UNIVERSITY
OF OSLO

Master's thesis

Analytic description for synchronization of two-level quantum systems

Johan Andreas Fløisand

CS: Physics
60 ECTS study points

Department of Physics
Faculty of Faculty of Mathematics and Natural Sciences

Spring 2023

Johan Andreas Fløisand

Analytic description for synchronization of two-level quantum systems

Supervisor:
Joakim Bergli

Abstract

Synchronization of quantum mechanical systems have in recent years become a well-discussed topic. The most explored systems are the quantum van der Pol oscillator [26][12][27] and two-level system (TLS) synchronizing to an external field [29] and synchronizing to another TLS [1]. However, it was claimed that two-level systems do not have a valid limit cycle [23], and they can therefore not be synchronized. It was suggested that we could circumvent this problem by realizing a mixed state as an ensemble of pure states [18]. Using quantum trajectory theory (QTT), numerical evidence was found for the synchronization of a TLS [13]. We want to find an analytic expression for the synchronization observed numerically. In this thesis we develop the framework to do so: We give results from QTT which helps analyze the model used, and we show how the flow of the state of a TLS from a QTT model can be easily visualized. Furthermore, we give an explicit expression for the space of all Hamiltonians giving the same Lindblad equation for the QTT model used in [13], and give a master equation which can be solved to find the analytic expression for the synchronization observed numerically.

Contents

1 Introduction **3**
1.1 Outline of thesis 4
1.2 Prior knowledge, notation and abbreviations 5

2 A brief overview of synchronization **6**
2.1 Historical example of synchronization 6
2.2 Necessary conditions for synchronization 7
2.3 Visual description of synchronization, phase slips and synchronization in presence of noise 11

3 Overview and results from quantum trajectory theory **13**
3.1 What is quantum trajectory theory? 14
3.2 Why we choose to use quantum trajectory theory 15
3.3 Weak measurement and POVMs 15
3.4 Time evolution in quantum trajectory theory coincides with the Lindblad equation 18
3.4.1 Dimension of θ 21
3.5 Other results in quantum trajectory theory 21
3.5.1 We can always choose the environment to be in the state $|0\rangle$ 22
3.5.2 The choice of measurement basis will decide the jump 22
3.5.3 n-level environment can only give $n-1$ Lindblad operators 23
3.5.4 Recovering the Lindblad equation with Hamiltonian of the system . 24
3.5.5 Independence in choice of Hamiltonian representation 25
3.5.6 Easy calculation of Hamiltonian 27

4 Quantum synchronization of two-level systems **29**
4.1 Can two-level systems be synchronized? 29
4.2 Limit cycle for two-level system constructed from pure states 31
4.3 Using quantum trajectory theory to construct the ensemble 32

5 Analysis on synchronization of two-level systems using quantum trajectory theory **37**
5.1 Deriving a master equation for the stochastic process araising from the quantum trajectory theory 37
5.1.1 A single step of QTT for a specific model 38
5.1.2 Master equation for the model proposed by Longva 39

5.2 Flow of the state of the two-level system on the Bloch sphere when applying quantum trajectory theory . . . 40
5.2.1 State flow for the model proposed by Longva . . . 41
5.2.2 Visualization of state flow on the sphere and in the plane . . . 43
5.3 Dimension analysis of the stochastic processes giving rise to the same Lindblad equation . . . 47
5.3.1 Example: The model proposed by Longva . . . 49

6 Discussion **51**
6.1 Future work . . . 52

Appendices **54**

A Rotating frame in quantum mechanics and entropy calculations **55**
A.1 Unitary operator and rotations . . . 55
A.2 Rotation of the Lindblad equation . . . 58
A.3 Shannon entropy for a two-level system . . . 60

B Different ways of recovering the Lindblad equation **65**
B.1 Time evolution with sum of two operators . . . 65
B.2 Recovering the Lindblad equation for both time steps δt and θ . . . 70
B.3 Recovering the Lindblad equation for general case . . . 73
B.4 Examples showing that the representation of the Hamiltonian does not matter 75
B.4.1 Same basis, but not "fully reduced" . . . 75
B.4.2 Different basis . . . 76

C Master equation **80**
C.1 Time evolution - direct computations . . . 80
C.1.1 What happens if we switch convention . . . 81
C.1.2 First step for actual system . . . 82
C.1.3 How many states can evolve to a fixed state? . . . 83
C.2 Diffusion from random walk . . . 85

D Flow of a TLS state on the Bloch sphere **89**
D.1 Bloch vector flow for model proposed by Longva . . . 90
D.2 The full Bloch vector for the model proposed by Longva . . . 94
D.3 Mapping state flow from the sphere onto the plane . . . 101
D.3.1 Stereographic projection . . . 102
D.3.2 Change of coordinates . . . 104
D.3.3 Winkel Tripel projection . . . 106
D.4 Full figures of state flow . . . 109

E Dimension analysis for the model proposed by Longva **113**

Chapter 1

Introduction

Synchronization is the word we use for how oscillating objects adjust their rhythm to be the same. It is a concept most of us should be familiar with from everyday life: Examples include fireflies lighting up at the same time, pacemakers keeping the rhythm of those with irregular heartbeats, and many more. However, not all objects can be synchronized. The objects that can be synchronized are called self-sustained oscillators and are mathematically described as a dynamical system with a so-called limit cycle.

In recent years there has been interest in synchronization of quantum mechanical systems. From the classical theory of synchronization, the van der Pol oscillator is one of the prime examples [21, section 7.1.1]. It has therefore been a natural choice to see if one could possibly synchronize a quantum mechanical van der Pol oscillator [26][12][27]. Besides the van der Pol oscillator, it is natural to study the synchronization of the least complex quantum system, namely a two-level system (TLS). Both two-level systems synchronizing to an external field [29] and synchronizing to another two-level systems [1]. However, it has later been claimed that two-level systems do not have a valid limit cycle [23], and they can therefore not be synchronized. A counter claim was then made [18] where it was argued that a mixed state is an ensemble of pure states, and such an ensemble can make up a valid limit cycle. This has later been verified experimentally [28].

Quantum trajectory theory (QTT) gives a framework for extracting possible trajectories all resulting in the same Lindblad equation. Using quantum trajectory theory, synchronization of a two-level system coupled to an external field was observed numerically in [13]. It would now be interesting to see if we can find an expression for the synchronization observed numerically. In this thesis we develop the framework to do so:

- We give an overview of QTT as described in [4] before we give our own results in QTT. These results will help analyze the model we want to use. The results we give tell us that we can always choose the environment in a QTT model with two-level environment to be in the state $|0\rangle$. We also show how we can achieve small jumps in a QTT model, and we show that for a two-level environment, we can only get a single Lindblad operator. We also show how to recover the Lindblad equation with a non-zero system Hamiltonian, as well as how the representation of the interaction Hamiltonian will not effect the results of a QTT model. Finally, we give an easy way of reading out the interaction Hamiltonian for a QTT model, whenever we know what Lindblad equation we want.

- To help better understand how the state of a TLS flows for a QTT model, we have visualized the flow for the model given in [13]. We also show how this flow can be projected down from the Bloch sphere to the plane through two different projection, the stereographic projection and the Winkel tripel projection.

- We realized that there are multiple Hamiltonians that give the same Lindblad equation in a QTT model. We give an explicit expression for the space of all Hamiltonians giving the same Lindblad equation for the QTT model used in [13]. We also discuss how the space of Hamiltonians would look in a scenario with a different choice of Lindblad equation.

- Most importantly, we give a master equation which can be solved to find the analytic expression for the synchronization observed numerically. The master equation is a difference equation which relates the probability density of the state of a TLS at a given time step. We hope that we can take the limit of the time step going to zero, and then end up with a differential equation for the probability density.

1.1 Outline of thesis

This thesis is about synchronization of two-level quantum systems. To understand what this means, we need to go through what we mean by synchronization and how we can observe synchronization in a quantum system. We begin by explaining what we mean by synchronization in chapter 2. In this chapter, we will work classically and only give an overview as the field is both large and demanding in terms of mathematical background.

Next, we will explain in chapter 3 why we can use *quantum trajectory theory* to observe synchronization in quantum systems. The quantum trajectory theory is based on a review article by Brun [4]. We also give results of our own, which are not covered by Brun's article.

After this, we will in chapter 4 go through what other researchers have done. We will focus on two articles and an earlier master thesis. The first article by Roulet and Bruder [23] argues that synchronization of a two-level system is impossible, and say that one has to at least have a three-level system. The next article by Parra-López and Bergli [18] argues that it is possible to synchronize a two-level system. The master thesis by Longva [13] builds on the idea by Parra-López and Bergli that a two-level system can indeed be synchronized, and implements numerically the ideas from the article by Brun [4]. Longva argues for why he has numerically observed synchronization of a two-level system.

In chapter 5 we give the expression for a master equation for the probability density of the state of a TLS. The dynamics of the TLS is given by the QTT model described in the thesis by Longva. For the same model, we visualize the flow of the state of the TLS on the Bloch sphere. We also visualize the flow when projected through a stereographic projection and a Winkel tripel projection. We end the chapter by analyzing the dimensionality of the space of Hamiltonians giving rise to the Lindblad equation used in the thesis by Longva.

Finally, in chapter 6 we summarize what we have found and give ideas for what to look at next.

1.2 Prior knowledge, notation and abbreviations

Throughout the text, it is assumed that the reader has a good understanding of two-level quantum systems (often written as TLS on short form), and a basic understanding of open quantum systems and the Lindblad equation. If this is not the case, I do not have any extensive literature other than what I have used myself, such as [24] and [17].

The following abbreviations are used throughout the thesis:
TLS - two-level system
POVM - positive operator-valued measure
QTT - quantum trajectory theory

The following notations are used throughout the thesis:

- If we have two systems A and B, which are in the states $|n\rangle$ and $|m\rangle$ respectively, then we denote the state of the combined system AB by any of the following: $|nm\rangle = |n\rangle\,|m\rangle = |n\rangle \otimes |m\rangle$.

- For a TLS, the eigenstates of σ_x are denoted by $|\uparrow_x\rangle = |x_+\rangle$ and $|\downarrow_x\rangle = |x_-\rangle$, where we assume $\sigma_x |\uparrow_x\rangle = |\uparrow_x\rangle$ and $\sigma_x |\downarrow_x\rangle = -|\downarrow_x\rangle$. Equivalently, the eigenstates of σ_y are denoted by $|\uparrow_y\rangle = |y_+\rangle$ and $|\downarrow_y\rangle = |y_-\rangle$, where we assume $\sigma_y |\uparrow_y\rangle = |\uparrow_y\rangle$ and $\sigma_y |\downarrow_y\rangle = -|\downarrow_y\rangle$.

- For a separable Hilbert space, an (orthonormal) basis will be denoted by any of the following: $\{|k\rangle\} = \{|k\rangle\}_k = \{|k\rangle\}_{k\in I}$ for an index set I.

- $:=$ means left side defined as right side, and $=:$ means right side defined as left side.

Chapter 2

A brief overview of synchronization

Synchronization is a quite general concept, and one everyone should be accustomed with. Intuitively, we understand synchronization as an adjustment of rhythms of oscillating objects due to their weak interaction. Examples are many and varied, which puts into perspective how general the theory of synchronization is. Examples include circadian rhythms in animals, unison applause at a concert, pacemakers for those with irregular heartbeats and many more. We will in this chapter define what we mean by synchronization, give examples and non-examples of synchronization, and explain how we can achieve synchronization in presence of noise. We begin by looking at Christiaan Huygens' observation of synchronization of two pendulum clocks, before we proceed to define the necessary conditions for synchronization to occur. We then go through a way of visualizing when we achieve synchronization, explaining frequency locking, phase locking and going through what an Arnold tongue is. We end the chapter by looking at phase slips and what we would mean by synchronization in presence of noise. The entirety of this chapter is based on the book *Synchronization: A Universal Concept in Nonlinear Sciences* by Pikovsky, Rosenblum and Kurths [21].

2.1 Historical example of synchronization

We can actually get some insight to what synchronization should be by looking at the etymology of the word. It comes from the greek words χρόνος (*chronos*, meaning time) and σύν (*syn*, meaning together). It therefore means "happening at the same time". We are therefore looking at phenomena that in some sense have dynamics that makes us think of them as "occurring at the same time".

As told in the book by Pikovsky, Rosenblum and Kurths [21], the Dutch researcher Christiaan Huygens was (probably) the first scientist to observe and describe the phenomenon of synchronization. His first mention of synchronization was in a letter he wrote to his father, in 1665, where he wrote that he observed how two pendulum clocks synchronized to each other. In his memoirs, *Horologium Oscillatorium* [9], he describes how two pendulum clocks hanging from a common support beam have oscillations that coincide, but move in opposite directions:

> ...It is quite worth noting that when we suspended two clocks so constructed from two hooks imbedded in the same wooden beam, the motions of each

> pendulum in opposite swings were so much in agreement that they never receded the least bit from each other and the sound of each was always heard simultaneously. Further, if this agreement was disturbed by some interference, it reestablished itself in a short time. For a long time I was amazed at this unexpected result, but after a careful examination finally found that the cause of this is due to the motion of the beam, even though this is hardly perceptible. The cause is that the oscillations of the pendula, in proportion to their weight, communicate some motion to the clocks. This motion, impressed onto the beam, necessarily has the effect of making the pendula come to a state of exactly contrary swings if it happened that they moved otherwise at first, and from this finally the motion of the beam completely ceases. But this cause is not sufficiently powerful unless the opposite motions of the clocks are exactly equal and uniform.

Huygens had described mutual synchronization, and he had also explained why it happened: The two pendulum clocks were coupled through the beam, synchronizing them in anti-phase due to the weak interaction between the clocks. Moreover, when the two clocks were synchronized, a small perturbation of any one of them would not effect the long term dynamics, i.e. the two clocks fall back to their synchronized state.

After Huygens, many other examples of synchronization have been described (see [21] for many detailed examples). The two pendulum clocks Huygens observed were in other words not special. As we want to describe synchronization, we first need to find out which objects can actually be synchronized. This is what we will do in the next section.

2.2 Necessary conditions for synchronization

From the example in the previous section (mutual synchronization of two pendulum clocks), we can try to give a verbose meaning to the phenomenon that is synchronization: We understand synchronization as an adjustment of frequency of oscillating objects due to their weak interaction. We now need to describe what we mean by an oscillating object, the frequency of an oscillating object, adjustment of frequency, and weak interaction of oscillating systems.

Oscillating object and frequency

An oscillating system is intuitively a system that swings back and forth. To be more precise, the system has some state which it always comes back to: It can pass the state quickly or stay in (or close to) the state for a longer time, but it always returns. A natural example is a pendulum: Let us imagine a pendulum moving back and forth in a plane, and choose a coordinate system (x, y) such that if the pendulum does not move at all, it has the x-coordinate zero. If the pendulum is put into motion and only experiences gravitational force, it will keep on moving in positive and negative x-direction, always passing $x = 0$. If it also experiences drag force(s), then the energy will dissipate over time, but it will still continue to oscillate around the point $x = 0$.

It turns out that not every oscillator can be synchronized. If we take the example with the pendulum that only experience gravitational force, then a perturbation (in form of for instance a push) which increases the total energy of the system will change the amplitude of the oscillations. This is different from that of the pendulum clocks Huygens observed. This is closer to that of someone building up speed on a swing. Systems like these can experience resonance, which is easily confused with synchronization. To get the amplitude back to the original value, we need a dissipation of energy. Taking the pendulum as example again, this would mean including drag force(s). This would, however, again differ from the pendulum clocks as all the energy will dissipate and the system will come to a halt. We therefore need to add some kind of internal energy source of the system, in such a way that the oscillations stays stable. By stable we mean that the energy of the pendulum does not keep increasing up to infinity, neither does it decrease such that the motion stops. Instead, the pendulum keeps on oscillating. Moreover, we want the system to fall back to this stable oscillation after any small[1] perturbation.

The type of system we have now described is close to those we are after: We are after systems that have stable oscillations, i.e. oscillatory objects experiencing dissipation, with an internal source of energy, such that after any small perturbation the object returns to the same stable oscillatory motion after some time. Since synchronization is all about how the *natural frequency* of the oscillatory object changes according to an external signal with a different frequency, we need the oscillatory object to have a well-defined frequency. We therefore restrict ourselves to look at oscillatory objects with periodic oscillations. Finally, we would not want the oscillations to be dependent on the initial conditions. That is, the periodic oscillatory motion should only depend on the internal parameters of the system.

We summarize all the traits we have discussed. We are after systems with the following traits:

1. The system is an oscillating object with periodic oscillations.
2. The oscillating object experiences dissipation of energy.
3. The oscillating object has an internal energy source which keeps the oscillations stable.
4. For any (small) perturbation of the objects motion, the system will fall back into the stable periodic oscillation it had originally.
5. Lastly, the object's stable periodic oscillation is independent of initial conditions and only depends on the internal parameters of the object.

When all this is satisfied, we have what is called an **self-sustained oscillator**.

[1]The word small is not well-defined here. If we again use the pendulum clock as an example, the idea is that if we force the pendulum of the clock to be stationary at $x = 0$ and then release it, there will be no non-trivial oscillation: The clock will stay still with the pendulum at $x = 0$. We have thus forced the clock to not move. We want to avoid this type of perturbation and will therefore be vague and say "small".

Mathematically, we can describe a self-sustained oscillator in the following way: Let

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} = (x_1, \ldots, x_M)$$

be an M-dimensional system of ordinary differential equations, such that it is dissipative and autonomous. Suppose that the system has a stable periodic solution $\boldsymbol{x}_0$, i.e. a period T_0 such that $\boldsymbol{x}_0(t + T_0) = \boldsymbol{x}_0(t)$ for all time points t. If we look at the trajectory of the stable periodic solution $\boldsymbol{x}_0$ in *phase space*,[2] it will be an isolated, closed and attractive curve. This is called a **limit cycle**. By isolated we mean that there are no other closed and attractive trajectories infinitely close to it. By attractive we mean that any point in phase space close enough to the limit cycle will converge (but never touch) to the limit cycle[3]. A point moving along the limit cycle will represent the self-sustained oscillator.

Weak interaction of oscillating systems

Synchronization happens when a self-sustained oscillator (or several of them) interact with an external force (or if they are weakly coupled, and thus "feel" each other). In the example with two pendulum clocks hanging from a common support beam observed by Hyugens, the beam could bend. That is it could vibrate slightly, moving from left to right. Thus the motion of one pendulum was transmitted through the beam to the other pendulum. This is the type of weak interaction we are after.

It is not easy to define what we mean by "weak interaction", but we can give an example from Pikovsky, Rosenblum and Kurths [21] where the interaction is two strong. Say, for instance, that the pendulums of two pendulum clocks are connected with a rigid rod (see Figure 2.1). Then the two pendulums are forced to oscillate with the same frequency. They move synchronously, but we do not want to call this synchronization as they trivially have the same frequency. We instead think of the two pendulum clocks as non-decomposable. By non-decomposable we mean that we cannot think of them as two separate self-sustained oscillators. Another example by Pikovsky, Rosenblum and Kurths [21] of a non-decomposable system is the hare–lynx cycle. The population of both the hare and the lynx oscillates with an approximate frequency, but it is not possible to separate the two systems. In other words, we must think of them as two components of an oscillator that vary synchronously.

Adjustment of frequency

We think of synchronization as adjustment of rhythms due to an interaction. Let us again look at the example with two pendulum clocks. If we isolate the clocks, we can measure their natural frequency f_1 and f_2, respectively. Even if the clocks are made to have the same frequency, there is always a tiny difference such that $\Delta f = f_1 - f_2 \neq 0$.

[2]By phase space we mean the space of all the variables $\boldsymbol{x}$. This will for us be equivalent to the state space.

[3]We have not defined what we mean by close enough, but intuitively it mean that if there is only a single limit cycle for our system, then any point not on the limit cycle is close enough. If we have more than two isolated closed trajectories, where at least one of them is a limit cycle, it could be the case that there is only a subset of points in phase space that are attracted.

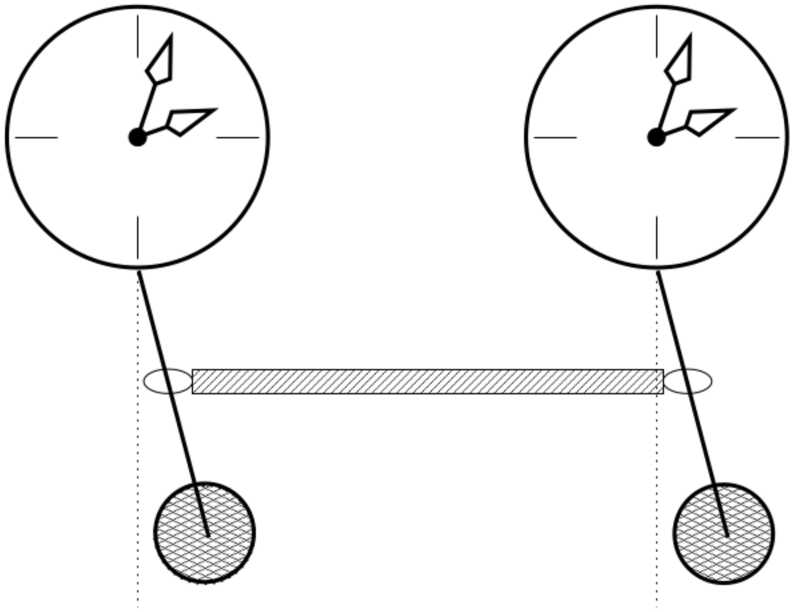


Figure 2.1: Two pendulum clocks connected with a rigid rod. This is an example of to strong coupling, and we do not call this synchronization. The figure is taken from the book by Pikovsky, Rosenblum and Kurths [21].

We then hang the clocks from the same beam. We can then measure their frequencies, F_1 and F_2, when they are hanging from the beam. If the beam is not rigid, so that the clocks are weakly coupled, and the frequency detuning Δf is not to large, we will see that the measured detuning $\Delta F \approx 0$. We call this **frequency locking** and say that the pendulum clocks are synchronized. We can plot the measured detuning ΔF against the natural detuning Δf. This will give us a region for Δf where ΔF is zero, which is the synchronization region. We generally expect this region to increase with the coupling strenght. (This is described in more detail later and in Figure 2.2.)

We can define another quantity which helps analyse synchronization, namely the phase. The phase of a self-sustained oscillator is defined to be a function ϕ of the state of the oscillator (i.e. a point in the phase space of the oscillator) to the real numbers, such that it is linear in time, and whenever we take a point on the limit cycle and return after one period, the phase has increased by 2π. Mathematically, if we denote the phase space (i.e. the state space) of the self-sustained oscillator by M, then we have $\phi : M \to \mathbb{R}$ such that $\frac{d\phi}{dt} = \omega t$ (i.e. linear in time) and $\phi(x(t+T_0)) = \phi(x(t)) + 2\pi$ whenever $x(t) \in M$ is on the limit cycle and T_0 is the period of the self-sustained oscillator. The phase can also be defined for a periodic force $F_e = \epsilon \sin(\omega t + \phi_0)$. Here, ϵ is the strength of the force, ω is the angular frequency and ϕ_0 is the initial phase. We would then say that the phase of the force is $\phi_e = \omega t + \phi_0$. If we have a self-sustained oscillator with phase ϕ, perturbed by this external force, then frequency locking would mean that the phase difference $\Delta\phi = \phi - \phi_e$ would become constant. This is called **phase locking**. A nice property of the phase can be seen if we go back to the example with the two pendulum clocks again. When the two clocks are synchronized, we can observe either *in-phase synchronization* or *anti-phase synchronization*. In-phase synchronization means that both pendulum attains their right

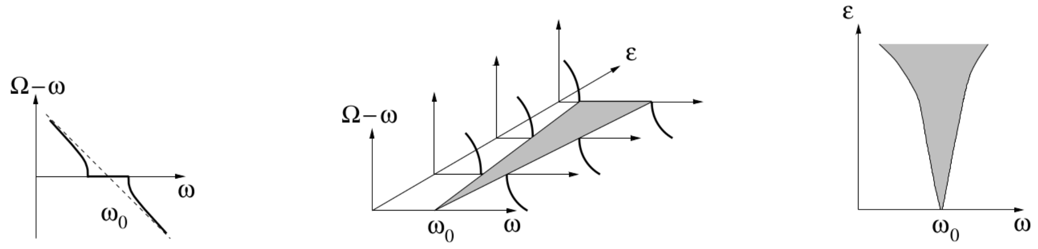


Figure 2.2: Figure 2.2a shows the difference of the frequencies of the driven oscillator Ω and the external force ω. The strength of the force ϵ is held constant. We see that for small detuning $\Delta\omega = \omega_0 - \omega$, we have frequency locking $\Omega - \omega = 0$. The dashed line shows $\omega_0 - \omega$ vs. ω). When synchronization breaks down we see that the force is too weak to synchronize the oscillator, but it pushes the frequency of the system towards its own frequency. Figure 2.2b shows the synchronization region as a function of both the external frequency ω and the strength of the force ϵ. The region of synchronization is marked in Figure 2.2c and is referred to as the Arnold tongue. The figure is taken from the book by Pikovsky, Kurths and Rosenblum [21].

maximum amplitude at the same time, then their left maximum amplitude at the same time, and so on. In other words, the motion is the exact same. For the phase this means that $\Delta\phi = 0$, modulo 2π. Anti-phase synchronization means that when one attains the right maximum amplitude, the other attains the left maximum amplitude at the same time. This mean that $\Delta\phi = \pi$, modulo 2π, so the phase detuning is shifted by π. This cannot be observed by just looking at the frequency detuning and we are thus able to distinguish between different synchronization regimes.

2.3 Visual description of synchronization, phase slips and synchronization in presence of noise

We would like to have an easy way of determining if we have synchronization. We have mentioned how synchronization is dependent on the frequency detuning and the coupling/interaction strength. Let us assume that we have an external periodic force with angular frequency ω. Let ω_0 be the natural (angular) frequency of the self-sustained oscillator. If we call the oscillator under influence of the external force the *driven oscillator*, and let Ω be the observed angular frequency of the driven oscillator, then we have synchronization whenever $\Omega = \omega$. In the terms introduced earlier, the driven oscillator is frequency locked to the external force.

If we keep the interaction strength, denoted by ϵ, of the external fixed, we can experiment with different detuning $\Delta\omega = \omega_0 - \omega$ to see when the synchronization regime stops. This will give us Figure 2.2a. If we instead visualize the synchronization region as a function of ω and ϵ, we will get the Arnold tongue in Figure 2.2c. Putting both figures together, we get Figure 2.2b. These figures help visualize when we have synchronization.

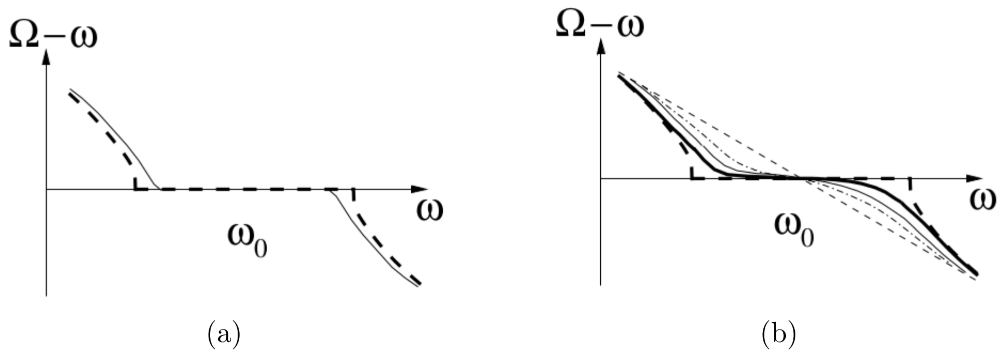


Figure 2.3: Frequency–detuning curves for noisy oscillators. The angular frequency of the driven oscillator Ω is the average observed frequency. Figure 2.3a shows how a self-sustained oscillator in presence of bounded noise which is weaker than the external force can still be synchronized. The region of synchronization will however be smaller than without the noise. On the other side, Figure 2.3b shows how unbounded noise or noise stronger than the force means that frequency locking only happens when the detuning $\Delta\omega = \omega_0 - \omega = 0$. If the noise is not to strong, we can almost get synchronization (the bold line), but when the noise gets stronger, synchronization is completely destroyed. The figure is taken from the book by Pikovsky, Kurths and Rosenblum [21].

It is interesting to note what happens at the end of the synchronization region. We imagine moving from outside the synchronization region of the Arnold tongue towards the synchronization region in Figure 2.2c. Outside the synchronization region, the phase difference grows to infinity, however this growth is not uniform. It turns out that when we are close to the region of synchronization, the phase stays almost constant before it rapidly changes by 2π. This jump is what we call a **phase slip**: The driven oscillator is almost phase locked, but ends up slipping and rapidly increasing by 2π. The closer we get to the region of synchronization, the longer the phase difference seems constant.

Phase slips are also common occurrences in the presence of noise. A self-sustained oscillator in presence of noise will not have a proper limit cycle. However, if we take the limit when the strength of the noise goes to zero, it does have a limit cycle. This means that in the presence of noise, the phase will not grow linearly. The randomness will accumulate in the phase. If we think of the driven oscillator again, the phase difference will stay close to constant, but it now has a stochastic contribution. It the strength of the noise is bounded and less than the strength of the force, then the phase will only fluctuate about a stable constant value. On the other hand, if the strength of the noise is unbounded (e.g. gaussian) or bounded but stronger than the force, then phase slips can occur. We note that even tough the self-sustained oscillator has a well-defined angular frequency ω_0 (defined from the limit cycle when there is no noise), we can only talk about an average observed angular frequency Ω for the driven oscillator. Figure 2.3 shows what happens in the two cases: If the noise is smaller than the force, we will have a region of synchronization which is smaller than the one without noise. If the noise is unbounded/stronger than the force, we will only have full frequency locking when $\omega_0 = \omega$. If the noise is not to large we can get close to synchronization, and when the noise grows we see that synchronization is destroyed.

Chapter 3

Overview and results from quantum trajectory theory

When we try to synchronize a quantum mechanical system, we need to add dissipation. A natural way to do this is by letting the system be open, such that the system interacts with the environment. If we do not have a simple model for the environment, the interaction quickly becomes complicated. If we let $\rho(t)$ denote the mixed state of the system at time t, and assume that the *Markovian approximation*[1] holds, then we can find the equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \sum_k \frac{\Gamma_k}{2} \left(2L_k \rho L_k^\dagger - L_k^\dagger L_k \rho - \rho L_k^\dagger L_k \right),$$

often called the Lindblad equation, for the time evolution of the mixed state [22, section 3.5]. The first term $-\frac{i}{\hbar}[H, \rho]$ is the normal unitary time evolution of the isolated system (with Hamiltonian H). The other terms describe interaction with the environment. The operators L_k are know as Lindblad operators, and $\Gamma_k \geq 0$ are the rate of which the interaction happens. The sum goes over all k Lindblad operators.

We can think of the Lindblad equation as a dynamical system, where the mixed state of the system ρ is the variable. The problem with using the mixed state is that it only correspond to statistical ensemble of pure states. Moreover, there is not a unique way of writing the mixed state [17, theorem 2.6]. When talking about synchronization, this means that not only can we only talk about synchronization of a statistical ensemble (i.e. a convex combination of pure states), but the ensemble is not unique. This is where the idea of trajectory theory comes in: Each trajectory can be thought of as the time evolution of the system, which on average can trace out a limit cycle of a self-sustained oscillator in the presence of noise. For further details, see [13]. We will base the theory in this section solely on the review article by Brun [4] and our own derivations.

[1] We will say that a quantum system is *Markovian* if it is local in time, i.e. $\rho(t + dt)$ is completely determined by $\rho(t)$. The *Markovian approximation* is that we can neglect the memory of the environment. In other words, it is not possible to see the effects of the system on the environment [22, section 3.5].

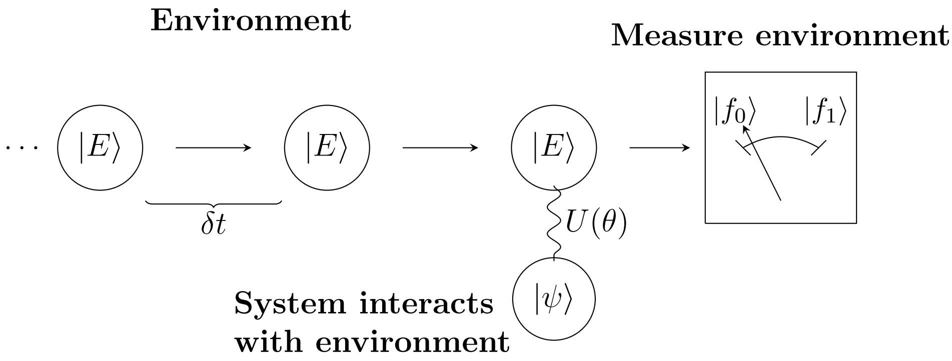


Figure 3.1: Schematic diagram of the QTT model we use. The environment consist of infinitely many two-level systems, all in the same state $|E\rangle$, which do not interact with each other. The system is also a two-level system, but in a state $|\psi\rangle$. The system time evolves over some transient time δt, before it interacts with a single environment TLS. After the interaction, the environment is immediately measured in a chosen basis $\{|f_0\rangle, |f_1\rangle\}$, and then discarded. This process is then repeated for as long as we desire: The system is time evolved, then it interacts with the environment, then the environment is measured, and the cycle repeats. The transient time δt is always the same. The Hamiltonian governing the transient time evolution of the system, H_S, is also the same for every time step. The interaction is for every step modelled by a unitary transformation $U(\theta)$, which depends on a parameter θ. Before the process begins, the common environment state, $|E\rangle$, must be decided. The measurement basis $\{|f_0\rangle, |f_1\rangle\}$ must also be the same for every time step.

3.1 What is quantum trajectory theory?

The quantum trajectory theory we present is based on the review article by Brun [4]. The theory is more vast and general than what we present, and Brun give references to other literature in his article.

We begin with our system $|\psi\rangle$ placed in a very specific environment: The environment will be an ensemble of quantum systems in the same state $|E\rangle$. After some transient time δt, a single environment state will interact with the system. This happens over a short time θ. (Right now, the interaction time will be denoted by θ, but note that the meaning of θ will change later on.) Then, the single environment state is measured in a chosen basis $\{f_k\}_{k\in I}$. This will collapse the state of the system. The state then evolves for the same transient time δt before again interacting with a single environment state $|E\rangle$ for some time θ, before the environment is measured in the same basis as before. This cycle is repeated indefinitely. Figure 3.1 visualizes this when both the system and the environment consists of two-level systems. The time evolution during the transient time is governed by a system Hamiltonian H_S, which gives some unitary time evolution $\widetilde{U}(\delta t)$. The interaction with the environment is modeled to happen so fast that we can neglect the system Hamiltonian fully, and only focus on the interaction Hamiltonian H_I. This interaction will give some unitary time evolution $U(\theta)$.

3.2 Why we choose to use quantum trajectory theory

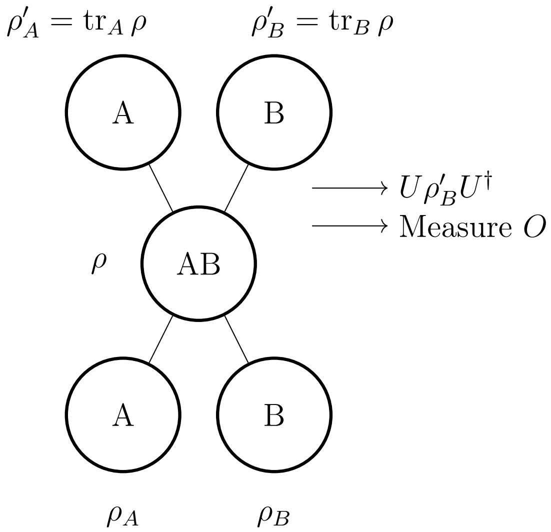


Figure 3.2: System A and B interact and become an entangled system AB. Whatever is done with system B after entanglement, on average, measurement on system A will yelled the same result.

When using quantum trajectory theory, we make a weak measurement as to not disturb the system "to much". Still, the system will "jump" to a pure state, tracing out a path on the surface of the Bloch sphere. We are not able to see this solving the Lindblad equation, but we will explain how the average will give the same result.

See figure 3.2 for visual representation of the following explanation. We have a two-level system, A, which interact with the environment; another two-level system B. The interaction results in a density matrix ρ for the entangled system. After the interaction, we can either measure system B, or just discard it. Say we measure the environment B and use this knowledge to update the reduced density matrix ρ'_A. A measurement on system A after this update will give us some result $\tilde{a}$. We can repeat this procedure to obtain an average $\langle\tilde{a}\rangle$.

On the other hand, if we let the environment B evolve and measure system A, we will obtain a result $\tilde{b}$. Repeating this procedure to obtain an average $\langle\tilde{b}\rangle$, we will have $\langle\tilde{b}\rangle = \langle\tilde{a}\rangle$. Thus, on average, we will measure the same.

In our case, QTT will trace out different paths each time we repeat the experiment. On average, these paths will give the same result as solving the Lindblad equation. More important is the fact that the representation of the density matrix is not unique. This means that even if it contains all possible information about measurement results, it does not contain the information about paths. This is the reason we use trajectory theory: To get the statistic of all the possible paths the system can trace out for a given Lindblad equation.

3.3 Weak measurement and POVMs

A nice property of the quantum trajectory theory is that each measurement can be thought of as being *weak*. By weak, we will mean the same as Brun describes in his article [4], namely that the information gained is in average small, but the disturbance of the state is also small. As a measure of information, we use the Shannon entropy [17, chapter 11],

$$S = -\sum_i p_i \log_2 p_i,$$

where p_i is the probability of i-th measurement outcome[2]. To describe what we mean by *weak measurement*, we will first calculate the Shannon entropy for a projective measurement on a TLS. Next, we take a quick detour through the topic of *Positive Operator-Valued Measure* (abbreviated POVM) [20, section 9.5][17, section 2.2.6].

A projective measurement[3] is written mathematically as a set of (orthogonal) projection operators $\{P_i\}_{i\in I}$ such that $\sum_i P_i = \mathbb{1}$. The index set I has cardinality less than or equal to the dimension of the Hilbert space of the system. If we choose to measure a TLS in a basis, the projective measurement is thus given by $\{P_1, P_2\}$, where $P_2 := \mathbb{1} - P_1$. Denoting the probability of getting outcome 1 and 2 by p_1 and p_2, respectively, the Shannon entropy is given by

$$S_{\text{proj}} = -p_1 \log_2 p_1 - p_2 \log_2 p_2 = -p_1 \log_2 p_1 - (1 - p_1) \log_2 (1 - p_1).$$

Elementary calculus gives us that this Shannon entropy is a concave function which is minimal for $p_1 = 0, 1$ with value $S_{\text{proj}} = 0$, and maximal for $p_1 = p_2 = \frac{1}{2}$ with value $S_{\text{proj}} = 1$ (see appendix A.3). After the measurement, the state is left in an eigenstate of P_1 or P_2 (depending on the outcome of the measurement), so repeating the measurement will result in the same outcome. In other words, measuring the state will make the state jump to one of the eigenstates. If the state of the system before measurement is far from either eigenstates (i.e. $p_1 = p_2 = \frac{1}{2}$), then the information gained is large, but the jump is also large. We want to (on average) avoid these large jumps such that we disturb the trajectory as little as possible. This is where POVMs come in to the picture.

Although the name *positive operator valued measure* suggests a broader theory, we will only be interested in the finite case described in the book by Nielsen and Chuang [17, section 2.2.6]. Nielsen and Chuang defines a POVM to be a set of positive operators $\{E_n\}_{n\in I}$ such that $\sum_{n\in I} E_n = \mathbb{1}$ (the index set I is taken to be finite). If the system we are looking at is in a state $|\psi\rangle$, then the probability of measurement outcome n will be $p_n = \langle\psi|E_n|\psi\rangle$. The state after measurement can be interpreted in different ways: If the POVM comes from a set of measurement operators, i.e. $E_n = M_n^\dagger M_n$, then the state after measurement will be $|\psi\rangle_n = M_n |\psi\rangle / p_n$. If we are only given the POVM (without any set of measurement operators), we can always define $M_n = \sqrt{E_n}$ (since E_n is a positive operator) and use the set of M_n's as measurement operators. Then the state after measurement will be $|\psi\rangle_n = \sqrt{E_n} |\psi\rangle / p_n$. A more general explanation of POVM can be found in [15, section 13.2.2].

We will go through two examples given in the article by Brun [4] which illustrates weak measurements. The intermediate steps and arguments can be found in appendix A.3. The first one is the POVM $\{E_1, E_2\}$ where

$$E_1 = |0\rangle\langle 0| + (1-\epsilon)|1\rangle\langle 1|, \quad E_2 = \epsilon |1\rangle\langle 1|,$$

and $\epsilon \ll 1$. We see that both E_1 and E_2 are positive and $E_1 + E_2 = \mathbb{1}$, so this is a POVM. Letting the system be in a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we can calculate the probabilities

$$p_1 = \langle\psi|E_1|\psi\rangle = 1 - \epsilon|\beta|^2 \quad \text{and} \quad p_2 = \langle\psi|E_2|\psi\rangle = \epsilon|\beta|^2,$$

[2]The Shannon entropy we use in this text is only defined for discrete probability distributions.

[3]Projective measurements are far better described in the book by Nieslen and Chuang [17, section 2.2.5], and we will only use the bare minimum needed to describe the Shannon entropy of a two-level system.

and Shannon entropy $S_{\text{POVM}} = -(1-\epsilon|\beta|^2)\log_2(1-\epsilon|\beta|^2) - \epsilon|\beta|^2\log_2(\epsilon|\beta|^2)$. The Shannon entropy is always close to zero, so we gain very little information. We let the actual measurements be given by the square root of the elements in the POVM, i.e

$$M_1 = |0\rangle\langle 0| + \sqrt{1-\epsilon}\,|1\rangle\langle 1|\,, \quad M_2 = \sqrt{\epsilon}\,|1\rangle\langle 1|\,.$$

The state after measurement will then be either

$$\begin{aligned}|\psi\rangle_1 &= \frac{M_1|\psi\rangle}{\sqrt{\langle\psi|E_1|\psi\rangle}} = \frac{\alpha|0\rangle + \sqrt{1-\epsilon}\beta|1\rangle}{\sqrt{1-\epsilon|\beta|^2}} \quad \text{or} \\ |\psi\rangle_2 &= \frac{M_2|\psi\rangle}{\sqrt{\langle\psi|E_2|\psi\rangle}} = |1\rangle \quad \text{(up to a phase factor)}\end{aligned}$$

with probability $p_1 = 1-\epsilon|\beta|^2$ and $p_2 = \epsilon|\beta|^2$, respectively. On average the state changes only slightly, but every so often, we expect a large jump to the state $|1\rangle$.

The second example is the POVM $\{E_1', E_2'\}$ where

$$E_1' = \frac{1+\epsilon}{2}|0\rangle\langle 0| + \frac{1-\epsilon}{2}|1\rangle\langle 1|\,, \quad E_2' = \frac{1-\epsilon}{2}|0\rangle\langle 0| + \frac{1+\epsilon}{2}|1\rangle\langle 1|$$

and $\epsilon \ll 1$. We again see that both E_1' and E_2' are positive and $E_1' + E_2' = \mathbb{1}$, so this is also a POVM. Letting the system again be in a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we can calculate the new probabilities

$$\begin{aligned}p_1' &= \langle\psi|E_1'|\psi\rangle = \frac{1+\epsilon(|\alpha|^2-|\beta|^2)}{2}, \\ p_2' &= \langle\psi|E_2'|\psi\rangle = \frac{1+\epsilon(|\beta|^2-|\alpha|^2)}{2}.\end{aligned}$$

and new Shannon entropy

$$\begin{aligned}S'_{\text{POVM}} =& -\frac{1+\epsilon(|\alpha|^2-|\beta|^2)}{2}\log_2\frac{1+\epsilon(|\alpha|^2-|\beta|^2)}{2} - \frac{1+\epsilon(|\beta|^2-|\alpha|^2)}{2}\log_2\frac{1+\epsilon(|\beta|^2-|\alpha|^2)}{2} \\ =& 1 - \frac{1}{2}\log_2(1-\epsilon^2(2|\alpha|^2-1)^2) - \frac{\epsilon(2|\alpha|^2-1)}{2}\log_2\frac{1+\epsilon(2|\alpha|^2-1)}{1-\epsilon(2|\alpha|^2-1)}.\end{aligned}$$

This Shannon entropy is, contrary to the previous example, always close to one. We again let the actual measurements be given by the square root of the elements in the POVM, i.e

$$M_1' = \sqrt{\frac{1+\epsilon}{2}}|0\rangle\langle 0| + \sqrt{\frac{1-\epsilon}{2}}|1\rangle\langle 1|\,, \quad M_2' = \sqrt{\frac{1-\epsilon}{2}}|0\rangle\langle 0| + \sqrt{\frac{1+\epsilon}{2}}|1\rangle\langle 1|\,.$$

The state after measurement will then be either

$$\begin{aligned}|\psi'\rangle_1 &= \frac{M_1'|\psi\rangle}{\sqrt{\langle\psi|E_1'|\psi\rangle}} = \frac{\alpha\sqrt{1+\epsilon}|0\rangle + \beta\sqrt{1-\epsilon}|1\rangle}{\sqrt{1+\epsilon(|\alpha|^2-|\beta|^2)}} \quad \text{or} \\ |\psi'\rangle_2 &= \frac{M_2'|\psi\rangle}{\sqrt{\langle\psi|E_2'|\psi\rangle}} = \frac{\alpha\sqrt{1-\epsilon}|0\rangle + \beta\sqrt{1+\epsilon}|1\rangle}{\sqrt{1+\epsilon(|\beta|^2-|\alpha|^2)}}\end{aligned}$$

with probability $p'_1 = \frac{1+\epsilon(|\alpha|^2-|\beta|^2)}{2}$ and $p'_2 = \frac{1+\epsilon(|\beta|^2-|\alpha|^2)}{2}$, respectively. Both of the POVMs we have looked at are considered by Brun as being weak. The first example only changes the state slightly on average, but has a tendency to do large jumps with a very small probability. The information gained is always close to zero. The second example always changes the state, but never in large jumps. The information gained is always close to one. This might seem contradictory to what we stated before: The information gained from a weak measurement should on average be small, but here it is comparable to the projective measurement. One way to interpret this is to use Neumark's theorem explained below: The POVM is realized as a projective measurement on an extension of the Hilbert space (e.g. an environment). The information gained from the environment is large, but the information about the system is still small.

So far we have seen that if we use a POVM, we can get weak measurements. We have yet to explain what this has to do with trajectory theory. The relation between the two comes from a mathematical theorem sometimes called Neumark's theorem (or Naimark's dilation theorem). The mathematical statement is (as always) densely formulated [19, theorem 4.6], but the main idea is that any POVM can be realized as a projective measurement on an extension of the Hilbert space [22, section 3.1.4] [20, chapter 9.5]. In other words, we let our system interact with an environment and then do a projective measurement on the environment. This is exactly what we are doing in quantum trajectory theory! Letting the interaction between the system and environment be weak, and happen over a short time span, we can measure the environment and only perturb the system very slightly.

3.4 Time evolution in quantum trajectory theory coincides with the Lindblad equation

We will now go into the actual dynamics of the model and show that time evolution in quantum trajectory theory coincides with the Lindblad equation. We begin by assuming the system Hamiltonian is zero in this section, and in the next section we will include a non-zero Hamiltonian for the system. The setup is as follows (see also Figure 3.1): Our system is in a state $|\psi\rangle$ which lies in a Hilbert space $\mathcal{H}_S$. The system is in contact with an environment. The environment consists of equal non-interacting systems all in the same state $|E\rangle$. Each single environment system is thus described by a Hilbert space $\mathcal{H}_E$. An environment system interacts with the actual system over a short time, before the environment system is measured. The actual system then evolves over some transient time δt before a new environment system interacts with the actual system over a short time again. The interaction time, and transient time between interactions, are always the same. The measurement done on the environment system after interaction is always the same. The interaction is described by a Hamiltonian $H \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_E)$, which can be written

$$H = H_S \otimes \mathbb{1} + H_I + \mathbb{1} \otimes H_E.$$

To get as quickly as possible to the result, we assume that both $H_S = 0$ and $H_E = 0$. The time evolution is then given by $U(\theta) = \exp(-i\theta H_I)$. The parameter θ takes into account both time and interaction strength. We Taylor expand the time evolution

$U(\theta) = \mathbb{1} - i\theta H_I - \frac{\theta^2}{2} H_I^2 + \mathcal{O}(\theta^3)$. Let $|\Psi\rangle = |\psi\rangle \otimes |E\rangle$ be the state of the system and environment before interaction, and $\rho_S = |\psi\rangle \langle\psi|$ and $\rho = |\Psi\rangle \langle\Psi|$ be the density matrix of the system and system coupled with environment, respectively. As we are trying to recover the Lindblad equation from the time evolution, we calculate the reduced density matrix of the system after time evolution

$$\rho_S' = \mathrm{tr}_{\mathrm{env}} \,(U \rho U^\dagger).$$

We find

$$\begin{aligned} U\rho U^\dagger &= \left(\mathbb{1} - i\theta H_I - \frac{\theta^2}{2} H_I^2 + \mathcal{O}(\theta^3)\right) \rho \left(\mathbb{1} - i\theta H_I - \frac{\theta^2}{2} H_I^2 + \mathcal{O}(\theta^3)\right)^\dagger \\ &= \rho - i\theta(H_I\rho - \rho H_I^\dagger) + \frac{\theta^2}{2}\left(2H_I\rho H_I^\dagger - H_I H_I \rho - \rho H_I^\dagger H_I^\dagger\right) + \mathcal{O}(\theta^3) \\ &= \rho - i\theta[H_I, \rho] + \frac{\theta^2}{2}\left(2H_I\rho H_I - H_I^2\rho - \rho H_I^2\right) + \mathcal{O}(\theta^3), \end{aligned}$$

which already has a familiar shape close to the Lindblad equation. The partial trace is easily computed if we can separate the system and environment. We can achieve this separation by writing the interaction as a sum of operators[4] $H_I = \sum_j A_j \otimes B_j$, where $A_j \in \mathcal{B}(\mathcal{H}_S)$ and $B_j \in \mathcal{B}(\mathcal{H}_E)$. Hence, exploiting the hermiticity of H_I,

$$\begin{aligned} \rho_S' =& \mathrm{tr}_{\mathrm{env}}\,(U\rho U^\dagger) \\ =& \rho_S - i\theta \sum_j \mathrm{tr}_{\mathrm{env}}([A_j \otimes B_j, \rho]) \\ &+ \frac{\theta^2}{2} \sum_{jl} \mathrm{tr}_{\mathrm{env}} \left(2(A_j \otimes B_j)\rho(A_l \otimes B_l)^\dagger - (A_j^\dagger A_l \otimes B_j^\dagger B_l)\rho - \rho(A_j^\dagger A_l \otimes B_j^\dagger B_l)\right) + \mathcal{O}(\theta^3). \end{aligned}$$

To calculate the partial trace[5], let $\{|n\rangle\}_{n\in I}$ be an orthonormal basis for $\mathcal{H}_E$ with $|E\rangle$ being one of the basis vectors. We see that

$$\begin{aligned} \mathrm{tr}_{\mathrm{env}}([A_j \otimes B_j, \rho]) &= \sum_n \left[(A_j |\psi\rangle \langle n|B_j|E\rangle) \langle\psi| \overbrace{\langle E|n\rangle}^{=\delta_{En}} - |\psi\rangle \overbrace{\langle n|E\rangle}^{=\delta_{nE}} (\langle\psi| A_j \langle E|B_j|n\rangle) \right] \\ &= \left(A_j \overbrace{|\psi\rangle\langle\psi|}^{=\rho_S} \langle E|B_j|E\rangle \right) - \left(\overbrace{|\psi\rangle\langle\psi|}^{=\rho_S} A_j \langle E|B_j|E\rangle \right) \\ &= [A_j, \rho_S] \langle E|B_j|E\rangle \,. \end{aligned}$$

Similar calculation gives

$$\begin{aligned} \mathrm{tr}_{\mathrm{env}}[(A_j^\dagger A_l \otimes B_j^\dagger B_l)\rho] &= A_j^\dagger A_l \rho_S \left\langle E|B_j^\dagger B_l|E \right\rangle, \\ \mathrm{tr}_{\mathrm{env}}[\rho(A_j^\dagger A_l \otimes B_j^\dagger B_l)] &= \rho_S A_j^\dagger A_l \left\langle E|B_j^\dagger B_l|E \right\rangle. \end{aligned}$$

[4]This is always possible as $\mathcal{B}(\mathcal{H})$ is a vector space for any Hilbert space $\mathcal{H}$ and thus has a basis which we can expand each operator in. To avoid limit arguments, we note that we will only need the result we find for finite dimensional Hilbert spaces.

[5]We again note that we let the Hilbert spaces be finite dimensional to avoid limit arguments.

