# VERTEX AND PATH ATTRIBUTES OF THE GENERALIZED HYPERCUBE WITH SIMPLEX CUTS

PAUL C. KETTLER

PURE MATHEMATICS E-PRINT SERIES, NO. 7
ISSN 0806–2439

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OSLO
JANUARY 2008

# Contents

**Preface** **ii**
I Introduction . . . ii
II Literature review . . . iii
III Acknowledgments . . . iii

**1 Fundamental cases: one and two cuts of a simplex** **1**
1 One cut of a simplex . . . 1
2 Two cuts of a simplex . . . 10

**2 General cases: the polytope after several cuts to a simplex** **16**
1 Geometry of a multiply-cut simplex . . . 16
2 An existence theorem . . . 19

**3 Vertex and edge counts of a multiply-cut simplex** **25**
1 Two cuts intersecting . . . 25
2 Two cuts, pairwise intersecting . . . 30

**4 The beta problem** **31**
1 Description . . . 31
2 Basic solutions of the uncut polytope . . . 32
3 Edges of the uncut polytope . . . 34
4 Basic solutions of the cut polytope . . . 36

**A Quotient distributions** **40**

**Bibliography** **46**

# Preface

## I Introduction

This monograph sets forth the results of a study of the geometry of a simplex, multiply-cut by hyperplanes, and its implications for constrained optimization. Chapter 1 presents basic cases about the polytope which results when a simplex is cut by either one or two hyperplanes, having designated feasible and infeasible sides. The polytope of study is the part of the simplex remaining feasible to all cuts.

Proceeding in Chapter 2, the study examines the geometry of a polytope after multiple cuts of a simplex, and specifies a test in the form of an existence theorem to determine whether or not $k+1$ cuts each deleting one vertex of a simplex $S_k$ in $k$ dimensions, together delete the entire simplex. This result is important in the subsequent determination of the vertex and edge counts of a polytope with arbitrary cuts. As well, this Chapter presents an efficient algorithm for determining the status — uncut, partially cut, or completely cut — for any sub-simplex of the original simplex.

Chapter 3 addresses attention to the vertex and edge counts of a polytope resulting from cutting a simplex. At first, two cuts operate, and either they intersect within the simplex, or they do not. In the former case, the number of vertices and edges is a simple consequence of a result in Chapter 1. In the latter case, the incremental number emerges in the presentation. An extension to the many cut scenario ensues, with pairwise intersecting allowed within a simplex, to provide a closed formula for vertices and edges.

Chapter 4 discusses the 'beta problem,' an optimization problem involving a feasible simplex constrained by inequalities among disjoint sets of the variables. The theory developed earlier for a simplex with cuts naturally finds a way to illuminate the features of this problem. For added interest, coefficients of the defining matrix are considered uncertain, and thus subject to probability distributions. The gist of the Chapter comprises these and related ideas, with examples.

As one commonly applies the simplex method to solve such 'beta problems,' he necessarily considers quotient distributions on the variables appearing in the pivot equations. For reference, therefore, the study attaches an Appendix which provides statistics for a few of the more commonly encountered quotient distributions.

## II Literature review

The contributions of many authors converge to the present work. Recommended reading includes these studies.

Books and monographs by these authors are the first and best comprehensive source on ideas related to the present work. (Gale 1960; Ford, Jr. and Fulkerson 1962; Isaacson and Keller 1966; Feller 1967; Mangasarian 1969; Tutte 1969; Stoer and Witzgall 1970; Feller 1971; Tutte 1971; Operations Research 1973; Hall, Jr. 1986; Grünbaum 2003). As well, these conference proceedings are deserving of study. (IBM Corporation, Data Processing Division 1964; University of Oklahoma 1971). Further, this thesis merits attention. (Telgen 1979)

These papers address various aspects of polytope geometry, including analysis of vertices, edges, and paths. (Gale 1956; Balinski 1961; Klee 1964; Klee 1966b; Klee and Witzgall 1968; Manas and Nedoma 1968; Grünbaum 1970; Balinski and Russakoff 1972; Bolker 1972; Philip 1972; Mattheiss 1973; Silverman 1973a; Silverman 1973b; Adler, Dantzig, and Murty 1974; Balinski 1974; Burdet 1974; Klee 1974; Greenberg 1975; Zaslavsky 1975; Bolker 1976; Dyer and Proll 1977; Klee 1978; Epelman 1979)

These papers address polytope diameters, heights, maximal path lengths. (Klee 1965; Klee and Walkup 1967; Saigal 1969; Adler 1974; Adler and Dantzig 1974; Padberg and Rao 1974; Lawrence, Jr. 1978; Von Hohenbalken 1978; Walkup 1978)

These papers look to linear programming and related topics. (Klee 1966a; Klee 1967; Klee and Minty 1972; Niedringhaus and Steiglitz 1978; Telgen 1978; Orden 1980)

This paper examines probabilistic aspects of polytope selection. (Kelly and Tolle 1981)

## III Acknowledgments

Basic research for this monograph began at the Graduate School of Business, University of Chicago, during the fall of 1979, concluding with a draft in January 1980. The work has been amended and revised during the intervening years, with the current version having been prepared during December 2007, while the author was in residence at the Centre of Mathematics for Applications and Department of Mathematics, University of Oslo. The author gratefully acknowledges the considerable assistance of these institutions, and their faculty and staff, in facilitating this research.

# Chapter 1

# Fundamental cases: one and two cuts of a simplex

This chapter discusses the geometry of an $n$-dimensional simplex $S_n$ intersected by one, then two, cuts of hyperplanes.

## 1 One cut of a simplex

To begin, consider the case of one cut. Two geometric objects are of interest — the closed convex polytope $P$, which is the intersection of $S_n$ and the cut $C$, and the closed polytope $Q_p$ remaining after the truncation of $p$ simplex vertices.

The principal results of this Chapter are embodied in the twelve theorems and related corollaries. The first two theorems discuss the basic geometry of $P$ and $Q_p$; the third discusses the number of components of the various dimensions of the class of polytopes of which both $P$ and $Q_p$ are examples; the fourth discusses the net number of additional components when passing from $S_n$ to $Q_p$ by means of the hyperplane cut. Theorem 5 establishes the number of vertices and edges of the resulting polytope after several mutually independent cuts, that is, for cuts which do not intersect among themselves within $S_n$. Theorems 6, 7, and 8, relate to the cardinality of vertices and edges when $S_n$ is cut by two hyperplanes, allowing for the possibility that the cutting planes may themselves intersect within $S_n$. Theorem 9 observes the indifference of cutting order on the computation of vertices and edges. Theorems 11 and 12, which are converses, present in greater detail, the set of possible polytopes when two planes cleave a simplex.

To continue, let $S_n$ be a simplex of $n$ dimensions, consisting of the convex closure of $n + 1$ points. Assume that a cutting hyperplane $C$ intersects $S_n$ with $p$ points on one side, and consequently with $n - p + 1$ points on the other side. By convention, call the side of the $p$ points the infeasible halfspace $H_I$, and the other side, including the cut, the feasible halfspace $H_F$. The $p$ points will be said to be made infeasible, deleted, excised, truncated, cut, or cleaved, by $C$ equivalently.

The first result is that the topology of $S_n \cap C = P$ is the cross product topology of $S_{p-1}$ and $S_{n-p}$. Formally,

**Theorem 1.1.**

$$P = S_{p-1} \times S_{n-p}$$

*Proof.* Let $x = \sum_0^n a_i p_i$ be a point of C given in terms of its barycentric coordinates on the vertices $\{p_i\}$ of $S_n$. Then $x \in P \iff a_i \geqslant 0, \forall i$. Let $b := \sum_0^{p-1} a_i$, implying $1 - b = \sum_p^n a_i$. Now, b lies in $(0, 1)$ for x in some open neighborhood N of P, with $C \supset N \supset P$, because otherwise $x \in S_{n-p}$ or $x \in S_{p-1}$. Let the projections of x into $R_{p-1}$ and $R_{n-p}$, the hyperplanes containing $S_{p-1}$ and $S_{n-p}$, respectively, be

$$(v, w) = \left( \frac{1}{b} \sum_0^{p-1} a_i p_i, \frac{1}{1-b} \sum_p^n a_i p_i \right)$$

Then

$$F : N \to R_{p-1} \times R_{n-p},$$

defined by

$$F(x) = (v, w)$$

is a bicontinuously differentiable function (a diffeomorphism) of N onto the image of N. Furthermore, F restricted to P is $S_{p-1} \times S_{n-p}$. Observe that F is linear if and only if b is constant on C, a condition equivalent to parallelism among $R_{p-1}$, C, and $R_{n-p}$. □

The second result is that the topology of $S_n \cap H_F = Q_p$ is the cross product topology $S_p \times S_{n-p}$. Formally,

**Theorem 1.2.**

$$Q_p = S_p \times S_{n-p}$$

*Proof.* Let the cut C delete $p + 1$ vertices from $S_{n+1}$. Then $P = S_{n+1} \cap C = S_p \times S_{n-p}$ by Theorem 1.1. Choose one of the deleted vertices of $S_{n+1}$, say $p_0$, and call the convex closure of the remaining $n+1$ vertices $S_n$. Let $H_F$ be the feasible halfspace of C, restricted to the hyperplane of $S_n$. It remains to demonstrate that $Q_p = S_n \cap H_F$ is diffeomorphic to $S_{n+1} \cap C$. To that end, let $x = \sum_0^{n+1} a_i p_i$ be a point of C in barycentric coordinates on the vertices $\{p_i\}$ of $S_{n+1}$. Then $x \in P \iff a_i \geqslant 0, \forall i$. Now, $a_0 < 1$ for x in some open neighborhood N of P, $C \supset N \supset P$, because otherwise $x = p_0$. Let the projection of x from $p_0$ into $R_n$, the hyperplane containing $S_n$, be

$$w = \frac{1}{1 - a_0} \sum_1^{n+1} a_i p_i$$

Then

$$G : N \to R_n,$$

defined by

$$G(x) = w$$

is a bicontinuously differentiable function (a diffeomorphism) of N onto the image of N. Furthermore, G restricted to P is $S_n \times H_F$. Observe that G is linear — in fact is an expansion mapping by the factor $1/(1 - a_0)$ — if and only if $a_0$ is constant on C, a condition equivalent to parallelism between C and $R_n$, which can obtain only if $p = 0$. □

**Corollary 1.3.**

$$Q_p = Q_{n-p}$$

*Proof.* Observe that the cross product is commutative. □

The next concern is to investigate the geometry of P and $Q_p$, that is, to determine the nature and arrangement and count of the components of the various dimensions. The task is essentially the same for both P and $Q_p$, insofar as they are both cross products of simplexes, differing only in the dimension of one factor.

**Theorem 1.4.** *There are*

$$\begin{aligned} A_n(p,j) &= \sum_{i=0}^{j} [C(p+1, i+1)C(n-p+1, j-i+1)] \\ &= \frac{1}{j!} \frac{d^j}{ds^j} h_A(s) \bigg|_{0+} \end{aligned}$$

*faces of dimension* j *in* $Q_p = S_p \times S_{n-p}$*, where the terms of the summation give the count of faces* $S_i \times S_{j-i}$*, and*

$$H_A(s) = \frac{1}{s^2} \left[(1+s)^{p+1} - 1\right] \left[(1+s)^{n-p+1} - 1\right]$$

*Furthermore,*

$$\sum_{j=0}^{n} A_n(p,j) = \left[2^{p+1} - 1\right] \left[2^{n-p+1} - 1\right]$$

*Proof.* Let $Q_p = S_p \times S_{n-p}$ be the general case. A j-dimensional face of $Q_p$ is a set $S_i \times S_{j-i}$, for $0 \leqslant i \leqslant j$, where $S_i \subset S_p$ and $S_{j-i} \subset S_{n-p}$. There are $C(p+1, i+1)$ ways that one may select $S_i$ within $S_p$, and $C(n-p+1, j-i+1)$ ways that one may select $S_{j-i}$ within $S_{n-p}$. Hence there are $C(p+1, i+1)C(n-p+1, j-i+1)$ ways to generate $S_i \times S_{j-i}$, and

$$A_n(p, j) = \sum_{i=0}^{j} C(p+1, i+1)C(n-p+1, j-i+1)$$

ways to generate a j-dimensional face. But this summation is the j-th term (starting with the zeroth) of the convolution of the two series $\{C(p+1, i+1)\}$ and $\{C(n-p+1, i+1)\}$, the terms of which represent the ways of selecting $S_i$ from $S_p$ and $S_{n-p}$, respectively. Now,

$$f_s = \frac{1}{s}\left[(1+s)^{p+1} - 1\right]$$

and

$$g_s = \frac{1}{s}\left[(1+s)^{n-p+1} - 1\right]$$

are the generating functions of the cited series, hence

$$\begin{aligned} h_A(s) &= f_s g_s \\ &= \frac{1}{s^2}\left[(1+s)^{p+1} - 1\right]\left[(1+s)^{n-p+1} - 1\right] \end{aligned}$$

is the generating function of the convolution. Consequently, the series of component counts

$$\{A_n(p, j)\} = \left\{ \frac{1}{j!} \frac{d^j}{ds^j} h_A(s) \bigg|_{0+} \right\},$$

and the total number of components of all dimensions

$$\begin{aligned} \sum_{j=0}^{n} A_n(p, j) &= h_A(1) \\ &= \left[2^{p+1} - 1\right]\left[2^{n-p+1} - 1\right] \end{aligned}$$ □

As an alternative to this iterative development of $A_n(p, j)$ one can opt for a recursive development. This latter approach and its reconciliation with the former offer additional insights into the structure of $S_p \times S_{n-p}$.

