$ be a non-empty spectrahedron. Then S is closed and convex.*

Proof. First we show that S is closed. For each point $y \in \mathbb{R}^m \backslash S = S^c$, we know that $z^T A(y) z < 0$ for at least one $z \in \mathbb{R}^m$. Let $k = z^T A(y) z$, then the ball $B(y, \epsilon)$ consist of the points $u = y + \epsilon r$ with $r \in \mathbb{R}^m$ such that $|r| = 1$. For the given $z \in \mathbb{R}^m$ we get

$$\begin{aligned} z^T A(y + \epsilon r) z &= z^T A_0 z + \sum_{i=1}^m y_i z^T A_i z + \epsilon \sum_{i=1}^m r_i z^T A_i z \\ &= z^T A(y) z + \epsilon \sum_{i=1}^m r_i z^T A_i z \\ &\leq k + \epsilon \sum_{i=1}^m 1 \cdot z^T A_i z \\ &= k + \epsilon \cdot M \end{aligned}$$

We see that for a sufficiently small ϵ, $B(y, \epsilon) \subset S^c$, hence S^c is open, and S is closed.

To show that S is convex, let $x, y \in S$, and $0 \leq \lambda \leq 0$. Then

$$\begin{aligned} A((1-\lambda)x + \lambda y) &= A_0 + \sum_{i=1}^m ((1-\lambda)x_i + \lambda y_i) A_i \\ &= (1-\lambda)A_0 + \lambda A_0 + (1-\lambda) \sum_{i=1}^m x_i A_i + \lambda \sum_{i=1}^m y_i A_i \\ &= (1-\lambda)A(x) + \lambda A(y) \end{aligned}$$

So if $z^T A(x) z \geq 0$ and $z^T A(y) z \geq 0$ for all $z \in \mathbb{R}^m$, then $z^T A((1-\lambda)x + \lambda y) z = (1-\lambda) z^T A(x) z + \lambda z^T A(y) z \geq 0$ for all $z \in \mathbb{R}^m$. ■

It will often be convenient to use a monic LMI representation of a spectrahedron, which we recall to have $A_0 = I$. In [BPT12] the folowing argument shows how each LMI can be represented by a *monic* LMI. This assures that every spectrahedra with non-empty interior can without loss of generality be represented by a monic LMI. The first step to see this is by assuming $A_0 \succ 0$. Then we know that for some non-singular matrix B, $A_0 = BB^T$. Proposition 2.5.4 implies that $A(x) \succeq 0$ if and only if $BA(x)B^T = I + x_1 B^{-1} A_1 B^{-T} + x_2 B^{-1} A_2 B^{-T} + \ldots + x_m B^{-1} A_m B^{-T} \succeq 0$. If $A_0 \not\succ 0$, then choose a point $x_0 \in S$ such that $A(x_0) \succ 0$, which we later in the chapter will see is exactly the interior points of S(Lemma 3.2.2). Let $S_t = \{x \in \mathbb{R}^m : A'(x) = A(x + x_0) \succeq 0\}$, be the translated spectrahedron. Now we get $A'_0 = A'(0) = A(0 + x_0) \succ 0$, so we can apply the congruence transformation described above.

Boundary and interior

The proof of this lemma is inspired by the proof of property 5 of spectrahedra in Tim Netzer's doctoral thesis "Spectrahedra and their shadows".

Lemma 3.2.2. *For all interior points of a spectrahedron , i. e. for all $x \in S^{\circ}$, we have $A(x) \succ 0$.*

Proof. Let be a $S \subseteq \mathbb{R}^m$ a full dimensional spectrahedron. Let $x_0 \in S^{\circ}$ and assume for contradiction that $A(x_0) \not\succ 0$, i.e., $\det A(x_0) = 0$ for the defining linear pencil $A(x)$ of S. Then there exist a vector $v \in \mathbb{R}^n$, where $n = \text{size}A(x)$, such that $v^T A(x_0)v = 0$, and consequently $v^T A_0 v = 0$. Now let $B = B(x_0, r)$ be an open ball of radius $r > 0$ around x_0 such that for every $x \in B$, $A(x) \succeq 0$. This implies that for every $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$ such that $|\epsilon| < r$, we have

$$A_0 + \sum_{i=1}^{m} \epsilon_i A_i \succeq 0.$$

Then for the $v \in \mathbb{R}^m$ above, $v^T(A_0 + \sum_{i=1}^m \epsilon_i A_i)v = v^T(\sum_{i=1}^m \epsilon_i A_i)v \geq 0$. This holds for all $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$ such that $|\epsilon| < r$, where each ϵ_i is not necessarily positive, so $v^T A_i v = 0$ for all A_i. Thus for all $x \in \mathbb{R}^m$, it holds that $v^T(A_0 + \sum_{i=1}^m x_i A_i)v = 0$, which implies $S = \mathbb{R}^m$, which is a contradiction. Thus $\det A(x_0) \neq 0$, and $A(x_0) \succ 0$ for $x_0 \in S^{\circ}$. ■

The following theorem is based on another claim, found in both [Ott+15] and [BPT12], where a proof seem to be excluded in most texts about spectrahedra.

Theorem 3.2.3. *Assume $S \subset \mathbb{R}^m$ is a non-empty spectrahedron with defining linear pencil $A(x)$. Then the boundary of S is defined by the equation $det(A(x)) = 0$. i.e., if $x \in S$, then $x \in \partial S$ if and only if $\det A(x) = 0$.*

Proof. From Lemma 3.2.2 it follows that if If $x \in \mathcal{S}$ and $\det A(x) = 0$, then $x \in \partial S$. Assume that $x \in \partial S$, then each neighbourhood containing x includes a point $x_1 \in \mathbb{R}^m \backslash S$ and a point $x_2 \in S$. In the point x_1, at least one eigenvalue of the linear pencil, $\lambda_i(A(x_1)$ is negative. For x_2 all $\lambda_i(A(x_2)) \geq 0$ for all i. By continuity of the eigenvalue, there exist a point y on the line segment between x_1 and x_2 such that $\lambda_i(A(y)) = 0$. As this holds for all neighbourhoods of x we get that $y = x$. Furthermore, $\lambda_i(A(x)) = 0$ for some i, if and only if $det(A(x)) = 0$, which concludes the proof. ■

Proposition 3.2.4. *For a point $x^* \in \partial S$, the nullity of $A(x^*)$ equals the root multiplicity of $\det A(x)$ in the point x^*.*

Proof. Let $x^* \in \partial S$, then $A^* := A(x^*)$ is a symmetric positive semidefinite matrix, with non trivial null space. From the spectral theorem for real symmetric matrices Theorem 2.5.1, $A^* = U^T D^* U$ where D_0 is diagonal with the eigenvalues of A_0 and the columns of U are the corresponding eigenvalues, which also form an orthonormal basis for R^m. Consequently there is one unique eigenvector of A^* for each diagonal element of D^* which is equal to zero. Thus the nullity of A^* equals the eigenvalue multiplicity of $\lambda = 0$. Furthermore, the characteristic polynomial of A^* can be expressed as $p_{A^*}(\lambda) = f(\lambda) - \det A^*$ for some univariate polynomial f. Thus the eigenvalue multiplicity of $\lambda = 0$ of A^* equals the root multiplicity of $\det A(x)$ in the point x^* which concludes the proof. ■

Example 3.2.5. The determinant of the linear pencil defines the boundary of a spectrahedron, however, non-negativity of the determinant is far from sufficient for describing a spectrahedron. The determinant is simply one of the set of principal minors of the linear pencil, and to see the importance of all the principal minors, we consider two spectrahedra in $\mathbb{R}^2$. Let

$$S_1 = \left\{ (x, y) \in \mathbb{R}^2 : \begin{pmatrix} x+2 & 1 \\ 1 & y+1 \end{pmatrix} \succeq 0 \right\},$$

depicted in Figure 3.2.

The principal minors of S_1 are $\{\det A_1 = xy + x + 2y + 1, x + 2, y + 1\}$. The determinant is non-negative for two separate convex subsets of the plane, but the two principal minors is only be non-negative for one of these sets. In this example, the principal minors are strictly positive on the boundary of S_1.
Let

$$S_2 = \left\{ (x, y) \in \mathbb{R}^2 : \begin{pmatrix} x+y-1 & 0 & 0 \\ 0 & x+y-1 & 0 \\ 0 & 0 & y-x+1 \end{pmatrix} \succeq 0 \right\},$$

depicted in Figure 3.3. The principal minors of S_2 are $\{\det A_2 = (x + y - 1)^2(y - x + 1), (x + y - 1)(y - x + 1), x + y - 1, y - x + 1\}$. In this case, the determinant is non-negative for all (x, y) such that $(y - x + 1) \geq 0$, but it is zero on a line "cutting through" the middle of this set. It is the lower degree principal minors that ensures that the boundary of the spectrahedra is defined by the zero set of the determinant.

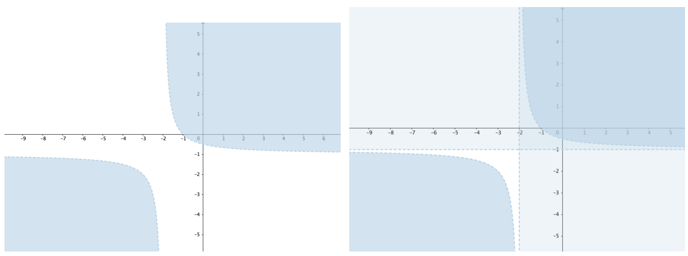

Figure 3.2: The non-negative set of the determinant on the left, and of all the principal minors, for which the intersection is S_1.

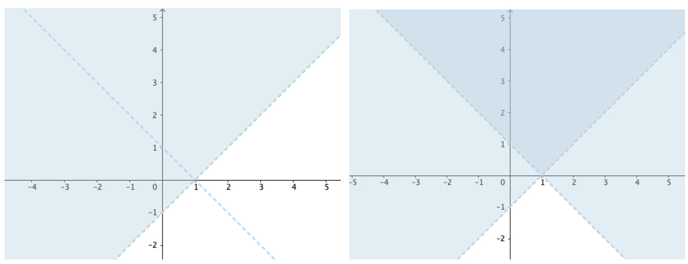

Figure 3.3: The non-negative set of the determinant on the left, and of all the principal minors, for which the intersection is S_2.

What about spectrahedra which are not full dimensional, i.e., spectrahedra with empty interior?

Example 3.2.6. Recall the two-dimensional unit disc in $\mathbb{R}^3$Example 2.2.2. This is a spectrahedron defined by the LMI

$$\left\{(x,y,z)\in\mathbb{R}^3 : \begin{pmatrix} 1 & x & y & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ y & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & -z \end{pmatrix} \succeq 0\right\}.$$

The linear pencil is PSD if and only if both the first 3×3 block matrix, B_1, and the last 2×2 block matrix, B_2, are PSD. We recognize B_1 to be the unit disc in $\mathbb{R}^2$ by taking the Schur complement, and with no restrictions on z, it forms a sylinder. B_2 simply ensures that $z=0$. As a consequence S is a spectrahedron with *empty* interior.

If we take a look at the determinant of this spectrahedron we see that $\det A(x,y,z) = \det B_1(x,y,z)\cdot \det B_2(x,y,z) = -z^2\cdot(1-x^2-y^2)\le 0$ for all $x\in S$. Thus we have that $\det A(x,y,z)=0$ for all $x\in S$ so $S=\partial S$. The determinant does still define the boundary of the spectrahedron, but since it is a convex set which is not full dimensional, the boundary is all of the set.

Minimal description LMI

Definition 3.2.7. The LMI $A(x)\succeq 0$ is a **minimal description** of a spectrahedron S with non-empty interior if $A(x)$ has the smallest possible size to describe the spectrahedron, i.e., if there does not exist $A'(x)$ with size $A' = n' < n =$ size A such that $S=\{x\in\mathbb{R}^m : A'(x)\succeq 0\}$.

Lemma 3.2.8. *[HV03] For each polynomial $p(x) = p_1(x)^{r_1}p_2(x)^{r_2}p_3(x)^{r_3}\dots p_k(x)^{r_k}$ where $\{p_i\}$ are irreducible polynomials over $\mathbb{R}$, the connected subset of $B=\{x\in\mathbb{R}^m : p(x)\neq 0\}$ are exactly the connected subsets of $B_0=\{x\in\mathbb{R}^m : p_0(x)=p_1(x)p_2(x)p_3(x)\dots p_k(x)\neq 0\}$.*

We have seen that the boundary of a spectrahedron is defined by the equation $\det A(x)=0$. The determinant is a polynomial, hence we get from Lemma 3.2.8 that the boundary of a spectrahedron can be expressed as a square free product of irreducible polynomials. We refer to this as the **minimal defining polynomial of S**. Furthermore, $\det A(x)>0$ on the interior of a spectrahedron, so without loss of generality we can assume all factors in any factorization of a defining polynomial are also positive on the interior of the spectrahedron S. In Chapter 4, we will return to the concept of defining polynomials, and see for which polynomials we can "create" spectrahedra.