However

$$\mathrm{tr}_{\mathrm{env}}[(A_j \otimes B_j)\rho(A_l \otimes B_l)^\dagger] = \sum_n (A_j |\psi\rangle \langle n|B_j|E\rangle) \left(\langle\psi| A_l^\dagger \left\langle E|B_l^\dagger|n\right\rangle\right)$$
$$= A_j |\psi\rangle \langle\psi| A_l^\dagger \left\langle E \middle| B_l^\dagger \sum_n |n\rangle \langle n| B_j \middle| E\right\rangle$$
$$= A_j |\psi\rangle \langle\psi| A_l^\dagger \left\langle E|B_l^\dagger B_j|E\right\rangle,$$

where we have used that $\sum_n |n\rangle \langle n| = \mathbb{1}$. In total, the time evolved density matrix is given by

$$\rho_S' = \rho_S - i\theta \sum_j [A_j, \rho_S] \langle E|B_j|E\rangle$$
$$+ \frac{\theta^2}{2} \sum_{jl} \left(2A_l\rho_S A_j^\dagger - A_j^\dagger A_l \rho_S - \rho_S A_j^\dagger A_l\right) \left\langle E|B_j^\dagger B_l|E\right\rangle + \mathcal{O}(\theta^3).$$

We now define a matrix M with entries $M_{lj} = \left\langle E|B_j^\dagger B_l|E\right\rangle$. We immediately have that M is hermitian and therefore diagonalizable with real spectrum. We write $M = \sum_k \lambda_k \boldsymbol{\mu}_k \boldsymbol{\mu}_k^\dagger$ where $\lambda_k \in \mathbb{R}$ and $\boldsymbol{\mu}_k$ are the k-th eigenvalue and eigenvector, respectively. We then define the operators $L_k = \sum_j (\boldsymbol{\mu}_k)_j A_j$ (where $(\boldsymbol{\mu}_k)_j$ denotes the j-th element of the eigenvector $\boldsymbol{\mu}_k$) and factors $\Gamma_k(\theta) = \frac{\theta^2 \lambda_k}{\delta t}$. If we assume that[6] $\sum_j [A_j, \rho_S] \langle E|B_j|E\rangle = 0$, then

$$\rho_S' - \rho_S = \frac{\theta^2}{2} \sum_{jl} \left(2A_l\rho_S A_j^\dagger - A_j^\dagger A_l \rho_S - \rho_S A_j^\dagger A_l\right) M_{lj} + \mathcal{O}(\theta^3)$$
$$= \frac{\theta^2}{2} \sum_{jl} \left[2A_l\rho_S A_j^\dagger \left(\sum_k \lambda_k \boldsymbol{\mu}_k \boldsymbol{\mu}_k^\dagger\right)_{lj} - A_j^\dagger A_l \rho_S \left(\sum_k \lambda_k \boldsymbol{\mu}_k \boldsymbol{\mu}_k^\dagger\right)_{lj} \right.$$
$$\left. - \rho_S A_j^\dagger A_l \left(\sum_k \lambda_k \boldsymbol{\mu}_k \boldsymbol{\mu}_k^\dagger\right)_{lj}\right] + \mathcal{O}(\theta^3)$$
$$= \frac{1}{2} \sum_k \theta^2 \lambda_k \left[2 \left(\sum_l (\boldsymbol{\mu}_k)_l A_l\right) \rho_S \left(\sum_j (\boldsymbol{\mu}_k)_j A_j\right)^\dagger \right.$$
$$- \left(\sum_j (\boldsymbol{\mu}_k)_j A_j\right)^\dagger \left(\sum_l (\boldsymbol{\mu}_k)_l A_l\right) \rho_S$$
$$\left. - \rho_S \left(\sum_j (\boldsymbol{\mu}_k)_j A_j\right)^\dagger \left(\sum_l (\boldsymbol{\mu}_k)_l A_l\right)\right] + \mathcal{O}(\theta^3)$$

and thus

$$\frac{\rho_S' - \rho_S}{\delta t} = \sum_k \frac{\Gamma_k(\theta)}{2} \left(2L_k^\dagger \rho_S L_k - L_k L_k^\dagger \rho_S - \rho_S L_k L_k^\dagger\right) + \mathcal{O}(\theta^3).$$

[6] This can always be done, as we show in later (see section 3.5.3).

We have thus recovered the Lindblad equation if $\Gamma_k(\theta)$ converges to some constant rate Γ_k for each k, i.e. $\lim_{\delta t\to 0}\Gamma_k(\theta) = \Gamma_k$.

3.4.1 Dimension of θ

We will briefly go through the possible dimensionality of θ. We know that the time evolution governed by a time independent Hamiltonian can be written as an exponential of the Hamiltonian operator. In our case, we have either $\exp(-i\theta H)$ or $\exp(-i\theta H/\hbar)$. The exponent must be dimensionless, and since the dimensions of $\hbar$ are energy times time, θH must either be dimensionless or have dimension energy times time. Let us look at the two different cases.

The Hamiltonian always has dimensions energy, so if θH is dimensionless, then θ must have dimensions 1/energy. Since we can write $H = \sum_j A_j \otimes B_j$ in QTT, we can choose to put all the dimensions into the B_j's. That is, we let A_j be dimensionless and B_j have dimensions energy. Then $M_{ij} = \left\langle E|B_j^\dagger B_i|E\right\rangle$ will have dimensions energy squared, and we can therefore say that the eigenvalues of the matrix M, λ_k, have dimensions energy squared. Thus $\frac{\theta^2}{\delta t}\lambda_k$ will have dimension 1/time, i.e. dimensions of a rate.

If $\theta H/\hbar$ is dimensionless, then θ must have dimensions time. The matrix M is in this case multiplied by $\hbar^2$, and hence $\frac{\lambda_k}{\hbar^2}$ will have dimensions 1/time squared. This means that $\frac{\theta^2}{\delta t}\frac{\lambda_k}{\hbar^2}$ has dimensions 1/time again. We therefore have a rate, as expected.

To be explicit, the parameter $\theta = \theta(t)$ always has an implicit time dependence such that $\theta \sim \sqrt{\delta t}$, but has dimensions either time or 1/energy. This is what we mean when we say that the parameter θ decides the strength of the interaction, as well as the time the interaction takes.

3.5 Other results in quantum trajectory theory

In the previous section we assumed that both $H_S = 0$ and $H_E = 0$. We will in this section show some results not covered in the article by Brun [4]. Among the results we show, we will go into detail on how we can recover the effective Hamiltonian of the system. We relay on the fact that $\{\sigma_i\}_{i=0}^3 = \{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ is a basis for $\mathcal{B}(\mathbb{C}^2)$, i.e. the bounded linear operators on two-level systems. These operators have the important property of being both unitary and hermitian. Moreover, $\{\sigma_i\}_{i=1}^3$ are traceless. Using the fact that they form a basis, we will write the interaction Hamiltonian as $H_I = \sum_{i,j=1}^3 h_{ij}\sigma_i \otimes \sigma_j$. Since the interaction Hamiltonian is hermitian, we immediately have $h_{ij} \in \mathbb{R}$.

The article by Brun [4] has more theory and many more examples than what we present here. We have only presented what we think is necessary. We have also excluded some details. For instance, Brun notes that the discarded environment still has entanglement which can effect the system. We are only interested in the ideal case in this text and have therefore chosen not to emphasize points like these.

3.5.1 We can always choose the environment to be in the state $|0\rangle$

In general, we can let the two-level environment of the model be in any initial state $|E\rangle$, as long as everyone of them is in the same state. This might first look like we have an independent parameter, but we can always assume the environment is in the state $|E\rangle = |0\rangle$. If this was not the case, we know that $|E\rangle = U|0\rangle$ with $U = |E\rangle\langle 0| + |E^\perp\rangle\langle 1|$. To see how the dynamics change, we let the interaction Hamiltonian be given by $H_I = \sum_{i,j=1}^3 h_{ij}\sigma_i \otimes \sigma_j$. We have not included terms of the form $\sigma_i \otimes \mathbb{1}$ or $\mathbb{1} \otimes \sigma_i$ as these are encapsulated in the Hamiltonians of the system and environment, respectively. Now, this can again be written as $H_I = \sum_i A_i \otimes B_i$ where $A_i = \sigma_i$ and $B_i = \sum_j h_{ij}\sigma_j$. Then $B_i|E\rangle = B_iU|0\rangle$, so we can define a new operator $B'_i = B_iU$. Since any unitary operator on a TLS can be written as $U = \mathbb{1}\cos(\phi) + i\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin(\phi)$ (see appendix A.1) and $\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k$, we can end up picking up a term proportional to the identity $\mathbb{1}$. This in undesirable as it should be encapsulated in the Hamiltonian of the environment. We therefore redefine the interaction Hamiltonian to be $H'_I = \sum_{i,j=1}^3 h'_{ij}\sigma_i\sigma_j$, where h'_{ij} are taken from $B_iU = \sum_{i=0}^3 h'_{ij}\sigma_i$ (taking $\sigma_0 := \mathbb{1}$). The leftover terms $h'_{0j}\mathbb{1}\otimes\sigma_j$ are added to the Hamiltonian of the environment.

3.5.2 The choice of measurement basis will decide the jump

In section 3.3 on weak measurement we saw two different examples of weak measurement: One where we on average only change the state slightly, but occasionally the state makes a larger jump, and an other where we only change the state slightly. We want to see when we get the different cases for our model, a two-level system interacting with an environment consisting of only two-level-systems. Let the system be in a state $|\psi\rangle = a|0\rangle + b|1\rangle$, the environment in the state $|0\rangle$ (as explained in the previous section, we can always assume this), and the interaction be $U(\theta) = e^{-i\theta H_I}$ with H_I being the interaction Hamiltonian. After interaction the entangled state will be

$$U(\theta)|\psi\rangle \otimes |0\rangle = a'|00\rangle + b'|10\rangle + c|01\rangle + d|11\rangle .$$

As the interaction is supposed to be weak/happen in a short time span (which is modelled by the parameter θ), we can compute the Taylor expansion of $U(\theta) \approx \mathbb{1} - i\theta H_I$, to find that the entangled state should only have changed slightly. That is, we approximately have

$$a' \propto a(1-\theta), \quad b' \propto b(1-\theta), \quad c \propto \theta, \quad d \propto \theta.$$

If we choose a (orthonormal) measurement basis $\{|f_0\rangle, |f_1\rangle\}$, we can find the change of basis matrix $B = \begin{pmatrix}\alpha_0 & \alpha_1 \\ \beta_0 & \beta_1\end{pmatrix}$, such that $|f_k\rangle = B|k\rangle$ for $k = 0, 1$. Expressing the environment in the measurement basis, the time evolved state is given by

$$\begin{aligned}U(\theta)|\psi\rangle \otimes |0\rangle &= (a'|0\rangle + b'|1\rangle)(\alpha_0|f_0\rangle + \alpha_1|f_1\rangle) + (c|0\rangle + d|1\rangle)(\beta_0|f_0\rangle + \beta_1|f_1\rangle) \\ &= [\alpha_0(a'|0\rangle + b'|1\rangle) + \beta_0(c|0\rangle + d|1\rangle)]|f_0\rangle \\ &\quad + [\alpha_1(a'|0\rangle + b'|1\rangle) + \beta_1(c|0\rangle + d|1\rangle)]|f_1\rangle .\end{aligned}$$

The state after measurement is therefore either

$$|\psi\rangle_0 = \frac{\alpha_0(a'|0\rangle + b'|1\rangle) + \beta_0(c|0\rangle + d|1\rangle)}{\sqrt{p_0}}, \text{ or}$$
$$|\psi\rangle_1 = \frac{\alpha_1(a'|0\rangle + b'|1\rangle) + \beta_1(c|0\rangle + d|1\rangle)}{\sqrt{p_1}},$$

with $p_k = [\alpha_k(a'|0\rangle + b'|1\rangle) + \beta_k(c|0\rangle + d|1\rangle)]^\dagger[\alpha_k(a'|0\rangle + b'|1\rangle) + \beta_k(c|0\rangle + d|1\rangle)]$ for $k = 0, 1$. As $a'|0\rangle + b'|1\rangle$ is close to $|\psi\rangle$ (as we assume that θ is small), it is the term $\beta_k(c|0\rangle + d|1\rangle)$ that will give us jumps.

Let us look at the very specific case when

$$\alpha_0 = \sqrt{1-\nu^2}, \quad \alpha_1 = \nu, \quad \beta_0 = \nu, \quad \beta_1 = -\sqrt{1-\nu^2},$$

where $0 \le \nu \le 1$. Then we have

$$|\psi\rangle_0 \approx \frac{\sqrt{1-\nu^2}(a(1-\theta)|0\rangle + b(1-\theta)|1\rangle) + \nu(\theta|0\rangle + \theta|1\rangle)}{N_0}, \text{ or}$$
$$|\psi\rangle_1 \approx \frac{\nu(a(1-\theta)|0\rangle + b(1-\theta)|1\rangle) - \sqrt{1-\nu^2}(\theta|0\rangle + \theta|1\rangle)}{N_1},$$

where N_0 and N_1 are normalization constants. Hence, if $\nu \ll \theta \ll 1$, then

$$\sqrt{1-\nu^2}(1-\theta) \approx 1, \quad \nu\theta \approx 0, \quad \nu(1-\theta) \approx \nu, \quad \sqrt{1-\nu^2}\theta \approx \theta,$$

so

$$|\psi\rangle_0 \approx a|0\rangle + b|1\rangle \quad \text{and} \quad |\psi\rangle_1 \approx \frac{c|0\rangle + d|1\rangle}{\widetilde{N}_1},$$

where $\widetilde{N}_1$ is a new normalization constant. This can be a large jump depending on the constants c and d. If we want to avoid large jumps, we need to have $\theta \ll \nu$. When this is the case, both

$$|\psi\rangle_0 \approx a|0\rangle + b|1\rangle \quad \text{and} \quad |\psi\rangle_1 \approx a|0\rangle + b|1\rangle.$$

3.5.3 n-level environment can only give $n-1$ Lindblad operators

Let the environment be an n-level system and choose any initial state $|E\rangle$. Let time evolution be given by $U(\theta) = e^{-i\theta H}$, where $H = \sum_j A_j \otimes B_j$ is the full Hamiltonian of the system, $H = H_S \otimes \mathbb{1} + H_I + \mathbb{1} \otimes H_E$. We will first show that we can redefine the Hamiltonian such that we can assume $\langle E|B_j|E\rangle = 0$, and therefore $\sum_j [A_j, \rho_S] \langle E|B_j|E\rangle = 0$.

Define $A := -\sum_j A_j \langle E|B_j|E\rangle$. As we are only interested in the interaction with the environment, we can then redefine $H' = H + A \otimes \mathbb{1} = \sum_j A'_j \otimes B'_j$. Then $\langle E|B'_j|E\rangle = 0$, and thus $\sum_j [A'_j, \rho_S] \langle E|B'_j|E\rangle = 0$.

We now show how we in specific cases can restrict the number of possible Lindblad operators. We can find an orthogonal basis $\{|E\rangle, |k\rangle : k = 1, \dots, n-1\}$ for the environment, such that

$$M_{ij} = \langle E|(B'_j)^\dagger B'_i|E\rangle = \sum_{k=1}^{n-1} \overbrace{\langle E|(B'_j)^\dagger|k\rangle}^{=:v^*_{kj}} \overbrace{\langle k|B'_i|E\rangle}^{=:v_{ki}} + \langle E|(B'_j)^\dagger|E\rangle \overbrace{\langle E|B'_i|E\rangle}^{=0}.$$

We define the vectors $\left(v_{k,1}, \quad \ldots, \quad v_{k,n-1}\right)^T$. Denoting these vectors by $|v_k\rangle$ and exploiting the (Dirac-)notation, we see that the matrix M can be represented as an outer product

$$M = \sum_k |v_k\rangle \langle v_k| = \sum_k \langle v_k|v_k\rangle |v'_k\rangle \langle v'_k| ,$$

where $|v'_k\rangle = |v_k\rangle / \sqrt{\langle v_k|v_k\rangle}$. We cannot right away assume $\langle v'_k|v'_l\rangle = 0$, but whenever this is the case, we have diagonalized the matrix M. Hence, whenever M is diagonalized, M can maximally have $n-1$ non-zero eigenvalues. For a TLS,

$$M_{ij} = \left\langle E|(B'_j)^\dagger B'_i|E\right\rangle = \overbrace{\left\langle E|(B'_j)^\dagger|E^\perp\right\rangle}^{=:v_j^*} \overbrace{\left\langle E^\perp|B'_i|E\right\rangle}^{=:v_i} + \left\langle E|(B'_j)^\dagger|E\right\rangle \overbrace{\left\langle E|B'_i|E\right\rangle}^{=0} .$$

In other words, $M = |v\rangle \langle v|$ is already diagonalized. Hence, for a TLS, we therefore conclude that we always have only a single non-zero eigenvalue, and therefore only a single Lindblad operator.

3.5.4 Recovering the Lindblad equation with Hamiltonian of the system

We want to end this chapter by elaborating on the assumption that $\sum_j A_j \langle E|B_j|E\rangle = 0$ and say how we can recover the Lindblad equation including the Hamiltonian of the system

$$\frac{d\rho_S}{dt} = -i[H_S, \rho_S] + \frac{1}{2}\sum_k \Gamma_k \left[2L_k\rho_S L_k^\dagger - L_k^\dagger L_k \rho_S - \rho_S L_k^\dagger L_k\right]. \tag{3.1}$$

The calculations are given in detail in appendix B. If $\sum_j A_j \langle E|B_j|E\rangle = 0$, then equation (3.1) is easily recovered (see section B.2). However, this assumption is not necessarily true. We argued that we could use an effective interaction Hamiltonian $H'_I = H_I - A$, but what happens to the dynamics of the system? It is not easy to answer this question in general. We therefore look at a specific example: Choose $|E\rangle = |0\rangle$, $H_I = \sum_{ij} h_{ij}\sigma_i \otimes \sigma_j$, $A_i = \sigma_i$, $B_i = \sum_j h_{ij}\sigma_j$, and $A = \sum_i A_i \langle 0|B_i|0\rangle$. If we go through the calculations, we end up with

$$\begin{aligned}\rho'_S :=& \operatorname{tr}_{\text{env}}\left(U(\theta)\,|\Psi\rangle\langle\Psi|\,U(\theta)^\dagger\right)\\ =&\rho_S - i\theta\sum_i h_{i3}[A_i, \rho_S] + \theta^2\left[L\rho_S L^\dagger - \frac{1}{2}L^\dagger L\rho_S - \frac{1}{2}\rho_S L^\dagger L\right]\\ &+ \theta^2\sum_{ij} h_{i3}h_{j3}\left[A_i\rho_S A_j - \frac{1}{2}A_iA_j\rho_S - \frac{1}{2}\rho_S A_iA_j\right] + \mathcal{O}(\theta^3).\end{aligned}$$

The calculations are given in section B.1.

The problem is now to handle the two terms containing A, namely $-i\theta\sum_i h_{i3}[A_i, \rho_S]$ and $\theta^2\sum_{ij} h_{i3}h_{j3}\left[A_i\rho_S A_j - \frac{1}{2}A_iA_j\rho_S - \frac{1}{2}\rho_S A_iA_j\right]$. To ensure that

$$\lim_{\delta t\to 0}\frac{\theta^2}{\delta t} \neq 0, \pm\infty,$$

we must have

$$\lim_{\delta t\to 0}\theta \propto \sqrt{\delta t}.$$

This again means that $-i\frac{\theta}{\delta t}\sum_i h_{i3}[A_i,\rho_S] \xrightarrow{\delta t\to 0} \pm\infty$ unless either $\sum_i h_{i3}[A_i,\rho_S] = 0$ or each h_{i3} is dependent on the time step δt. This is the same as demanding $h_{i3} = 0$ for all i or letting them be dependent on the time step δt. The first case, $h_{i3} = 0$ for all i, would mean that both problematic terms are zero. We look at what happens if we let them depend on the time step δt. Then we must have $\lim_{\delta t\to 0}\frac{\theta}{\delta t}h_{i3} = 0$, and as $\lim_{\delta t\to 0}\theta \propto \sqrt{\delta t}$, we have to demand $\lim_{\delta t\to 0} h_{i3} \propto (\delta t)^a$ where $a \geq \frac{1}{2}$. If we have $a = \frac{1}{2}$, then the first term stays and the latter disappears. If not, then both disappears.

It is therefore possible to recover the Lindblad equation, at least in this very specific case. We can even combine this result with the time evolution during the transient time, and we will still recover equation (3.1). This is shown in section B.3.

3.5.5 Independence in choice of Hamiltonian representation

It is actually possible to show that the Lindblad operator is independent of the representation of H. It is namely possible to define the partial trace of an operator in such a way that it is both linear and well-defined [2]. Let therefore $H = \sum_j A_j \otimes B_j$ and $H = \sum_k C_k \otimes D_k$ be two representations of H. Then

$$\mathrm{tr}_{\mathrm{env}}(H) = \mathrm{tr}_{\mathrm{env}}\left(\sum_j A_j \otimes B_j\right) = \sum_j \mathrm{tr}_{\mathrm{env}}(A_j \otimes B_j) = \sum_j A_j\,\mathrm{tr}(B_j),$$
$$\mathrm{tr}_{\mathrm{env}}(H) = \mathrm{tr}_{\mathrm{env}}\left(\sum_k C_k \otimes D_k\right) = \sum_k \mathrm{tr}_{\mathrm{env}}(C_k \otimes D_k) = \sum_k C_k\,\mathrm{tr}(D_k),$$

which means that all representations will give the same result. For $U = \exp(-i\theta H)$ we have

$$\begin{aligned} U\left|\Psi\right\rangle\left\langle\Psi\right|U^\dagger &= \left(\mathbb{1} - i\theta H - \frac{\theta^2}{2}H^2 + \mathcal{O}(\theta^3)\right)\left|\Psi\right\rangle\left\langle\Psi\right|\left(\mathbb{1} + i\theta H - \frac{\theta^2}{2}H^2 + \mathcal{O}(\theta^3)\right) \\ &= \left|\Psi\right\rangle\left\langle\Psi\right| + i\theta\left[\left|\Psi\right\rangle\left\langle\Psi\right|, H\right] + \theta^2\left(H\left|\Psi\right\rangle\left\langle\Psi\right|H - \frac{1}{2}H^2\left|\Psi\right\rangle\left\langle\Psi\right| - \frac{1}{2}\left|\Psi\right\rangle\left\langle\Psi\right|H^2\right) + \mathcal{O}(\theta^3). \end{aligned}$$

If the Hamiltonian has a representation $H = \sum_j A_j \otimes B_j$, we get (using the notation from [2] with $\mathcal{K}$ as the Hilbert space of the environment and $|\Psi\rangle = |\psi\rangle \otimes |E\rangle$)

$$
\begin{aligned}
\mathrm{tr}_{\mathrm{env}}\left([|\Psi\rangle\langle\Psi|, H]\right) &= \mathrm{tr}_{\mathrm{env}}\left(|\Psi\rangle\langle\Psi| \sum_j A_j \otimes B_j - |\Psi\rangle\langle\Psi| \sum_j A_j \otimes B_j\right) \\
&= \sum_j \mathrm{tr}_{\mathrm{env}}\left(|\Psi\rangle\langle\Psi| A_j \otimes B_j - (A_j \otimes B_j)|\Psi\rangle\langle\Psi|\right) \\
&= \sum_j \sum_k \Big({}_{\mathcal{K}}\langle k|(|\psi\rangle \otimes |E\rangle)(\langle\psi| \otimes \langle E|)A_j \otimes B_j |k\rangle_{\mathcal{K}} \\
&\qquad\qquad - {}_{\mathcal{K}}\langle k| (A_j \otimes B_j)(|\psi\rangle \otimes |E\rangle)(\langle\psi| \otimes \langle E|)|k\rangle_{\mathcal{K}} \Big) \\
&= \sum_j \Big(|\psi\rangle (\langle\psi| A_j \otimes \langle E| B_j) |E\rangle_{\mathcal{K}} \\
&\qquad\quad - {}_{\mathcal{K}}\langle E| (A_j |\psi\rangle \otimes B_j |E\rangle) \langle\psi| \Big) \\
&= \sum_j \left(|\psi\rangle\langle\psi| A_j \langle E|B_j|E\rangle - \langle E|B_j|E\rangle A_j |\psi\rangle\langle\psi|\right) \\
&= \sum_j \langle E|B_j|E\rangle \left[|\psi\rangle\langle\psi|, A_j\right],
\end{aligned}
$$

where we have used that $\{|k\rangle\}$ is an orthonormal basis for $\mathcal{K}$ containing $|E\rangle$, and that $(\langle\Psi|)\,|k\rangle_{\mathcal{K}}\,|f\rangle = \langle\Psi|\,(|f\rangle \otimes |k\rangle) = \langle\psi|f\rangle\langle E|k\rangle$ for all system states $|f\rangle$, so $(\langle\Psi|)\,|k\rangle_{\mathcal{K}} = \langle E|k\rangle\langle\psi|$, and finally that ${}_{\mathcal{K}}\langle k|(|\psi\rangle \otimes |E\rangle) = \langle k|E\rangle\,|\psi\rangle$. The same argument also gives

$$
\begin{aligned}
\mathrm{tr}_{\mathrm{env}}\left(H^2 |\Psi\rangle\langle\Psi|\right) &= \sum_{ij} \langle E|B_i B_j|E\rangle A_i A_j |\psi\rangle\langle\psi|, \\
\mathrm{tr}_{\mathrm{env}}\left(|\Psi\rangle\langle\Psi| H^2\right) &= \sum_{ij} \langle E|B_i B_j|E\rangle |\psi\rangle\langle\psi| A_i A_j.
\end{aligned}
$$

For the last term, we get

$$
\begin{aligned}
\mathrm{tr}_{\mathrm{env}}\left(H |\Psi\rangle\langle\Psi| H\right) &= \sum_{ij} \sum_k {}_{\mathcal{K}}\langle k|(A_i |\psi\rangle \otimes B_i |E\rangle)(\langle\psi| A_j \otimes \langle E| B_j)|k\rangle_{\mathcal{K}} \\
&= \sum_{ij} \sum_k (A_i |\psi\rangle \langle k|B_i|E\rangle)(\langle\psi| A_j \langle E|B_j|k\rangle) \\
&= \sum_{ij} A_i |\psi\rangle\langle\psi| A_j \left\langle E \,\middle|\, B_j \sum_k |k\rangle\langle k| B_i \,\middle|\, E \right\rangle \\
&= \sum_{ij} \langle E|B_j B_i|E\rangle A_i |\psi\rangle\langle\psi| A_j.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\operatorname{tr}_{\text{env}}\left(U\left|\Psi\right\rangle\left\langle\Psi\right|U^{\dagger}\right)\\
&\quad=\left|\psi\right\rangle\left\langle\psi\right|+i\theta\operatorname{tr}_{\text{env}}\left(\left[\left|\Psi\right\rangle\left\langle\Psi\right|,H\right]\right)\\
&\qquad+\theta^{2}\operatorname{tr}_{\text{env}}\left(H\left|\Psi\right\rangle\left\langle\Psi\right|H-\frac{1}{2}H^{2}\left|\Psi\right\rangle\left\langle\Psi\right|-\frac{1}{2}\left|\Psi\right\rangle\left\langle\Psi\right|H^{2}\right)+\mathcal{O}(\theta^{3})\\
&\quad=\left|\psi\right\rangle\left\langle\psi\right|+i\theta\sum_{j}\left[\left|\psi\right\rangle\left\langle\psi\right|,A_{j}\right]\left\langle E\middle|B_{j}\middle|E\right\rangle\\
&\qquad+\theta^{2}\sum_{ij}\left\langle E\middle|B_{j}B_{i}\middle|E\right\rangle\left(A_{i}\left|\psi\right\rangle\left\langle\psi\right|A_{j}-\frac{1}{2}A_{j}A_{i}\left|\psi\right\rangle\left\langle\psi\right|-\frac{1}{2}\left|\psi\right\rangle\left\langle\psi\right|A_{j}A_{i}\right)+\mathcal{O}(\theta^{3}).
\end{aligned}
\tag{3.2}
$$

Suppose now that the first order term was zero, i.e. $\sum_j \left[\left|\psi\right\rangle\left\langle\psi\right|, A_j\right]\left\langle E\middle|B_j\middle|E\right\rangle = 0$, and let $H = \sum_k C_k \otimes D_k$ be another representation of the Hamiltonian. Then we must have

$$
0=\sum_{j}\left[\left|\psi\right\rangle\left\langle\psi\right|,A_{j}\right]\left\langle E\middle|B_{j}\middle|E\right\rangle=\operatorname{tr}_{\text{env}}\left(\left[\left|\Psi\right\rangle\left\langle\Psi\right|,H\right]\right)=\sum_{k}\left[\left|\psi\right\rangle\left\langle\psi\right|,C_{k}\right]\left\langle E\middle|D_{k}\middle|E\right\rangle.
$$

In other words, if one representation gives zero, all other representations must also give zero. Contrapositively, if one representation gives non-zero, all other representations must also give non-zero.

We have included two explicit examples in appendix B.4 showing that the representation does not matter.

3.5.6 Easy calculation of Hamiltonian

We go back to our starting point, $H = \sum_{ij} h_{ij}\sigma_i \otimes \sigma_j = \sum_i \sigma_i \otimes \left(\sum_j h_{ij}\sigma_j\right)$, $\sigma_z\left|0\right\rangle = \left|0\right\rangle$, $A_i = \sigma_i$, and $B_i = \sum_j h_{ij}\sigma_j$. Since we want the first order term in equation (3.2) to be zero, we compute $\left\langle 0\middle|B_i\middle|0\right\rangle = h_{i3}$ and define $A := -\sum_i A_i \left\langle 0\middle|B_i\middle|0\right\rangle = -\sum_i A_i h_{i3}$. Then, as explained in the beginning of this section, we change to the Hamiltonian

$$
\begin{aligned}
H' = H - A\otimes\mathbb{1} &= \sum_i \left(A_i\otimes B_i - A_i\otimes\mathbb{1}h_{i3}\right)\\
&= \sum_i A_i \otimes \begin{pmatrix} h_{i3}-h_{i3} & h_{i1}-ih_{i2} \\ h_{i1}+ih_{i2} & -h_{i3}-h_{i3}\end{pmatrix} = \sum_i A_i \otimes \begin{pmatrix} 0 & h_{i1}-ih_{i2} \\ h_{i1}+ih_{i2} & -2h_{i3}\end{pmatrix}\\
&= \sum_i A_i \otimes \overbrace{\left[h_{i1}\sigma_x + h_{i2}\sigma_y - 2h_{i3}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right]}^{=:B_i'}.
\end{aligned}
$$

We have now expressed the transformed Hamiltonian with a representation of the form $H' = \sum_{ij} h'_{ij}\sigma_i \otimes s_j$, where $s_1 = \sigma_x$, $s_2 = \sigma_y$ and $s_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We compute

$$\begin{aligned} B'_j B'_i &= \begin{pmatrix} 0 & h'_{j1} - ih'_{j2} \\ h'_{j1} + ih'_{j2} & -2h'_{j3} \end{pmatrix} \begin{pmatrix} 0 & h'_{i1} - ih'_{i2} \\ h'_{i1} + ih'_{i2} & -2h'_{i3} \end{pmatrix} \\ &= \begin{pmatrix} (h'_{j1} - ih'_{j2})(h'_{i1} + ih'_{i2}) & h'_{i3}(h'_{j1} - ih'_{j2}) \\ h'_{j3}(h'_{i1} + ih'_{i2}) & (h'_{i1} - ih'_{i2})(h'_{j1} + ih'_{j2}) + h'_{j3}h'_{i3} \end{pmatrix} \end{aligned}$$

and hence $\langle 0|B'_j B'_i|0\rangle = \overbrace{(h'_{j1} - ih'_{j2})}^{=:k_j^*}\overbrace{(h'_{i1} + ih'_{i2})}^{=:k_i} = M_{ij}$. We therefore have

$$\begin{pmatrix} |k_1|^2 & k_1 k_2^* & k_1 k_3^* \\ k_2 k_1^* & |k_2|^2 & k_2 k_3^* \\ k_3 k_1^* & k_3 k_2^* & |k_3|^2 \end{pmatrix} = M = \lambda \begin{pmatrix} |\mu_1|^2 & \mu_1\mu_2^* & \mu_1\mu_3^* \\ \mu_2\mu_1^* & |\mu_2|^2 & \mu_2\mu_3^* \\ \mu_3\mu_1^* & \mu_3\mu_2^* & |\mu_3|^2 \end{pmatrix}$$

and we can therefore read straight out what h'_{ij} must be.

Chapter 4

Quantum synchronization of two-level systems

As we mentioned in the introduction, there has in recent years been interest in synchronization of quantum mechanical systems. The quantum mechanical van der Pol oscillator is one of the more studied examples [26][12][27], along side two-level systems [29][1].

In this chapter, we will look at two articles and one master thesis. The first article by Roulet and Bruder [23], titled "Synchronizing the Smallest Possible System", finds that the smallest quantum system that can be synchronized is a three-level system. In particular, they conclude that it is impossible to synchronize a TLS. The next article by Parra-López and Bergli [18] shows how it actually does make sense to synchronize a TLS. This is done by interpreting the mixed state as an ensemble of pure states, where the ensemble can have a valid limit cycle. Finally, the thesis by Longva [13] uses quantum trajectory theory to numerically compute the synchronization regime of a TLS.

4.1 Can two-level systems be synchronized?

The article by Roulet and Bruder [23] concludes that two-level systems cannot be synchronized. Their goal is to find the smallest possible quantum system that can be synchronized. As is standard, they represent a TLS on the Bloch sphere and use that any unitary operation on the TLS will be a rotation around some axis $\boldsymbol{n}$ on the Bloch sphere, visualised in Figure 4.1 (see appendix A.1 for mathematics behind this). They argue that if one tries to add a dissipative map to the Hamiltonian of the TLS, to get a self-sustained oscillator, then we will only end up with the trivial limit cycle that stays at either $\boldsymbol{n}$ or $-\boldsymbol{n}$. We quote the first part of this argument here before we try to break it down:

> ...To make contact with the standard paradigm of synchronization, we first need to establish a valid limit cycle for the self-sustained oscillator. Specifically, adding loss and gain to the dynamics of the qubit, we look for a fixed point of the dissipative map that does not possess any phase preference. That the phase of the limit cycle needs to be free is a sine qua non condition that ensures that any perturbation neither grows nor decays, which is the essence of synchronization.

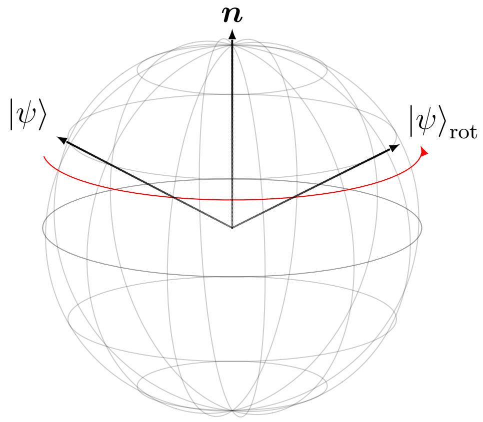


Figure 4.1: Rotation of a state $|\psi\rangle$ around an axis $\boldsymbol{n}$ on the Bloch sphere due to a unitary operator, e.g. a time-evolution.

To add loss and gain to the dynamics of the TLS is a correct idea, but it is not easy to understand what is meant by "... look for a fixed point of the dissipative map that does not possess any phase preference. ...". The argument continues in the following way:

> In the case of a qubit, any state belonging to the space $\mathcal{H}_{\text{qubit}}$ can be written as $\hat{\rho}_{\text{qubit}} = (\hat{\mathbb{1}} + \vec{m} \cdot \vec{\sigma})/2$, where $|\vec{m}| \leq 1$ is maximized for pure states that lie on the surface of the Bloch sphere. This implies that the set of states invariant under rotations with respect to the axis $\vec{n}$ satisfy $\vec{m} = \lambda\vec{n}$ with $-1 \leq \lambda \leq 1$. In other words, they correspond to probabilistic mixtures of the eigenstates $|\pm\vec{n}\rangle$, which are lying exactly on the rotation axis where the phase variable is not defined. Any attempt with qubits is thus bound to fail due to the absence of a phase-symmetric state that is different from the extremal eigenstates. ...

It is true as they say that the set of states invariant under rotations with respect to the axis $\boldsymbol{n}$ is $\{\lambda\vec{n} : -1 \leq \lambda \leq 1\}$, and of these, only $\pm|\boldsymbol{n}\rangle$ are pure. What is confusing is that the states on a limit cycle need not be independent of the phase. It is therefore not obvious why they are looking for a "phase-symmetric state". Moreover, they have excluded all possibility of mixed states. This is the idea behind the article by Parra-López and Bergli.

Before we move on to the article by Parra-López and Bergli, we will try to look a bit closer at what Roulet and Bruder might have meant. When they say "... look for a fixed point of the dissipative map that does not possess any phase preference. ...", they might have meant that they look for a limit cycle of the dynamic, when including dissipation, such that the phase defined from the limit cycle is linear in time. If this is the case, we must assume that the dissipative map adds both loss and gain. If this is not the case, we would only have dissipation of energy, so we obviously do not have a self-sustained oscillator. It is also not obvious what is meant by "... the phase of the limit cycle needs to be free ...". They do not say what it needs to be "free" from, and we know that the phase of a limit cycle is defined to be linear. That is, the phase of a limit cycle is defined **after** the limit cycle of a system is defined, and the phase is then **defined** to be a function ϕ from the phase space to $\mathbb{R}_+$, i.e. $(\boldsymbol{x}(t)) \mapsto \phi(\boldsymbol{x}(t))$, such that if we start at some point $\boldsymbol{x}(0)$ in phase space at time $t = 0$, and return to this point $\boldsymbol{x}(T) = \boldsymbol{x}(0)$ at time $t = T$, then $\phi(\boldsymbol{x}(0)) = 0$ and $\phi(\boldsymbol{x}(T)) = \phi(\boldsymbol{x}(0)) = 2\pi$. The function ϕ should also be linear in time [21, section 7.1]. Pikovsky, Kurths and Rosenblum do talk about the phase being free [21, section 2.2.2], but they focus on the fact that a perturbation just adds (or subtracts) a constant term to the phase. That is, the perturbation in the phase neither grows nor decays.

As a final remark, from the text in the article by Roulet and Bruder, it could seem like their interpretation of phase space is different from how we interpret it. Even though the book by Pikovsky, Kurths and Rosenblum works classically (instead of quantum mechanically), they explicitly state that phase space and state space is the same thing [21, section 2.1.2], and that phase space is the space of all variables $\boldsymbol{x}$ in the differential equation giving rise to the limit cycle [21, section 7.1.1]. For a TLS under unitary time-evolution, the phase space will therefore be the same as the state space, namely $\{|\psi\rangle \in \mathbb{C}^2 : \langle\psi|\psi\rangle = 1\}$. There is no doubt that the Bloch sphere is a representation of the state space of a TLS [17]. However, the use of the spin equivalent of the Husimi Q representation looks similar to treatments in phase-space formulation of quantum mechanics [6].

4.2 Limit cycle for two-level system constructed from pure states

The article by Parra-López and Bergli [18] discusses how a two-level system can be synchronized, contrary to the conclusion drawn by Roulet and Bruder [23].

They begin by explaining how a TLS can be thought of as having a valid limit cycle: To make the calculations easier, they choose a system Hamiltonian $\hat{H}_0 = \frac{\hbar}{2}\omega_0\hat{\sigma}_z$, where ω_0 is the *natural frequency* that the Bloch vector precesses around the z-axis with. Letting $\hat{\rho}' = \frac{1}{2}(\mathbb{1} + \boldsymbol{m}' \cdot \hat{\boldsymbol{\sigma}})$ (where $\boldsymbol{m}'$ is the Bloch vector of the system), they change to a rotating frame $\hat{\rho} = \hat{T}_{\omega_0}\hat{\rho}'\hat{T}^\dagger_{\omega_0} = \frac{1}{2}(\mathbb{1} + \boldsymbol{m} \cdot \hat{\boldsymbol{\sigma}})$, where $\hat{T}_{\omega_0} = e^{i\frac{\omega_0}{2}\hat{\sigma}_z t}$. The Lindblad equation in the rotating frame is

$$\frac{d\hat{\rho}}{dt} = \frac{\Gamma_g}{2}\mathcal{D}[\hat{\sigma}_+]\hat{\rho} + \frac{\Gamma_d}{2}\mathcal{D}[\hat{\sigma}_-]\hat{\rho}$$

where Γ_g and Γ_d are the gain and damping rates, $\mathcal{D}[\hat{O}]\hat{\rho} = \hat{O}\hat{\rho}\hat{O}^\dagger - \frac{1}{2}\{\hat{O}^\dagger\hat{O}, \hat{\rho}\}$ is the Lindblad superoperator, and $\hat{\sigma}_+$ and $\hat{\sigma}_-$ are the ladder operators for the system, $\hat{\sigma}_\pm = \frac{1}{2}(\hat{\sigma}_x \pm i\hat{\sigma}_y)$. Solving for a fixed point, i.e. $\dot{\hat{\rho}} = 0$, $\dot{\boldsymbol{m}} = 0$, they find

$$m_x = 0; \quad m_y = 0; \quad m_z = \frac{\Gamma_g - \Gamma_d}{\Gamma_g + \Gamma_d}.$$

The extremal points, i.e. the points $m_z = 1$ (corresponding to $\Gamma_g \neq 0$ and $\Gamma_d = 0$) and $m_z = -1$ (corresponding to $\Gamma_g = 0$ and $\Gamma_d \neq 0$), are excluded as these are pure states and will only give trivial limit cycles (same argument as Roulet and Bruder). However, Parra-López and Bergli suggest that the mixed states can be used. To quote, the mixed state is "... not a superposition and our system is for sure in any of those pure states, but *only in one of them at the same time.*" They go on explaining that if we build up the mixed state as mixture of pure states on the plane orthogonal to the z-axis and at the same height as the mixed state (see Figure 4.2). Letting this be their choice of limit cycle, they continue

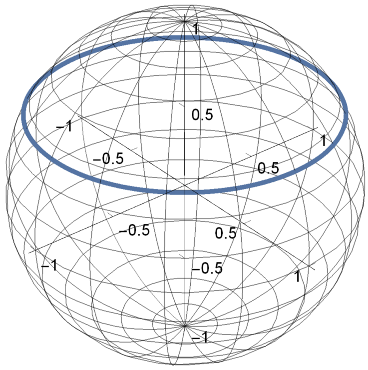


Figure 4.2: Limit cycle of a TLS. A mixed state on the z-axis can be thought of as a convex combination of the pure states with the same z-value. Taken from [18].

by adding a periodic signal, $\hat{H}_{\text{signal}} = i\hbar\frac{\epsilon}{4}(e^{i\omega t}\hat{\sigma}_- - e^{-i\omega t}\hat{\sigma}_+)$. This signal is the same that Roulet and Bruder uses [23], i.e. a classical external force with frequency ω and strength ϵ. Taking the original Lindblad equation (the non-rotated frame corresponding to $\hat{\rho}'$), they rotate the Lindblad equation by $\hat{T}_\omega = e^{i\frac{\omega}{2}\hat{\sigma}_z t}$ to find (see appendix A.2)

$$\frac{\hat{\rho}}{dt} = -\frac{i}{2}\left[\Delta\hat{\sigma}_z + \epsilon\hat{\sigma}_y, \hat{\rho}\right] + \frac{\Gamma_g}{2}\mathcal{D}[\hat{\sigma}_+]\hat{\rho} + \frac{\Gamma_d}{2}\mathcal{D}[\hat{\sigma}_-]\hat{\rho},$$

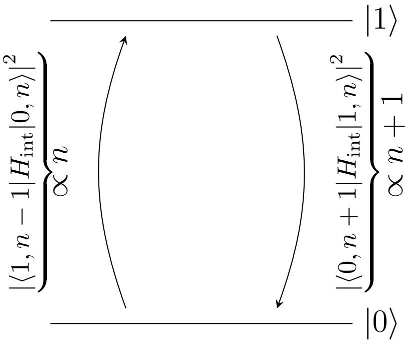
where $\Delta := \omega_0 - \omega$ and $\hat{\rho} = \hat{T}_\omega\hat{\rho}\hat{T}_\omega^\dagger$. They solve for the steady state $\dot{\rho} = 0$, and when transforming back to the non-rotating frame they find that m_x' and m_y' will vary in time with the frequency of the signal, ω. They conclude that this means that the system phase locks to the external signal.

Parra-López and Bergli have thus describe how a TLS can in fact be synchronized, contrary to the analysis by Roulet and Bruder. We will now move on to the master thesis by Longva [13].

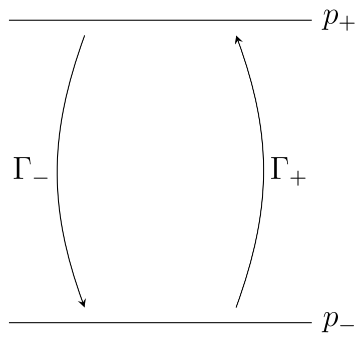
4.3 Using quantum trajectory theory to construct the ensemble

The work done by Longva [13] continues on the result by Parra-López and Bergli [18], i.e. that two-level systems can indeed be synchronized. Although Parra-López and Bergli have argued that a TLS can be synchronized, they state that they are working with mixed states. Since a non-pure mixed states does not tell us which pure states it is made up of, there could be doubts as to how the synchronization can actually be achieved. Longva's idea is therefore to use quantum trajectory theory (from chapter 3) to explicitly find the limit cycle numerically.

To get both gain and dampening, Longva needs at least a three-level system environment in his quantum trajectory (as explained in section 3.5.3). He instead chooses to use a two-level system as environment, but he lets the interaction alternate between gain



(a) Our TLS is in contact with a bosonic field.



(b) We can study the transition directly.

Figure 4.3: We model the transitions with a bosonic field to the left, and to the right we do not assume any model for the environment. **Left figure**: The probability of transitioning to the excited state is given by $|\langle 1, n-1|H_{\text{int}}|0, n\rangle|^2$, and the probability of transitioning to the ground state is given by $|\langle 0, n+1|H_{\text{int}}|1, n\rangle|^2$. Here, n is the number of excitations in the field of the mode with correct energy, and H_{int} models the interaction between the field and the TLS. We have $H_{\text{int}} = g_- a + g_+ a^\dagger$ where a and $a^\dagger$ are ladder operators and $g_\pm \propto \sigma_\pm$. As $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$ and $a^\dagger\,|n\rangle = \sqrt{n+1}\,|n\rangle$, the transition probabilities will be proportional to n and $n+1$. (See [11, chapter 15.4] for more details.)
Right figure: Let p_- and p_+ be the probabilities of being in the ground- and excited state, respectively, and Γ_+ and Γ_- be the transition rate of absorption (excitation) and emission of the TLS, respectively. If we want equilibrium, we must have detailed balance, $p_-\Gamma_+ = p_+\Gamma_-$.

and loss. Since Longva is after the same type of Lindblad equation as Parra-López and Bergli [18], he wants Lindblad operators $L_+ \propto \sigma_+$ and $L_- \propto \sigma_-$ (where plus and minus refers to gain and loss, respectively). Longva therefore calculates that he needs interaction Hamiltonians $H_{\text{int}_\pm} = \frac{1}{4}(\sigma_x \otimes \sigma_x \pm \sigma_y \otimes \sigma_y)$.

The Lindblad operators from the QTT, using the Hamiltonians $H_{\text{int}_\pm}$, take the form $L_\pm = \sqrt{\frac{\theta_\pm^2}{2\delta t}}\sigma_\pm$. As the variable $\theta_\pm$ depends on both time and the rate in the Lindblad equation, Longva needs to make sure that it takes the correct value to represent the Lindblad equation he is after,

$$\dot{\rho} = \frac{\Gamma_+}{2}\mathcal{D}[\sigma_+]\rho + \frac{\Gamma_-}{2}\mathcal{D}[\sigma_-]\rho.$$

Longva chooses to set $\theta_\pm = \sqrt{\Gamma_\pm}\theta$, where θ is purely a time variable which satisfy $\lim_{\delta t\to 0}\sqrt{\frac{\theta^2}{\delta t}} = 1$. To choose $\Gamma_\pm$ in a reasonable way, he uses a statistical mechanical approach: He lets E_0 and E_1 denote the energy of the ground- and excited state, respectively, $\Delta E := E_1 - E_0$, and $\beta = \frac{1}{k_B T}$ where k_B is the Boltzmann constant and T is the temperature of the system. Longva then choose $p_0 = \frac{1}{Z}e^{-\beta E_0}$ and $p_1 = \frac{1}{Z}e^{-\beta E_1}$ to be the probability of finding the system in the ground- and excited state, respectively, where $Z = e^{-\beta E_0} + e^{-\beta E_1}$ is the canonical partition function. He then assumes that the environment can be modeled as a bosonic field, e.g. photons as excitation of the field in some mode. The energy needed to excite the TLS/that the TLS can excite the field with will be ΔE. Since we are working with bosons, the number, n, of excitations of the mode(s) with the correct energy will be Bose-Einstein distributed

$$n = \frac{1}{e^{\beta \Delta E} - 1}.$$

We also have that the transition rates Γ_+ and Γ_- are proportional to n and $n-1$, respectively (with the same proportionality constant). Figure 4.3a tries to illustrate the transition. Hence,

$$\Gamma_+ = \frac{1}{e^{\beta \Delta E} - 1} \quad \text{and} \quad \Gamma_- = \frac{e^{\beta \Delta E}}{e^{\beta \Delta E} - 1}.$$

(Most of the derivation can be found in chapter 15 of the book by Bellac [11].) We could alternatively follow the diagram in Figure 4.3b. Here, p_- and p_+ are the probabilities of being in the ground- and excited state, respectively, and Γ_+ and Γ_- are the transition rate of absorption (excitation) and emission of the TLS, respectively. If we want equilibrium, we must have detailed balance

$$p_- \Gamma_+ = p_+ \Gamma_-.$$

Moreover, we have that $p_+ = p_1$ and $p_- = p_0$.

Longva goes on to describe how we should update the state. After interaction with $H_\pm$, the state of the system and environment will be

$$\begin{aligned} |\Psi'\rangle &= c_{00}\,|00\rangle + c_{01}\,|01\rangle + c_{10}\,|10\rangle + c_{11}\,|11\rangle \\ &= \left|\psi'_+\right\rangle \otimes |x_+\rangle + \left|\psi'_-\right\rangle \otimes |x_-\rangle\,, \end{aligned}$$

where $c_{ij} \in \mathbb{C}$ are constants dependent on the time-evolution with $H_\pm$, and $\left|\psi'_+\right\rangle := \frac{1}{\sqrt{2}}[(c_{00}+c_{01})\,|0\rangle + (c_{10}+c_{11})\,|1\rangle]$ and $\left|\psi'_-\right\rangle := \frac{1}{\sqrt{2}}[(c_{00}-c_{01})\,|0\rangle + (c_{10}-c_{11})\,|1\rangle]$. Note that the subscript $\pm$ in $\left|\psi'_\pm\right\rangle$ correspond to the measurement $|x_\pm\rangle$, not the interaction $H_\pm$! Longva continues finding the probabilities of measuring $|x_\pm\rangle$ by tracing out the system, finding that he will measure $|x_\pm\rangle$ with probability $p_\pm = \left\langle \psi'_\pm \middle| \psi'_\pm \right\rangle$. Thus, the system state will after the interaction be updated to

$$|\psi'\rangle = \frac{1}{\sqrt{p_\pm}} \left|\psi'_\pm\right\rangle$$

when the environment is measured to be in $|x_\pm\rangle$ with probability $p_\pm$. Longva notes that the same analysis can be done if we measure in y- or z-basis, or any other measurement basis for that matter.

Longva then explains how he simulates the evolution of the system as follows:

1. He begins with the system in an initial state $|\psi_0\rangle$, which he connects to the environment, $|\Psi\rangle = |\psi_0\rangle \otimes |E\rangle$.

2. The time-evolution is then calculated for the interaction $H_\pm$ (H_+ is used if we are on an even number of previous interactions, and H_- is used if we are on an odd number of previous interactions). That is, $|\Psi'\rangle = U_\pm(\theta)\,|\Psi\rangle$ is calculated. (Here $U_\pm(\theta) = e^{i\theta H_{\text{int}\pm}}$.)

3. The environment is measured in the basis $\{|f_+\rangle, |f_-\rangle\}$, giving an outcome $|\psi'_0\rangle$.

4. Finally, time-evolution is then calculated for the system Hamiltonian H (this can include the external signal) giving the new state $|\psi_1\rangle$.

5. We return to the first step with $|\psi_1\rangle$, and go through the algorithm until we have calculated the desired number of steps $|\psi_N\rangle$.

This only gives one trajectory, and since the system is inherently stochastic, Longva simulates multiple trajectories to gather statistics.

Longva has now laid out the groundwork for his numerical simulations. As a proof of concept, he compares it to the analytical solution from the corresponding Lindblad equation. He does this by calculating the average density matrix of the trajectory simulations, $\rho(n) = \frac{1}{S}\sum_j^S \rho_j(n)$, where the index j is the j-th trajectory out of the S trajectories simulated, and $\rho_j(n) = |\psi_n\rangle\langle\psi_n|$ where $|\psi_n\rangle$ is from the j-th trajectory simulation. He finds that the average density matrix matches the density matrix computed from the Lindblad equation better and better when increasing the number S of trajectories simulated.

Confident that the QTT approach works, Longva introduces the same signal Hamiltonian as Parra-López and Bergli, $H_{\text{signal}} = i\hbar\frac{\epsilon}{4}(e^{i\omega t}\sigma_- - e^{-i\omega t}\sigma_+)$, and transforms to the rotating frame using $T_\omega = e^{i\frac{\omega}{2}\sigma_z t}$. Simulating the system, he finds that the system synchronizes the same way as a classical system with noise: At no point does he get full synchronizations, as phase slips occur, but he finds clear regions where the measured frequency of the system is very close to the frequency of the signal. One of his best result is shown in Figure 4.4.