**Corollary 1.5.**

$$\begin{aligned} A_n(p,j) &= C(n+2,j+2) - [C(p+1,j+2) + C(n-p+1,j+2)] \\ &= A_{n+1}(0,j+1) - [A_p(0,j+1) + A_{n-p}(0,j+1)] \end{aligned}$$

*Proof.* As before, let a hyperplane cut truncate $p$ vertices of $S_n$. Then by Theorems 1.1 and 1.2, the intersection with the cut is $P = S_{p-1} \times S_{n-p}$, and the polytope remaining after the cut is $Q_p = S_p \times S_{n-p}$. One is therefore able to generate a formula for $A_n(p,j)$, the count of components of dimension $j$ in $P$, in terms of $A_{n-1}(p-1,j)$. Specifically,

$$\begin{aligned} A_n(p,j) &= C(n+1,j+1) + A_{n-1}(p-1,j) - C(p,j+1) \\ &= A_n(0,j) + A_{n-1}(p-1,j) - A_{p-1}(0,j), \end{aligned}$$

where the first term of the summation is the number of components of dimension $j$ in $S_n$, the second term is the number of components of dimension $j$ added by the cut, and the third term is the number of components of dimension $j$ deleted by the cut. The recursions begin with

$$A_n(0,j) = C(n+1,j+1)$$

and

$$A_0(1,j) = 0,$$

which by the recursion implies

$$A_n(n+1,j) = 0$$

By Theorem 1.4, however, it follows that

$$\begin{aligned} &\sum_{i=0}^{j} C(p+1,i+1)C(n-p+1,j-i+1) \\ &= C(n+1,j+1) + \sum_{i=0}^{j} C(p,i+1)C(n-p+1,j-i+1) - C(p,j+1) \end{aligned}$$

must be a binomial coefficient identity. Demonstration of this fact relies on the more common identity (couched in the incumbent notation)

$$\sum_{i=-1}^{j+1} C(p+1,i+1)C(n-p+1,j-i+1) = C(n+2,j+2),$$

readily proved by induction. This summation is, however, simply the expression for $A_n(p,j)$, with an additional term at the beginning and at the end. Hence an alternate form for $A_n(p,j)$ is

$$C(n+2,j+2) - [C(p+1,j+2) + C(n-p+1,j+2)],$$

from which the more complex identity above follows directly. □

An alternative argument for this formula lends some insight into the process of determining the count of extreme points. Let $Q_p$ be given by the barycentric coordinates induced by those of its factors, $S_p$ and $S_{n-p}$. Specifically, let $(a_0, a_1, \ldots, a_p)$ and $(b_0, b_1, \ldots, b_{n-p})$ have all non-negative coordinates, with

$$\sum_{i=0}^{p} a_i = \sum_{j=0}^{n-p} b_j = 1,$$

be representations of $S_p$ and $S_{n-p}$. Then the $(n+2)$-tuple

$$(a_0, a_1, \ldots, a_p, b_0, b_1, \ldots, b_{n-p})$$

is a point of $Q_p$. The j-faces of $Q_p$, consequently, are the linear combinations of the coefficients, with $n - j$ of them held at zero. Now there are

$$C(n+2, n-j) = C(n+2, j+2)$$

ways to distribute $n - j$ zeros among $n + 2$ coefficients. However, any way which distributes only zeros to either the first $p+1$ coefficients or the last $n-p+1$ coefficients (meaning all the variable entries to either the first or last group) is not allowable, because the groups respectively must sum to one. Therefore, there are

$$C(n-p+1, j+2) + C(p+1, j+2)$$

disallowed combinations.

**Corollary 1.6.**

$$\begin{aligned} A_n(p,j) &= C(n+2, j+2) \\ &= A_{n+1}(0, j+1) \quad \textit{if} \quad j \geqslant \max(p, n-p) \end{aligned}$$

*Otherwise,*

$$A_n(p,j) < A_{n+1}(0, j+1)$$

*Proof.* This assertion follows directly from Corollary 1.5, noting that the terms in the brackets are zero if $j \geqslant p$, or if $j \geqslant n - p$, respectively. □

This result has paradoxical overtones, because it states that $A_n(p,j)$ is diminished from the simple expression $A_{n+1}(0, j+1)$ if and only if a component of dimension j exists properly in at least one of the factors $S_p$ or $S_{n-p}$.

Six additional results derive directly from Theorem 1.4. They are listed here as Corollaries.

**Corollary 1.7.**

$$A_n(p,j) = A_n(n-p, j)$$

**Corollary 1.8.**

$$A_n(p,0) = (p+1)(n-p+1)$$

**Corollary 1.9.**

$$A_n(p,0) = (n+1) + A_{n-2}(p-1,0)$$

**Corollary 1.10.**

$$\begin{aligned} A_n(p,1) &= \frac{n}{2}(p+1)(n-p+1) \\ &= \frac{n}{2} A_n(p,0) \end{aligned}$$

**Corollary 1.11.**

$$A_n(p,1) = C(n+1,2) + \frac{n}{n-2} A_{n-2}(p-1,1)$$

**Corollary 1.12.**

$$A_n(1,1) = n^2$$

**Corollary 1.13.** *The mean dimension of components of* $Q_p = S_p \times S_{n-p}$*, is*

$$\frac{\partial}{\partial s} h_A(s)\bigg|_{\frac{1}{h_A(1)}}$$

*or*

$$\big((n-2)2^{n+1}-2\big) - \frac{(p-3)2^p + (n-p-3)2^{n-p}}{(2^{p+1}-1)\,(2^{n-p+1}-1)}$$

Corollaries 1.8 and 1.10, giving the counts of vertices and of edges, respectively, in $Q_p$, are important to the sequel.

The next business of this Chapter concerns the development of formulas parallel to those of Theorem 1.4 and its Corollaries, relating to the net number of components added when passing from $S_n$ to $Q_p = S_p \times S_{n-p}$ by means of a hyperplane cut C.

**Theorem 1.14.** *There are*

$$\begin{aligned} B_n(p,j) &= C(n+1,j+2) - [C(p+1,j+2) + C(n-p+1,j+2)] \\ &= A_n(0,j+1) - [A_p(0,j+1) + A_{n-p}(0,j+1)] \\ &= \frac{1}{j!}\frac{d^j}{ds^j} h_B(s)\bigg|_{0+} \end{aligned}$$

*faces of dimension* $j$ *in net addition when passing from* $S_n$ *to* $Q_p = S_p \times S_{n-p}$*, where*

$$H_B(s) = \frac{1+s}{s^2}\,[(1+s)^p - 1]\,[(1+s)^{n-p} - 1]$$

*Furthermore,*

$$\sum_{j=0}^{n} B_n(p,j) = 2(2^p - 1)(2^{n-p} - 1)$$

*Proof.* The net addition of components of dimension $j$ when cutting $p$ vertices from $S_n$ is

$$\begin{aligned} B_n(p,j) &= A_n(p,j) - C(n+1,j+1) \\ &= A_{n-1}(p-1,j) - C(p,j+1) \end{aligned}$$

by the recursion relation. In words, $B_n(p,j)$ equals either the j-faces of $Q_p$ less the j-faces of $S_n$, or the j-faces of $P = S_n \cap C$ less the j-faces of the truncated $S_{p-1}$. By Corollary 1.5,

$$\begin{aligned} B_n(p,j) &= C(n+2,j+2) - [C(p+1,j+2) + C(n-p+1,j+2)] - C(n+1,j+1) \\ &= C(n+1,j+2) - [C(p+1,j+2) + C(n-p+1,j+2)] \end{aligned}$$

The generating function $h_B(s)$ of $\{B_n(p,j)\}$ is likewise the difference of $h_A(s)$ and

$$\frac{1}{s}\left[(1+s)^{n+1} - 1\right],$$

which is the generating function of $\{C(n+1,j+1)\}$. Thus,

$$\begin{aligned} h_B(s) &= \frac{1}{s^2}\left[(1+s)^{p+1} - 1\right]\left[(1+s)^{n-p+1} - 1\right] - \frac{1}{s}(1+s)^{n+1} \\ &= \frac{1+s}{s^2}\left[(1+s)^p - 1\right]\left[(1+s)^{n-p} - 1\right] \end{aligned}$$

Consequently, the total net number of components added of all dimensions

$$\begin{aligned} \sum_{j=0}^{n} B_n(p,j) &= h_B(1) \\ &= 2[2^p - 1][2^{n-p} - 1] \end{aligned}$$

□

Following now is a series of Corollaries parallel to those to Theorem 1.4, starting with the second. The parallel development of the first has been included in the just completed proof of Theorem 1.14.

**Corollary 1.15.**

$$\begin{aligned} B_n(p,j) &= C(n+1,j+2) \\ &= A_n(0,j+1) \quad \textit{if} \quad j \geqslant \max(p, n-p) \end{aligned}$$

*Otherwise,*

$$B_n(p,j) < A_n(0,j+1)$$

*Proof.* Note that

$$A_p(0, j+1) + A_{n-p}(0, j+1) = 0 \iff j \geqslant \max(p, n-p),$$

exactly as in Corollary 1.5. □

A comment parallel to that of Corollary 1.6, about paradoxical overtones, applies here. $B_n(p, j)$ is diminished from the simple expression $A_n(0, j+1)$ if and only if a component of dimension j exists properly in at least one of the factors $S_p$ or $S_{n-p}$.

Six additional results derive directly from Theorem 1.14. They are listed here as Corollaries.

**Corollary 1.16.**

$$B_n(p, j) = B_n(n-p, j)$$

**Corollary 1.17.**

$$B_n(p, 0) = p(n-p)$$

**Corollary 1.18.**

$$B_n(p, 0) = (n-1) + B_{n-2}(p-1, 0)$$

**Corollary 1.19.**

$$\begin{aligned} B_n(p, 1) &= \frac{n}{2} p(n-p) \\ &= \frac{n}{2} B_n(p, 0) \end{aligned}$$

**Corollary 1.20.**

$$B_n(p, 1) = C(n, 2) + \frac{n}{n-2} B_{n-2}(p-1, 1)$$

**Corollary 1.21.**

$$B_n(1, 1) = C(n, 2)$$

**Corollary 1.22.** *The mean dimension of components added in passing from* $S_n$ *to* $Q_p = S_p \times S_{n-p}$*, is*

$$\left.\frac{\partial}{\partial s} h_B(s)\right|_{\frac{1}{h_B(1)}}$$

*or*

$$\big((n-3)2^n - 3\big) - \frac{(p-3)2^p + (n-p-3)2^{n-p}}{(2^p - 1)\,(2^{n-p} - 1)}$$

Corollaries 1.17 and 1.19, giving the net additions of vertices and of edges, respectively, in passing from $S_n$ to $Q_p$, are important to the sequel.

The results of the first four theorems extend easily to a simple class of polytopes formed by multiple cuts of $S_n$. The class is defined by the condition that no point of $S_n$ be infeasible to more than one of the cuts. Henceforth, cuts which satisfy this criterion will be termed *vertex independent*. Under this assumption, the various cuts have independent effect on the number of components of any dimension in the resulting polytope. Formally, one has

**Theorem 1.23.** *If* $k$ *hyperplanes, no pair of which intersect within* $S_n$*, cut* $\{p_i\}$ *vertices, respectively, from* $S_n$*, then the resulting polytope* $P_{n,k}(\{p_i\})$ *has*

$$A_n(0,j) + \sum_{i=1}^{k} B_n(p_i, j)$$

*components of dimension* $j$*,* $0 \leqslant j \leqslant n$*.*

*Proof.* The terms of the summation are the independent increments induced by the respective cuts, by Theorem 1.14. □

**Corollary 1.24.** $P_{n,k}(\{p_i\})$ *has* $(n+1) + \sum_{i=1}^{k} p_i(n-p_i)$ *vertices.*

**Corollary 1.25.** $P_{n,k}(\{p_i\})$ *has* $\frac{n}{2}\left[(n+1) + \sum_{i=1}^{k} p_i(n-p_i)\right]$ *edges.*

# 2 Two cuts of a simplex

At this juncture it is natural to extend the theory of cutting a simplex to the case of two cuts, without the restriction that the cuts not intersect within the simplex. One approach is to consider an arbitrary cut of $Q_p = S_p \times S_{n-p}$, which, as has been developed, is the result of cutting $S_n$ once. The questions arise: How many new vertices appear in the cutting hyperplane?, and how many net vertices accrue in the course of truncating $Q_p$? The remainder of this Chapter addresses these and related questions.

Recall that the 0-faces (points) and 1-faces (edges) of $Q_p$ are of the form $S_0 \times S_0$, and of the form $S_0 \times S_1$ or $S_1 \times S_0$, respectively. This seemingly trivial comment is important because it distinguishes these lower dimensional components as the only ones guaranteed to lie wholly within a copy of one of the factors of $S_p \times S_{n-p}$. In turn, this guarantee allows a simpler analysis to determine the vertex and edge counts of the truncated $Q_p$, than would be required to determine the counts of higher dimensional components. To derive the benefits of this streamlined theory, the discussion henceforth focuses on vertices and edges.

**Theorem 1.26.** *The number of vertices in the polytope* $Q(p,V)$*, which is the intersection of* $Q_p$ *and a hyperplane cut, is*

$$M(p,V) = \sum_{i=0}^{p} B_{n-p+1}(r_i, 0) + \sum_{j=0}^{n-p} B_{p+1}(s_j, 0),$$

*where* $V$ *is an incidence matrix of vertices of* $Q_p$ *deleted by the cut, and*

$$r_i = \sum_{j=0}^{n-p} v_{i,j}$$

*and*

$$s_j = \sum_{i=0}^{p} v_{i,j}$$

*Proof.* The polytope $Q(p)$, as stated in Corollary 1.8, has $(p+1)(n-p+1)$ vertices, which viewed symmetrically, are either the $p+1$ copies of the vertices of $S_{n-p}$ or the $n-p+1$ copies of the vertices of $S_p$. The cut of $Q_p$, therefore, defines new geometry by specifying which of the $(p+1)(n-p+1)$ vertices become infeasible, and by which remain feasible. One may display this information in an incidence matrix $V$, of size $p+1$ by $n-p+1$, containing 1's and 0's according to whether or not the cut deletes a given vertex of $Q_p$. The order of the rows and columns of $V$ is immaterial in this instance, reflecting an arbitrary ordering of the vertices of $S_P$ and of $S_{n-p}$.