Lemma 3.2.9. *For a spectrahedron $S\subseteq\mathbb{R}^m$, and any describing LMI, $A(x)\succeq 0$,*

$$deg(p(x)) \le deg(\det A(x)) \le n_A$$

where $p(x)$ is the minimal defining polynomial of S, and n_A is the size of $A(x)$.

Proof. The first inequality is due to the determinant of any describing LMI also being a defining polynomial. The second is clear, as $A(x)$ only includes linear terms, and the determinant is the sum of products of n terms from $A(x)$. ∎

3.3 Faces and façades of spectrahedra

Faces

Recall that a face $F \subseteq C$ of a convex set, is a subset such that whenever the open line segment between two points $x_1, x_2 \in C$ lies in F, the endpoints x_1 and x_2 also belongs to F.

Proposition 3.3.1. *[RG95] Every face of a spectrahedron is also an exposed face. In other words, the two notions coincide for spectrahedra.*

For spectrahedra of higher dimension than 2, there are no complete characterization of which convex sets are spectrahedra. For this reason, results like this can be useful in deciding whether or not a given set is a spectrahedron. An example is the set in Example 2.3.10. We saw that there are points of the boundary which are faces, but not exposed faces. Hence the set is *not* a spectrahedron and can not be represented by an LMI.

Façades

From the definition of faces of convex sets, faces include line segments, which colloquially means they are "flat", or they are single points. The polynomial inequalities defining a spectrahedron indicates that the boundary is mostly curved, not flat, thus most of the boundary are single point faces. Nevertheless, there are several interesting properties of the parts of the boundary which are the curved analogues to faces. To describe these, we introduce the new concept of *façades*. Before giving the definition, we establish some essential theory.

Given two points $x, y \in \mathbb{R}^m$, a **path** between them is is a continuous function $f : [a, b] \to \mathbb{R}^m$ such that $f(a) = x$ and $f(b) = y$. Two points are in the same **path component** of $S \subseteq \mathbb{R}^m$ if and only if there exist a path in S between them. It is worth noting that all convex sets in $\mathbb{R}^m$ are path connected, since the convexity property defines a path and holds for all pair of points in the convex set.

Proposition 3.3.2. *The boundary of a line-free spectrahedron S is path connected.*

Proof. Let $x, y \in \partial S$. Let H_x, H_y be the supporting hyperplanes of S in x, y respectively. First consider the case where $H_x \cap H_y \neq \emptyset$. As hyperplanes are convex, they are path connected, and there is a path p from x to y in $H_x \cup H_y$. As supporting hyperplanes only intersect the convex set in boundary points, every point on the path is either in ∂S or in S^c. Spectrahedra are closed and convex, so the path in $H_x \cup H_y$ projects continuously onto S with the projection π_C defined in Definition 2.4.6. The projection of the path onto S is thus a path in ∂S form x to y.

Consider now the case where $H_x \cap H_y = \emptyset$, i.e., the supporting hyperplanes are parallel, moreover, we can assume as well that $H_{x'} \cap H_{y'} = \emptyset$ for every x' in the path component of x and every y' in the path component of y'. Choose a point x_0 in the interior of S, and define the line $x_0 + \lambda v$ such that $\lambda \in \mathbb{R}$ and

v is a vector parallel to the the hyperplanes H_x and H_y. Then, since S is line free, there exist a point $z \in x_0 + \lambda v$ such that $z \in \partial S$. Consequently there is a supporting hyperplane of S in z, which is not parallel to H_x and H_y, which implies that $H_z \cap H_x \neq \emptyset$ and $H_z \cap H_y \neq \emptyset$ so we can define a path between x and y in ∂S as in the first case. ■

Lemma 3.3.3. *Let S be a full dimensional spectrahedron. Assume ∂S have more than one path component. Then the path components of ∂S are parallel hyperplanes, and there are exactly two path components.*

Proof. Assume P_1, P_2 are two path components of ∂S, $x_1 \in P_1$, and $x_2 \in P_2$, and let H_i be the supporting hyperplane of S in x_i for $i = 1, 2$. Then H_1 is parallel to H_2, for if not, we could define a path as in the proof of Proposition 3.3.2. Since this holds for every pair of points $x_1 \in P_1$ and $x_2 \in P_2$, the path components are parallel affine sets. The spectrahedron is full dimensional, so the boundary is an affine set of dimension $m - 1$. Moreover, since ∂S is not path connected, there exist a line in the interior of S, so S is unbounded. Since S is closed, the boundary is also unbounded, and hence consists of hyperplanes. ■

Faces of convex sets are contained in affine subsets of $\mathbb{R}^m$ and since this is a translated linear space, there is also an orthogonal complement. By examining several spectrahedra, it becomes apparent that they are somewhat divided in separate curved parts, and that there is some sort of orthogonal complement which is constant for each part. From the definition of vertices of convex set, we see how normal cones can be used to describe such "orthogonal complement" which motivated the use of normal cones in the definition of façades. Moreover, faces of convex sets are defined by an open line segment, which is a type of path. The endpoints of the open line segments are by definition always in the face, and for façades the endpoints of every path is in the façade because the closure of each path is included.

Definition 3.3.4. Let C be a closed convex set in a euclidean space, then an **r-façade** is the closure of a path component of the set $\{x \in \partial C : \dim N_C(x) = \dim C - r\}$.

We let F^r be the path component we have before taking the closure. The set of all r-façades for all $r = 1, 2, \dots, \dim C - 1$ are the **façades** of C, and if r is unspecified or without importance we might say only façade. The 0-façades are defined as the same points on the boundary as the vertices, and as this is a more common term, we will use *vertex/vertices.* For every subset S of a real topological space, we can define the subspace topology by letting a set be open in the subspace topology if and only if it is the intersection of S and a open set in the real space. Thus we can define a topology on the boundary of a spectrahedron.

A façades is defined as the closure of some subset of ∂S, which can be open or closed, i.e, F^r is not in general open or closed for a façade F. If it is closed, we have information about all points in the façade, however if it is open, there are points $x \in F \setminus F^r$ which we know less about. Upon further study, we attain the result that all these points have normal cone in S of strictly larger dimension than $m - r$, i.e., strictly larger than for the points in the path component F^r. To prove this, we need some lemmas.

Lemma 3.3.5. *Let $M \subseteq \mathbb{R}^3$ be a closed set defined by all points $(x, y, z) \in \mathbb{R}^3$ such that $y \geq 0$, and $z \geq f(x, y)$. The function $f(x, y)$ is given by*

$$f(x,y) = \begin{cases} 0 & x \leq 0 \\ a(y)x & x > 0 \end{cases} \tag{3.5}$$

Here $a(y) > 0$ for $y > 0$ and $a(y) = 0$ for $y = 0$. Then M is not convex.

Proof. Consider the points $(0, 1, 0), (1, 0, 0)$. Then for $(0, 1, 0)$, $x = 0$, and $z = 0$, which is equal $f(0, y)$, and $y = 0$, so the point $(0, 1, 0) \in M$. For the point $(1, 0, 0)$, $x > 0$, but $y = 0$, so $f(x, y) = 0 \cdot x = 0 = z$. Thus $(1, 0, 0) \in M$. Now consider the point

$$\frac{1}{2}(0,1,0) + \frac{1}{2}(1,0,0) = (\frac{1}{2}, \frac{1}{2}, 0),$$

i.e., a point on the line segment between the two points in M. In this point $f(x, y) = f(\frac{1}{2}, \frac{1}{2}) = a(\frac{1}{2}) \cdot \frac{1}{2} > 0$. Since $z = 0$, $f(x, y) > z$, so this point is not in M, and M is not convex. ∎

If we expand M and consider the set $M' = M \cup \{(x, y, z) \in \mathbb{R}^3 : z \geq 0 \text{ and } y < 0\}$, the only vector in the normal cone of M' in $(0, 0, 0)$ is $(0, 0, z)$. For all $(0, y, 0)$ and $y > 0$, the normal cone of M' have two directions, as the line $(0, y, 0)$ forms a "ridge" in M. The boundary of M' can be modelled by using a sheet of paper. Make a fold on one edge of the sheet, but keep the opposite side straight. It is not an accurate representation, but it looks somewhat similar.

Lemma 3.3.6. *Let $C \subseteq \mathbb{R}^m$ be a closed convex set. Let F be an r-façade of C. Then for each $x \in F \setminus F^r$, and each sequence $\{x_n\} \subseteq F^r$ such that $\lim_{n\to\infty} x_n = x$, we have*

$$\dim N_C(x_n) \leq \dim N_S(x).$$

Note that the lemma does not address the existence of an $x \in F \setminus F^r$.

Proof. Let x be as given, and assume $\{x_n\}$ is contained in a small neighborhood of x. Assume $\dim N_S(x) > \dim N_S(x_n)$ for some, and thus for all, x_n in the sequence. Then there exist two normal vectors of each point x_n that converge to one normal vector in the point x. We can choose the neighborhood to be sufficiently small to assume linearity, and furthermore, without loss of generality, restrict C to $\mathbb{R}^3$. Then two of the normal vectors of C collapsing to one in x defines a set like M in Lemma 3.3.5. Thus C is not convex, which is a contradiction. Hence, $\dim N_S(x_n) \leq \dim N_S(x)$. ∎

Proposition 3.3.7. *Let $S \subseteq \mathbb{R}^m$ be a spectrahedron, and F an r-façade of S, then for each $x \in F \setminus F^r$,*

$$\dim N_S(x) > m - r.$$

Proof. Spectrahedra are closed convex sets, so by Lemma 3.3.6 we know that for each $x \in F \setminus F^r$, $\dim N_S(x) \geq m - r$, since $\dim N_S(x_n) = m - r$ for every element of $\{x_n\} \subseteq F^r$ by definition.

If x is in the closure of a path component of ∂S, then x is in the same path component. This is a consequence of Lemma 3.3.3, as each path component of ∂S is closed, and must include the closure of all open sets contained in the path component. This implies that x is path connected to F^r in ∂S.

Let $\{x_n\} \in F^r$ such that $\lim_{n\to\infty} x_n = x$ and each x_n is path connected to x_{n+1} and we can define paths $\gamma_n : [t_n, t_{n+1}) \to F$ such that $\gamma_n(t_n) = x_n$ and $\gamma_n(t_{n+1}) = x_{n+1}$ for $n \in \mathbb{N}$. Furthermore, let $t_0 = 0$ and all $t_n < 1$. Now we can easily define a path γ by concatenating the paths. Then $\gamma : [0, 1) \to F$, and since $\{x_n\} \to x$ we can let $\gamma(1) = x$. Thus x is path connected to F^r in F, and must have $\dim N_S(x) \neq m - r$, as it is in $F \setminus F^r$. ∎

Though it is an abuse of topological terms, we will refer to F^r as the interior of the façade, and the points $x \in F \setminus F^r$ as the boundary points of the façade. It is clear that it is an abuse of terms if one consider a vertex. The "interior" of the façade is the point, and there are no "boundary" points. This is in contrast to one-point-sets being closed in $\mathbb{R}^m$ with the usual topology, which is the one we use.

Proposition 3.3.8. *Let $\{p_1, p_2, \ldots, p_k\}$ be the distinct irreducible factor of the determinant of the linear pencil $A(x)$ of a full dimensional spectrahedron $S \subseteq \mathbb{R}^m$. Then each $(m-1)$-façade is contained in the set $\{x \in \partial S : p_i(x) = 0\}$ for exactly one i. We say that this is the **corresponding irreducible factor** of the façade.*

Proof. Note that we can without loss of generality assume that all factors $p_i(x) \geq 0$ for all $x \in S$.