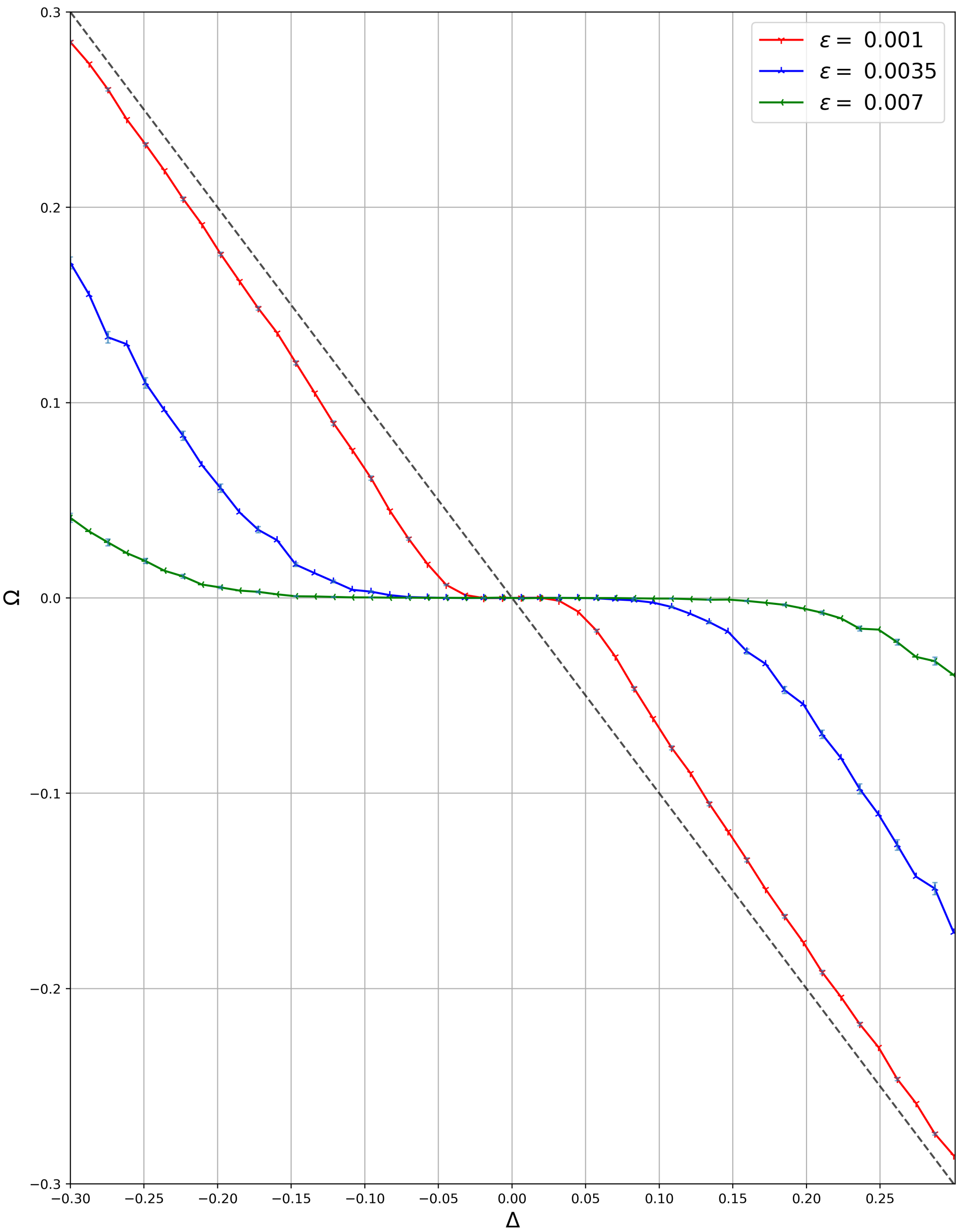


Figure 4.4: Taken from the master thesis by Longva [13]. See the thesis for full explanation of the figure. The figure shows how a TLS is almost frequency locked to an external field. The figure is created by doing simulations of the system for a QTT model.

Chapter 5

Analysis on synchronization of two-level systems using quantum trajectory theory

In this chapter we will present our results and analysis. There are three main topics we want to discuss

1. Is there an analytic expression for the synchronization we observe numerically (cf. what Longva finds in his thesis [13])?
2. How can we most easily visualize the trajectory traced out by the quantum trajectory theory?
3. When recovering the Lindblad equation from the trajectory theory, there seems to be a freedom in choice of interaction Hamiltonian, initial state and measurement basis. Does there exist a clever description of the space of stochastic processes (i.e. interaction Hamiltonian, initial state and measurement basis) that give rise to the same Lindblad equation?

The analysis we do will only concern the system, without any external signal. As in the chapter on quantum trajectory theory, we assume the system is a two-level system in a state $|\psi\rangle$, and the environment consists of two-level systems all in the same state $|E\rangle$.

5.1 Deriving a master equation for the stochastic process araising from the quantum trajectory theory

Quantum trajectory theory, as described in chapter 3, gives us a stochastic process for the time evolution of the state of our system. We want to describe this stochastic process through a master equation. The setup is the same as before: We have a system (not necessarily a TLS in general) in a state $|\psi\rangle$, which interacts with an environment consisting of systems all in the same state $|E\rangle$. We denote the time evolution during the transient time between interactions by $\widetilde{U}(\delta t)$ and the time evolution during the interactions by $U(\theta)$. After each interaction the environment system is measured in a basis $\{|f_i\rangle\}_{i\in I}$.

Up to now, we have only formulated the QTT for environments with finite dimensional Hilbert space. We therefore assume that the index set I is finite.

If we stay this general to begin with, we do not get a lot of information: Letting the system be an m-level system and denoting the state of the system after the n-th measurement by $|\psi\rangle_n$, we know that

$$U(\theta)\left(\widetilde{U}(\delta t)\,|\psi\rangle_{n-1} \otimes |E\rangle\right) = \sum_{k=1}^{m} \sum_{i\in I} c_{ki}\,|m\rangle\,|i\rangle\,.$$

We can therefore derive a general expression for $|\psi\rangle_n$, but to compute anything we need to know $U(\theta)$, $\widetilde{U}(\delta t)$, $|E\rangle$ and the initial state of the system $|\psi\rangle$. We will therefore look at the model proposed by Longva in his thesis [13]. Computations and minor details are given in appendix C.

5.1.1 A single step of QTT for a specific model

We begin by looking at the first step of the QTT algorithm for the model proposed in the thesis by Longva [13].

In his thesis, Longva find evidence for synchronization of a TLS numerically. He chooses to model the TLS using quantum trajectory theory: The TLS is in an environment consisting of two-level systems all in the state $|E\rangle = |0\rangle$. The system Hamiltonian is $H_S = \frac{\hbar}{2}\omega_0\sigma_z$ and the environment Hamiltonian is $H_E = 0$. As we saw in section 3.5.3, choosing the environment to be a TLS forces the recovered Lindblad equation to have only a single Lindblad operator. This means that his environment can only either give or take energy from the system, and not both. This can therefore not be a self-sustained oscillator. Longva thus chooses to a model switching between two interaction Hamiltonians,

$$H_{I^\pm} = \frac{1}{4}\left(\sigma_x \otimes \sigma_x \pm \sigma_y \otimes \sigma_y\right), \tag{5.1}$$

between each interaction. These interaction Hamiltonians are shown to give rise to Lindblad operators $L_+ = \sigma_+$ and $L_- = \sigma_-$. The measurement basis is chosen to be the x-basis[1], $|\uparrow_x\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|\downarrow_x\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$.

We begin by looking at the case $H_S = 0$. The time evolution is given by the unitary operators

$$U_\pm(\theta) = \exp\left(-i\theta H_{I^\pm}\right).$$

To find the eigenstates and corresponding eigenvalues of the interaction Hamiltonians $H_{I^\pm}$, we can use functional calculus [17, section 2.1.8] to compute $U_\pm(\theta)\,|\psi\rangle \otimes |E\rangle$. Here $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ is the state of our TLS. After measurement of the environment in the

[1] Note that we here have chosen the convention $\sigma_z\,|0\rangle = |0\rangle$.

x-basis, the system will be in the state

$$|\psi'\rangle = \begin{cases} \frac{\left(\alpha - i\beta\sin\frac{\theta}{2}\right)|0\rangle + \left(\beta\cos\frac{\theta}{2}\right)|1\rangle}{\sqrt{1+2\,\mathrm{Im}(\alpha^*\beta)\sin\frac{\theta}{2}}}, & \text{if interaction } H_{\text{int}}^{+} \text{ and measure } |\uparrow_x\rangle\,, \\ \frac{\left(\alpha\cos\frac{\theta}{2}\right)|0\rangle + \left(\beta - i\alpha\sin\frac{\theta}{2}\right)|1\rangle}{\sqrt{1+2\,\mathrm{Im}(\alpha\beta^*)\sin\frac{\theta}{2}}}, & \text{if interaction } H_{\text{int}}^{+} \text{ and measure } |\downarrow_x\rangle\,, \\ \frac{\left(\alpha + i\beta\sin\frac{\theta}{2}\right)|0\rangle + \left(\beta\cos\frac{\theta}{2}\right)|1\rangle}{\sqrt{1+2\,\mathrm{Im}(\alpha\beta^*)\sin\frac{\theta}{2}}}, & \text{if interaction } H_{\text{int}}^{-} \text{ and measure } |\uparrow_x\rangle\,, \\ \frac{\left(\alpha\cos\frac{\theta}{2}\right)|0\rangle + \left(\beta + i\alpha\sin\frac{\theta}{2}\right)|1\rangle}{\sqrt{1+2\,\mathrm{Im}(\alpha^*\beta)\sin\frac{\theta}{2}}}, & \text{if interaction } H_{\text{int}}^{-} \text{ and measure } |\downarrow_x\rangle\,. \end{cases} \tag{5.2}$$

We have here used that $\sigma_z|0\rangle = |0\rangle$, but if we use $\sigma_z|0\rangle = -|0\rangle$ we will only get a phase factor -1 on the two states corresponding to the interaction H_{I^+}. The calculations can be found in appendix C.1

5.1.2 Master equation for the model proposed by Longva

We now want to find a master equation for the model proposed by Longva, i.e.

$$H_{I^\pm} = \frac{1}{4}\left(\sigma_x \otimes \sigma_x \pm \sigma_y \otimes \sigma_y\right).$$

This is not to be confused with the Lindblad master equation: We already know that quantum trajectory theory with $H_{I^\pm}$ will give back the Lindblad equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_S,\rho] + \frac{\Gamma_+}{2}\left(2L_+^\dagger\rho L_+ - \rho L_+L_+^\dagger - L_+L_+^\dagger\rho\right) + \frac{\Gamma_-}{2}\left(2L_-^\dagger\rho L_- - \rho L_-L_-^\dagger - L_-L_-^\dagger\rho\right),$$

where $H_S = \frac{\hbar}{2}\omega_0\sigma_z$, $L_\pm = \sigma_\pm$ and $\Gamma_\pm$ are the respective rates whose values are decided by the parameter θ. We are instead after an equation telling us the probability of being in a state after a given time. This is closer to how we recover the diffusion equation from random walk: For random walk in one dimension, we let $u(x,t)$ be the probability of a random walker being at position x at a time t. If the time step Δt and physical step Δx possible for the random walker are discretized, then the following differential equation holds true:

$$u(x, t+\Delta t) = pu(x-\Delta x, t) + qu(x+\Delta x, t).$$

Here, p is the probability of the walker taking a step Δx, and $q = 1-p$ is the probability of the walker taking a step $-\Delta x$. Taking the limit of Δt and Δx going to zero, in such a way that $\frac{(\Delta x)^2}{\Delta t}$ goes to a constant, we can recover the diffusion equation in one dimension,

$$\frac{\partial u(x,t)}{\partial t} = -2C\frac{\partial u(x,t)}{\partial x} + D\frac{\partial^2 u(x,t)}{\partial x^2}.$$

The constants C and D arises when taking the limit. Details and calculations are given in appendix C.2.

Given the state after $n+1$ measurements, $|\psi\rangle_{n+1}$, we therefore want to see how many possible states $|\psi\rangle_n$ that can have evolved to this state. It is not difficult to show that if each step must follow equation (5.2), then there are only two possible states that can give $|\psi\rangle_{n+1}$. The calculation is given in appendix C.1.3 and give

$$\alpha_n = \frac{\alpha_{n+1} \pm i\beta_{n+1}\tan\frac{\theta}{2}}{\sqrt{\left|\alpha_{n+1} \pm i\beta_{n+1}\tan\frac{\theta}{2}\right|^2 + \frac{|\beta_{n+1}|^2}{\cos^2\frac{\theta}{2}}}} \qquad \beta_n = \frac{\frac{\beta_{n+1}}{\cos\frac{\theta}{2}}}{\sqrt{\left|\alpha_{n+1} \pm i\beta_{n+1}\tan\frac{\theta}{2}\right|^2 + \frac{|\beta_{n+1}|^2}{\cos^2\frac{\theta}{2}}}} \tag{5.3}$$

if the interaction is H_{I^+}, and

$$\alpha_n = \frac{\frac{\alpha_{n+1}}{\cos\frac{\theta}{2}}}{\sqrt{\left|\beta_{n+1} \pm i\alpha_{n+1}\tan\frac{\theta}{2}\right|^2 + \frac{|\alpha_{n+1}|^2}{\cos^2\frac{\theta}{2}}}} \qquad \beta_n = \frac{\beta_{n+1} \pm i\alpha_{n+1}\tan\frac{\theta}{2}}{\sqrt{\left|\beta_{n+1} \pm i\alpha_{n+1}\tan\frac{\theta}{2}\right|^2 + \frac{|\alpha_{n+1}|^2}{\cos^2\frac{\theta}{2}}}} \tag{5.4}$$

if the interaction is H_{I^-}. The $\pm$ corresponds to measuring $|\uparrow_x\rangle$ and $|\downarrow_x\rangle$, respectively.

Let $u(|\psi\rangle, t)$ be the probability of the system being in the state $|\psi\rangle$ at time t. Similarly to a random walk, we can write the different equation for each step in the QTT model. They are

$$u(|\psi\rangle_{n+1}, t + (\delta t + \theta)) = p_{+,0}u(\left|\psi^{(0)}\right\rangle_n, t) + p_{+,1}u(\left|\psi^{(1)}\right\rangle_n, t), \text{ and}$$
$$u(|\psi\rangle_{n+1}, t + (\delta t + \theta)) = p_{-,0}u(\left|\psi^{(0)}\right\rangle_n, t) + p_{-,1}u(\left|\psi^{(1)}\right\rangle_n, t),$$

and correspond to the interaction H_{I^+} and H_{I^-}, respectively. If $|\psi\rangle_{n+1} = \alpha_{n+1}|0\rangle + \beta_{n+1}|1\rangle$ is known, then $\left|\psi^{(0)}\right\rangle_n$ and $\left|\psi^{(1)}\right\rangle_n$ are given by equation (5.3) and (5.4). The probabilities are given by $p_{\pm,i} = {}_n\langle\psi^{(i)}|\psi^{(i)}\rangle_n$ for $i = 0, 1$. If we look at two consecutive interactions, H_{I^+} followed by H_{I^-}, we find

$$u(|\psi\rangle_{n+2}, t + 2(\delta t + \theta)) = p_{+,0}u(\left|\psi^{(0)}\right\rangle_{n+1}, t + (\delta t + \theta)) + p_{+,1}u(\left|\psi^{(1)}\right\rangle_{n+1}, t + (\delta t + \theta)) \tag{5.5}$$
$$= p_{+,0}\left[p_{-,0}u\left(\left|\psi^{(0,0)}\right\rangle_n, t\right) + p_{-,1}u\left(\left|\psi^{(0,1)}\right\rangle_n, t\right)\right] \tag{5.6}$$
$$+ p_{+,1}\left[p_{-,0}u\left(\left|\psi^{(1,0)}\right\rangle_n, t\right) + p_{-,1}u\left(\left|\psi^{(1,1)}\right\rangle_n, t\right)\right]. \tag{5.7}$$

Next would be to do as shown in appendix C.2 with the diffusion equation, to find a differential equation when taking the limit $\delta t \to 0$. As of now this has not been done yet, but it would be a perfect thing to do in the future.

5.2 Flow of the state of the two-level system on the Bloch sphere when applying quantum trajectory theory

To get an intuition for how the TLS evolves as we trace out its trajectory, we would like to visualize the flow of the state. That is, we would like to visualize how the state changes similarly to visualizing the velocity field of a flow. This also limits what type of systems we can visualize: The state space of a two-level system can be visualized as a unit sphere, known as the Bloch sphere [17, exercise 2.72]. For an n-level system with $n > 2$ however, the state space has more than three complex dimensions. We are therefore not able to visualize the state space in its entirety, even when taking phase factors and normalization into account.

For a general QTT model of a TLS (i.e. a sequence of n-level environments all in the same state $|E\rangle$, interaction Hamiltonian H_I and unitary evolution $U(\theta) = \exp(-i\theta H_I)$, and a chosen measurement basis $\{|f_i\rangle\}_{i\in I}$), we find the flow of the TLS on the Bloch sphere by calculating $\frac{d\boldsymbol{n}}{d\theta}$. Here $\boldsymbol{n}$ is the Bloch vector of a state $|\psi\rangle$, where $|\psi\rangle$ is the state after a step of the QTT model. In other words, we take a state $|\phi\rangle$, let the state $|\phi\rangle|E\rangle$ evolve to $U(\theta)|\phi\rangle|E\rangle$, and are then left with the TLS in the state $|\psi\rangle$ after measurement of the environment. Each possible measurement will give its own Bloch vector, so each measurement will correspond to a unique flow. We will now calculate the vector flow for the model presented in the thesis by Longva [13]. Various steps in the calculations and minor details can be found in appendix D.

5.2.1 State flow for the model proposed by Longva

In the QTT model proposed by Longva [13], we let the environment be in the state $|E\rangle$ and let the interaction Hamiltonian be given by $H_{I^\pm} = \frac{1}{4}(\sigma_x \otimes \sigma_x \pm \sigma_y \otimes \sigma_y)$, where we alternate between the positive and negative interaction between each step. The measurement basis is $|f_k\rangle = \begin{cases} |x_+\rangle, & k=0 \\ |x_-\rangle, & k=1 \end{cases}$. We let $\boldsymbol{n} = \langle\psi|\boldsymbol{\sigma}|\psi\rangle$ be the Bloch vector of the state of the TLS (before evolution). For unitary evolution $U(\theta) = \exp(-i\theta H_I)$, the Bloch vector after a measurement of $|f_k\rangle$ will in general have the form

$$\boldsymbol{n}_k = \frac{U + \theta V + \mathcal{O}(\theta^2)}{\sqrt{|U|^2 + 2\theta V \cdot U + \mathcal{O}(\theta^2)}},$$

where the index k tells us the measurement result and

$$U = |\langle f_k|E\rangle|^2\boldsymbol{n}, \qquad V = 2\,\mathrm{Re}\left(i\sum_j \langle f_k|B_j|E\rangle^* \left\langle\psi|A_j^\dagger\boldsymbol{\sigma}|\psi\right\rangle \langle f_k|E\rangle\right).$$

Here we assume that the interaction Hamiltonian is written on the form $H_{I^\pm} = \sum_j A_j^\pm \otimes B_j^\pm$ for some operators $A_j^\pm$ and $B_j^\pm$. Taking the derivative with respect to θ we find that the flow evaluated at $\theta = 0$ is given by

$$\frac{d\boldsymbol{n}_k}{d\theta} = \frac{1}{|U|}(V - n(V \cdot n)). \tag{5.8}$$

Defining $A_1^\pm = \frac{1}{4}\sigma_x$, $A_2^\pm = \pm\frac{1}{4}\sigma_y$, $B_1^\pm = \sigma_x$ and $B_2^\pm = \sigma_y$, it is not difficult to show that $U = \frac{1}{2}\boldsymbol{n}$ and

$$V^T = \frac{1}{4}\begin{cases} \begin{pmatrix}0, & -n_z+1, & n_y\end{pmatrix}, & k=0 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix}0, & n_z-1, & -n_y\end{pmatrix}, & k=1 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix}0, & -n_z-1, & n_y\end{pmatrix}, & k=0 \text{ and } H_{\text{int}_-} \\ \begin{pmatrix}0, & n_z+1, & -n_y\end{pmatrix}, & k=1 \text{ and } H_{\text{int}_-} \end{cases}.$$

The normalized the Bloch vector, given by equation (5.8), is then

$$\left(\frac{d\boldsymbol{n}_k}{d\theta}\right)^T = \frac{1}{2}\begin{cases}\Big(-n_y n_x, \quad -n_z + 1 - n_y^2, \quad n_y(1-n_z)\Big), & k=0 \text{ and } H_{\text{int}_+}\\ \Big(n_y n_x, \quad n_z - 1 + n_y^2, \quad n_y(-1+n_z)\Big), & k=1 \text{ and } H_{\text{int}_+}\\ \Big(n_y n_x, \quad -n_z - 1 + n_y^2, \quad n_y(1+n_z)\Big), & k=0 \text{ and } H_{\text{int}_-}\\ \Big(-n_y n_x, \quad n_z + 1 - n_y^2, \quad n_y(-1-n_z)\Big), & k=1 \text{ and } H_{\text{int}_-}\end{cases}. \tag{5.9}$$

The details of this argument can be found in section D.1.

We will most often simulate our QTT model, which means that setting $\theta = 0$ would not work: We need a finite non-zero value for θ. We can still approximate the flow by assuming it follows equation (5.9), but we would like to have control over the error we are making. We can therefore compute the full Bloch vector. Although not difficult, it is time consuming to compute the actual Bloch vector, i.e. without approximating. This is done in appendix D.2. We then find that the non-normalized Bloch vector is

$$\widetilde{\boldsymbol{n}}_k = \frac{1}{2}\begin{cases}\begin{pmatrix}(\alpha^*\beta + \alpha\beta^*)\cos\frac{\theta}{2}\\ i(\alpha\beta^* - \alpha^*\beta)\cos\frac{\theta}{2} + |\beta|^2\sin\theta\\ |\alpha|^2 - |\beta|^2\cos\theta - i(\alpha^*\beta - \alpha\beta^*)\sin\frac{\theta}{2}\end{pmatrix}, & k=0 \text{ and } H_{\text{int}_+}\\ \begin{pmatrix}(\alpha^*\beta + \alpha\beta^*)\cos\frac{\theta}{2}\\ i(\alpha\beta^* - \alpha^*\beta)\cos\frac{\theta}{2} - |\beta|^2\sin\theta\\ |\alpha|^2 - |\beta|^2\cos\theta + i(\alpha^*\beta - \alpha\beta^*)\sin\frac{\theta}{2}\end{pmatrix}, & k=1 \text{ and } H_{\text{int}_+}\\ \begin{pmatrix}(\alpha^*\beta + \alpha\beta^*)\cos\frac{\theta}{2}\\ i(\alpha\beta^* - \alpha^*\beta)\cos\frac{\theta}{2} - |\alpha|^2\sin\theta\\ |\alpha|^2\cos\theta - |\beta|^2 - i(\alpha^*\beta - \alpha\beta^*)\sin\frac{\theta}{2}\end{pmatrix}, & k=0 \text{ and } H_{\text{int}_-}\\ \begin{pmatrix}(\alpha^*\beta + \alpha\beta^*)\cos\frac{\theta}{2}\\ i(\alpha\beta^* - \alpha^*\beta)\cos\frac{\theta}{2} + |\alpha|^2\sin\theta\\ |\alpha|^2\cos\theta - |\beta|^2 + i(\alpha^*\beta - \alpha\beta^*)\sin\frac{\theta}{2}\end{pmatrix}, & k=1 \text{ and } H_{\text{int}_-}\end{cases}$$

with norm

$$\|\widetilde{\boldsymbol{n}}_k\| = \frac{1}{2}\begin{cases}1 - i\sin\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k=0 \text{ and } H_{\text{int}_+}\\ 1 + i\sin\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k=1 \text{ and } H_{\text{int}_+}\\ 1 + i\sin\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k=0 \text{ and } H_{\text{int}_-}\\ 1 - i\sin\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k=1 \text{ and } H_{\text{int}_-}\end{cases}.$$

The derivatives are give by

$$\frac{d\widetilde{\boldsymbol{n}}_k}{d\theta} = \frac{1}{2}\begin{cases} \begin{pmatrix} -(1/2)(\alpha^*\beta + \alpha\beta^*)\sin\frac{\theta}{2} \\ -(1/2)i(\alpha\beta^* - \alpha^*\beta)\sin\frac{\theta}{2} + |\beta|^2\cos\theta \\ |\beta|^2\sin\theta - (1/2)i(\alpha^*\beta - \alpha\beta^*)\cos\frac{\theta}{2} \end{pmatrix}, & k = 0 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix} -(1/2)(\alpha^*\beta + \alpha\beta^*)\sin\frac{\theta}{2} \\ -(1/2)i(\alpha\beta^* - \alpha^*\beta)\sin\frac{\theta}{2} - |\beta|^2\cos\theta \\ |\beta|^2\sin\theta + (1/2)i(\alpha^*\beta - \alpha\beta^*)\cos\frac{\theta}{2} \end{pmatrix}, & k = 1 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix} -(1/2)(\alpha^*\beta + \alpha\beta^*)\sin\frac{\theta}{2} \\ -(1/2)i(\alpha\beta^* - \alpha^*\beta)\sin\frac{\theta}{2} - |\alpha|^2\cos\theta \\ -|\alpha|^2\sin\theta - (1/2)i(\alpha^*\beta - \alpha\beta^*)\cos\frac{\theta}{2} \end{pmatrix}, & k = 0 \text{ and } H_{\text{int}_-} \\ \begin{pmatrix} -(1/2)(\alpha^*\beta + \alpha\beta^*)\sin\frac{\theta}{2} \\ -(1/2)i(\alpha\beta^* - \alpha^*\beta)\sin\frac{\theta}{2} + |\alpha|^2\cos\theta \\ -|\alpha|^2\sin\theta + (1/2)i(\alpha^*\beta - \alpha\beta^*)\cos\frac{\theta}{2} \end{pmatrix}, & k = 1 \text{ and } H_{\text{int}_-} \end{cases}$$

and

$$\frac{\|\widetilde{\boldsymbol{n}}_k\|}{d\theta} = \frac{1}{2}\begin{cases} -i\cos\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k = 0 \text{ and } H_{\text{int}_+} \\ i\cos\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k = 1 \text{ and } H_{\text{int}_+} \\ i\cos\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k = 0 \text{ and } H_{\text{int}_-} \\ -i\cos\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k = 1 \text{ and } H_{\text{int}_-} \end{cases}.$$

We can put all of this together to find the actual Bloch vector

$$\frac{d\boldsymbol{n}_k}{d\theta} = \frac{\frac{d\widetilde{\boldsymbol{n}}_k}{d\theta}\|\widetilde{\boldsymbol{n}}_k\| - \widetilde{\boldsymbol{n}}_k\frac{d\|\widetilde{\boldsymbol{n}}_k\|}{d\theta}}{\|\widetilde{\boldsymbol{n}}_k\|^2}.$$

The final expressions are not very readable and do not give much insight. The interested reader can themselves put everything together. What is interesting to note however, is that if we put $\theta = 0$ we do indeed recover equation (5.9) as we should. It is therefore no problem to approximate $\boldsymbol{n}_k$ with equation (5.9) for small θ.

We now move on to visualizing the flow of the state given in equation (5.9).

5.2.2 Visualization of state flow on the sphere and in the plane

Figure 5.1 shows the flow of the state given in equation (5.9) for the interaction H_{I^+} and measurement of $|\uparrow_x\rangle$. We have here assumed that $\boldsymbol{n} = \begin{pmatrix}0, & 0, & 1\end{pmatrix}$ corresponds to the state $|0\rangle$, so $\sigma_z|0\rangle = |0\rangle$. The state flow for the other interactions and measurements can be found in appendix D.4. As the Bloch sphere is a three dimensional object, it is not trivial to interpret the flow from the figure. For interaction H_{I^+} and measurement $|\uparrow_x\rangle$, the state will flow in circles first going towards the state $|\downarrow_y\rangle$, then towards the state $|1\rangle$, next towards the state $|\uparrow_y\rangle$, and then finally towards the state $|0\rangle$. We see that the state moves slowly close to $|0\rangle$, and fast close to $|1\rangle$. For a state at $|0\rangle$ we have $\boldsymbol{n} = \begin{pmatrix}0, & 0, & 1\end{pmatrix}$. Plugging this into the flow equation (5.9) gives $\frac{d\boldsymbol{n}}{d\theta} = 0$. It is interesting to note that if the state is not exactly $|0\rangle$, but closer to $|\downarrow_y\rangle$, then it will actually move around the entire

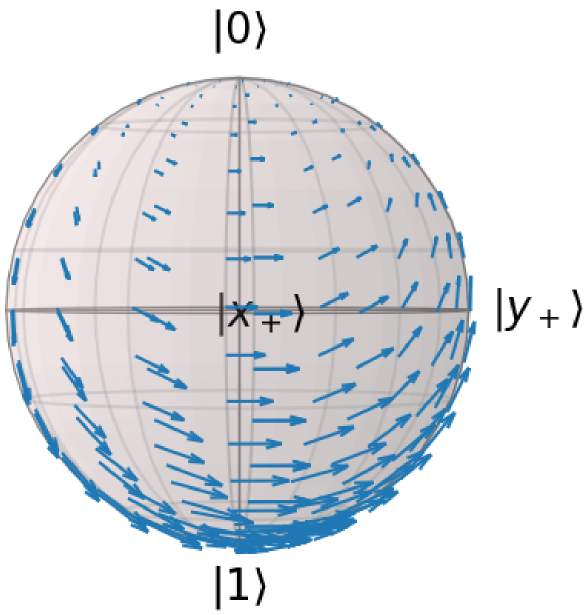


Figure 5.1: The flow of the state given by equation (5.9) for $k = 1$ and H_{I^+}, i.e. interaction Hamiltonian H_{I^+} and environment measured to be in the state $|\uparrow_x\rangle$. The flow goes in circles always ending up at the state $|0\rangle$. The state $|0\rangle$ is a fixed point where there is no flow.

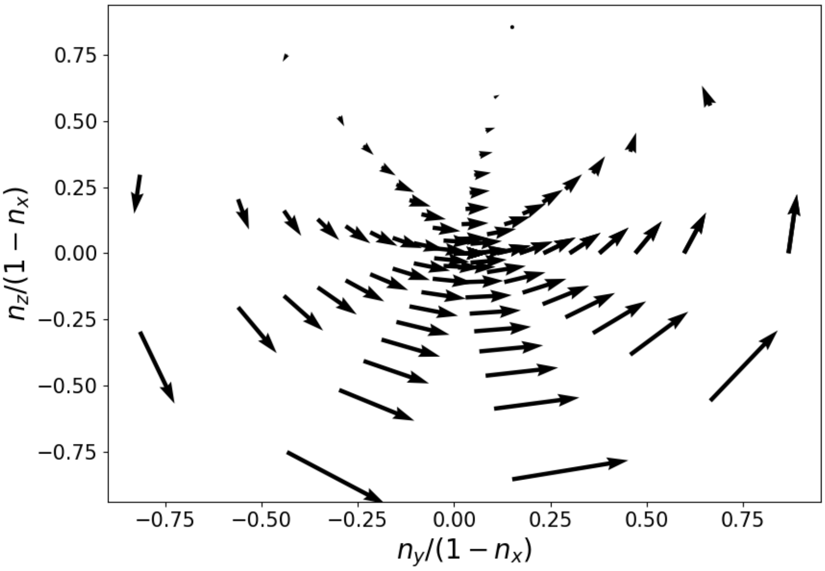


Figure 5.2: Stereographic projection of the flow of the state given by equation (5.9) for $k = 1$ and H_{I^+}, i.e. interaction Hamiltonian H_{I^+} and environment measured to be in the state $|\uparrow_x\rangle$. This is the same flow as depicted in Figure 5.1. We have here used $\boldsymbol{n} = (1, 0, 0)$ as the north pole and only show the south half of the sphere. We will get the exact same figure using $\boldsymbol{n} = (-1, 0, 0)$ as the north pole and depicting only the south half of the sphere in this case. It is now easier to see the path which a state flows: It will move in a circle towards the state $|0\rangle$ where it stops completely. The flow is fast close to $|1\rangle$ and slow close to $|0\rangle$.

Bloch sphere to reach the state $|0\rangle$. That is, it will not take the shortest path to $|0\rangle$. The state will instead evolve around the Bloch sphere. The figures for the other interactions and measurements show the same trend, but the flow changes direction and fixed point.

As we noted above, interpreting a three dimensional flow is not always obvious. Since the surface of a sphere is a two dimensional object, we will instead project the surface down to the plane. The procedure for doing this is explained in appendix D.3.

The first projection we have done is a stereographic projection [7, chapter 1, section 3]. The calculations and explenation for stereographic projection is given in appendix D.3.1. Figure 5.2 shows how the flow in Figure 5.1 with $n_x \geq 0$ is mapped down to the unit disc through a stereographic projection. The flow on the Bloch sphere at $n_x \leq 0$ turns out to be the exact same after projecting. It is now easier to see that a state will go in a circle and end up at the fixed point $|0\rangle$. The flow is slow close to the state $|0\rangle$ and increases until it reaches a maximum at $|1\rangle$. For the other interactions and measurements we see a similar trend. Figures for the other interactions and measurements can be found in appendix D.4.

Next we did Winkel tripel projection [10]. The calculations and explenation for the Winkel tripel is given in appendix D.3.2 and D.3.3. The Winkel tripel uses a parameter φ_0 which specifies the line of latitude that has true scale. Figure 5.2 shows how the flow

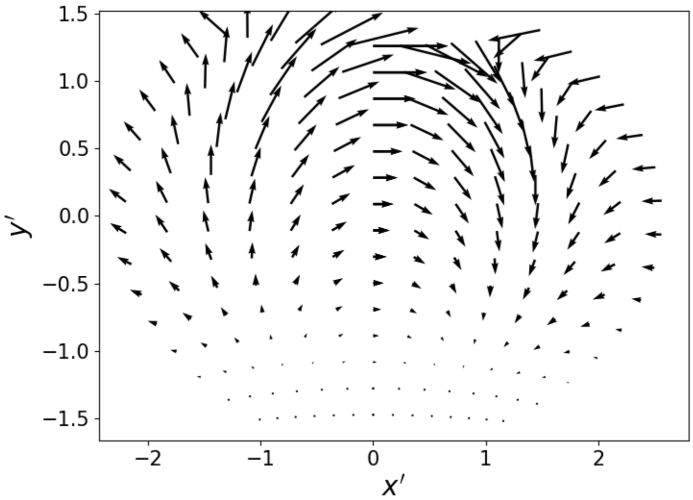


Figure 5.3: Winkel tripel projection of the flow of the state given by equation (5.9) for $k = 1$ and H_{I^+}, i.e. interaction Hamiltonian H_{I^+} and environment measured to be in the state $|\uparrow_x\rangle$. This is the same flow as depicted in Figure 5.1 and we have used $\cos(\varphi_0) = 2/\pi$, where φ_0 is the standard parallel for the equirectangular projection (see appendix D.3.3 for full explanation). We now get the full flow, contrary to the stereographic projection we showed in Figure 5.2. We again see the circular flow in the middle of the figure. The flow to the left in the figure also follows a circular motion which continues at the right side.

in Figure 5.1 for $\cos(\varphi_0) = 2/\pi$. Again, we see in the middle that a state will flow in a circle towards the state $|0\rangle$. On each side of the figure we see how a state will again flow in a circle towards the state $|0\rangle$, but this time we must understand that the states jumps from the left edge to the right edge of the figure. The flow is slow close to the state $|0\rangle$ and increases until it reaches a maximum at $|1\rangle$. For the other interactions and measurements we see a similar trend. Figures for the other interactions and measurements can be found in appendix D.4.

5.3 Dimension analysis of the stochastic processes giving rise to the same Lindblad equation

There are two main ideas we want to explore:

- Are we able to describe the space of Hamiltonians giving rise to the same dynamics governed by the the same Lindblad equation.
- Does there exist different Hamiltonians and measurement basis which gives the same stochastic process?

For a given Lindblad operator L we are interested in knowing all sets $\{H, |f_E\rangle\}$, which generates all possible paths (i.e. stochastic processes) that give the same result as the Lindblad equation (see Figure 5.4). Note that $e^{i\phi}L$ will give the same Lindblad equation as L:

$$\begin{aligned}\frac{d\rho}{dt} &= -i[H_S, \rho] + \frac{1}{2}\left[2(e^{i\phi}L)\rho(e^{i\phi}L)^\dagger - (e^{i\phi}L)^\dagger(e^{i\phi}L)\rho - \rho(e^{i\phi}L)^\dagger(e^{i\phi}L)\right] \\ &= -i[H_S, \rho] + \frac{1}{2}\left[2L\rho L^\dagger - L^\dagger L\rho - \rho L^\dagger L\right].\end{aligned}$$

We have not taken this into account as of yet, but it means that even more stochastic processes can be thought of as giving the same dynamics as the Lindblad equation.

We restrict ourselves to look at the situation when both system and environment are two-level systems. Let a Lindblad operator of the form $L = \sqrt{\frac{\theta^2\lambda}{\delta t}}\sum_i \mu_i\sigma_i$ be given. We know $H = \sum_j A_j \otimes B_j$. We choose to work in the basis $\{\mathbb{1}, \sigma_i : i = 1, 2, 3\}$, and as we are not interested in term of the form $\mathbb{1} \otimes \sigma_i$ (evolution concerning only the environment) or $\sigma_i \otimes \mathbb{1}$ (evolution concerning only the system), the Hamiltonian is given by $H = \sum_{i,j=1}^{3} h_{ij}\sigma_i \otimes \sigma_j$. As we want the given Lindblad operator L we choose $A_i = \sigma_i$ and $B_i = \sum_j h_{ij}\sigma_j$.

We can always assume the environment is in the state $|E\rangle = |0\rangle$. If this was not the case, we know that $|E\rangle = U|0\rangle$ with $U = |E\rangle\langle 0| + |E^\perp\rangle\langle 1|$. Hence, $M_{ij} = \left\langle E|B_j^\dagger B_i|E\right\rangle = \left\langle 0|U^\dagger B_j^\dagger B_i U|0\right\rangle$ and we can thus define $B_i' = B_iU = \sum_j h_{ij}'\sigma_j$. We then have, since

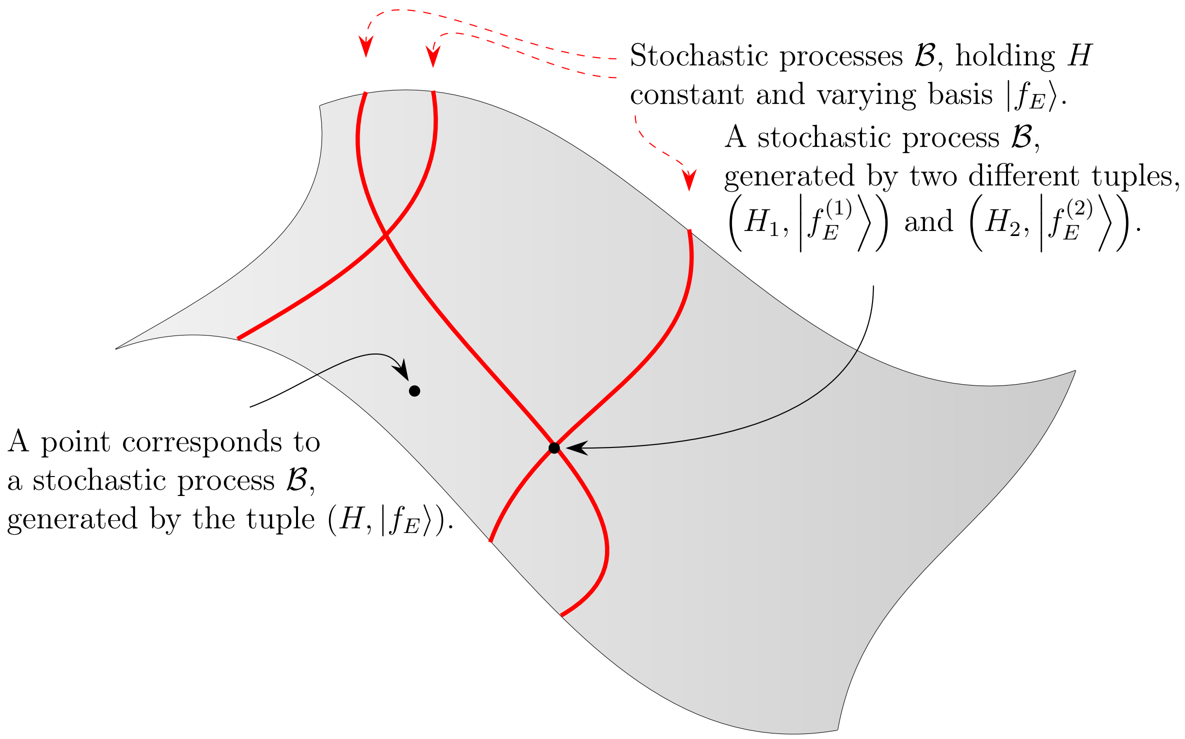


Figure 5.4: Hypersurface of all stochastic processes $\mathcal{B}$ "equivalent" to the Lindblad equation with a given Lindblad operator L. Each process is generated by a Hamiltonian H and measurement basis $\{|f_E\rangle\}$. For now, we assume that environment is a TLS; This means that we only need to specify $|f_E\rangle$, as the basis then will be given by $\{|f_E\rangle, |f_E\rangle^{\perp}\}$.

$B_i^\dagger = B_i$ and $\sigma_k\sigma_l = \delta_{kl}\mathbb{1} + i\epsilon_{klm}\sigma_m$,

$$\begin{aligned}\langle 0|B_jB_i|0\rangle &= \left\langle 0|(\sum_k h_{jk}\sigma_k)(\sum_l h_{il}\sigma_l)|0\right\rangle = \sum_{kl}\langle 0|h_{jk}h_{il}\sigma_k\sigma_l|0\rangle \\ &= \sum_{kl} h_{jk}h_{il}(\delta_{kl}\mathbb{1} + i\epsilon_{klm}\langle 0|\sigma_m|0\rangle) = \sum_k h_{jk}h_{ik} + i(h_{j1}h_{i2} - h_{j2}h_{i1}),\end{aligned}$$

where we have used that $\langle 0|\sigma_x|0\rangle = \langle 0|\sigma_y|0\rangle = 0$ and $\langle 0|\sigma_z|0\rangle = 1$. Furthermore,

$$M = \lambda|\mu\rangle\langle\mu| = \lambda\begin{pmatrix}|\mu_1|^2 & \mu_1\mu_2^* & \mu_1\mu_3^* \\ \mu_2\mu_1^* & |\mu_2|^2 & \mu_2\mu_3^* \\ \mu_3\mu_1^* & \mu_3\mu_2^* & |\mu_3|^2\end{pmatrix}$$

as well as $M_{ij} = \langle 0|B_jB_i|0\rangle = \sum_k h_{jk}h_{ik} + i(h_{j1}h_{i2} - h_{j2}h_{i1})$. As L is known, M is also known. Hence, we have nine equation for the nine unknowns h_{ij}: We demand that M is equal to

$$\begin{pmatrix}\sum_k h_{1k}h_{1k} & \sum_k h_{2k}h_{1k} + i(h_{21}h_{12} - h_{22}h_{11}) & \sum_k h_{3k}h_{1k} + i(h_{31}h_{12} - h_{32}h_{11}) \\ \sum_k h_{1k}h_{2k} + i(h_{22}h_{11} - h_{21}h_{12}) & \sum_k h_{2k}h_{2k} & \sum_k h_{3k}h_{2k} + i(h_{31}h_{22} - h_{32}h_{21}) \\ \sum_k h_{1k}h_{3k} + i(h_{32}h_{11} - h_{31}h_{12}) & \sum_k h_{2k}h_{3k} + i(h_{32}h_{21} - h_{31}h_{22}) & \sum_k h_{3k}h_{3k}\end{pmatrix}. \tag{5.10}$$

5.3.1 Example: The model proposed by Longva

In his thesis [13], Longva shows that the Hamiltonian $H_{I^\pm} = \frac{1}{4}(\sigma_x\otimes\sigma_x \pm \sigma_y\otimes\sigma_y)$ corresponds to the Lindblad operator $L_\pm = \sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$ with rate $\Gamma_\pm = \sqrt{\frac{\theta^2}{\delta t}}$ in the QTT framework. We will now start with the Lindblad operator $L = \sigma_+ = \frac{1}{2}(\sigma_x \pm i\sigma_y)$ and rate $\Gamma = \frac{\theta^2}{\delta t}2$, and see exactly which Hamiltonians H corresponds to L. From the QTT framework we demand (see section 3.4)

$$\Gamma = \frac{\theta^2}{\delta t}\lambda, \qquad L = \sum_j \boldsymbol{\mu}_j A_j,$$

where $\lambda\boldsymbol{\mu}\boldsymbol{\mu}^\dagger = M$ and $M_{ij} = \left\langle\psi\left|B_j^\dagger B_i\right|\psi\right\rangle$. As we noted above, we will try the representation $H_I = \sum_{ij} h_{ij}\sigma_i\otimes\sigma_j = \sum_j A_j\otimes B_j$ with $A_j = \sigma_j$ and $B_j = \sum_i H_{ij}\sigma_i$. Then $\mu \propto \begin{pmatrix}1, & i, & 0\end{pmatrix}^T$, and thus $\mu = \frac{1}{\sqrt{2}}\begin{pmatrix}1, & i, & 0\end{pmatrix}^T$ and $\lambda = 2$. This gives

$$M = \lambda|\mu\rangle\langle\mu| = \begin{pmatrix}1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0\end{pmatrix}.$$

Setting M equal to equation (5.10), we go through the process of solving for the different values of h_{ij}. This is done in appendix E. We find that the Lindblad equation with Lindblad operator L and rate Γ must come from a Hamiltonian on the form

$$H = h(\sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y) \pm \sqrt{1-h^2}(\sigma_x\otimes\sigma_y - \sigma_y\otimes\sigma_x),$$

where $h \in [-1, 1]$. This means that the subspace of the Hamiltonians generating this Lindblad operator is one dimensional, even though we have nine possible parameters to tweak and only six equation to satisfy.

From the equations in (5.10), we see that we have nine non-linear equations with nine unknowns. Furthermore, for the Lindblad operator we have three complex numbers μ_i and one real number λ. Since $\|\mu\| = 1$, we should have six numbers that determines the matrix M. It therefore sounds like we have nine unknown parameters with six numbers determined by them. This should give us three independent parameters. However, we have seen from the above example that we can find at least one case where we only have a single degree of freedom. It is not difficult to find other cases with decreased degree of freedom. Take for instance $L = 0$. This can only be achieved by $h_{ij} = 0$ for all i, j. This is due to the diagonal equations $\sum_j h_{ij}^2 = 0$ and that $h_{ij} \in \mathbb{R}$. As we have seen, the diagonal equations are quite restricting whenever they are equal to zero. Take for instance $\sum_j h_{2j}^2 = \sum_j h_{3j}^2 = 0$ and $\sum_j h_{1j}^2 = \lambda$. Then $h_{2j} = h_{3j} = 0$ for $j = 1, 2, 3$ and we get a two parameter family of solutions, $h_{11} \in [-\lambda, \lambda]$, $h_{12} \in [|h_{11}| - \lambda, \lambda - |h_{11}|]$ and $h_{13} = \pm\sqrt{1 - h_{11}^2 - h_{12}^2}$.

As far as we can see, there is no obvious way of telling the dimensionality of solutions for a given choice of Lindblad operator.

Chapter 6

Discussion

When we began working on this thesis, our goal was to continue the work by Longva [13] and find an analytical expression for the synchronization he observed numerically. As we saw in section 5.1 we found a master equation for the probability density of state of a TLS modeled the same way as in the thesis by Longva. Our idea is to use the same approach as when deriving the diffusion equation, and solve a differential equation instead.

Although we have not yet found an analytical expression for synchronization in the model given by Longva, we have developed the groundwork to do so. In section 3.5 we have given several results making computation easier and building understanding for the QTT framework. We saw that for a QTT model with two-level environment, we can always choose the environment to be in the state $|0\rangle$. This means that we can remove the freedom in the choice of the environment state. We also saw that for a TLS, to achieve small jumps of the system state when measuring the environment, we should choose a measurement basis not containing the environment state $|E\rangle$. We did a specific example where the environment was in the state $|0\rangle$. For this example we saw that choosing the measurement basis to be $\{|0\rangle, |1\rangle\}$ would occasionally result in large jumps of the system state. Next, we showed that an two-level can only give a single Lindblad operator in the QTT framework. For an n-level environment we saw that the matrix M, which was used to determine the Lindblad operators, would maximally give $n-1$ Lindblad operators whenever we could argue that it was diagonalized. We then went on to show how we can recover the Lindblad equation when we have a system Hamiltonian $H_S \neq 0$. This was not done in the article by Brun [4], and we saw that we need the interaction Hamiltonian to have a specific form as to not pick up an extra term not in the Lindblad equation. After this, we showed that the representation of the interaction Hamiltonian does not matter. In other words, both $H_I = \sum_j A_j \otimes B_j$ and $H_I = \sum_j C_j \otimes D_j$ will give the same Lindblad equation. This also means that for a QTT model of a TLS with two-level environment, the representation $H_I = \sum_{i=1}^{3} h_{ij}\sigma_i \otimes \sigma_j$ can always be chosen. Finally, we ended by showing an easy way of determining the interaction Hamiltonian whenever we had a specific Lindblad operator in mind. This was shown when we had a TLS and a two-level environment.

Going back to chapter 5 visualized how the state of a TLS, modeled the same way as in the thesis by Longva, would flow on the Bloch sphere. We realized that a three-dimensional flow is not trivial to interpret and therefore showed how we could map the flow on the Bloch sphere down to the plane. We gave both a stereographic projection as well as a Winkel tripel projection. From these projections it is easy to see that the model chosen by Longva results in the state flowing in circles on the Bloch sphere for all four different flow regimes. If we looks at a single flow in particular, we see that it flows to a fixed point. The flow is slow close to this point, and fast the further away from the point we are.

Lastly, we realized how several interaction Hamiltonians can give rise to the same Lindblad equation. We looked at the model proposed by Longva in particular and found all the possible Hamiltonians giving the same Lindblad equation, as the model by Longva. We saw that the dimensionality of the space of Hamiltonians giving the same Lindblad equation is not necessarily given by counting the number of free variables. As and example, for a TLS with two-level environment, the space of Hamiltonians can have dimension 0, 1 and 2, depending on the chosen Lindblad equation.

6.1 Future work

Now that we have laid down the groundwork, the next thing to look at would be a solution to the master equation (5.5) given in section 5.1. We would need to take the limit when the interaction time goes to zero. When doing this for the random walk/diffusion case, we need to make sure that the probabilities have a specific form (see appendix C.2). This should also be the case for our master equation. The restriction on the probabilities will most likely turn out naturally when taking the limit. If we manage to find the limit when time goes to zero, we are left with a differential equation for the probability density. We would then need to solve this differential equation. This differential equation would also hopefully give insight to the limit cycle of the system. When all of this is done, we will still need to add the system Hamiltonian H_S. Furthermore, to observe synchronization, we would also need to add the signal Hamiltonian described by Longva. We can then compare our results with those Longva obtain numerically.

There are many minor details we would have liked to explore. These are not necessarily important, but interesting nonetheless:

- Although intuitive, it would be nice to confirm that QTT does indeed give a Markov process. We say intuitive as each step in a QTT model is determined from the current state. There should therefore be no dependence on the previous state of the system, i.e. we have a Markovian process.

- The QTT framework described in the article by Brun [4] seem to suggest that the matrix M, defined in section 3.4, should be positive. Since the eigenvalues of the matrix determines the rate, it would make sense that they are positive. However, this was never shown or explicitly claimed in the article by Brun.

- In the QTT framework we say that all environment states are in the same state $|E\rangle$, but experimentally we know that there will be noise which can interfere with this assumption. It could therefore be an idea to build the framework up again, but this time with a stochastic element in each environment state. We could then see how this will change the dynamics of the system, and if we still recover the Lindblad equation.

- In both the article by Rolet and Bruder [23], and the article by Parra-López and Bergli [18], it is claimed that the solution to the Lindblad equation is a fixed point. Although this is true, it is not shown that this fixed point in attractive in the sense that all other density matrices will converge to this fixed point. This means that neither article has excluded the possibility that the dynamical system given by their Lindblad equation has a valid limit cycle. It would therefore be interesting to see if it is possible to determine if there is a valid limit cycle.

- In the QTT model proposed by Longva [13], he determines the rates of the Lindblad equation by looking at excitations in a field. This lets him determine the rates by the temperature. This is a valid approach, but it implicitly assumes that the environment can be modeled as a field. However, he know exactly how the environment is modeled: It is described by two-level systems all in the state $|0\rangle$, and not a field. It would therefore make sense to find the rates determined by this environment instead.

- It feels intuitive that switching between interaction Hamiltonian $H_{I^\pm} = \frac{1}{4}(\sigma_x \otimes \sigma_x \pm \sigma_y \otimes \sigma_y)$ should give the same Lindblad equation as if we chose an interaction Hamiltonian for a four-level system instead. It is therefore interesting to verify that this is actually the case.

