Hydrocarbon production optimization in multi-reservoir fields - tools for enhanced value chain analysis

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At Norwegian Computing Center I would also like to thank my co-author Managing Director Lars Holden for his interest in my project and for his important contributions. At Det Norske Veritas (DNV) I am grateful to my co-authors Principal Consultant Steinar Lyngroth, Associate Director Frank Børre Pedersen, Principal Specialist Jingzhen Xu and Consultant Tore Aasheim for their positive attitude, their contributions and for sharing their extensive knowledge of the energy industry. At the Statistics Division I would like to thank all PhD research fellows for good spirit and friendly atmosphere. In particular I thank my friend Pål N. Henriksen for entertaining and interesting discussions and lunch breaks. I would also like to thank PhD research fellow Trond Reitan for useful contributions in the fields of physics, computer science, mathematics and statistics and entertaining discussions.

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Preface

This thesis has been prepared in partial fulfillment of the requirements for the Ph.D. degree at the Departments of Mathematics, Faculty of Mathematics and Natural Sciences at the University Of Oslo. Associate Professor Arne B. Huseby, University Of Oslo has been thesis supervisor, while Senior Advisor Eivind Damsleth, AGR Petroleum Services and Chief Research Scientist Xeni K. Dimakos, Norwegian Computing Center have been co- supervisors. The work started in April 2005 and was finalized in August 2008.

This PhD project is a part of the project *Statistical Analysis of Risk* at the Department of Mathematics, University of Oslo, in collaboration with Norwegian Computing Center. The project has received financial support from the Norwegian Research Council, grant number 154079.

The thesis contains an introductory chapter and four papers. The introduction is prepared for this thesis exclusively, while the four papers are published or submitted for publication.

Paper information

In the following we present the authors and indicate the publication status of each paper. The roles of each author are described. Summaries of the papers are given in the introductory chapter, where a broad outline of the PhD project is presented.

Paper I: Multi-segment production profiles - a tool for enhanced total value chain analysis

Authors: Nils F. Haavardsson, Arne B. Huseby

Publication details: Journal of Petroleum Science and Engineering, Vol. 58, No. 2 (2007), p. 325-338. Available as technical research report - Statistical Research Report No 2, 2007, Statistical Research Report Series (www.math.uio.no).

Comments: Arne B. Huseby had the project idea, developed the concept and developed a generic, object-oriented computer program structure. Nils F. Haavardsson contributed in the development of the concept, the programming and created examples. The manuscript work was shared among the authors.

Paper II: Multi-reservoir Production Optimization

Authors: Arne B. Huseby, Nils F. Haavardsson

Publication details: Sent for review to European Journal of Operational Research on April, 24th 2008. Available as technical research report - Statistical Research Report No 4, 2008, Statistical Research Report Series (www.math.uio.no).

Comments: The major parts of this work is done by Arne B. Huseby. Nils F. Haavardsson contributed in the development of the concept, worked with implementation, created examples and did some parts of the manuscript work.

Paper III: A parametric class of production strategies for multi-reservoir production optimization.

Authors: Nils F. Haavardsson, Arne B. Huseby and Lars Holden[1]

[1] Norwegian Computing Center

Publication details: Sent for review to *Journal of Petroleum Science and Engineering* on September, 3rd 2008. Available as technical research report - Statistical Research Report No 8, 2008, Statistical Research Report Series (www.math.uio.no).

Comments: Lars Holden had the project idea. Mr. Holden's contribution lead to what is referred to as *first-order fixed-weight* strategies. Nils F. Haavardsson and Arne B. Huseby developed the concept, developed new theory that lead to what is referred to as *higher-order fixed-weight* strategies, implemented the methods and extended the object-oriented prototype developed in earlier papers. Arne B. Huseby wrote some parts of the manuscript. The rest of the manuscript was written by Nils F. Haavardsson.

Paper IV: Hydrocarbon production optimization in fields with different ownership and commercial interests.

Authors: Nils F. Haavardsson, Steinar Lyngroth[1], Frank Børre Pedersen[1], Jingzhen Xu[1], Tore I. Aasheim[1] and Arne B. Huseby

[1] Det Norske Veritas

Publication details: Will be sent for review to *SPE Reservoir Evaluation & Engineering*. Available as technical research report - Statistical Research Report No 9, 2008, Statistical Research Report Series (www.math.uio.no).

Comments: Steinar Lyngroth, Nils F. Haavardsson and Arne Bang Huseby had the project idea. Steinar Lyngroth and Frank Børre Pedersen provided data and case material. Nils F. Haavardsson developed the concept, in cooperation with the other authors. Nils F. Haavardsson extended the framework from earlier papers to handle the increased complexity. Nils F. Haavardsson did the programming and wrote the manuscript.

Contents

Chapter 1

Introduction

The capital expenditure involved in an offshore field development is typically huge, and many investment decisions are irreversible and finance is committed for the long-term. Therefore the oil companies invest heavily in geophysical interpretations, geological models, flow simulation models and economic models to obtain satisfactory decision support.

Economic models used in decision support typically utilize simplified production models, often referred to as *fast models* to be able to create a broad, overall perspective of large offshore fields. The fast models enable analyzes of the economic impact of different scenarios and concepts. Decisions are made analyzing the trade off between the cost (operational costs and capital expenditure and the offered functionality (production capacities, infrastructure flexibility, technology contributions etc.). A fast model typically incorporates knowledge about the properties of the reservoir. The framework needs to be flexible so that economic and strategic factors may to be taken into account. The primary focus of this PhD project has been to develop a generic production optimization framework that uses simplified production profile models. The framework is designed for total value chain analyzes of large offshore field development projects. In such an analysis all revenues, costs and investments in the oil and gas value chain are modelled to obtain assessments of project profitability and different strategies. The revenues and tariffs are calculated from the oil, gas and water production profiles. The cash flow is calculated deducting the capital expenditure, the operational costs and the tariffs from the revenues. The net present value of the project is calculated from the cash flows, and the after tax profit can be assessed. For a discussion of total value chain analysis, see Huseby & Brækken (2000) or Hollund et al. (2007).