Let F be an $(m-1)$-façade of S, and $x_0 \in F$ such that x_0 is in the zero set of exactly one irreducible factor, which we denote p_0. Thus we have $p_0(x_0) = 0$, and $p_i(x_0) > 0$ for all $p_i \neq p_0$. Let $x_1 \in \{x \in \partial S : p_0(x) \neq 0\}$, so $p_0(x_1) > 0$, and there is some other factor p^*, such that $p^*(x_1) = 0$. Then for each path in ∂S from x_0 to x_1, there is either at least one point z such that $p_0(z) = p^*(z) = 0$, or there are points $z_1, z_2, \ldots, z_l$ and irreducible factors $\{p_i\}_{i=1,\ldots,l-1}$ of $\det A(x)$ distinct from p_0, such that $p_i(z_i) = p_{i-1}(z_i) = 0$ for $i = 0, 1, \ldots, l$ and $p^*(z_l) = 0$. First consider the case where for each path from x_0 to x_1, there exist a z such that $p_0(z) = 0$ and $p^*(z) = 0$. If $\nabla p_0(z) \neq k \cdot \nabla p^*(z)$, then since both gradients are contained in the normal cone of S in z, $\dim N_S(z) > 1$, and x_0, x_1 are not on the same façade.

If we now assume $\nabla p_0(z) = k \cdot \nabla p^*(z)$, the gradients span a one dimensional vector space, and the normal cone is one dimensional. Define the polynomial $p(x) = p_0(x) - p^*(x)$. Then, using the Taylor polynomial of degree 2, $p(x) = p(z) + (x-z)^T \nabla p(z) + \frac{1}{2}(x-z)^T \nabla^2 p(z)(x-z)$ where ∇^2 is the hessian matrix. If $p_0(z) = p^*(z) = 0$ and $\nabla p_0(z) = \nabla p^*(z)$ then $\nabla p(z) = 0$ then $p(x) = \frac{1}{2}(x-z)^T \nabla^2 p(z)(x-z)$. We recognize this to be a quadratic form, so $p(x)$ is either positive for all x in a sufficiently small neighborhood of z, or negative for all such x. Since $p = p_0 - p^*$, we have either that $p_0(x) > p^*(x)$ or

$p_0(x) < p^*(x)$ for all x in a neighborhood of z. Thus if z is the only point on the path between x_0 and x_1 such that both $p_0(x) = 0$ and $p^*(x) = 0$, then either $p_0(x_0) > p^*(x_0)$ and $p_0(x_1) > p^*(x_1)$ or $p_0(x_0) < p^*(x_0)$ and $p_0(x_1) < p^*(x_1)$. Therefore, only one of p_0 and p^* can define the boundary on any path from x_0 to x_1, which is a contradiction.

Now consider the case where there are points $z_1, z_2, \ldots, z_l$ and irreducible factors $\{p_i\}_{i=1,\ldots,l-1}$ of $\det A(x)$ distinct from p_0, such that $p_i(z_i) = p_{i-1}(z_i) = 0$ for $i = 0, 1, \ldots, l$ and $p^*(z_l) = 0$. Let $p(x) = \prod_{i=1}^{l-1} p_i(x)$. Then $p(x_1) = 0$ and $p(x_0) > 0$, and $p_0(z_1) = p(z_1) = 0$. Thus we can apply the argument from the first case. This shows that if $x_1 \in \partial S$ with $p_0(x_1) > 0$, then $x_1 \notin F$, which implies that for every $x \in F$, it holds that $p_0(x) = 0$. ■

Example 3.3.9. Let

$$S = \left\{ (x, y) \in R^2 : \begin{pmatrix} 1 & x & y & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ y & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} - x & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} + x \end{pmatrix} \succeq 0 \right\}$$

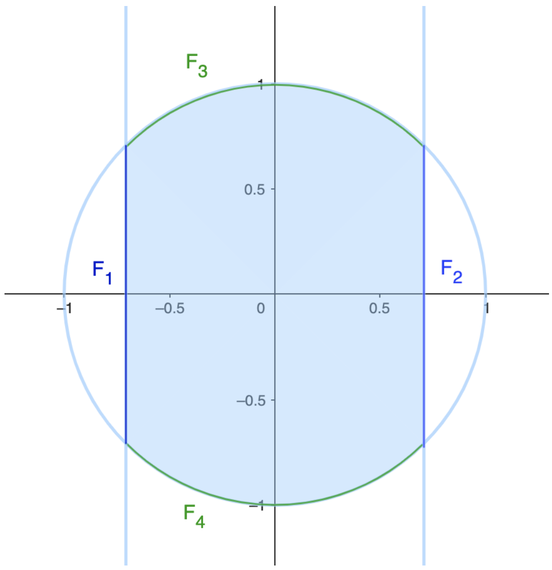

Figure 3.4: The figure shows the zero set of the determinant in Example 3.3.9 and the 1-façades of the S.

We recognize the first 3×3 block to be the unit disc in $\mathbb{R}^2$, and two next 1×1 blocks to define the halfspaces $x \geq -\frac{\sqrt{2}}{2}$ and $x \leq \frac{\sqrt{2}}{2}$. The intersection of these sets give the set in Figure 3.4, with minimal defining polynomial $p(x, y) = (1 - x^2 - y^2)(\frac{\sqrt{2}}{2} - x)(\frac{\sqrt{2}}{2} + x)$. From the figure it is easy to see that all the points $\{(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})\}$ are vertices and have two-dimensional normal cones.

Consider now the 1-façades $F_1,\ F_2,\ F_3,\ F_4$. We see that $F_1 \subset \{(x, y) \in \mathbb{R}^2 : \frac{\sqrt{2}}{2} + x = 0\}$, and $F_2 \subset \{x \in \mathbb{R}^2 : \frac{\sqrt{2}}{2} - x = 0\}$. Thus for F_1, F_2 they are the unique $(m-1)$-façade contained in the zero set of the respective corresponding zero set.

F_3 and F_4 are are both contained in $\{(x, y) \in \mathbb{R}^2 : 1 - x^2 - y^2 = 0\}$. This shows that even though each $(m-1)$-façade is contained in the zero set of exactly one irreducible factor of the minimal defining polynomial, the zero sets may contain more than one $(m-1)$-façade.

From Proposition 3.3.7 and Proposition 3.3.8 we can deduce some properties of façades similar to some of the ones we have for faces of convex sets in Proposition 2.3.13. The first property of faces in the proposition is that every faces is contained in a supporting hyperplane, for façades this holds for a hypersurface i.e., the zero set of some polynomial. The second property is that faces are closed, which façades are by definition.

Corollary 3.3.10. *If F_1, F_2 are respectively r_1- and r_2-façades of a spectrahedron S, and $F_1 \subseteq F_2$, then $r_1 \leq r_2$.*

Proof. If $F_1^{r_1} \cap F_2^{r_2} \neq \emptyset$, then $r_1 = r_2$ and $F_1^{r_1}$ is path connected to $F_2^{r_2}$ so $F_1 = F_2$. So if $F_1 \subsetneq F_2$, the path components are not path connected with the same r, and $F_1 \subseteq F_2 \setminus F_2^{r_2}$. Proposition 3.3.7 implies that $m - r_2 < m - r_1$, so $r_1 < r_2$. ∎

Example 3.3.11. Consider again the elliptope given in Example 3.1.3 and depicted in Figure 3.1. The determinant of the linear pencil is the irreducible polynomial $1 - x^2 - y^2 - z^2 + 2xyz$, thus the whole boundary is contained in one zero set, and defined one $m-1$-façade. Nevertheless, there are 4 vertices. So even though the intersection of $m-1$-façades define smaller façades, they can also occur without being an intersection of other façades.

Proposition 3.3.12. *Let $S \subseteq \mathbb{R}^m$ be a full-dimensional spectrahedron, then the intersection of k distinct $(m-1)$-façades is an $(m-k)$-façade.*

Proof. Let $F_1, F_2, \ldots, F_k$ be $(m-1)$-façades of S. Then for each F_i there is an irreducible factor p_i of the determinant of the linear pencil of S such that F_i is contained in the zero set of p_i. Moreover, the gradient of p_i is in the normal cone of S at x for all $x \in F_i$. Then if $x \in \cap_{i=1}^k F_i$, we have that $\pm\nabla f_1(x), \ldots, \pm\nabla f_k(x) \in N_S(x)$, where $\pm$ refers to either $+$ or $-$. Since the façades are distinct, the irreducible polynomials are distinct. Moreover the façades can not be parallel in x as only the "inner" zero set would be a façade in that case. Thus all gradients are distinct, and the $\dim N_S(x) = k$. As this holds for all $x \in \cap_{i=1}^k F_i$, the dimension of the normal cone equals m minus the number of intersecting $m-1$-façades. ∎

Example 3.3.13. Consider the spectrahedron

$$S = \{(x, y, z) \in \mathbb{R}^3 : \begin{pmatrix} 1 & x & y & z \\ x & 1 & x & y \\ y & x & 1 & x \\ z & y & x & 1 \end{pmatrix} \succeq 0\}$$

The determinant of the linear pencil is

$$\det A(x,y,z) = (x^2+2xy-xz-x+y^2-z-1)(x^2-2xy-xz+x+y^2+z-1).$$

The two factors, which we denote $f_1(x,y,z), f_2(x,y,z)$ are irreducible, and define the two 2-façades of the spectrahedron. For each point on the boundary which is only on one of these, one of the factors have a simple zero, i.e., $f_i(x,y,z) = 0$ and $\nabla f_i(x,y,z) \neq (0,0,0)$.

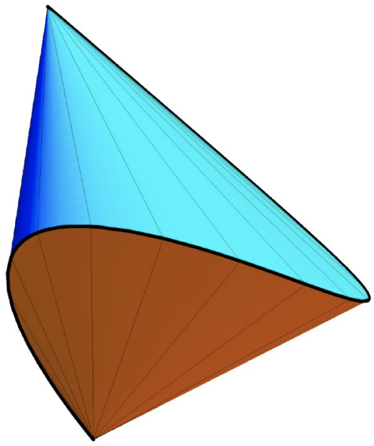

Figure 3.5: The spectrahedron S, known as the Toeplitz spectrahedron. On the "backside" of the spectrahedron, there is a line segment between the two vertices separating the blue and orange surfaces.

The intersection of the two 2-façades form two 1-façades, where one of them is also a face. The interior of the 1-façades are exactly the points where $f_1 = f_2 = 0$ and $\nabla f_i(x,y,z) \neq (0,0,0)$ for $i = 1,2$. The gradients of the two irreducible factors are $\nabla f_1(x,y,z) = (2x+2y-z-1, 2x+2y, -x-1), \nabla f_2(x,y,z) = (2x-2y-z+1, 2x+2y, -x+1)$. The interior of the 1-façade which is also a face is is the open line segment $(1-2\lambda, 1, 1-2\lambda)$ for $0 < \lambda < 1$. We check that it has the properties we have claimed.

$$\begin{aligned}
&f_1(1-2\lambda, 1, 1-2\lambda)\\
&= (1-2\lambda)^2 + 2(1-2\lambda) - (1-2\lambda)^2 - (1-2\lambda) + 1^2 - (1-2\lambda) - 1\\
&= 0
\end{aligned}$$

$$\begin{aligned}
&f_2(1-2\lambda, 1, 1-2\lambda)\\
&= (1-2\lambda)^2 - 2(1-2\lambda) - (1-2\lambda)^2 + (1-2\lambda) + 1^2 + (1-2\lambda) - 1\\
&= 0
\end{aligned}$$

$$\begin{aligned}
&\nabla f_1(1-2\lambda, 1, 1-2\lambda)\\
&= (2(1-2\lambda) + 2 - (1-2\lambda) - 1, 2(1-2\lambda) + 2, -(1-2\lambda) - 1)
\end{aligned}$$

$$\neq (0,0,0)$$

$$\begin{aligned}
&\nabla f_2(1-2\lambda, 1, 1-2\lambda) \\
&= (2(1-2\lambda) - 2 - (1-2\lambda) + 1, 2(1-2\lambda) + 2, -(1-2\lambda) + 1) \\
&\neq (0,0,0)
\end{aligned}$$

If we allow λ to be 1, then $\nabla f_1 = (0,0,0)$, and if $\lambda = 0$, then $\nabla f_2 = (0,0,0)$. Thus the points $(1,1,1), (-1,1,-1)$ are zeros of both f_1, f_2, and a zero of one of the gradients, i.e., they have normal cone of dimension 3, and are vertices of the spectrahedron.

Façades are strongly related to the faces of a convex set, and for many objects there are subsets of the boundary that are both façades and faces. For example we will see that for polyhedra, all faces and façades coincide. For other objects, the sets of façades and faces are disjoint. For example are all the points on the boundary of a closed ball in $\mathbb{R}^m$ faces of dimension 0, but the whole boundary is one $(m-1)$-façade. So none of the faces of the ball are façades, and the façade is not a face. Yet, there is one clear connection between the two notions, which is the following.