- In the beginning of chapter 3 we calculated the entropy of different measurements. As of now, we have not calculated the entropy for the QTT model. We could therefore get new insight from the entropy. For instance, we could verify that our choice of measurement basis gives small jumps, and that it is the best basis to choose.

- We discuss in appendix D.3 how our choice of mapping is not the only one. It is possible there are better ways of visualizing the flow by projecting it down to a plane. What is more, we saw that when computing the Jacobian of the coordinate transform, we let the radius r of the Bloch sphere vary, even though we knew $r = 1$. We would therefore like to find out what we actually mean when we choose to let r be free, before we choose $r = 1$.

Appendices

Appendix A

Rotating frame in quantum mechanics and entropy calculations

A.1 Unitary operator and rotations

We will here who that a unitary operator on a two-level system will always correspond to a rotation around an axis of rotation $\boldsymbol{n}$ on the Bloch sphere. This is (as we will see later on) the same as showing that the most general unitary operator on a single q-bit can be written as

$$U = \mathbb{1}\cos(\varphi) + i\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin(\varphi) = e^{i\varphi\boldsymbol{n}\cdot\boldsymbol{\sigma}}, \tag{A.1}$$

modulo an irrelevant overall phase, where $\boldsymbol{n} = (n_x, n_y, n_z)$ is a three-vector with unit norm and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. We start by observing that

$$\begin{aligned}(\boldsymbol{n}\cdot\boldsymbol{\sigma})^2 =& n_x(n_x\sigma_x\sigma_x + n_y\sigma_x\sigma_y + n_z\sigma_x\sigma_z) + n_y(n_x\sigma_y\sigma_x + n_y\sigma_y\sigma_y + n_z\sigma_y\sigma_z)\\ &+ n_z(n_x\sigma_z\sigma_x + n_y\sigma_z\sigma_y + n_z\sigma_z\sigma_z)\\ =&(n_x^2 + n_y^2 + n_z^2)I + n_xn_y\{\sigma_x,\sigma_y\} + n_xn_z\{\sigma_x,\sigma_z\} + n_zn_y\{\sigma_z,\sigma_y\}\\ =&\mathbb{1},\end{aligned}$$

so

$$\begin{aligned}e^{i\varphi\boldsymbol{n}\cdot\boldsymbol{\sigma}} &= \sum_{k=0}^{\infty}\frac{1}{k!}(i\varphi\boldsymbol{n}\cdot\boldsymbol{\sigma})^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\varphi^{2k}(\boldsymbol{n}\cdot\boldsymbol{\sigma})^{2k} + i\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\varphi^{2k+1}(\boldsymbol{n}\cdot\boldsymbol{\sigma})^{2k+1}\\ &= \mathbb{1}\cos(\varphi) + i\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin(\varphi).\end{aligned}$$

Thus the right hand side of equation (A.1) is shown.

We know that any unitary operator U can be written as $U = e^{iH}$ (see theorem 2.5.8 in Murphy [16]). It is not hard to check that $\mathrm{span}\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\} = B(\mathbb{C}^2)_{\mathrm{sa}}$, i.e. the hermitian (same as self-adjoint) operators on the Hilbert space we are working on, $\mathbb{C}^2$. (If

$$H = \begin{pmatrix} a & b_1 + ib_2 \\ b_1 - ib_2 & c\end{pmatrix},\ a, b_1, b_2, c \in \mathbb{R},$$

choose numbers $c_0 = \frac{a+c}{2}$, $c_3 = \frac{a-c}{2}$, $c_1 = b_1$ and $c_2 = -b_2$, and observe that $H = c_0\mathbb{1} + c_1\sigma_x + c_2\sigma_y + c_3\sigma_z$.) Let $\varphi := \sqrt{c_1^2 + c_2^2 + c_3^2}$ and define $\boldsymbol{n} = (c_1, c_2, c_3)/\varphi$. Then

$\|\boldsymbol{n}\| = 1$ and, as $[I, \sigma_i] = 0$, $U = e^{ic_0} e^{i\varphi \boldsymbol{n}\cdot\boldsymbol{\sigma}}$. This was exactly what we wanted to show in equation (A.1). Next, we will see that this actually corresponds to a rotation of 2φ around the axis $\boldsymbol{n}$. First a direct and convoluted approach, and then an abstract but perhaps more elegant way.

Rodrigues' rotation formula

We know that the density operator of a two-level system can be written as

$$\rho = \frac{1}{2}\left(\mathbb{1} + \boldsymbol{r}\cdot\boldsymbol{\sigma}\right)$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z) = (\sigma_1, \sigma_2, \sigma_3)$ is a vector containing the Pauli matrices, and $\boldsymbol{r}$ is the Bloch vector, and with time-evolution $U\rho U^\dagger$ [17, section 2.4 and exercise 2.72]. We will calculate the time-evolution for $U = e^{i\varphi \boldsymbol{n}\cdot\boldsymbol{\sigma}}$ directly and end up with Rodrigues' rotation formula [5] (called rotation formula in [8]),

$$U\rho U^\dagger = \left(\mathbb{1}\cos(\varphi) + i\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin(\varphi)\right)\left(\frac{\mathbb{1} + \boldsymbol{r}\cdot\boldsymbol{\sigma}}{2}\right)\left(\mathbb{1}\cos(\varphi) - i\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin(\varphi)\right).$$

Using the relation $\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k$, we begin by calculating

$$\begin{aligned}
(\boldsymbol{n}\cdot\boldsymbol{\sigma})(\boldsymbol{r}\cdot\boldsymbol{\sigma}) =& \overbrace{(r_1n_1\sigma_1^2 + r_2n_2\sigma_2^2 + r_3n_3\sigma_3^2)}^{(\boldsymbol{r}\cdot\boldsymbol{n})\mathbb{1}} + (r_1n_2\overbrace{\sigma_1\sigma_2}^{i\sigma_3} + r_2n_1\overbrace{\sigma_2\sigma_1}^{-i\sigma_3}) \\
&+ (r_1n_3\underbrace{\sigma_1\sigma_3}_{-i\sigma_2} + r_3n_1\underbrace{\sigma_3\sigma_1}_{i\sigma_2}) + (r_2n_3\underbrace{\sigma_2\sigma_3}_{i\sigma_1} + r_3n_2\underbrace{\sigma_3\sigma_2}_{-i\sigma_1}) \\
=& (\boldsymbol{r}\cdot\boldsymbol{n})\mathbb{1} + i\left[(r_2n_3 - r_3n_2)\sigma_1 - (r_1n_3 - r_3n_1)\sigma_2 + (r_1n_2 - r_2n_1)\sigma_3\right] \\
=& (\boldsymbol{r}\cdot\boldsymbol{n})\mathbb{1} + i(\boldsymbol{r}\times\boldsymbol{n})\cdot\boldsymbol{\sigma},
\end{aligned}$$

which gives

$$[(\boldsymbol{n}\cdot\boldsymbol{\sigma})(\boldsymbol{r}\cdot\boldsymbol{\sigma}), (\boldsymbol{r}\cdot\boldsymbol{\sigma})(\boldsymbol{n}\cdot\boldsymbol{\sigma})] = (\boldsymbol{r}\cdot\boldsymbol{n})\mathbb{1} + i(\boldsymbol{r}\times\boldsymbol{n})\cdot\boldsymbol{\sigma} - (\boldsymbol{r}\cdot\boldsymbol{n})\mathbb{1} - i(\boldsymbol{n}\times\boldsymbol{r})\cdot\boldsymbol{\sigma} = 2i(\boldsymbol{r}\times\boldsymbol{n})\cdot\boldsymbol{\sigma}$$

and

$$\begin{aligned}
(\boldsymbol{n}\cdot\boldsymbol{\sigma})(\boldsymbol{r}\cdot\boldsymbol{\sigma})(\boldsymbol{n}\cdot\boldsymbol{\sigma}) &= (\boldsymbol{n}\cdot\boldsymbol{\sigma})\left((\boldsymbol{r}\cdot\boldsymbol{n})\mathbb{1} + i(\boldsymbol{r}\times\boldsymbol{n})\cdot\boldsymbol{\sigma}\right) \\
&= (\boldsymbol{r}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{\sigma}) + i(\boldsymbol{n}\cdot\boldsymbol{\sigma})\left((\boldsymbol{r}\times\boldsymbol{n})\cdot\boldsymbol{\sigma}\right) \\
&= (\boldsymbol{r}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{\sigma}) + i\left(\overbrace{(\boldsymbol{n}\cdot(\boldsymbol{r}\times\boldsymbol{n}))}^{\boldsymbol{r}\cdot(\boldsymbol{n}\times\boldsymbol{n})=0}\mathbb{1} + i\overbrace{(\boldsymbol{n}\times(\boldsymbol{r}\times\boldsymbol{n}))}^{(\boldsymbol{n}\cdot\boldsymbol{n})\boldsymbol{r}-(\boldsymbol{n}\cdot\boldsymbol{r})\boldsymbol{n}}\cdot\boldsymbol{\sigma}\right) \\
&= (\boldsymbol{r}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{\sigma}) - (((\boldsymbol{n}\cdot\boldsymbol{n})\boldsymbol{r} - (\boldsymbol{n}\cdot\boldsymbol{r})\boldsymbol{n})\cdot\boldsymbol{\sigma}) \\
&= 2(\boldsymbol{r}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{\sigma}) - (\boldsymbol{r}\cdot\boldsymbol{\sigma}).
\end{aligned}$$

Here we have used that $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$, $\boldsymbol{n}\times\boldsymbol{n} = 0$, and $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$ for three dimensional vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ [14].

Hence,

$$\begin{aligned}
U\left(\boldsymbol{r}\cdot\boldsymbol{\sigma}\right)U^{\dagger} &= \left(\mathbb{1}\cos\left(\varphi\right)+i\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin\left(\varphi\right)\right)\left(\boldsymbol{r}\cdot\boldsymbol{\sigma}\right)\left(\mathbb{1}\cos\left(\varphi\right)-i\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin\left(\varphi\right)\right)\\
&= \left(\boldsymbol{r}\cdot\boldsymbol{\sigma}\right)\cos^2\left(\varphi\right)+i\sin\left(\varphi\right)\cos\left(\varphi\right)[(\boldsymbol{n}\cdot\boldsymbol{\sigma})(\boldsymbol{r}\cdot\boldsymbol{\sigma}),(\boldsymbol{r}\cdot\boldsymbol{\sigma})(\boldsymbol{n}\cdot\boldsymbol{\sigma})]\\
&\quad+\left(\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)\left(\boldsymbol{r}\cdot\boldsymbol{\sigma}\right)\left(\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)\sin^2\left(\varphi\right)\\
&= \left(\boldsymbol{r}\cdot\boldsymbol{\sigma}\right)\cos^2\left(\varphi\right)+2i^2\sin\left(\varphi\right)\cos\left(\varphi\right)(\boldsymbol{r}\times\boldsymbol{n})\cdot\boldsymbol{\sigma}\\
&\quad+\left(2(\boldsymbol{r}\cdot\boldsymbol{n})\left(\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)-\left(\boldsymbol{r}\cdot\boldsymbol{\sigma}\right)\right)\sin^2\left(\varphi\right)\\
&= \left(\boldsymbol{r}\cdot\boldsymbol{\sigma}\right)\cos\left(2\varphi\right)-\sin\left(2\varphi\right)(\boldsymbol{r}\times\boldsymbol{n})\cdot\boldsymbol{\sigma}+2(\boldsymbol{r}\cdot\boldsymbol{n})\left(\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)\sin^2\left(\varphi\right)\\
&= \left(\boldsymbol{r}\cos\left(2\varphi\right)+(1-\cos\left(2\varphi\right))(\boldsymbol{r}\cdot\boldsymbol{n})\boldsymbol{n}+(\boldsymbol{r}\times\boldsymbol{n})\sin\left(2\varphi\right)\right)\cdot\boldsymbol{\sigma}.
\end{aligned}$$

The expression which is dotted with $\boldsymbol{\sigma}$ is sometimes referred to as Rodrigues' rotation formula [5], and corresponds to a 2φ rotation of $\boldsymbol{r}$ around $\boldsymbol{n}$.

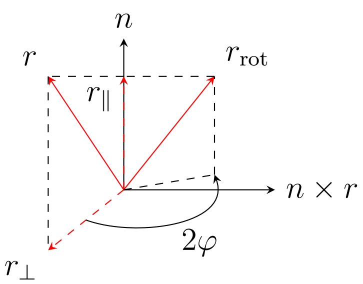


Figure A.1: Help figure for Rodrigues' rotation formula. The vector $\boldsymbol{r}$ is rotated by 2φ around $\boldsymbol{n}$. $\boldsymbol{r}_{\perp}$ and $\boldsymbol{r}_{\parallel}$ are the perpendicular and the parallel component of $\boldsymbol{r}$, respectively, when compared to $\boldsymbol{n}$. $\boldsymbol{r}_{\text{rot}}$ is the rotation of $\boldsymbol{r}$ around $\boldsymbol{n}$.

To see this we first decompose $\boldsymbol{r}$ into components parallel and perpendicular to $\boldsymbol{n}$,

$$\boldsymbol{r}=\boldsymbol{r}_{\parallel}+\boldsymbol{r}_{\perp},$$

where the component parallel to $\boldsymbol{n}$ is $\boldsymbol{r}_{\parallel}=(\boldsymbol{r}\cdot\boldsymbol{n})\boldsymbol{n}$, and the component perpendicular to $\boldsymbol{n}$ is

$$\boldsymbol{r}_{\perp}=\boldsymbol{r}-\boldsymbol{r}_{\parallel}=\boldsymbol{r}-(\boldsymbol{r}\cdot\boldsymbol{n})\boldsymbol{n}=-\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{r}).$$

(To keep track of the different variables and what happens to them, we have tried to visualize them in Figure A.1.) Both $\boldsymbol{n}\times\boldsymbol{r}$ and $\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{r})$ can be thought of as $\boldsymbol{r}_{\perp}$, but rotated anticlockwise by $\frac{\pi}{2}$ and π radians, respectively about $\boldsymbol{n}$, so that their magnitudes are equal to $\boldsymbol{r}_{\perp}$. Denoting the rotated vector $\boldsymbol{r}$ by $\boldsymbol{r}_{\text{rot}}$, we know that the component parallel to $\boldsymbol{n}$ stays constant, i.e. $\boldsymbol{r}_{\parallel,\text{rot}}=\boldsymbol{r}_{\parallel}$. The perpendicular component will however be rotated by

$$\boldsymbol{r}_{\perp,\text{rot}}=\cos\left(2\varphi\right)\boldsymbol{r}_{\perp}+\sin\left(2\varphi\right)\boldsymbol{n}\times\boldsymbol{r}_{\perp}.$$

Moreover, since $\boldsymbol{n}$ and $\boldsymbol{r}_{\parallel}$ are parallel, we have $\boldsymbol{n}\times\boldsymbol{r}_{\parallel}=0$ so

$$\boldsymbol{n}\times\boldsymbol{r}_{\perp}=\boldsymbol{n}\times\left(\boldsymbol{r}-\boldsymbol{r}_{\parallel}\right)=\boldsymbol{n}\times\boldsymbol{r}-\boldsymbol{n}\times\boldsymbol{r}_{\parallel}=\boldsymbol{n}\times\boldsymbol{r}.$$

Now the full rotated vector is

$$\boldsymbol{r}_{\text{rot}}=\boldsymbol{r}_{\parallel,\text{rot}}+\boldsymbol{r}_{\perp,\text{rot}}.$$

Hence

$$\begin{aligned}
\boldsymbol{r}_{\text{rot}} &= \boldsymbol{r}_{\parallel}+\cos\left(2\varphi\right)\boldsymbol{r}_{\perp}+\sin\left(2\varphi\right)\boldsymbol{n}\times\boldsymbol{r}\\
&= \boldsymbol{r}_{\parallel}+\cos\left(2\varphi\right)(\boldsymbol{r}-\boldsymbol{r}_{\parallel})+\sin\left(2\varphi\right)\boldsymbol{n}\times\boldsymbol{r}\\
&= \cos\left(2\varphi\right)\boldsymbol{r}+(1-\cos\left(2\varphi\right))\boldsymbol{r}_{\parallel}+\sin\left(2\varphi\right)\boldsymbol{n}\times\boldsymbol{r}\\
&= \cos\left(2\varphi\right)\boldsymbol{r}+(1-\cos\left(2\varphi\right))(\boldsymbol{n}\cdot\boldsymbol{r})\boldsymbol{n}+\sin\left(2\varphi\right)\boldsymbol{n}\times\boldsymbol{r},
\end{aligned}$$

which is the formula we were after.

Another approach

We will now take another approach. Assume that $\boldsymbol{n} = \begin{pmatrix}0, & 0, & 1\end{pmatrix}^T$. Then we can compute that

$$\begin{aligned} U\left(\boldsymbol{r}\cdot\boldsymbol{\sigma}\right)U^\dagger &= \left(\boldsymbol{r}\cos\left(2\varphi\right) + \left(1-\cos\left(2\varphi\right)\right)\left(\boldsymbol{r}\cdot\boldsymbol{n}\right)\boldsymbol{n} + \left(\boldsymbol{r}\times\boldsymbol{n}\right)\sin\left(2\varphi\right)\right)\cdot\boldsymbol{\sigma} \\ &= \left(\begin{pmatrix}r_x\\ r_y\\ r_z\end{pmatrix}\cos\left(2\varphi\right) + \begin{pmatrix}0\\ 0\\ r_z\end{pmatrix}\left(1-\cos\left(2\varphi\right)\right) + \begin{pmatrix}r_y\\ -r_x\\ 0\end{pmatrix}\sin\left(2\varphi\right)\right)\cdot\boldsymbol{\sigma} \\ &= \begin{pmatrix}r_x\cos\left(2\varphi\right) + r_y\sin\left(2\varphi\right)\\ r_y\cos\left(2\varphi\right) - r_x\sin\left(2\varphi\right)\\ r_z\end{pmatrix}\cdot\boldsymbol{\sigma}, \end{aligned}$$

which we recognize as a rotation of 2φ around the z-axis. For a general $\boldsymbol{n}$, we just create an orthonormal coordinate system $\boldsymbol{n}_x$, $\boldsymbol{n}_y$, $\boldsymbol{n}_z$, with $\boldsymbol{n}_z := \boldsymbol{n}$. The rotation will then look like

$$U\left(\boldsymbol{r}\cdot\boldsymbol{\sigma}\right)U^\dagger = \begin{pmatrix}r_x\cos\left(2\varphi\right) + r_y\sin\left(2\varphi\right)\\ r_y\cos\left(2\varphi\right) - r_x\sin\left(2\varphi\right)\\ r_z\end{pmatrix}\cdot\begin{pmatrix}\sigma_{\boldsymbol{n}_x}\\ \sigma_{\boldsymbol{n}_y}\\ \sigma_{\boldsymbol{n}_z}\end{pmatrix},$$

where $\sigma_{\boldsymbol{n}_i} := \boldsymbol{n}_i\cdot\boldsymbol{\sigma}$.

A.2 Rotation of the Lindblad equation

We often change reference frame to rotating reference frames. We want to see what the Lindblad equation looks like in the rotated reference frame. Let therefore $T = e^{i\varphi\boldsymbol{n}\cdot\boldsymbol{\sigma}}$ be the rotation we want, and recall the Lindblad equation [11, chapter 15]

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \sum_k \frac{\Gamma_k}{2}\mathcal{D}[L_k]\rho,$$

where $\rho = \frac{1}{2}(\mathbb{1} + \boldsymbol{r}\cdot\boldsymbol{\sigma})$ is our mixed state, $\dot{\rho} = \frac{d\rho}{dt}$ is the time derivative, H is the system Hamiltonian, $\hbar$ is Planck's constant divided by 2π, L_k is the k-th Lindblad operator, Γ_k is the rate of the k-th Lindblad operator, and $\mathcal{D}[L_k]\rho := 2L_k\rho L_k^\dagger - L_k^\dagger L_k\rho - \rho L_k^\dagger L_k$ is the Lindblad superoperator.

The mixed state in the rotated frame is given by $T\rho T^\dagger =: \rho'$, and hence

$$\frac{d\rho'}{dt} = \dot{T}\rho T^\dagger + T\dot{\rho}T^\dagger + T\rho\dot{T}^\dagger.$$

We calculate

$$\begin{aligned} \frac{d}{dt}e^{\pm i\varphi(t)\boldsymbol{n}\cdot\boldsymbol{\sigma}} &= \frac{d}{dt}\left(\mathbb{1}\cos\varphi(t) \pm i\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin\varphi(t)\right) = -\mathbb{1}\dot{\varphi}(t)\sin\varphi(t) \pm i\boldsymbol{n}\cdot\boldsymbol{\sigma}\dot{\varphi}(t)\cos\varphi(t) \\ &= \pm i\dot{\varphi}(t)\boldsymbol{n}\cdot\boldsymbol{\sigma}\left(\mathbb{1}\cos\varphi(t) \pm i\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin\varphi(t)\right) = \pm i\dot{\varphi}(t)\boldsymbol{n}\cdot\boldsymbol{\sigma}e^{\pm i\varphi(t)\boldsymbol{n}\cdot\boldsymbol{\sigma}}, \end{aligned}$$

where we have used that $(\boldsymbol{n}\cdot\boldsymbol{\sigma})^2 = \mathbb{1}$. Thus

$$\dot{T}\rho T^\dagger + T\rho\dot{T}^\dagger = i\dot{\varphi}(t)\boldsymbol{n}\cdot\boldsymbol{\sigma}\overbrace{T\rho T^\dagger}^{=\rho'} - i\dot{\varphi}(t)\boldsymbol{n}\cdot\boldsymbol{\sigma}\overbrace{T\rho T^\dagger}^{=\rho'} = i\dot{\varphi}(t)[\boldsymbol{n}\cdot\boldsymbol{\sigma},\rho'].$$

For the final term, $T\dot{\rho}T^\dagger$, we need to calculate $T[H,\rho]T^\dagger$ and $T(\mathcal{D}[L_k]\rho)T^\dagger$. Observe that we for any operator O have

$$TO\rho T^\dagger = TOT^\dagger T\rho T^\dagger = TOT^\dagger\rho',$$

and likewise $T\rho OT^\dagger = \rho' TOT^\dagger$ since $TT^\dagger = T^\dagger T = \mathbb{1}$. This means that $T[H,\rho]T^\dagger = [THT^\dagger,\rho']$ and

$$T(\mathcal{D}[L_k]\rho)T^\dagger = 2TL_kT^\dagger\rho' TL_k^\dagger T^\dagger - TL_k^\dagger T^\dagger TL_kT^\dagger\rho' - \rho' TL_k^\dagger T^\dagger TL_kT^\dagger.$$

The rotated Lindblad equation is therefore

$$\begin{aligned}\dot{\rho}' =& i\dot{\varphi}(t)[\boldsymbol{n}\cdot\boldsymbol{\sigma},\rho'] - \frac{i}{\hbar}[THT^\dagger,\rho'] \\ &+ \sum_k \frac{\Gamma_k}{2}\left(2TL_kT^\dagger\rho' TL_k^\dagger T^\dagger - TL_k^\dagger T^\dagger TL_kT^\dagger\rho' - \rho' TL_k^\dagger T^\dagger TL_kT^\dagger\right).\end{aligned}$$

Let us now calculate a specific example from the article by Parra-López and Bergli [18]: Let $H = H_0 + H_{\text{signal}}$, where $H_0 = \frac{\hbar}{2}\omega_0\sigma_z$ and $H_{\text{signal}} = i\hbar\frac{\epsilon}{4}(e^{i\omega t}\sigma_- - e^{-i\omega t}\sigma_+)$, and $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$. Let the original Lindblad equation be

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \frac{\Gamma_g}{2}\mathcal{D}[\sigma_+]\rho + \frac{\Gamma_d}{2}\mathcal{D}[\sigma_-]\rho,$$

where Γ_g and Γ_d models gain and damping rates, respectively. Finally, let the rotation be given by $T = e^{i\frac{\omega}{2}\sigma_z t}$. Then, using Rodrigues' rotation formula found in appendix A.1,

$$\begin{aligned}T\sigma_\pm T^\dagger &= T\frac{1}{2}(\sigma_x \pm i\sigma_y)T^\dagger = \frac{1}{2}(T\sigma_xT^\dagger \pm iT\sigma_yT^\dagger) \\ &= \frac{1}{2}\left(\left[\begin{pmatrix}1\\0\\0\end{pmatrix}\cos(\omega t) - \begin{pmatrix}0\\1\\0\end{pmatrix}\sin(\omega t)\right]\cdot\boldsymbol{\sigma} \pm i\left[\begin{pmatrix}0\\1\\0\end{pmatrix}\cos(\omega t) + \begin{pmatrix}1\\0\\0\end{pmatrix}\sin(\omega t)\right]\cdot\boldsymbol{\sigma}\right) \\ &= \frac{1}{2}\left([\sigma_x\cos(\omega t) - \sigma_y\sin(\omega t)] \pm i[\sigma_y\cos(\omega t) + \sigma_x\sin(\omega t)]\right) \\ &= \frac{1}{2}(\cos(\omega t) \pm i\sin(\omega t))\sigma_x \pm \frac{1}{2}i(\sigma_y\cos(\omega t) \pm i\sin(\omega t))\sigma_y \\ &= e^{\pm i\omega t}\sigma_\pm,\end{aligned}$$

and hence $T(\mathcal{D}[\sigma_\pm]\rho)T^\dagger = \mathcal{D}[\sigma_\pm]\rho'$. This calculation also gives

$$THT^\dagger = T\left(\frac{\hbar}{2}\omega_0\sigma_z + i\hbar\frac{\epsilon}{4}(e^{i\omega t}\sigma_- - e^{-i\omega t}\sigma_+)\right)T^\dagger = \frac{\hbar}{2}\omega_0\sigma_z + i\hbar\frac{\epsilon}{4}(\sigma_- - \sigma_+) = \frac{\hbar}{2}\omega_0\sigma_z + i\hbar\frac{\epsilon}{2}\sigma_y.$$

Putting everything together, we get

$$\begin{aligned}\dot{\rho}' &= i\frac{\omega}{2}[\sigma_z,\rho'] - \frac{i}{\hbar}\left[\frac{\hbar}{2}\omega_0\sigma_z + i\hbar\frac{\epsilon}{2}\sigma_y,\rho'\right] + \frac{\Gamma_g}{2}\mathcal{D}[\sigma_+]\rho' + \frac{\Gamma_d}{2}\mathcal{D}[\sigma_-]\rho' \\ &= -\frac{i}{2}[\Delta\sigma_z + \epsilon\sigma_y,\rho'] + \frac{\Gamma_g}{2}\mathcal{D}[\sigma_+]\rho' + \frac{\Gamma_d}{2}\mathcal{D}[\sigma_-]\rho',\end{aligned}$$

where we have defined $\Delta := \omega_0 - \omega$.

A.3 Shannon entropy for a two-level system

Projective measurement

The Shannon entropy for a (finite) discrete probability distribution with probabilities $(p_1, \dots, p_n)$ is given by [17, section 11.1]

$$S = -\sum_{i=1}^{n} p_i \log_2 p_i.$$

If we measure an observable for a two-level system, we will mathematically apply one of the two (orthogonal) projection operators P_1 and $P_2 := \mathbb{1} - P_1$. The projection operator will be $P_1 = |\phi\rangle\langle\phi|$ where $|\phi\rangle$ is an eigenstate of the observable. The probability for outcome 1 and 2 can be written as p_1 and $p_2 := 1 - p_1$, respectively. Thus the Shannon entropy is given by

$$S_{\text{proj}} = -p_1 \log_2 p_1 - p_2 \log_2 p_2 = -p_1 \log_2 p_1 - (1-p_1)\log_2(1-p_1).$$

Before we analyze the Shannon entropy, we show that $\log_a x = \frac{\log_b x}{\log_b a}$ for $x, a, b \in (0, \infty)$: As $\log_a x = c$ for some $c \in \mathbb{R}$, we have

$$\begin{aligned} x &= a^c \\ \log_b a^c &= \log_b x \\ c \log_b a &= \log_b x \\ c &= \frac{\log_b x}{\log_b a}. \end{aligned}$$

Hence, $\log_a x = \frac{\log_b x}{\log_b a}$, and for $b = e$ and $a = 2$ we get $\log_2 x = \frac{\ln x}{\ln 2}$.

We therefore have

$$S_{\text{proj}} = -p_1 \frac{\ln p_1}{\ln 2} - (1-p_1)\frac{\ln(1-p_1)}{\ln 2} = \frac{1}{\ln 2}\left[p_1 \ln \frac{1-p_1}{p_1} - \ln(1-p_1)\right]. \tag{A.2}$$

We calculate the derivatives

$$\frac{dS_{\text{proj}}}{dp_1} = \frac{1}{\ln 2}\left[\frac{1}{1-p_1} + \ln\frac{1-p_1}{p_1} + p_1\left(-\frac{1}{1-p_1} - \frac{1}{p_1}\right)\right] = \frac{1}{\ln 2}\ln\frac{1-p_1}{p_1}$$

and

$$\frac{d^2 S_{\text{proj}}}{dp_1^2} = \frac{1}{\ln 2}\left(-\frac{1}{1-p_1} - \frac{1}{p_1}\right) = -\frac{1}{p_1(1-p_1)\ln 2}.$$

The extreme point is given by $\frac{dS_{\text{proj}}}{dp_1} = 0$, i.e. $p_1 = p_2 = \frac{1}{2}$. As $p_1 \in [0, 1]$, we see that $\frac{d^2 S_{\text{proj}}}{dp_1^2} < 0$ for $p_1 \in (0, 1)$, which means that the Shannon entropy is strictly concave. Furthermore,

$$S_{\text{proj}}(p_1 = 0) = S_{\text{proj}}(p_1 = 1) = -\frac{1}{\ln 2}\lim_{p_1 \to 0} p_1 \ln p_1 = -\frac{1}{\ln 2}\lim_{p_1 \to 0}\frac{\ln p_1}{1/p_1} = -\frac{1}{\ln 2}\lim_{p_1 \to 0}\frac{1/p_1}{-1/p_1^2} = 0,$$

where we have used L'Hôpital's rule, and

$$S_{\text{proj}}(p_1 = 1/2) = \frac{1}{\ln 2}\left[\frac{1}{2}\ln\frac{1-\frac{1}{2}}{\frac{1}{2}} - \ln\left(1 - \frac{1}{2}\right)\right] = -\frac{1}{\ln 2}\ln\frac{1}{2} = 1.$$

POVM measurement

We look at the examples given in section 3.3. The first example of a POVM was $\{E_1, E_2\}$ where

$$E_1 = |0\rangle\langle 0| + (1-\epsilon)|1\rangle\langle 1|, \quad E_2 = \epsilon |1\rangle\langle 1|$$

and $\epsilon \ll 1$. To see that they are positive operators, we let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ be a general state of a TLS and compute [17, section 2.1.6]

$$\begin{aligned} p_1 &= \langle\psi|E_1|\psi\rangle = |\alpha|^2 + (1-\epsilon)|\beta|^2 = 1 - \epsilon|\beta|^2 \geq 0 \\ p_2 &= \langle\psi|E_2|\psi\rangle = \epsilon|\beta|^2 \geq 0. \end{aligned}$$

The Shannon entropy is calculated by

$$\begin{aligned} S_{\text{POVM}} &= -(1-\epsilon|\beta|^2)\log_2(1-\epsilon|\beta|^2) - \epsilon|\beta|^2\log_2(\epsilon|\beta|^2) \\ &= -\frac{1}{\ln 2}\left[(1-\epsilon|\beta|^2)\ln(1-\epsilon|\beta|^2) + \epsilon|\beta|^2\ln(\epsilon|\beta|^2)\right] \\ &= \frac{1}{\ln 2}\left[\epsilon|\beta|^2\ln\left(\frac{1-\epsilon|\beta|^2}{\epsilon|\beta|^2}\right) - \ln(1-\epsilon|\beta|^2)\right]. \end{aligned} \tag{A.3}$$

For $0 < x \leq 2$ we have $\ln x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n} = (x-1) + \mathcal{O}((x-1)^2)$, and thus we can approximate

$$\begin{aligned} S_{\text{POVM}} &\approx -\frac{1}{\ln 2}\left[(1-\epsilon|\beta|^2)(1-\epsilon|\beta|^2-1) + \epsilon|\beta|^2\ln(\epsilon|\beta|^2)\right] \\ &= -\frac{\epsilon|\beta|^2}{\ln 2}\left[-\overbrace{(1-\epsilon|\beta|^2)}^{\approx 1} + \ln(\epsilon|\beta|^2)\right] \\ &\approx \frac{\epsilon|\beta|^2}{\ln 2}\left[1 - \ln(\epsilon|\beta|^2)\right]. \end{aligned}$$

Since $\lim_{x\to 0} x\ln x = 0$, we have $S_{\text{POVM}} \ll 1$ for $\epsilon \ll 1$. We can also see that the Shannon entropy is small by finding the maximum information: Comparing equation (A.2) for S_{proj} with equation (A.3) for S_{POVM}, we see that they are equal if we put $p_1 = \epsilon|\beta|^2$. Thus

$$\frac{dS_{\text{POVM}}}{d|\beta|^2} = \frac{dS_{\text{POVM}}}{d(\epsilon|\beta|^2)}\frac{d(\epsilon|\beta|^2)}{d|\beta|^2} = \frac{\epsilon}{\ln 2}\ln\frac{1-\epsilon|\beta|^2}{\epsilon|\beta|^2}.$$

Since $0 \leq |\beta|^2 \leq 1$ and we assume $\epsilon \ll 1$, we can safely assume that $|\beta|^2 < \frac{1}{2\epsilon}$, which means that S_{POVM} has maximum and minimum for $|\beta|^2 = 0, 1$, i.e.

$$\begin{aligned} S_{\text{POVM}}(|\beta|^2 = 0) &= -\frac{1}{\ln 2}\lim_{|\beta|^2\to 0}\epsilon|\beta|^2\ln(\epsilon|\beta|^2) = 0, \\ S_{\text{POVM}}(|\beta|^2 = 1) &= -(1-\epsilon)\log_2(1-\epsilon) - \epsilon\log_2\epsilon. \end{aligned}$$

Since $0 < -\ln(1-\epsilon) \ll 1$ and $\lim_{\epsilon\to 0}\epsilon\ln\epsilon = 0$, we can again conclude that the information gained is very small.

The small information gain is good, but we need to know where the state ends up after measurement. This is unfortunately one of the downsides with a POVM. Following postulate 3 in the book by Nielsen and Chuang [17], we need to build up each POVM with measurement operators. In his article [4], Brun states that this can be done by knowing a set of operators A_{nk} such that each element in a POVM $\{E_n\}_n$ is given by $E_n = \sum_k A^\dagger_{nk} A_{nk}$. As $A^\dagger A$ is a positive operator for any operator A [17, exercise 2.25], the construction Brun suggest will be valid as long as $\sum_n \sum_k A^\dagger_{nk} A_{nk} = \mathbb{1}$. However, it is not obvious how a measurement should (mathematically) be done (at least not in the context given in the literature [17][20][22][11]. On the other hand, we do know that any positive operator can be written uniquely as the square of another positive operator, $E_n = A_n A_n$ where $A_n = \sqrt{E_n}$ is positive [16, theorem 2.2.1]. Following the book by Nielsen and Chuang [17, section 2.2.6], the set $\{A_n\}$ describes a measurement with POVM $\{E_n\}$, and the state after measurement will be

$$|\psi\rangle_n = \frac{A_n |\psi\rangle}{\sqrt{\langle\psi|E_n|\psi\rangle}}.$$

For the POVM in our example, it is easy to check that

$$A_1 = |0\rangle\langle 0| + \sqrt{1-\epsilon}\,|1\rangle\langle 1|, \quad A_2 = \sqrt{\epsilon}\,|1\rangle\langle 1|,$$

are positive and the square root of E_1 and E_2, respectively, and hence the state after measurement will be either

$$|\psi\rangle_1 = \frac{A_1|\psi\rangle}{\sqrt{\langle\psi|E_1|\psi\rangle}} = \frac{\alpha|0\rangle + \sqrt{1-\epsilon}\beta|1\rangle}{\sqrt{1-\epsilon|\beta|^2}} \quad \text{or}$$
$$|\psi\rangle_2 = \frac{A_2|\psi\rangle}{\sqrt{\langle\psi|E_2|\psi\rangle}} = \frac{\sqrt{\epsilon}\beta|1\rangle}{\sqrt{\epsilon}|\beta|} \simeq |1\rangle \quad \text{(up to a phase factor)}$$

with probability $p_1 = 1-\epsilon|\beta|^2$ and $p_2 = \epsilon|\beta|^2$, respectively. On average the state changes only slightly, but every so often, we expect a large jump to the state $|1\rangle$.

The second example is the POVM $\{E'_1, E'_2\}$ where

$$E'_1 = \frac{1+\epsilon}{2}|0\rangle\langle 0| + \frac{1-\epsilon}{2}|1\rangle\langle 1|, \quad E'_2 = \frac{1-\epsilon}{2}|0\rangle\langle 0| + \frac{1+\epsilon}{2}|1\rangle\langle 1|$$

with square roots

$$A'_1 = \sqrt{E'_1} = \sqrt{\frac{1+\epsilon}{2}}|0\rangle\langle 0| + \sqrt{\frac{1-\epsilon}{2}}|1\rangle\langle 1|, \quad A'_2 = \sqrt{E'_1} = \sqrt{\frac{1-\epsilon}{2}}|0\rangle\langle 0| + \sqrt{\frac{1+\epsilon}{2}}|1\rangle\langle 1|$$

and $\epsilon \ll 1$. It is again easy to see that A'_1 and A'_2 are positive and the square root of E'_1 and E'_2, respectively. We again let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ be a general state of a TLS and compute the probabilities

$$p'_1 = \langle\psi|E'_1|\psi\rangle = \frac{1+\epsilon}{2}|\alpha|^2 + \frac{1-\epsilon}{2}|\beta|^2 = \frac{1+\epsilon(|\alpha|^2-|\beta|^2)}{2},$$
$$p'_2 = \langle\psi|E'_2|\psi\rangle = \frac{1-\epsilon}{2}|\alpha|^2 + \frac{1+\epsilon}{2}|\beta|^2 = \frac{1+\epsilon(|\beta|^2-|\alpha|^2)}{2}.$$

The Shannon entropy is

$$
\begin{aligned}
S'_{\mathrm{POVM}} =& -\frac{1+\epsilon(|\alpha|^2-|\beta|^2)}{2}\overbrace{\log_2\frac{1+\epsilon(|\alpha|^2-|\beta|^2)}{2}}^{\log_2(1+\epsilon(|\alpha|^2-|\beta|^2))-1}-\frac{1+\epsilon(|\beta|^2-|\alpha|^2)}{2}\overbrace{\log_2\frac{1+\epsilon(|\beta|^2-|\alpha|^2)}{2}}^{\log_2(1+\epsilon(|\beta|^2-|\alpha|^2))-1}\\
=& -\frac{1+\epsilon(|\alpha|^2-|\beta|^2)}{2}\log_2(1+\epsilon(|\alpha|^2-|\beta|^2))\\
& -\frac{1+\epsilon(|\beta|^2-|\alpha|^2)}{2}\log_2(1+\epsilon(|\beta|^2-|\alpha|^2))+\frac{\overbrace{1+\epsilon(|\alpha|^2-|\beta|^2)+1+\epsilon(|\beta|^2-|\alpha|^2)}^{=1}}{2}\\
=&1-\frac{1}{2}\log_2\left[\overbrace{(1+\epsilon(|\alpha|^2-|\beta|^2))(1+\epsilon(|\beta|^2-|\alpha|^2))}^{1-\epsilon^2(|\alpha|^2-|\beta|^2)^2}\right]\\
& -\frac{\epsilon(|\alpha|^2-|\beta|^2)}{2}\log_2\frac{1+\epsilon(|\alpha|^2-|\beta|^2)}{1+\epsilon(|\beta|^2-|\alpha|^2}\\
=&1-\frac{1}{2}\log_2(1-\epsilon^2(2|\alpha|^2-1)^2)-\frac{\epsilon(2|\alpha|^2-1)}{2}\log_2\frac{1+\epsilon(2|\alpha|^2-1)}{1-\epsilon(2|\alpha|^2-1)},
\end{aligned}
$$

where we have used that $|\alpha|^2+|\beta|^2=1$. We calculate the first and second derivative,

$$
\begin{aligned}
\frac{dS'_{\mathrm{POVM}}}{d|\alpha|^2} =& -\frac{1}{2\ln 2}\frac{1}{1-\epsilon^2(2|\alpha|^2-1)^2}\left[-2\epsilon^2(2|\alpha|^2-1)2\right]\\
& -\epsilon\log_2\frac{1+\epsilon(2|\alpha|^2-1)}{1-\epsilon(2|\alpha|^2-1)}-\frac{\epsilon(2|\alpha|^2-1)}{2\ln 2}\left[\frac{2\epsilon}{1+\epsilon(2|\alpha|^2-1)}-\frac{-2\epsilon}{1-\epsilon(2|\alpha|^2-1)}\right]\\
=&\frac{2\epsilon^2}{\ln 2}\frac{(2|\alpha|^2-1)}{1-\epsilon^2(2|\alpha|^2-1)^2}-\epsilon\log_2\frac{1+\epsilon(2|\alpha|^2-1)}{1-\epsilon(2|\alpha|^2-1)}\\
& -\frac{\epsilon^2}{\ln 2}\frac{(2|\alpha|^2-1)}{1-\epsilon^2(2|\alpha|^2-1)^2}\left[1-\epsilon(2|\alpha|^2-1)+1+\epsilon(2|\alpha|^2-1)\right]\\
=& -\epsilon\log_2\frac{1+\epsilon(2|\alpha|^2-1)}{1-\epsilon(2|\alpha|^2-1)}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d^2S'_{\mathrm{POVM}}}{d(|\alpha|^2)^2} &= -\frac{\epsilon}{\ln 2}\left[\frac{2\epsilon}{1+\epsilon(2|\alpha|^2-1)}-\frac{-2\epsilon}{1-\epsilon(2|\alpha|^2-1)}\right]\\
&= -\frac{2\epsilon^2}{\ln 2}\left[\frac{1-\epsilon(2|\alpha|^2-1)+1+\epsilon(2|\alpha|^2-1)}{1-\epsilon^2(2|\alpha|^2-1)^2}\right]\\
&= -\frac{4\epsilon^2}{\ln 2}\frac{1}{1-\epsilon^2(2|\alpha|^2-1)^2}.
\end{aligned}
$$

As $0\leq|\alpha|^2\leq 1$ we immediately see that S'_{POVM} is symmetric about $|\alpha|^2=\frac{1}{2}$ and $\frac{d^2S'_{\mathrm{POVM}}}{d(|\alpha|^2)^2}<0$ for all values of $|\alpha|^2$, so the Shannon entropy is a concave function. Next, we find that

$$
\begin{aligned}
\frac{dS'_{\text{POVM}}}{d|\alpha|^2} &= 0 \\
\frac{1+\epsilon(2|\alpha|^2-1)}{1-\epsilon(2|\alpha|^2-1)} &= 1 \\
\epsilon(2|\alpha|^2-1) &= -\epsilon(2|\alpha|^2-1) \\
4|\alpha|^2 &= 2,
\end{aligned}
$$

so $|\alpha|^2 = \frac{1}{2}$ is a maximum (since the Shannon entropy is concave). We calculate

$$
\begin{aligned}
&S'_{\text{POVM}}\left(|\alpha|^2 = \frac{1}{2}\right) = 1 \quad \text{and} \\
&S'_{\text{POVM}}\left(|\alpha|^2 = 1\right) = S'_{\text{POVM}}\left(|\alpha|^2 = 0\right) = 1 - \frac{1}{2}\log_2(1-\epsilon) - \frac{\epsilon}{2}\log_2\frac{1+\epsilon}{1-\epsilon}.
\end{aligned}
$$

Since $\epsilon \ll 1$ we see that the entropy is always very close to 1.

The state after this measurement will be either

$$
\begin{aligned}
|\psi'\rangle_1 &= \frac{A'_1|\psi\rangle}{\sqrt{\langle\psi|E'_1|\psi\rangle}} = \frac{\sqrt{\frac{1+\epsilon}{2}}\alpha|0\rangle + \sqrt{\frac{1-\epsilon}{2}}\beta|1\rangle}{\sqrt{\frac{1+\epsilon(|\alpha|^2-|\beta|^2)}{2}}} = \frac{\alpha\sqrt{1+\epsilon}|0\rangle + \beta\sqrt{1-\epsilon}|1\rangle}{\sqrt{1+\epsilon(|\alpha|^2-|\beta|^2)}} \quad \text{or} \\
|\psi'\rangle_2 &= \frac{A'_2|\psi\rangle}{\sqrt{\langle\psi|E'_2|\psi\rangle}} = \frac{\sqrt{\frac{1-\epsilon}{2}}\alpha|0\rangle + \sqrt{\frac{1+\epsilon}{2}}\beta|1\rangle}{\sqrt{\frac{1+\epsilon(|\beta|^2-|\alpha|^2)}{2}}} = \frac{\alpha\sqrt{1-\epsilon}|0\rangle + \beta\sqrt{1+\epsilon}|1\rangle}{\sqrt{1+\epsilon(|\beta|^2-|\alpha|^2)}}.
\end{aligned}
$$

Both these POVMs are considered by Brun as being weak. One of them only changes the state slightly on average, but has a tendency to do large jumps with a very small probability. The other one always changes the state, but never in large jumps.

Appendix B

Different ways of recovering the Lindblad equation

The Taylor series, around the point $t = 0$, of a time dependent operator $A(t)$ is given by

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{d^k A(t)}{dt^k}|_{t=0}.$$

In our case, we have $U(\theta) = \exp(-i\theta H)$, and hence the Taylor series will be

$$\sum_{k=0}^{\infty} \frac{(-iH)^k}{k!} \theta^k.$$

Up to second order we get $U(\theta) = \mathbb{1} - iH\theta - \frac{H^2}{2}\theta^2 + \mathcal{O}(\theta^3)$.

B.1 Time evolution with sum of two operators

General approach

Let us begin completely general and let the time evolution be given by $U(\theta) = e^{-i\theta(H_1+H_2)}$, where H_1 and H_2 are just two operators. We will later let them be hermitian, but for now, they need not be. The system is in a state $|\psi\rangle$ and coupled with an environment in state $|E\rangle$. The coupled state is denoted by $|\Psi\rangle = |\psi\rangle \otimes |E\rangle$. Then we have

$$\begin{aligned} U(\theta)|\Psi\rangle\langle\Psi|U(\theta)^\dagger = &|\Psi\rangle\langle\Psi| - i\theta\left[(H_1+H_2)|\Psi\rangle\langle\Psi| - |\Psi\rangle\langle\Psi|(H_1+H_2)^\dagger\right] \\ &+ \frac{\theta^2}{2}\Big[2(H_1+H_2)|\Psi\rangle\langle\Psi|(H_1+H_2)^\dagger \\ &\qquad - (H_1+H_2)^2|\Psi\rangle\langle\Psi| - |\Psi\rangle\langle\Psi|(H_1^\dagger+H_2^\dagger)^2\Big] + \mathcal{O}(\theta^3). \end{aligned} \tag{B.1}$$

Next would be to take the partial trace over the environment. To get an explicit expression, we assume $H_1 = \sum_j A_j \otimes B_j$, $H_2 = \sum_l C_l \otimes D_l$, and let $\{n\}_{n\in I}$ be an orthonormal basis for the environment containing $|E\rangle$. Then the partial trace of the first order term in θ will be

$$\begin{aligned}
&\sum_j \sum_n (A_j |\psi\rangle \langle n |B_j| E\rangle) \langle\psi| \langle E|n\rangle + \sum_l \sum_n (C_l |\psi\rangle \langle n |D_l| E\rangle) \langle\psi| \langle E|n\rangle \\
&- \sum_j \sum_n |\psi\rangle \langle n|E\rangle \left(\langle\psi| A_j^\dagger \left\langle E \middle| B_j^\dagger \middle| n \right\rangle\right) - \sum_l \sum_n |\psi\rangle \langle n|E\rangle \left(\langle\psi| C_l^\dagger \left\langle E \middle| D_l^\dagger \middle| n \right\rangle\right) \\
=& \sum_j A_j |\psi\rangle \langle\psi| \langle E |B_j| E\rangle + \sum_l C_l |\psi\rangle \langle\psi| \langle E |D_l| E\rangle \\
&- \sum_j |\psi\rangle \langle\psi| A_j^\dagger \left\langle E \middle| B_j^\dagger \middle| E \right\rangle - \sum_l |\psi\rangle \langle\psi| C_l^\dagger \left\langle E \middle| D_l^\dagger \middle| E \right\rangle .
\end{aligned}$$

The partial trace of the second order term in θ will be

$$2\,\mathrm{tr}_{\mathrm{env}} \left[(H_1 + H_2) |\Psi\rangle \langle\Psi| (H_1 + H_2)^\dagger\right] - \mathrm{tr}_{\mathrm{env}} \left[(H_1 + H_2)^2 |\Psi\rangle \langle\Psi|\right] - \mathrm{tr}_{\mathrm{env}} \left[|\Psi\rangle \langle\Psi| (H_1^\dagger + H_2^\dagger)^2\right] .$$

The first of these three terms gives

$$\begin{aligned}
&\mathrm{tr}_{\mathrm{env}} \left[(H_1 + H_2) |\Psi\rangle \langle\Psi| (H_1 + H_2)^\dagger\right] \\
&= \mathrm{tr}_{\mathrm{env}} \left[H_1 |\Psi\rangle \langle\Psi| H_1^\dagger + H_1 |\Psi\rangle \langle\Psi| H_2^\dagger + H_2 |\Psi\rangle \langle\Psi| H_1^\dagger + H_2 |\Psi\rangle \langle\Psi| H_2^\dagger\right] \\
&= \sum_{jk} \sum_n (A_j |\psi\rangle \langle n |B_j| E\rangle) \left(\langle\psi| A_j^\dagger \left\langle E \middle| B_j^\dagger \middle| n \right\rangle\right) \\
&\qquad + \sum_{jl} \sum_n (A_j |\psi\rangle \langle n |B_j| E\rangle) \left(\langle\psi| C_l^\dagger \left\langle E \middle| D_l^\dagger \middle| n \right\rangle\right) \\
&\quad + \sum_{lj} \sum_n (C_l |\psi\rangle \langle n |D_l| E\rangle) \left(\langle\psi| A_j^\dagger \left\langle E \middle| B_j^\dagger \middle| n \right\rangle\right) \\
&\qquad + \sum_{lk} \sum_n (C_l |\psi\rangle \langle n |D_l| E\rangle) \left(\langle\psi| C_l^\dagger \left\langle E \middle| D_l^\dagger \middle| n \right\rangle\right) \\
&= \sum_{jk} A_j |\psi\rangle \langle\psi| A_j^\dagger \left\langle E \middle| B_j^\dagger \sum_n |n\rangle \langle n| B_j \middle| E \right\rangle \\
&\qquad + \sum_{jl} A_j |\psi\rangle \langle\psi| C_l^\dagger \left\langle E \middle| D_l^\dagger \sum_n |n\rangle \langle n| B_j \middle| E \right\rangle \\
&\quad + \sum_{lj} C_l |\psi\rangle \langle\psi| A_j^\dagger \left\langle E \middle| B_j^\dagger \sum_n |n\rangle \langle n| D_l \middle| E \right\rangle \\
&\qquad + \sum_{lk} C_l |\psi\rangle \langle\psi| C_l^\dagger \left\langle E \middle| D_l^\dagger \sum_n |n\rangle \langle n| D_l \middle| E \right\rangle \\
&= \sum_{jk} A_j |\psi\rangle \langle\psi| A_j^\dagger \left\langle E \middle| B_j^\dagger B_j \middle| E \right\rangle + \sum_{jl} A_j |\psi\rangle \langle\psi| C_l^\dagger \left\langle E \middle| D_l^\dagger B_j \middle| E \right\rangle \\
&\quad + \sum_{lj} C_l |\psi\rangle \langle\psi| A_j^\dagger \left\langle E \middle| B_j^\dagger D_l \middle| E \right\rangle + \sum_{lk} C_l |\psi\rangle \langle\psi| C_l^\dagger \left\langle E \middle| D_l^\dagger D_l \middle| E \right\rangle .
\end{aligned}$$

The second of these three terms gives

$$\begin{aligned}
&\mathrm{tr}_{\mathrm{env}}\left[(H_1+H_2)^2\left|\Psi\right\rangle\left\langle\Psi\right|\right] \\
&\quad= \mathrm{tr}_{\mathrm{env}}\left[H_1^2\left|\Psi\right\rangle\left\langle\Psi\right| + H_1H_2\left|\Psi\right\rangle\left\langle\Psi\right| + H_2H_1\left|\Psi\right\rangle\left\langle\Psi\right| + H_2^2\left|\Psi\right\rangle\left\langle\Psi\right|\right] \\
&\quad= \sum_{jk} A_jA_k\left|\psi\right\rangle\left\langle\psi\right|\left\langle E\left|B_jB_k\right|E\right\rangle + \sum_{jl} A_jC_l\left|\psi\right\rangle\left\langle\psi\right|\left\langle E\left|B_jD_l\right|E\right\rangle \\
&\qquad+ \sum_{lj} C_lA_j\left|\psi\right\rangle\left\langle\psi\right|\left\langle E\left|D_lB_j\right|E\right\rangle + \sum_{lk} C_lC_k\left|\psi\right\rangle\left\langle\psi\right|\left\langle E\left|D_lD_k\right|E\right\rangle .
\end{aligned}$$

The third and last of the three terms will similarly give

$$\begin{aligned}
&\mathrm{tr}_{\mathrm{env}}\left[\left|\Psi\right\rangle\left\langle\Psi\right|(H_1^\dagger+H_2^\dagger)^2\right] \\
&\quad= \sum_{jk}\left|\psi\right\rangle\left\langle\psi\right| A_j^\dagger A_k^\dagger\left\langle E\left|B_j^\dagger B_k^\dagger\right|E\right\rangle + \sum_{jl}\left|\psi\right\rangle\left\langle\psi\right| A_j^\dagger C_l^\dagger\left\langle E\left|B_j^\dagger D_l^\dagger\right|E\right\rangle \\
&\qquad+ \sum_{lj}\left|\psi\right\rangle\left\langle\psi\right| C_l^\dagger A_j^\dagger\left\langle E\left|D_l^\dagger B_j^\dagger\right|E\right\rangle + \sum_{lk}\left|\psi\right\rangle\left\langle\psi\right| C_l^\dagger C_k^\dagger\left\langle E\left|D_l^\dagger D_k^\dagger\right|E\right\rangle .
\end{aligned}$$

Putting all this into equation (B.1) we see that there is not much information to be gained. If we want to recover the Lindblad equation we need to assume more about the operators A_j, B_j, C_l, D_l, H_1 and H_2. We therefore move on to a much more specific case.