We start in Section 1.1 by presenting some new challenges in the oil and gas value chain. In Section 1.2 we continue by presenting the traditions of related, relevant research and some important features of the works done. In particular, this PhD project is placed within these research traditions. Section 1.3 introduces some basic concepts and ideas, before a broad outline of the PhD project is given in Section 1.4. Section 1.5 gives guidelines for future research. An interesting extension of the framework developed in this PhD project involves decision making under uncertainty and capital budgeting decisions under uncertainty. This area of research is often referred to as *real options* and is also discussed in Section 1.5. In hydrocarbon development projects risks are large and there is considerable uncertainty associated to important parameters affecting the production. By appropriate modelling of uncertainty in a physical model, we are able to model the information we receive as the reservoirs are produced. However, caution and skill are needed in the modelling of uncertainty in a physical model, which is the topic of Section 1.6.

In Section 1.7 we review some relevant, canonical optimization techniques we have considered using in this PhD project. However, we found that these techniques were not so well-suited for our application. Instead we have developed our own framework, founded on convex optimization principles, see Boyd & Vandeberghe (2004*a*). Quoting Boyd & Vandeberghe (2004*a*):

> *The idea that convex optimization problems are tractable is not new. It has long been recognized that the theory of convex optimization is far more straightforward (and complete) than the theory of general nonlinear optimization. In this context Rockafellar stated, in his 1993 SIAM review survey paper Rockafeller (1993):*
>
> > *In fact the great watershed in optimization isn't between linearity and nonlinearity, but convexity and non-convexity.*

Numerical methods and algorithms are developed in parallel with the theoretical developments. An object-oriented prototype in Java is developed for this purpose, of which the most important features are described in Section 1.8.

1.1 New challenges in the oil and gas value chain

The oil and gas industry plays a vital role in the Norwegian economy. Natural resources has made Norway a wealthy nation. However, the oil fields on the Norwegian continental shelf are maturing. This development is a part of a global trend in the western world. Production optimization and increased outtake become even more important facing these new challenges, see Meling (2006). This section is based on Meling (2006), which presents a quantitative approach to assess oil supply and demand in the world today.

Due to a shift in the oil demand from western world countries to emerging economies like China, India, Brazil and Russia the oil demand is expected to increase in the coming years. To meet the increasing demand global oil production needs to be increased in the future. In contrast to the belief of industry experts based on more qualitative assessments, Meling (2006) states that exploration and development are less important than increased outtake from existing fields to increase the global oil production.

1.1.1 The global supply and demand of oil

Future oil demand

Meling (2006) comments that the demand for oil in the last 100 years has mirrored economic and political history. The demand growth is oscillating fiercely, especially before the 1970s. This period is often referred to as "The Golden Age". Energy-intensive industries, such as steel and car production and the cold war contributed to considerable oil demand in this period. After the oil embargo in 1973-1974 and the oil crises of the early 1980s, yearly demand growth has been moderately stable, on average slightly above 1.6 %.

One reason for this development is that rich economies are less dependent on oil than they used to be. Energy conservation, a shift to other fuels and a decline in the importance of heavy, energy-intensive industries have reduced oil consumption. Software, consultancy and mobile telephones use far less oil than steel or car production. For each dollar of GDP (in constant prices) rich economies now use nearly 50% less oil than in 1973.

In recent years, from around 2002, there has been a surging demand from developing countries, such as China. If we assume an economic growth of some 7 % for China, a gross national product of 3,000 USD per capita and an oil consumption of 6 barrels per capita per year, China will demand 25 millions barrels of oil per day in 2020, see Meling (2006) for details. This demand equals the US demand in 2020, basing our estimate of US demand on demographic development only. With India and other emerging economies following it is evident that a serious oil supply challenge will arise. Some simple calculations reveal that such volume growth is impossible, there is not enough oil, neither now nor in the future.

Future oil supply

Reserves and resources. According to Meling (2006), approximately 2.2 trillions of barrels of oil has been discovered and approximately 1 trillion barrels has been produced, leaving some 1.2 trillion barrels of remaining oil to be produced. The reserves are very unevenly distributed; while Non-OPEC countries have produced the equivalent of more than three quarters of their aggregated reserves, the same number for the OPEC countries is below 30 %. Of remaining developed resources, non-OPEC countries hold some 25 %. In addition, Canada holds almost 200 billion barrels of ultra heavy crude produced by mining, representing considerable environmental challenges.

Reserve replacement by exploration and development. According to Meling (2006), field size is 10 times more important than the number of exploration wells drilled in explaining discovered volumes. Thus, the declining exploration volumes are strongly related to reduced field size. Due to reduced field sizes on discovered fields in recent years, future exploration alone will be insufficient to grow production to satisfy future demand.

Outtake. In Meling (2006) outtake is referred to as a measure of the yearly production of the remaining developed reserves. In the same paper it is remarked that oil market analysts have ignored the importance and magnitude of field redevelopment and production optimization. Non-OPEC countries have had a larger outtake growth than the OPEC countries since 1910. Since the early 1930s the difference has increased. New technology, such as horizontal drilling, deep water technology and the introduction of 3D and 4D seismic and interpretation, has been an important contributor to this increased outtake for non-OPEC countries, as discussed by Meling (2006). Offshore producers have stronger incentives than onshore producers to optimize production, since their operational costs are far greater.