Proposition 3.3.14. *Let $C \subseteq \mathbb{R}^m$ be a closed convex set. Then every (proper) face of C is contained in a façade of C.*

Proof. Let $x \in \operatorname{rint}(F)$ for some face $F \subseteq C$, and let $z \in N_C(x)$, i.e., in the normal cone of C at x. Moreover, since F is a subset of C, $N_C(x) \subseteq N_F(x)$. Since x is in the relative interior of F, each point in F can be written as $x + \lambda l$ where $l \in L$, the associated linear space of aff F, and $\lambda \in \mathbb{R}$ is sufficiently small. Thus $z \in N_F(x)$ implies that

$$\begin{aligned}
\langle z, x\rangle &\geq \langle z, x + \lambda l\rangle \\
0 &\geq \langle z, \lambda l\rangle \\
0 &\geq \lambda\langle z, l\rangle
\end{aligned}$$

for all $l \in \operatorname{aff} F - x$ and some sufficiently small $\lambda \in \mathbb{R}$. Assume for contradiction that $z \in L$, then for each λ there exist some $l \in L$ such that $0 < \lambda\langle z, l\rangle$. Thus $z \in L^\perp$, for all choices of λ. If $z \in L^\perp$, then $\langle z, x - y\rangle = 0$ for all $x, y \in \operatorname{aff} F$, and $N_F(x) = L^\perp$. Moreover, as the inclusions of z does not depend on λ, so $N_{\operatorname{aff} F}(x) = L^\perp$.

If there exist a $z \in N_F(x) \backslash N_C(x) = L^\perp \backslash N_C(x)$, then $\langle z, x\rangle < \langle z, y\rangle$ for some $y \in C \backslash F$. In this case, $\langle -z, x\rangle > \langle -z, y\rangle$, and consequently $\langle -\lambda z, x\rangle \geq \langle -\lambda z, y\rangle$ for all $\lambda \geq 0$. Thus for each line in $L^\perp$, there is a halfline with the same direction in $N_C(x)$. This implies that $\operatorname{aff} N_{\operatorname{aff} F}(x) = \operatorname{aff} N_C(x)$ and the normal cones share the same dimension. Since $x \in \operatorname{rint}(F)$ is arbitrary, we have that $\operatorname{rint}(F) \subseteq \{x \in \partial C : \dim N_C(x) = \dim C - r\}$, for some r. In addition, the relative interior of a face is path connected by the definition of relative interior. By taking the closure, we get that the face F is included in a façade. ■

Lemma 3.3.15. *A $(m-1)$-façade is a face if and only if the corresponding irreducible factor is linear. Moreover, if an $(m-1)$-façade is a face, it has dimension $m-1$.*

Proof. Let F_i be the façade corresponding to the factor $f_i(x)$ of the $\det A(x)$, where $A(x)$ is the defining linear pencil of some spectrahedra $S \subseteq \mathbb{R}^m$.
Assume first that F_i is a face, then there exist a hyperplane H such that $H \cap S = F_i$. Hyperplanes in $\mathbb{R}^m$ are given by the zero set of some linear polynomial in m variables, for H we denote this polynomial $h(x)$. Then $H \cap S = \{x \in \mathbb{R}^m : h(x) = 0\} \cap \{x \in S : \det A(x) = 0\}$ and $F_i \subseteq \{x \in \mathbb{R}^m : f_i(x) = 0\} \cap \{x \in S : \det A(x) = 0\}$. This shows that $\{x \in \mathbb{R}^m : h(x) = 0\} \subseteq \{x \in \mathbb{R}^m : f_i(x) = 0\}$, and consequently $h(x)$ is a factor of $f_i(x)$, but since $f_i(x)$ is an irreducible polynomial, they are equal.

For the converse statement, let $f_i(x)$ be linear. Then $F_i \subseteq \{x \in \mathbb{R}^m : f_i(x) = 0\} \cap \{x \in S : \det A(x) = 0\}$ which is the intersection between a hyperplane and S, hence F_i is a face.

If F_i is a face, and a $(m-1)$-façade. Recall from Lemma 2.3.5 that for an affine set, the normal cone at any point equals the orthogonal complement of the linear subspace of $\mathbb{R}^m$ corresponding to the affine space. Moreover, a proof earlier in the section showed that for a face F of a convex set C and a point x in the relative interior of F, $\dim N_{\text{aff } F}(x) = \dim N_C(x)$. Thus

$$\dim F = m - \dim N_S(x) = m - 1.$$

■

Corollary 3.3.16. *Let S be a spectrahedron with minimal defining linear pencil $A(x)$. Then S is a polyhedron if and only if all irreducible factors of* $\det A(x)$ *are linear.*

Proof. By the the definition of polyhedra (Definition 2.7.2), S is a polyhedron if and only if it can be written on the form $S = \{x \in \mathbb{R}^m : A'x - b \geq 0\}$, for some matrix $A' \in \mathbb{R}^{n \times m}$ or $S = \{x \in \mathbb{R}^m : l_1(x) \geq 0, l_2(x) \geq 0, \ldots, l_n(x) \geq 0$ where all l_i are linear polynomials$\}$. The boundary of the latter is clearly the set where $l_i(x) = 0$ for at least one i, which is exactly when the product $\Pi_{i=1}^n l_i(x) = 0$. Consequently S is a polyhedron if and only if the minimal defining polynomial is a product of linear factors, i.e. if and only if the irreducible factors of $\det A(x)$ are linear. ■

The last proposition of this section is of importance to validate that façades successfully describes the "curved faces" of a spectrahedron, as one would want the terms to coincide when the curved parts are in fact flat.

Proposition 3.3.17. *If a full dimensional spectrahedron $S \subseteq \mathbb{R}^m$ with defining linear pencil $A(x)$ is a polyhedra, then the r-façades and the faces of dimension r coincide.*

Proof. Assume S is a polyhedron of dimension n. Then the boundary of S is the union of all faces of dimension $m-1$. From Proposition 3.3.14, every face of a spectrahedron is contained in a façade. Furthermore, when S is a polyhedron, the corresponding irreducible factor of each $(m-1)$-façade is a linear polyonomial.

Assume now that F is a face of S with dimension $m-1$. Then we first note that F is also an exposed faces, so $F = H \cap S$ for some hyperplane H. We also have that $F \subseteq F$ç for some façade Fç. Since the dimension of the orthogonal complement of the corresponding linear subspace of the affine hull of F is 1, the normal cone of F is of dimension 1 for all points in the relative interior. Thus Fç must be a $(m-1)$-façade. We also now that $F\text{ç} \subseteq H' \cap \mathcal{S}$ for some hyperplane H', since the corresponding linear factor of Fç is linear, and hence the zero set defines a hyperplane. This gives us the following

$$H \cap S = F \subseteq F\text{ç} \subseteq H' \cap S$$

But then H' is a supporting hyperplane of S in F, and since F is of dimension $m-1$, $H = H'$, and $F = F$ç. Thus the $(m-1)$-façades and faces of dimension $m-1$ coincide.

From Proposition 3.3.12, the intersection of k distinct $(m-1)$-façades is a $(m-k)$-façade. AS the gradient of linear polynomials are constant, the zero multiplicity of a point in S, and hence the dimension of the normal cone, is the number of linear factors with intersecting zero sets in this point. Thus when S is a polyhedron, then every $(m-k)$-façade is the intersection of k distinct $(m-1)$-façades. Consider now a r-façade for $r < m-1$, we denote it G. Then $G = \cap_{i=1}^{m-r} F\text{ç}_i$ where $F\text{ç}_i$ are distinct $(m-1)$-façades. We then have that $F = (\cap_{i=1}^{m-r} H_i) \cap S$, also the points in the interior of G has normal cone of dimension $m-r$, and since G is a subspace of an affine set, G has dimension r. Furthermore, if $x, y \in S$ such that the open line segments between them is in G, then the line segment is also in $\cap_{i=1}^{m-r} H_i$, and hence $x, y \in \cap_{i=1}^{m-r} H_i$. AS we choose $x, y \in S$, and S is convex, this implies $x, y \in S \cap (\cap_{i=1}^{m-r} H_i)$, and G is a face of S, which we already showed has dimension r. ∎

Faces of h-spectrahedra

The faces of an h-spectrahedron correspond to the faces of the spectrahedron in $\mathbb{R}^m$ it is affinely equivalent to. This can be seen by using the steps described in Section 3.1.

Definition 3.3.18. [LV16] The smallest face of an h-spectrahedron $S' \in \mathcal{S}^n$ (Equation (3.2)) containing an element $A \in S'$ is a subset

$$F_{S'}(A) = \Big\{ X \in S' : \ker X \supseteq \ker A. \Big\}$$

The following example suggests that the dimension of a face of an h-spectrahedron is the same as the dimension a corresponding face in the corresponding spectrahedron in $\mathbb{R}^m$.

Example 3.3.19. Let $S' = \{X \in \mathcal{S}^2_+ : \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, X \right\rangle = 2\}\}$. Then all $X \in S'$ are on the form $\begin{pmatrix} x_1 & 1 \\ 1 & x_2 \end{pmatrix}$.

Let $A = \begin{pmatrix} a_1 & 1 \\ 1 & a_2 \end{pmatrix} \in S'$, then A can be row reduced to $\begin{pmatrix} 1 & \frac{1}{a_1} \\ 0 & a_2 - \frac{1}{a_1} \end{pmatrix}$, thus if $a_2 \neq \frac{1}{a_1}$, A has a trivial kernel, and $\ker X \supseteq \ker A$ for all $X \in S'$. So the smallest face containing A is the trivial face which is all of the h-spectrahedron. If however $a_2 = \frac{1}{a_1}$, the kernel of A is spanned by the vector $\begin{pmatrix} -1 \\ a_1 \end{pmatrix}$. Let $X \in S'$

and $k \in \mathbb{R}$, then

$$X \begin{pmatrix} -k \\ ka_1 \end{pmatrix} = \begin{pmatrix} -kx_1 + ka_1 \\ -k + ka_1x_2 \end{pmatrix}$$

equals $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ if and only if $x_1 = a_1$ and $x_2 = \frac{-1}{a_1}$, which is exactly the matrix A. This shows that A is a face. Furthermore, since the face consist of exactly one element, it is zero dimensional, so it is an extreme point. This holds for all elements of S'.

The corresponding spectrahedron $S \subseteq \mathbb{R}^2$ is the set defined by $xy - 1 \geq 0$ and $x \geq 0$. Thus the boundary is curved and the faces are each of the points on the curve. Points are one dimensional, so the dimension of the faces are the same as for $S' \subseteq \mathcal{S}^2_+$.

3.4 Spectrahedra in semidefinite optimization

Spectrahedra were first defined to have a name for the feasible set of solutions for semidefinite optimization problems. Thus is seems only fair that we give a proper introduction to have they occur, and how their properties are utilized in optimization.

In semidefinite optimization, there is a *primal* problem, and a *dual* problem which are

$$\begin{aligned} (P) \quad \inf_{X \in \mathcal{S}^n} \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle A_i, X \rangle = b_i, \quad i = 1, \dots, m \\ & X \succeq 0 \end{aligned}$$

$$\begin{aligned} (D) \quad \sup_{y \in \mathbb{R}^m} \quad & b^T y \\ \text{s.t.} \quad & \sum_{j=1}^{m} y_i A_i + Y = C \\ & Y \succeq 0 \end{aligned}$$

From the primal problem (P), we immediately recognize the constraints to be a spectrahedron on the form in Definition 3.1.4. For the dual, rearrange the constraint

$$\sum_{j=1}^{m} y_i A_i + Y = C, \quad Y \succeq 0$$

$$Y = C - \sum_{j=1}^{m} y_i A_i \succeq 0$$

considering that the objective function is optimizing over a variable $y \in \mathbb{R}^m$, this is the spectrahedron defined in Definition 3.1.1.

Theorem 3.4.1. *[AL11] Let (P) and (D) be a primal-dual semidefinite optimization problem as defined above. If X^* is feasible for (P) and y^*, B^* for (D), then*

$$\langle C, x\rangle \leq b^T y^*$$

.