Choosing a basis

Back in chapter 3 on trajectory theory we said we could assume $\sum_j [A_j, \rho_S]\left\langle E|B_j|E\right\rangle = 0$ for $H = \sum_j A_j \otimes B_j$. We will now take a close look at this.

Let $H = \sum_{i,j=1}^3 h_{ij}\sigma_i \otimes \sigma_j$ and define $A_i = \sigma_i$. If we let the environment be in the state $\left|E\right\rangle = \left|0\right\rangle$ and $\sigma_3\left|0\right\rangle = \left|0\right\rangle$, then $\sum_i A_i \left\langle 0\right|\sum_j h_{ij}\sigma_j\left|0\right\rangle = \sum_i A_i h_{i3}$ and we can define $A := \sum_i h_{i3}A_i \otimes \mathbb{1}$. Let us define

$$H' := H - A = \sum_i A_i \otimes \overbrace{\left[h_{i1}\sigma_1 + h_{i2}\sigma_2 - 2h_{i3}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right]}^{=:B_i} .$$

H' is the dynamics we want, so we look at the time evolution with H, $U(\theta) = e^{-i\theta H} = e^{-i\theta(H'+A)}$, to see if we can remove A somehow. If $[H', A] = 0$, then we could split the exponential, but

$$[H', A] = \left[\sum_i A_i \otimes B_i, \sum_j h_{j3}A_j \otimes \mathbb{1}\right] = \sum_{ij} h_{j3}[A_i \otimes B_i, A_j \otimes \mathbb{1}]$$

and

$$\begin{aligned}
[A_i \otimes B_i, A_j \otimes \mathbb{1}] &= (A_i \otimes B_i)(A_j \otimes \mathbb{1}) - (A_j \otimes \mathbb{1})(A_i \otimes B_i) \\
&= A_iA_j \otimes B_i - A_jA_i \otimes B_i = [A_i, A_j] \otimes B_i \\
&= [\sigma_i, \sigma_j] \otimes B_i = 2i\epsilon_{ijk}\sigma_k \otimes B_i.
\end{aligned}$$

This means that

$$
\begin{aligned}
[H', A] &= \sum_{ij} h_{j3} 2i\epsilon_{ijk}\sigma_k \otimes B_i \\
&= 2i\left(h_{13}(\sigma_3 - \sigma_2) \otimes B_1 + h_{23}(\sigma_1 - \sigma_3) \otimes B_2 + h_{33}(\sigma_2 - \sigma_1) \otimes B_3\right) \neq 0.
\end{aligned}
$$

We again denote the system state by $|\psi\rangle$ and the coupled state by $|\Psi\rangle := |\psi\rangle \otimes |E\rangle$. As $U(\theta) = \mathbb{1} - i\theta(H' + A) - \frac{\theta^2}{2}(H' + A)^2 + \mathcal{O}(\theta^3)$ and $(H' + A)^\dagger = H^\dagger = H = H' + A$, we get

$$
\begin{aligned}
U(\theta)\,|\Psi\rangle\langle\Psi|\,U(\theta)^\dagger =& |\Psi\rangle\langle\Psi| - i\theta([H', |\Psi\rangle\langle\Psi|] + [A, |\Psi\rangle\langle\Psi|]) \\
& + \theta^2 (H' + A)\,|\Psi\rangle\langle\Psi|\,(H' + A) - \frac{\theta^2}{2}(H' + A)^2\,|\Psi\rangle\langle\Psi| \\
& - \frac{\theta^2}{2}\,|\Psi\rangle\langle\Psi|\,(H' + A)^2 + \mathcal{O}(\theta^3) \\
=& |\Psi\rangle\langle\Psi| - i\theta([H', |\Psi\rangle\langle\Psi|] + [A, |\Psi\rangle\langle\Psi|]) \\
& + \theta^2 \Big[(H'\,|\Psi\rangle\langle\Psi|\,H' + H'\,|\Psi\rangle\langle\Psi|\,A + A\,|\Psi\rangle\langle\Psi|\,H' + A\,|\Psi\rangle\langle\Psi|\,A) \\
& \quad - \frac{1}{2}\left((H')^2\,|\Psi\rangle\langle\Psi| + H'A\,|\Psi\rangle\langle\Psi| + AH'\,|\Psi\rangle\langle\Psi| + A^2\,|\Psi\rangle\langle\Psi|\right) \\
& \quad - \frac{1}{2}\left(|\Psi\rangle\langle\Psi|\,(H')^2 + |\Psi\rangle\langle\Psi|\,H'A + |\Psi\rangle\langle\Psi|\,AH' + |\Psi\rangle\langle\Psi|\,A^2\right) \Big] \\
& + \mathcal{O}(\theta^3).
\end{aligned}
$$

H' was chosen such that $\mathrm{tr}_{\mathrm{env}}[H', |\Psi\rangle\langle\Psi|] = 0$ and $\mathrm{tr}_{\mathrm{env}}[A, |\Psi\rangle\langle\Psi|] = \sum_j [A_j, |\Psi\rangle\langle\Psi|] h_{j3}$. This was done to ensure that $\sum_j A_j \langle E|B_i|E\rangle = 0$. Moreover, since $\{H', A\}\,|\psi\rangle\langle\psi|$ occurs in the equation above, we calculate

$$
\{H', A\} = \sum_{ij} h_{j3}\{A_i \otimes B_i, A_j \otimes \mathbb{1}\} = \sum_{ij} h_{j3}\{\sigma_i, \sigma_j\} \otimes B_i = \sum_{ij} h_{j3} 2\delta_{ij}\mathbb{1} \otimes B_i = 2\sum_i h_{i3}\mathbb{1} \otimes B_i.
$$

Hence

$$
\begin{aligned}
&\mathrm{tr}_{\mathrm{env}}\left(U(\theta)\,|\Psi\rangle\langle\Psi|\,U(\theta)^\dagger\right) \\
&= \rho_S - i\theta(\overbrace{\mathrm{tr}_{\mathrm{env}}[H', |\Psi\rangle\langle\Psi|]}^{=0} + \mathrm{tr}_{\mathrm{env}}[A, |\Psi\rangle\langle\Psi|]) \\
&\quad + \theta^2 \Big[\mathrm{tr}_{\mathrm{env}}\,(H'\,|\Psi\rangle\langle\Psi|\,H' + H'\,|\Psi\rangle\langle\Psi|\,A + A\,|\Psi\rangle\langle\Psi|\,H' + A\,|\Psi\rangle\langle\Psi|\,A) \\
&\qquad - \frac{1}{2}\,\mathrm{tr}_{\mathrm{env}}\left((H')^2\,|\Psi\rangle\langle\Psi| + H'A\,|\Psi\rangle\langle\Psi| + AH'\,|\Psi\rangle\langle\Psi| + A^2\,|\Psi\rangle\langle\Psi|\right) \\
&\qquad - \frac{1}{2}\,\mathrm{tr}_{\mathrm{env}}\left(|\Psi\rangle\langle\Psi|\,(H')^2 + |\Psi\rangle\langle\Psi|\,H'A + |\Psi\rangle\langle\Psi|\,AH' + |\Psi\rangle\langle\Psi|\,A^2\right) \Big] \\
&\quad + \mathcal{O}(\theta^3).
\end{aligned}
$$

Grouping term only containing A, only containing H', and mixing A and H', we get

$$\begin{aligned}
&\operatorname{tr}_{\text{env}}\left(U(\theta)\left|\Psi\right\rangle\left\langle\Psi\right|U(\theta)^\dagger\right)\\
=&\rho_S - i\theta\operatorname{tr}_{\text{env}}[A,\left|\Psi\right\rangle\left\langle\Psi\right|] + \theta^2\operatorname{tr}_{\text{env}}\left(H'\left|\Psi\right\rangle\left\langle\Psi\right|H' - \frac{1}{2}(H')^2\left|\Psi\right\rangle\left\langle\Psi\right| - \frac{1}{2}\left|\Psi\right\rangle\left\langle\Psi\right|(H')^2\right)\\
&+\theta^2\operatorname{tr}_{\text{env}}\left(A\left|\Psi\right\rangle\left\langle\Psi\right|A - \frac{1}{2}A^2\left|\Psi\right\rangle\left\langle\Psi\right| - \frac{1}{2}\left|\Psi\right\rangle\left\langle\Psi\right|A^2\right)\\
&-\frac{\theta^2}{2}\operatorname{tr}_{\text{env}}\left(\{H',A\}\left|\Psi\right\rangle\left\langle\Psi\right| + \left|\Psi\right\rangle\left\langle\Psi\right|\{H',A\}\right)\\
&+\theta^2\operatorname{tr}_{\text{env}}\left(H'\left|\Psi\right\rangle\left\langle\Psi\right|A + A\left|\Psi\right\rangle\left\langle\Psi\right|H'\right) + \mathcal{O}(\theta^3)\\
=&\rho_S - i\theta\sum_i h_{i3}[A_i,\rho_S] + \theta^2\,\overbrace{\lambda\left[L\rho_S L^\dagger - \frac{1}{2}L^\dagger L\rho_S - \frac{1}{2}\rho_S L^\dagger L\right]}^{\text{Lindblad operator from } H'}\\
&+\theta^2\left[\sum_{ij}h_{i3}A_i\rho_S h_{j3}A_j - \frac{1}{2}\sum_{ij}h_{i3}h_{j3}A_iA_j\rho_S - \frac{1}{2}\rho_S\sum_{ij}h_{i3}h_{j3}A_iA_j\right]\\
&-\theta^2\operatorname{tr}_{\text{env}}\left(\{H',A\}\left|\Psi\right\rangle\left\langle\Psi\right|\right)\\
&+\theta^2\left(\overbrace{\sum_i\left\langle E|B_i|E\right\rangle A_i}^{=0}\rho_S\sum_j h_{j3}A_j + \sum_i h_{i3}A_i\rho_S\overbrace{\sum_j\left\langle E|B_j|E\right\rangle A_j}^{=0}\right) + \mathcal{O}(\theta^3).
\end{aligned}$$

Setting in for the anti-commutator gives

$$\begin{aligned}
\operatorname{tr}_{\text{env}}\left(U(\theta)\left|\Psi\right\rangle\left\langle\Psi\right|U(\theta)^\dagger\right) =&\rho_S - i\theta\sum_i h_{i3}[A_i,\rho_S] + \theta^2\left[L\rho_S L^\dagger - \frac{1}{2}L^\dagger L\rho_S - \frac{1}{2}\rho_S L^\dagger L\right]\\
&+\theta^2\sum_{ij}h_{i3}h_{j3}\left[A_i\rho_S A_j - \frac{1}{2}A_iA_j\rho_S - \frac{1}{2}\rho_S A_iA_j\right]\\
&-2\theta^2\sum_i h_{i3}\left\langle E|B_i|E\right\rangle\rho_S + \mathcal{O}(\theta^3).
\end{aligned}$$

We know $\sum_i \left\langle E|B_i|E\right\rangle A_i = 0$, but since $A_i = \sigma_i$ are linearly independent, we must have $\left\langle E|B_i|E\right\rangle = 0$ for each i. In other words,

$$\begin{aligned}
\rho_S' :=&\operatorname{tr}_{\text{env}}\left(U(\theta)\left|\Psi\right\rangle\left\langle\Psi\right|U(\theta)^\dagger\right)\\
=&\rho_S - i\theta\sum_i h_{i3}[A_i,\rho_S] + \theta^2\left[L\rho_S L^\dagger - \frac{1}{2}L^\dagger L\rho_S - \frac{1}{2}\rho_S L^\dagger L\right]\\
&+\theta^2\sum_{ij}h_{i3}h_{j3}\left[A_i\rho_S A_j - \frac{1}{2}A_iA_j\rho_S - \frac{1}{2}\rho_S A_iA_j\right] + \mathcal{O}(\theta^3).
\end{aligned}$$

The problem is now to handle the two terms containing A, namely $-i\theta\sum_i h_{i3}[A_i,\rho_S]$ and $\theta^2\sum_{ij}h_{i3}h_{j3}\left[A_i\rho_S A_j - \frac{1}{2}A_iA_j\rho_S - \frac{1}{2}\rho_S A_iA_j\right]$. We need $\lim_{\delta t\to 0}\frac{\theta^2}{\delta t} \neq 0, \pm\infty$, and thus $\lim_{\delta t\to 0}\theta \propto \sqrt{\delta t}$. This again means that $-i\frac{\theta}{\delta t}\sum_i h_{i3}[A_i,\rho_S] \xrightarrow{\delta t\to 0} = \pm\infty$ unless either

$\sum_i h_{i3}[A_i, \rho_S] = 0$ or is dependent on the time step δt. This is the same as demanding $h_{i3} = 0$ for all i or letting them be dependent on the time step δt. The first case, $h_{i3} = 0$ for all i, would mean that both problematic terms are zero. We look at what happens if we let them depend on the time step δt. Then we must have $\lim_{\delta t \to 0} \frac{\theta}{\delta t} h_{i3} = 0$, and as $\lim_{\delta t \to 0} \theta \propto \sqrt{\delta t}$, we have to demand $\lim_{\delta t \to 0} h_{i3} \propto (\delta t)^a$ where $a \geq \frac{1}{2}$. If we have $a = \frac{1}{2}$, then the first term stays and the latter disappears. If not, then both disappears.

B.2 Recovering the Lindblad equation for both time steps δt and θ

We now compute the full time evolution of one single step, i.e. time evolution of TLS only, followed by interaction between environment and system. Let $U(\theta) = \exp(-i\theta H_I)$ and $\widetilde{U}(\delta t) = \exp(-i\delta t H_S)$ be the time evolution operator for the interaction and the transient time, respectively. Here, $H_S := H_S \otimes \mathbb{1}$ is the system Hamiltonian and $H_I = \sum_j A_j \otimes B_j$ is the interaction Hamiltonian between system and environment. Let $|\Psi\rangle = |\psi\rangle \otimes |E\rangle$ be the initial state. We assume that we apply the two operators in succession and calculate the reduced density matrix of the system after time translation, $\mathrm{tr}_{\mathrm{env}} \left[U(\theta)\widetilde{U}(\delta t) |\Psi\rangle \langle\Psi| \widetilde{U}(\delta t)^\dagger U(\theta)^\dagger \right]$. To recover the Lindblad equation, we know that we need the second order Taylor expansions of $U(\theta)$. For the time evolution of the system alone, we only need the first order Taylor expansion of $U(\delta t)$. These are given by

$$\begin{aligned} U(\theta) &= \mathbb{1} - i\theta H_I + \frac{\theta^2}{2}(-iH_I)^2 + \mathcal{O}(\theta^3) \\ &= \mathbb{1} - i\theta \sum_j A_j \otimes B_j - \frac{\theta^2}{2} \sum_{ij} A_i A_j \otimes B_i B_j + \mathcal{O}(\theta^3) \end{aligned}$$

and

$$U(\delta t) = \mathbb{1} - i\delta t H_S + \mathcal{O}(\delta t^2).$$

Hence

$$\begin{aligned} U(\theta)U(\delta t) =& \mathbb{1} - i\delta t H_S - i\theta \sum_j A_j \otimes B_j - \theta\delta t \sum_j A_j \otimes B_j H_S \\ &- \frac{\theta^2}{2} \sum_{ij} A_i A_j \otimes B_i B_j + \frac{\theta^2 \delta t}{2} i \sum_{ij} A_i A_j \otimes B_i B_j H_S + \mathcal{O}(\theta^3, \delta t^2) \end{aligned}$$

and

$$
\begin{aligned}
&U(\theta)\widetilde{U}(\delta t)\left|\Psi\right\rangle\left\langle\Psi\right|\widetilde{U}(\delta t)^{\dagger}U(\theta)^{\dagger} \\
&\quad= \left|\Psi\right\rangle\left\langle\Psi\right| + i\delta t\left|\Psi\right\rangle\left\langle\Psi\right| H_S + i\theta\sum_j \left|\Psi\right\rangle\left\langle\Psi\right| A_j^{\dagger}\otimes B_j^{\dagger} - \theta\delta t\sum_j \left|\Psi\right\rangle\left\langle\Psi\right| H_S A_j^{\dagger}\otimes B_j^{\dagger} \\
&\qquad\quad - \frac{\theta^2}{2}\sum_{ij}\left|\Psi\right\rangle\left\langle\Psi\right| A_i^{\dagger}A_j^{\dagger}\otimes B_i^{\dagger}B_j^{\dagger} - \frac{\theta^2\delta t}{2}i\sum_{ij}\left|\Psi\right\rangle\left\langle\Psi\right| H_S A_i^{\dagger}A_j^{\dagger}\otimes B_i^{\dagger}B_j^{\dagger} \\
&\qquad - i\delta t H_S\left|\Psi\right\rangle\left\langle\Psi\right| + \mathcal{O}(1,\delta t^2) + \theta\delta t H_S\left|\Psi\right\rangle\left\langle\Psi\right|\sum_j A_j^{\dagger}\otimes B_j^{\dagger} \\
&\qquad\quad + \mathcal{O}(\theta,\delta t^2) + i\frac{\theta^2\delta t}{2}H_S\left|\Psi\right\rangle\left\langle\Psi\right|\sum_{ij}A_i^{\dagger}A_j^{\dagger}\otimes B_i^{\dagger}B_j^{\dagger} + \mathcal{O}(\theta^2,\delta t^2) \\
&\qquad - i\theta\sum_j A_j\otimes B_j\left|\Psi\right\rangle\left\langle\Psi\right| + \theta\delta t\sum_j A_j\otimes B_j\left|\Psi\right\rangle\left\langle\Psi\right| H_S \\
&\qquad\quad + \theta^2\sum_j A_j\otimes B_j\left|\Psi\right\rangle\left\langle\Psi\right|\sum_j A_j^{\dagger}\otimes B_j^{\dagger} + i\theta^2\delta t\sum_j A_j\otimes B_j\left|\Psi\right\rangle\left\langle\Psi\right|\sum_j H_S A_j^{\dagger}\otimes B_j^{\dagger} \\
&\qquad\quad + \mathcal{O}(\theta^3,1) + \mathcal{O}(\theta^3,\delta t) \\
&\qquad - \theta\delta t\sum_j A_j\otimes B_j H_S\left|\Psi\right\rangle\left\langle\Psi\right| - i\theta(\delta t)^2\sum_j A_j\otimes B_j H_S\left|\Psi\right\rangle\left\langle\Psi\right| H_S \\
&\qquad\quad - i\theta^2\delta t\sum_j A_j\otimes B_j H_S\left|\Psi\right\rangle\left\langle\Psi\right|\sum_j A_j^{\dagger}\otimes B_j^{\dagger} + \mathcal{O}(\theta^2,\delta t^2) + \mathcal{O}(\theta^3,\delta t) + \mathcal{O}(\theta^4,\delta t^2) \\
&\qquad - \frac{\theta^2}{2}\sum_{ij}A_iA_j\otimes B_iB_j\left|\Psi\right\rangle\left\langle\Psi\right| - i\frac{\theta^2\delta t}{2}\sum_{ij}A_iA_j\otimes B_iB_j\left|\Psi\right\rangle\left\langle\Psi\right| H_S \\
&\qquad\quad + \mathcal{O}(\theta^3,1) + \mathcal{O}(\theta^3,\delta t) + \mathcal{O}(\theta^4,1) + \mathcal{O}(\theta^4,\delta t) \\
&\qquad + \frac{\theta^2\delta t}{2}i\sum_{ij}A_iA_j\otimes B_iB_j H_S\left|\Psi\right\rangle\left\langle\Psi\right| + \mathcal{O}(\theta^2,\delta t^2) + \mathcal{O}(\theta^3,\delta t) \\
&\qquad\quad + \mathcal{O}(\theta^3,\delta t^2) + \mathcal{O}(\theta^4,\delta t) + \mathcal{O}(\theta^4,\delta t^2) \\
&\qquad + \mathcal{O}(\theta^3,\delta t^2).
\end{aligned}
$$

This might look long and ugly, but it is nothing more than writing out all terms that are of interest to us. Letting ρ_S' denote the reduced density matrix after the time evolution, and letting $\{n\}_{n\in I}$ be an orthonormal basis for the environment containing $|E\rangle$, we have

$$
\begin{aligned}
\rho_S' =& \operatorname{tr}_{\text{env}}\left(U(\theta)\widetilde{U}(\delta t)\left|\Psi\right\rangle\left\langle\Psi\right|\widetilde{U}(\delta t)^\dagger U(\theta)^\dagger\right)\\
=& \left|\psi\right\rangle\left\langle\psi\right| + i\delta t\left|\psi\right\rangle\left\langle\psi\right| H_S + i\theta\sum_j \left|\psi\right\rangle\left\langle\psi\right| A_j^\dagger \left\langle E|B_j^\dagger|E\right\rangle\\
&- \theta\delta t\sum_j \left|\psi\right\rangle\left\langle\psi\right| H_S A_j^\dagger\left\langle E|B_j^\dagger|E\right\rangle - \frac{\theta^2}{2}\sum_{ij}\left|\psi\right\rangle\left\langle\psi\right| A_i^\dagger A_j^\dagger\left\langle E|B_i^\dagger B_j^\dagger|E\right\rangle\\
&- \frac{\theta^2\delta t}{2} i\sum_{ij}\left|\psi\right\rangle\left\langle\psi\right| H_S A_i^\dagger A_j^\dagger\left\langle E|B_i^\dagger B_j^\dagger|E\right\rangle\\
&- i\delta t H_S\left|\psi\right\rangle\left\langle\psi\right| + \theta\delta t H_S\left|\psi\right\rangle\left\langle\psi\right|\sum_j A_j^\dagger\left\langle E|B_j^\dagger|E\right\rangle\\
&+ i\frac{\theta^2\delta t}{2} H_S\left|\psi\right\rangle\left\langle\psi\right|\sum_{ij} A_i^\dagger A_j^\dagger\left\langle E|B_i^\dagger B_j^\dagger|E\right\rangle\\
&- i\theta\sum_j\left\langle E|B_j|E\right\rangle A_j\left|\psi\right\rangle\left\langle\psi\right| + \theta\delta t\sum_j\left\langle E|B_j|E\right\rangle A_j\left|\psi\right\rangle\left\langle\psi\right| H_S\\
&+ \theta^2\sum_{i,j}\sum_n\left\langle n|B_i|E\right\rangle A_i\left|\psi\right\rangle\left\langle\psi\right| A_j^\dagger\left\langle E|B_j^\dagger|n\right\rangle\\
&+ i\theta^2\delta t\sum_{i,j}\sum_n\left\langle n|B_i|E\right\rangle A_i\left|\psi\right\rangle\left\langle\psi\right| H_S A_j^\dagger\left\langle E|B_j^\dagger|n\right\rangle\\
&- \theta\delta t\sum_j\left\langle E|B_j|E\right\rangle A_j H_S\left|\psi\right\rangle\left\langle\psi\right| - i\theta(\delta t)^2\sum_j\left\langle E|B_j|E\right\rangle A_j H_S\left|\psi\right\rangle\left\langle\psi\right| H_S\\
&- i\theta^2\delta t\sum_{i,j}\sum_n\left\langle n|B_i|E\right\rangle A_i H_S\left|\psi\right\rangle\left\langle\psi\right| A_j^\dagger\left\langle E|B_j^\dagger|n\right\rangle\\
&- \frac{\theta^2}{2}\sum_{i,j}\left\langle E|B_iB_j|E\right\rangle A_iA_j\left|\psi\right\rangle\left\langle\psi\right| - i\frac{\theta^2\delta t}{2}\sum_{i,j}\left\langle E|B_iB_j|E\right\rangle A_iA_j\left|\psi\right\rangle\left\langle\psi\right| H_S\\
&+ \frac{\theta^2\delta t}{2} i\sum_{i,j}\left\langle E|B_iB_j|E\right\rangle A_iA_j H_S\left|\psi\right\rangle\left\langle\psi\right|\\
&+ \mathcal{O}(1,\delta t^2) + \mathcal{O}(\theta,\delta t^2) + \mathcal{O}(\theta^2,\delta t^2) + \mathcal{O}(\theta^3,\delta t^2) + \mathcal{O}(\theta^4,\delta t^2)\\
&+ \mathcal{O}(\theta^3,1) + \mathcal{O}(\theta^3,\delta t) + \mathcal{O}(\theta^4,\delta t) + \mathcal{O}(\theta^4,1).
\end{aligned}
$$

Up to this point there has not been many assumptions. However, to not get in trouble like in the previous section, we assume $\sum_j A_j\left\langle E|B_j|E\right\rangle = 0$. We will again use the fact that

$$
\sum_n\left\langle n|B_i|E\right\rangle\left\langle E|B_j|n\right\rangle = \sum_n\left\langle E|B_j|n\right\rangle\left\langle n|B_i|E\right\rangle = \left\langle E\right| B_j\sum_n\left|n\right\rangle\left\langle n\right| B_i\left|E\right\rangle = \left\langle E|B_jB_i|E\right\rangle.
$$

Finally, as we noted in section 3.4, we must have $H_I^\dagger = H_I$. We can therefore mix adjoints to recover what we want. Then, excluding the terms $\mathcal{O}(\theta^m, \delta t^n)$,

$$
\begin{aligned}
\rho_S' = &\left|\psi\right\rangle\left\langle\psi\right| + i\delta t \left|\psi\right\rangle\left\langle\psi\right| H_S + 0 - 0 - \frac{\theta^2}{2}\sum_{ij}\left|\psi\right\rangle\left\langle\psi\right| A_i A_j \left\langle E|B_i B_j|E\right\rangle \\
&\quad - \frac{\theta^2\delta t}{2} i \sum_{ij}\left|\psi\right\rangle\left\langle\psi\right| H_S A_i^\dagger A_j \left\langle E|B_i^\dagger B_j|E\right\rangle \\
&- i\delta t H_S \left|\psi\right\rangle\left\langle\psi\right| + 0 + i\frac{\theta^2\delta t}{2} H_S \left|\psi\right\rangle\left\langle\psi\right| \sum_{ij} A_i A_j \left\langle E|B_i B_j|E\right\rangle \\
&- 0 + 0 + \theta^2 \sum_{i,j}\left\langle E|B_j^\dagger B_i|E\right\rangle A_i \left|\psi\right\rangle\left\langle\psi\right| A_j^\dagger \\
&\quad + i\theta^2\delta t \sum_{i,j}\left\langle E|B_j B_i|E\right\rangle A_i \left|\psi\right\rangle\left\langle\psi\right| H_S A_j \\
&- 0 - 0 - i\theta^2\delta t \sum_{i,j}\left\langle E|B_j B_i|E\right\rangle A_i H_S \left|\psi\right\rangle\left\langle\psi\right| A_j \\
&- \frac{\theta^2}{2}\sum_{i,j}\left\langle E|B_i^\dagger B_j|E\right\rangle A_i^\dagger A_j \left|\psi\right\rangle\left\langle\psi\right| - i\frac{\theta^2\delta t}{2}\sum_{i,j}\left\langle E|B_i B_j|E\right\rangle A_i A_j \left|\psi\right\rangle\left\langle\psi\right| H_S \\
&+ \frac{\theta^2\delta t}{2} i \sum_{i,j}\left\langle E|B_i B_j|E\right\rangle A_i A_j H_S \left|\psi\right\rangle\left\langle\psi\right| .
\end{aligned}
$$

We will eventually divide by δt and take the limit $\delta t \to 0$. Since we want $\lim_{\delta t\to 0}\frac{\theta^2}{\delta t}$ to be equal to some non-zero constant, all the terms proportional to $\theta^2\delta t$ will go to zero. We therefore exclude these from now on. Defining $\rho_S := \left|\psi\right\rangle\left\langle\psi\right|$ and $\left\langle E|B_j^\dagger B_i|E\right\rangle =: M_{ij} = \sum_k \lambda_k \mu_{ki}\mu_{kj}^*$ as we did in section 3.4, with $L_k := \sum_i \mu_{ki} A_i$, we get

$$
\begin{aligned}
\rho_S' &= \rho_S - i\delta t[H_S, \rho_S] - \frac{\theta^2}{2}\sum_{i,j}\left[\rho_S A_i^\dagger A_j M_{ji} - 2A_i\rho_S A_j^\dagger M_{ji} + A_i^\dagger A_j \rho_S M_{ji}\right] \\
&= \rho_S - i\delta t[H_S, \rho_S] - \frac{\theta^2}{2}\sum_k \lambda_k \left[\rho_S L_k^\dagger L_k - 2L_k\rho_S L_k^\dagger + L_k^\dagger L_k \rho_S\right]
\end{aligned}
$$

and

$$
\frac{\rho_S' - \rho_S}{\delta t} = -i[H_S, \rho_S] + \frac{\theta^2}{2\delta t}\sum_k \lambda_k \left[2L_k\rho_S L_k^\dagger - L_k^\dagger L_k\rho_S - \rho_S L_k^\dagger L_k\right].
$$

Taking the limit $\delta t \to 0$ we get the Lindblad equation back:

$$
\frac{d\rho_S}{dt} = -i[H_S, \rho_S] + \frac{1}{2}\sum_k \Gamma_k \left[2L_k\rho_S L_k^\dagger - L_k^\dagger L_k\rho_S - \rho_S L_k^\dagger L_k\right].
$$

B.3 Recovering the Lindblad equation for general case

We are now interested in putting the two previous results together: We saw that we could recover the Lindblad equation for a redefined interaction Hamiltonian H_I' in section B.1, and we saw that the full time translation under only the redefined Hamiltonian would give the full Lindblad equation in section B.2. However, we would like to see that we get the same result if we add the system Hamiltonian in the time evolution $U(\theta)$, and that the result from section B.1 can be put into the framework of section B.2.

Let $U(\theta) = \exp\left(-i\theta(H'_S + H'_I + A)\right)$, where H'_I and A are the same as in section B.1 and H'_S is the system Hamiltonian (possibly modified in such a way that the limit works out). If we define $A' := A + H'_S$, then the calculations in section B.1 gives us

$$\begin{aligned}
&\operatorname{tr}_{\text{env}}\left(U(\theta)\left|\Psi\right\rangle\left\langle\Psi\right|U(\theta)^\dagger\right)\\
&\quad=\rho_S + \theta\operatorname{tr}_{\text{env}}[A', \left|\Psi\right\rangle\left\langle\Psi\right|]\\
&\qquad+\theta^2\operatorname{tr}_{\text{env}}\left(H'_I\left|\Psi\right\rangle\left\langle\Psi\right|H'_I - \frac{1}{2}(H'_I)^2\left|\Psi\right\rangle\left\langle\Psi\right| - \frac{1}{2}\left|\Psi\right\rangle\left\langle\Psi\right|(H'_I)^2\right)\\
&\qquad+\theta^2\operatorname{tr}_{\text{env}}\left(A'\left|\Psi\right\rangle\left\langle\Psi\right|A' - \frac{1}{2}(A')^2\left|\Psi\right\rangle\left\langle\Psi\right| - \frac{1}{2}\left|\Psi\right\rangle\left\langle\Psi\right|(A')^2\right)\\
&\qquad-\frac{\theta^2}{2}\operatorname{tr}_{\text{env}}\left(\{H'_I, A'\}\left|\Psi\right\rangle\left\langle\Psi\right| + \left|\Psi\right\rangle\left\langle\Psi\right|\{H'_I, A'\}\right)\\
&\qquad+\theta^2\operatorname{tr}_{\text{env}}\left(H'_I\left|\Psi\right\rangle\left\langle\Psi\right|A' + A'\left|\Psi\right\rangle\left\langle\Psi\right|H'_I\right) + \mathcal{O}(\theta^3).
\end{aligned}$$

We know that we need to choose h_i3 in such a way that the terms

$$\begin{aligned}
&\operatorname{tr}_{\text{env}}\left(A\left|\Psi\right\rangle\left\langle\Psi\right|A - \frac{1}{2}A^2\left|\Psi\right\rangle\left\langle\Psi\right| - \frac{1}{2}\left|\Psi\right\rangle\left\langle\Psi\right|A^2\right),\\
&\operatorname{tr}_{\text{env}}\left(\{H'_I, A\}\left|\Psi\right\rangle\left\langle\Psi\right| + \left|\Psi\right\rangle\left\langle\Psi\right|\{H'_I, A\}\right),\ \text{and}\\
&\operatorname{tr}_{\text{env}}\left(H'_I\left|\Psi\right\rangle\left\langle\Psi\right|A + A\left|\Psi\right\rangle\left\langle\Psi\right|H'_I\right)
\end{aligned}$$

disappears. We are therefore left with

$$\begin{aligned}
&\operatorname{tr}_{\text{env}}\left(U(\theta)\left|\Psi\right\rangle\left\langle\Psi\right|U(\theta)^\dagger\right)\\
&\quad=\rho_S + \theta\operatorname{tr}_{\text{env}}[H'_S, \left|\Psi\right\rangle\left\langle\Psi\right|] + \theta\operatorname{tr}_{\text{env}}[A, \left|\Psi\right\rangle\left\langle\Psi\right|]\\
&\qquad+\theta^2\operatorname{tr}_{\text{env}}\left(H'_I\left|\Psi\right\rangle\left\langle\Psi\right|H'_I - \frac{1}{2}(H'_I)^2\left|\Psi\right\rangle\left\langle\Psi\right| - \frac{1}{2}\left|\Psi\right\rangle\left\langle\Psi\right|(H'_I)^2\right)\\
&\qquad+\theta^2\operatorname{tr}_{\text{env}}\left(A'\left|\Psi\right\rangle\left\langle\Psi\right|A' - \frac{1}{2}(A')^2\left|\Psi\right\rangle\left\langle\Psi\right| - \frac{1}{2}\left|\Psi\right\rangle\left\langle\Psi\right|(A')^2\right)\\
&\qquad-\frac{\theta^2}{2}\operatorname{tr}_{\text{env}}\left(\{H'_I, H'_S\}\left|\Psi\right\rangle\left\langle\Psi\right| + \left|\Psi\right\rangle\left\langle\Psi\right|\{H'_I, H'_S\}\right)\\
&\qquad+\theta^2\operatorname{tr}_{\text{env}}\left(H'_I\left|\Psi\right\rangle\left\langle\Psi\right|H'_S + H'_S\left|\Psi\right\rangle\left\langle\Psi\right|H'_I\right) + \mathcal{O}(\theta^3).
\end{aligned}$$

Denoting $a := \lim_{\delta t\to 0}\frac{\theta^2}{\delta t}$, and the interaction time by δt_I, we see that choosing $H'_S = \frac{\theta}{a}\delta t_I H_S$ gives

$$\rho'_S = \rho_S + \frac{\theta^2}{a}\delta t_I[H'_S, \rho_S] + \theta[A, \rho_S] + \theta^2\,\overbrace{\lambda\left[L\rho_S L^\dagger - \frac{1}{2}L^\dagger L\rho_S - \frac{1}{2}\rho_S L^\dagger L\right]}^{\text{Lindblad operator from } H'} + \mathcal{O}(\theta^3).$$

This gives us a good indication that we will get back the correct Lindblad equation when we also do the time evolution of $\widetilde{U}(\delta t)$.

We have

$$U(\theta)\widetilde{U}(\delta t)|\Psi\rangle\langle\Psi|\widetilde{U}(\delta t)^\dagger U(\theta)^\dagger = \widetilde{U}(\delta t)|\Psi\rangle\langle\Psi|\widetilde{U}(\delta t)^\dagger - i\delta t H_S\widetilde{U}(\delta t)|\Psi\rangle\langle\Psi|\widetilde{U}(\delta t)^\dagger + i\delta t\widetilde{U}(\delta t)|\Psi\rangle\langle\Psi|\widetilde{U}(\delta t)^\dagger H_S + \mathcal{O}((\delta t)^2).$$

As all terms except ρ_S in $\widetilde{U}(\delta t)|\Psi\rangle\langle\Psi|\widetilde{U}(\delta t)^\dagger$ are $\mathcal{O}(\theta)$, they will all disappear in the limit when multiplied with δt. In other words,

$$\begin{aligned}\rho_S' &= \mathrm{tr}_{\mathrm{env}}\left(U(\theta)\widetilde{U}(\delta t)|\Psi\rangle\langle\Psi|\widetilde{U}(\delta t)^\dagger U(\theta)^\dagger\right)\\ &= \mathrm{tr}_{\mathrm{env}}\left(\widetilde{U}(\delta t)|\Psi\rangle\langle\Psi|\widetilde{U}(\delta t)^\dagger\right) - i\delta t[H_S,\rho_S] + \mathcal{O}(\theta^2\delta t) + \mathcal{O}((\delta t)^2)\\ &= \rho_S - i\left(\frac{\theta^2}{a}\delta t_I + \delta t\right)[H_S',\rho_S] - i\theta[A,\rho_S] + \theta^2\lambda\left[L\rho_S L^\dagger - \frac{1}{2}L^\dagger L\rho_S - \frac{1}{2}\rho_S L^\dagger L\right]\\ &\quad + \mathcal{O}(\theta^3) + \mathcal{O}(\theta^2\delta t) + \mathcal{O}((\delta t)^2).\end{aligned}$$

When the limit is taken over the sum of interaction time and transient time going to zero, we see that we recover the Lindblad equation

$$\frac{d\rho_S}{dt} = -i[H_S,\rho_S] + \frac{1}{2}\sum_k \Gamma_k\left[2L_k\rho_S L_k^\dagger - L_k^\dagger L_k\rho_S - \rho_S L_k^\dagger L_k\right].$$

B.4 Examples showing that the representation of the Hamiltonian does not matter

B.4.1 Same basis, but not "fully reduced"

Say we want a Lindblad equation with the Lindblad operator $L = \sigma_x + i\sigma_y = 2\sigma_+$ with rate $\Gamma = \frac{\theta^2}{\delta t}$. As always, we start with a unitary operator of the form $U(\theta) = e^{-i\theta\sum_j A_j\otimes B_j}$. We know that $L = \sum_j \mu_{kj}A_j$ and $\Gamma = \frac{\theta^2}{\delta t}\lambda_k$, where λ_k and μ_k are the eigenvalues and normalized eigenvectors, respectively, of the matrix $M = \sum_k \lambda_k\mu_k\mu_k^*$, where we define $M_{ij} = \left\langle E|B_j^\dagger B_i|E\right\rangle$. Furthermore, since we have two-level environment we know that there is only one (non-zero) eigenvalue with corresponding eigenvector. Choosing $A_1 = B_1 = \sigma_x$ and $A_2 = B_2 = \sigma_y$, we calculate

$$\langle 0|\sigma_x\sigma_x|0\rangle = \langle 0|\mathbb{1}|0\rangle = 1, \qquad \langle 0|\sigma_y\sigma_x|0\rangle = \langle 0|-i\sigma_z|0\rangle = -i,$$
$$\langle 0|\sigma_x\sigma_y|0\rangle = \langle 0|i\sigma_z|0\rangle = i, \qquad \langle 0|\sigma_y\sigma_y|0\rangle = \langle 0|\mathbb{1}|0\rangle = 1.$$

Hence

$$M = \begin{pmatrix}1 & -i\\ i & 1\end{pmatrix},$$

which we quickly check has normalized eigenvectors and eigenvalues

$$\mu_1 = \begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix}, \qquad \mu_2 = \begin{pmatrix}1/\sqrt{2}\\ -i/\sqrt{2}\end{pmatrix}, \qquad \lambda_1 = 2, \qquad \lambda_2 = 0$$

respectively. Hence, we only get a single non-zero Lindblad operator

$$L' = \sum_j \mu_{1j} A_j = \frac{1}{\sqrt{2}}(\sigma_x + i\sigma_y)$$

with rate $\Gamma' = \frac{\theta^2 \lambda_1}{\delta t} = 2\frac{\theta^2}{\delta t}$. This gives the Lindblad equation we where after.

We now ask the following: *What if* $H = \sum_j A_j \otimes B_j$ *has another representation,* $H = \sum_{j'} A_{j'} \otimes B_{j'}$*? Will we still get the same Lindblad operator?* Let us look at a "silly" example: Take $A_1 = A_2 = \sigma_x$, $A_3 = \sigma_y$, $B_1 = 2\sigma_x$, $B_2 = -\sigma_x$ and $B_3 = \sigma_y$. Then

$$H = \sum_{j'} A_{j'} \otimes B_{j'} = \sigma_x \otimes (2\sigma_x) + \sigma_x \otimes (-\sigma_x) + \sigma_y \otimes \sigma_y = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y,$$

which is the same Hamiltonian as before, and should therefore give the same dynamic. We calculate

$$\begin{aligned}
&\langle 0|(2\sigma_x)(2\sigma_x)|0\rangle = 4\langle 0|\mathbb{1}|0\rangle = 4, && \langle 0|(-\sigma_x)(2\sigma_x)|0\rangle = -2\langle 0|\mathbb{1}|0\rangle = -2, && \langle 0|\sigma_y(2\sigma_x)|0\rangle = 2\langle 0| - i\sigma_z|0\rangle = -2i,\\
&\langle 0|(2\sigma_x)(-\sigma_x)|0\rangle = -2\langle 0|\mathbb{1}|0\rangle = -2, && \langle 0|(-\sigma_x)(-\sigma_x)|0\rangle = \langle 0|\mathbb{1}|0\rangle = 1, && \langle 0|\sigma_y(-\sigma_x)|0\rangle = -\langle 0| - i\sigma_z|0\rangle = i,\\
&\langle 0|(2\sigma_x)\sigma_y|0\rangle = 2\langle 0|i\sigma_z|0\rangle = 2i, && \langle 0|(-\sigma_x)\sigma_y|0\rangle = -\langle 0|i\sigma_z|0\rangle = -i, && \langle 0|\sigma_y\sigma_y|0\rangle = \langle 0|\mathbb{1}|0\rangle = 1.
\end{aligned}$$

Hence

$$M = \begin{pmatrix} 4 & -2 & -2i \\ -2 & 1 & i \\ 2i & -i & 1 \end{pmatrix},$$

which we quickly check has eigenvectors and eigenvalues

$$\mu_1 = \begin{pmatrix} -2/\sqrt{6} \\ 1//\sqrt{6} \\ -i/\sqrt{6} \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -i/\sqrt{3} \end{pmatrix}, \quad \lambda_1 = 6, \quad \lambda_2 = 0, \quad \lambda_3 = 0$$

respectively. Again, we only get a single non-zero Lindblad operator

$$L'' = \sum_{j'} \mu_{1j'} A_{j'} = \frac{1}{\sqrt{6}}(-2\sigma_x + \sigma_x - i\sigma_y) = \frac{1}{\sqrt{6}}(-\sigma_x - i\sigma_y),$$

with rate $\Gamma'' = \frac{\theta^2 \lambda_1}{\delta t} = 6\frac{\theta^2}{\delta t}$. Since the Lindblad equation does not care about the phase factor of the Lindblad operator, we are again left with the same Lindblad equation.

B.4.2 Different basis

As above, we take the same Hamiltonian, representation and Lindblad operator, and ask the same question. That is $L = \sigma_x + i\sigma_y = 2\sigma_+$, $\Gamma = \frac{\theta^2}{\delta t}$, $H = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y$ and the question "*What if* $H = \sum_j A_j \otimes B_j$ *has another representation,* $H = \sum_{j'} A_{j'} \otimes B_{j'}$*? Will we still get the same Lindblad operator?*"

Define $S_1 := \sigma_y + \sigma_z$ and $S_2 := \sigma_y - \sigma_z$. Then $\{\mathbb{1}, \sigma_x, S_1, S_2\}$ is a basis of hermitian operators (since $S_i^\dagger = \sigma_y^\dagger \pm \sigma_z^\dagger = \sigma_y \pm \sigma_z = S_i$) and

$$H = \sigma_x \otimes \sigma_x + \frac{1}{2}(S_1 + S_2) \otimes \frac{1}{2}(S_1 + S_2) = \sigma_x \otimes \sigma_x + \frac{1}{4}(S_1 \otimes S_1 + S_2 \otimes S_1 + S_1 \otimes S_2 + S_2 \otimes S_2)$$

gives the Hamiltonian expressed in this new basis. Let us choose

$$
\begin{aligned}
&A_1 = \sigma_x, \qquad && A_2 = A_4 = S_1/4, \qquad && A_3 = A_5 = S_2/4, \\
&B_1 = \sigma_x, && B_2 = B_3 = S_1, && B_4 = B_5 = S_2.
\end{aligned}
$$

As $\sigma_i \sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k$ and $[\sigma_i \sigma_j] = 2i\epsilon_{ijk}\sigma_k$, we get

$$
\begin{aligned}
&S_1 S_2 = (\sigma_y + \sigma_z)(\sigma_y - \sigma_z) = \sigma_y^2 + [\sigma_z, \sigma_y] - \sigma_z^2 = -2i\sigma_x, \qquad && S_2 S_1 = (S_1 S_2)^\dagger = 2i\sigma_x, \\
&\sigma_x S_1 = \sigma_x(\sigma_y + \sigma_z) = i\sigma_z - i\sigma_y = -iS_2, && S_1 \sigma_x = (\sigma_x S_1)^\dagger = iS_2, \\
&\sigma_x S_2 = \sigma_x(\sigma_y - \sigma_z) = i\sigma_z + i\sigma_y = iS_1, && S_2 \sigma_x = (\sigma_x S_2)^\dagger = -iS_1.
\end{aligned}
$$

Moreover, $S_i |0\rangle = \sigma_y |0\rangle \pm \sigma_z |0\rangle = i|1\rangle \pm |0\rangle$ and $S_i^2 = \sigma_y^2 \pm \{\sigma_y, \sigma_z\} + \sigma_z^2 = 2\mathbb{1}$, since $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$. Hence,

$$
\begin{array}{lllll}
\langle 0|\sigma_x\sigma_x|0\rangle = 1 & \langle 0|S_1\sigma_x|0\rangle = i\,\langle 0|S_2|0\rangle = -i & \langle 0|S_1\sigma_x|0\rangle = -i & \langle 0|S_2\sigma_x|0\rangle = -i\,\langle 0|S_1|0\rangle = -i & \langle 0|S_2\sigma_x|0\rangle = -i \\
\langle 0|\sigma_x S_1|0\rangle = -i\,\langle 0|S_2|0\rangle = i & \langle 0|S_1S_1|0\rangle = 2 & \langle 0|S_1S_1|0\rangle = 2 & \langle 0|S_2S_1|0\rangle = 2i\,\langle 0|\sigma_x|0\rangle = 0 & \langle 0|S_2S_1|0\rangle = 0 \\
\langle 0|\sigma_x S_1|0\rangle = -i\,\langle 0|S_2|0\rangle = i & \langle 0|S_1S_1|0\rangle = 2 & \langle 0|S_1S_1|0\rangle = 2 & \langle 0|S_2S_1|0\rangle = 2i\,\langle 0|\sigma_x|0\rangle = 0 & \langle 0|S_2S_1|0\rangle = 0 \\
\langle 0|\sigma_x S_2|0\rangle = i\,\langle 0|S_1|0\rangle = i & \langle 0|S_1S_2|0\rangle = -2i\,\langle 0|\sigma_x|0\rangle = 0 & \langle 0|S_1S_2|0\rangle = 0 & \langle 0|S_2S_2|0\rangle = 2 & \langle 0|S_2S_2|0\rangle = 2 \\
\langle 0|\sigma_x S_2|0\rangle = i\,\langle 0|S_1|0\rangle = i & \langle 0|S_1S_2|0\rangle = -2i\,\langle 0|\sigma_x|0\rangle = 0 & \langle 0|S_1S_2|0\rangle = 0 & \langle 0|S_2S_2|0\rangle = 2 & \langle 0|S_2S_2|0\rangle = 2
\end{array}
$$

and thus

$$
M = \begin{pmatrix} 1 & -i & -i & -i & -i \\ i & 2 & 2 & 0 & 0 \\ i & 2 & 2 & 0 & 0 \\ i & 0 & 0 & 2 & 2 \\ i & 0 & 0 & 2 & 2 \end{pmatrix}.
$$

We check that M has normalized eigenvectors

$$
\mu_1 = \frac{1}{\sqrt{4}}\begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad
\mu_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -i \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad
\mu_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad
\mu_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad
\mu_5 = \frac{1}{2\sqrt{5}}\begin{pmatrix} 4i \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},
$$

with respective eigenvalues

$$
\lambda_1 = 4, \qquad \lambda_2 = 5, \qquad \lambda_3 = 0, \qquad \lambda_4 = 0, \qquad \lambda_5 = 0.
$$

It would seem like this contradicts the fact that we can only have a single non-zero eigenvalue, and therefore only a single Lindblad operator for a two-level system environment. However, we are left with

$$
L_1 = \sum_j \mu_{1j} A_j = \frac{1}{\sqrt{4}}\left(0 \cdot \sigma_x + \frac{1}{4}(-S_1 - S_2 + S_1 + S_2)\right) = 0
$$

and

$$
L_2 = \sum_j \mu_{2j} A_j = \frac{1}{\sqrt{5}}\left(-i\sigma_x + \frac{1}{4}(S_1 + S_2 + S_1 + S_2)\right) = \frac{1}{\sqrt{5}}(-i\sigma_x + \sigma_y),
$$

with rate $\Gamma_2 = \frac{\theta^2 5}{\delta t}$. This again gives back the same Lindblad equation.

To get only a single non-zero eigenvalue for M, we must have $\langle E|B_j|E\rangle = 0$. In our case, we have

$$\langle 0|\sigma_x|0\rangle = 0 \qquad \langle 0|S_1|0\rangle = 1 \qquad \langle 0|S_2|0\rangle = -1,$$

but $\sum_j A_j \langle E|B_j|E\rangle = (S_1 + S_2 - S_1 - S_2)/4 = 0$. We therefore need to change to

$$\begin{aligned} A_1 &= \sigma_x, & A_2 &= A_4 = S_1/4, & A_3 &= A_5 = S_2/4, \\ B_1' &= \sigma_x, & B_2' &= B_3' = S_1 - \mathbb{1}, & B_4' &= B_5' = S_2 + \mathbb{1}. \end{aligned}$$

We have

$$\begin{aligned} (S_1 - \mathbb{1})(S_2 + \mathbb{1}) &= S_1 S_2 + S_1 - S_2 - \mathbb{1} = -2i\sigma_x + 2\sigma_z - \mathbb{1}, \\ (S_2 + \mathbb{1})(S_1 - \mathbb{1}) &= ((S_1 - \mathbb{1})(S_2 + \mathbb{1}))^\dagger = 2i\sigma_x + 2\sigma_z - \mathbb{1}, \\ (S_1 - \mathbb{1})^2 &= S_1^2 - 2S_1 + \mathbb{1} = 3\mathbb{1} - 2S_1, \\ (S_2 + \mathbb{1})^2 &= S_2^2 + 2S_2 + \mathbb{1} = 3\mathbb{1} + 2S_2, \end{aligned}$$

$$\begin{aligned} \sigma_x(S_1 - \mathbb{1}) &= -iS_2 - \sigma_x, & (S_1 - \mathbb{1})\sigma_x &= (\sigma_x(S_1 - \mathbb{1}))^\dagger = iS_2 - \sigma_x, \\ \sigma_x(S_2 + \mathbb{1}) &= iS_1 + \sigma_x, & (S_2 + \mathbb{1})\sigma_x &= -iS_1 + \sigma_x. \end{aligned}$$

Hence, the missing inner products are

$$\begin{aligned} \langle 0|\sigma_x(S_1 - \mathbb{1})|0\rangle &= \langle 0| - iS_2 - \sigma_x|0\rangle = -i\langle 0|S_2|0\rangle = i, \\ \langle 0|\sigma_x(S_2 + \mathbb{1})|0\rangle &= \langle 0|iS_1 + \sigma_x|0\rangle = i\langle 0|S_1|0\rangle = i, \\ \langle 0|(S_1 - \mathbb{1})(S_2 + \mathbb{1})|0\rangle &= \langle 0| - 2i\sigma_x + 2\sigma_z - \mathbb{1}|0\rangle = \langle 0|2\sigma_z|0\rangle - 1 = 1, \\ \left\langle 0|(S_1 - \mathbb{1})^2|0\right\rangle &= \langle 0|3\mathbb{1} - 2S_1|0\rangle = 3 - 2\langle 0|S_1|0\rangle = 1, \\ \left\langle 0|(S_2 + \mathbb{1})^2|0\right\rangle &= \langle 0|3\mathbb{1} + 2S_2|0\rangle = 3 + 2\langle 0|S - 2|0\rangle - 1 = 1, \end{aligned}$$

and thus

$$M' = \begin{pmatrix} 1 & -i & -i & -i & -i \\ i & 1 & 1 & 1 & 1 \\ i & 1 & 1 & 1 & 1 \\ i & 1 & 1 & 1 & 1 \\ i & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We check that M has normalized eigenvectors

$$\mu_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} -i \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mu_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad \mu_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mu_4 = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mu_5 = \frac{1}{2\sqrt{5}}\begin{pmatrix} 4i \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

with respective eigenvalues

$$\lambda_1 = 5, \qquad \lambda_2 = 0, \qquad \lambda_3 = 0, \qquad \lambda_4 = 0, \qquad \lambda_5 = 0.$$

This shows that there was no contradiction as we get a single Lindblad operator

$$L = \sum_j \mu_{1j} A_j = \frac{1}{\sqrt{5}} \left(-i\sigma_x + \frac{1}{4}(S_1 + S_2 + S_1 + S_2) \right) = \frac{-i}{\sqrt{5}} \left(\sigma_x + i\sigma_y \right)$$

with rate $\Gamma = \frac{\theta^2 \lambda_1}{\delta t} = \frac{\theta^2 5}{\delta t}$.