1.1.2 Summary

The global supply and demand of oil is a complex, controversial, macro economic issue of considerable public and professional interest. Predictions about the future will seldom entail complete objectivity. Different analyzes differ and contradict in conclusions due to model choice, input choice, methodology, opinions and convictions. A less controversial issue is the supply situation for Norway, which face maturing oil fields in the North Sea. From a commercial point of view any operator would also be interested in an optimal production strategy for the reservoirs in its portfolio. Therefore, we believe it is relevant from an industry perspective with the national interests of Norway and sound business principles in mind to develop methods for production optimization.

1.2 Optimization in oil and gas recovery

1.2.1 Contributions in petroleum engineering

Optimization in upstream oil and gas recovery in petroleum engineering has generally been focused in three areas: (a) production scheduling, (b) well placement, and (c) production facilities design, see Horne (2002).

Production scheduling. Multiple wells penetrate the reservoir, including both injectors and producers. Injecting water can enhance the recovery by increasing pressure and sweeping oil through the reservoir. However, when water is produced at the production wells the recovery efficiency is dramatically reduced. In the optimal approach the oil is swept uniformly through the reservoir. Consequently all the production wells start to produce water at the same time.

In the early attempts in the 50's linear programming techniques were used, see Aronofsky (1983). For a detailed treatment of linear programming methods in petroleum engineering, see Aronofsky & Lee (1958). In the 60's production scheduling was modelled using a optimal control theory framework, see Rowan & Warren (1967) or O'Dell et al. (1973). Later approaches linked numerical simulation with linear programming models, see Wattenbarger (1970) and Lang & Horne (1983). For examples of field applications see Asheim (1978) and Nesvold et al. (1996). In Davidson & Beckner (2003) sequential quadratic programming methods are used to set well rates in a facility network of a reservoir simulator so that production objectives are maximized subject to constraints on pressures, flow rates and stream compositions. This algorithm automatizes the very time-consuming setting of operating conditions in reservoir simulators traditionally handled by the modeler. For a related approach see Wang et al. (2002). Uncertainty was not considered in these works.

Well placement. Optimal placement of production and injection wells is a complex problem that depends on reservoir and fluid properties, well and surface equipment specifications, as well as economic parameters. The optimization problem is complex because the number of variables is high, and because their interaction is complex and non-linear. Function evaluation through reservoir simulations is difficult since every reservoir simulation is very computer intensive. Often the number of reservoir simulations is reduced by utilizing the knowledge of the reservoir physics.

Aanonsen et al. (1995) applied interpolation techniques to provide a substitute for the search space with a smooth function. The method is based on response surfaces and experimental design developed by Damsleth et al. (1992). A similar approach was also used by Pan & Horne (1998). Randomized search methods have also been used in well placement problems. For applications with simulated annealing and the genetic algorithm, see Holland (1975), Bittencourt & Horne (1997) and Goldberg (1989). Beckner & Song (1995) used simulated annealing for both well placement and scheduling problems.

Production facilities design. The production facilities need to be optimized to maximize recovery at minimum cost. The configuration of well and surface equipment also contributes in the optimization of a petroleum development. For applications see Carroll III & Horne (1992) and Fujii & Horne (1995).

1.2.2 Contributions in supply chain management

In works in supply chain management the objective is to construct models for the planning and scheduling of hydrocarbon fields. Grossmann et al. (2002) provides an overview of methods design to optimize planning and scheduling decisions simultaneously. van den Heever et al. (2001) classify decisions made in reservoir management in two main categories, design decisions and operational decisions. Design decisions comprise selecting the type of platform, the staging of compression and assessing the number of wells to be drilled in a reservoir. These decisions are discrete in nature. In operational decisions production rates from individual reservoirs and wells are assessed. In contrast to design decisions, operational decisions are continuous in nature. Neiro & Pinto (2004) propose a framework for modelling the entire petroleum supply chain.

Ivyer & Grossmann (1998) present a multi-period mixed-integer linear programming formulation for the planning and scheduling of investment and operation in offshore oilfields. In this work an offshore oil field is considered for development. The problem consisted of determining the actual number and location of platforms, actual wells to be drilled and their interconnection to the platform, and the production planning and scheduling of the oil field.

In van den Heever et al. (2001) a multi-period mixed-integer non-linear programming model is presented for the long-term design and planning of offshore hydrocarbon field infrastructures with complex economic objectives. This model requires increased computational resources, and to address this problem a specialized heuristic algorithm relying on the concept of Lagrangian decomposition is proposed.

Neiro & Pinto (2004) propose a framework for modelling the entire petroleum supply chain. In this work the planning and scheduling of the most important subsystems of the petroleum supply chain, such as oilfield infrastructure, crude oil supply, refinery operations and product transportation, can be integrated into one framework.

1.2.3 Contributions where uncertainty is taken into account

An exploration and production (E & P) asset's value chain consists of many components. In order to make good decisions many sources of uncertainty should be taken into account. The most important contributors are typically:

- Uncertainty about the available amount of resources, i.e. how much oil or gas can be produced from the field.
- Uncertainty about the oil and gas price, i.e. how much profit can we gain from producing the resources.
- Uncertainty about the capital expenditure, i.e. how much will we have to invest, mainly before production can begin (facilities, drilling).
- Uncertainty about the operating costs, i.e. how much will it cost to run the field during the production phases.

Taking into account uncertainty of all components at all levels in one framework continues to be a challenge for the industry. Narayanan et al. (2003) proposes a technology that fully integrates rigorous reservoir modelling, flow simulation and economics within a decision optimization framework that explicitly manages risk. In this work a system is proposed which integrates reservoir simulation, an economic model, and a Monte Carlo algorithm with a global search algorithm to identify more optimal reservoir planning and management decision alternatives under uncertainty.