Suppose, now, that the cut of $Q_p$ is passed, and therefore each entry of

$$V = [v_{i,j}],\ (0,0) \leqslant (i,j) \leqslant (p, n-p)$$

is assigned the value 1 or 0. Then one may ascertain the number of vertices in the cutting plane by the following reasoning. Look at a row, say row $i$, of $V$, which contains $r_i$ instances of the digit 1. Before the cut, the positions of this row represented vertices of a simplex $S_{n-p}$ which contained $C(n-p+1,2) = A_{n-p}(0,1)$ edges. After the cut, however, a simplex $S_{r_i-1}$ containing

$$C(r_i,2) = A_{r_i-1}(0,1)$$

edges, became entirely infeasible, and a simplex $S_{n-p-r_i}$ containing

$$C(n-p+1-r_i,2) = A_{n-p-r_i}(0,1)$$

edges, remained entirely feasible. Therefore, the residual of edges

$$\begin{aligned}
&A_{n-p}(0,1) - [A_{r_i-1}(0,1) + A_{n-p-r_i}(0,1)] \\
&= r_i(n-p+1-r_i) \\
&= A_{n-p-1}(r_i-1,0) \\
&= B_{n-p+1}(r_i,0)
\end{aligned}$$

were cut by the hyperplane, generating one new vertex at each intersection. This number is, of course, the count of edges connecting the $r_i$ vertices of $S_{r_i-1}$ with the $n-p+1-r_i$ vertices of $S_{n-p-r_i}$. Similar reasoning applies to a typical column of $V$, say column j, which contains $s_j$ instances of the digit 1. In this direction, the residual of edges is

$$A_{p-1}(s_j - 1, 0) = B_{p+1}(s_j, 0),$$

representing the count of edges between $S_{s_j - 1}$ and $S_{p-s_j}$. □

**Corollary 1.27.** *The number of edges in* $Q(p, V)$ *is* $\dfrac{n-1}{2} M(p, V)$.

*Proof.* The statement is a consequence of the Theorem and of a general result for simple polytopes, that the number of edges is half the dimension $n$ times the number of vertices. Demonstration of that fact follows easily from the observations that $n$ edges emanate from every vertex, and that every edge terminates with two vertices. □

The next step is to determine the net effect of the cut of $Q_p$ on the count of vertices and edges in the remaining polytope. To ascertain this number it is only necessary to subtract from $M(p, V)$ the count of vertices deleted by the cut, which in turn equals either $\sum_{i=0}^{p} r_i$ or $\sum_{j=0}^{n-p} s_j$.

**Theorem 1.28.** *The net addition to the count of vertices in the formation of the polytope* $R(p, V)$ *by intersecting* $Q_p$ *with the feasible halfspace* $H_F$ *of a hyperplane cut is*

$$\begin{aligned} N(p, V) &= \sum_{i=0}^{p} B_{n-p}(r_i, 0) + \sum_{j=0}^{n-p} B_{p+1}(s_j, 0) \\ &= \sum_{i=0}^{p} B_{n-p+1}(r_i, 0) + \sum_{j=0}^{n-p} B_p(s_j, 0) \end{aligned}$$

*Proof.*

$$\begin{aligned} B_{n-p+1}(r_i, 0) - r_i &= r_i(n - p + 1 - r_i) - r_i \\ &= r_i(n - p - r_i) \\ &= B_{n-p}(r_i, 0) \\ B_{p+1}(s_j, 0) - s_j &= s_j(p + 1 - s_j) - s_j \\ &= s_j(p - s_j) \\ &= B_p(s_j, 0) \end{aligned}$$

□

**Corollary 1.29.** *The net addition of edges in forming* $R(p, V)$ *is* $\dfrac{n}{2} N(p, V)$.

The next result is the focal theorem of this sequence.

**Theorem 1.30.** $T(p, V) = A_n(p, 0) + N(p, V)$ *is the total number of vertices in* $R(p, V)$.

*Proof.* $A_n(p, 0)$ is the number of vertices in $Q_p$, and $N(p, V)$ is the increment by means of the second cut. □

**Corollary 1.31.** $L(p, V) = \frac{n}{2} T(p, V)$ *is the total number of edges in* $R(p, V)$.

Observe, now, that $R(p, V)$ is independent of the order in which one considers the two cuts of $S_n$. Thus if the second cut, considered above, alone deletes q vertices of $S_n$, and if the first cut, considered above, induces the incidence matrix $W$ on the resulting $Q_q$ in producing $R(q, W)$, then one has,

**Theorem 1.32.** $R(q, W) = R(p, V)$

*Proof.* Omitted

**Corollary 1.33.** $T(q, W) = T(p, V)$

**Corollary 1.34.** $L(q, W) = L(p, V)$

Owing to considerations of linearity, it is not in general possible arbitrarily to select vertices of $Q_p$ for deletion or inclusion by a cut. The discussion continues by examining the necessary restrictions imposed on the incidence matrix V by these considerations. The first of two relevant observations is that the deletion of any two vertices of differing row and column be accompanied by the deletion of at least one of the vertices at the junctions of the respective rows and columns.

A consequence of this analysis, and the second of the two observations, is that of any two rows (or of any two columns) the union of two rows (columns) is one of the rows (columns) and the intersection of the two is the other. In this sense union is understood to mean element-by-element logical 'or,' and intersection is understood to mean element-by-element logical 'and.'

**Lemma 1.35.** *If, in an incidence matrix* $V$, $v_{i_1,j_1} = v_{i_2,j_2} = 1$, *with* $i_1 \neq i_2$ *and* $j_1 \neq j_2$, *then either* $v_{i_1,j_2} = 1$ *or* $v_{i_2,j_1} = 1$, *or both. Equivalently, if* $v_{i_1,j_1} = v_{i_2,j_2} = 0$, *with* $i_1 \neq i_2$ *and* $j_1 \neq j_2$, *then either* $v_{i_1,j_2} = 0$ *or* $v_{i_2,j_1} = 0$, *or both.*

*Proof.* Indirect

Assume that $v_{i_1,j_1}$ and $v_{i_2,j_2}$ of the incidence matrix V are deleted, with $i_1 \neq i_2$ and $j_1 \neq j_2$. Assume also that $v_{i_1,j_2}$ and $v_{i_2,j_1}$ are not deleted. It follows that the lines (which are not edges of the polytope) connecting $v_{i_1,j_1}$ with $v_{i_2,j_2}$, and $v_{i_1,j_2}$ with $v_{i_2,j_1}$, intersect at a point which is both feasible and infeasible. Simply stated, the face defined by these four vertices is a [two-dimensional, topological] square, and it is impossible to cut diagonally opposite corners of a square (recalling the assumption of convexity) without excising a third corner. □

**Theorem 1.36.** *One can order the rows (columns) of an incidence matrix* $V$ *to satisfy a transitive inclusion relation, by which is meant that if row (column)* $V_{i_1}$ *includes row (column)* $V_{i_2}$, *then*

$$v_{i_2,j} = 1 \Longrightarrow v_{i_1,j} = 1, \forall j$$

*and*

$$\nu_{i,j_2} = 1 \Longrightarrow \nu_{i,j_1} = 1, \forall i;$$

*equivalently,*

$$\nu_{i_1,j} = 0 \Longrightarrow \nu_{i_2,j} = 0, \forall j$$

*and*

$$\nu_{i,j_1} = 0 \Longrightarrow \nu_{i,j_2} = 0, \forall i$$

*Alternatively, one can order the rows (columns) so that*

$$\nu_{i_1,j_1} = 1 \Longrightarrow \nu_{i_2,j_2} = 1, \quad \textit{for} \quad (i_2, j_2) \leqslant (i_1, j_1);$$

*equivalently,*

$$\nu_{i_1,j_1} = 0 \Longrightarrow \nu_{i_2,j_2} = 0, \quad \textit{for} \quad (i_2, j_2) \geqslant (i_1, j_1)$$

*Proof.* Indirect

If the hypothesized relationship were not true, then there would exist a situation of

$$\nu_{i_1,j_1} = \nu_{i_2,j_2} = 1$$

and

$$\nu_{i_1,j_2} = \nu_{i_2,j_1} = 0,$$

by which line of analysis cannot occur. □

*Remark.* This relationship between rows (columns) induces an order relation among all rows (columns) with those of a greater number of 1′s including (conventionally — or else excluding) those of a lesser or equal number of 1′s. This transitive relation is anti-symmetric to that implied in noting the number of 0′s in a row (column.) This observation is necessarily true because the number of 1′s and 0′s in a row (column) is fixed. Furthermore, any analysis which holds for the subset of 1′s must also hold for the subset of 0′s, because it is a matter of choice which side of a cut is called feasible.

The discussion concludes by developing the converse to Theorem 1.36, a result which is of interest in its own right.

**Theorem 1.37 (Converse of Theorem 1.36).** *If* $V$ *is a* $p+1$ *by* $n-p+1$ *matrix of ones and zeros such that* $\nu_{i_1,j_1} = 1$ *implies* $\nu_{i_2,j_2} = 1$ *for* $(i_2, j_2) \leqslant (i_1, j_1)$ *[equivalently, such that* $\nu_{i_1,j_1} = 0$ *implies* $\nu_{i_2,j_2} = 0$ *for* $(i_2, j_2) \geqslant (i_1, j_1)$*,] or if* $V$ *can be put into this form by a permutation of rows and columns, then* $V$ *is an incidence matrix.*

*Proof.* Assume that the $p+1$ by $n-p+1$ matrix $V$ of ones and zeros is such that $v_{i_1,j_1}=1$ implies $v_{i_2,j_2}=1$ for $(i_2,j_2)\leqslant(i_1,j_1)$ [equivalently, such that $v_{i_1,j_1}=0$ implies $v_{i_2,j_2}=0$ for $(i_2,j_2)\geqslant(i_1,j_1)$.] The task is to produce the parameters of a hyperplane which when cutting $S_p\times S_{n-p}$ induces the incidence matrix $V$. Let the points of $S_p$ and $S_{n-p}$ have barycentric coordinates $(a_0,a_1,\dots,a_p)$ and $(b_0,b_1,\dots,b_{n-p})$, respectively. Then the points of $S_p\times S_{n-p}$ are $(n+2)$-tuples $(a_0,a_1,\dots,a_p,b_0,b_1,\dots,b_{n-p})$, with the coordinates summing to 2. One additional independent linear relation on these points, for example, the relation

$$\sum_{i=0}^{p} c_i a_i + \sum_{j=0}^{n-p} d_j b_j = 2$$

determines a hyperplane. In the case that $(c_0,c_1,\dots,c_p,d_0,d_1,\dots,d_{n-p})$ consists of all 1's except for a single 0, the hyperplane is an $(n-1)$-face of $Q_p$; for the points of $Q_p$ the above sum does not exceed 2. Now, since the vertices of $S_p\times S_{n-p}$ have $a_i=b_j=1$ for some pair $(i,j)$, with other coordinates zero, the task reduces to finding coefficients $\{c_i\}$ and $\{d_j\}$ such that $v_{i,j}=1$ implies $c_i+d_j>2$, and such that $v_{i,j}=0$ implies $c_i+d_j<2$. A solution is provided by $c_i=1-i$ and $d_j=i\left(V_j^T\right)+\frac{1}{2}$, where $i\left(V_j^T\right)$ is the index of the first zero in the j-th column $V_j^T$ of $V$, (and where $i\left(V_j^T\right)=p+1$ if $V_j^T$ consists entirely of ones.) $\square$

# Chapter 2

# General cases: the polytope after several cuts to a simplex

This Chapter delves into the geometric description, or specification, of the polytope remaining after cutting a simplex by several hyperplanes. As heretofore, it is assumed that cuts of $S_n$ are vertex independent. The challenge is to convert an algebraic description of a simplex and cuts into information about the inclusion, partial inclusion, or deletion, of the various components of $S_n$ by the cuts.

## 1 Geometry of a multiply-cut simplex

To begin, assume $m + 1$ cutting hyperplanes of $S_n$, with no redundancy. Assume further that $t$ vertices of $S_n$ remain feasible to all cuts, so that the correspondingly spanned simplex $S_{t-1}$ remains feasible, along with all of its components. Since $t$ vertices of $S_n$ are within this region of feasibility to all cuts, and consequently since $n - t + 1$ vertices are outside the region, the specification of the polytope depends on the partial inclusion or deletion of the components of positive dimension belonging to the simplex $S_{n-t}$ spanned by these latter vertices. There are

$$\left(s^{n-t+1} - 1\right) - (n - t + 1) = 2^{n-t+1} - (n - t + 2)$$

such components.

The task of determining the partial inclusion or deletion of the components of $S_{n-t}$ is rendered easier by the observation of two facts. The first is the trivial point that if a component $S_k \subset S_{n-t}$ is deleted, then so are the components of $S_k$ of all lower dimensions. The second, deserving of more rigor, is that under some conditions the elimination of the boundary of $S_k$ implies the elimination of $S_k$ itself. Formally, then,

**Lemma 2.1.** *If fewer than $k + 1$ cuts eliminate the boundary of $S_k$, then these cuts together eliminate $S_k$.*

*Proof.* Indirect

Assume to the contrary that there exists a feasible point interior to $S_k$. Then convexity assures that the entire feasible region is interior to $S_k$. Furthermore, the feasible region must be bounded solely by the cuts, insofar as the boundary of $S_k$ is infeasible. But this condition is impossible because at least $k + 1$ hyperplanes of dimension $k - 1$ are needed to bound a compact region. □

This Lemma suggests another, more powerful, result.

**Lemma 2.2.** *If fewer than* $j + 1$ *cuts eliminate the boundaries of all components* $\{S_j\} \subset S_k$, i.e., *eliminate all components* $\{S_{j-1}\} \subset S_k$*, then these cuts together eliminate* $S_k$*.*

*Proof.* By Lemma 2.1 the cuts eliminate all components $\{S_j\}$. Iterating the argument $k-j$ additional times provides the desired conclusion. □

The task at hand, in consequence of these facts, reduces to the verification of whether or not components of $S_{n-t}$, typically $S_k$, partially excluded by $k+1$ different cuts of the $k + 1$ vertices, are in fact totally excluded.

An algorithm for the systematic efficient determination of the specification of the polytope now suggests itself. Begin by examining the components of dimension $m$ in $S_{n-t}$. These components have their vertices cut by $m + 1$ distinct hyperplanes. Those components which are totally excluded by the respective cuts will, of course, have totally excluded components of all lower dimensions. For those which are not totally excluded, continue by examining the $m+1$ boundary components of dimension $m-1$, the vertices of each of which are cut by $m$ separate hyperplanes. Proceed through the tree structure, examining all components of lower dimensions. Upon completion of the analysis for dimension one, therefore, the inclusion status of all components for which the vertices are separately deleted will be known, from which the status of all remaining components of $S_{n-t}$, and thus of $S_n$, may be inferred.