Appendix C

Master equation

C.1 Time evolution - direct computations

Let us assume that the interaction between the two systems is given by the Hamiltonian $H_{\text{int}}^{\pm} = \frac{1}{4}(\sigma_x \otimes \sigma_x \pm \sigma_y \otimes \sigma_y)$. Then the time evolution is given by aplying the unitary operator

$$U(\theta) = e^{-i\theta H_{\text{int}}^{\pm}} = e^{-\frac{i\theta}{4}(\sigma_x \otimes \sigma_x \pm \sigma_y \otimes \sigma_y)}.$$

As $\sigma_z |0\rangle = -|0\rangle$ and $\sigma_z |1\rangle = |1\rangle$, we get

$$\sigma_x |0\rangle = |1\rangle, \quad \sigma_x |1\rangle = |0\rangle, \quad \sigma_y |0\rangle = -i|1\rangle, \quad \sigma_y |1\rangle = i|0\rangle,$$

so

$$\begin{aligned}
(\sigma_x \otimes \sigma_x \pm \sigma_y \otimes \sigma_y)|00\rangle &= |11\rangle \pm (-i)^2 |11\rangle, \\
(\sigma_x \otimes \sigma_x \pm \sigma_y \otimes \sigma_y)|11\rangle &= |00\rangle \pm i^2 |00\rangle, \\
(\sigma_x \otimes \sigma_x \pm \sigma_y \otimes \sigma_y)|01\rangle &= |10\rangle \pm (-i)i |10\rangle, \\
(\sigma_x \otimes \sigma_x \pm \sigma_y \otimes \sigma_y)|10\rangle &= |01\rangle \pm i(-i) |01\rangle,
\end{aligned}$$

Hence, eigenstates for H_{int}^{+} are $\{|00\rangle, |11\rangle, \frac{|01\rangle+|10\rangle}{\sqrt{2}}, \frac{|01\rangle-|10\rangle}{\sqrt{2}}\}$ with respective eigenvalues $\{0, 0, \frac{1}{2}, -\frac{1}{2}\}$, and eigenstates for H_{int}^{-} are $\{\frac{|11\rangle+|00\rangle}{\sqrt{2}}, \frac{|11\rangle-|00\rangle}{\sqrt{2}}, |01\rangle, |10\rangle\}$ with respective eigenvalues $\{\frac{1}{2}, -\frac{1}{2}, 0, 0\}$. Both sets of eigenstates for orthonormal bases.

To see the time evolution of a state $|\phi\rangle$, we write $|\phi\rangle = \sum_{j=1}^{4} a_j |j\rangle$, where the states $|j\rangle$ refer to the orthonormal eigenstates of either H_{int}^{+} or H_{int}^{-}. Then, by functional calculus [17, section 2.1.8],

$$U_{\pm}(\theta)|\Psi\rangle = e^{-i\theta H_{\text{int}}^{\pm}} |\Psi\rangle = \sum_{i=1}^{4} a_j e^{-i\theta\lambda_j} |j\rangle,$$

where λ_j is the eigenvalue corresponding to $|j\rangle$. Thus,

$$U_{+}(\theta)|\Psi\rangle = a_1|00\rangle + a_2|11\rangle + a_3 e^{-\frac{i\theta}{2}} \frac{|01\rangle + |10\rangle}{\sqrt{2}} + a_4 e^{\frac{i\theta}{2}} \frac{|01\rangle - |10\rangle}{\sqrt{2}}, \tag{C.1a}$$

$$U_{-}(\theta)|\Psi\rangle = a_1 e^{-\frac{i\theta}{2}} \frac{|11\rangle + |00\rangle}{\sqrt{2}} + a_2 e^{\frac{i\theta}{2}} \frac{|11\rangle - |00\rangle}{\sqrt{2}} + a_3|01\rangle + a_4|10\rangle. \tag{C.1b}$$

After interaction we measure the environment with σ_x. We rewrite the time evolved state as

$$\begin{aligned}
U_{\pm}(\theta)\left|\Psi\right\rangle &= c_{00}^{\pm}\left|00\right\rangle + c_{11}^{\pm}\left|11\right\rangle + c_{01}^{\pm}\left|01\right\rangle + c_{10}^{\pm}\left|10\right\rangle \\
&= \left(c_{00}^{\pm}\left|0\right\rangle + c_{10}^{\pm}\left|1\right\rangle\right)\left|0\right\rangle + \left(c_{11}^{\pm}\left|1\right\rangle + c_{01}^{\pm}\left|0\right\rangle\right)\left|1\right\rangle \\
&= \left(c_{00}^{\pm}\left|0\right\rangle + c_{10}^{\pm}\left|1\right\rangle\right)\frac{1}{\sqrt{2}}(\left|\uparrow_x\right\rangle - \left|\downarrow_x\right\rangle) + \left(c_{11}^{\pm}\left|1\right\rangle + c_{01}^{\pm}\left|0\right\rangle\right)\frac{1}{\sqrt{2}}(\left|\uparrow_x\right\rangle + \left|\downarrow_x\right\rangle) \\
&= \frac{1}{\sqrt{2}}\left[\left(c_{00}^{\pm}\left|0\right\rangle + c_{10}^{\pm}\left|1\right\rangle\right) + \left(c_{11}^{\pm}\left|1\right\rangle + c_{01}^{\pm}\left|0\right\rangle\right)\right]\left|\uparrow_x\right\rangle \\
&\quad + \frac{1}{\sqrt{2}}\left[-\left(c_{00}^{\pm}\left|0\right\rangle + c_{10}^{\pm}\left|1\right\rangle\right) + \left(c_{11}^{\pm}\left|1\right\rangle + c_{01}^{\pm}\left|0\right\rangle\right)\right]\left|\downarrow_x\right\rangle) \\
&= \frac{1}{\sqrt{2}}\left[\left(c_{00}^{\pm} + c_{01}^{\pm}\right)\left|0\right\rangle + \left(c_{10}^{\pm} + c_{11}^{\pm}\right)\left|1\right\rangle\right]\left|\uparrow_x\right\rangle \\
&\quad + \frac{1}{\sqrt{2}}\left[\left(-c_{00}^{\pm} + c_{01}^{\pm}\right)\left|0\right\rangle + \left(-c_{10}^{\pm} + c_{11}^{\pm}\right)\left|1\right\rangle\right]\left|\downarrow_x\right\rangle) \\
&=: \left|\psi_{\uparrow_x}^{\pm}\right\rangle\left|\uparrow_x\right\rangle + \left|\psi_{\downarrow_x}^{\pm}\right\rangle\left|\downarrow_x\right\rangle .
\end{aligned}$$

The state we transition to after time evolution and measurement on environment is therefor given by

$$\left|\psi'\right\rangle = \begin{cases}
\frac{1}{\sqrt{\langle\psi_{\uparrow_x}^{+},\psi_{\uparrow_x}^{+}\rangle}}\left|\psi_{\uparrow_x}^{+}\right\rangle, & \text{if interaction } H_{\text{int}}^{+} \text{ and measure } \left|\uparrow_x\right\rangle, \\
\frac{1}{\sqrt{\langle\psi_{\downarrow_x}^{+},\psi_{\downarrow_x}^{+}\rangle}}\left|\psi_{\downarrow_x}^{+}\right\rangle, & \text{if interaction } H_{\text{int}}^{+} \text{ and measure } \left|\downarrow_x\right\rangle, \\
\frac{1}{\sqrt{\langle\psi_{\uparrow_x}^{-},\psi_{\uparrow_x}^{-}\rangle}}\left|\psi_{\uparrow_x}^{-}\right\rangle, & \text{if interaction } H_{\text{int}}^{-} \text{ and measure } \left|\uparrow_x\right\rangle, \\
\frac{1}{\sqrt{\langle\psi_{\downarrow_x}^{-},\psi_{\downarrow_x}^{-}\rangle}}\left|\psi_{\downarrow_x}^{-}\right\rangle, & \text{if interaction } H_{\text{int}}^{-} \text{ and measure } \left|\downarrow_x\right\rangle.
\end{cases} \tag{C.2}$$

C.1.1 What happens if we switch convention

Let us now assume $\sigma_z\left|0\right\rangle = \left|0\right\rangle$ and $\sigma_z\left|1\right\rangle = -\left|1\right\rangle$. Then

$$\sigma_x\left|0\right\rangle = \left|1\right\rangle, \quad \sigma_x\left|1\right\rangle = \left|0\right\rangle, \quad \sigma_y\left|1\right\rangle = -i\left|0\right\rangle, \quad \sigma_y\left|0\right\rangle = i\left|1\right\rangle,$$

so

$$\begin{aligned}
(\sigma_x\otimes\sigma_x \pm \sigma_y\otimes\sigma_y)\left|00\right\rangle &= \left|11\right\rangle \pm i^2\left|11\right\rangle, \\
(\sigma_x\otimes\sigma_x \pm \sigma_y\otimes\sigma_y)\left|11\right\rangle &= \left|00\right\rangle \pm (-i)^2\left|00\right\rangle, \\
(\sigma_x\otimes\sigma_x \pm \sigma_y\otimes\sigma_y)\left|01\right\rangle &= \left|10\right\rangle \pm i(-i)\left|10\right\rangle, \\
(\sigma_x\otimes\sigma_x \pm \sigma_y\otimes\sigma_y)\left|10\right\rangle &= \left|01\right\rangle \pm (-i)i\left|01\right\rangle .
\end{aligned}$$

Hence, the eigenstates are still $\{\left|00\right\rangle, \left|11\right\rangle, \frac{\left|01\right\rangle+\left|10\right\rangle}{\sqrt{2}}, \frac{\left|01\right\rangle-\left|10\right\rangle}{\sqrt{2}}\}$ for the positive case and $\{\frac{\left|11\right\rangle+\left|00\right\rangle}{\sqrt{2}}, \frac{\left|11\right\rangle-\left|00\right\rangle}{\sqrt{2}}, \left|01\right\rangle, \left|10\right\rangle\}$ for the negative case. The eigenvalues are here $\{0, 0, \frac{1}{2}, -\frac{1}{2}\}$ and $\{\frac{1}{2}, -\frac{1}{2}, 0, 0\}$ for H_{int}^{+} and H_{int}^{-}, respectively. This means that the time evolution given by equation (C.1) will be the same in both conventions. However,

$$\begin{aligned}
U_{\pm}(\theta)\left|\Psi\right\rangle &= c_{00}^{\pm}\left|00\right\rangle + c_{11}^{\pm}\left|11\right\rangle + c_{01}^{\pm}\left|01\right\rangle + c_{10}^{\pm}\left|10\right\rangle \\
&= \left(c_{00}^{\pm}\left|0\right\rangle + c_{10}^{\pm}\left|1\right\rangle\right)\left|0\right\rangle + \left(c_{11}^{\pm}\left|1\right\rangle + c_{01}^{\pm}\left|0\right\rangle\right)\left|1\right\rangle \\
&= \left(c_{00}^{\pm}\left|0\right\rangle + c_{10}^{\pm}\left|1\right\rangle\right)\frac{1}{\sqrt{2}}\left(\left|\uparrow_x\right\rangle + \left|\downarrow_x\right\rangle\right) + \left(c_{11}^{\pm}\left|1\right\rangle + c_{01}^{\pm}\left|0\right\rangle\right)\frac{1}{\sqrt{2}}\left(\left|\uparrow_x\right\rangle - \left|\downarrow_x\right\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left[\left(c_{00}^{\pm}\left|0\right\rangle + c_{10}^{\pm}\left|1\right\rangle\right) + \left(c_{11}^{\pm}\left|1\right\rangle + c_{01}^{\pm}\left|0\right\rangle\right)\right]\left|\uparrow_x\right\rangle \\
&\quad + \frac{1}{\sqrt{2}}\left[\left(c_{00}^{\pm}\left|0\right\rangle + c_{10}^{\pm}\left|1\right\rangle\right) - \left(c_{11}^{\pm}\left|1\right\rangle + c_{01}^{\pm}\left|0\right\rangle\right)\right]\left|\downarrow_x\right\rangle) \\
&= \frac{1}{\sqrt{2}}\left[\left(c_{00}^{\pm} + c_{01}^{\pm}\right)\left|0\right\rangle + \left(c_{10}^{\pm} + c_{11}^{\pm}\right)\left|1\right\rangle\right]\left|\uparrow_x\right\rangle \\
&\quad + \frac{1}{\sqrt{2}}\left[\left(c_{00}^{\pm} - c_{01}^{\pm}\right)\left|0\right\rangle + \left(c_{10}^{\pm} - c_{11}^{\pm}\right)\left|1\right\rangle\right]\left|\downarrow_x\right\rangle) \\
&=: \left|\psi_{\uparrow_x}^{\pm}\right\rangle\left|\uparrow_x\right\rangle + \left|\psi_{\downarrow_x}^{\pm}\right\rangle\left|\downarrow_x\right\rangle,
\end{aligned}$$

which shows that the dynamics have slightly changed. Equation (C.2) can be written in the exact same way, but when unpacking $\left|\psi_{\uparrow_x}^{\pm}\right\rangle$ and $\left|\psi_{\downarrow_x}^{\pm}\right\rangle$ one needs to specify which convention is used.

C.1.2 First step for actual system

The choice of $H_{\text{int}}^{\pm}$ relies on the assumption that the environment bit is $\left|E\right\rangle = \left|0\right\rangle$. Let the system be in the state $\left|\psi\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle$. Then

$$\left|\Psi\right\rangle = \left|\psi\right\rangle\left|E\right\rangle = \alpha\left|00\right\rangle + 0\left|11\right\rangle + 0\left|01\right\rangle + \beta\left|10\right\rangle.$$

Following the equations for time evolution (C.1), this means

$$a_1 = \alpha, \quad a_2 = 0, \quad a_3 = \beta/\sqrt{2}, \quad a_4 = -\beta/\sqrt{2}$$

for H_{int}^{+} and

$$a_1 = \alpha/\sqrt{2}, \quad a_2 = -\alpha/\sqrt{2}, \quad a_3 = 0, \quad a_4 = \beta$$

for H_{int}^{-}. Thus

$$c_{00}^{+} = a_1 = \alpha, \quad c_{11}^{+} = a_2 = 0, \quad c_{01}^{+} = \frac{a_3e^{-i\theta/2} + a_4e^{i\theta/2}}{\sqrt{2}} = -i\beta\sin\frac{\theta}{2}, \quad c_{10}^{+} = \frac{a_3e^{-i\theta/2} - a_4e^{i\theta/2}}{\sqrt{2}} = \beta\cos\frac{\theta}{2},$$
$$c_{00}^{-} = \frac{a_1e^{-i\theta/2} - a_2e^{i\theta/2}}{\sqrt{2}} = \alpha\cos\frac{\theta}{2}, \quad c_{11}^{-} = \frac{a_1e^{-i\theta/2} + a_2e^{i\theta/2}}{\sqrt{2}} = -i\alpha\sin\frac{\theta}{2}, \quad c_{01}^{-} = a_3 = 0, \quad c_{10}^{-} = a_4 = \beta,$$

which means

$$\begin{aligned}
\left|\psi_{\uparrow_x}^{+}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left(\alpha - i\beta\sin\frac{\theta}{2}\right)\left|0\right\rangle + \left(\beta\cos\frac{\theta}{2}\right)\left|1\right\rangle\right], \\
\left|\psi_{\uparrow_x}^{-}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left(\alpha\cos\frac{\theta}{2}\right)\left|0\right\rangle + \left(\beta - i\alpha\sin\frac{\theta}{2}\right)\left|1\right\rangle\right], \\
\left|\psi_{\downarrow_x}^{+}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left(-\alpha - i\beta\sin\frac{\theta}{2}\right)\left|0\right\rangle + \left(-\beta\cos\frac{\theta}{2}\right)\left|1\right\rangle\right], \\
\left|\psi_{\downarrow_x}^{-}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left(-\alpha\cos\frac{\theta}{2}\right)\left|0\right\rangle + \left(-\beta - i\alpha\sin\frac{\theta}{2}\right)\left|1\right\rangle\right],
\end{aligned}$$

if we use the convention $\sigma_z |0\rangle = -|0\rangle$. If we instead use the convention $\sigma_z |0\rangle = |0\rangle$, then

$$\begin{aligned}
\left|\psi^+_{\uparrow_x}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left(\alpha - i\beta\sin\frac{\theta}{2}\right)|0\rangle + \left(\beta\cos\frac{\theta}{2}\right)|1\rangle\right], \\
\left|\psi^-_{\uparrow_x}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left(\alpha\cos\frac{\theta}{2}\right)|0\rangle + \left(\beta - i\alpha\sin\frac{\theta}{2}\right)|1\rangle\right], \\
\left|\psi^+_{\downarrow_x}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left(\alpha + i\beta\sin\frac{\theta}{2}\right)|0\rangle + \left(\beta\cos\frac{\theta}{2}\right)|1\rangle\right], \\
\left|\psi^-_{\downarrow_x}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left(\alpha\cos\frac{\theta}{2}\right)|0\rangle + \left(\beta + i\alpha\sin\frac{\theta}{2}\right)|1\rangle\right].
\end{aligned}$$

We see that the only difference is that $\left|\psi^\pm_{\downarrow_x}\right\rangle$ picks up a phase factor of $e^{i\pi} = -1$. So the convention does not matter up to a phase factor. We calculate the different probabilities:

$$\begin{aligned}
\left\langle\psi^+_{\uparrow_x}|\psi^+_{\uparrow_x}\right\rangle &= \frac{1}{2}\left(\left|\alpha - i\beta\sin\frac{\theta}{2}\right|^2 + |\beta|^2\cos^2\frac{\theta}{2}\right) \\
&= \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\sin\frac{\theta}{2} + |\beta|^2\cos^2\frac{\theta}{2} - i(\alpha^*\beta - \alpha\beta^*)\sin\frac{\theta}{2}\right) = \frac{1}{2} + \mathrm{Im}(\alpha^*\beta)\sin\frac{\theta}{2}, \\
\left\langle\psi^-_{\uparrow_x}|\psi^-_{\uparrow_x}\right\rangle &= \frac{1}{2}\left(|\alpha|^2\cos^2\frac{\theta}{2} + \left|\beta - i\alpha\sin\frac{\theta}{2}\right|^2\right) \\
&= \frac{1}{2}\left(|\beta|^2 + |\alpha|^2\sin\frac{\theta}{2} + |\alpha|^2\cos^2\frac{\theta}{2} - i(\alpha\beta^* - \alpha^*\beta)\sin\frac{\theta}{2}\right) = \frac{1}{2} + \mathrm{Im}(\alpha\beta^*)\sin\frac{\theta}{2}, \\
\left\langle\psi^+_{\downarrow_x}|\psi^+_{\downarrow_x}\right\rangle &= \frac{1}{2}\left(\left|\alpha + i\beta\sin\frac{\theta}{2}\right|^2 + |\beta|^2\cos^2\frac{\theta}{2}\right) \\
&= \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\sin\frac{\theta}{2} + |\beta|^2\cos^2\frac{\theta}{2} - i(\alpha\beta^* - \alpha^*\beta)\sin\frac{\theta}{2}\right) = \frac{1}{2} + \mathrm{Im}(\alpha\beta^*)\sin\frac{\theta}{2}, \\
\left\langle\psi^-_{\downarrow_x}|\psi^-_{\downarrow_x}\right\rangle &= \frac{1}{2}\left(|\alpha|^2\cos^2\frac{\theta}{2} + \left|\beta + i\alpha\sin\frac{\theta}{2}\right|^2\right) \\
&= \frac{1}{2}\left(|\beta|^2 + |\alpha|^2\sin\frac{\theta}{2} + |\alpha|^2\cos^2\frac{\theta}{2} - i(\alpha^*\beta - \alpha\beta^*)\sin\frac{\theta}{2}\right) = \frac{1}{2} + \mathrm{Im}(\alpha^*\beta)\sin\frac{\theta}{2}.
\end{aligned}$$

We can now update equation (C.2) for the state after measurement,

$$|\psi'\rangle = \begin{cases}
\frac{\left(\alpha - i\beta\sin\frac{\theta}{2}\right)|0\rangle + \left(\beta\cos\frac{\theta}{2}\right)|1\rangle}{\sqrt{1+2\,\mathrm{Im}(\alpha^*\beta)\sin\frac{\theta}{2}}}, & \text{if interaction } H^+_{\text{int}} \text{ and measure } |\uparrow_x\rangle, \\
\frac{\left(\alpha\cos\frac{\theta}{2}\right)|0\rangle + \left(\beta - i\alpha\sin\frac{\theta}{2}\right)|1\rangle}{\sqrt{1+2\,\mathrm{Im}(\alpha\beta^*)\sin\frac{\theta}{2}}}, & \text{if interaction } H^+_{\text{int}} \text{ and measure } |\downarrow_x\rangle, \\
\frac{\left(\alpha + i\beta\sin\frac{\theta}{2}\right)|0\rangle + \left(\beta\cos\frac{\theta}{2}\right)|1\rangle}{\sqrt{1+2\,\mathrm{Im}(\alpha\beta^*)\sin\frac{\theta}{2}}}, & \text{if interaction } H^-_{\text{int}} \text{ and measure } |\uparrow_x\rangle, \\
\frac{\left(\alpha\cos\frac{\theta}{2}\right)|0\rangle + \left(\beta + i\alpha\sin\frac{\theta}{2}\right)|1\rangle}{\sqrt{1+2\,\mathrm{Im}(\alpha^*\beta)\sin\frac{\theta}{2}}}, & \text{if interaction } H^-_{\text{int}} \text{ and measure } |\downarrow_x\rangle.
\end{cases} \tag{C.3}$$

C.1.3 How many states can evolve to a fixed state?

We have seen what the state looks like after measurement. Now we ask ourselves the following question: *If we after* $n+1$ *steps have a system in the state* $|\psi\rangle_{n+1} = \alpha_{n+1}|0\rangle +$

$\beta_{n+1}|1\rangle$, *what are the possible states* $|\psi\rangle_n = \alpha_n|0\rangle + \beta_n|1\rangle$ *that could have evolved to the state* $|\psi\rangle_{n+1}$*?* We know that each step must follow equation (C.3). In other words, state $|\psi\rangle_{n+1}$ comes for the interaction $H_{I^\pm}$ and measurement of either $|{\uparrow_x}\rangle$ or $|{\downarrow_x}\rangle$. We go through the four different possibilities.

Assume $|\psi\rangle_{n+1} = \frac{1}{\sqrt{\langle\psi^+_{\uparrow_x},\psi^+_{\uparrow_x}\rangle}}|\psi^+_{\uparrow_x}\rangle$, where we have used the notation for equation (C.2). If we define $p^+_\uparrow := \langle\psi^+_{\uparrow_x},\psi^+_{\uparrow_x}\rangle$, we get

$$\alpha_{n+1} = \frac{\alpha_n - i\beta_n \sin\frac{\theta}{2}}{\sqrt{2p^+_\uparrow}}, \quad \beta_{n+1} = \frac{\beta_n\cos\frac{\theta}{2}}{\sqrt{2p^+_\uparrow}}.$$

We then solve for α_n and β_n:

$$\beta_n = \frac{\sqrt{2p^+_\uparrow}\beta_{n+1}}{\cos\frac{\theta}{2}}, \quad \alpha_n = \sqrt{2p^+_\uparrow}\alpha_{n+1} + i\beta_n\sin\frac{\theta}{2} = \sqrt{2p^+_\uparrow}\left(\alpha_{n+1} + i\beta_{n+1}\tan\frac{\theta}{2}\right).$$

Moreover,

$$\begin{aligned} 1 = |\alpha_n|^2 + |\beta_n|^2 &= 2p^+_\uparrow \overbrace{\left[\left|\alpha_{n+1} + i\beta_{n+1}\tan\frac{\theta}{2}\right|^2 + \frac{|\beta_{n+1}|^2}{\cos^2\frac{\theta}{2}}\right]}^{=:f(\alpha_{n+1},\beta_{n+1};\theta)} \\ p^+_\uparrow &= \frac{1}{2f(\alpha_{n+1},\beta_{n+1};\theta)}. \end{aligned}$$

We have thus written the probability in terms of α_{n+1} and β_{n+1}. As we only have one choice for the values of α_n and β_n, this shows that we only have one possible choice for $|\psi\rangle_n$.

We can do the exact same computation for the other choices: Assume $|\psi\rangle_{n+1} = \frac{1}{\sqrt{\langle\psi^+_{\downarrow_x},\psi^+_{\downarrow_x}\rangle}}|\psi^+_{\downarrow_x}\rangle$. Defining $p^+_\downarrow := \langle\psi^+_{\downarrow_x},\psi^+_{\downarrow_x}\rangle$, we get

$$\alpha_{n+1} = \frac{\alpha_n + i\beta_n \sin\frac{\theta}{2}}{\sqrt{2p^+_\downarrow}}, \quad \beta_{n+1} = \frac{\beta_n\cos\frac{\theta}{2}}{\sqrt{2p^+_\downarrow}},$$

and thus

$$\beta_n = \frac{\sqrt{2p^+_\downarrow}\beta_{n+1}}{\cos\frac{\theta}{2}}, \quad \alpha_n = \sqrt{2p^+_\downarrow}\alpha_{n+1} - i\beta_n\sin\frac{\theta}{2} = \sqrt{2p^+_\downarrow}\left(\alpha_{n+1} - i\beta_{n+1}\tan\frac{\theta}{2}\right).$$

Moreover,

$$\begin{aligned} 1 = |\alpha_n|^2 + |\beta_n|^2 &= 2p^+_\downarrow \overbrace{\left[\left|\alpha_{n+1} - i\beta_{n+1}\tan\frac{\theta}{2}\right|^2 + \frac{|\beta_{n+1}|^2}{\cos^2\frac{\theta}{2}}\right]}^{=:f(\alpha_{n+1},\beta_{n+1};\theta)} \\ p^+_\downarrow &= \frac{1}{2f(\alpha_{n+1},\beta_{n+1};\theta)}. \end{aligned}$$

Next, assume $|\psi\rangle_{n+1} = \frac{1}{\sqrt{\langle \psi^-_{\uparrow_x}, \psi^-_{\uparrow_x} \rangle}} \left|\psi^-_{\uparrow_x}\right\rangle$. Defining $p^-_\uparrow := \left\langle \psi^-_{\uparrow_x}, \psi^-_{\uparrow_x} \right\rangle$, we get

$$\alpha_{n+1} = \frac{\alpha_n \cos\frac{\theta}{2}}{\sqrt{2p^-_\uparrow}}, \quad \beta_{n+1} = \frac{\beta_n - i\alpha_n \sin\frac{\theta}{2}}{\sqrt{2p^-_\uparrow}},$$

and thus

$$\alpha_n = \frac{\sqrt{2p^-_\uparrow}\alpha_{n+1}}{\cos\frac{\theta}{2}}, \quad \beta_n = \sqrt{2p^-_\uparrow}\beta_{n+1} + i\alpha_n \sin\frac{\theta}{2} = \sqrt{2p^-_\uparrow}\left(\beta_{n+1} + i\alpha_{n+1}\tan\frac{\theta}{2}\right).$$

Moreover,

$$\begin{aligned} 1 = |\alpha_n|^2 + |\beta_n|^2 &= 2p^-_\uparrow \overbrace{\left[\frac{|\alpha_{n+1}|^2}{\cos^2\frac{\theta}{2}} + \left|\beta_{n+1} + i\alpha_{n+1}\tan\frac{\theta}{2}\right|^2\right]}^{=:f(\alpha_{n+1},\beta_{n+1};\theta)} \\ p^-_\uparrow &= \frac{1}{2f(\alpha_{n+1},\beta_{n+1};\theta)}. \end{aligned}$$

Finally, assume $|\psi\rangle_{n+1} = \frac{1}{\sqrt{\langle \psi^-_{\downarrow_x}, \psi^-_{\downarrow_x} \rangle}} \left|\psi^-_{\downarrow_x}\right\rangle$. Defining $p^-_\downarrow := \left\langle \psi^-_{\downarrow_x}, \psi^-_{\downarrow_x} \right\rangle$, we get

$$\alpha_{n+1} = \frac{\alpha_n \cos\frac{\theta}{2}}{\sqrt{2p^-_\downarrow}}, \quad \beta_{n+1} = \frac{\beta_n + i\alpha_n \sin\frac{\theta}{2}}{\sqrt{2p^-_\downarrow}},$$

and thus

$$\alpha_n = \frac{\sqrt{2p^-_\downarrow}\alpha_{n+1}}{\cos\frac{\theta}{2}}, \quad \beta_n = \sqrt{2p^-_\downarrow}\beta_{n+1} - i\alpha_n \sin\frac{\theta}{2} = \sqrt{2p^-_\downarrow}\left(\beta_{n+1} - i\alpha_{n+1}\tan\frac{\theta}{2}\right).$$

Moreover,

$$\begin{aligned} 1 = |\alpha_n|^2 + |\beta_n|^2 &= 2p^-_\downarrow \overbrace{\left[\frac{|\alpha_{n+1}|^2}{\cos^2\frac{\theta}{2}} + \left|\beta_{n+1} - i\alpha_{n+1}\tan\frac{\theta}{2}\right|^2\right]}^{=:f(\alpha_{n+1},\beta_{n+1};\theta)} \\ p^-_\downarrow &= \frac{1}{2f(\alpha_{n+1},\beta_{n+1};\theta)}. \end{aligned}$$

We see that in all cases we have a unique state $|\psi\rangle_n$.

C.2 Diffusion from random walk

Following [3, section 14.2], we give a heuristic argument for why the random walk converges to a diffusion process when time and space are taken infinitesimal.

1D case: Let $(X_n)_{n\in\mathbb{N}}$ be a stochastic process consisting of i.i.d. Bernoulli r.v.'s with parameter p, i.e. $P(X_n = l) = p$ and $P(X_n = -l) = 1 - p =: q$, where $l > 0$ is a distance. Append $X_0 = 0$ and define $S_n := \sum_{i=0}^{n} X_i$. Then

$$\begin{aligned}
E[S_n] &= \sum_{i=0}^{n} E[X_i] = \sum_{i=1}^{n} (lp - lq) = nl(p-q) = nl(2p-1), \\
E[S_n^2] &= E[\sum_{i=0}^{n}\sum_{j=0}^{n} X_i X_j] = E[\sum_{i=1}^{n} X_i^2] + 2\sum_{i<j} E[X_i]E[X_j] = nl^2 + n(n-1)l^2(2p-1)^2, \\
\text{Var}[S_n] &= E[S_n^2] - E[S_n]^2 = 4nl^2pq.
\end{aligned}$$

We define a time $t := n\Delta t$ and a distance $\Delta x = l$. We want $\lim_{\Delta t, \Delta x \to 0} S_n$ to have both first and second moment. Since the time t should be fixed, we see that $n = t/\Delta t$ goes to infinity when Δt goes to zero. Hence, if $\text{Var}[S_n]$ is to converge, we must have $\lim_{\Delta t, \Delta x \to \infty} \frac{(\Delta x)^2}{\Delta t} = 2D$. However, then $\lim_{\Delta t, \Delta x \to \infty} E[S_n] = \lim_{\Delta t, \Delta x \to \infty} \frac{\Delta x}{\Delta t} = \lim_{\Delta t, \Delta x \to \infty} \frac{(\Delta x)^2}{\Delta t} \frac{1}{\Delta x} = \infty$. We must therefore demand $p = a + b\Delta x + \mathcal{O}((\Delta x)^2)$, where $a, b \in \mathbb{R}$. As all higher order terms disappear, we simply assume $p = \frac{1}{2} + \frac{C}{2D}\Delta x$, where $C \in \mathbb{R}$. Then

$$\begin{aligned}
\lim_{\Delta t, \Delta x \to \infty} E[S_n] &= \lim_{\Delta t, \Delta x \to \infty} nl\frac{C}{D}\Delta x = \frac{C}{D}D = C, \\
\lim_{\Delta t, \Delta x \to \infty} \text{Var}[S_n] &= \lim_{\Delta t, \Delta x \to \infty} 4nl^2(\frac{1}{4} - \frac{C^2}{4D^2}(\Delta x)^2) = 2D.
\end{aligned}$$

We can now look at the probability density of being at position x at time t, denoted by $u(x,t)$. We must have

$$u(x, t + \Delta t) = pu(x - \Delta x, t) + qu(x + \Delta x, t).$$

Hence, as $q - p = -\frac{C}{D}\Delta x$,

$$\begin{aligned}
&u(x,t) + \frac{\partial u(x,t)}{\partial t}\Delta t + \mathcal{O}\left((\Delta t)^2\right) \\
&\qquad = p\left[u(x,t) + \frac{\partial u(x,t)}{\partial x}(-\Delta x) + \frac{\partial^2 u(x,t)}{\partial x^2}\frac{(-\Delta x)^2}{2!} + \mathcal{O}\left((-\Delta x)^3\right)\right] \\
&\qquad\quad + q\left[u(x,t) + \frac{\partial u(x,t)}{\partial x}\Delta x + \frac{\partial^2 u(x,t)}{\partial x^2}\frac{(\Delta x)^2}{2!} + \mathcal{O}\left((\Delta x)^3\right)\right] \\
&\qquad = u(x,t) + (q-p)\frac{\partial u(x,t)}{\partial x}\Delta x + \frac{\partial^2 u(x,t)}{\partial x^2}\frac{(\Delta x)^2}{2} + \mathcal{O}\left((\Delta x)^3\right),
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} + \mathcal{O}(\Delta t) &= (q-p)\frac{\partial u(x,t)}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial^2 u(x,t)}{\partial x^2}\frac{(\Delta x)^2}{2\Delta t} + \mathcal{O}\left(\frac{(\Delta x)^3}{\Delta t}\right) \\
\frac{\partial u(x,t)}{\partial t} &= -2C\frac{\partial u(x,t)}{\partial x} + D\frac{\partial^2 u(x,t)}{\partial x^2},
\end{aligned}$$

where we in the last line have taken the limit when $\Delta t, \Delta x \to 0$. We have found the diffusion equation!

3D case: This time we have

$$P(X_n = \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix}) = p_1, P(X_n = \begin{pmatrix} -l \\ 0 \\ 0 \end{pmatrix}) = p_2, \dots, P(X_n = \begin{pmatrix} 0 \\ 0 \\ -l \end{pmatrix}) = p_6.$$

Hence,

$$E[S_n] = \sum_{i=1}^{n} E[X_i] = nl \begin{pmatrix} p_1 - p_2 \\ p_3 - p_4 \\ p_5 - p_6 \end{pmatrix}$$

and

$$\begin{aligned}
E[S_n S_n^T] &= E[\sum_{i,j=0}^{n} X_i X_j^T] = \sum_{i,j=0}^{n} E[X_i X_j^T] = \sum_{i=1}^{n} E[X_i X_i^T] + 2\sum_{i<j} E[X_i]E[X_j]^T \\
&= nE[X_1 X_1^T] + n(n-1)E[X_1]E[X_1]^T \\
&= nE[\begin{pmatrix} X_{1,1}^2 & X_{1,1}X_{1,2} & X_{1,1}X_{1,3} \\ X_{1,2}X_{1,1} & X_{1,2}^2 & X_{1,2}X_{1,3} \\ X_{1,3}X_{1,1} & X_{1,3}X_{1,2} & X_{1,3}^2 \end{pmatrix}] + n(n-1)l^2 \begin{pmatrix} p_1 - p_2 \\ p_3 - p_4 \\ p_5 - p_6 \end{pmatrix} \begin{pmatrix} p_1 - p_2 \\ p_3 - p_4 \\ p_5 - p_6 \end{pmatrix}^T \\
&= nl^2 \begin{pmatrix} p_1 + p_2 & 0 & 0 \\ 0 & p_3 + p_4 & 0 \\ 0 & 0 & p_5 + p_6 \end{pmatrix} + n(n-1)l^2 \begin{pmatrix} p_1 - p_2 \\ p_3 - p_4 \\ p_5 - p_6 \end{pmatrix} \begin{pmatrix} p_1 - p_2 \\ p_3 - p_4 \\ p_5 - p_6 \end{pmatrix}^T .
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{Var}[S_n] &= E[S_n S_n^T] - E[S_n]E[S_n^T] \\
&= nl^2 \begin{pmatrix} p_1 + p_2 & 0 & 0 \\ 0 & p_3 + p_4 & 0 \\ 0 & 0 & p_5 + p_6 \end{pmatrix} - nl^2 \begin{pmatrix} p_1 - p_2 \\ p_3 - p_4 \\ p_5 - p_6 \end{pmatrix} \begin{pmatrix} p_1 - p_2 \\ p_3 - p_4 \\ p_5 - p_6 \end{pmatrix}^T .
\end{aligned}$$

As before, we have

$$\begin{aligned}
u(x,y,z;t+\Delta t) =& p_1 u(x-\Delta x, y, z; t) + p_2 u(x+\Delta x, y, z; t) \\
&+ p_3 u(x, y-\Delta y, z; t) + p_4 u(x, y, z+\Delta z; t) \\
&+ p_5 u(x, y-\Delta y, z; t) + p_6 u(x, y, z+\Delta z; t),
\end{aligned}$$

so

$$
\begin{aligned}
u(\boldsymbol{x};t) + \frac{\partial u}{\partial t}\Delta t + \mathcal{O}((\Delta t)^2) =& p_1 \left(u(\boldsymbol{x};t) - \frac{\partial u}{\partial x}\Delta x + \frac{\partial^2 u}{\partial x^2}\frac{(\Delta x)^2}{2!} + \mathcal{O}((\Delta x)^3) \right) \\
&+ p_2 \left(u(\boldsymbol{x};t) - \frac{\partial u}{\partial x}\Delta x + \frac{\partial^2 u}{\partial x^2}\frac{(\Delta x)^2}{2!} + \mathcal{O}((\Delta x)^3) \right) \\
&+ p_3 \left(u(\boldsymbol{x};t) - \frac{\partial u}{\partial y}\Delta y + \frac{\partial^2 u}{\partial y^2}\frac{(\Delta y)^2}{2!} + \mathcal{O}((\Delta y)^3) \right) \\
&+ p_4 \left(u(\boldsymbol{x};t) - \frac{\partial u}{\partial y}\Delta y + \frac{\partial^2 u}{\partial y^2}\frac{(\Delta y)^2}{2!} + \mathcal{O}((\Delta y)^3) \right) \\
&+ p_5 \left(u(\boldsymbol{x};t) - \frac{\partial u}{\partial z}\Delta z + \frac{\partial^2 u}{\partial z^2}\frac{(\Delta z)^2}{2!} + \mathcal{O}((\Delta z)^3) \right) \\
&+ p_6 \left(u(\boldsymbol{x};t) - \frac{\partial u}{\partial z}\Delta z + \frac{\partial^2 u}{\partial z^2}\frac{(\Delta z)^2}{2!} + \mathcal{O}((\Delta z)^3) \right) \\
\frac{\partial u}{\partial t} + \mathcal{O}(\Delta t) =& \begin{pmatrix} (p_2 - p_1)\frac{\Delta x}{\Delta t} \\ (p_4 - p_3)\frac{\Delta y}{\Delta t} \\ (p_6 - p_5)\frac{\Delta z}{\Delta t} \end{pmatrix} \cdot \nabla u + \begin{pmatrix} \frac{p_1+p_2}{2}\frac{(\Delta x)^2}{\Delta t} \\ \frac{p_3+p_4}{2}\frac{(\Delta y)^2}{\Delta t} \\ \frac{p_5+p_6}{2}\frac{(\Delta z)^2}{\Delta t} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial^2 u}{\partial z^2} \end{pmatrix} \\
&+ \mathcal{O}(\frac{(\Delta x)^3}{\Delta t}) + \mathcal{O}(\frac{(\Delta y)^3}{\Delta t}) + \mathcal{O}(\frac{(\Delta z)^3}{\Delta t}).
\end{aligned}
$$

Defining $2D_i = \lim_{\Delta x_i, \Delta t \to 0} \frac{(\Delta x_i)^2}{\Delta t}$ and $p_1 = \frac{1}{2} + \frac{C_1}{2D_1}\Delta x$, $p_2 = \frac{1}{2} - \frac{C_1}{2D_1}\Delta x$, and so on, we get

$$
\frac{\partial u}{\partial t} = -2 \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} \cdot \nabla u + \begin{pmatrix} (p_1 + p_2)D_1 \\ (p_3 + p_4)D_2 \\ (p_5 + p_6)D_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial^2 u}{\partial z^2} \end{pmatrix}.
$$

Appendix D

Flow of a TLS state on the Bloch sphere

This part of the appendix is dedicated to finding the flow of the state of a TLS for the model proposed in the thesis by Longva [13]. As we want to represent the flow on the Bloch sphere, we need to compute the Bloch vector of a TLS. Recall that

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the computations that follow, we will use the convention $\sigma_z |0\rangle = |0\rangle$.

Given a mixed state $\rho = \frac{1}{2}(\mathbb{1} + \boldsymbol{n} \cdot \boldsymbol{\sigma})$, where $\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x, & \sigma_y, & \sigma_z \end{pmatrix}^T$, we define the vector $\boldsymbol{n}$ as the Bloch vector of the TLS. Observe that

$$\operatorname{tr}(\rho\sigma_j) = \operatorname{tr}\left(\frac{1}{2}(\mathbb{1} + \sum_i n_i \sigma_i)\sigma_j\right) = \frac{1}{2}\operatorname{tr}\left(\sigma_j + \sum_i n_i \sigma_i \sigma_j\right) = \frac{1}{2} 2 n_j = n_j,$$

as $\sigma_i \sigma_j = \mathbb{1}\delta_{ij} + i\epsilon_{ijk}\sigma_k$ and $\operatorname{tr}(\sigma_j) = 0$. Hence, for a pure state $|\psi\rangle\langle\psi|$, the Bloch vector can be computed by

$$\boldsymbol{n} = \operatorname{tr}(|\psi\rangle\langle\psi| \boldsymbol{\sigma}) = \langle\psi|\boldsymbol{\sigma}|\psi\rangle.$$

Moreover, if $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is a pure state such that $\rho = |\psi\rangle\langle\psi| = |\alpha|^2 |0\rangle\langle 0| + \alpha\beta^* |0\rangle\langle 1| + \alpha^*\beta |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|$, then the Bloch vector can be expressed by the following:

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + n_z & n_x - i n_y \\ n_x + i n_y & 1 - n_z \end{pmatrix} = \frac{1}{2}(\mathbb{1} + \boldsymbol{n} \cdot \boldsymbol{\sigma}).$$

Hence

$$\begin{aligned} 1 + n_z &= 2|\alpha|^2 & n_x - i n_y &= 2\alpha\beta^* & n_x + i n_y &= 2\alpha^*\beta, \\ 1 - n_z &= 2|\beta|^2 & n_x &= \alpha\beta^* + \alpha^*\beta & i n_y &= -\alpha\beta^* + \alpha^*\beta, \\ n_z &= 2|\alpha|^2 - |\beta|^2 & n_x &= 2\operatorname{Re}(\alpha\beta^*) & n_y &= 2\operatorname{Im}(\alpha^*\beta). \end{aligned} \tag{D.1}$$

Before we go on to the actual computations, we note that if the state $|\psi\rangle$ is not normalized, then we can choose to either normalize the state before computing the Bloch vector, or first compute a non-normalized Bloch vector before normalizing the Bloch vector. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ be a state which is not normalized, i.e. $|\alpha|^2 + |\beta|^2 \neq 1$. Define $\widetilde{\boldsymbol{n}} := \langle\psi|\boldsymbol{\sigma}|\psi\rangle$. Then

$$\widetilde{n}_x = (\alpha^* \langle 0| + \beta^* \langle 1|)\sigma_x(\alpha|0\rangle + \beta|1\rangle) = (\alpha^* \langle 0| + \beta^* \langle 1|)(\alpha|1\rangle + \beta|0\rangle) = \alpha^*\beta + \alpha\beta^*, \tag{D.2a}$$

$$\widetilde{n}_y = (\alpha^* \langle 0| + \beta^* \langle 1|)\sigma_y(\alpha|0\rangle + \beta|1\rangle) = (\alpha^* \langle 0| + \beta^* \langle 1|)i(\alpha|1\rangle - \beta|0\rangle) = i(-\alpha^*\beta + \alpha\beta^*), \tag{D.2b}$$

$$\widetilde{n}_z = (\alpha^* \langle 0| + \beta^* \langle 1|)\sigma_z(\alpha|0\rangle + \beta|1\rangle) = (\alpha^* \langle 0| + \beta^* \langle 1|)(\alpha|0\rangle - \beta|1\rangle) = |\alpha|^2 - |\beta|^2. \tag{D.2c}$$

As all terms are real, the norm is given by

$$\begin{aligned}
\|\widetilde{\boldsymbol{n}}\|^2 &= (\alpha^*\beta + \alpha\beta^*)^2 - (-\alpha^*\beta + \alpha\beta^*)^2 + (|\alpha|^2 - |\beta|^2)^2 \\
&= (\alpha^*\beta)^2 + 2|\alpha|^2|\beta|^2 + (\alpha\beta^*)^2 - \big((\alpha^*\beta)^2 - 2|\alpha|^2|\beta|^2 + (\alpha\beta^*)^2\big) + |\alpha|^4 - 2|\alpha|^2|\beta|^2 + |\beta|^4 \\
&= 2|\alpha|^2|\beta|^2 + |\alpha|^4 + |\beta|^4 \\
&= (|\alpha|^2 + |\beta|^2)^2.
\end{aligned} \tag{D.3}$$

Since $\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2$, we have

$$\boldsymbol{n} = \frac{\langle\psi|\boldsymbol{\sigma}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\widetilde{\boldsymbol{n}}}{\|\widetilde{\boldsymbol{n}}\|}.$$

In other words, it does not matter if we normalize the state before finding the Bloch vector, or normalize the Bloch vector of the non-normalized state.

D.1 Bloch vector flow for model proposed by Longva

Before we calculate the flow of the Bloch vector for the model proposed by Longva in his thesis [13], we will go through the general method. Using QTT, we assume that our system is a TLS[1] in a state $|\psi\rangle$, the environment consists of n-level systems all in the same state $|E\rangle$, the interaction Hamiltonian is on the form $H_I = \sum_j A_j \otimes B_j$ and the time evolution is given by $U(\theta) = \exp(-i\theta H_I)$, and we have chosen a (orthonormal) measurement basis $\{|f_k\rangle\}_{1\leq k\leq n}$. By writing the Taylor expansion of $U(\theta)$, the non-normalized state after measurement outcome k is then given by

$$\left|\psi^k\right\rangle := |\psi\rangle\langle f_k|E\rangle - i\theta\sum_j A_j|\psi\rangle\langle f_k|B_j|E\rangle + \mathcal{O}(\theta^2).$$

[1] As we noted in section 5.2 it is not possible to visualise the entire state space of an n-level system for $n > 2$. The argument can be generalized to n-level systems, but this will only give an expression for the flow of the state. One can only hope to visualize the flow in a subspace of the state space.

The (non-normalized) Bloch vector of the k-th measurement outcome is then given by

$$
\begin{aligned}
\widetilde{\boldsymbol{n}}_k &= \left\langle \psi^k \middle| \boldsymbol{\sigma} \middle| \psi^k \right\rangle && \text{(D.4)} \\
&= \left| \langle f_k | E \rangle \right|^2 \overbrace{\langle \psi | \boldsymbol{\sigma} | \psi \rangle}^{=\boldsymbol{n},\ \text{Bloch vector before time evolution}} && \text{(D.5)} \\
&\quad + i\theta \sum_j \langle f_k | B_j | E \rangle^* \left\langle \psi \middle| A_j^\dagger \boldsymbol{\sigma} \middle| \psi \right\rangle \langle f_k | E \rangle && \text{(D.6)} \\
&\quad - i\theta \sum_j \langle f_k | B_j | E \rangle \langle \psi | \boldsymbol{\sigma} A_j | \psi \rangle \langle f_k | E \rangle^* + \mathcal{O}(\theta^2) && \text{(D.7)} \\
&=: U + \overbrace{\theta v + \theta v^*}^{\theta 2 \operatorname{Re}(v) =: \theta V} + \mathcal{O}(\theta^2) = U + \theta V + \mathcal{O}(\theta^2). && \text{(D.8)}
\end{aligned}
$$

Before we can take the derivative to find the flow of the state, we need to normalize:

$$
\boldsymbol{n}_k = \frac{\widetilde{\boldsymbol{n}}_k}{\|\widetilde{\boldsymbol{n}}_k\|} = \frac{\widetilde{\boldsymbol{n}}_k}{\sqrt{\widetilde{\boldsymbol{n}}_k \cdot \widetilde{\boldsymbol{n}}_k}} = \frac{U + \theta V + \mathcal{O}(\theta^2)}{\sqrt{|U|^2 + 2\theta V \cdot U + \mathcal{O}(\theta^2)}}
$$

Taking the derivative with respect to θ we find that the flow is given by

$$
\frac{d\boldsymbol{n}_k}{d\theta} = \frac{V\sqrt{|U|^2 + 2\theta V \cdot U + \mathcal{O}(\theta^2)} - (U + \theta V + \mathcal{O}(\theta^2)) \frac{1}{2} \frac{2V \cdot U + \mathcal{O}(\theta)}{\sqrt{|U|^2 + 2\theta V \cdot U + \mathcal{O}(\theta^2)}}}{|U|^2 + 2\theta V \cdot U + \mathcal{O}(\theta^2)}.
$$

Since $\theta \ll 1$ we put $\theta = 0$ to find the main contribution. With this assumption, and knowing that $\|n\| = 1$ and $|U| = |\langle f_k | E \rangle|^2 |\boldsymbol{n}| = |\langle f_k | E \rangle|^2$, the flow can be written as

$$
\frac{d\boldsymbol{n}_k}{d\theta} = \frac{1}{|U|}(V - n(V \cdot n)). \tag{D.9}
$$

We are now ready to tackle the model proposed by Longva.