Floris & Peersmann (2000) introduces an E & P Decision Support System that combines the data and information from earth modeling, surface engineering and economics into one integrated asset model. The system offers decision tree scenario analysis and Monte Carlo simulation in conjunction with utility function analysis. Simplified production profile models, or 'fast' models are used.

Begg et al. (2001) proposes the Stochastic Integrated Asset Model (SIAM), that incorporates all components of the E & P value chain into one framework. This is obtained by proposing simplified models for some of the components contributing to an investment decision. In SIAM simplified models for fast scenario, Monte Carlo and value-of-information analyzes are integrated.

To model and quantify the uncertainties in reservoir simulation studies Damsleth et al. (1992) use experimental design, response surface and Monte Carlo simulation. Dejean & Blanc (1999) proposed the integration of experimental design, response surface and Monte Carlo methods to optimize the production scheme.

In Zhang et al. (2007) an Integrated Reservoir Simulation System (IRSS) is developed. A framework that distributes multiple reservoir simulations on a cluster of CPUs for fast and efficient process optimization studies is developed. This platform utilizes several commercial reservoir simulators for flow simulations, an experimental design and a Monte Carlo algorithm with a global optimization search engine to identify the optimum combination of reservoir decision factors under uncertainty. The framework is applied on a field-scale development exercise involving a well placement design.

1.2.4 Optimization in petroleum-related PhD projects

Examples of relevant PhD projects related to oil field development projects include Lund (1997) and Jonsbraaten (1998). In his PhD thesis Lund (1997) aims to identify the value of flexibility in offshore oil development projects. By developing a prototype for an oilfield development project Lund (1997) wants to replicate the life of the project. The main objective is to mirror the major decisions and the information the operator receives throughout the different phases of the project in the prototype. Different kinds of flexibility are discussed. In particular, the flexibility to postpone a project as well as the flexibility to terminate a project are discussed. Further, start/stop flexibility and capacity flexibility are treated. Stochastic dynamic programming is used to evaluate the project. The prototype is applied on a case study.

The thesis of Jonsbraaten (1998) consists of two parts where the first part presents various reservoir models. The second part presents four papers, of which the three first papers aim to solve various stochastic optimization problems. The first paper aims to optimize an oil field under price uncertainty. In the second article Jonsbraaten (1998) develops a class of stochastic programs with decision dependent random elements. The third paper deals with optimal selection and sequencing. A Bayesian model for updating the a priori probability distribution over reservoir characteristics is proposed. The last paper uses game theory to analyze oil extraction on a block with different owners.

1.2.5 Optimization in this PhD project

Figure 1.1 shows an overview of the hydrocarbon value chain. The contributions discussed in the sections 1.2.1, 1.2.2, 1.2.3 and 1.2.4 can be divided in two categories:

- The focus is on the problem of modelling the entire hydrocarbon value chain. Since the value chain is very complex, as we observe from Figure 1.1, many aspects of it needs to be simplified to be able to construct such comprehensive models.

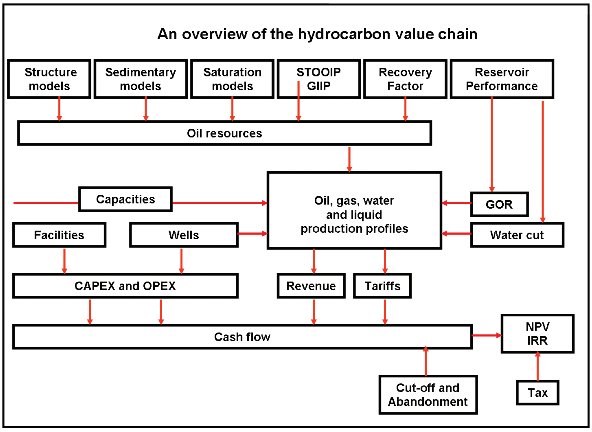

Figure 1.1: *An overview of the hydrocarbon value chain.*

- The focus of the optimization is to solve a petroleum engineering problem. Relating this to Figure 1.1 it involves *Wells*, *Facilities*, *Capacities* or *Reservoir performance*. Often a simulator is used directly, some times in conjunction with other software modules or as a part of an integrated reservoir simulation system.

The main purpose of this PhD project is focus on optimization of production in an oil or gas field with many reservoirs, which constitutes an important component in the hydrocarbon value chain. Relating this to Figure 1.1 we focus on *oil, gas, water and liquid production profiles*. We do not use a reservoir simulator directly, we only use the simulation output that provides us with state-of-the-art production profiles. To explain the work process roughly, key properties of the reservoirs are assessed by geologists, geophysicists, petroleum engineers and other specialists. This knowledge is then assembled and quantified into a reservoir model. Our analysis starts at the stage where a reservoir simulation has been performed, and the output from this simulation is given. Simplified production models can then be constructed based on this output. The objective is then to assess how the entire field should be produced to maximize total discounted production[1], using the simplified production profiles as proxies for the potential production from the reservoirs. By focusing on this important component in the hydrocarbon value chain only we are able to develop a generic framework that provides insight into how a large oil or gas field should be produced. The approach of this PhD project enables focus on coordination of the production of a large oil or gas field that consists of many reservoirs, possibly with several owners with different commercial interests. The following passage from Horne (2002) puts our effort in perspective:

[1]Other objective functions will also be considered.