The analysis also suggests a classification system for polytopes so generated, wherein a polytope is included in Class $K_i$ if no component of dimension $i$ is completely eliminated from $S_n$. Clearly, $K_i \subset K_j$ if $i \leqslant j$.

Consider an algebraic description of a system of a simplex and cuts. For the purposes of this discussion, it will be assumed that the system is in the following standard barycentric format.

1. The vertices of $S_n$ are the unit vectors in $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$, with the simplex given as the convex linear closure of these points. If $X$ be a vector in the space, therefore, then the simplex is defined as the set

$$\left\{X \mid X \geqslant \varnothing \text{ and } E^T X = 1\right\},$$

where $E$ is the vector of all ones. Conventionally, an order relation between vectors shall be assumed to apply to each pair of elements.

2. The cuts are the inequalities of the vector relation $CX \leqslant E$, where $C$ is $m+1$ by $n+1$. Observe that even though the right hand side is in normalized form, the matrix $C$ is not necessarily unique, for

$$B_i X = \frac{1}{1-a}\left(C_i - aE^T\right) X \leqslant 1$$

also suffices as an inequality, for any real parameter $a < 1$. A canonical form for $C$ may be imposed by the requirement $CE = E$, which is the condition that all cuts of $S_n$ pass through $E$.

If a system be presented for analysis in general rectangular coordinates, however, a straightforward transformation identifies an equivalent barycentric system. Specifically, let $\{Y^0, Y^1, \ldots, Y^n\}$ be the vertices of the simplex $S_n \subset R_n$. Then the linear transformation $Y = F(X)$ defined by $F = \left(Y^0 Y^1 \cdots Y^n\right) : \mathbb{R}^{n+1} \to \mathbb{R}^n$, maps the unit vectors of $\mathbb{R}^{n+1}$ onto the vertices of $S_n$. Assuming that these vertices are in general position, *i.e.*, that $F$ is of full rank, then $Y = G(X)$ is invertible, where $G$ is $F$ restricted to the hyperplane of the unit vectors. In particular, $S = G^{-1}(Y)$ is given by the solution to $\left(EY^T\right)^T = \left(EF^T\right)^T X$. For proof that $\left(EF^T\right)^T$ is invertible, assume without loss of generality that $\{Y^0, Y^1, \ldots, Y^n\}$ are linearly independent. Then the null space of $\left(EF^T\right)^T$ consists only of the zero vector, because together $E^T X = 0$ and $FX = \varnothing$ imply

$$\left(Y^1 - Y^0\right) x_1 + \left(Y^2 - Y^0\right) x_2 + \cdots + \left(Y^n - Y^0\right) x_n = \varnothing,$$

which implies $x_1 = x_2 = \cdots = x_n = 0$, which in turn implies $x_0 = 0$.

Next assume that the cuts in the same rectangular coordinates are given by $DY \leqslant K$. Then the induced cuts are $DFX \leqslant K$ with $E^T X = 1$. Equivalently, a representative transformed cut is

$$\left(D_i F - (k_i - 1)E^T\right) X = C_i X \leqslant 1$$

with $E^T X = 1$, which is a cut in standard form. Letting

$$C = \left({C_0}^T {C_1}^T \cdots {C_m}^T\right)^T,$$

the full standard set is then $CX \leqslant E$ with $E^T X = 1$. Now, according to the hypothesized form of the system, not more than one cut can excise a particular vertex of $S_n$. This condition is evidenced in the standard barycentric format by the requirement that only one entry in any given column of $C$ may be greater than 1. Assuming, then, that the cuts are in general position, meaning both that $C$ is of full rank and that no cut passes through a vertex of $S_n$, all entries of $C$ not greater than 1 are properly less than 1.

The attention of the development now passes to the central issue of the Chapter — the construction of a test which informs whether or not the $k+1$ cuts of a simplex $S_k$, cutting one vertex each, together delete the entire $S_k$. To simplify notation, $S_k$ is abstracted from its role as a component of $S_n$ by looking to the restriction of the system to the subspace $\mathbb{R}^{k+1}$ of $\mathbb{R}^{n+1}$ spanned by $S_k$ and the origin. The vertices are labeled 0 through $k$, as are the corresponding cuts. In standard barycentric coordinates, then, one has the following system.

1. $x_1 + x_2 + \cdots + x_k = 1$, written $E^T X = 1$. This equation defines a hyperplane in $(k+1)$-dimensional space.

2. $x_0 \geqslant 0, x_1 \geqslant 0, \ldots, x_k \geqslant 0$, written $X \geqslant \varnothing$. These inequalities define a simplex in the above hyperplane.

3. Additionally,

$$
\begin{aligned}
&c_{0,0}x_0 + c_{0,1} + \cdots + c_{0,k} \leqslant 1 \\
&c_{1,0}x_0 + c_{1,1} + \cdots + c_{1,k} \leqslant 1 \\
&\cdots \\
&c_{k,0}x_0 + c_{k,1} + \cdots + c_{k,k} \leqslant 1,
\end{aligned}
$$

the system written $CX \leqslant E$. These inequalities are the cuts of the simplex. On occasion, the $i$-th cut will be denoted as $C_i X \leqslant e_i = 1$.

Assume that the $i$-th cut deletes the $i$-th vertex of the simplex, and no other. Hence, $c_{i,i} > 1$, and $c_{i,j} < 1$ for $i \neq j$. Further, assume that $C^{-1}$ exists, a condition equivalent to the requirement that the columns (or rows) of $C$ be in general position, *i.e.*, be non-co[hyper]planar. Observe that the system may be written more compactly in block matrix notation as

$$\left(EIC^T\right)^T X \quad \{= \geqslant \leqslant\} \quad \left(1\ \varnothing^T\ E^T\right)^T$$

or as

$$\left(E(\neg I)C^T\right)^T X \quad \{= \leqslant \leqslant\} \quad \left(1\ E^T\ E^T\right)^T,$$

where $(\neg I)$ in the latter representation symbolizes the complement identity matrix, with diagonal elements of zero and off-diagonal elements of one. The representations are equivalent because $x_i \geqslant 0$ and $\sum_{j=0}^{k} x_j = 1 \iff \sum_{j=0}^{\hat{\imath},k} s_j \leqslant 1$ and $\sum_{j=0}^{k} x_j = 1$, wherein the appearance of $\hat{\imath}$ indicates omission of the term having index $i$.

# 2 An existence theorem

A preliminary result, presented as a Lemma, concerns the issue of possible values for $W = C^{-1}E$.

**Lemma 2.3.** *If* $E^T W = E^T C^{-1} E \leqslant 1$*, and if* $W \geqslant \varnothing$*, then* $W > \varnothing$*.*

*Proof.* Indirect

Without loss of generality, assume $W_0 = 0$. Then satisfaction of the zeroth constraint provides

$$\sum_{j=0}^{k} c_{0,j} w_j = \sum_{j=1}^{k} c_{0,j} w_j = 1$$

But $c_{0,j} < 1$ for $j \neq 0$, and each $w_j \geqslant 0$. Therefore,

$$\sum_{j=1}^{k} c_{0,j} w_j < \sum_{j=1}^{k} w_j \leqslant 1,$$

implying $1 < 1$, an absurdity. □

The next result, an existence theorem, also specifies the test of whether or not the cuts delete all of $S_k$.

**Theorem 2.4.** *If* $W = C^{-1}E$, *then no point satisfies the system*

$$\left(EIC^{T}\right)^{T} X \quad \{= \geqslant \leqslant\} \quad \left(1\ \varnothing^{T}\ E^{T}\right)^{T} \iff 0 \leqslant E^{T}W = E^{T}C^{-1}E < 1 \quad \textit{and} \quad W > \varnothing$$

*Proof.* The proof is segmented into three disjoint and exhaustive cases.

$$\begin{cases} 1. & E^{T}W \geqslant 1 \\ 2. & E^{T}W < 1 \quad \text{and some } w_i < 0 \\ 3. & E^{T}W < 1 \quad \text{and } W \geqslant \varnothing \end{cases}$$

In Case 3, $0 \leqslant E^{T}W$, and thus Lemma 2.3 implies $W > \varnothing$. In Cases 1 and 2, a point $Y$ satisfying the system will be specified. In Case 3, it will be demonstrated indirectly that no such point exists.

*Case 1* —

Define $V$ as follows. Let $V = \max(W, \varnothing)$. That is, if $w_j \leqslant 0$, let $v_j = 0$; if $w_j > 0$, let $v_j = w_j$. Then let $Y = \left[(1/\left(E^{T}V\right)\right] V$. That is, let $y_j = \left[1/\sum_{i=0}^{k} v_i\right] v_j, \forall j$. Note that $E^{T}V \geqslant 1$, and $E^{T}Y = 1$. Furthermore, $Y \geqslant \varnothing$.

Two categories of constraints must be satisfies: those corresponding to indices $\{j\}$ for which $v_j = 0$, and those corresponding to indices $\{j\}$ for which $v_j > 0$. (Recall that the $i$-th constraint is the one which deletes the $i$-th vertex.) First, assume without loss of generality, that $v_0 = 0$. Then

$$\begin{aligned} \sum_{j=0}^{k} c_{0,j} y_j &= c_{0,0} y_0 + \sum_{j=1}^{k} c_{0,j} y_j \\ &= 0 + \sum_{j=1}^{k} c_{0,j} y_j \end{aligned}$$

But, $c_{0,j} < 1$ for $1 \leqslant j \leqslant k$, and $y_j \geqslant 0, \forall j$. Therefore,

$$\sum_{j=0}^{k} c_{0,j} y_j < \sum_{j=1}^{k} y_j = 1$$

Next, assume without loss of generality, that $\nu_j = 0$, $0 \leqslant j \leqslant l-1$, and that $\nu_j > 0$, $l \leqslant j \leqslant k$, for $1 \leqslant l \leqslant k$. In particular, observe that $\nu_k > 0$. Then,

$$\begin{aligned}\sum_{j=0}^{k} c_{k,j} y_j &= \sum_{j=0}^{l-1} c_{k,j} y_j + \sum_{j=l}^{k} c_{k,j} y_j \\ &= 0 + \sum_{j=l}^{k} c_{k,j} y_j \\ &= \left[1 \Big/ \sum_{i=l}^{k} w_i\right] \sum_{j=l}^{k} c_{k,j} w_j \\ &= \left[1 \Big/ \sum_{i=l}^{k} w_i\right] \left[1 - \sum_{j=0}^{l-1} c_{k,j} w_j\right]\end{aligned}$$

(because $\sum_{j=0}^{k} c_{k,j} w_j = 1$)

$$\leqslant \left[1 \Big/ \sum_{i=l}^{k} w_i\right] \left[1 - \sum_{j=0}^{l-1} w_j\right]$$

(because $\sum_{i=l}^{k} w_i > 0$, and both $c_{k,j} < 1$ and $w_j \leqslant 0$ for $0 \leqslant j \leqslant l-1$)

$$\leqslant 1$$

(because $\sum_{j=0}^{k} w_j \geqslant 1$ and $\sum_{i=l}^{k} w_i > 0$.)

*Case 2* —

Define V as follows. Let $V = \min(W, \varnothing)$. That is, if $w_j \geqslant 0$, let $\nu_j = 0$; if $w_j < 0$, let $\nu_j = w_j$. Then let $Y = \left[(1/\left(E^T V\right)\right] V$. That is, let $y_j = \left[1/\sum_{i=0}^{k} \nu_i\right] \nu_j, \forall j$. Note that $E^T V < 1$, and $E^T Y = 1$. Furthermore, $Y \geqslant \varnothing$.

Two categories of constraints must be satisfies: those corresponding to indices $\{j\}$ for which $\nu_j = 0$, and those corresponding to indices $\{j\}$ for which $\nu_j < 0$. (Recall that the $i$-th constraint is the one which deletes the $i$-th vertex.) First, assume without loss of generality, that $\nu_0 = 0$. Then

$$\begin{aligned}\sum_{j=0}^{k} c_{0,j} y_j &= c_{0,0} y_0 + \sum_{j=1}^{k} c_{0,j} y_j \\ &= 0 + \sum_{j=1}^{k} c_{0,j} y_j\end{aligned}$$

But, $c_{0,j} < 1$ for $1 \leqslant j \leqslant k$, and $y_j \geqslant 0, \forall j$. Therefore,

$$\sum_{j=0}^{k} c_{0,j} y_j < \sum_{j=1}^{k} y_j = 1$$

Next, assume without loss of generality, that $v_j = 0$, $0 \leqslant j \leqslant l-1$, and that $v_j < 0$, $l \leqslant j \leqslant k$, for $1 \leqslant l \leqslant k$. In particular, observe that $v_k < 0$. Then,

$$\begin{aligned}
\sum_{j=0}^{k} c_{k,j} y_j &= \sum_{j=0}^{l-1} c_{k,j} y_j + \sum_{j=l}^{k} c_{k,j} y_j \\
&= 0 + \sum_{j=l}^{k} c_{k,j} y_j \\
&= \left[ 1 \Big/ \sum_{i=l}^{k} w_i \right] \sum_{j=l}^{k} c_{k,j} w_j \\
&= \left[ 1 \Big/ \sum_{i=l}^{k} w_i \right] \left[ 1 - \sum_{j=0}^{l-1} c_{k,j} w_j \right]
\end{aligned}$$

(because $\sum_{j=0}^{k} c_{k,j} w_j = 1$)

$$\leqslant \left[ 1 \Big/ \sum_{i=l}^{k} w_i \right] \left[ 1 - \sum_{j=0}^{l-1} w_j \right]$$

(because $\sum_{i=l}^{k} w_i < 0$, and both $c_{k,j} < 1$ and $w_j \geqslant 0$ for $0 \leqslant j \leqslant l-1$)

$$< 1$$

(because $\sum_{j=0}^{k} w_j < 1$ and $\sum_{i=l}^{k} w_i < 0$.)