Let $H_I^{\pm} = \frac{1}{4}(\sigma_x \otimes \sigma_x \pm \sigma_y \otimes \sigma_y)$ be the interaction Hamiltonian, $|E\rangle = |0\rangle$ be the state of the environment, and choose measurement basis $\{|x_+\rangle, |x_-\rangle\}$, i.e. $|f_k\rangle = \begin{cases} |x_+\rangle, & k = 0 \\ |x_-\rangle, & k = 1 \end{cases}$.
We define $A_1 = \frac{1}{4}\sigma_x, A_2 = \pm\frac{1}{4}\sigma_y, B_1 = \sigma_x, B_2 = \sigma_y$, such that $H_I = A_1 \otimes B_1 + A_2 \otimes B_2$. This gives us

$$
\langle f_k | E \rangle = \begin{cases} \langle x_+ | 0 \rangle = \frac{1}{\sqrt{2}}(\langle 0 | + \langle 1 |) |0\rangle, & k = 0 \\ \langle x_- | 0 \rangle = \frac{1}{\sqrt{2}}(\langle 0 | - \langle 1 |) |0\rangle, & k = 1 \end{cases} = \frac{1}{\sqrt{2}},
$$

$$
\langle f_k | B_1 | E \rangle = \langle x_\pm | \overbrace{\sigma_x |0\rangle}^{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle} = \frac{1}{\sqrt{2}}(\langle 0 | \pm \langle 1 |) |1\rangle = \begin{cases} 1/\sqrt{2}, & k = 0 \\ -1/\sqrt{2}, & k = 1 \end{cases},
$$

$$
\langle f_k | B_2 | E \rangle = \langle x_\pm | \overbrace{\sigma_y |0\rangle}^{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle} = \frac{i}{\sqrt{2}}(\langle 0 | \pm \langle 1 |) |1\rangle = \begin{cases} i/\sqrt{2}, & k = 0 \\ -i/\sqrt{2}, & k = 1 \end{cases},
$$

$$\left\langle \psi | A_1^\dagger \boldsymbol{\sigma} | \psi \right\rangle = \frac{1}{4} \langle \psi | \sigma_x^\dagger \boldsymbol{\sigma} | \psi \rangle = \frac{1}{4} \langle \psi | \begin{pmatrix} \mathbb{1}, & \sigma_x \sigma_y, & \sigma_x \sigma_z \end{pmatrix} | \psi \rangle$$
$$= \frac{1}{4} \begin{pmatrix} 1, & i \langle \psi | \sigma_z | \psi \rangle, & -i \langle \psi | \sigma_y | \psi \rangle \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1, & i n_z, & -i n_y \end{pmatrix},$$

$$\left\langle \psi | A_2^\dagger \boldsymbol{\sigma} | \psi \right\rangle = \frac{1}{4} \langle \psi | \pm \sigma_y^\dagger \boldsymbol{\sigma} | \psi \rangle = \pm \frac{1}{4} \langle \psi | \begin{pmatrix} \sigma_y \sigma_x, & \mathbb{1}, & \sigma_y \sigma_z \end{pmatrix} | \psi \rangle$$
$$= \pm \frac{1}{4} \begin{pmatrix} -i \langle \psi | \sigma_z | \psi \rangle, & 1, & i \langle \psi | \sigma_x | \psi \rangle \end{pmatrix} = \pm \frac{1}{4} \begin{pmatrix} -i n_z, & 1, & i n_x \end{pmatrix},$$

where we have used that $\sigma_j = \sigma_j^\dagger = \sigma_j^{-1}$, $\sigma_i \sigma_j = \mathbb{1} \delta_{ij} + i \epsilon_{ijk} \sigma_k$ and $\langle \psi | \sigma_i | \psi \rangle = n_i$. We calculate the variables in equation (D.4),

$$v^T = \begin{cases} i \left[\frac{1}{\sqrt{2}} \frac{1}{4} \begin{pmatrix} 1, & i n_z, & -i n_y \end{pmatrix} \frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}} \frac{1}{4} \begin{pmatrix} -i n_z, & 1, & i n_x \end{pmatrix} \frac{1}{\sqrt{2}} \right], & k = 0 \text{ and } H_{\text{int}_+} \\ i \left[\frac{-1}{\sqrt{2}} \frac{1}{4} \begin{pmatrix} 1, & i n_z, & -i n_y \end{pmatrix} \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \frac{1}{4} \begin{pmatrix} -i n_z, & 1, & i n_x \end{pmatrix} \frac{1}{\sqrt{2}} \right], & k = 1 \text{ and } H_{\text{int}_+} \\ i \left[\frac{1}{\sqrt{2}} \frac{1}{4} \begin{pmatrix} 1, & i n_z, & -i n_y \end{pmatrix} \frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}} (-\frac{1}{4}) \begin{pmatrix} -i n_z, & 1, & i n_x \end{pmatrix} \frac{1}{\sqrt{2}} \right], & k = 0 \text{ and } H_{\text{int}_-} \\ i \left[\frac{-1}{\sqrt{2}} \frac{1}{4} \begin{pmatrix} 1, & i n_z, & -i n_y \end{pmatrix} \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} (-\frac{1}{4}) \begin{pmatrix} -i n_z, & 1, & i n_x \end{pmatrix} \frac{1}{\sqrt{2}} \right], & k = 1 \text{ and } H_{\text{int}_-} \end{cases}$$

$$= \frac{i}{8} \begin{cases} \begin{pmatrix} 1 + (-i)^2 n_z, & i n_z + (-i), & -i n_y + (-i) i n_x \end{pmatrix}, & k = 0 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix} -1 + i(-i) n_z, & -i n_z + i, & -(-i) n_y + i^2 n_x \end{pmatrix}, & k = 1 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix} 1 - (-i)^2 n_z, & i n_z - (-i), & -i n_y - (-i) i n_x \end{pmatrix}, & k = 0 \text{ and } H_{\text{int}_-} \\ \begin{pmatrix} -1 - i(-i) n_z, & -i n_z - i, & -(-i) n_y - i^2 n_x \end{pmatrix}, & k = 1 \text{ and } H_{\text{int}_-} \end{cases}$$

$$= \frac{i}{8} \begin{cases} \begin{pmatrix} 1 - n_z, & i(n_z - 1), & -i n_y + n_x \end{pmatrix}, & k = 0 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix} -1 + n_z, & i(-n_z + 1), & i n_y - n_x \end{pmatrix}, & k = 1 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix} 1 + n_z, & i(n_z + 1), & -i n_y - n_x \end{pmatrix}, & k = 0 \text{ and } H_{\text{int}_-} \\ \begin{pmatrix} -1 - n_z, & i(-n_z - 1), & i n_y + n_x \end{pmatrix}, & k = 1 \text{ and } H_{\text{int}_-} \end{cases},$$

and $U = \frac{1}{2} \boldsymbol{n} = \frac{1}{2} \begin{pmatrix} n_x, & n_y, & n_z \end{pmatrix}^T$. Next we compute

$$V^T = 2 \operatorname{Re}(v^T)$$
$$= \frac{1}{4} \begin{cases} \begin{pmatrix} \operatorname{Re}(i(1 - n_z)), & \operatorname{Re}(i^2(n_z - 1)), & \operatorname{Re}(i(-i n_y + n_x)) \end{pmatrix}, & k = 0 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix} \operatorname{Re}(i(-1 + n_z)), & \operatorname{Re}(i^2(-n_z + 1)), & \operatorname{Re}(i(i n_y - n_x)) \end{pmatrix}, & k = 1 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix} \operatorname{Re}(i(1 + n_z)), & \operatorname{Re}(i^2(n_z + 1)), & \operatorname{Re}(i(-i n_y - n_x)) \end{pmatrix}, & k = 0 \text{ and } H_{\text{int}_-} \\ \begin{pmatrix} \operatorname{Re}(i(-1 - n_z)), & \operatorname{Re}(i^2(-n_z - 1)), & \operatorname{Re}(i(i n_y + n_x)) \end{pmatrix}, & k = 1 \text{ and } H_{\text{int}_-} \end{cases}$$

$$= \frac{1}{4} \begin{cases} \begin{pmatrix} 0, & -n_z + 1, & n_y \end{pmatrix}, & k = 0 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix} 0, & n_z - 1, & -n_y \end{pmatrix}, & k = 1 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix} 0, & -n_z - 1, & n_y \end{pmatrix}, & k = 0 \text{ and } H_{\text{int}_-} \\ \begin{pmatrix} 0, & n_z + 1, & -n_y \end{pmatrix}, & k = 1 \text{ and } H_{\text{int}_-} \end{cases}.$$

The normalized the Bloch vector for $\theta = 0$, given by equation (D.9), is then

$$\frac{d\boldsymbol{n}_k}{d\theta} = 2(V - \boldsymbol{n}(V \cdot \boldsymbol{n})).$$

Since

$$V \cdot n_S = \frac{1}{4}\begin{cases} 0 \cdot n_x + (-n_z + 1)n_y + n_y n_z, & k = 0 \text{ and } H_{\text{int}+} \\ 0 \cdot n_x + (n_z - 1)n_y - n_y n_z, & k = 1 \text{ and } H_{\text{int}+} \\ 0 \cdot n_x + (-n_z - 1)n_y + n_y n_z, & k = 0 \text{ and } H_{\text{int}-} \\ 0 \cdot n_x + (n_z + 1)n_y - n_y n_z, & k = 1 \text{ and } H_{\text{int}-} \end{cases} = \frac{1}{4}\begin{cases} n_y, & k = 0 \text{ and } H_{\text{int}+} \\ -n_y, & k = 1 \text{ and } H_{\text{int}+} \\ -n_y, & k = 0 \text{ and } H_{\text{int}-} \\ n_y, & k = 1 \text{ and } H_{\text{int}-} \end{cases},$$

we get

$$\begin{aligned} \left(\frac{d\boldsymbol{n}_k}{d\theta}\right)^T &= \frac{1}{2}\begin{cases} \begin{pmatrix} 0, & -n_z + 1, & n_y \end{pmatrix} - n_y \begin{pmatrix} n_x, & n_y, & n_z \end{pmatrix}, & k = 0 \text{ and } H_{\text{int}+} \\ \begin{pmatrix} 0, & n_z - 1, & -n_y \end{pmatrix} + n_y \begin{pmatrix} n_x, & n_y, & n_z \end{pmatrix}, & k = 1 \text{ and } H_{\text{int}+} \\ \begin{pmatrix} 0, & -n_z - 1, & n_y \end{pmatrix} + n_y \begin{pmatrix} n_x, & n_y, & n_z \end{pmatrix}, & k = 0 \text{ and } H_{\text{int}-} \\ \begin{pmatrix} 0, & n_z + 1, & -n_y \end{pmatrix} - n_y \begin{pmatrix} n_x, & n_y, & n_z \end{pmatrix}, & k = 1 \text{ and } H_{\text{int}-} \end{cases} \\ &= \frac{1}{2}\begin{cases} \begin{pmatrix} -n_y n_x, & -n_z + 1 - n_y^2, & n_y(1 - n_z) \end{pmatrix}, & k = 0 \text{ and } H_{\text{int}+} \\ \begin{pmatrix} n_y n_x, & n_z - 1 + n_y^2, & n_y(-1 + n_z) \end{pmatrix}, & k = 1 \text{ and } H_{\text{int}+} \\ \begin{pmatrix} n_y n_x, & -n_z - 1 + n_y^2, & n_y(1 + n_z) \end{pmatrix}, & k = 0 \text{ and } H_{\text{int}-} \\ \begin{pmatrix} -n_y n_x, & n_z + 1 - n_y^2, & n_y(-1 - n_z) \end{pmatrix}, & k = 1 \text{ and } H_{\text{int}-} \end{cases}. \end{aligned} \tag{D.10}$$

As a final remark, we note that the flow vector $\frac{d\boldsymbol{n}_k}{d\theta}$ is indeed orthogonal to the Bloch sphere:

$$\begin{aligned} \frac{d\boldsymbol{n}_k}{d\theta} \cdot \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} &= \frac{1}{2}\begin{cases} -n_y n_x^2 + (-n_z + 1 - n_y^2)n_y + n_z n_y(1 - n_z), & k = 0 \text{ and } H_{\text{int}+} \\ n_y n_x^2 + (n_z - 1 + n_y^2)n_y + n_z n_y(-1 + n_z), & k = 1 \text{ and } H_{\text{int}+} \\ n_y n_x^2 + (-n_z - 1 + n_y^2)n_y + n_z n_y(1 + n_z), & k = 0 \text{ and } H_{\text{int}-} \\ -n_y n_x^2 + (n_z + 1 - n_y^2)n_y + n_z n_y(-1 - n_z), & k = 1 \text{ and } H_{\text{int}-} \end{cases} \\ &= \frac{1}{2}\begin{cases} -n_y n_x^2 + (-n_z + n_x^2 + n_z^2)n_y + n_z n_y(1 - n_z), & k = 0 \text{ and } H_{\text{int}+} \\ n_y n_x^2 + (n_z - n_x^2 - n_z^2)n_y + n_z n_y(-1 + n_z), & k = 1 \text{ and } H_{\text{int}+} \\ n_y n_x^2 + (-n_z - n_x^2 - n_z^2)n_y + n_z n_y(1 + n_z), & k = 0 \text{ and } H_{\text{int}-} \\ -n_y n_x^2 + (n_z + n_x^2 + n_z^2)n_y + n_z n_y(-1 - n_z), & k = 1 \text{ and } H_{\text{int}-} \end{cases} \\ &= \frac{1}{2}\begin{cases} n_z n_y(-1 + n_z) + n_z n_y(1 - n_z), & k = 0 \text{ and } H_{\text{int}+} \\ n_z n_y(1 - n_z) + n_z n_y(-1 + n_z), & k = 1 \text{ and } H_{\text{int}+} \\ n_z n_y(-1 - n_z) + n_z n_y(1 + n_z), & k = 0 \text{ and } H_{\text{int}-} \\ n_z n_y(1 + n_z) + n_z n_y(-1 - n_z), & k = 1 \text{ and } H_{\text{int}-} \end{cases} \\ &= 0. \end{aligned}$$

This fact will be useful when we try to map the flow from the sphere to the plane later in the appendix.

D.2 The full Bloch vector for the model proposed by Longva

Although we have found an approximation for the Bloch vector for small θ, we could also calculate the full Bloch vector without approximation. We could then get an exact expression for the error we get by assuming all higher order terms are zero.

We are in the same setting as the previous section. That is, we assume that our system is a TLS in a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, the environment consists of two-level systems all in the same state $|E\rangle = |0\rangle$, the interaction Hamiltonian is on the form

$$H_{I^\pm} = \frac{1}{4}(\sigma_x \otimes \sigma_x \pm \sigma_y \otimes \sigma_y) = \sum_j A_j^\pm \otimes B_j^\pm$$

where $A_1^\pm = \frac{1}{4}\sigma_x$, $A_2^\pm = \pm\frac{1}{4}\sigma_y$, $B_1^\pm = \sigma_x$, $B_2^\pm = \sigma_y$. The time evolution is given by $U(\theta) = \exp(-i\theta H_{I^\pm})$, and we have chosen a measurement basis $\{|\uparrow_x\rangle, |\downarrow_x\rangle\}$, i.e. $|f_k\rangle = \begin{cases} |\uparrow_+\rangle, & k=0 \\ |\downarrow_-\rangle, & k=1 \end{cases}$. From appendix C we found that the unitary evolution could be written as

$$\begin{aligned} U_\pm(\theta)|\psi\rangle &= c_{00}^\pm|00\rangle + c_{11}^\pm|11\rangle + c_{01}^\pm|01\rangle + c_{10}^\pm|10\rangle \\ &= \frac{1}{\sqrt{2}}\left[\left(c_{00}^\pm + c_{01}^\pm\right)|0\rangle + \left(c_{10}^\pm + c_{11}^\pm\right)|1\rangle\right]|\uparrow_x\rangle \\ &\quad + \frac{1}{\sqrt{2}}\left[\left(c_{00}^\pm - c_{01}^\pm\right)|0\rangle + \left(c_{10}^\pm - c_{11}^\pm\right)|1\rangle\right]|\downarrow_x\rangle) \\ &=: \left|\psi_{\uparrow_x}^\pm\right\rangle|\uparrow_x\rangle + \left|\psi_{\downarrow_x}^\pm\right\rangle|\downarrow_x\rangle, \end{aligned}$$

where

$$\begin{aligned} c_{00}^+ &= \alpha, \quad c_{11}^+ = 0, \quad c_{01}^+ = -i\beta\sin\frac{\theta}{2}, \quad c_{10}^+ = \beta\cos\frac{\theta}{2}, \\ c_{00}^- &= \alpha\cos\frac{\theta}{2}, \quad c_{11}^- = -i\alpha\sin\frac{\theta}{2}, \quad c_{01}^- = 0, \quad c_{10}^- = \beta. \end{aligned}$$

The non-normalized Bloch vector after measurement k, given by equation (D.2), will then be

$$\begin{aligned} \tilde{\boldsymbol{n}}_k &= \left\langle\psi^k|\boldsymbol{\sigma}|\psi^k\right\rangle \\ &= \begin{cases} \frac{1}{2}\left[\left(c_{00}^\pm + c_{01}^\pm\right)^*\langle 0| + \left(c_{10}^\pm + c_{11}^\pm\right)^*\langle 1|\right]\boldsymbol{\sigma}\left[\left(c_{00}^\pm + c_{01}^\pm\right)|0\rangle + \left(c_{10}^\pm + c_{11}^\pm\right)|1\rangle\right], & k=0 \\ \frac{1}{2}\left[\left(c_{00}^\pm - c_{01}^\pm\right)^*\langle 0| + \left(c_{10}^\pm - c_{11}^\pm\right)^*\langle 1|\right]\boldsymbol{\sigma}\left[\left(c_{00}^\pm - c_{01}^\pm\right)|0\rangle + \left(c_{10}^\pm - c_{11}^\pm\right)|1\rangle\right], & k=1 \end{cases} \\ &= \frac{1}{2}\begin{cases} \begin{pmatrix} \left(c_{00}^\pm + c_{01}^\pm\right)^*\left(c_{10}^\pm + c_{11}^\pm\right) + \left(c_{10}^\pm + c_{11}^\pm\right)^*\left(c_{00}^\pm + c_{01}^\pm\right) \\ -i\left(c_{00}^\pm + c_{01}^\pm\right)^*\left(c_{10}^\pm + c_{11}^\pm\right) + i\left(c_{10}^\pm + c_{11}^\pm\right)^*\left(c_{00}^\pm + c_{01}^\pm\right) \\ \left|c_{00}^\pm + c_{01}^\pm\right|^2 - \left|c_{10}^\pm + c_{11}^\pm\right|^2 \end{pmatrix}, & k=0 \\ \begin{pmatrix} \left(c_{00}^\pm - c_{01}^\pm\right)^*\left(c_{10}^\pm - c_{11}^\pm\right) + \left(c_{10}^\pm - c_{11}^\pm\right)^*\left(c_{00}^\pm - c_{01}^\pm\right), \\ -i\left(c_{00}^\pm - c_{01}^\pm\right)^*\left(c_{10}^\pm - c_{11}^\pm\right) + i\left(c_{10}^\pm - c_{11}^\pm\right)^*\left(c_{00}^\pm - c_{01}^\pm\right), \\ \left|c_{00}^\pm - c_{01}^\pm\right|^2 - \left|c_{10}^\pm - c_{11}^\pm\right|^2 \end{pmatrix}, & k=1 \end{cases}. \end{aligned}$$

We now need to calculate all the different terms:

$$\left(c_{00}^{+}+c_{01}^{+}\right)^{*}\left(c_{10}^{+}+c_{11}^{+}\right)=(\alpha-i\beta\sin\frac{\theta}{2})^{*}\beta\cos\frac{\theta}{2}=\alpha^{*}\beta\cos\frac{\theta}{2}+i|\beta|^{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$
$$\left(c_{00}^{+}-c_{01}^{+}\right)^{*}\left(c_{10}^{+}-c_{11}^{+}\right)=(\alpha+i\beta\sin\frac{\theta}{2})^{*}\beta\cos\frac{\theta}{2}=\alpha^{*}\beta\cos\frac{\theta}{2}-i|\beta|^{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$
$$\left(c_{00}^{-}+c_{01}^{-}\right)^{*}\left(c_{10}^{-}+c_{11}^{-}\right)=\alpha^{*}\cos\frac{\theta}{2}(\beta-i\alpha\sin\frac{\theta}{2})=\alpha^{*}\beta\cos\frac{\theta}{2}-i|\alpha|^{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$
$$\left(c_{00}^{-}-c_{01}^{-}\right)^{*}\left(c_{10}^{-}-c_{11}^{-}\right)=\alpha^{*}\cos\frac{\theta}{2}(\beta+i\alpha\sin\frac{\theta}{2})=\alpha^{*}\beta\cos\frac{\theta}{2}+i|\alpha|^{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

and

$$\begin{aligned}\left|c_{00}^{+}+c_{01}^{+}\right|^{2}-\left|c_{10}^{+}+c_{11}^{+}\right|^{2}&=\left|\alpha-i\beta\sin\frac{\theta}{2}\right|^{2}-|\beta|^{2}\cos^{2}\frac{\theta}{2}\\&=|\alpha|^{2}+|\beta|^{2}(\sin^{2}\frac{\theta}{2}-\cos^{2}\frac{\theta}{2})-i(\alpha^{*}\beta-\alpha\beta^{*})\sin\frac{\theta}{2},\\\left|c_{00}^{+}-c_{01}^{+}\right|^{2}-\left|c_{10}^{+}-c_{11}^{+}\right|^{2}&=\left|\alpha+i\beta\sin\frac{\theta}{2}\right|^{2}-|\beta|^{2}\cos^{2}\frac{\theta}{2}\\&=|\alpha|^{2}+|\beta|^{2}(\sin^{2}\frac{\theta}{2}-\cos^{2}\frac{\theta}{2})+i(\alpha^{*}\beta-\alpha\beta^{*})\sin\frac{\theta}{2},\\\left|c_{00}^{-}+c_{01}^{-}\right|^{2}-\left|c_{10}^{-}+c_{11}^{-}\right|^{2}&=|\alpha|^{2}\cos^{2}\frac{\theta}{2}-\left|\beta-i\alpha\sin\frac{\theta}{2}\right|^{2}\\&=|\alpha|^{2}(\cos^{2}\frac{\theta}{2}-\sin^{2}\frac{\theta}{2})-|\beta|^{2}+i(\beta^{*}\alpha-\beta\alpha^{*})\sin\frac{\theta}{2},\\\left|c_{00}^{-}-c_{01}^{-}\right|^{2}-\left|c_{10}^{-}-c_{11}^{-}\right|^{2}&=|\alpha|^{2}\cos^{2}\frac{\theta}{2}-\left|\beta+i\alpha\sin\frac{\theta}{2}\right|^{2}\\&=|\alpha|^{2}(\cos^{2}\frac{\theta}{2}-\sin^{2}\frac{\theta}{2})-|\beta|^{2}-i(\beta^{*}\alpha-\beta\alpha^{*})\sin\frac{\theta}{2}.\end{aligned}$$

Then

$$\begin{aligned}&\left(c_{00}^{+}\pm c_{01}^{+}\right)^{*}\left(c_{10}^{+}\pm c_{11}^{+}\right)+\left(c_{10}^{+}\pm c_{11}^{+}\right)^{*}\left(c_{00}^{+}\pm c_{01}^{+}\right)\\&\qquad=(\alpha^{*}\beta\cos\frac{\theta}{2}\pm i|\beta|^{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2})+(\alpha\beta^{*}\cos\frac{\theta}{2}\mp i|\beta|^{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2})\\&\qquad=(\alpha^{*}\beta+\alpha\beta^{*})\cos\frac{\theta}{2},\\&\left(c_{00}^{-}\pm c_{01}^{-}\right)^{*}\left(c_{10}^{-}\pm c_{11}^{-}\right)+\left(c_{10}^{-}\pm c_{11}^{-}\right)^{*}\left(c_{00}^{-}\pm c_{01}^{-}\right)\\&\qquad=\alpha^{*}\beta\cos\frac{\theta}{2}\mp i|\alpha|^{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}+(\alpha\beta^{*}\cos\frac{\theta}{2}\pm i|\alpha|^{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2})\\&\qquad=(\alpha^{*}\beta+\alpha\beta^{*})\cos\frac{\theta}{2},\end{aligned}$$

and

$$
\begin{aligned}
&\left(c_{00}^{+} \pm c_{01}^{+}\right)^{*}\left(c_{10}^{+} \pm c_{11}^{+}\right)-\left(c_{10}^{+} \pm c_{11}^{+}\right)^{*}\left(c_{00}^{+} \pm c_{01}^{+}\right) \\
&\qquad=(\alpha^{*} \beta \cos \frac{\theta}{2} \pm i|\beta|^{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2})-(\alpha \beta^{*} \cos \frac{\theta}{2} \mp i|\beta|^{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2}) \\
&\qquad=(\alpha^{*} \beta-\alpha \beta^{*}) \cos \frac{\theta}{2} \pm 2 i|\beta|^{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2}, \\
&\left(c_{00}^{-} \pm c_{01}^{-}\right)^{*}\left(c_{10}^{-} \pm c_{11}^{-}\right)-\left(c_{10}^{-} \pm c_{11}^{-}\right)^{*}\left(c_{00}^{-} \pm c_{01}^{-}\right) \\
&\qquad=\alpha^{*} \beta \cos \frac{\theta}{2} \mp i|\alpha|^{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2}-(\alpha \beta^{*} \cos \frac{\theta}{2} \pm i|\alpha|^{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2}) \\
&\qquad=(\alpha^{*} \beta-\alpha \beta^{*}) \cos \frac{\theta}{2} \mp 2 i|\alpha|^{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2}.
\end{aligned}
$$

Hence, using that $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ and $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$$
\begin{aligned}
-i\left[\left(c_{00}^{+} \pm c_{01}^{+}\right)^{*}\left(c_{10}^{+} \pm c_{11}^{+}\right)-\left(c_{10}^{+} \pm c_{11}^{+}\right)^{*}\left(c_{00}^{+} \pm c_{01}^{+}\right)\right] &= -i(\alpha^{*}\beta-\alpha\beta^{*})\cos\frac{\theta}{2} \pm |\beta|^{2}\sin\theta, \\
-i\left[\left(c_{00}^{-} \pm c_{01}^{-}\right)^{*}\left(c_{10}^{-} \pm c_{11}^{-}\right)-\left(c_{10}^{-} \pm c_{11}^{-}\right)^{*}\left(c_{00}^{-} \pm c_{01}^{-}\right)\right] &= -i(\alpha^{*}\beta-\alpha\beta^{*})\cos\frac{\theta}{2} \mp |\alpha|^{2}\sin\theta.
\end{aligned}
$$

The non-normalized Bloch vector is therefore

$$
\widetilde{\boldsymbol{n}}_k = \frac{1}{2}\begin{cases}
\begin{pmatrix} (\alpha^*\beta+\alpha\beta^*)\cos\frac{\theta}{2} \\ i(\alpha\beta^*-\alpha^*\beta)\cos\frac{\theta}{2} + |\beta|^2\sin\theta \\ |\alpha|^2-|\beta|^2\cos\theta - i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2} \end{pmatrix}, & k=0 \text{ and } H_{\text{int}_+} \\
\begin{pmatrix} (\alpha^*\beta+\alpha\beta^*)\cos\frac{\theta}{2} \\ i(\alpha\beta^*-\alpha^*\beta)\cos\frac{\theta}{2} - |\beta|^2\sin\theta \\ |\alpha|^2-|\beta|^2\cos\theta + i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2} \end{pmatrix}, & k=1 \text{ and } H_{\text{int}_+} \\
\begin{pmatrix} (\alpha^*\beta+\alpha\beta^*)\cos\frac{\theta}{2} \\ i(\alpha\beta^*-\alpha^*\beta)\cos\frac{\theta}{2} - |\alpha|^2\sin\theta \\ |\alpha|^2\cos\theta-|\beta|^2 - i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2} \end{pmatrix}, & k=0 \text{ and } H_{\text{int}_-} \\
\begin{pmatrix} (\alpha^*\beta+\alpha\beta^*)\cos\frac{\theta}{2} \\ i(\alpha\beta^*-\alpha^*\beta)\cos\frac{\theta}{2} + |\alpha|^2\sin\theta \\ |\alpha|^2\cos\theta-|\beta|^2 + i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2} \end{pmatrix}, & k=1 \text{ and } H_{\text{int}_-}
\end{cases}. \tag{D.11}
$$

As a sanity check, we check that the norm of the Bloch vector is the same as the norm of the state after measurement. For the interaction H_{i^+},

$$
\begin{aligned}
\langle\psi^k|\psi^k\rangle &= \frac{1}{2}\left[(\alpha \mp i\beta\sin\frac{\theta}{2})^*\langle 0| + \beta^*\cos\frac{\theta}{2}\langle 1|\right]\left[(\alpha \mp i\beta\sin\frac{\theta}{2})|0\rangle + \beta\cos\frac{\theta}{2}|1\rangle\right] \\
&= \frac{1}{2}\left[\left|\alpha \mp i\beta\sin\frac{\theta}{2}\right|^2 + \left|\beta\cos\frac{\theta}{2}\right|^2\right] \\
&= \frac{1}{2}\left[|\alpha|^2 \mp i\sin\frac{\theta}{2}[\alpha^*\beta-\alpha\beta^*] + |\beta|^2\sin^2\frac{\theta}{2} + |\beta|^2\cos^2\frac{\theta}{2}\right] \\
&= \frac{1}{2}\left[1 \mp i\sin\frac{\theta}{2}[\alpha^*\beta-\alpha\beta^*]\right].
\end{aligned}
$$

The norm of the Bloch vector for interaction H_{I^+} is

$$
\begin{aligned}
\|2\widetilde{\boldsymbol{n}}_k\|^2 =&(\alpha^*\beta+\alpha\beta^*)^2\cos^2\frac{\theta}{2}+\left[i(\alpha\beta^*-\alpha^*\beta)\cos\frac{\theta}{2}\pm|\beta|^2\sin\theta\right]^2\\
&+\left(|\alpha|^2-|\beta|^2\cos\theta\mp i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}\right)^2\\
=&(\alpha^*\beta+\alpha\beta^*)^2\cos^2\frac{\theta}{2}\\
&-(\alpha\beta^*-\alpha^*\beta)^2\cos^2\frac{\theta}{2}\mp 2i(\alpha\beta^*-\alpha^*\beta)\cos\frac{\theta}{2}|\beta|^2\sin\theta+|\beta|^4\sin^2\theta\\
&+|\alpha|^4+2|\alpha|^2\left[-|\beta|^2\cos\theta\mp i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}\right]+\left[|\beta|^2\cos\theta\pm i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}\right]^2,
\end{aligned}
$$

and as $\left[|\beta|^2\cos\theta\pm i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}\right]^2 = |\beta|^4\cos^2\theta\pm 2i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}|\beta|^2\cos\theta-(\alpha^*\beta-\alpha\beta^*)^2\sin^2\frac{\theta}{2}$,

$$
\begin{aligned}
\|2\widetilde{\boldsymbol{n}}_k\|^2 =&(\alpha^*\beta+\alpha\beta^*)^2\cos^2\frac{\theta}{2}-(\alpha\beta^*-\alpha^*\beta)^2\cos^2\frac{\theta}{2}-(\alpha^*\beta-\alpha\beta^*)^2\sin^2\frac{\theta}{2}\\
&+|\alpha|^4+|\beta|^4\cos^2\theta+|\beta|^4\sin^2\theta+2|\alpha|^2\left[-|\beta|^2\cos\theta\mp i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}\right]\\
&\mp 2i(\alpha\beta^*-\alpha^*\beta)|\beta|^2\left[\cos\frac{\theta}{2}\sin\theta-\sin\frac{\theta}{2}\cos\theta\right]\\
=&4|\alpha|^2|\beta|^2\cos^2\frac{\theta}{2}-((\alpha^*\beta)^2-2|\alpha|^2|\beta|^2+(\alpha\beta^*)^2)\sin^2\frac{\theta}{2}-2|\alpha|^2|\beta|^2\cos\theta\\
&+|\alpha|^4+|\beta|^4\mp 2i|\alpha|^2(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}\mp 2i(\alpha\beta^*-\alpha^*\beta)|\beta|^2\sin\frac{\theta}{2}\\
=&2|\alpha|^2|\beta|^2+2|\alpha|^2|\beta|^2\cos^2\frac{\theta}{2}-((\alpha^*\beta)^2+(\alpha\beta^*)^2)\sin^2\frac{\theta}{2}-2|\alpha|^2|\beta|^2\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)\\
&+|\alpha|^4+|\beta|^4\mp 2i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}\\
=&(|\alpha|^2+|\beta|^2)^2+2|\alpha|^2|\beta|^2\sin^2\frac{\theta}{2}-((\alpha^*\beta)^2+(\alpha\beta^*)^2)\sin^2\frac{\theta}{2}\\
&\mp 2i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}\\
=&1\mp 2i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}-((\alpha^*\beta)^2-2|\alpha|^2|\beta|^2+(\alpha\beta^*)^2)\sin^2\frac{\theta}{2}\\
=&\left(1\mp i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}\right)^2.
\end{aligned}
$$

They are therefore equal, just as we wanted. For H_{int_-} we have

$$\begin{aligned}\left\langle\psi^k|\psi^k\right\rangle &= \frac{1}{2}\left[\alpha^*\cos\frac{\theta}{2}\left\langle 0\right| + (\beta\mp i\alpha\sin\frac{\theta}{2})^*\left\langle 1\right|\right]\left[\alpha\cos\frac{\theta}{2}\left\langle 0\right| + (\beta\mp i\alpha\sin\frac{\theta}{2})\left\langle 1\right|\right]\\ &= \frac{1}{2}\left[\left|\alpha\cos\frac{\theta}{2}\right|^2 + \left|\beta\mp i\alpha\sin\frac{\theta}{2}\right|^2\right]\\ &= \frac{1}{2}\left[|\beta|^2 \mp i\sin\frac{\theta}{2}[\alpha\beta^* - \alpha^*\beta] + |\alpha|^2\sin^2\frac{\theta}{2} + |\alpha|^2\cos^2\frac{\theta}{2}\right]\\ &= \frac{1}{2}\left[1\mp i\sin\frac{\theta}{2}[\alpha\beta^* - \alpha^*\beta]\right]\end{aligned}$$

and

$$\begin{aligned}\|2\widetilde{\boldsymbol{n}}_k\|^2 =&(\alpha^*\beta+\alpha\beta^*)^2\cos^2\frac{\theta}{2} + \left[i(\alpha\beta^*-\alpha^*\beta)\cos\frac{\theta}{2}\mp|\alpha|^2\sin\theta\right]^2\\ &+\left(|\alpha|^2\cos\theta - |\beta|^2 \mp i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}\right)^2\\ =&(\alpha^*\beta+\alpha\beta^*)^2\cos^2\frac{\theta}{2}\\ &-(\alpha\beta^*-\alpha^*\beta)^2\cos^2\frac{\theta}{2}\mp 2i(\alpha\beta^*-\alpha^*\beta)\cos\frac{\theta}{2}|\alpha|^2\sin\theta + |\alpha|^4\sin^2\theta\\ &+|\beta|^4 - 2|\beta|^2\left[|\alpha|^2\cos\theta\mp i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}\right] + \left[|\alpha|^2\cos\theta\mp i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}\right]^2.\end{aligned}$$

As $\left[|\alpha|^2\cos\theta\mp i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}\right]^2 = |\alpha|^4\cos^2\theta\mp 2i(\alpha^*\beta-\alpha\beta^*)\sin\frac{\theta}{2}|\alpha|^2\cos\theta - (\alpha^*\beta -$

$\alpha\beta^*)^2 \sin^2 \frac{\theta}{2}$,

$$\begin{aligned}
\|2\widetilde{\boldsymbol{n}}_k\|^2 =&(\alpha^*\beta + \alpha\beta^*)^2 \cos^2 \frac{\theta}{2} - (\alpha\beta^* - \alpha^*\beta)^2 \cos^2 \frac{\theta}{2} - (\alpha^*\beta - \alpha\beta^*)^2 \sin^2 \frac{\theta}{2} \\
&+ |\beta|^4 + |\alpha|^4 \cos^2\theta + |\alpha|^4 \sin^2\theta - 2|\beta|^2 \left[|\alpha|^2 \cos\theta \mp i(\alpha^*\beta - \alpha\beta^*) \sin \frac{\theta}{2} \right] \\
&\mp 2i(\alpha\beta^* - \alpha^*\beta)|\alpha|^2 \left[\cos\frac{\theta}{2} \sin\theta - \sin\frac{\theta}{2} \cos\theta \right] \\
=&4|\alpha|^2|\beta|^2 \cos^2 \frac{\theta}{2} - ((\alpha^*\beta)^2 - 2|\alpha|^2|\beta|^2 + (\alpha\beta^*)^2) \sin^2 \frac{\theta}{2} - 2|\alpha|^2|\beta|^2 \cos\theta \\
&+ |\alpha|^4 + |\beta|^4 \pm 2i|\beta|^2(\alpha^*\beta - \alpha\beta^*) \sin\frac{\theta}{2} \mp 2i(\alpha\beta^* - \alpha^*\beta)|\alpha|^2 \sin\frac{\theta}{2} \\
=&2|\alpha|^2|\beta|^2 + 2|\alpha|^2|\beta|^2 \cos^2 \frac{\theta}{2} - ((\alpha^*\beta)^2 + (\alpha\beta^*)^2) \sin^2 \frac{\theta}{2} - 2|\alpha|^2|\beta|^2 \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \right) \\
&+ |\alpha|^4 + |\beta|^4 \pm 2i(\alpha^*\beta - \alpha\beta^*) \sin\frac{\theta}{2} \\
=&(|\alpha|^2 + |\beta|^2)^2 + 2|\alpha|^2|\beta|^2 \sin^2\frac{\theta}{2} - ((\alpha^*\beta)^2 + (\alpha\beta^*)^2) \sin^2 \frac{\theta}{2} \\
&\pm 2i(\alpha^*\beta - \alpha\beta^*) \sin\frac{\theta}{2} \\
=&1 \pm 2i(\alpha^*\beta - \alpha\beta^*) \sin\frac{\theta}{2} - ((\alpha^*\beta)^2 - 2|\alpha|^2|\beta|^2 + (\alpha\beta^*)^2) \sin^2 \frac{\theta}{2} \\
=&\left(1 \pm i(\alpha^*\beta - \alpha\beta^*) \sin\frac{\theta}{2} \right)^2,
\end{aligned}$$

which is what we expected. We have thus found

$$\|\widetilde{\boldsymbol{n}}_k\| = \frac{1}{2} \begin{cases} 1 - i\sin\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k=0 \text{ and } H_{\text{int}_+} \\ 1 + i\sin\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k=1 \text{ and } H_{\text{int}_+} \\ 1 + i\sin\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k=0 \text{ and } H_{\text{int}_-} \\ 1 - i\sin\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k=1 \text{ and } H_{\text{int}_-} \end{cases}.$$

Moreover,

$$\frac{d\widetilde{\boldsymbol{n}}_k}{d\theta} = \frac{1}{2} \begin{cases} \begin{pmatrix} -(1/2)(\alpha^*\beta + \alpha\beta^*)\sin\frac{\theta}{2} \\ -(1/2)i(\alpha\beta^* - \alpha^*\beta)\sin\frac{\theta}{2} + |\beta|^2\cos\theta \\ |\beta|^2\sin\theta - (1/2)i(\alpha^*\beta - \alpha\beta^*)\cos\frac{\theta}{2} \end{pmatrix}, & k=0 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix} -(1/2)(\alpha^*\beta + \alpha\beta^*)\sin\frac{\theta}{2} \\ -(1/2)i(\alpha\beta^* - \alpha^*\beta)\sin\frac{\theta}{2} - |\beta|^2\cos\theta \\ |\beta|^2\sin\theta + (1/2)i(\alpha^*\beta - \alpha\beta^*)\cos\frac{\theta}{2} \end{pmatrix}, & k=1 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix} -(1/2)(\alpha^*\beta + \alpha\beta^*)\sin\frac{\theta}{2} \\ -(1/2)i(\alpha\beta^* - \alpha^*\beta)\sin\frac{\theta}{2} - |\alpha|^2\cos\theta \\ -|\alpha|^2\sin\theta - (1/2)i(\alpha^*\beta - \alpha\beta^*)\cos\frac{\theta}{2} \end{pmatrix}, & k=0 \text{ and } H_{\text{int}_-} \\ \begin{pmatrix} -(1/2)(\alpha^*\beta + \alpha\beta^*)\sin\frac{\theta}{2} \\ -(1/2)i(\alpha\beta^* - \alpha^*\beta)\sin\frac{\theta}{2} + |\alpha|^2\cos\theta \\ -|\alpha|^2\sin\theta + (1/2)i(\alpha^*\beta - \alpha\beta^*)\cos\frac{\theta}{2} \end{pmatrix}, & k=1 \text{ and } H_{\text{int}_-} \end{cases}$$

and

$$\frac{\|\widetilde{\boldsymbol{n}}_k\|}{d\theta} = \frac{1}{2}\begin{cases} -i\cos\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k=0 \text{ and } H_{\mathrm{int}_+} \\ i\cos\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k=1 \text{ and } H_{\mathrm{int}_+} \\ i\cos\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k=0 \text{ and } H_{\mathrm{int}_-} \\ -i\cos\frac{\theta}{2}[\alpha^*\beta - \alpha\beta], & k=1 \text{ and } H_{\mathrm{int}_-} \end{cases}$$

and

$$\frac{d\boldsymbol{n}_k}{d\theta} = \frac{\frac{d\widetilde{\boldsymbol{n}}_k}{d\theta}\,\|\widetilde{\boldsymbol{n}}_k\| - \widetilde{\boldsymbol{n}}_k\frac{d\|\widetilde{\boldsymbol{n}}_k\|}{d\theta}}{\|\widetilde{\boldsymbol{n}}_k\|^2}.$$

Putting $\theta = 0$ we find

$$\widetilde{\boldsymbol{n}}_k = \frac{1}{2}\begin{cases} \begin{pmatrix} (\alpha^*\beta + \alpha\beta^*) \\ i(\alpha\beta^* - \alpha^*\beta) \\ |\alpha|^2 - |\beta|^2 \end{pmatrix}, \\ \begin{pmatrix} (\alpha^*\beta + \alpha\beta^*) \\ i(\alpha\beta^* - \alpha^*\beta \\ |\alpha|^2 - |\beta|^2 \end{pmatrix}, \\ \begin{pmatrix} (\alpha^*\beta + \alpha\beta^*) \\ i(\alpha\beta^* - \alpha^*\beta \\ |\alpha|^2 - |\beta|^2 \end{pmatrix}, \\ \begin{pmatrix} (\alpha^*\beta + \alpha\beta^*) \\ i(\alpha\beta^* - \alpha^*\beta \\ |\alpha|^2 - |\beta|^2 \end{pmatrix}, \end{cases} \qquad \frac{d\widetilde{\boldsymbol{n}}_k}{d\theta} = \frac{1}{2}\begin{cases} \begin{pmatrix} 0 \\ |\beta|^2 \\ -(1/2)i(\alpha^*\beta - \alpha\beta^*) \end{pmatrix}, & k=0 \text{ and } H_{\mathrm{int}_+} \\ \begin{pmatrix} 0 \\ -|\beta|^2 \\ (1/2)i(\alpha^*\beta - \alpha\beta^*) \end{pmatrix}, & k=1 \text{ and } H_{\mathrm{int}_+} \\ \begin{pmatrix} 0 \\ -|\alpha|^2 \\ -(1/2)i(\alpha^*\beta - \alpha\beta^*) \end{pmatrix}, & k=0 \text{ and } H_{\mathrm{int}_-} \\ \begin{pmatrix} 0 \\ |\alpha|^2 \\ (1/2)i(\alpha^*\beta - \alpha\beta^*) \end{pmatrix}, & k=1 \text{ and } H_{\mathrm{int}_-} \end{cases},$$

$$\|\widetilde{\boldsymbol{n}}_k\| = \frac{1}{4} \quad \text{and} \quad \frac{\|\widetilde{\boldsymbol{n}}_k\|}{d\theta} = \frac{1}{2}\begin{cases} -i[\alpha^*\beta - \alpha\beta], & k=0 \text{ and } H_{\mathrm{int}_+} \\ i[\alpha^*\beta - \alpha\beta], & k=1 \text{ and } H_{\mathrm{int}_+} \\ i[\alpha^*\beta - \alpha\beta], & k=0 \text{ and } H_{\mathrm{int}_-} \\ -i[\alpha^*\beta - \alpha\beta], & k=1 \text{ and } H_{\mathrm{int}_-} \end{cases}.$$

Thus

$$\frac{d\boldsymbol{n}_k}{d\theta} = \begin{cases} \begin{pmatrix} 0 \\ |\beta|^2 \\ -(1/2)i(\alpha^*\beta - \alpha\beta^*) \end{pmatrix} - \begin{pmatrix} (\alpha^*\beta + \alpha\beta^*) \\ i(\alpha\beta^* - \alpha^*\beta) \\ |\alpha|^2 - |\beta|^2 \end{pmatrix} \frac{-1}{2}i[\alpha^*\beta - \alpha\beta], & k=0 \text{ and } H_{\mathrm{int}_+} \\ \begin{pmatrix} 0 \\ -|\beta|^2 \\ (1/2)i(\alpha^*\beta - \alpha\beta^*) \end{pmatrix} - \begin{pmatrix} (\alpha^*\beta + \alpha\beta^*) \\ i(\alpha\beta^* - \alpha^*\beta \\ |\alpha|^2 - |\beta|^2 \end{pmatrix} \frac{1}{2}i[\alpha^*\beta - \alpha\beta], & k=1 \text{ and } H_{\mathrm{int}_+} \\ \begin{pmatrix} 0 \\ -|\alpha|^2 \\ -(1/2)i(\alpha^*\beta - \alpha\beta^*) \end{pmatrix} - \begin{pmatrix} (\alpha^*\beta + \alpha\beta^*) \\ i(\alpha\beta^* - \alpha^*\beta \\ |\alpha|^2 - |\beta|^2 \end{pmatrix} \frac{1}{2}i[\alpha^*\beta - \alpha\beta], & k=0 \text{ and } H_{\mathrm{int}_-} \\ \begin{pmatrix} 0 \\ |\alpha|^2 \\ (1/2)i(\alpha^*\beta - \alpha\beta^*) \end{pmatrix} - \begin{pmatrix} (\alpha^*\beta + \alpha\beta^*) \\ i(\alpha\beta^* - \alpha^*\beta \\ |\alpha|^2 - |\beta|^2 \end{pmatrix} \frac{-1}{2}i[\alpha^*\beta - \alpha\beta], & k=1 \text{ and } H_{\mathrm{int}_-} \end{cases}.$$

As $|\alpha|^2 + |\beta|^2 = 1$ we know from equation (D.1) and (D.2) that

$$\frac{d\boldsymbol{n}_k}{d\theta} = \begin{cases} \begin{pmatrix} 0 \\ (1-n_z)/2 \\ (1/2)n_y \end{pmatrix} - \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \frac{1}{2} n_y, \\ \begin{pmatrix} 0 \\ -(1-n_z)/2 \\ -(1/2)n_y \end{pmatrix} - \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \frac{-1}{2} n_y, \\ \begin{pmatrix} 0 \\ -(1+n_z)/2 \\ (1/2)n_y \end{pmatrix} - \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \frac{-1}{2} n_y, \\ \begin{pmatrix} 0 \\ (1+n_z)/2 \\ -(1/2)n_y \end{pmatrix} - \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \frac{1}{2} n_y, \end{cases} = \begin{cases} \begin{pmatrix} -n_x n_y/2 \\ (1-n_z-n_y^2)/2 \\ n_y(1-n_z)/2 \end{pmatrix}, & k=0 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix} n_x n_y/2 \\ -(1-n_z-n_y^2)/2 \\ -n_y(1-n_z)/2 \end{pmatrix}, & k=1 \text{ and } H_{\text{int}_+} \\ \begin{pmatrix} n_x n_y/2 \\ -(1+n_z-n_y^2)/2 \\ n_y(1+n_z)/2 \end{pmatrix}, & k=0 \text{ and } H_{\text{int}_-} \\ \begin{pmatrix} -n_x n_y/2 \\ (1+n_z-n_y^2)/2 \\ -n_y(1+n_z)/2 \end{pmatrix}, & k=1 \text{ and } H_{\text{int}_-} \end{cases}$$

which, as it should be, is exactly the same as we got in equation (D.10).

D.3 Mapping state flow from the sphere onto the plane

We want to map the vector flow from the Bloch sphere to the plane. Since the vectors describing the flow are tangent to the surface of the sphere (cf. end of section D.1), they live in the tangent space of the sphere. We will take a brief detour through the theory of manifolds so that we easily can see how we transform vector flow from one surface to another. The notation and ideas will be based on the book by Tu [25].

Let N and M be two manifolds, and let $F : N \to M$ be a C^∞-map. Given a tangent vector X_p at the point $p \in N$, we wish to describe X_p on M. This we do through the *push-forward* $F_{*,p} : T_pN \to T_{F(p)}M$, given by [25, section 8.2]

$$F_{*,p}(X_p)(g) = X_p(g \circ F) \quad \text{for all } g \in C^\infty(F(p)).$$

Given a basis $\{\frac{\partial}{\partial x^i}|_p\}_{i\in I}$ for T_pN and a basis $\{\frac{\partial}{\partial y^j}|_{F(p)}\}_{j\in J}$ for $T_{F(p)}M$, the push-forward can be described using the Jacobian,

$$[F_{*,p}]_{i,j} = \frac{\partial F^j}{\partial x^i}\Big|_p.$$

Hence, the new tangent vector $Y_{F(p)} := F_{*,p}(X_p)$ is given by

$$Y_{F(p)} = \begin{pmatrix} \frac{\partial F^1}{\partial x^1}\big|_p & \frac{\partial F^1}{\partial x^2}\big|_p & \cdots \\ \frac{\partial F^2}{\partial x^1}\big|_p & \frac{\partial F^2}{\partial x^2}\big|_p & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} X_p.$$

With these ideas, we are now ready to transform the vector flow

D.3.1 Stereographic projection

Let N be the Bloch sphere, i.e. $N = S^2 \subseteq \mathbb{R}^3$. We want to map the sphere onto a subset of the plane $M \subseteq \mathbb{R}^2$. We begin by exploring the *stereographic projection*. A stereographic projection maps the sphere onto $\mathbb{R}^2$ by drawing a line through the north pole. The point on the sphere intersecting the line is mapped to the point in $\mathbb{R}^2$ which also intersects the line. Hence, the points below the equator is mapped to the unit disk in $\mathbb{R}^2$, while the upper half of the sphere is mapped to the rest of $\mathbb{R}^2$. We think of the north pole as being mapped to infinity. This projection is common in complex analysis, and an introductory book such as [7, chapter 1, section 3] will explain it in further detail.

By looking at the different flows in Figure D.2, we choose to divide the sphere in half along the great circle intersecting $|0\rangle$ and $|\uparrow_y\rangle$. This way, we can visualize the flow on two discs, and we will see that the flow never leaves the disc where it starts. The projection mapping with $|\uparrow_x\rangle$ as north pole is given by

$$F : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y/(1-x) \\ z/(1-x) \end{pmatrix}.$$

We represent the push-forward by the Jacobian

$$\begin{aligned} F_{*,p} &= \begin{pmatrix} \frac{\partial}{\partial x} y/(1-x)\big|_p & \frac{\partial}{\partial y} y/(1-x)\big|_p & \frac{\partial}{\partial z} y/(1-x)\big|_p \\ \frac{\partial}{\partial x} z/(1-x)\big|_p & \frac{\partial}{\partial y} z/(1-x)\big|_p & \frac{\partial}{\partial z} z/(1-x)\big|_p \end{pmatrix} \\ &= \begin{pmatrix} y/(1-x)^2\big|_p & 1/(1-x)\big|_p & 0 \\ z/(1-x)^2\big|_p & 0 & 1/(1-x)\big|_p \end{pmatrix}. \end{aligned}$$

In our specific case, the flow at $p = \boldsymbol{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$ is given by equation (D.10),

$$X_p = \frac{1}{2} \begin{cases} \Big(-n_y n_x, \;\; -n_z + 1 - n_y^2, \;\; n_y(1-n_z)\Big)^T, & k = 0 \text{ and } H_{\text{int}_+} \\ \Big(n_y n_x, \;\; n_z - 1 + n_y^2, \;\; n_y(-1+n_z)\Big)^T, & k = 1 \text{ and } H_{\text{int}_+} \\ \Big(n_y n_x, \;\; -n_z - 1 + n_y^2, \;\; n_y(1+n_z)\Big)^T, & k = 0 \text{ and } H_{\text{int}_-} \\ \Big(-n_y n_x, \;\; n_z + 1 - n_y^2, \;\; n_y(-1-n_z)\Big)^T, & k = 1 \text{ and } H_{\text{int}_-} \end{cases}. \tag{D.12}$$

For H_{I^+} and $k = 0$, the flow after the stereographic projection is given by

$$
\begin{aligned}
Y_{F(p)} &= \begin{pmatrix} n_y/(1-n_x)^2 & 1/(1-n_x) & 0 \\ n_z/(1-n_x)^2 & 0 & 1/(1-n_x) \end{pmatrix} \frac{1}{2} \begin{pmatrix} -n_y n_x \\ -n_z + 1 - n_y^2 \\ n_y(1-n_z) \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} -\frac{n_x n_y^2}{(1-n_x)^2} + \frac{-n_z+1-n_y^2}{1-n_x} \\ -\frac{n_x n_y n_z}{(1-n_x)^2} + \frac{n_y(1-n_z)}{1-n_x} \end{pmatrix} = \frac{1}{2(1-n_x)^2} \begin{pmatrix} -n_x n_y^2 + (-n_z + 1 - n_y^2)(1-n_x) \\ -n_x n_y n_z + n_y(1-n_z)(1-n_x) \end{pmatrix} \\
&= \frac{1}{2(1-n_x)^2} \begin{pmatrix} -n_x n_y^2 + n_x n_y^2 + 1 - n_z - n_x - n_y^2 + n_x n_z \\ -n_x n_y n_z + n_y(1 - n_z - n_x + n_x n_z) \end{pmatrix} \\
&= \frac{1}{2(1-n_x)^2} \begin{pmatrix} 1 - n_x - n_z - n_y^2 + n_x n_z \\ n_y(1 - n_x - n_z) \end{pmatrix}. \qquad \text{(D.13)}
\end{aligned}
$$

If we let $|\downarrow_x\rangle$ represent the north pole instead, then the stereographic projection is given by

$$
G : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y/(1+x) \\ z/(1+x) \end{pmatrix}
$$

with Jacobian

$$
\begin{aligned}
G_{*,p} &= \begin{pmatrix} \frac{\partial}{\partial x} y/(1+x)\big|_p & \frac{\partial}{\partial y} y/(1+x)\big|_p & \frac{\partial}{\partial z} y/(1+x)\big|_p \\ \frac{\partial}{\partial x} z/(1+x)\big|_p & \frac{\partial}{\partial y} z/(1+x)\big|_p & \frac{\partial}{\partial z} z/(1+x)\big|_p \end{pmatrix} \\
&= \begin{pmatrix} -y/(1+x)^2\big|_p & 1/(1+x)\big|_p & 0 \\ -z/(1+x)^2\big|_p & 0 & 1/(1+x)\big|_p \end{pmatrix}.
\end{aligned}
$$

For H_{I^+} and $k = 0$, the flow at $p = \boldsymbol{n}$ after the stereographic projection is given by

$$
\begin{aligned}
Y_{G(p)} &= \begin{pmatrix} -n_y/(1+n_x)^2 & 1/(1+n_x) & 0 \\ -n_z/(1+n_x)^2 & 0 & 1/(1+n_x) \end{pmatrix} \frac{1}{2} \begin{pmatrix} -n_y n_x \\ -n_z + 1 - n_y^2 \\ n_y(1-n_z) \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} \frac{n_x n_y^2}{(1+n_x)^2} + \frac{-n_z+1-n_y^2}{1+n_x} \\ \frac{n_x n_y n_z}{(1+n_x)^2} + \frac{n_y(1-n_z)}{1+n_x} \end{pmatrix} = \frac{1}{2(1+n_x)^2} \begin{pmatrix} n_x n_y^2 + (-n_z + 1 - n_y^2)(1+n_x) \\ n_x n_y n_z + n_y(1-n_z)(1+n_x) \end{pmatrix} \\
&= \frac{1}{2(1+n_x)^2} \begin{pmatrix} n_x n_y^2 - n_x n_y^2 + 1 - n_z + n_x - n_y^2 - n_x n_z \\ n_x n_y n_z + n_y(1 - n_z + n_x - n_x n_z) \end{pmatrix} \\
&= \frac{1}{2(1+n_x)^2} \begin{pmatrix} 1 + n_x - n_z - n_y^2 - n_x n_z \\ n_y(1 + n_x - n_z) \end{pmatrix}. \qquad \text{(D.14)}
\end{aligned}
$$

Note that equations (D.13) and (D.14) are equal if we change the sign of n_x. This means that the flow on one side of the sphere mirror the flow on the opposite side. Moreover, we see that the flow for H_{I^+} and $k = 1$ is just the negative of the flow for H_{I^+} and $k = 0$. The direction of the flow is therefore reversed.