> *Petroleum engineers face a wide variety of optimization problems. Every time a production strategy is designed or a location for a well is chosen, a decision problem has been solved. Yet it is surprising how rarely a formal optimization technique is used to solve these problems. In fact, the word optimization in the oil recovery industry is misused widely in the sense of analyzing a few cases and choosing the best one. Perhaps the explanation for this is that petroleum optimization problems are extremely difficult.* [...] *However, with the improvement of computer modeling over time, it is now becoming feasible to apply optimization techniques to address several petroleum recovery issues.*

As the calculations using the simplified production profiles can be done very efficiently, uncertainty may be added to the framework using Monte Carlo simulation. Thus, robustness and sensitivity analysis of different production strategies can be performed. The proposed framework constitutes an important building block in total value chain analysis, that may be incorporated in a full-scale analysis of a project.

1.3 The model framework

1.3.1 Model and notation

A fundamental model assumption is that the *potential production rate* of oil from a reservoir can be expressed as a function of the remaining producible volume, or equivalently as a function of the volume produced. Thus, if $Q(t)$ denotes the cumulative production at time $t \geq 0$, and $f(t)$ denotes the potential production rate at the same point in time, we assume that $f(t) = f(Q(t))$. This assumption implies that the total producible volume from a reservoir does not depend on the production schedule. In particular, if we delay the production from a reservoir, we can still produce the same volume at a later time. We refer to the function f as the *potential production rate function* or *PPR-function* of the reservoir. If a reservoir is produced without any production constraint from time $t = 0$, the cumulative production function will satisfy the following autonomous differential equation:

$$\frac{dQ(t)}{dt} = f(Q(t)), \tag{1.3.1}$$

with the boundary condition $Q(0) = 0$. Due to various kinds of restrictions, including possible time-dependent constraints, the actual production rate will typically be less than or equal to $f(t)$.

We consider oil production from n reservoirs that share a processing facility with a constant process capacity of K units (typically kSm^3) per day. Let $\boldsymbol{Q}(t) = (Q_1(t), \ldots, Q_n(t))$ denote the vector of cumulative production functions for the n reservoirs, and $\boldsymbol{f}(t) = (f_1(t), \ldots, f_n(t))$ be the corresponding vector of PPR-functions. We assume that the PPR-functions can be written as

$$f_i(t) = f_i(Q_i(t)), \quad t \geq 0, \quad i = 1, \ldots, n, \tag{1.3.2}$$

implying that the potential production rate of one reservoir does not depend on the volumes produced from the other reservoirs. We will also assume that f_i is non-negative and strictly decreasing as a function of $Q_i(t)$ for all t and $i = 1, \ldots, n$. These assumptions reflect the natural properties that the production rate cannot be negative, and that reservoir pressure typically decreases as more and more oil is produced. Finally, to ensure uniqueness of potential production profiles we will also assume that f_i is Lipschitz continuous in Q_i, $i = 1, \ldots, n$, see Dettman (1986) for details.

A *production strategy* is defined by a vector valued function $\boldsymbol{b} = \boldsymbol{b}(t) = (b_1(t), \ldots, b_n(t))$, defined for all $t \geq 0$, where $b_i(t)$ represents the *choke factor* applied to the ith reservoir at time $t, i = 1, \ldots, n$. We refer to the individual b_i-functions as the *choke factor functions* of the production strategy. The *actual production rates* from the reservoirs, after the production is choked is given by

$$\boldsymbol{q}(t) = (q_1(t), \ldots, q_n(t)),$$

where

$$q_i(t) = \frac{dQ_i(t)}{dt} = b_i(t) f_i(Q_i(t)), \quad i = 1, \ldots, n. \tag{1.3.3}$$

We also introduce the total production rate function $q(t) = \sum_{i=1}^{n} q_i(t)$ and the total cumulative production function $Q(t) = \sum_{i=1}^{n} Q_i(t)$. To reflect that $\boldsymbol{q}$ and $\boldsymbol{Q}$ depend on the chosen productions strategy $\boldsymbol{b}$, we sometimes indicate this by writing $\boldsymbol{q}(t) = \boldsymbol{q}(t, \boldsymbol{b})$ etc.

To satisfy the physical constraints of the reservoirs and the process facility, we require that

$$0 \leq q_i(t) \leq f_i(Q_i(t)), \quad t \geq 0, \quad i = 1, \ldots, n, \tag{1.3.4}$$

and that

$$q(t) = \sum_{i=1}^{n} q_i(t) \leq K, \quad t \geq 0. \tag{1.3.5}$$

Let $\mathcal{B}$ denote the class of production strategies that satisfy the physical constraints (1.3.4) and (1.3.5). We refer to production strategies $\boldsymbol{b} \in \mathcal{B}$ as *valid production strategies.*

For a given production strategy $\boldsymbol{b} \in \mathcal{B}$ the *plateau length* is defined as

$$T_K = T_K(\boldsymbol{b}) = \sup\{t \geq 0 : \sum_{i=1}^{n} f_i(Q_i(t)) \geq K\}. \tag{1.3.6}$$

An *admissible production strategy* is defined as a production strategy $\boldsymbol{b} \in \mathcal{B}$ satisfying the following constraint:

$$q(t) = \sum_{i=1}^{n} q_i(t) = \sum_{i=1}^{n} b_i(t) f_i(Q_i(t)) = \min\{K, \sum_{i=1}^{n} f_i(Q_i(t))\}. \tag{1.3.7}$$

Moreover, we let $\mathcal{B}' \subseteq \mathcal{B}$ denote the class of admissible strategies.