Proof. We have that

$$b^T y^* - \langle C, X^*\rangle = \sum_{i=1}^{m} y_j^* \langle X^*, A_i, \rangle - \langle X^*, C\rangle = \langle X^*, \sum_{i=1}^{m} y_i^* A_i - C\rangle = \langle X^*, Y^*\rangle.$$

Since X^*, Y^* are feasible, they are positive semidefinite. The positive semidefinite cone is self dual, so $\langle X^*, Y^*\rangle \geq 0$, which concludes the proof. ■

Contrary to linear programming, semidefinite programming does not have strong duality in general, i.e., the optimal value of the primal problem may be different from the optimal value of the dual problem. If we however have a feasible solutions to the primal and dual problem, we can easily check if the solutions are optimal.

Lemma 3.4.2. *If (X^*, Y^*) is a feasible pair for a primal and dual SDP problem, and*

$$X^* Y^* = 0,$$

then (X^*, Y^*) is optimal.

The lemma follows directly from Theorem 3.4.1, and the fact that $\langle X, Y\rangle = 0$ if and only if $XY = YX = 0$.

Example 3.4.3. [BPT12] Consider the primal-dual pair

$$\begin{array}{lllll}
\textit{minimize} & \alpha X_{11} & \textit{maximize} & y_2 & \\
\text{subject to} & X_{22} = 0 & & & \\
 & X_{11} + 2X_{23} = 1 & \text{subject to} & \begin{pmatrix} \alpha - y_2 & 0 & 0 \\ 0 & -y_1 & -y_2 \\ 0 & -y_2 & 0 \end{pmatrix} \succeq 0 & \\
 & X \succeq 0 & & &
\end{array}$$

If $X_{22} = 0$, then all elements in the same row and column must be 0 for X to be positive semidefinite. Thus $X_{11} = 1$, and the optimal solution to the primal problem is attained equals α. For the dual problem, $y_2 = 0$ for all y_1, y_2 such that the constraint matrix is positive semidefinite. This shows that the duality gap is equal to α for all feasible solutions, end hence for all optimal solutions, to the primal-dual problem.

With some extra *constraint qualifications*, we can ensure strong duality also for semidefinite optimization. These are extra conditions regarding the set of feasible solution. One simple and common constraint qualification is the *Slater's condition.* For a semidefinite optimization problem, the Slater condition holds if there exist a *strictly feasible* solution to the primal or/and the dual problem, i.e., $X \succ 0$ and or $Y \succ 0$.

Theorem 3.4.4. *[BPT12] Assume both the primal (P) and the dual (D) of a positive semidefinite optimization problem are strictly feasible. Then both problems have optimal values, and there is no duality gap.*

Example 3.4.5. Consider the primal-dual semidefinite optimization problem given by

$$\begin{aligned}
\textit{minimize} \quad & \langle \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X \rangle \\
\text{subject to} \quad & \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X \rangle = 1 \\
& \langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X \rangle = 1 \\
& \langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, X \rangle = 1 \\
& X \succeq 0
\end{aligned}
\qquad
\begin{aligned}
\textit{maximize} \quad & y_1 + y_2 + y_3 \\
\text{subject to} \quad & \begin{pmatrix} -y_1 & c & 0 \\ c & -y_2 & 0 \\ 0 & 0 & -y_3 \end{pmatrix} \succeq 0
\end{aligned}$$

The primal problem is strictly feasible as the identity matrix is a feasible solution, and is positive definite. For the dual, consider a Y such that $y_i < 0$ for $i = 1, 2, 3$ and $y_2 \neq \frac{-c^2}{-y_1}$. Then $Y \succ 0$ and the dual is also strictly feasible. Then by Theorem 3.4.4, there exist an optimal solution for both problems which give the same value.

Semidefinite relaxation

In optimization, if one is able to express the problem as a semidefinite optimization problem, there are good and effective way of solving the problem. Unfortunately, most problems can not be directly expressed as SDP problems, but it can be approximated by one, which we say is the *semidefinite relaxation* of the problem.

Linear optimization relaxations has been widely used in optimization theory since it was introduced in the 40's by Dantzig. However, in the last 30 years, semidefinite relaxations have taken more over. As linear optimization is a special case of semidefinite optimization, it is only natural that semidefinite relaxations can create tighter bounds of optimization problems. This allow us to approximate a wider range of problems and improve the precision and accuracy of our solutions. Semidefinite optimization is anticipated to be applied in more fields in near future, but we provide some examples of where it is already applied.

The elliptope, which we have seen in several examples, is a semidefinite relaxation to the maximum cut problem. The max cut problem involves dividing the vertices of an undirected graph into two sets to maximize the number of edges that connect vertices from different sets. It seeks to find a partition that cuts the most edges in the graph. There are also other graph problems, like the *graph coloring problem*, which have semidefinite relaxations.

In [Luo+10], semidefinite relaxations to non-convex quadratically constrained quadratic programs(QCQP). Quadratic programs have applications in finance, agriculture, economics, production operations, marketing, and public policy. A type of QCQP which is non homogeneous is the problem

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & x^TCx \\ \text{subject to} \quad & x^TA_ix \trianglerighteq_i b_i,\ i = 1,\dots,m \end{aligned}$$

where $\trianglerighteq_i$ represent $\geq$, $\leq$ or $=$ for each i, $C, A_1, \dots A_m \in \mathcal{S}^n$, and $b_1, \dots, b_m \in \mathbb{R}$. [Luo+10] shows that this problem has semidefinite relaxation, which is exactly the standard form for a primal semidefinite program given in **??**. A standard semidefinite problem can be solved, in arbitrary accuracy in an efficient and reliable way using software created to solve SDP problems. And thus give strong bounds on the QCQP problem.

Another field where semidefinite relaxations is of importance is in control theory. About a hundred years before the term spectrahedron was introduced, Lyapunov showed that all trajectories of the differential equation

$$\frac{d}{dt}x(t) = Ax(t)$$

converge to zero if and only if there exist a positive semidefinite matrix P such that $A^TP + PA$ is negative definite definite.[Boy+94]

One last example of a semidefinite relaxation is from [Das+04]. Here they define a semidefinite relaxation of a *principal component analysis problem*, which is the problem of approximating a symmetric matrix by a rank one matrix with an upper bound on the cardinality of its eigenvector. To find a lower bound on this problem, they first formulate it as a variational problem, and then construct a semidefinite relaxation. Efficient solvers will the provide a lower bound. Without diving too deep in the first variational formulation, we state the semidefinite relaxation to see an example of a semidefinite formulation which is not on standard form.

Let A be the symmetric matrix we want to approximate with a rank one matrix, I_n be the identity matrix, and $\mathbf{1}$ be a vector of all ones. Then the semidefinite relaxation of the PCA-problem is given by

$$\begin{aligned} \max_{X\in\mathcal{S}^n} \quad & \langle A, X\rangle \\ \text{s.t.} \quad & \langle I_n, X\rangle = 1 \\ & \mathbf{1}^TX\mathbf{1} \leq k \\ & X \succeq 0 \end{aligned}$$

[Das+04] further states that the optimal value of the semidefinite relaxation will be an upper bound of the optimal value of the first variational problem formulation, which is then a lower bound to the original PCA-problem. The optimal solution X to the semidefinite relaxation is not always of rank 1, but a small adjustment and specific choice of eigenvector of X will give an approximate solution to the original problem.

CHAPTER 4

Spectrahedra in the plane

While a comprehensive characterization of spectrahedra is still under exploration, the most significant result to date has been obtained by J. Wiliam Helton and Victor Vinnikov. Their work presents a property that holds for all spectrahedra and is as a sufficient condition for classifying some two-dimensional set as a spectrahedron. In this chapter about spectrahedra in the plane, it is appropriate to begin by presenting this fundamental result.

Throughout the chapter, we let $S = \{(x, y) \in \mathbb{R}^2 : A_0 + xA_1 + yA_2 \succeq 0\}$, and $A(x, y)$ the corresponding linear pencil. We let $p_I(x, y)$, where $I \in \mathcal{P}(\{1, \dots, n\})$, denote the principal minors of $A(x, y)$ including the intersection of the rows and columns corresponding to the indices in I.

Before we introduce the Vinnikov-Helton-theorem, there are two necessary concepts we need to know, algebraic interiors and real zero polynomials.

Definition 4.0.1. [HV03] A closed set $C \subseteq \mathbb{R}^m$ is an **algebraic interior**, if there is a polynomial p in m variables such that C equals the closure of a connected component of the set $\{x \in \mathbb{R}^m : p(x) > 0\}$. We denote the algebraic interior including x^0 with defining polynomial p by $C_p(x^0)$

From Lemma 3.2.8 we easily derive that, that all algebraic interiors C, have a minimal defining polynomial with no repeated factors, which we denote p_0.

Example 4.0.2. The set $C = \{(x, y) : x^3 + x - y^2 \geq 0\}$ is the closure of the only component of $\{x \in \mathbb{R}^m : p(x) > 0\}$, and hence an algebraic interior. Note that C is a non-convex algebraic interior. $(\frac{1}{10}, 0), (3, 5) \in C$, but there are points on the line segment between them that are not in C. E.g., for the point $\frac{1}{2}(\frac{1}{10}, 0) + \frac{1}{2}(3, 5)$, $x^3 + x - y^2 < 0$.

Definition 4.0.3. [HV03] A polynomial p is **real zero polynomial** in x^0 (RZ_{x^0}) if it satisfy the condition; for each $x \in \mathbb{R}^n$,

$$p(\mu x + x^0) = 0 \text{ implies that } \mu \text{ is real .} \tag{4.1}$$

If $x^0 = 0$, then p is simply a real zero polynomial (RZ)

In [HV03] they refer to the polynomials that are real zeros in origo as simply a real zero polynomial.

Example 4.0.4. Let $p : \mathbb{R}^m \to \mathbb{R}$ be a linear polynomial. Then

$$\begin{aligned} p(\mu x + x^0) &= a_0 + a_1(\mu x_1 + x_1^0) + \ldots + a_m(\mu x_m + x_m^0) \\ &= a_0 + a_1 x_1^0 + \ldots a_m x_m^0 + \mu(a_1 x_1 + \ldots + a_m x_m) \end{aligned}$$

So if $p(\mu x + x^0) = 0$, then since a_i is real for $i = 0, 1, \ldots, m$ and x_i, x_i^0 are real for $i = 1, 2, \ldots, m$, then μ must be real. Thus all linear polynomials are real zero polynomials.

Example 4.0.5. [BPT12] The polynomial $p(x, y) = 1 - (x^4 + y^4)$ is not a real zero polynomial for any $x_0 \in \mathbb{R}^2$. Consider $x_0 = 0$. Then for every $(x, y) \in \mathbb{R}^2$, the univariate polynomial in μ

$$p(\mu(x, y)) = (1 - \mu^2(x^4 + y^4)^{\frac{1}{2}})(1 + \mu^2(x^4 + y^4)^{\frac{1}{2}}),$$

has two non-real zeros.

Theorem 4.0.6 ([HV03])**.** *Some properties of real zero polynomials are the following*

(i) *The product of RZ_{x^0} polynomials is a RZ_{x^0} polynomial*

(ii) *If a RZ_{x^0} polynomial p factors as $p = p_1 p_2 \ldots p_k$, then all factors $p_1, p_2, \ldots p_k$ are RZ_{x^0} polynomials.*

(iii) *If C is an algebraic interior and p is RZ_{x^0} for $x^0 \in C$, then p is RZ_x for all $x \in C^o$.*

With these two definition established, we state the Theorem by Helton and Vinnikov.

Theorem 4.0.7. *[HV03] If a set $C \subseteq \mathbb{R}^m$ with x^0 in the interior have an LMI representation, then it is a convex algebraic interior, and the minimal defining polynomial p for C satisfies the RZ_{x^0} condition, 4.1.*

Conversely, when $n = 2$, if p is a polynomial of degree d, satisfying the RZ_{x^0} condition, 4.1 and $p(x^0) > 0$, then $C_p(x^0)$ has an LMI representation of dimension $d \times d$.

The proofs can be found in [HV03], but we will see how properties of spectrahedra we have already described imply parts of the theorem.

The boundary of a spectrahedron is defined by the determinant of its linear pencil, i.e. a polynomial in m variables, moreover the determinant is non-negative on the interior. So a spectrahedron is a subset of the non-negative set of some polynomial. Moreover, a spectrahedron is closed and convex.

For a spectrahedron, the determinant of its linear pencil is $\det A(x) = \det(I + A_1 x_1 + \ldots A_m x_m)$, where all A_i are symmetric. Then we can further derive that det $(\frac{x}{\lambda})$ = det $(\frac{1}{\lambda}(\lambda I + A_1 x_1 + \ldots + A_m x_m) = \frac{1}{\lambda^m} det\ (\lambda I + A_1 x_1 + \ldots + A_m x_m)$ which is the characteristic polynomial of the symmetric matrix $-A_1 x_1 - \ldots - A_m x_m$, hence it has only real roots. Consequently the polynomial $\det A(x)$ has only real zeros and is a RZ polynomial. This argument is from [HV03].