*Case 3* —

Assume the existence of a point $Y$, with $Y \geqslant \varnothing$ and $E^T Y = 1$, such that $CY \leqslant E$. Look at any point $Z = aW + (1-a)Y$ for which $a > 1$. Then $Z$ is infeasible to all constraints; that is, $CZ = aCW + (1-a)CY \geqslant aE + (1-a)E = E$, insofar as $CW = E$, $1 - a < 0$, and $CY \leqslant E$. However, if the coefficient $a$ is chosen so that

$$a = \min_{j | y_j > x_j} \frac{y_j}{y_j - w_j},$$

then $a > 1$, and the resulting Z is feasible to any constraint $C_i$ such that

$$a = \frac{y_i}{y_i - w_i}$$

Furthermore, $Z \geqslant \varnothing$ and $E^T Z < 1$, implying $0 \leqslant E^T Z < 1$.

The demonstration begins by observing that the domain of definition for the coefficient $a$ is non-void, for otherwise $Y \leqslant W$, implying $E^T Y \leqslant E^T W < 1$. Next observe that $a > 1$, since $y_j > w_j > 0$ gives $y_j > y_j - w_j > 0$, which in turn gives

$$\frac{y_j}{y_j - w_j} > 1$$

Now, $Z > \varnothing$, with $z_i = 0$ if

$$a = \frac{y_i}{y_i - w_i}$$

Note that

$$z_j = \begin{cases} a w_j + (1-a) y_j = (y_j - w_j)\left[\dfrac{y_j}{y_j - w_j} - a\right] & \text{if } y_j > w_j \\ y_j + a(w_j - y_j) & \text{if } y_j \leqslant w_j \end{cases}$$

The former expression is non-negative because $y_j - w_j > 0$, and because the coefficient $a$ is the minimum of expressions of the form

$$\frac{y_j}{y_j - w_j}$$

The latter expression is non-negative because $y_j \geqslant 0$, $a > 0$, and $y_j - w_j \leqslant 0$. Furthermore,

$$\begin{aligned} z_i &= a w_i + (1-a) y_i \\ &= \frac{y_i}{y_i - w_i} w_i - \frac{w_i}{y_i - w_i} y_i \\ &= 0 \end{aligned}$$

Additionally, $E^T Z < 1$, because $E^T Z = a E^T W + (1-a) E^T Y$, and $a > 0$, $E^T W < 1$, and $E^T Y = 1$, giving $E^T Z < a + (1-a) = 1$.

Finally, $C_i Z < e_i = 1$, establishing the contradiction, because

$$\begin{aligned} \sum_{j=0}^{k} c_{i,j} z_j &\leqslant \sum_{j=0}^{\hat{\imath},k} c_{i,j} z_j \\ &\leqslant \sum_{j=0}^{\hat{\imath},k} z_j \\ &< 1 \end{aligned}$$

insofar as $z_j \geqslant 0$ and $c_{i,j} < 1$, for $i \neq j$. □

**Corollary 2.5.** *If* $k = 1$*, the determining condition for the deletion of an edge of* $S_n$ *is*

$$0 < w_0 + w_1 < \frac{1}{|C|}(c_{0,0} - c_{0,1} - c_{1,0} + c_{1,1})$$

# Chapter 3

# Vertex and edge counts of a multiply-cut simplex

The machinery now exists to enable the determination of the number of vertices and edges in the polytope remaining after multiple hyperplane cuts of disjoint vertex sets from a simplex. The discussion first turns to the familiar case of two cuts of $S_n$, then later addresses the general case of many cuts using the results and insights of the earlier analysis for inspiration. In course, the development presents an algorithm for determining these vertex and edge counts. Subsequently appear closed formulas in the special case of a Class $K_2$ polytope.

## 1 Two cuts intersecting

Consider again the case of two cuts of $S_n$ as developed following Theorem 1.23. At that time the concept of the incidence matrix $V$ was introduced, describing the result of cutting $Q_p = S_p \times S_{n-p}$ with a hyperplane cut. Insofar as $Q_p$ itself is the result of cutting $p$ vertices from $S_n$, $V$ provided the necessary information about cutting $S_n$ twice. No restrictions were placed on the manner in which $Q_p$ could be cut, and thenceforth formulas were developed showing the counts of vertices and edges of the residual polytope $R_{p,V}$. Specifically, Theorem 1.30 and its Corollary gave this information.

At that stage of analysis, though, two lingering problems deterred the quest for further generalization. First, after the cut of $Q_p$, the convenient cross product structure was lost, leaving no ready format for cataloging the vertices of $R_{p,V}$, such as $V$ was used for $Q_p$. Second, each cut of $S_n$ was treated in sequential fashion, a fact which ignored a desired indifference in the order of cutting. Theorem 1.32 and its Corollaries duly noted the irrelevance of the cutting order, yet the two descriptive matrices $V$ and $W$ of $Q_p$ and $Q_q$, respectively, were generally incomparable. They were not even the same size, transposition notwithstanding, unless either $q = p$ or $q = n - p$. Given these difficulties, then, the analysis of further arbitrary cutting of $S_n$ beyond two cuts would seem intractable by this methodology. A way clear exists, however, by accepting the assumption of vertex independent cuts.

In this configuration, only disjoint sets of vertices of $S_n$ can be deleted by the several cuts. Two or more cuts are, however, permitted to intersect within $S_n$. In fact, most of the interesting examples of the theory include some such intersection. This simplifying assumption permits the adoption of a more concise array representation for the information displayed by the incidence matrix V, and in the process allows a symmetric treatment of the cuts in a manner amenable to generalization in the number of cuts. This concise form will be called the reduced incidence matrix (occasionally henceforth *RIM*,) and will be denoted by the symbol U. Now, this matrix U will be shown to be an invertible function of the incidence matrix V in the case of two cuts (the only condition under which V is defined.) Hence no information shall be lost in going to the new format. Additionally, U has the property that it is independent of the order in which the two cuts have been placed, except for insignificant transposition owing to the order in which the cuts are identified. Furthermore, U is easily generalized to cases of more than two cuts in a way that retains all of the necessary information for the complete determination of the geometry of the residual polytope.

The development of U begins with an investigation into the effect on V of the restriction that the two cuts of $S_n$ delete no common vertex. Insofar as V is a matrix of information about the inclusion/exclusion status of the vertices of $Q_p$ following a cut, and since $Q_p$ is a polytope resulting from the truncation of $p$ vertices from $S_n$, it is reasonable to initiate this inquiry by taking a closer look at the construction in Theorem 1.2, which established the geometry of $Q_p$ as $S_p \times S_{n-p}$. In this construction, $p+1$ vertices were excised from $S_{n+1}$ by a hyperplane cut, creating $P = S_p \times S_{n-p}$ in the cutting plane, by Theorem 1.1. After selecting an arbitrary $p_0$ from the deleted vertices, it was then demonstrated that the polytope $Q_p$ remaining in the opposing $S_n$ after the cut was a homeomorphic image of P. In the ensuing discourse relating to $Q_p$ and V, it was neither necessary to recognize the distinction in $Q_p$ between those vertices original to $S_n$ and those introduced by the cut, nor to recall the vertices deleted by that cut.

In the present circumstance, however, it becomes desirable to retain the geometry of $Q_p$ embedded in $S_n$, and to discriminate among those three disjoint sets of vertices. Consider $S_{n+1}$ as in the construction, with vertices $\{p_0, p_1, \ldots, p_{n+1}\}$, wherein the first $p+1$ of these are deleted by the cut C. Now, the vertices of P are the intersections of C with the various edges terminating in one infeasible vertex and one feasible vertex. For convenience, relabel the feasible vertices of $S_n$ as $\{q_0, q_1, \ldots, q_{n+1}\}$. Then a vertex of P, say $w_{i,j}$, $(0,0) \leqslant (i,j) \leqslant (p, n-p)$, is the intersection of C with the edge connecting $p_i$ with $q_j$. By the homeomorphism, the vertices are mapped into the corresponding vertices $\{v_{i,j}\}$ of $Q_p$ in the simplex $S_n$ opposite $p_0$. Observe, however, that the vertices $\{w_{0,j}\}$, $0 \leqslant j \leqslant n-p$, pass to $q_j$, and the vertices $\{w_{i,j}\}$, $(1,0) \leqslant (i,j) \leqslant (p, n-p)$ remain fixed, because the homeomorphism is a projection of P from $p_0$ into $S_n$. Consequently, it is possible to identify a specific vertex of $Q_p$ with a position of a $p+1$ by $n-p+1$ incidence matrix V destined to hold information about a cut of $Q_p$.

The $\{(0,j)\}$ positions, $0 \leqslant j \leqslant n-p$, correspond to the feasible vertices $\{(q_j)\}$, original to $S_n$, and the $\{(i,j)\}$ positions, $(1,0) \leqslant (i,j) \leqslant (p, n-p)$, correspond to the vertices at the intersections of the cut of $S_n$ and the edges joining the infeasible $\{p_i\}$ with the feasible $\{q_j\}$. The feasible vertices of $S_n$ remaining in $Q_p$, and the corresponding upper row of the incidence matrix $V$, will be called interchangeably the *simplex base*.

The first significant result of this chapter is that if a second cut of $S_n$, deleting a set of vertices disjoint from those infeasible to the first cut, deletes a vertex $v_{i,j}$, $(1,0) \leqslant (i,j) \leqslant (p, n-p)$, then that cut also deletes $v_{0,j}$. That is, if a second cut deletes any vertex introduced by the first cut, then the feasible remaining vertex of $S_n$ identified with that introduced vertex, is also deleted. The result is formally stated as a Lemma.

**Lemma 3.1.** *Let two cuts of $S_n$ delete disjoint sets of vertices $\{p_i\}$ and $\{q_j\}$. Further, let $Q_p$ be the polytope feasible to one of the cuts, with a corresponding incidence matrix $V$. Then if the other cut deletes $v_{i,j}$, $(1,0) \leqslant (i,j) \leqslant (p, n-p)$, it also deletes $v_{0,j}$.*

*Proof.* Indirect

Assume otherwise, that $v_{0,j}$ is feasible to the second cut. Since $p_i$ is known to be feasible to that cut, it follows by convexity that the entire edge connecting $p_i$ and $v_{0,j} = q_j$ must be feasible, contrary to hypothesis. □

This Lemma allows the abbreviation of $V$ to the reduced incidence matrix $U$, as follows. First, it is only necessary to retain the columns corresponding to the vertices of $S_n$ deleted by the second cut, because all other columns must contain only zeros, and hence hold no information. Second, it is not necessary to retain the initial (zeroth) row, because it specifies the vertices of $S_n$ deleted by the second cut, and now contains only ones, and hence holds no information. The matrix thusly trimmed is the reduced incidence matrix. If $q$ be the number of vertices of $S_n$ cut by the second cut, then the RIM is $p$ by $q$. Insofar as no information (save the insignificant loss of column ordering) was lost in the trimming from $V$ to $U$, the following result obtains.

**Theorem 3.2.** $U$ *is an invertible function of* $V$.

*Proof.* Omitted

Thus we have developed a condensed format for representing the geometry of a doubly-cut simplex. The real usefulness of the RIM, though, rests upon two additional insights relating to the interpretation of its information. The first insight concerns the indifference of cutting order on the determination of $U$.

**Theorem 3.3.** $U$ *is independent of cutting order.*

*Proof.* Observe that an entry in $U$ — say at position $u_{i,j}$ — specifies whether or not a vertex in the cutting plane of one cut is deleted or not by the other cut. This determination is equivalent to specifying whether or not the entire edge connecting vertex $i$ (deleted by one cut) with vertex $j$ (deleted by the other cut) is completely excised. Accordingly, the timing of the cuts is irrelevant in the generation of $U$, with the understanding, of course, that $U$ represents the equivalence class of matrices under row or column permutation, or transposition, as these specifications are consequences of labeling choice only. □

The second insight concerns the ability directly to generate $U$ without recourse to $V$.

**Theorem 3.4.** $U$ *is completely determined by the exclusion status — complete or partial — of the component simplexes having their defining vertices cut by distinct hyperplanes.*

*Proof.* Insofar as the entries of $U$ depend on the complete or incomplete excision of edges connecting excised vertices, one may use the test of Theorem 2.4 (specifically, its Corollary 2.5) for any system in standard barycentric coordinates. The hypothesis is satisfied insofar as each edge is a one-dimensional simplex, with the two defining vertices cut by separate hyperplanes. □

Note that special care has been taken in the statement of this theorem to remove specific references to the case of two cuts of $S_n$. The reason is that for an appropriately generalized RIM, accommodating more than two cuts, the statement remains true. The expanded definition of $U$ and proof of the generalized theorem follow the reanalysis, in terms of the reduced incidence matrix, of the results of Theorem 1.30 and its Corollary, relating to the counts of vertices and edges in a doubly-cut simplex.

**Theorem 3.5 (Symmetric analog of Theorem 1.30).**

$$\begin{gathered} T_{p,q} := T_{p,V} = (n+1) + [p(n-p) + q(n-q)] \\ + \left\{ (n-1) \sum_{i=1}^{p} \sum_{j=1}^{q} u_{i,j} - \left[ \sum_{i=1}^{p} x_i^2 + \sum_{j=1}^{q} y_j^2 \right] \right\}, \\ \textit{where} \quad x_i := \sum_{j=1}^{q} u_{i,j} \quad \textit{and} \quad y_j := \sum_{i=1}^{p} u_{i,j} \end{gathered}$$

*Proof.* Begin with a statement of Theorem 1.30, then expand.

$$\begin{aligned} T(p,V) &= A_n(p,0) + N(p,V) \\ &= (p+1)(n-p+1) + \sum_{i=0}^{p} B_{n-p}(r_i,0) + \sum_{j=0}^{n-p} B_{p+1}(s_j,0) \end{aligned}$$

by Corollary 1.8 and Theorem 1.28

$$= (p+1)(n-p+1) + \sum_{i=0}^{p} r_i(n-p-r_i) + \sum_{j=0}^{n-p} s_j(p+1-s_j)$$

by Corollary 1.17.