1.3.2 Objective functions

To evaluate production strategies we introduce an *objective function*, i.e., a mapping $\phi : \mathcal{B} \to \mathbb{R}$ representing some sort of a performance measure. If $\boldsymbol{b}^1, \boldsymbol{b}^2 \in \mathcal{B}$, we prefer $\boldsymbol{b}^2$ to $\boldsymbol{b}^1$ if $\phi(\boldsymbol{b}^2) \geq \phi(\boldsymbol{b}^1)$. Moreover, an *optimal production strategy* with respect to ϕ is a production strategy $\boldsymbol{b}^{opt} \in \mathcal{B}$ such that $\phi(\boldsymbol{b}^{opt}) \geq \phi(\boldsymbol{b})$ for all $\boldsymbol{b} \in \mathcal{B}$.

If $\boldsymbol{b}^1, \boldsymbol{b}^2 \in \mathcal{B}$ are two production strategies such that $\boldsymbol{Q}(t, \boldsymbol{b}^1) \leq \boldsymbol{Q}(t, \boldsymbol{b}^2)$ for all $t \geq 0$, one would most likely prefer $\boldsymbol{b}^2$ to $\boldsymbol{b}^1$. Thus, a sensible objective function should have the property that $\phi(\boldsymbol{b}^1) \leq \phi(\boldsymbol{b}^2)$ whenever $\boldsymbol{Q}(t, \boldsymbol{b}^1) \leq \boldsymbol{Q}(t, \boldsymbol{b}^2)$ for all $t \geq 0$. Objective functions satisfying this property will be referred to as *monotone objective functions*.

In general the revenue generated by the production may vary between the reservoirs. This may occur if e.g., the quality of the oil, or the average production cost per unit are different from reservoir to reservoir. Such differences should then be reflected in the chosen objective function. On the other hand, if all the reservoirs are similar, we could restrict ourselves to considering objective functions depending on the production strategy $\boldsymbol{b}$ only through the total production rate function $q(\cdot, \boldsymbol{b})$ (or equivalently through $Q(\cdot, \boldsymbol{b})$). We refer to such objective functions as *symmetric*.

In this PhD thesis we will often consider the following monotone, symmetric objective function:

$$\phi_{C,R}(\boldsymbol{b}) = \int_0^\infty I\{q(u) \geq C\} q(u) e^{-Ru} du, \quad 0 \leq C \leq K, \quad R \geq 0. \qquad (1.3.8)$$

The parameter R may be interpreted as a discount factor, while C is a threshold value reflecting the minimum acceptable production rate. If we insert $C = 0$ and $R > 0$ in (1.3.8), the resulting value of the objective function is simply the *discounted production*. On the other hand if we insert $C = K$ in (1.3.8), the integrand is positive only when $q(u) = K$. When $R = 0$ we obtain that $\phi_{C,0}(\boldsymbol{b}) = \phi_{K,0}(\boldsymbol{b}) = KT_K(\boldsymbol{b})$. It also follows from the definition of $\phi_{C,R}$ in (1.3.8) and T_K in (1.3.6) that $\phi_{K,0}(\boldsymbol{b}) = KT_K(\boldsymbol{b}) = \sum_{i=1}^n Q_i(T_K(\boldsymbol{b}))$.

1.4 A broad outline of the PhD project

Two of the most important profitability drivers in an offshore development project are the available amounts of resources and their market prices. In this PhD thesis we have devoted all the attention to the first driver. We start this section by briefly discussing the rationale behind this prioritization. For a comprehensive discussion of modelling of the oil price, see Lund (1997). Like many other financial variables, the oil price is to a large extent driven by politics, macroeconomics and human psychology. As illustrated in Section 1.1 these factors are complex. Furthermore, they are to a large extent unpredictable and hard to quantify and incorporate in our model framework. To obtain a complete model coverage of the financial risks from a Norwegian perspective, currency risk and interest rate risk should also be taken into account, complicating the modelling even further. The incorporation of all contributing factors in a complex problem as ours also introduces noise that makes it harder to draw conclusions and learn something.

An underlying principle of this PhD thesis is that the available resources should be produced as fast as possible. This issue is controversial, as the fluctuations in the hydrocarbon market prices may have profound effects on the offshore project profitability. In hindsight, if the Norwegian oil adventure were postponed, say 10 or 20 years, the revenues to the Norwegian State would undoubtedly have been much larger. Here different stake holders may have conflicting interests. The Norwegian State that represents the community may have a long-term perspective, while the oil companies, having to stay attractive in the capital markets, may have a shorter-term perspective.

The purpose of the following section is to present a broad outline of the PhD project and the thread of the PhD thesis. Figure 1.2 illustrates how the four papers are connected and their main themes. The first paper constitutes a building block for the subsequent papers, which deals with multi-reservoir production. In the second, third and fourth paper oil, water and gas flow from each reservoir to a common processing facility. The processing facility is only capable of handling limited amounts of oil, gas and water per unit of time. In order to satisfy the resulting constraints, the production needs to be *choked*. In the second and third paper we focus on *single phase* production optimization, meaning we consider the production of a *primary hydrocarbon phase* - oil or gas. *Multi-phase* production optimization of primary and *associated* hydrocarbon phases - oil, gas and water - is treated in the fourth paper and in a forth-coming research project.

Paper I. The first paper focuses on the problem of constructing simplified production models based on the output from a reservoir simulator. Such simplified production models are a necessary component in the multi-reservoir production framework developed in the subsequent papers.

Single Arps curves, introduced by Arps (1945) model the production rate function and the cumulative production function mathematically through a one-way, causal relation. In the first paper this approach is extended to multiple segments so that a combination of Arps curves may be used to get a satisfactory fit to a specific set of production data.

To also take into account various production delays, the dynamic two-way relation between the production rate function and the cumulative production is modelled in terms of a differential equation. The relation between the production rate function, q, and the cumulative production function, Q, should be of the following form:

$$q(t) = f(Q(t)), \qquad \text{for all } t \geq 0, \tag{1.4.1}$$

with $Q(t_0) = 0$ as a boundary condition.