Corollary 4.0.8. *Let $S = \{x \in \mathbb{R}^m : A(x) \succeq 0\}$, with $A(x)$ a linear pencil of size n, (be a spectrahedra with x^0 in the interior). Then if the minimal defining polynomial of S is of degree n, then $A(x) \succeq 0$ is a minimal LMI of S. If $m = 2$, the converse statement also holds.*

Proof. First assume that the minimal defining polynomial of the set S is of degree n. From Lemma 3.2.9 the determinant of $A(x)$ has degree n, so it is the minimal defining polynomial of S. Then all $A'(x, y)$ of size $n' \times n'$ and $n' < n$ will have $\det A'(x, y)$ of degree less than n and hence it cannot be a LMI of S. Thus $A(x, y) \succeq 0$ is the minimal LMI of S.

Now let $A(x, y) \succeq 0$ be a minimal LMI of $S \subseteq \mathbb{R}^2$. Then $\det A(x, y)$ is a defining polynomial of S with degree less than or equal to n. From theorem Theorem 4.0.7 we know that the minimal defining polynomial of a spectrahedron is a RZ, and conversely that for two variables, any RZ has an LMI representation of the same degree. So if the minimal defining polynomial of S has a smaller degree d than $\det A(x, y)$, there exist an LMI representation of S of size $d \times d$ wich contradicts that $A(x, y) \succeq 0$ is a minimal LMI for S. ∎

As a consequence of the above results, we can define a **minimal standard form** of spectrahedra in the plane. Let $S \in \mathbb{R}^2$ be a spectrahedron with non-empty interior and $\det A(x, y) = f_1(x, y)f_2(x, y) \dots f_k(x, y)$, where the factors $f_i(x, y)$ are polynomials of degree n_i, irreducible over $\mathbb{R}$. In other words, each f_i defines a 1-façade of S. Then the spectrahedron can be expressed as

$$S_m = \left\{ (x, y) \in \mathbb{R}^2 : A(x, y) = \begin{pmatrix} A_1(x, y) & 0 & \dots & 0 \\ 0 & A_2(x, y) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_k(x, y) \end{pmatrix} \right\} \quad (4.2)$$

where $A_i(x, y)$ is of size n_i with $\sum_{i=1}^{k} n_i = n = \text{size } A(x, y)$ and $\det A_i(x, y) = f_i(x, y)$.

In [HV03], Theorem 4.0.7 is also presented as a geometric version through a property called *rigid convexity*. They show that for an algebraic interior $C_p(x^0)$, rigid convexity of $C_p(x^0)$ is the same as p having the real zero condition in x^0.

Definition 4.0.9. An algebraic interior $C \in \mathbb{R}^m$ of degree d with minimal defining polynomial p is **rigid convex** provided for every $x^0 \in C^\circ$ and generic line l through x^0, l intersects the (affine) real algebraic hypersurface $p(x) = 0$ in exactly d points.

An example of a rigid convex algebraic interior is the unit ball. The minimal defining polynomial of the unit ball in any dimension is a degree 2 polynomial, and every line through the interior of the unit ball intersects the boundary in two points. An example of an algebraic interior which is not rigid convex is one defined by the polynomial $x^3 - 3xy^2 - (x^2 + y^2)^2$. Figure 4.1 shows how a line through $(0.5, 0)$ only intersect the zero set of the polynomial in two points. As the polynomial is of degree 4, it would have to intersect four points for the algebraic interior to be rigid convex.

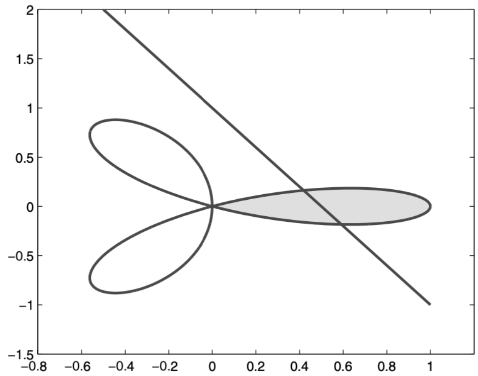

Figure 4.1: An algebraic interior which is not rigid convex.(Figure 6.2 in [BPT12].

4.1 Particular cases of plane spectrahedra

We know that spectrahedra are closed and convex sets of $\mathbb{R}^2$, and typically they will appear as a "rounded" polyhedra with some bounded part and possibly including a convex cone. In this section we will have a look at some less of the less conventional spectrahedra in the plane.

$S = \mathbb{R}^2$

The first case we consider is whether or not the whole plane can be a spectrahedron. If we let $S = \{(x, y) \in \mathbb{R}^2 : A_0 \succeq 0\}$ we clearly get $S = \mathbb{R}^2$ for any positive semidefinite matrix A_0. In this case, all points of the spectrahedron are interior points, or all points are boundary points, only depending on the rank of A_0. This is because a positive semidefinite matrix A is positive definite if and only if $\det A \neq 0$.

For any $S = \{(x, y) \in \mathbb{R}^2 : A_0 + xA_1 + yA_2 \succeq 0\}$ with either $A_1 \neq 0$ or $A_2 \neq 0$, $S \subsetneq \mathbb{R}^2$. Each diagonal element of $A(x, y)$ is a linear polynomial, and if the coefficient of x or y is non-zero in at least one diagonal element, then it can be both greater than and less than zero, and hence the spectrahedra is not equal to $\mathbb{R}^2$. If all diagonal elements are constant, there has to be elements off the diagonal with non-zero coefficient for either x or y. Then every element off the diagonal is included in the principal minor $PM_{ij} = (A(x, y))_{ii}(A(x, y))_{jj} - (A(x, y))^2_{ij}$. The polynomial $(A(x, y))^2_{i,j}$ for a non-constant linear polynomial $(A(x, y))_{i,j}$, is positive and unbounded, so the principal minor $PM_{ij} < 0$ for some $(x, y) \in \mathbb{R}^2$, and $S \neq \mathbb{R}^2$.

$S = \emptyset$

On the contrary, we have the case of an empty spectrahedron, $S = \emptyset$. It is easy to create examples of empty spectrahedra, but in general it is challenging to find algorithms deciding whether a given spectrahedra is in fact non-empty. As a spectrahedron is the feasible solutions to a semidefinite program, deciding whether it is non-empty, coincide with deciding feasibility of a semidefinite program. In literature, this problem is thus referred to as the *semidefinite feasibility problem.*

Two examples of empty spectrahedra are the following

$$S_1 = \left\{ (x, y) \in \mathbb{R}^2 : \begin{pmatrix} x+y-1 & 0 & 0 \\ 0 & -x-y-1 & 0 \\ 0 & 0 & -x+y \end{pmatrix} \succeq 0 \right\}$$

$$S_2 = \left\{ (x, y) \in \mathbb{R}^2 : \begin{pmatrix} -x-y-1 & x+y \\ x+y & 1 \end{pmatrix} \succeq 0 \right\}$$

For S_1, the sets such that the 1×1 principal minors are positive are not intersecting, hence $S_1 = \emptyset$. For S_2 determinant of the linear pencil in S_2 is always negative.

$\boldsymbol{S^\circ = \emptyset}$

We have seen in the earlier chapters that spectrahedra have empty interior if and only if they are not full dimensional. This means that in $\mathbb{R}^2$, the spectrahedra with empty interior are the spectrahedra that consist of one single point,a line segment, a halfline or a line. As the interior is empty, the determinant of the defining linear pencil will be zero on all of S. If $S = \{x\}$, then there are no restrictions on the size of the blocks in the standard minimal form of the linear pencil. The zero sets of the irreducible factors must intersect in one point, and this can be done with both linear irreducible factors and higher degree irreducible factors. If the spectrahedron on the other hand is a line segment, halfline or line, the determinant can only have linear factors, as the determinant defines the boundary.

$$S_3 = \left\{ (x, y) \in \mathbb{R}^2 : \begin{pmatrix} 1+x & y & 0 & 0 \\ y & 1-x & 0 & 0 \\ 0 & 0 & 1+x & y-2 \\ 0 & 0 & y-2 & 1-x \end{pmatrix} \succeq 0 \right\}$$

$$S_4 = \left\{ (x, y) \in \mathbb{R}^2 : \begin{pmatrix} x & 0 & 0 \\ 0 & -x & 0 \\ 0 & 0 & y \end{pmatrix} \succeq 0 \right\}$$

S_3 defines two circles intersecting in the point $(0, 1)$, where all principal minors are greater or equal to zero for both circles. This shows that we can define a one-point-spectrahedron where the determinant of $\mathrm{t}A(x, y)$ has only higher degree irreducible factors. In S_4, the determinant is greater or equal to zero, if and only if $x = 0$, so the determinant defines the y-axis. In addition, y is a principal minor, thus the spectrahedron is the halfline defined by the non-negative part of the y-axis.

4.2 Number of vertices

The boundary of a plane spectrahedron is in algebraic geometry referred to as a plane algebraic curve. Plane algebraic curves are widely studied, and many of the results of them are also relevant in the study of spectrahedra and for applications in SDP.

The vertices of a plane spectrahedra are **singular points** in the subset of a plane algebraic curve that is the boundary of the spectrahedron. The singular points of a plane curve are the points where the curve has higher multiplicity than one, or in other words, it is a point where the curve fails to have a well defined tangent line. This subsection is based on results from [Kun05].

Lemma 4.2.1. *The vertices of a plane spectrahedra are singular points of the zero set of its minimal defining polynomial.*

Proof. For any vertex (x_0, y_0) of a spectrahedron $S \subseteq \mathbb{R}^2$, the normal cone of S in (x_0, y_0) has dimension 2. A tangent line is perpendicular to a vector in the normal cone. Consequently, the tangent line of S in a vertex is not well defined, and is a singular point. ■

The lemmas 4.2.2,4.2.4, 4.2.5 and4.2.6 follow directly from the result in [Kun05] they are referring to. In the reference text, the results are given as number of singular points on plane algebraic curves over an *algebraically closed* field. I.e., a field such that each univariate polynomial with coefficients in the given field have at least one root in that field. As this is not the case for $\mathbb{R}$, the singular points of an algebraic curve over $\mathbb{R}$ may all be outside $\mathbb{R}^2$.

As each vertex of a spectrahedron is a singular point of the curve defined by the determinant of its defining linear pencil, the number of vertices are less than or equal to the bounds in the lemmas underneath. Moreover, two curves may have more intersection points than the ones that are vertices of the spectrahedra they define. For example if one curve is a hyperbola and the other is a line intersection both components of the hyperbola. Then as a spectrahedron is a connected convex component, only one component of the hyperbola define the boundary of the spectrahedron, and thus only one of the intersections with the line is a vertex.

The first result is a weak form of the famous "Bézout’s theorem".

Lemma 4.2.2 ([Kun05], corollary 3.10)**.** *Let $f(x, y)$ and $g(x, y)$ be polynomials of degree n and m with no factors in common. Then the zero sets $\{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$ and $\{(x, y) \in \mathbb{R}^2 : g(x, y) = 0\}$ intersect in at most $m \cdot n$ points in $\mathbb{R}^2$.*

Corollary 4.2.3. *Let $f(x, y) = f_1(x, y) \cdot f_2(x, y) \cdot \ldots \cdot f_k(x, y)$ be a polynomial. If the degree of each factor $f_i(x, y)$ is n_i and $\sum_{i=1}^{k} n_i = n$, then there are at most*

$$n_1(n - n_1) + n_2(n - n_1 - n_2) + \ldots + n_{k-1}(n - n_1 - n_2 - \ldots - n_{k-1})$$

points where the zero set of two or more factors intersect.