In the transition from $V$ to $U$ the row counts remain the same, but the column counts decrease by one, owing to the omission of the zeroth row. Re-index $j$ over the $q$ vertices truncated by the second cut, and let $x_i = r_i$, $y_j = s_j - 1$, $(1,1) \leqslant (i,j) \leqslant (p,q)$, be the respective row and column counts in $U$. Then segregating the contribution from the zeroth row of $V$,

$$\begin{aligned} T_{p,V} &= (p+1)(n-p+1) + q(n-p-q) \\ &\quad + \sum_{i=1}^{p} x_i(n-p-x_i) + \sum_{j=1}^{q} (y_j+1)\big(p+1-(y_j+1)\big) \\ &= (p+1)(n-p+1) + q(n-p-q) \\ &\quad + (n-p)\sum_{i=1}^{p} x_i - \sum_{i=1}^{p} x_i^2 + pq + (p-1)\sum_{j=1}^{q} y_j - \sum_{j=1}^{q} y_j^2 \\ &= (n+1) + [p(n-p) + q(n-q)] \\ &\quad + (n-p)\sum_{i=1}^{p} x_i + (p-1)\sum_{j=1}^{q} y_j - \left[\sum_{i=1}^{p} x_i^2 + \sum_{j=1}^{q} y_j^2\right] \end{aligned}$$

Now

$$\sum_{i=1}^{p} x_i = \sum_{j=1}^{q} y_j,$$

as both summations represent the count of unit entries in $U$. Indicating this value as the double sum

$$\sum_{i=1}^{p}\sum_{j=1}^{q} u_{i,j},$$

and renaming the function $T_{p,q}$ to reflect the new symmetry, produces the assertion. □

**Corollary 3.6.**

$$L_{p,q} := L_{p,V} = \frac{n}{2} T_{p,q}$$

It is instructive to examine the terms of the expression for $T_{p,q}$ in Theorem 3.5, because these terms reveal something of the nature in which the count of extreme points builds with increased encroachment of the cutting hyperplanes. Imagine the cutting hyperplanes positioned outside the simplex, and then moved, or encroached, upon $S_n$. Before any encroachment, $S_n$ has, in its pristine form $n+1$ vertices. This quantity is duly recorded in the first (parenthesized) term of the expression for $T_{p,q}$. When the planes are passed through the $p$ and $q$ vertices, respectively, of $S_n$, but as yet do not intersect between themselves within $S_n$, the net additional count of vertices in the residual polytope is $p(n-p) + q(n-q)$ by Corollary 1.17. This quantity is duly recorded in the second [bracketed] term of $T_{p,q}$. In consequence, the third {braced} term of $T_{p,q}$ must represent the net additional vertices resulting from the next phase of encroachment — that of allowing the cutting planes a non-void intersection within $S_n$.

The ensuing discussion addresses an extension of Theorem 3.5 to encompass vertex counts for a Class $K_2$ polytope, so developed.

# 2 Two cuts, pairwise intersecting

Let $k$ vertex-independent cuts delete sets of vertices of $S_n$ containing, respectively, $\{p_1, \ldots, p_k\}$ points. Necessarily, $\sum_{l=1}^{k} p_l \leqslant n+1$. Consider first that no pairwise intersections among the cutting hyperplanes occur within $S_n$. The polytope is now of Class $K_1$, and the count of vertices is

$$(n+1) + \sum_{l=1}^{k} p_l(n - p_l) \tag{3.1}$$

by Corollary 1.24. By assumption for a Class $K_2$ polytope only pairs of hyperplanes can intersect within $S_n$, not triples, nor quadruples, nor any greater number. Thus the net addition of vertices to the Class $K_1$ polytope by virtue of such binary interactions is itself additive over such pairs. So the totality of vertices is the sum of the terms of Equation (3.1) and terms like the third term of the statement of Theorem 3.5, one for each pair of intersecting hyperplanes within $S_n$. In consequence one has the following statement for the number of vertices in a Class $K_2$ polytope, with its Corollary showing the number of edges. Herein $\mu$ and $\nu$ are the indices of the cutting hyperplanes, and $u_{i,j}^{\mu,\nu}$ is an entry in the corresponding RIM.

**Theorem 3.7 (Class $K_2$ Extension).**

$$T_{p_1,\ldots,p_k} := (n+1) + \sum_{l=1}^{k} p_l(n-p_l)$$
$$+ \sum_{1\leqslant\mu<\nu\leqslant k} \left\{ (n-1)\sum_{i=1}^{p_\mu}\sum_{j=1}^{p_\nu} u_{i,j}^{\mu,\nu} - \left[ \sum_{i=1}^{p_\mu} \left(x_i^{\mu,\nu}\right)^2 + \sum_{j=1}^{p_\nu} \left(y_j^{\mu,\nu}\right)^2 \right] \right\},$$
$$\textit{where} \quad x_i^{\mu,\nu} := \sum_{j=1}^{p_\nu} u_{i,j}^{\mu,\nu} \quad \textit{and} \quad y_j^{\mu,\nu} := \sum_{i=1}^{p_\mu} u_{i,j}^{\mu,\nu}$$

*is the number of vertices of the Class $K_2$ polytope.*

*Proof.* Omitted

*Remark.* Observe that there are $\binom{k}{2}$ terms within the third term, corresponding to the number of pairs of cuts.

**Corollary 3.8.**

$$L_{p_1,\ldots,p_k} := \frac{n}{2} T_{p_1,\ldots,p_k}$$

*is the number of edges of the Class $K_2$ polytope.*

Formulas for further extensions of these vertex and edge counts to Classes $K_3$, $K_4$, and higher polytopes are possible, but the results are combinatorially expensive, requiring corresponding incidence arrays of dimensions 3, 4, and higher, and are here omitted. These results are best found by recursion, using the stated algorithm.

# Chapter 4

# The beta problem and basic solutions

## 1 Description of the beta problem

The *beta problem* is a generic term pertaining to the complex of questions about the following convex set of non-negative variables. This set of relations has $n+1$ variables and $m+1$ inequality constraints. The blocks, respectively, have $k_0, k_1, \ldots, k_m$ variables. For simplicity, index values on the $\{b_j\}$, ranging from 1 through $n+1$, are omitted from the matrix.

$$\begin{array}{ccccl} 11\ldots1 & 11\ldots1 & \ldots\ldots & 11\ldots1 & = 1 \\ bb\ldots b & & & & \leqslant a_0 \\ & bb\ldots b & & & \leqslant a_1 \\ & & \ldots\ldots & & \\ & & & bb\ldots b & \leqslant a_m \end{array}$$

Throughout, the $\{a_i\}$ and the $\{b_j\}$ will be assumed drawn independently from probability distributions. The distributions of the $\{a_i\}$ will be alike, as will those of the $\{b_j\}$. However, the 'a' and 'b' distributions may differ.

The convex set without the equality constraint, which specifies a simplex, shall be termed the *uncut polytope*. The set is clearly bounded, *i.e.*, is a polytope, if the $\{b_j\}$ and the $\{a_i\}$ are positive, conditions henceforth assumed, except possibly on sample subsets of measure zero.

The convex set including the equality constraint shall be termed the *cut polytope*. Observe that $k_i$ of the $\{b_j\}$ appear in block $i$, $0 \leqslant i \leqslant m$, corresponding to the $i$-th inequality constraint. Obviously, $\sum_{i=0}^{m} k_i = n+1$.

## 2 Basic solutions of the uncut polytope

A cursory inspection of the uncut polytope constraint set indicates that there are $k_i + 1$ choices of a basic variable associated with inequality constraint $i$, $0 \leqslant i \leqslant m$. Any of the $k_i$ original (instrumental) variables qualifies, as does the slack variable of the $i$-th row. Consequently, there are

$$\prod_{i=0}^{m}(k_i + 1) \tag{4.1}$$

basic solutions. Further, each of these solutions is feasible, given positive $\{a_i\}$ and $\{b_j\}$, insofar as a given instrumental variable appears in only one row.

The basic solutions may be partitioned into subsets according to the number of slack variables which are non-basic. This attribute of a solution is called the rank $r$. Insofar as there are $m+1$ slacks, clearly there are $m+2$ ranks, $0 \leqslant r \leqslant m+1$. Further, within each rank, solutions may be again partitioned, according to which of the slacks are non-basic. In rank $r$, therefore, there are $\binom{n+1}{r}$ of these subsets, each called a *node*. Finally, within each node appear the basic solutions, which number

$$\prod_{i=0}^{m} k_i, \tag{4.2}$$

where the index set I is composed of the indices of non-basic slacks at the node. Locally, the nodes are the cross products of $r$ simplexes of respective dimensions $\{k_i - 1\}$.

The first result of significance is that the function

$$\Phi(t) = \prod_{i=0}^{m}(k_i t + 1) \tag{4.3}$$

generates the number of basic solutions by rank. Simply expand

$$\Phi(t) = \sum_{i=0}^{m}(a_i t^i) \tag{4.4}$$

as a polynomial, and observe that the coefficient of $t^r$ is the sum of the $r$-fold products of the $\{k_i\}$, taken over the index subsets of cardinality $r$. The terms of this summation on rank $r$ are the various products like that of Equation (4.2), and therefore represent the numbers of basic solutions at the various nodes. $\Phi(1)$ is, of course, the total of basic solutions, as in Equation (4.1).

Next, consider the neighbors of a chosen basic solution, that is, the set of other basic solutions which are connected to the chosen solution by an edge of the polytope. To recognize these connections as edges of the polytope, note that it is necessary to have observed that all the basic solutions are feasible.

Neighbors are of two distinct types — those in adjacent ranks and those in the same rank. Those in adjacent ranks are connected by an edge which is traversed when a slack variable leaves or enters the basis. A basic solution involving one fewer slack has higher rank, whereas one involving an additional slack has lower rank. These edges are called transient edges. The neighbors within the same rank are connected by an edge which is traversed when an instrumental variable is exchanged for another in the basis. These edges, called *local edges*, in addition to lying within a rank, lie also within a node; otherwise, the instrumental variables would correspond to different blocks, and the change of basic status of one would require the simultaneous exchange of slack variables, the transient edge case already considered. Observe that the totality of edges must be

$$\frac{n+1}{2}\,\Phi(1) \tag{4.5}$$

insofar as the embedding of the polytope in $\mathbb{R}^{n+1}$ insures that $n+1$ edges emanate from each basic solution, but a counting of these edges by basic solution tallies each edge twice.

At this point it may be useful to have some pictorial representation of the set of basic solutions. For example, take the case of

$$[k_1, k_2, k_3] = [2, 3, 4]$$

The value of $n$ therefore is 8. For this case

$$\Phi(1) = 3 \cdot 4 \cdot 5 = 60$$

basic solutions, so the total of edges is

$$\frac{8+1}{2} \cdot 60 = 270$$

Further,

$$\begin{aligned}\Phi(t) &= 1 + 9t + 26t^2 + 24t^3 \\ &= (2t+1)(3t+1)(4t+1)\end{aligned}$$

shows the breakdown of basic solutions by rank.

Local neighbors (connected by local edges) are visited by exchanging the first of two variables for the other one, or by exchanging the third of four variables for any of the other three. There is a total therefore of four local edges connecting the focal solution to others within the node.

Transient neighbors are located by including or excluding a slack variable into or from the basis. Going first to the lower rank, rank 1, involves the entry of either slack 2 or slack 0 to the basis, and respectively, the exit of either the third variable in the block of 4, or the first variable in the block or 2.

Next, going to the higher rank, rank 3, slack 1 must leave the basis. In so doing, any of the three variables in the block of three may enter. There is a total therefore of five transient edges connecting the focal solution to those in adjacent ranks. The sum of four local neighbors and five transient neighbors equals nine, the necessary number of all neighbors.

# 3 Edges of the uncut polytope

Next is an analysis of the number of transient edges between ranks and the number of local edges within the ranks. Generating functions for each shall be presented. Adjoining each solution in rank $i$ are $i$ solutions in rank $i-1$, because a solution in rank $i$ has $i$ non-basic slacks, and there are consequently $i$ choices of slack to enter the basis.

Recalling that

$$\Phi(t) = \sum_{i=0}^{m+1} a_i t^i$$

generates these solutions by rank, it follows that

$$\Psi(t) = \sum_{i=0}^{m+1} i a_i t^i = t\Phi'(t) \tag{4.6}$$

generates the transient edges between ranks $i$ and $i-1$, or that $\Phi'(t)$ alone generates those between ranks $i$ and $i+1$. Further,

$$\Psi(1) = \Phi'(1)$$

is the total of transient edges.

The local edges can be represented as a residual, considering that of all the edges connecting with solutions in a rank, some come from the higher rank, and some go to the lower rank.

Now, $(n+1)a_i$ edges connect totally with rank $i$ solutions, or more carefully stated, emanate from these solutions. Insofar as $(i+1)a_{i+1}$ descend from the higher rank, and $ia_i$ descend to the lower rank, it follows that

$$(n+1)a_i - (i+1)a_{i+1} - ia_i$$

edges leave and enter within rank $i$. But these are doubly counted, so half that amount,

$$b_i = \frac{1}{2}[(n+1)a_i - (i+1)a_{i+1} - ia_i],$$

is the correct count of edges within the rank. Thus

$$\mathrm{P}(t) = \sum_{i=1}^{m+1} b_i t^i$$

generates these numbers. Note symbol 'P' is uppercase Greek Rho.

$$\mathrm{P}(t) = \frac{n+1}{2} \sum_{i=0}^{m+1} a_i t^i - \frac{1}{2} \sum_{i=0}^{m} (i+1) a_{i+1} t^i - \frac{1}{2} \sum_{i=0}^{m+1} i a_i t^i$$

(recognizing in the second term that $a_{m+2} = 0$.)

$$= \frac{n+1}{2} \Phi(t) - \frac{1}{2t} \sum_{i=0}^{m+1} i a_i t^i - \frac{1}{2} \Psi(t)$$

(recognizing in the second term that $0 \cdot a_0 t^0 = 0$.)

$$\begin{aligned} &= \frac{n+1}{2} \Phi(t) - \frac{1}{2t} \Psi(t) - \frac{1}{2} t \Phi'(t) \\ &= \frac{n+1}{2} \Phi(t) - \frac{1}{2t} t \Phi'(t) - \frac{1}{2} t \Phi'(t) \\ &= \frac{n+1}{2} \Phi(t) - \frac{1}{2} \Phi'(t)(1+t) \qquad (4.7) \end{aligned}$$

Further,

$$\mathrm{P}(1) = \frac{n+1}{2} \Phi(1) - \Phi'(1)$$

is the total number of local edges.