The differential equation approach can also be extended to the more general situation where the production rate function consists of s segments. For each segment we assume that we have fitted a model in terms of a differential equation on the form given in (1.4.1). In order to connect these segment models, we need to specify a *switching rule* describing when to switch from one segment model to the next one.

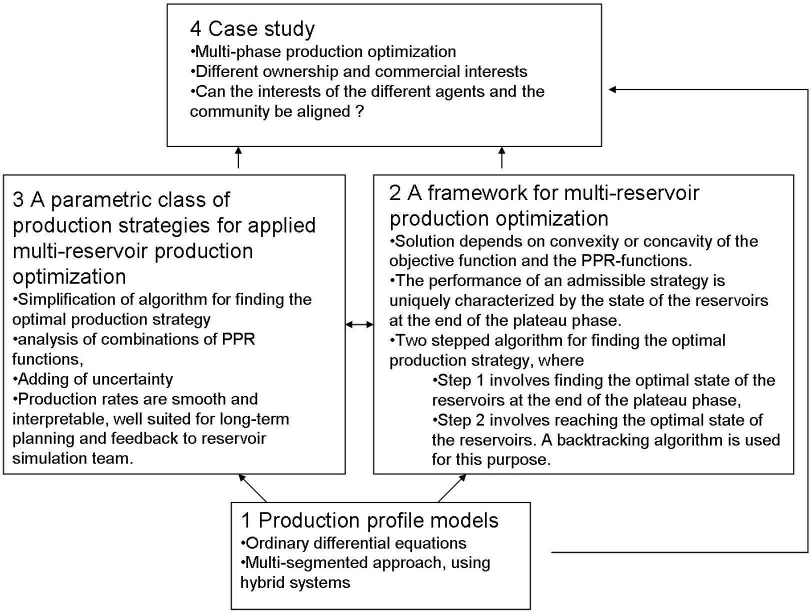

Figure 1.2: *An overview of the papers in the PhD project and how they are connected. Papers that are foundations for other papers are indicated by a one-way arrow. Papers that complement each other are connected with two-way arrows.*

We define a switching rule based on the produced volume. By using this switching rule, we obtain a model for the combined differential equation.

Uncertainty is added to the production model by modelling some of the key parameters as stochastic variables. A large sample, N, of the key parameters is generated, and every simulated vector of key parameters produces one simulated production profile. A Monte Carlo simulation algorithm is thus developed.

Paper II. The second paper deals with single phase optimization from a theoretical point of view. A general framework for optimizing the production strategies defined in Section 1.3.1 with respect various types of objective functions is developed. This paper brings much insight into the optimization problem and serves as a pillar for later papers.

An important result in the paper is that the performance of an admissible strategy is uniquely characterized by the state of the reservoirs at the end of the plateau phase. Thus, finding an optimal admissible production strategy, is essentially equivalent to finding the optimal state at the end of the plateau phase. Given the optimal state a backtracking algorithm can then used to derive an optimal production strategy.

To explain this, consider the set of all possible cumulative production vectors for the given field, denoted by $\mathcal{Q}$:

$$\mathcal{Q} = [0, V_1] \times \cdots \times [0, V_n], \tag{1.4.2}$$

where $V_1, \ldots, V_n$ are the recoverable volumes from the n reservoirs. Furthermore, we divide the hypercube $\mathcal{Q}$ in two subsets. In the *plateau region* the total production rate can be sustained at plateau level, while in the *decline region* the total production rate cannot be sustained at plateau level. Let $\boldsymbol{b}$ be any production strategy, and consider the points in $\mathcal{Q}$ generated by $\boldsymbol{Q}(t) = \boldsymbol{Q}(t, \boldsymbol{b})$ as t increases. From the boundary conditions we know that $\boldsymbol{Q}(0) = \mathbf{0}$. By the continuity of the PPR-functions, $\boldsymbol{Q}(t)$ will move along some path in the plateau region until the *plateau boundary* is reached.

If $\boldsymbol{b} \in \mathcal{B}$, the resulting path is said to be a *valid path*, while if $\boldsymbol{b} \in \mathcal{B}'$, the path is called an *admissible path*. In general only a subset of the plateau region can be reached by admissible paths. We refer to this subset as the *admissible plateau region*. An admissible path will move along some path in the admissible plateau region with a total production rate equal to K until the *admissible plateau boundary* is reached.

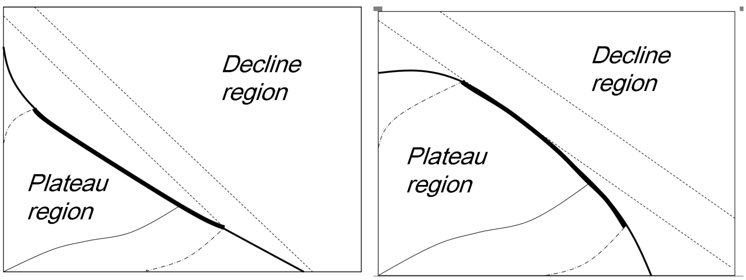

Figure 1.3: *The plateau region and the decline region in two important cases. The plateau boundary separating the plateau region and the decline region is marked with a boldfaced line. The admissible plateau boundary is marked with the thickest line. The dotted lines are iso-curves of the chosen linear objective function* $\phi_{K,0}$. *The panel to the left illustrates the situation for convex PPR-functions, while the panel to the right illustrates the corresponding situation for concave PPR-functions.*

Figure 1.3 provides an illustration of two important cases with two reservoirs, i.e., $n = 2$. We see from the figure that $\boldsymbol{Q}(t)$ will move along some path in the plateau region, chosen to be admissible in both cases, until the plateau boundary is reached. In both panels we have also displayed two paths, marked with semi-dotted lines, that end up in the *boundary of the admissible plateau boundary.* Note that for $n = 2$ the boundary of the admissible plateau boundary is simply two points in $\mathcal{Q}$. However, in higher dimensions this set is much more complex.

