DEPT. OF MATH./CMA UNIV. OF OSLO
PURE MATHEMATICS No. 2
ISSN 0806–2439 January 2009

TIGHT PACKING A HYPERSPHERE WITH OTHERS OF LIKE KIND

PAUL C. KETTLER

ABSTRACT. All are familiar with the exercise of surrounding a circle with six circles of the same size. A similar exercise — surrounding a hypersphere with hyperspheres of the same size tightly packed — is not possible in any higher dimension. This note demonstrates the equivalence of this exercise to that of constructing specialized simplicial complexes, and then provides a necessary condition for surrounding a hypersphere, proving the condition is violated for hyperspheres of dimension greater than 1. An implication for the general packing of hyperspheres follows.

1. INTRODUCTION

Modern theory of packing spheres began with the two papers (Levenshtein 1979) and (Odlyzko and Sloane 1979). More recent developments are chronicled in (Conway and Sloane 1999), (Cohn and Elkies 2003), and (Cohn 2002). The last two are a sequence, which, like Beethoven's first two piano concerti, became published in reverse order. These are good foundational studies for the present œuvre, and are recommended with references within.

We begin with some rudimentary definitions. The n-sphere is the sphere of dimension n considered embedded in $\mathbb{R}^{n+1}$. The 1-sphere is the circle; the 2-sphere is the common sphere; all others carry the dimension as part of the name. Unless otherwise qualified, all spheres have radius one.

Think of the circle S_1 with six circles around it. One could form six equilateral triangles — 2-simplexes — joining the center of S_1 with the centers of the surrounding circles, joining those centers to form a hexagon. The insight looking forward is that the interior angles of these triangles — $\pi/3$ each — exactly complete a circle. In this case note that $n = 1$, and that $\operatorname{arcsec}(n+1)$ divides 2π six times. The concept of tightness appears, meaning that any circle touching S_1, say T_1, also touches two others.

Think next of the 2-sphere S_2, trying first to surround another 2-sphere T_2 touching S_2, with others of like kind also touching S_2. One finds by simple examination that five spheres will fit, but that a sixth will not. The curvature of S_2 now plays a role. The concept of tightness here is that any sphere touching T_2 and S_2 also touches two others.

As in the first example, construct now a series of 3-simplexes, or tetrahedra, joining the center of S_2 with the center of T_2 and with the centers of any two adjacent spheres to it. In analogy to the fitting of spheres, five of these 3-simplexes will fit without consuming a complete circle. Here the angle of relevance is the one joining the center of the common edge of the 3-simplexes to the outer vertices of any one of the simplexes. Such vertices lie in a plane, for all are equidistant from the two common vertices of the tetrahedra, and thus lie in

Date: 16 January 2009.
2000 *Mathematics Subject Classification.* 51M15, 51M20, 52C17.
Key words and phrases. Packing in n dimensions, simplicial complexes, geometric constructions.
Thanks to Professor Bernt Øksendal for inspiring this research.

the intersection of two 2-spheres, which is a circle. This angle, easily determined from the basic geometry of simplexes and an appeal to the law of cosines, is $\operatorname{arcsec}(3) = \operatorname{arcsec}(n+1)$, which does not divide 2π, so T_2 cannot be surrounded exactly by spheres of like kind.

2. The general case

Now the plan for generalizing becomes apparent, but first here is a brief digression on the geometry of simplexes. An inductive argument reveals that the height of a standard n-simplex, the convex hull of $(n+1)$ points joined by edges of unit length, is $\sqrt{(n+1)/(2n)}$. Here the height is measured as the distance from any vertex to the barycenter of the $(n-1)$-simplex opposite. Also, the angle between any two vertices to the barycenter of the $(n-2)$-simplex opposite is $\operatorname{arcsec}(n)$. In the examples above the simplexes have edges of length two, but our primary interest is the angle inscribed, which is invariant to scaling.

Following is the principal result, stated formally.

Theorem 2.1. *Tight packing a hypersphere with others of like kind is not possible in dimensions greater than 1.*

Proof. It shall be demonstrated that a single sphere attached to another of like kind cannot be surrounded by an integer number of spheres of like kind in dimensions greater than 1, implying the result.

In the general case, one attempts to surround an n-sphere T_n attached to another, S_n, by starting with n other n-spheres, $\{U_n^1, \dots, U_n^n\}$, all touching. Together their centers describe an $(n+1)$-simplex Δ_{n+1}^1. At each iteration one attaches another $(n+1)$-simplex to the previous one on a common n-simplex boundary by adding a new vertex. Let these simplexes be $\{\Delta_{n+1}^2, \Delta_{n+1}^3, \dots\}$.

Vertices added this way are co-planar, meaning in the same 2-plane, because they are equidistant from a set of n distinct vertices, to wit, the centers of $\{S_n, T_n, U_n^1 \dots, U_n^{n-2}\}$ (reducing to $\{S_1\}$ and $\{S_2, T_2\}$ for $n = 1$ and $n = 2$, respectively,) and thus are in the intersection of n – n-spheres, which is a circle.

The angle of relevance within this circle is the one joining two vertices of a simplex by the barycenter of the common $(n-1)$-simplex opposite. This angle is $\operatorname{arcsec}(n+1)$. If and only if this angle divides 2π can one surround T_n by spheres of like kind.

The issue then devolves upon finding a solution in natural numbers n and k such that

$$k \cdot \operatorname{arcsec}(n+1) = 2\pi$$

The only solution to this equation is the familiar $n = 1$ and $k = 6$, the circle with six circles around. □

Remark. An interesting conclusion involves finding the [irrational] dimension for which $k = 5$. We know the solution for $k = 6$, and $k = 4$ cannot occur because

$$\lim_{n \to \infty} \operatorname{arcsec}(n+1) = \frac{\pi}{2},$$

never attained. An easy analysis reveals that $k = 5$ at $n = \sqrt{5}$. Implications for comparisons with the golden ratio and the Fibonacci numbers tempt, but will be set aside.

Finally, one sees that a tight packing of n-spheres into any $(n+1)$-dimensional manifold is not possible if the manifold admits the inclusion of an n-sphere of radius 4. The demonstration is left to the reader.

References

Cohn, H. (2002). New upper bounds on sphere packings II. *Geom. Topol. (electronic) 6*, 329–353.

Cohn, H. and N. Elkies (2003). New upper bounds on sphere packings I. *Ann. of Math. (2) 157*(2), 689–714.

Conway, J. H. and N. J. A. Sloane (1999). *Sphere packings, lattices and groups* (3rd ed.). Number 290 in Grundlehren der Mathematischen Wissenschaften. New York: Springer-Verlag.

Levenshtein, V. I. (1979). Boundaries for packings in n-dimensional Euclidean space. *Dokl. Akad. Nauk SSSR 245*(6), 1299–1303.

Odlyzko, A. M. and N. J. A. Sloane (1979). New bounds on the number of unit spheres that can touch a unit sphere in n dimensions. *J. Combin. Theory Ser. A 26*(2), 210–214.

Paul C. Kettler
Centre of Mathematics for Applications
Department of Mathematics
University of Oslo
P.O. Box 1053, Blindern
N–0316 Oslo
Norway
E-mail address: paulck@math.uio.no
URL: http://www.math.uio.no/~paulck/