In the example of the preceding section, *i.e.*, $[k_1, k_2, k_3] = [2, 3, 4]$, the generated transient and local edges are as follows.

$$\begin{aligned} \Phi(t) &= 1 + 9t + 26t^2 + 24t^3 \\ \Phi(1) &= 60 \quad \text{basic solutions} \end{aligned}$$

$$\begin{aligned} \Psi(t) &= 9t + 52t^2 + 72t^3 \\ \Psi(1) &= 133 \quad \text{transient edges} \end{aligned}$$

$$\begin{aligned} \mathrm{P}(t) &= 10t + 55t^2 + 72t^3 \\ \mathrm{P}(1) &= 137 \quad \text{local edges} \end{aligned}$$

Therefore, there are

$$\frac{n+1}{2} \Phi(1) = \Psi(1) + \mathrm{P}(1) = 270 \quad \text{total edges.}$$

# 4 Basic solutions of the cut polytope

The initial observation is that the edges of the uncut polytope correspond to the basic solutions of the cut polytope. This is because the simplex of the top row of the beta problem constraint set intersects all of the edges [extended] of the uncut polytope. The only necessary assumption is that all of the $\{b_j\}$ within a row block differ from each other, so that there is no edge parallel to the cut. In this section we shall assume probability distributions on the $\{b_j\}$ insuring that two or more of these random variables never be equal, except possibly on sample sets of measure zero.

In the cutting of a local edge within rank $r$, one is interested in the probability that the sum of $r$ random variables of the form $q = a/b$ be less than 1 before the exchange of one variable for another, and be greater than 1 after the exchange, or *vice versa*. The reasoning is that the basic solutions must be on opposite sides of the simplex for the simplex to cut the intervening edge. Notice that this probability depends only on the number of variables $\{q\}$, so that the probability within a rank of any edge's being cut is the same as that for any other edge. Assume the $r+1$ variables are coordinate variables in $\mathbb{R}^{r+1}$. We may restrict our attention to the non-negative orthant, because we are herein assuming that the $a$ and $b$ variables are non-negative. The regions of interest over which we would care to integrate a product density are twofold, recognizing that indexing is arbitrary.

$$I(1): \quad \sum_{i=1}^{\hat{1},r+1} q_i < 1 \quad \text{and} \quad \sum_{i=1}^{\hat{2},r+1} q_i > 1$$

and

$$I(2): \quad \sum_{i=1}^{\hat{1},r+1} q_i > 1 \quad \text{and} \quad \sum_{i=1}^{\hat{2},r+1} q_i < 1$$

The two hyperplanes, $\sum_{i=1}^{\hat{1},r+1} q_i$ and $\sum_{i=1}^{\hat{2},r+1} q_i$, of course, partition the orthant into four regions, so we may identify the others and give them names. Let

$$I(0): \quad \sum_{i=1}^{\hat{1},r+1} q_i < 1 \quad \text{and} \quad \sum_{i=1}^{\hat{2},r+1} q_i < 1$$

This region includes the origin.

$$I(3): \quad \sum_{i=1}^{\hat{1},r+1} q_i > 1 \quad \text{and} \quad \sum_{i=1}^{\hat{2},r+1} q_i > 1$$

The analysis is aided by the observation that $I(0)$ can be further partitioned — in fact cut in two equal pieces — by passing the hyperplane $q_1 = q_2$. The resulting two pieces are simplexes. To see this fact, consider their boundaries. In the one sub-case the boundaries are the $r+2$ hyperplanes

$$\sum_{i=1}^{\hat{1},r+1} q_i = 1,\ q_1 = 0,\ q_2 = q_1,\ q_3 = 0,\ \ldots,\ q_{r+1} = 0$$

In the other sub-case the boundaries are the $r+2$ hyperplanes

$$\sum_{i=1}^{\hat{2},r+1} q_i = 1,\ q_1 = q_2,\ q_2 = 0,\ q_3 = 0,\ \ldots,\ q_{r+1} = 0$$

Further, the volume of each simplex is $1/(r+1)!$.

Now, assume that each $q$ variable is the quotient of two uniform variables of the same parameter. (See Appendix A.)

We shall proceed to compute the probability associated with regions $I(1)$ and $I(2)$. Note that it is not at this point necessary to compute the entire distribution of the sum of $r$ of these variables. In fact, it would be insufficient to do so, insofar as we are interested in the interaction between two such summation variables.

From Appendix A, it is known that the product density over the unit hypercube is uniform, and is $1/2^{r+1}$. The regions of interest, $I(1)$ and $I(2)$, are the cross products of $r$-simplexes and half lines, less the volumes of the two $(r+1)$-simplexes which touch the origin. Each of $I(1)$ and $I(2)$ for reasons of symmetry has the same probability.

For either, the mass of the base $r$-simplex is $(1/2^r)(1/r!)$. The mass of the half-line is 1, so the probability of $I(0) \cap I(1)$ or of $I(0) \cap I(2)$ is also $(1/2^r)(1/r!)$.

Now the probability of $I(0)$ (recognizing the two $(r+1)$-simplex components) is

$$2 \cdot \frac{1}{2^{r+1}(r+1)!} = \frac{1}{2^r(r+1)!}$$

So that of $I(1)$, or of $I(2)$ is

$$\frac{1}{2^r}\frac{1}{r!} - \frac{1}{2^r}\frac{1}{(r+1)!}$$

So that of $I(0) \cap I(2)$ is

$$p(r) = \frac{1}{2^{r-1}}\left(\frac{1}{r!} - \frac{1}{(r+1)!}\right) \tag{4.8}$$

With knowledge of this probability, and of the generating function $P(t)$ for local edges, and invoking the classical theorem that the expectation of the sum of random variables is the sum of the expectations, it will be possible to compute the expected number of cut local edges. Note that the expectation theorem applies even to variables which are dependent, which these clearly are (speaking of those variables which are 1 if an edge is cut, 0 otherwise.)

Before that calculation, though, let us perform the similar development for transient edges. This case is a bit easier, because there is only one region to consider, this because we are interested in the probability that, for an edge between ranks $r-1$ and $r$,

$$\sum_{i=1}^{r-1} q_i < 1 \quad \text{and} \quad \sum_{i=1}^{r} q_i > 1$$

Insofar as we are only considering non-negative variables, it is not necessary to address the complementary region, which is void.

The probability of this region is

$$\frac{1}{2^{r-1}}\frac{1}{(r-1)!} - \frac{1}{2^r}\frac{1}{r!} = \frac{1}{2^r}\frac{2}{(r-1)!} - \frac{1}{r!} =: q(r)$$

As noted for local edges, knowledge of this probability, along with the generating function $\Psi(t)$ for transient edges, allows us to compute the expected number of cut transient edges.

Of more immediate interest, though, is the expected number of all cut edges, for this is the expected number of feasible basic solutions to the full beta problem.

This total is

$$E := \sum_{r=1}^{m+1} [b_r p_r + r a_r q_r],$$

recalling that the $\{b_r\}$ are the coefficients in $P(t)$, and that the $\{ra_r\}$ are the coefficients in $t\Phi'(t) = \Psi(t)$, respectively the counts of local and transient edges in and immediately below rank $r$. The summation can begin with rank 1 because there are no local edges in rank 0, and no transient edges below rank 0.

Expanded,

$$E = \sum_{r=1}^{m+1} \left\{ \frac{1}{2}[(n+1)a_r - (r+1)a_{r+1} - ra_r] \cdot \frac{1}{2^{r-1}} \left[\frac{1}{r!} - \frac{1}{(r+1)!}\right] \right.$$
$$\left. + ra_r \frac{1}{2^r} \left[\frac{2}{(r-1)!} - \frac{1}{r!}\right] \right\}$$

Regrouping terms and simplifying,

$$= (n+2) \sum_{r=1}^{m+1} \frac{r}{2^r (r+1)!} a_r$$

or,

$$E = (n+2)\sum_{r=1}^{m+1} \frac{r}{r+1} \cdot \frac{1}{2^r r!} a_r \tag{4.9}$$

Note that the terms are 'almost' the volumes of the $(r-1)$-simplexes cut from hypercubes of dimension $1/2$.

A trite calculation shows that the expectation of feasible basic solutions for the problem of Section 2 is $47\frac{11}{12}$.

As a final note, consider that the terms of Equation (4.9) might possibly relate to the expected number of cut edges emanating from the various basic solutions of the uncut polytope, insofar as the coefficients $\{a_r\}$ appear in Equation (4.9). Such is not the case, as a convenient counterexample illustrates.

Take the basic solution of the Section 2 problem. A simple calculation shows that the expectation of cut edges emanating from this solution, weighted by $s$ to the lower rank and by $(1-s)$ to the upper rank is,

$$s \cdot \frac{3}{8} + \frac{1}{2} \cdot \frac{1}{6} + (1-s) \cdot \frac{5}{48}$$

The coefficient in Equation (4.9), however, is

$$10 \cdot \frac{1}{12} = \frac{5}{6}.$$

Hence no choice of $s$, $0 \leqslant s \leqslant 1$, provides an identical coefficient.

# Appendix A

# Quotient distributions

This appendix presents several results. First calculated are the distributions of four quotient variables defined on rectangles — specifically, those for which the numerator and denominator are each chosen independently from either uniform or exponential distributions. Customarily the distribution functions will be represented as $F(t)$, and the densities as $f(t) := F'(t)$. All of these quotient distributions have infinite expectations. Following, two additional distributions are calculated, these being the respective minima of several like variables, either quotients of two uniforms or quotients of two exponentials. These distributions are of interest in the selection of a pivot position in the operation of the simplex method on the general linear problem, wherein one seeks to locate the smallest of several ratios. It is noteworthy that when more than one quotient variable is involved in the choice of a minimum, the expectation does exist. Formulations are presented, along with specific mean values for a few initial dimensions.

Now look at the first case, the ratio of two uniform variables. Let $\alpha$ be uniform on $[0, a]$, and let $\beta$ be uniform on $[0, b]$. Then

$$\begin{aligned} F(t) &:= \Pr\left\{\frac{\alpha}{\beta} \leqslant t\right\} \\ &= \Pr\{\alpha \leqslant t\beta\} \end{aligned}$$

Easy calculations demonstrate that

$$F(t) = \begin{cases} \dfrac{bt}{2a} & \text{if } 0 \leqslant t \leqslant \dfrac{a}{b} \\ 1 - \dfrac{a}{2bt} & \text{if } \dfrac{a}{b} < t < \infty \end{cases}$$

and

$$f(t) = \begin{cases} \dfrac{b}{2a} & \text{if } 0 \leqslant t \leqslant \dfrac{a}{b} \\ \dfrac{a}{2bt^2} & \text{if } \dfrac{a}{b} < t < \infty \end{cases}$$

Note that the density is continuous, but not differentiable at $a/b$. Also, observe that the mean in infinite, because

$$\int_{a/b}^{\infty} t f(t)\, dt$$

is unbounded.

Next consider the case of the ratio of two exponential variables. To avoid possible confusion in the ensuing computations between $a$ and $\alpha$, and between $b$ and $\beta$, let $x$ and $y$ be the respective variables of the sample spaces for $\alpha$ and $\beta$. Then,

$$\begin{aligned} F(t) &= ab \int_0^{\infty} \int_0^{ty} \exp[-(ax + by)]\, dx\, dy \\ &= \frac{1}{1 + \frac{b}{at}} \end{aligned}$$

$$f(t) = \frac{\frac{a}{b}}{\left(1 + \frac{at}{b}\right)^2}$$

This distribution, as well, has infinite expectation.

Continuing, consider the ratio of a uniform variable to an exponential variable. Let $\alpha$ be uniform on $[0, a]$, and $\beta$ be exponential with density $b \exp(-bt)$. Then,

$$\begin{aligned} F(t) &= 1 - \frac{b}{a} \int_0^{a/t} \int_{ty}^{a} \exp(-by)\, dx\, dy \\ &= \frac{t}{ab}\left[1 - \exp\left(-\frac{ab}{t}\right)\right] \end{aligned}$$

$$f(t) = \frac{1}{ab} - \exp\left(-\frac{ab}{t}\right)\left(\frac{1}{t} + \frac{1}{ab}\right)$$

Expectation, again, is infinite.

Finally, consider the ratio of an exponential variable to a uniform variable. Let $\alpha$ be exponential with density $a\exp(-at)$, and $\beta$ be uniform on $[0,b]$.

$$F(t) = \frac{a}{b}\int_0^{tb}\int_{x/t}^{b} \exp(-ax)\,dy\,dx$$
$$= a + \frac{1}{abt}[\exp(-abt) - 1]$$

$$f(t) = \frac{1}{abt^2} - \exp(-abt)\left(\frac{1}{t} + \frac{1}{abt^2}\right)$$

Next, let $\gamma = \alpha/\beta$, wherein $\alpha$ and $\beta$ are independent and uniform, and consider the distribution of the smallest of j of these identical $\gamma$ variables.

$$\Pr\{\min(\gamma_1, \gamma_2, \ldots, \gamma_j) \leqslant t\} = 1 - \Pr\{\gamma_1 > t, \gamma_2 > t, \ldots, \gamma_j > t\}$$

Two cases must be considered —

$$0 \leqslant t \leqslant \frac{a}{b} \quad \text{and} \quad \frac{a}{b} < t < \infty$$

In the former domain,

$$F(t) = 1 - \left(1 - \frac{bt}{2a}\right)^j$$

This formulation is evident when one considers that the mass of a segment $[0,t]$ in any axis is

$$\int_0^t f(x)\,dx = \int_0^t \frac{b}{2a}\,dx$$
$$= \frac{bt}{2a}$$

In the latter domain,

$$F(t) = 1 - \left(\frac{a}{2bt}\right)^j$$

This formulation, again, is evident, considering the mass of a segment $[t,\infty)$, which is

$$\int_t^\infty f(x)\,dx = \int_t^\infty \frac{a}{2bx^2}\,dx$$
$$= \frac{a}{2bt}$$

To reprise the distribution function, now including the density function —

$$F(t) = \begin{cases} 1 - \left(1 - \dfrac{bt}{2a}\right)^j & \text{if } 0 \leqslant t \leqslant \dfrac{a}{b} \\ 1 - \left(\dfrac{a}{2bt}\right)^j & \text{if } \dfrac{a}{b} < t < \infty \end{cases}$$

and

$$f(t) = \begin{cases} \dfrac{b}{2a} \cdot j \left(1 - \dfrac{bt}{2a}\right)^{j-1} & \text{if } 0 \leqslant t \leqslant \dfrac{a}{b} \\ j \left(\dfrac{a}{2b}\right)^j \left(\dfrac{1}{t}\right)^{j+1} & \text{if } \dfrac{a}{b} < t < \infty \end{cases}$$

For $j = 1$, which is the minimum of a single $\gamma$ variable, or the variable itself, the mean does not exist, as previously observed. However, for $j > 1$, the mean does exist, and is calculated herein.