Proof. Each pair of factors f_i, f_j are polynomials as described in 4.2.2. Thus f_i, f_j intersect in at most $n_i n_j$ points. Summing over all pairs of factor we get

$$\begin{aligned}&n_1(n_2+n_3+\ldots+n_k)+n_2(n_3+n_4+\ldots+n_k)+\ldots+n_{k-1}n_k\\=&n_1(n-n_1)+n_2(n-n_1-n_2)+\ldots+n_{k-1}(n-n_1-n_2-\ldots-n_{k-1})\end{aligned}$$

∎

Lemma 4.2.4. *[[Kun05], corollary 7.10] Let* $Z=\{(x,y)\in\mathbb{R}^2 : f(x,y)=f_1(x,y)\cdot f_2(x,y)\cdot\ldots\cdot f_k(x,y)=0\}$ *where all* $f_i(x,y)$ *are irreducible and distinct. Then* Z *has at most* $\frac{n(n-1)}{2}$ *singular points. Equality holds if and only if all factors* f_i *are linear and non-parallel, i.e., if all façades are faces.*

The lemma is based on [Kun05], but the text does not provide a proof for the last statement on when equality holds.

proof of equality. It follows from the Corollary 4.2.3 that if $n_i=1$ for all $i=1,2,\ldots,k$, and no factors define parallel lines, then all intersections of the curves are in the real plane, and we get

$$1\cdot(n-1)+1\cdot(n-2)+\ldots+1\cdot 1=\frac{n(n-1)}{2}$$

intersections of factors.

Without loss of generality, asssume $n_1\geq n_2\geq\ldots\geq n_k$. If $n_1>1$ and $n_i=1$ for $i=1,2,\ldots,k=n-n_1$, then

$$\begin{aligned}&n_1(n-n_1)+\ldots+n_{k-1}(n-n_1-n_2-\ldots-n_{k-1})\\=&n_1(n-n_1)+1\cdot(n-n_1-1)+1\cdot(n-n_1-2)+\ldots1\cdot(n-n_1-(k-2))\\=&n_1(n-n_1)+\frac{(n-n_1-1+n-n_1-(k-2))}{2}\cdot(k-2)\\=&\frac{2n_1(n-n_1)+(2n-2n_1-1-(n-n_1-2))\cdot(n-n_1-2)}{2}\\=&\frac{2n_1(n-n_1)+(n-n_1+1)\cdot(n-n_1-2)}{2}.\\=&\frac{n^2-n-(n_1^2-n_1+2)}{2}\\<&\frac{n(n-1)}{2}\end{aligned}$$

The last inequality holds as $(n_1^2-n_1+2)>0$ for all n_1. Thus if a factor is of higher degree than 1, then we do not attain the maximum. Now we can do the same for the remaining linear factors, which will for each factor we change, create a smaller number of intersections. ∎

Lemma 4.2.5. *[[Kun05], corollary 7.17] Let* $Z=\{(x,y)\in\mathbb{R}^2 : f(x,y)=0,\ f\ \text{ is irreducible}\}$. *Then there are at most* $\frac{(n-1)(n-2)}{2}$ *singular points.*

Lemma 4.2.6. *[[Kun05], corollary 6.7] Let* $S\subseteq\mathbb{R}^2$ *be a non-empty spectrahedron with minimal defining linear pencil* $A(x,y)$. *If* S *has no vertices, then* $\det A(x,y)$ *is an irreducible polynomial, and* S *has only one façade.*

Proof. Suppose $\det A(x,y)$ has two factors $f_1(x,y), f_2(x,y)$. If the zero set of the factors do not intersect, then only one of the factors define the boundary of S, which contradicts $A(x,y)$ being a minimal defining linear pencil. If the zero sets of the factors do intersect in S, it will be a vertex of S which is a contradiction. ∎

Lemma 4.2.7. *Let $S \subseteq \mathbb{R}^2$ be a spectrahedron with minimal defining linear pencil $A(x,y)$. Then for each linear factor $l(x,y)$ of $\det A(x,y)$ the set $\{(x,y) \in \mathbb{R}^2 : l(x,y) = 0\} \cap S$ contains at most two vertices.*

Proof. Assume $v_1, v_2, v_3 \in \{(x,y) \in \mathbb{R}^2 : l(x,y) = 0\} \cap S$ are vertices of the spectrahedron S. As they are all in the zero set of the same linear factor, we can assume v_2 is on the line segment between v_1 and v_3. If both $(1-\lambda)v_1 + \lambda v_2$ for $\lambda \in [0,1]$ and $(1-\gamma)v_2 + \gamma v_3$ for $\gamma \in [0,1]$ are in S, then v_2 is not an extreme point, and hence not a vertex, which is a contradiction. If on the other hand, at least one of the described line segments are in S, S can not be convex, which is also a contradiction. Thus the zero set of a linear factor contains at most 2 vertices of a spectrahedron in $\mathbb{R}^2$. ∎

Consider a polyhedra defined by a diagonal minimal defining linear pencil $A(x,y)$ of size n. We know that $\det A(x,y)$ is a product of n unique linear polynomials. By Lemma 4.2.4, the polyhedra may have $\frac{n(n-1)}{2}$ vertices. However, for linear factors, every vertex belongs to at least two zero sets, and since we have a minimal description, exactly two zero sets. From this we deduce the following corollary.

Corollary 4.2.8. *Any polyhedra $P \subseteq \mathbb{R}^2$ has at most n vertices.*

Another corollary is based on the number of vertices, and then deciding polyhedrality from that.

Corollary 4.2.9. *If a non-empty spectrahedron $S \subseteq \mathbb{R}^2$ with minimal defining pencil $A(x,y)$ of size n has $\frac{n(n-1)}{2}$ vertices, then it is a polyhedron. Moreover, size $n \leq 3$.*

Proof. In Lemma 4.2.4 we saw that the maximum number of singular points is $\frac{n(n-1)}{2}$ and is attained if and only if all factors are linear. From Corollary 3.3.16, a spectrahedron is a polyhedron if and only if all factors of the determinant are linear. Thus any spectrahedron with $\frac{n(n-1)}{2}$ vertices is a polyhedron. Moreover, since a polyhedron has at most n vertices, and $\frac{n(n-1)}{2} \leq n$ if and only if $n \leq 3$, the second statement holds. ∎

Example 4.2.10. Consider a spectrahdron with defining linear pencil of size $n = 4$. Then the possible degrees of the factors are $\{1,1,1,1\}$, $\{1,1,2\}$, $\{2,2\}$, $\{3,1\}, \{4\}$. By Lemma 4.2.2, the possible number of intersections between the curves defined by the zero set of the factors are respectively 10, 5, 4, 3 and 0, assuming each factor defines a non-singular curve. However, for factors of degrees $\{1,1,1,1\}$, all factors are linear so the maximum number of vertices for this factorization of a size $n = 4$ spectrahedron only gives 4 vertices. Also for $\{1,1,2\}$, each linear factor can only contain 2 vertices, so there is is a maximum of 4 vertices. For {3,1}, the linear factor can at most contain 2 vertices, but the third degree factor may have a singular point, so the maximum number of vertices is 3. If the determinant is irreducible of degree 4, the only vertices that can exist are the at most 3 singular points.

Example 4.2.11. An example of a spectrahedron with irreducible determinant of the linear pencil where the maximum number of vertices is attained is the spectrahedron

$$S\left\{(x,y) \in \mathbb{R}^2 : \begin{pmatrix} 1 & x & 0 \\ x & -x & y \\ 0 & y & -x \end{pmatrix} \succeq 0\right\}$$

The determinant of the linear pencil is $x^2 + x^3 - y^2$ and defines the curve in Figure 4.2. As we see, $(0,0)$ is a singular point. The linear pencil is of size

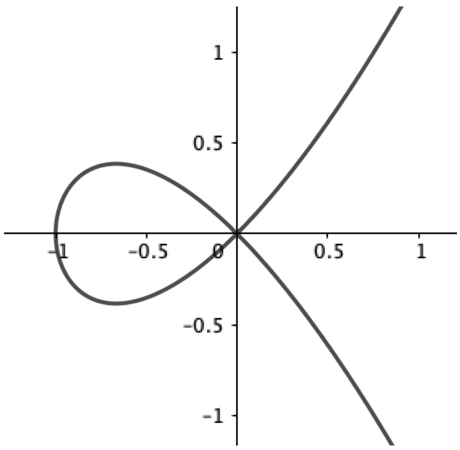

Figure 4.2: The curve $x^2 + x^3 - y^2 = 0$

$n = 3$, and the determiant is an irreducible polynomial, thus from Lemma 4.2.5, there are at most 1 singular points, which we have. Thus the spectrahedron attain the maximal number of vertices.

Even though the results from [Kun05] can give useful upper bounds on the number of vertices of some spectrahedra, they apply to general curves, and will in general not give sharp bundaries. The characterization by Helton and Vinnikov in [HV03] can in some cases offer a tighter upper bound. A geometric interpretation of Theorem 4.0.7 provides a description of the zero set of linear pencils as nested "ovaloids".

For a technical definition of the terms "ovaloid" and "pseudo-hyperplane" see [HV03]. In this paper we will only be interested in the ovaloids in $\mathbb{R}^2$, or **ovals**. These are smooth closed curves, i.e., curves with no endpoint, which completely encloses a convex area. We refer to the enclosed area as the interior of the oval, and the complement as the exterior. A **pseudo-hyperplane** in $\mathbb{R}^2$ is an unbounded curve.
The "standard example" of an oval is the set $\{(x,y) \in \mathbb{R}^2 : x^{2s} + y^{2t} - 1 = 0$. Note that this is not a spectrahedra as any line

Theorem 4.2.12. *[HV03] Let $S \subset \mathbb{R}^2$ be a spectrahedron with minimal defining linear pencil $A(x,y)$ of size n and $(x^0, y^0) \in S^\circ$. Then for each irreducible* *smooth (i.e., non-singular) factor* $p_i(x,y)$ *of* $\det A(x,y)$ *with degree* n_i*, the following holds*

(i) *If* $n_i = 2k$ *even, then* $\{(x,y) :\in \mathbb{R}^2 : p_i(x,y) = 0\}$ *consist of* k *disjoint ovals* $W_1, \ldots, W_k$ *such that* W_i *is contained in* W_{i+1} *for* $i = 1, \ldots, k-1$*, and* (x^0, y^0) *is in the interior of* W_1*.*

(ii) *If* $n_i = 2k+1$ *is odd, then* $\{(x,y) :\in \mathbb{R}^2 : p_i(x,y) = 0\}$ *consist of* k *disjoint ovals* $W_1, \ldots, W_k$ *such that* W_i *is contained in* W_{i+1} *for* $i = 1, \ldots, k-1$*, and* (x^0, y^0) *is in the interior of* W_1*, and a pseudo-hyperplane* W_{k+1} *is in the exterior of* W_k*.*

Instead of expressing the number of ovals by k, when $n = \{2k+1, 2k\}$, we can use the floor function on the fraction $\frac{n}{2}$. Then if n is odd, $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ and if n is even $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$, which is the same as k in both cases in Theorem 4.2.12.

[HV03] further suggests that whenever $\det A(x,y)$ is not smooth, i.e. has at least one singular point, then the zero set is still a collection of nested ovals, but they touch in said singular point.

Example 4.2.13. An example of a spectrahedron where Theorem 4.2.12 can come in handy for deciding the number of vertices of a spectrahedron is if

$$S = \left\{ (x,y) \in \mathbb{R}^2 : \begin{pmatrix} A_1(x,y) & 0 \\ 0 & A_2(x,y) \end{pmatrix} \succeq 0 \right\}$$

where $A_1(x,y)$ is of size 10 and defines a smooth curve, and A_2 is of size 2, and defines an unbounded curve. Then by Corollary 4.2.3, there are at most 20 vertices. However, based on Theorem 4.2.12, $\det A_1(x,y)$ defines 5 nested ovals, where the innermost is the spectrahedron. Then for every unbounded curve intersecting the inner oval, it also intersects the outer ovals. Furthermore, the curve has at least two intersections with each oval, as it is unbounded. Thus there are 10 intersections of the irreducible zero sets, but only 2 are vertices of the spectrahedron. If the degree two curve is e.g., a hyperbola and both component intersect the innermost oval, all 20 intersections are in the real plane, and only 4 of them are vertices of the spectrahedron. Figure 4.3 is an illustration of how such spectrahedra may appear.

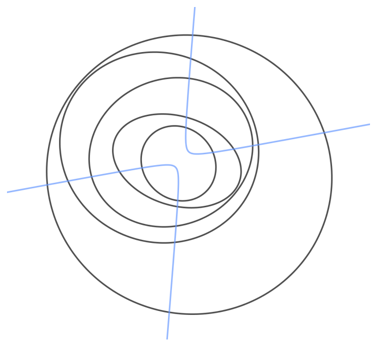

Figure 4.3: 5 nested ovals and a hyperbola, with both components intersecting the innermost oval.