We can go through the exact same calculations for H_{I^+} and $k = 0$ and find

$$Y'_{F(p)} = \begin{pmatrix} n_y/(1-n_x)^2 & 1/(1-n_x) & 0 \\ n_z/(1-n_x)^2 & 0 & 1/(1-n_x) \end{pmatrix} \frac{1}{2} \begin{pmatrix} n_y n_x \\ -n_z - 1 + n_y^2 \\ n_y(1+n_z) \end{pmatrix}$$
$$= \frac{1}{2(1-n_x)^2} \begin{pmatrix} -1 + n_x - n_z + n_y^2 + n_x n_z \\ n_y(1 - n_x + n_z) \end{pmatrix}.$$

and

$$Y'_{G(p)} = \begin{pmatrix} -n_y/(1+n_x)^2 & 1/(1+n_x) & 0 \\ -n_z/(1+n_x)^2 & 0 & 1/(1+n_x) \end{pmatrix} \frac{1}{2} \begin{pmatrix} n_y n_x \\ -n_z - 1 + n_y^2 \\ n_y(1+n_z) \end{pmatrix}$$
$$= \frac{1}{2(1+n_x)^2} \begin{pmatrix} -1 - n_x - n_z + n_y^2 - n_x n_z \\ n_y(1 + n_x + n_z) \end{pmatrix}.$$

We again see that the only difference between $Y'_{F(p)}$ and $Y'_{G(p)}$ is the sign of n_x. Moreover, we see that the flow for H_{I^-} and $k = 1$ is just the negative of the flow for H_{I^-} and $k = 0$. The direction of the flow is therefore reversed.

The figures depicting the flow after transformation can be found in the figure section D.4.

D.3.2 Change of coordinates

The other transformation we want to use (the Winkel tripel projection) assumes that the sphere is described by a polar angle ϕ and an (angle with respect to polar axis) and azimuthal angle θ (angle of rotation from the initial meridian plane). We will therefore find the coordinate transform and Jacobian for spherical coordinates. Letting r denote the radius of the sphere, cartesian coordinates can be described as

$$x = r\cos\theta\sin\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\phi.$$

As $\sin(\arccos t) = \sqrt{1-t^2}$,

$$\phi = \arccos\frac{z}{r} \qquad \theta = \operatorname{sgn}(y)\arccos\frac{x/r}{\sqrt{1-(z/r)^2}} = \operatorname{sgn}(y)\arccos\frac{x}{\sqrt{r^2-z^2}}, \tag{D.15}$$

The $\operatorname{sgn}(y)$ shows up here since $\arccos : [-1,1] \mapsto [0,\pi]$, and we therefore need to take into account the sign of y to get $[-\pi,\pi]$. We denote the coordinate transformation by $T : (x,y,z) \mapsto (\phi,\theta)$. If we also use that $r = \sqrt{x^2+y^2+z^2}$ we can write $\theta = \arccos\frac{x}{\sqrt{x^2+y^2}}$. We can calculate the different derivatives,

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\arccos\frac{z}{\sqrt{x^2+y^2+z^2}} = -\frac{1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}} z \frac{-1}{2}\frac{1}{(x^2+y^2+z^2)^{3/2}} 2x$$
$$= \frac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2-z^2}} = \frac{xz}{r^2\sqrt{x^2+y^2}},$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z} \arccos \frac{z}{\sqrt{x^2+y^2+z^2}} = -\frac{1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}} \frac{\sqrt{x^2+y^2+z^2} - z\frac{1}{2}\frac{1}{\sqrt{x^2+y^2+z^2}}2z}{x^2+y^2+z^2}$$

$$= -\frac{1-\frac{z^2}{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2-z^2}} = -\frac{x^2+y^2+z^2-z^2}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} = -\frac{x^2+y^2}{r^2\sqrt{x^2+y^2}}.$$

We immediately get $\frac{\partial \phi}{\partial y} = \frac{yz}{r^2\sqrt{x^2+y^2}}$. The partial derivative of θ with respect to x is

$$\frac{\partial \theta}{\partial x} = \operatorname{sgn}(y) \frac{\partial}{\partial x} \arccos \frac{x}{\sqrt{x^2+y^2}} = -\frac{\operatorname{sgn}(y)}{\sqrt{1-\frac{x^2}{x^2+y^2}}} \frac{\sqrt{x^2+y^2} - x\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}2x}{x^2+y^2}$$

$$= -\frac{1-\frac{x^2}{x^2+y^2}}{\sqrt{x^2+y^2-x^2}} \operatorname{sgn}(y) = -\frac{x^2+y^2-x^2}{(x^2+y^2)\sqrt{y^2}} \operatorname{sgn}(y) = -\frac{y}{x^2+y^2}.$$

The two others are

$$\frac{\partial \theta}{\partial y} = \operatorname{sgn}(y) \frac{\partial}{\partial y} \arccos \frac{x}{\sqrt{x^2+y^2}} = -\frac{\operatorname{sgn}(y)}{\sqrt{1-\frac{x^2}{x^2+y^2}}} x \frac{-1}{2} \frac{1}{(x^2+y^2)^{3/2}} 2y$$

$$= \frac{xy}{(x^2+y^2)\sqrt{x^2+y^2-x^2}} \operatorname{sgn}(y) = \frac{x}{x^2+y^2}$$

and $\frac{\partial \theta}{\partial z} = 0$. In total we get

$$T_{*,p} = \begin{pmatrix} \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{xz}{r^2\sqrt{x^2+y^2}} & \frac{yz}{r^2\sqrt{x^2+y^2}} & \frac{-(x^2+y^2)}{r^2\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix}. \tag{D.16}$$

Before we continue, we give the following observations. Firstly, we could equivalently have written $\theta = \operatorname{sgn}(x) \arcsin \frac{y}{\sqrt{r^2-z^2}}$, and therefore $\theta = \operatorname{sgn}(x) \arcsin \frac{y}{\sqrt{x^2+y^2}}$. The Jacobian would be the same:

$$\frac{\partial \theta}{\partial x} = \operatorname{sgn}(x) \frac{\partial}{\partial x} \arcsin \frac{y}{\sqrt{x^2+y^2}} = \frac{\operatorname{sgn}(x)}{\sqrt{1-\frac{y^2}{x^2+y^2}}} y \frac{-1}{2} \frac{1}{(x^2+y^2)^{3/2}} 2x$$

$$= -\frac{xy}{(x^2+y^2)\sqrt{x^2+y^2-y^2}} \operatorname{sgn}(x) = -\frac{y}{x^2+y^2}$$

and

$$\frac{\partial \theta}{\partial y} = \operatorname{sgn}(x) \frac{\partial}{\partial y} \arcsin \frac{y}{\sqrt{x^2+y^2}} = \frac{\operatorname{sgn}(x)}{\sqrt{1-\frac{y^2}{x^2+y^2}}} \frac{\sqrt{x^2+y^2} - y\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}2y}{x^2+y^2}$$

$$= \frac{1-\frac{y^2}{x^2+y^2}}{\sqrt{x^2+y^2-y^2}} \operatorname{sgn}(x) = \frac{x^2+y^2-y^2}{(x^2+y^2)\sqrt{x^2}} \operatorname{sgn}(x) = \frac{x}{x^2+y^2}.$$

The second observation is that we have not used that $r = 1$. The condition $r = 1$ means that r has no x-, y-, or z-dependence. We therefore have

$$\phi = \arccos z \qquad \theta = \operatorname{sgn}(y) \arccos \frac{x}{\sqrt{1-z^2}} = \operatorname{sgn}(y) \arccos \frac{x}{\sqrt{x^2+y^2}},$$

or equivalently $\theta = \operatorname{sgn}(x) \arcsin \frac{y}{\sqrt{1-z^2}} = \operatorname{sgn}(x) \arcsin \frac{y}{\sqrt{x^2+y^2}}$ instead. If we now proceed naively, we get

$$\frac{\partial \phi}{\partial x} = 0 \qquad \frac{\partial \phi}{\partial y} = 0 \qquad \frac{\partial \phi}{\partial z} = \frac{-1}{\sqrt{1-z^2}},$$

but for θ we have the following problem: We have four different expressions with different x-, y- and z-dependence. We can for instance find

$$\frac{\partial \theta}{\partial x} = \operatorname{sgn}(y) \frac{\partial}{\partial x} \arccos \frac{x}{\sqrt{1-z^2}} = -\frac{\operatorname{sgn}(y)}{\sqrt{1-\frac{x^2}{1-z^2}}} \frac{1}{\sqrt{1-z^2}} = -\frac{\operatorname{sgn}(y)}{\sqrt{1-z^2-x^2}},$$

and at the same time get

$$\frac{\partial \theta}{\partial x} = \operatorname{sgn}(x) \frac{\partial}{\partial x} \arcsin \frac{y}{\sqrt{1-z^2}} = 0.$$

To fix this we could try to introduce

$$x^2 = 1 - y^2 - z^2 \qquad y^2 = 1 - x^2 - z^2 \qquad z^2 = 1 - x^2 - y^2.$$

We could therefore say that x, y and z are dependent of each other. However, we then get in trouble when taking partial derivatives. For instance

$$\frac{\partial y^2}{\partial x} = \frac{\partial}{\partial x}(1 - x^2 - z^2) = -2x - \frac{\partial}{\partial x}(1 - x^2 - y^2) = -2x + 2x - \frac{\partial}{\partial x}(1 - x^2 - z^2) = \ldots.$$

It is not obvious how to introduce the dependence $r = 1$, so we left with our original find for the Jacobian given in equation (D.16).

D.3.3 Winkel Tripel projection

The Winkel Tripel projection (of the unit sphere) is given by [10]

$$x' = \frac{1}{2}\left[\lambda \cos \phi_0 + \frac{2\cos\phi \sin\frac{\lambda}{2}}{\operatorname{sinc}\alpha}\right], \quad y' = \frac{1}{2}\left[\phi + \frac{\sin\phi}{\operatorname{sinc}\alpha}\right],$$

where $\operatorname{sinc}\alpha := \frac{\sin\alpha}{\alpha}$, $\alpha := \arccos\left(\cos\phi \cos\frac{\lambda}{2}\right)$, (x', y') are the coordinate in the plane, λ is the longitude relative to the central meridian of the projection[2], ϕ is the latitude, ϕ_0 is the standard parallel for the equirectangular projection[3]. Figure D.1 show how longitude

[2]The central meridian is the meridian to which the sphere is rotated before projecting

[3]a standard parallel is a line of latitude that has true scale, and the equirectangular projection maps the angles straight down to the plane (i.e. $x' = \lambda$ and $y' = \phi$).

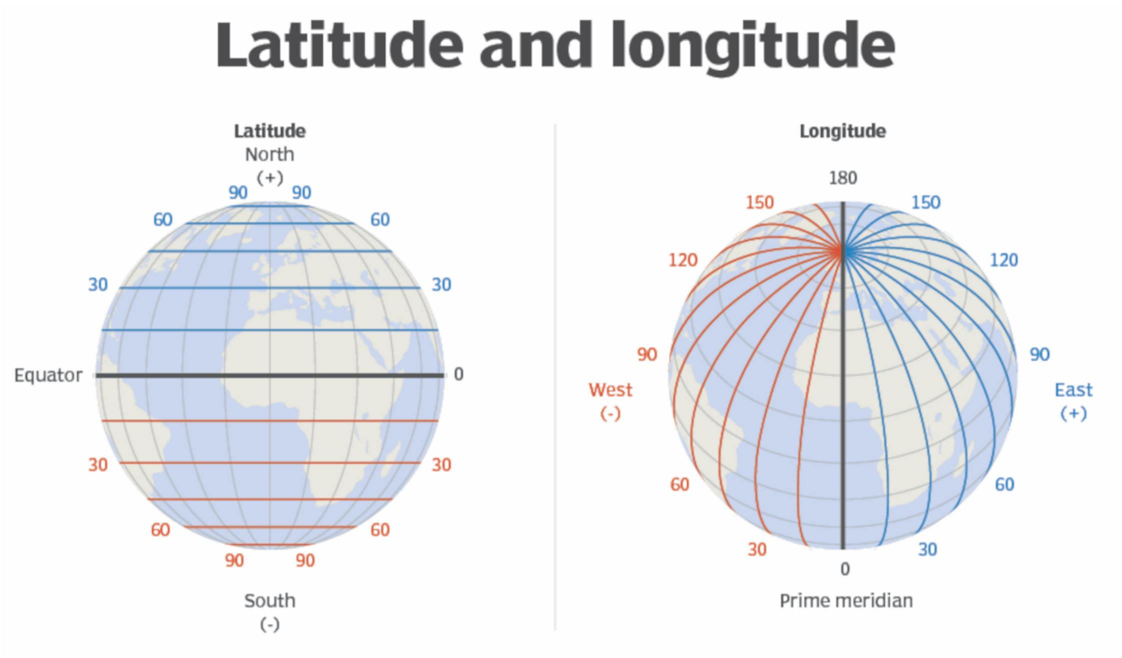


Figure D.1: Visual reference for how longitude and latitude are defined. Taken from https://www.techtarget.com/whatis/definition/latitude-and-longitude (accessed 27.05.23).

and latitude are defined on earth. Our projection will be $P : (\phi, \lambda) \mapsto (x', y')$. We will therefore need $\frac{\partial}{\partial\phi}\frac{1}{\operatorname{sinc}\alpha}$ and $\frac{\partial}{\partial\lambda}\frac{1}{\operatorname{sinc}\alpha}$. We have

$$\frac{1}{\operatorname{sinc}\alpha} = \frac{\alpha}{\sin\alpha} = \frac{\alpha}{\sqrt{1-\cos^2\alpha}} = \frac{\arccos\left(\cos\phi\cos\frac{\lambda}{2}\right)}{\sqrt{1-\cos^2\phi\cos^2\frac{\lambda}{2}}}$$

and

$$\frac{\partial}{\partial\phi}\arccos\left(\cos\phi\cos\frac{\lambda}{2}\right) = -\frac{1}{\sqrt{1-\cos^2\phi\cos^2\frac{\lambda}{2}}}(-\sin\phi)\cos\frac{\lambda}{2},$$

$$\frac{\partial}{\partial\lambda}\arccos\left(\cos\phi\cos\frac{\lambda}{2}\right) = -\frac{1}{\sqrt{1-\cos^2\phi\cos^2\frac{\lambda}{2}}}\cos\phi\left(-\sin\frac{\lambda}{2}\right)\frac{1}{2},$$

$$\frac{\partial}{\partial\phi}\sqrt{1-\cos^2\phi\cos^2\frac{\lambda}{2}} = \frac{1}{\sqrt{1-\cos^2\phi\cos^2\frac{\lambda}{2}}}\frac{1}{2}2\cos\phi\sin\phi\cos^2\frac{\lambda}{2},$$

$$\frac{\partial}{\partial\lambda}\sqrt{1-\cos^2\phi\cos^2\frac{\lambda}{2}} = \frac{1}{\sqrt{1-\cos^2\phi\cos^2\frac{\lambda}{2}}}\frac{1}{2}2\frac{1}{2}\cos^2\phi\cos\frac{\lambda}{2}\sin\frac{\lambda}{2}.$$

Hence

$$\frac{\partial}{\partial \phi}\frac{1}{\operatorname{sinc}\alpha} = \frac{\frac{\sin\phi\cos\frac{\lambda}{2}}{\sqrt{1-\cos^2\phi\cos^2\frac{\lambda}{2}}}\sqrt{1-\cos^2\phi\cos^2\frac{\lambda}{2}} - \arccos\left(\cos\phi\cos\frac{\lambda}{2}\right)\frac{\cos\phi\sin\phi\cos^2\frac{\lambda}{2}}{\sqrt{1-\cos^2\phi\cos^2\frac{\lambda}{2}}}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}}$$

$$= \frac{\sin\phi\cos\frac{\lambda}{2} - \frac{\cos\phi\sin\phi\cos^2\frac{\lambda}{2}}{\operatorname{sinc}\alpha}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}} = \frac{1-\frac{\cos\phi\cos\frac{\lambda}{2}}{\operatorname{sinc}\alpha}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}}\sin\phi\cos\frac{\lambda}{2},$$

$$\frac{\partial}{\partial \lambda}\frac{1}{\operatorname{sinc}\alpha} = \frac{\frac{\cos\phi\sin\frac{\lambda}{2}}{\sqrt{1-\cos^2\phi\cos^2\frac{\lambda}{2}}}\frac{1}{2}\sqrt{1-\cos^2\phi\cos^2\frac{\lambda}{2}} - \arccos\left(\cos\phi\cos\frac{\lambda}{2}\right)\frac{\cos^2\phi\cos\frac{\lambda}{2}\sin\frac{\lambda}{2}}{2\sqrt{1-\cos^2\phi\cos^2\frac{\lambda}{2}}}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}}$$

$$= \frac{\frac{1}{2}\cos\phi\sin\frac{\lambda}{2} - \frac{\cos^2\phi\cos\frac{\lambda}{2}\sin\frac{\lambda}{2}}{2\operatorname{sinc}\alpha}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}} = \frac{1}{2}\frac{1-\frac{\cos\phi\cos\frac{\lambda}{2}}{\operatorname{sinc}\alpha}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}}\cos\phi\sin\frac{\lambda}{2}.$$

Thus

$$\frac{dx'}{d\phi} = \frac{d}{d\phi}\frac{1}{2}\left[\lambda\cos\phi_0 + \frac{2\cos\phi\sin\frac{\lambda}{2}}{\operatorname{sinc}\alpha}\right] = \frac{d}{d\phi}\frac{1}{\operatorname{sinc}\alpha}\cos\phi\sin\frac{\lambda}{2}$$

$$= \left[-\frac{\sin\phi}{\operatorname{sinc}\alpha} + \frac{1-\frac{\cos\phi\cos\frac{\lambda}{2}}{\operatorname{sinc}\alpha}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}}\sin\phi\cos\frac{\lambda}{2}\cos\phi\right]\sin\frac{\lambda}{2}$$

$$= \left[\frac{\cos\frac{\lambda}{2}\cos\phi}{1-\cos^2\phi\cos^2\frac{\lambda}{2}} + \frac{-1+1-\cos^2\phi\cos^2\frac{\lambda}{2}}{\left(1-\cos^2\phi\cos^2\frac{\lambda}{2}\right)\operatorname{sinc}\alpha} - \frac{1}{\operatorname{sinc}\alpha}\right]\sin\phi\sin\frac{\lambda}{2}$$

$$= \left[\frac{\cos\frac{\lambda}{2}\cos\phi}{1-\cos^2\phi\cos^2\frac{\lambda}{2}} - \frac{1}{\left(1-\cos^2\phi\cos^2\frac{\lambda}{2}\right)\operatorname{sinc}\alpha}\right]\sin\phi\sin\frac{\lambda}{2}$$

$$= \frac{\sin 2\phi\sin\lambda}{1-\cos^2\phi\cos^2\frac{\lambda}{2}} - \frac{\sin\phi\sin\frac{\lambda}{2}}{\left(1-\cos^2\phi\cos^2\frac{\lambda}{2}\right)\operatorname{sinc}\alpha},$$

$$\frac{dx'}{d\lambda} = \frac{d}{d\lambda}\frac{1}{2}\left[\lambda\cos\phi_0 + \frac{2\cos\phi\sin\frac{\lambda}{2}}{\operatorname{sinc}\alpha}\right] = \frac{\cos\phi_0}{2} + \frac{d}{d\lambda}\frac{1}{\operatorname{sinc}\alpha}\cos\phi\sin\frac{\lambda}{2}$$

$$= \frac{\cos\phi_0}{2} + \left[\frac{\cos\frac{\lambda}{2}}{2\operatorname{sinc}\alpha} + \frac{1}{2}\frac{1-\frac{\cos\phi\cos\frac{\lambda}{2}}{\operatorname{sinc}\alpha}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}}\cos\phi\sin\frac{\lambda}{2}\sin\frac{\lambda}{2}\right]\cos\phi$$

$$= \frac{\cos\phi_0}{2} + \left[\frac{\cos\frac{\lambda}{2}}{\operatorname{sinc}\alpha} + \frac{1-\frac{\cos\phi\cos\frac{\lambda}{2}}{\operatorname{sinc}\alpha}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}}\cos\phi\sin^2\frac{\lambda}{2}\right]\frac{\cos\phi}{2}$$

$$= \frac{\cos\phi_0}{2} + \left[\frac{\cos\phi\sin^2\frac{\lambda}{2}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}} + \frac{\left(1-\cos^2\phi\cos^2\frac{\lambda}{2}\right)-\cos^2\phi\sin^2\frac{\lambda}{2}}{\left(1-\cos^2\phi\cos^2\frac{\lambda}{2}\right)\operatorname{sinc}\alpha}\cos\frac{\lambda}{2}\right]\frac{\cos\phi}{2}$$

$$= \frac{\cos\phi_0}{2} + \left[\frac{\cos\phi\sin^2\frac{\lambda}{2}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}} + \frac{\cos\frac{\lambda}{2}\sin^2\phi}{\left(1-\cos^2\phi\cos^2\frac{\lambda}{2}\right)\operatorname{sinc}\alpha}\right]\frac{\cos\phi}{2},$$

$$\frac{dy'}{d\phi} = \frac{d}{d\phi}\frac{1}{2}\left[\phi + \frac{\sin\phi}{\operatorname{sinc}\alpha}\right] = \frac{1}{2} + \frac{1}{2}\left[\frac{\cos\phi}{\operatorname{sinc}\alpha} + \sin\phi\frac{1-\frac{\cos\phi\cos\frac{\lambda}{2}}{\operatorname{sinc}\alpha}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}}\sin\phi\cos\frac{\lambda}{2}\right]$$
$$= \frac{1}{2} + \frac{1}{2}\left[\frac{\sin^2\phi\cos\frac{\lambda}{2}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}} + \frac{\left(1-\cos^2\phi\cos^2\frac{\lambda}{2}\right) - \cos^2\frac{\lambda}{2}\sin^2\phi}{\left(1-\cos^2\phi\cos^2\frac{\lambda}{2}\right)\operatorname{sinc}\alpha}\cos\phi\right]$$
$$= \frac{1}{2} + \frac{1}{2}\left[\frac{\sin^2\phi\cos\frac{\lambda}{2}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}} + \frac{\cos\phi\sin^2\frac{\lambda}{2}}{\left(1-\cos^2\phi\cos^2\frac{\lambda}{2}\right)\operatorname{sinc}\alpha}\right],$$

$$\frac{dy'}{d\lambda} = \frac{d}{d\lambda}\frac{1}{2}\left[\phi + \frac{\sin\phi}{\operatorname{sinc}\alpha}\right] = \frac{\sin\phi}{2}\frac{1}{2}\frac{1-\frac{\cos\phi\cos\frac{\lambda}{2}}{\operatorname{sinc}\alpha}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}}\cos\phi\sin\frac{\lambda}{2}$$
$$= \frac{\sin\phi\cos\phi\sin\frac{\lambda}{2}}{4}\left[\frac{1}{1-\cos^2\phi\cos^2\frac{\lambda}{2}} - \frac{\frac{\cos\phi\cos\frac{\lambda}{2}}{\operatorname{sinc}\alpha}}{1-\cos^2\phi\cos^2\frac{\lambda}{2}}\right]$$
$$= \frac{\sin 2\phi\sin\frac{\lambda}{2}}{8}\left[\frac{1}{1-\cos^2\phi\cos^2\frac{\lambda}{2}} - \frac{\cos\phi\cos\frac{\lambda}{2}}{\left(1-\cos^2\phi\cos^2\frac{\lambda}{2}\right)\operatorname{sinc}\alpha}\right].$$

Putting everything together, we see that we can project the Bloch sphere as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = (P \circ H \circ T)(\boldsymbol{n}),$$

where $H : (\phi, \theta) \mapsto (\phi - \pi/2, \theta - \pi) = (\varphi, \lambda)$ and T was defined in equation (D.15). As the push-forward of H is the identity, we have

$$\begin{pmatrix} \frac{dx'}{d\theta} \\ \frac{dy'}{d\theta} \end{pmatrix} = P_{*,H(T(\boldsymbol{n}))}T_{*,\boldsymbol{n}}\frac{d\boldsymbol{n}}{d\theta}.$$

Note that the θ we differentiate with respect to here is the time and not the angle in spherical coordinates. The push-forward of the coordinate transformation was given in equation (D.16) and the push-forward of the Winkel tripel projection will be

$$P_{*,p} = \begin{pmatrix} \left.\frac{dx'}{d\lambda}\right|_p & \left.\frac{dx'}{d\phi}\right|_p \\ \left.\frac{dy'}{d\lambda}\right|_p & \left.\frac{dy'}{d\phi}\right|_p \end{pmatrix}.$$

The figures depicting the flow after transformation can be found in the figure section D.4.

D.4 Full figures of state flow

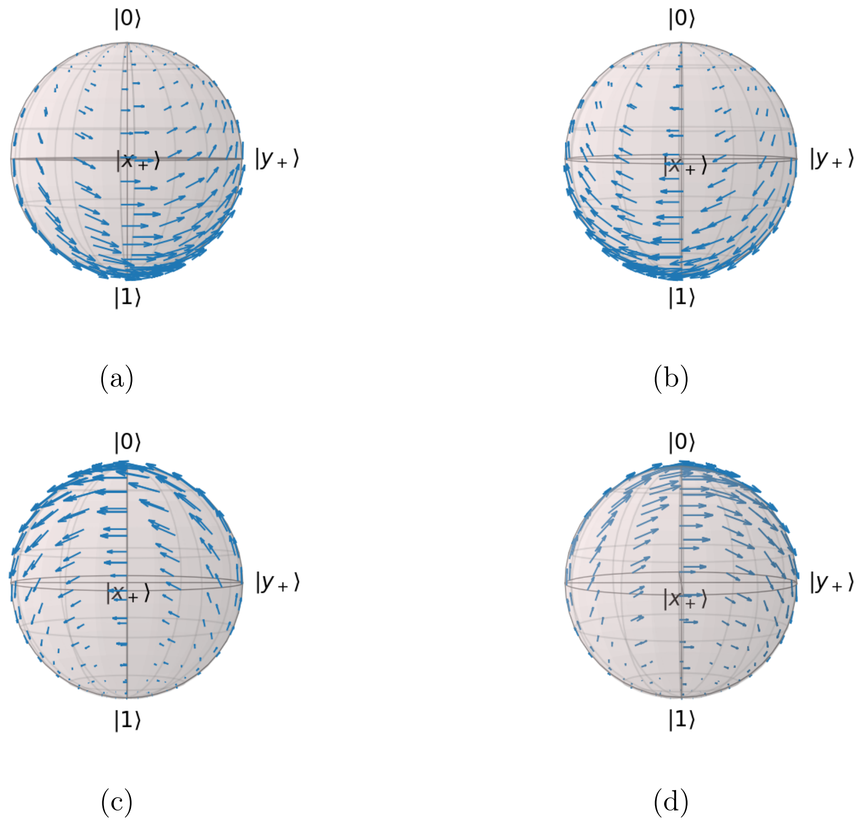
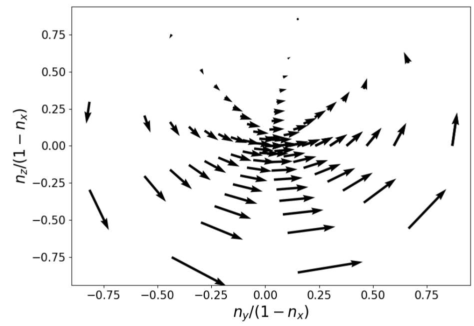
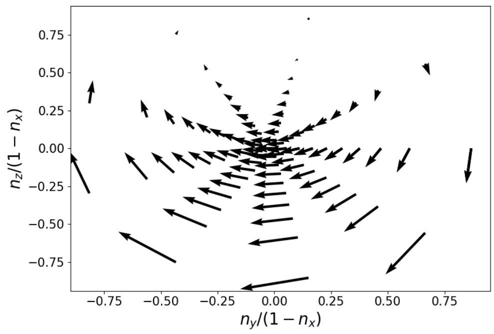


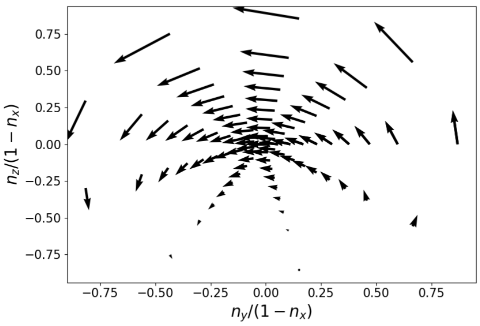
Figure D.2: The flow of the state given by equation (5.9) for H_{I^+}. Figure D.2a shows the flow for $k = 0$ and Figure D.2b shows the flow for $k = 1$. To the right in both subfigures we have removed the opaque color of the Bloch sphere and changed $|0\rangle = z$, $|\uparrow_x\rangle = x$ and $|\uparrow_y\rangle = y$. The flow goes in circles always ending up at the state $|0\rangle$. The state $|0\rangle$ is a fixed point where there is no flow. We see that the direction of the flow changes depending on which measurement outcome we get. Figure D.2c and D.2d shows the flow of the state, given by equation (5.9) for H_{I^-}, for $k = 0$ and $k = 1$, respectively. We now see that the fixed point has changed to $|1\rangle$. The flow goes in circles always ending up at the state $|1\rangle$ this time. We again see that the direction of the flow changes depending on which measurement outcome we get.



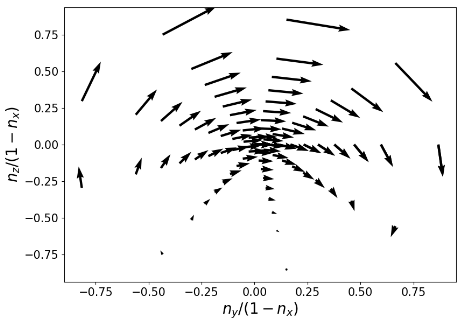
(a)



(b)



(c)



(d)

Figure D.3: A stereographic projection of the flow of the state given by equation (5.9) for H_{I^+}. This is the same flow as depicted in Figure D.2. Figure D.3a shows the flow for $k = 0$ and Figure D.3b shows the flow for $k = 1$. In both subfigures D.3a and D.3b we have used $\boldsymbol{n} = (1, 0, 0)$ as the north pole and only show the south half of the sphere. We would get the exact same figure using $\boldsymbol{n} = (-1, 0, 0)$ as the north pole and depicting only the south half of the sphere in this case. It is here easier to see the path which a state flows: It will move in a circle towards the state $|0\rangle$ where it stops completely. The flow is fast close to $|1\rangle$ and slow close to $|0\rangle$. Equation (D.14) gives an explicit expression for the flow in Figure D.3a. We see that the direction of the flow changes depending on which measurement outcome we get. Figure D.3c and D.3d shows the flow of the state, given by equation (5.9) for H_{I^-}, for $k = 0$ and $k = 1$, respectively. We now see that the fixed point has changed to $|1\rangle$. The flow goes in circles always ending up at the state $|1\rangle$ this time. We again see that the direction of the flow changes depending on which measurement outcome we get. In both subfigure D.3c and D.3d we have again used $\boldsymbol{n} = (1, 0, 0)$ as the north pole and only show the south half of the sphere. We would get the exact same figure using $\boldsymbol{n} = (-1, 0, 0)$ as the north pole and depicting only the south half of the sphere in this case.

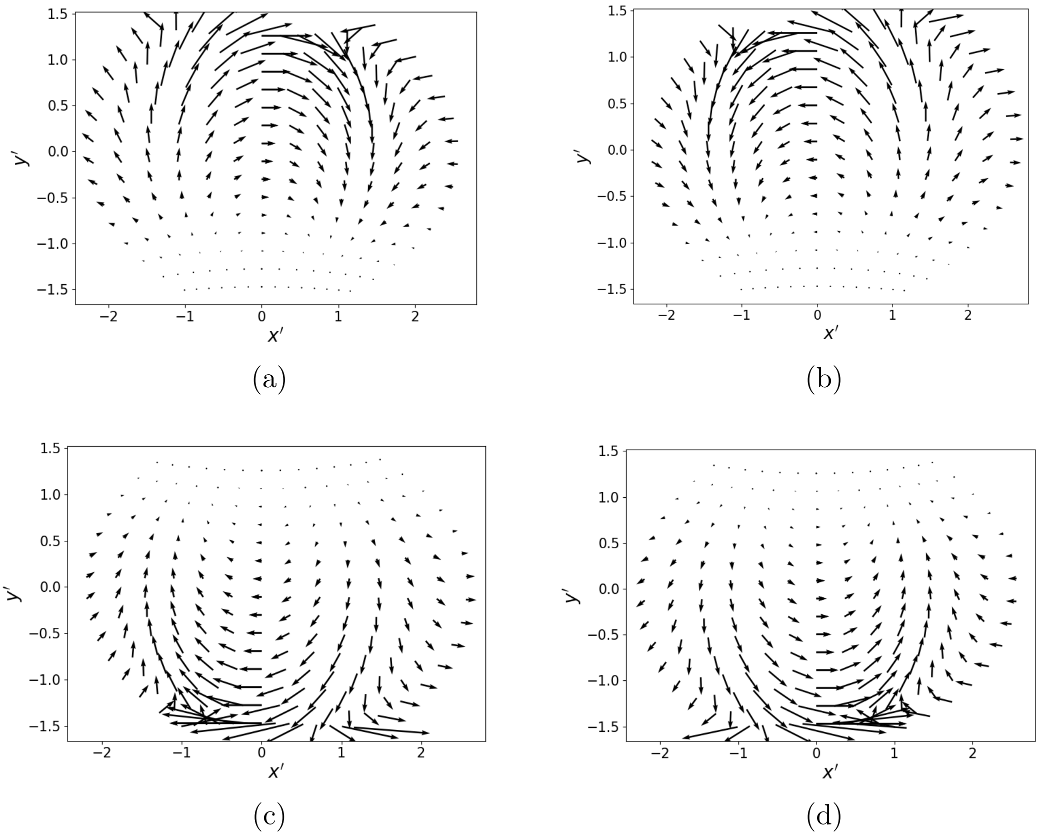


Figure D.4: Winkel tripel projection of the flow of the state given by equation (5.9) for H_{I^+}. This is the same flow as depicted in Figure D.2 and we have used $\cos(\varphi_0) = 2/\pi$, where φ_0 is the standard parallel for the equirectangular projection. Figure D.4a shows the flow for $k = 0$ and Figure D.4b shows the flow for $k = 1$. We now get the full flow, contrary to the stereographic projection we showed in Figure D.3. We again see the circular flow in the middle of the figures. The flow to the left in the figures also follows a circular motion which continues at the right side. Figure D.4c and D.4d shows the flow of the state, given by equation (5.9) for H_{I^-}, for $k = 0$ and $k = 1$, respectively. We now see that the fixed point has changed to $|1\rangle$. The flow goes in circles always ending up at the state $|1\rangle$ this time. We again see that the direction of the flow changes depending on which measurement outcome we get.

Appendix E

Dimension analysis for the model proposed by Longva

We want to find all interaction Hamiltonians H_I that describe the Linblad equation with rate $\Gamma = 2\frac{\theta^2}{\delta t}$ and Lindblad operator $L = \sigma_+ = \frac{1}{2}(\sigma_x + i\sigma_y)$. We assume $H_I = \sum_{ij} h_{ij}\sigma_i \otimes \sigma_j$. This then gives us $\mu = \frac{1}{\sqrt{2}}\begin{pmatrix}1, & i, & 0\end{pmatrix}^T$ and $\lambda = 2$. Hence,

$$M = \lambda \left|\mu\right\rangle \left\langle\mu\right| = \begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The system of equation is therefore (either the upper or lower triangle of the matrix in (5.10))

$$\begin{aligned} h_{11}^2 + h_{12}^2 + h_{13}^2 &= 1, & h_{21}h_{11} + h_{22}h_{12} + h_{23}h_{13} &= 0, & h_{21}h_{12} - h_{22}h_{11} &= -1, \\ h_{31}h_{11} + h_{32}h_{12} + h_{33}h_{13} &= 0, & h_{31}h_{12} - h_{32}h_{11} &= 0, & h_{21}^2 + h_{22}^2 + h_{23}^2 &= 1, \\ h_{31}h_{21} + h_{32}h_{22} + h_{33}h_{23} &= 0, & h_{31}h_{22} - h_{32}h_{21} &= 0, & h_{31}^2 + h_{32}^2 + h_{33}^2 &= 0. \end{aligned}$$

Observation: Equation nine, $\sum_{i=1}^3 h_{3i}^2 = 0$, means $h_{3i} = 0$ for $i = 1, 2, 3$ (since $h_{ij} \in \mathbb{R}$). This is even more general; If $\sum_i h_{ji}^2 = 0$ for any $j = 1, 2, 3$, then we must have $h_{ji} = 0$ for $i = 1, 2, 3$. In other word, we only have the following equations following equations:

$$h_{11}^2 + h_{12}^2 + h_{13}^2 = 1, \qquad h_{21}h_{11} + h_{22}h_{12} + h_{23}h_{13} = 0, \tag{E.1}$$

$$h_{21}h_{12} - h_{22}h_{11} = -1, \qquad h_{21}^2 + h_{22}^2 + h_{23}^2 = 1. \tag{E.2}$$

We will now proceed case by case. Assume first $h_{11} = 0$. The first part of equation (E.2) then gives $h_{21}h_{12} = -1$, which means we must have $h_{12}, h_{21} \neq 0$. The second part of equation (E.1) then reads $h_{22} = -\frac{h_{23}h_{13}}{h_{12}}$, and inserting into the second part of (E.2) gives

$$\frac{1}{h_{12}^2} + \frac{h_{23}^2 h_{13}^2}{h_{12}^2} + h_{23}^2 = 1 \quad \Rightarrow \quad h_{23} = \pm\sqrt{\frac{1 - \frac{1}{h_{12}^2}}{1 + \frac{h_{13}^2}{h_{12}^2}}} = \pm\sqrt{\frac{h_{12}^2 - 1}{h_{12}^2 + h_{13}^2}}.$$

As the relation $h_{12}^2 + h_{13}^2 = 1$ from equation (E.1) tells us that $h_{12}^2, h_{13}^2 \leq 1$, we must have $h_{12} = \pm 1$ for h_{23} to be real. This gives $h_{13} = 0$ and $h_{23} = 0$, which again means $h_{22} = 0$

from the last part of equation (E.2). All in all, we are left with $h_{12} = \pm 1$, $h_{21} = \mp 1$, and all others equal to zero. This means $H = \pm(\sigma_x \otimes \sigma_y - \sigma_y \otimes \sigma_x)$ will give the same dynamic.

Next, assume $h_{11} = \pm 1$. Then the first part of equation (E.1) gives $h_{12} = h_{13} = 0$. The last part of equation (E.1) then reads $h_{21} = 0$, and the first part of equation (E.2) reads $h_{22} = \pm 1$. Inserting into the last part of equation (E.2), we get $h_{23} = 0$. Hence, $H = \pm(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)$.

We now assume $0 < h_{11}^2 < 1$. Assume further that $h_{12} = 0$. The last part of equation (E.1) reads $h_{21} = -\frac{h_{23}h_{13}}{h_{11}}$, and the first part of equation (E.2) reads $h_{22} = \frac{1}{h_{11}}$. Putting this into the last part of equation (E.2) gives

$$\frac{h_{23}^2 h_{13}^2}{h_{11}^2} + \frac{1}{h_{11}^2} + h_{23}^2 = 1 \Rightarrow h_{23}^2 \overbrace{(h_{13}^2 + h_{11}^2)}^{=1} = h_{11}^2 - 1,$$

which is impossible as $h_{23} \in \mathbb{R}$ and the above equation means $h_{23} = \pm\sqrt{h_{11}^2 - 1} \in \mathbb{C} \setminus \mathbb{R}$ as $h_{11}^2 < 1$. In other words, we cannot have $h_{12} = 0$. We therefore look at the case $0 < h_{11}^2 < 1$, $h_{12} \neq 0$ and $h_{13} = 0$. Equation (E.1) and (E.2) then give

$$h_{12} = \pm\sqrt{1 - h_{11}^2}, \qquad h_{21}h_{11} = -h_{22}h_{12}, \qquad h_{21}h_{12} + 1 = h_{22}h_{11}.$$

Hence,

$$h_{21}h_{11} = -\frac{h_{21}h_{12} + 1}{h_{11}}h_{12} \Rightarrow h_{21} \overbrace{(h_{11}^2 + h_{12}^2)}^{=1} = -h_{12} \Rightarrow h_{21} = -h_{12}.$$

This then gives $h_{22} = h_{11}$, which means $h_{21}^2 + h_{22}^2 = 1$. Thus $h_{23} = 0$ and we are left with the solution

$$\begin{aligned} H &= h_{11}\sigma_x \otimes \sigma_x \pm \sqrt{1 - h_{11}^2}\sigma_x \otimes \sigma_y \mp \sqrt{1 - h_{11}^2}\sigma_y \otimes \sigma_x + h_{11}\sigma_y \otimes \sigma_y \\ &= h_{11}(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y) \pm \sqrt{1 - h_{11}^2}(\sigma_x \otimes \sigma_y - \sigma_y \otimes \sigma_x) \end{aligned}$$

for $0 < h_{11}^2 < 1$.

To finish of, we look at the final case: $0 < h_{11}^2 < 1$, $h_{12} \neq 0$ and $h_{13} \neq 0$. We know $h_{13} \neq \pm 1$ by the first part of equation (E.1). We have $h_{21}h_{12} = h_{22}h_{11} - 1$ from the first part of equation (E.2), which means the last part of equation (E.2) is $\frac{h_{22}h_{11}^2 - h_{11}}{h_{12}} + h_{22}h_{12} + h_{23}h_{13} = 0$. We rewrite and get

$$\frac{h_{22}\overbrace{(h_{11}^2 + h_{12}^2)}^{=1-h_{13}^2}}{h_{12}} + h_{23}h_{13} = \frac{h_{11}}{h_{12}} \Rightarrow h_{23} = \frac{h_{11} - h_{22}(1 - h_{13}^2)}{h_{12}h_{13}}.$$

We are therefore left with

$$
\begin{aligned}
1 = h_{21}^2 + h_{22}^2 + h_{23}^2 &= \left(\frac{h_{22}h_{11} - 1}{h_{12}}\right)^2 + h_{22}^2 + \left(\frac{h_{11} - h_{22}(1 - h_{13}^2)}{h_{12}h_{13}}\right)^2 \\
h_{12}^2 &= (h_{22}^2 h_{11}^2 - 2h_{22}h_{11} + 1) + h_{22}^2 h_{12}^2 + \frac{h_{11}^2 - 2h_{11}h_{22}(1 - h_{13}^2) + h_{22}^2(1 - h_{13}^2)^2}{h_{13}^2} \\
h_{12}^2 h_{13}^2 &= h_{22}^2 \overbrace{(h_{11}^2 + h_{12}^2)}^{1-h_{13}^2} h_{13}^2 - 2h_{22}h_{11}h_{13}^2 + h_{13}^2 + [h_{11}^2 - 2h_{11}h_{22}(1 - h_{13}^2) + h_{22}^2(1 - h_{13}^2)^2] \\
h_{12}^2 h_{13}^2 &= h_{22}^2(1 - h_{13}^2)(h_{13}^2 + 1 - h_{13}^2) - 2h_{22}h_{11}(h_{13}^2 + 1 - h_{13}^2) + (h_{13}^2 + h_{11}^2) \\
0 &= h_{22}^2(1 - h_{13}^2) - 2h_{22}h_{11} + (h_{13}^2 + h_{11}^2 - h_{12}^2 h_{13}^2).
\end{aligned}
$$

We solve this quadratic equation:

$$
\begin{aligned}
h_{22} &= \frac{2h_{11} \pm \sqrt{4h_{11}^2 - 4(1 - h_{13}^2)(h_{13}^2 + h_{11}^2 - h_{12}^2 h_{13}^2)}}{2(1 - h_{13}^2)} \\
&= \frac{h_{11} \pm \sqrt{(h_{12}^2 h_{13}^2 - h_{13}^2) + h_{13}^2(h_{13}^2 + h_{11}^2 - h_{12}^2 h_{13}^2)}}{1 - h_{13}^2} \\
&= \frac{h_{11} \pm h_{13}\sqrt{(h_{12}^2 - 1) + (h_{13}^2 + h_{11}^2 - h_{12}^2 h_{13}^2)}}{1 - h_{13}^2} \\
&= \frac{h_{11} \pm h_{13}\sqrt{h_{12}^2(1 - h_{13}^2) + \overbrace{(h_{13}^2 - 1)}^{=-h_{11}-h_{12}} + h_{11}^2}}{1 - h_{13}^2} \\
&= \frac{h_{11} \pm h_{13}\sqrt{h_{12}^2(1 - h_{13}^2) - h_{12}^2}}{1 - h_{13}^2} = \frac{h_{11} \pm h_{13}\sqrt{-h_{12}^2 h_{13}^2}}{1 - h_{13}^2},
\end{aligned}
$$

which gives us a contradiction as this would mean $h_{22} \notin \mathbb{R}$. We must either have $h_{12} = 0$ or $h_{13} = 0$, which means $h_{13} = 0$ by the above argumentation.

All in all, we have shown that $L = \sqrt{\frac{\theta^2}{\delta t}} 2\sigma_+$ must come from a Hamiltonian

$$
H = h(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y) \pm \sqrt{1 - h^2}(\sigma_x \otimes \sigma_y - \sigma_y \otimes \sigma_x),
$$

where $h \in [-1, 1]$. This means that the subspace of the Hamiltonians generating this Lindblad operator is one dimensional, even though we have nine possible parameters to tweak and only six equation to satisfy.

Bibliography

[1] Vahid Ameri, Mohammad Eghbali-Arani, Andrea Mari, Alessandro Farace, Fardin Kheirandish, Vittorio Giovannetti, and Rosario Fazio. Mutual information as an order parameter for quantum synchronization. *Phys. Rev. A*, 91:012301, January 2015. URL: https://link.aps.org/doi/10.1103/PhysRevA.91.012301, doi:10.1103/PhysRevA.91.012301.

[2] Stéphane Attal. Lectures in quantum noise theory, lecture 2, tensor products and partial trace. Accsessed on 27.03.2023. URL: http://math.univ-lyon1.fr/~attal/Partial_traces.pdf.

[3] Norman Thomas John Bailey. *The Elements of Stochastic Processes*. Wiley Classics Library. Wiley Interscience, 1990.

[4] Todd A. Brun. A simple model of quantum trajectories. *American Journal of Physics*, 70(7):719–737, 2002. arXiv:https://doi.org/10.1119/1.1475328, doi:10.1119/1.1475328.

[5] Hui Cheng and Krishna C. Gupta. An Historical Note on Finite Rotations. *Journal of Applied Mechanics*, 56(1):139–145, March 1989. doi:10.1115/1.3176034.

[6] Thomas L. Curtright, David B. Fairlie, and Cosmas Zachos. *A concise treatise on quantum mechanics in phase space*. World Scientific, 2014.

[7] Theodore William Gamelin. *Complex Analysis*. Undergraduate Texts in Mathematics. Springer Science+Business Media New York, 1 edition, 2001. doi:https://doi.org/10.1007/978-0-387-21607-2.

[8] Herbert Goldstein, Charles P. Poole, and John L. Safko. *Classical Mechanics*. Addison Wesley, 3 edition, 2001.

[9] Christiaan Huygens. *Horologium Oscillatorium: Sive de Motu Pendulorum ad Horologia Aptato Demonstrationes Geometricae*. Apud F. Muguet, Parisiis, France, 1673. English translation: Richard J. Blackwell. Christiaan Huygens' the Pendulum Clock or Geometrical Demonstrations Concerning the Motion of Pendula As Applied to Clocks. Iowa State University Press, Ames, 1986. URL: https://gallica.bnf.fr/ark:/12148/bpt6k1523597w/.

[10] Cengizhan Ipbüker and Ibrahim Öztug Bildirici. A general algorithm for the inverse transformation of map projections using jacobian matrices. *Proceedings of the Third*

International Symposium Mathematical & Computational Applications, pages 175–182, 2002. Accesses from the original (PDF) on 27.05.23. URL: https://web.archive.org/web/20151208051806/http://atlas.selcuk.edu.tr/paperdb/papers/130.pdf.

[11] Michel Le Bellac and Patricia de Forcrand-Millard. *Quantum Physics*. Cambridge University Press, 2006. Translated by Patricia de Forcrand-Millard. doi:10.1017/CBO9780511616471.

[12] Tony E. Lee and H. R. Sadeghpour. Quantum synchronization of quantum van der pol oscillators with trapped ions. *Phys. Rev. Lett.*, 111:234101, Dec 2013. URL: https://link.aps.org/doi/10.1103/PhysRevLett.111.234101, doi:10.1103/PhysRevLett.111.234101.

[13] Erlend Andreas Longva. Frequency Synchronisation of a Two-Level Quantum System. Master's thesis, University of Oslo, 2022. URL: http://hdl.handle.net/10852/97038.

[14] Paul Charles Matthews. *Vector Calculus*. Springer Undergraduate Mathematics Series. Springer London, 1998.

[15] Valter Moretti. *Spectral Theory and Quantum Mechanics*. UNITEXT. Springer Cham, 2 edition, 2017. doi:https://doi.org/10.1007/978-3-319-70706-8.

[16] Gerard John Murphy. *C*-Algebras and Operator Theory*. Academic Press, 1990.

[17] Michael Aaron Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, 2010. doi:10.1017/CBO9780511976667.

[18] Álvaro Parra-López and Joakim Bergli. Synchronization in two-level quantum systems. *Phys. Rev. A*, 101:062104, June 2020. URL: https://link.aps.org/doi/10.1103/PhysRevA.101.062104, doi:10.1103/PhysRevA.101.062104.

[19] Vern Paulsen. *Completely Bounded Maps and Operator Algebras*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2002. doi:https://doi.org/10.1017/CBO9780511546631.

[20] Asher Peres. *Quantum Theory: Concepts and Methods*. Fundamental Theories of Physics. Springer Dordrecht, 1 edition, 2002. doi:https://doi.org/10.1007/0-306-47120-5.

[21] Arkady Pikovsky, Jürgen Kurths, and Michael Rosenblum. *Synchronization: A Universal Concept in Nonlinear Sciences*. Cambridge Nonlinear Science Series. Cambridge University Press, 2001.

[22] John Preskill. Lecture notes for physics 229: Quantum information and computation. http://www.theory.caltech.edu/~preskill/ph229/notes/book.ps, September 1998. Accessed 3. May 2023.

[23] Alexandre Roulet and Christoph Bruder. Synchronizing the smallest possible system. *Phys. Rev. Lett.*, 121:053601, July 2018. URL: https://link.aps.org/doi/10.1103/PhysRevLett.121.053601, doi:10.1103/PhysRevLett.121.053601.

[24] Jun John Sakurai and Jim Napolitano. *Modern Quantum Mechanics*. Cambridge University Press, 2 edition, 2017. doi:10.1017/9781108499996.

[25] Loring W. Tu. *An Introduction to Manifolds*. Universitext. Springer New York, 2 edition, 2010. doi:10.1007/978-1-4419-7400-6.

[26] Stefan Walter, Andreas Nunnenkamp, and Christoph Bruder. Quantum synchronization of a driven self-sustained oscillator. *Phys. Rev. Lett.*, 112:094102, March 2014. URL: https://link.aps.org/doi/10.1103/PhysRevLett.112.094102, doi:10.1103/PhysRevLett.112.094102.

[27] Stefan Walter, Andreas Nunnenkamp, and Christoph Bruder. Quantum synchronization of two van der pol oscillators. *Annalen der Physik*, 527(1-2):131–138, 2015. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/andp.201400144, arXiv:https://onlinelibrary.wiley.com/doi/pdf/10.1002/andp.201400144, doi:https://doi.org/10.1002/andp.201400144.

[28] Liyun Zhang, Zhao Wang, Yucheng Wang, Junhua Zhang, Zhigang Wu, Jianwen Jie, and Yao Lu. Observing quantum synchronization of a single trapped-ion qubit, 2023. arXiv:2205.05936.

[29] Oleg V. Zhirov and Dima L. Shepelyansky. Synchronization and bistability of a qubit coupled to a driven dissipative oscillator. *Phys. Rev. Lett.*, 100:014101, January 2008. URL: https://link.aps.org/doi/10.1103/PhysRevLett.100.014101, doi:10.1103/PhysRevLett.100.014101.