Let $m = m_1 + m_2$ be the mean, where

$$\begin{aligned} m_1 &= \int_0^{a/b} t f(t)\, dt \\ &= \frac{a}{2b} \cdot j \sum_{i=0}^{j-1} C_{j-1,i} \left(\frac{1}{2}\right)^i \frac{1}{i+2} \end{aligned}$$

$$\begin{aligned} m_2 &= \int_{a/b}^{\infty} t f(t)\, dt \\ &= \frac{a}{b} \cdot \frac{j}{j-1} \left(\frac{1}{2}\right)^j, \quad \text{if} \quad j > 1, \quad \text{else} \quad \infty \end{aligned}$$

A few values of $m^{(j)}$ follow.

| $j$ | $m^{(j)}$ |
|---|---|
| 1 | $\infty$ |
| 2 | 0.8333 $(a/b)$ |
| 3 | 0.5313 $(a/b)$ |
| 4 | 0.4083 $(a/b)$ |
| 5 | 0.3359 $(a/b)$ |
| 6 | 0.2866 $(a/b)$ |

Lastly, consider that $\gamma_i$, $1 \leqslant i \leqslant j$ is the ratio of two exponential variables, and look to the analogous computation, that of the distribution of the smallest of these $\{\gamma_i\}$ variables.

$$\begin{aligned} F(t) &= 1 - \Pr\{\gamma_1 > t, \gamma_2 > t, \ldots, \gamma_j > t\} \\ &= 1 - \left(\frac{a}{b}\right)^j \int_t^\infty \cdots \int_t^\infty \prod_{i=1}^{j} \frac{1}{\left(\frac{ax_i}{b} + 1\right)^2} \, dx_1 \cdots dx_j \\ &= 1 - \frac{1}{\left(\frac{at}{b} + 1\right)^j} \end{aligned}$$

$$f(t) = \frac{a}{b} \cdot \frac{j}{\left(\frac{at}{b} + 1\right)^{j+1}}$$

In conclusion,

$$\begin{aligned} m &= \int_{j=0}^{\infty} t f(t) \, dt \\ &= \frac{b}{a} \cdot \frac{1}{j-1} \end{aligned}$$

A few values of $m^{(j)}$ follow. Alongside for comparison are the corresponding values for the quotient of two uniform variables.

| | exp / exp | unif / unif |
|---|---|---|
| $j$ | $m^{(j)}$ | $m^{(j)}$ |
| 1 | $\infty$ | $\infty$ |
| 2 | 1.0000 $(b/a)$ | 0.8333 $(a/b)$ |
| 3 | 0.5000 $(b/a)$ | 0.5313 $(a/b)$ |
| 4 | 0.3333 $(b/a)$ | 0.4083 $(a/b)$ |
| 5 | 0.2500 $(b/a)$ | 0.3359 $(a/b)$ |
| 6 | 0.2000 $(b/a)$ | 0.2866 $(a/b)$ |

Observe that the ratio of parameters for the exponentials is $b/a$, whereas for the uniforms is $a/b$. This observation conforms to intuition, for the parameters are just scale factors, with the low parameter exponential and high parameter uniform being more "spread out" (having higher means.) Note that when $a = b$, the mean is higher in the exponential case for two quotients, but is apparently lower for more than two.

Here is a final note, a preview of work to come. Let $Z = X \cap Y$, where X is an exponential random variable, and Y is the one just calculated — the minimum of j quotient variables, exponential over exponential.

If X is interpreted as a coefficient in a linear functional, corresponding to a pivot column in an implementation of the simplex method on a linear program, $Z = X \cap Y$ is the distribution of the improvement in the objective upon pivoting in that column. Its distribution is

$$\begin{aligned} H(t) &= \int_0^\infty f(x) \int_0^{t/x} g(y)\, dy\, dx \\ &= \int_0^\infty f(x) G\left(\frac{t}{x}\right) dx \end{aligned}$$

where

$$\begin{aligned} f(x) &= c \exp(-cx) \\ G(y) &= 1 - \frac{1}{\left(\frac{ay}{b} + 1\right)^j} \end{aligned}$$

So,

$$H(t) = c \int_0^\infty \left[ 1 - \left( \frac{bx}{at + bx} \right)^j \right] \exp(-cx)\, dx$$

This integral has no known closed form solution.

# Bibliography


Adler, I. (1974). Lower bounds for maximum diameters of polytopes. In M. L. Balinski (Ed.), *Mathematical Programming Study 1, Pivoting and Extensions*, pp. 11–19. New York: North-Holland.

Adler, I. and G. B. Dantzig (1974). Maximum diameter of abstract polytopes. In M. L. Balinski (Ed.), *Mathematical Programming Study 1, Pivoting and Extensions*, pp. 20–40. New York: North-Holland.

Adler, I., G. B. Dantzig, and K. G. Murty (1974). Existence of A-avoiding paths in abstract polytopes. In M. L. Balinski (Ed.), *Mathematical Programming Study 1, Pivoting and Extensions*, pp. 41–42. New York: North-Holland.

Balinski, M. L. (1961, Mar.). An algorithm for finding all vertices of convex polyhedral sets. *J. SIAM 9*(1), 72–88.

Balinski, M. L. (1974). On two special classes of transportation polytopes. In M. L. Balinski (Ed.), *Mathematical Programming Study 1, Pivoting and Extensions*, pp. 43–58. New York: North-Holland.

Balinski, M. L. and A. Russakoff (1972, Dec.). Some properties of the assignment polytope. *Math. Prog. 3*(1), 257–258.

Bolker, E. D. (1972). Transportation polytopes. *J. Comb. Thy., Series B 13*, 251–262.

Bolker, E. D. (1976, Mar.). Simplicial geometry and transportation polytopes. *Trans. Amer. Math. Soc. 217*, 121–142.

Burdet, C.-A. (1974, May). Generating all the faces of a polyhedron. *SIAM J. Appl. Math. 26*(3), 479–489.

Dyer, M. E. and L. G. Proll (1977, Dec.). An algorithm for determining all extreme points of a convex polytope. *Math. Prog. 12*(1), 81–96.

Epelman, M. S. (1979, Dec.). On a property of polyhedral sets. *Math. Prog. 16*(1), 373–373.

Feller, W. K. (1967). *Introduction to Probability Theory and Its Applications* (3rd ed.), Volume I. New York: Wiley.

Feller, W. K. (1971). *Introduction to Probability Theory and Its Applications* (2nd ed.), Volume II. New York: Wiley.

Ford, Jr., L. R. and D. R. Fulkerson (1962). *Flows in Networks*. Princeton: Princeton University.

Gale, D. (1956). Neighboring vertices on a convex polyhedron. In H. W. Kuhn and A. W. Tucker (Eds.), *Inequalities and Related Systems*, pp. 255–263. Princeton: Princeton University.

Gale, D. (1960). *The Theory of Linear Economic Models*. New York: McGraw-Hill.

Greenberg, H. (1975, Feb.). An algorithm for determining redundant inequalities and all solutions to convex polyhedra. *Numer. Math. 24*(1), 19–26.

Grünbaum, B. (1970, Nov.). Polytopes, graphs, and complexes. *Bull. Amer. Math. Soc. 76*(6), 1131–1201.

Grünbaum, B. (2003). Convex polytopes. In *Graduate Texts in Mathematics* (2nd ed.), Volume 221. New York: Wiley. 1st ed. 1967.

Hall, Jr., M. (1986). *Combinatorial Theory* (2nd ed.). New York: Wiley. 1st ed. 1967.

IBM Corporation, Data Processing Division (1964, Mar.). *Combinatorial Problems*, Proceedings of the IBM Scientific Computing Symposium, White Plains, New York. IBM Corporation, Data Processing Division.

Isaacson, E. and H. B. Keller (1966). *Applied Numerical Methods*. New York: Wiley.

Kelly, D. G. and J. W. Tolle (1981). Expected number of vertices of a random convex polytope. *SIAM J. Algebraic Discrete Methods 2*(4), 441–451.

Klee, V. (1964, Mar.). Convex polytopes and linear programming. In *Combinatorial Problems*, Proceedings of the IBM Scientific Computing Symposium, White Plains, New York, pp. 123–158. IBM Corporation, Data Processing Division.

Klee, V. (1965). Heights of convex polytopes. *J. Math. Anal. Appl. 11*, 176–190.

Klee, V. (1966a, Dec.). A comparison of primal and dual methods of linear programming. *Numer. Math. 9*(3), 227–235.

Klee, V. (1966b). Paths on polyhedra II. *Pacific J. Math. 17*(2), 249–262.

Klee, V. (1967). Program size in linear programming. In *Proceedings of the Colloquium on Convexity, Copenhagen, 1965*, pp. 177–184.

Klee, V. (1974, Oct.). Polytope pairs and their relationship to linear programming. *Acta Math. 133*(2), 1–25.

Klee, V. (1978). Adjoints of projective transformations and face-figures of convex polytopes. In M. L. Balinski and A. J. Hoffman (Eds.), *Mathematical Programming Study 8, Polyhedral Combinatorics*, pp. 209–216. New York: North-Holland. Paper no. 14.

Klee, V. and G. J. Minty (1972). How good is the simplex algorithm? In O. Shisha (Ed.), *Inequalities*, Volume III, pp. 159–175. New York: Academic.

Klee, V. and D. W. Walkup (1967). The d-step conjecture for polyhedra of dimension $d < 6$. *Acta Math. 117*, 53–78.

Klee, V. and C. J. Witzgall (1968). Facets and vertices of transportation polytopes. In G. B. Dantzig and A. F. Veinott, Jr. (Eds.), *Mathematics of the Decision Sciences*, pp. 257–282. Providence: Amer. Math. Soc.

Lawrence, Jr., J. A. (1978). Abstract polytopes and the Hirsch Conjecture. *Math. Prog. 15*, 100–104.

Manas, M. and J. Nedoma (1968). Finding all vertices of a convex polyhedron. *Numer. Math. 12*, 226–229.

Mangasarian, O. L. (1969). *Nonlinear Programming*. New York: McGraw-Hill.

Mattheiss, T. H. (1973, Jan.–Feb.). An algorithm for determining irrelevant constraints and all vertices in systems of linear inequalities. *Oper. Res. 21*(1), 247–260.

Niedringhaus, W. P. and K. Steiglitz (1978). Some experiments with the pathological linear programs of N. Zadeh. *Math. Prog. 15*, 352–354.

Operations Research (1973, Jan.–Feb.). Programming. *Oper. Res. 21*(1). Special issue.

Orden, A. (1980, Dec.). A step toward probabilistic analysis of simplex method convergence. *Math. Prog. 19*(1), 3–13.

Padberg, M. W. and M. R. Rao (1974, Dec.). The travelling salesman problem and a class of polyhedra of diameter two. *Math. Prog. 7*(1), 32–45.

Philip, J. (1972, Dec.). Plane sections of simplices. *Math. Prog. 3*(1), 312–325.

Saigal, R. (1969, Nov.). A proof of the Hirsch Conjecture on the polyhedron of the shortest route problem. *SIAM J. Appl. Math. 17*(6), 1232–1238.

Silverman, R. (1973a). Decomposition of plane convex sets I. *Pacific J. Math. 47*(2), 521–530.

Silverman, R. (1973b). Decomposition of plane convex sets II: Sets associated with a width function. *Pacific J. Math. 48*(2), 497–502.

Stoer, J. and C. J. Witzgall (1970). *Convexity and Optimization in Finite Dimensions*, Volume I. New York: Springer-Verlag.

Telgen, J. (1978). Redundant constraints in linear programming problems. *Oper. Res. Verf. 28*, 420–433.

Telgen, J. (1981 [1979]). *Redundancy and Linear Programs*. Ph. D. thesis, Mathematisch Centrum, Amsterdam. Mathematical Centre Tracts No. 137.

Tutte, W. T. (Ed.) (1969). *Recent Progress in Combinatorics*. New York: Academic.

Tutte, W. T. (1971). Introduction to the theory of matroids. In *Modern analytic and computational methods in science and mathematics*, Number 37. New York: Elsevier.

University of Oklahoma (1971, Jun.). *Conference on Convexity and Combinatorial Geometry*, Norman, Oklahoma. University of Oklahoma.

Von Hohenbalken, B. (1978, Dec.). Least distance methods for the scheme of polytopes. *Math. Prog. 15*(1), 1–11.

Walkup, D. W. (1978, Aug.). The Hirsch Conjecture fails for triangulated 27-spheres. *Math. Oper. Res. 3*(3), 224–230.

Zaslavsky, T. (1975, Jan.). Facing up to arrangements: face-count formulas for partitions of space by hyperplanes. In *Mem. Amer. Math. Soc.*, Volume 1(154). Providence: Amer. Math. Soc. Issue No. 1.

# Colophon

In December 2007 the author re-composed this paper from his original typescript of January 1980, as amended and revised. He used the standard $\mathcal{AMS}$ article class and packages of the American Mathematical Society, and compiled the bibliography with BibTeX, employing the University of Chicago Press style. He also included his own macros for specialized notation and convenience.

This original composition was set in the 12pt Palatino and Euler Virtual Math fonts of Hermann Zapf for an A4 formatted page, the standard for this printing.