The ovals are all of course convex, but can have different curvature, thus it is not trivial to present a general number of vertices given the degree of the irreducible factors of the determinant of the linear pencil. Still, as we see in the above example, knowing what the determinant of linear pencils "look like" in the plane, can be a tool for deciding the vertices of a spectrahedron.

4.3 Rank of spectrahedra

For a spectrahedron, the rank of a defining linear pencil will vary between the points in the spectrahedron. We know that for the interior points, $A(x) \succ 0$, hence $\operatorname{rk} A(x, y) = n$. For the points on the boundary, we will see that the rank of the linear pencil can be everything from 0 to $n - 1$. It is well known that for any square matrix A of size n

$$\operatorname{rk} A + \operatorname{nl} A = n.$$

Proposition 3.2.4 shows that the nullity of a linear pencil $A(x, y)$ at a point (x_0, y_0) equals the root multiplicity of $\det A(x, y)$ in (x_0, y_0). Thus we can decide the rank of the linear pencil at different points in the spectrahedron simply by examining the determinant.

Lemma 4.3.1. *Let $S \subseteq \mathbb{R}^2$ be a spectrahedron with defining linear pencil $A(x, y)$ of size n. Then $\operatorname{rk} A(x, y) = 0$ for some point in S, if and only if all elements of $A(x, y)$ intersect in one point (x_0, y_0). Furthermore, if $A(x, y)$ is minimal, then $n = 2$.*

Proof. Any matrix has rank 0 if and only if it is the zero matrix. Thus the linear pencil has rank 0 in (x_0, y_0) if and only if all elements are zero, which is if and only if all the lines defined by the zero set of each factor intersect in the point. Furthermore, if $A(x, y)$ is minimal, then all vertices belong to exactly two zero sets, hence there can only be 2 (linear) factors of the determinant, and $n = 2$. ■

Lemma 4.3.2. *Let A be a block diagonal symmetric matrix with k blocks, i.e., on the form*

$$A = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_k \end{pmatrix} \tag{4.3}$$

where each A_i is a symmetric matrix of size n_i, and $\sum_{i=1}^{k} n_i = n = sizeA$. Then $\operatorname{rk} A = \sum_{i=1}^{k} \operatorname{rk} A_i$.

Proof. Let A be a block matrix as presented in the lemma. Then each block

$A_i = U_i^T D_i U_i$ and

$$\begin{aligned}
&\begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_k \end{pmatrix} \\
&= \begin{pmatrix} U_1^T D_1 U_1 & 0 & \dots & 0 \\ 0 & U_2^T D_2 U_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & U_k^T D_k U_k \end{pmatrix} \\
&= \begin{pmatrix} U^1 & 0 & \dots & 0 \\ 0 & U_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & U_k \end{pmatrix}^T \begin{pmatrix} D_1 & 0 & \dots & 0 \\ 0 & D_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_k \end{pmatrix} \begin{pmatrix} U^1 & 0 & \dots & 0 \\ 0 & U_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & U_k \end{pmatrix}
\end{aligned}$$

Then the number of diagonal elements equal to zero in the block matrix including all D_i is equal the sum of eigenvalues equal to zero for each D_i. It follows that

$$\begin{aligned}
\operatorname{rk} A &= n - \operatorname{nl} A \\
&= \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} \operatorname{nl} A_i \\
&= \sum_{i=1}^{k} (n_i - \operatorname{nl} A_i) \\
&= \sum_{i=1}^{k} \operatorname{rk} A_i.
\end{aligned}$$

■

Since each spectrahedron $S \subseteq \mathbb{R}^2$ has a minimal description on the form Equation (4.2), the rank of each point in the spectrahedra can be determined by finding which rank it has in each block of the linear pencil on standard form. Thus we can examine the rank of the linear pencil in a point, by breaking it up in smaller parts, in particular, the parts connected to the façades of the spectrahedron.

The following two theorems are based on the assumption that Helton and Vinnikov's claim on the character of irreducible singular determinants of linear pencils holds. They claim that it is "intuitive" that also singular determinants of linear pencils consist of $\lfloor \frac{n}{2} \rfloor$ nested ovals and an additional pseudohyperplane if the degree is odd. The difference is that two or more ovals and possibly the pseudohyperplane touch in singular points. They don't give a proof, and if it is not the case, the theorems does not hold. To support the claim, we will give an explanation of the intuition.

Let S be a spectrahedron with a minimal defining linear pencil with an irreducible singular determinant. Then for a singular point with higher multiplicity than 2, the curve must "circle back" to the singular point, which calls for an extra two degrees of the determinant. We have seen in Figure 4.1 that when a curve create non-nested ovals, there are lines through the interior which does not intersect the other ovals, so this cannot be the case for our curve which defines the boundary of a spectrahedron. Consequently the ovals need to be nested. If the degree of the determinant is odd, there are also directions where the curve is unbounded, thus we get a pseudohyperplane.

Theorem 4.3.3. *Let $S \in \mathbb{R}^2$ be a spectrahedron with non-empty interior, and minimal defining linear pencil $A(x, y)$ of size n such that $\det A(x, y)$ is an irreducible polynomial over $\mathbb{R}$. Then if $\det A(x, y)$ is non-singular*

$$\operatorname{rk} A(x, y) = n - 1$$

And if $\det A(x, y)$ is a singular curve

$$\operatorname{rk} A(x, y) \geq \left\lfloor \frac{n}{2} \right\rfloor$$

for all $(x, y) \in S$

Proof. Assume first that $\det A(x, y)$ is non-singular, then S has no vertices, and all points on the boundary have a well defined tangent line. Since $\det A(x, y)$ is a polynomial, this implies that the gradient is non-zero, and $\operatorname{rk} A(x, y) = n-1$.

Assume now that $\det A(x, y)$ is singular, and let (x_0, y_0) be a singular point which is also a vertex, i.e., a singular point on the innermost ovals. Let l be a line including one point from the interior of S, which is thus the interior of the inner oval and the point (x_0, y_0), such that l does not intersect any other singular points of the curve $\det A(x, y) = 0$. Such line exists as there are finitely many singular points on the boundary, and since S is convex, infinitely many lines through the interior of S and the point (x_0, y_0).

By Theorem 4.0.7, or more precisely, the geometric version, the line must intersect the zero set of $\det A(x, y)$ in n points, counting with multiplicity. There are $\lfloor \frac{n}{2} \rfloor$ ovals, and the halfline from the interior of S which does not intersect any singular points, must intersect all the ovals. Thus this halfline intersect $\det A(x, y) = 0$ in $\lfloor \frac{n}{2} \rfloor$ points. Since the singular point can be a point where all the ovals and the pseudohyperplane intersect, the multiplicity is at most $n - \lfloor \frac{n}{2} \rfloor = \lceil \frac{sn}{2} \rceil$. Then the nullity in the singular point is $\operatorname{nl} A(x_0, y_0) \leq \lceil \frac{n}{2} \rceil$ and $\operatorname{rk} A(x_0, y_0) = n - \operatorname{nl} A(x_0, y_0) \geq \lfloor \frac{n}{2} \rfloor$ ■

Now we can deduce that for *any* minimal description, where each irreducible factor can have singularities or not, we have the following bounds for the rank of the linear pencil.

Theorem 4.3.4. *Let $S \in \mathbb{R}^2$ be a spectrahedron with non-empty interior and minimal defining linear pencil $A(x, y)$ of size n. The factorisation of $\det A(x, y)$ in irreducible factors is $p_1(x, y) \cdot p_2(x, y) \cdot \ldots \cdot p_k(x, y)$. For each (x, y) in ∂S, we have*

$$\left\lfloor \frac{n}{2} \right\rfloor - (k - 1) \leq \operatorname{rk} A(x, y) \leq n - 1 \tag{4.4}$$

Proof. From Lemma 4.3.2 $\operatorname{rk} A(x,y) = \operatorname{rk} A_1(x,y) + \operatorname{rk} A_2(x,y) + \ldots + \operatorname{rk} A_k(x,y)$ where for all submatrices A_i, the assumptions from Theorem 4.3.3 holds. Thus $\operatorname{rk} A(x,y) \geq \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \ldots + \left\lfloor \frac{n_k}{2} \right\rfloor$. Furthermore, $\lfloor r_1 \rfloor + \lfloor r_2 \rfloor \leq \lfloor r_1 + r_2 \rfloor + 1$ for any two $r_1, r_2 \in \mathbb{R}$. By induction on this inequality, $\left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \ldots + \left\lfloor \frac{n_k}{2} \right\rfloor \geq \left\lfloor \frac{n_1+n_2+\ldots+n_k}{2} \right\rfloor - (k-1) = \left\lfloor \frac{n}{2} \right\rfloor - (k-1)$. Thus the left inequality holds. The right inequality is because all boundary points are in the zero set of the $\det A(x,y)$, and hence the nullspace of $A(x,y)$ is non-trivial in all boundary points. Then $\operatorname{rk} A(x,y) = n - \operatorname{null} A(x,y) \leq n-1$. ■

As vertices are of extra importance, we provide the following result on the rank of vertices of spectrahedra.

Proposition 4.3.5. *Let $S \in \mathbb{R}^2$ be a spectrahedron then $\operatorname{rk} A(x,y) \leq n-2$ for all vertices.*

Proof. A vertex $(x_0, y_0) \in S$ has $\dim N_S(x_0, y_0) = 2$, and so the tangent line is not defined in (x_0, y_0). Sine $\det A(x,y)$ is a polynomial, the tangent line is not defined if and only if $\nabla \det A(x_0, y_0) = 0$, which is when the determinant has a multiple zero in the point. Then from Proposition 3.2.4, $\operatorname{nl} A(x_0, y_0) \geq 2$, and $\operatorname{rk} A(x_0, y_0) \leq n-2$. ■

Low rank points

For some spectrahedra, there are low rank points which are not vertices. If the describing linear pencil is not minimal, there can be repeated factors of the determinant of the linear pencil, or factors that do not define any parts of the boundary of the spectrahedron. However, also for minimal description representation there can be low rank points which are not vertices.

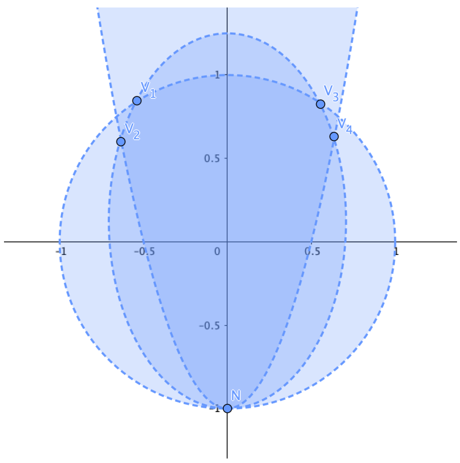

Figure 4.4: V_1, V_2, V_3, V_4 are all vertices where $\operatorname{rk} A(x,y) = n-2 = 4$. The point N is not a vertex, but $\operatorname{rk} A(x,y) = 3$

Example 4.3.6. Let $S \subseteq \mathbb{R}^2$ be the spectrahedron depicted in Figure 4.4 defined by

$$S = \left\{ (x, y) \in \mathbb{R}^2 : \begin{pmatrix} y+1 & 2x & 0 & 0 & 0 & 0 \\ 2x & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+y & x & 0 & 0 \\ 0 & 0 & x & 1-y & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\frac{8}{9}y-\frac{1}{9} & \sqrt{2}x \\ 0 & 0 & 0 & 0 & \sqrt{2}x & 1-\frac{8}{9}y-\frac{1}{9} \end{pmatrix} \succeq 0 \right\}$$

At the point $N = (0, -1)$, each block of the matrix have rank 1, so the linear pencil have rank 3. Furthermore, each block have a determinant with gradient either on the form $(k \cdot x, 1)$ or $(k \cdot x, c \cdot y)$, for some $k, c \in \mathbb{R}$. Thus in N, all gradients have the same direction, $(0, -1)$, so the normal cone is of dimension 1, and N is not a vertex.

For further studies, it could be interesting to take a closer look at what happens to the non-vertex low rank points under small perturbation of the linear pencil. Consider for example the point N in Figure 4.4. Shifting the inner curve, lets denote it f_0, slightly downwards. Then the zero provided in N by f_0 will split into two new vertices, and the rank in N will increase by one. Thus the perturbed spectrahedron has a different number of vertices, and a strictly higher rank for all points on the boundary. In applications, this can make a difference, considering it is often of interest to minimize or maximize the rank, or finding the value of an objective function in the vertices. This is just one example of a perturbation of a spectrahedron, and it would be interesting to dive deeper into what we can achieve by perturbing the linear pencils of spectrahedra.

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