The panel to the left in Figure 1.3 illustrates the case of convex PPR-functions and linear objective function. Since we assume that the production strategy does not alter the producible volume of a reservoir, we will benefit from leaving the plateau region as late as possible, for this will imply that the decline phase will be as brief as possible. Each chosen production strategy will have a specific state $\boldsymbol{Q}(T_K(\boldsymbol{b}))$ of the reservoirs at the end of plateau phase for a chosen production strategy $\boldsymbol{b}$, where the plateau length $T_K(\boldsymbol{b})$ is defined in (1.3.6). Imagine that we want to maximize plateau production, so that the chosen objective function is $\phi_{K,0}(\boldsymbol{b})$, where $\phi_{C,R}(\boldsymbol{b}) = \phi_{K,0}(\boldsymbol{b})$ is defined in (1.3.8). Then we know that $\phi_{K,0}(\boldsymbol{b}) = \sum_{i=1}^{n} Q_i(T_K(\boldsymbol{b}))$, as explained in Section 1.3.2. Consequently the objective function is a hyperplane that will intersect with the plateau region when the process $\boldsymbol{Q}(t, \boldsymbol{b})$ reaches the plateau boundary. The later in time the hyperplane generated by $\phi_{K,0}(\boldsymbol{b}) = \sum_{i=1}^{n} Q_i(T_K(\boldsymbol{b}))$ intersects the plateau boundary, the larger the plateau production. Consequently we want this hyperplane to intersect with the admissible plateau boundary as late as possible, for this ensures maximal plateau production and minimal decline production. From the left panel we see that we can obtain a higher value of the iso-curves of the objective function when the iso-curves intersect the boundary of the admissible plateau boundary. However, in the case of concave PPR-functions and linear objective function shown in the panel to the right, we will typically experience that the optimal $\boldsymbol{Q}^*$ is located in the interior of the admissible plateau boundary.

The two cases shown in Figure 1.3 constitute two main results, stated and proved in the second paper. The first result states that if the PPR-functions are convex and we impose some mild restrictions on the objective function ϕ, the optimal $\boldsymbol{Q}^*$ can be found within the boundary of the admissible plateau boundary. The extreme points of this set correspond to a certain class of admissible production strategies called *priority strategies* introduced in the second paper.

The second result treats the situation when the PPR-functions are concave and we impose some other, mild restrictions on the objective function ϕ. Then a solution to finding the optimal state of the reservoirs at the end of the plateau phase typically involves finding the separating hyperplane supporting the plateau region at the optimal $\boldsymbol{Q}^*$. If the PPR-functions and the extended ϕ-function are differentiable, the standard way to solve this is by using Lagrange multipliers.

When the optimal $\boldsymbol{Q}^*$ lies in the interior of the admissible plateau boundary, there is typically no unique production strategy that reaches the optimal state of the reservoirs at the end of the plateau phase. Typically there will be many admissible paths through the plateau region from $\boldsymbol{0}$ to $\boldsymbol{Q}^*$. In the second paper, when searching for such a path, a backtracking algorithm is developed for this purpose.

Paper III. The general framework developed in the second paper is of fundamental importance in order to gain insight into the general production optimization problem. However, the two-step optimization algorithm proposed in the second paper has some weaknesses. First, the backtracking algorithm proposed to derive an admissible production strategy to reach the optimal state at the end of the plateau phase is not guaranteed to work. To understand why, note that in the first step of the two-step optimization algorithm an optimum candidate $\boldsymbol{Q} = \boldsymbol{Q}^{opt}$ is found. If the optimum candidate $\boldsymbol{Q} = \boldsymbol{Q}^{opt}$ is located close to or in the boundary of the admissible plateau boundary, the backtracking algorithm may not work. If the optimum candidate $\boldsymbol{Q} = \boldsymbol{Q}^{opt}$ is located in the valid, inadmissible plateau boundary, the backtracking algorithm *will* not work. Second, it may not be straight-forward to find the optimal state at the end of the plateau phase. If the optimal $\boldsymbol{Q}^*$ is located in the boundary of the admissible plateau boundary, Lagrange multipliers may not be used. Third, it may be of interest to analyze the robustness of selected production strategies, including uncertainty in the framework for this purpose. It is not obvious whether or how uncertainty can be included in the framework developed in the second paper.

To address these issues a parametric class of admissible production strategies is proposed in the third paper. Production strategies from the proposed parametric class reach all points in the admissible plateau boundary. Hence, an optimal production strategy can always be found within the proposed parametric class.

A production strategy within the parametric class is defined assigning a fixed-weight w_i to each reservoir. The positive real number w_i reflects the priority we aim to give reservoir i. If $w_i > w_j$ then reservoir i is prioritized higher than reservoir j. To ensure admissibility each w_i must be multiplied with a function $c(t)$. To avoid that the choke factors exceed one, the final choke factor for reservoir i is given by $b_i(t) = \min\{1, w_i c(t)\}$. Strategies within this parametric class is referred to as *first-order fixed-weight strategies*.

A weakness of first-order fixed-weight strategies is that they do not allow strict priorities between the reservoirs. In an extension of the first-order fixed-weight strategies, the reservoirs may be divided in groups, consisting of at least one element each. In this extension the reservoirs from one group is given strict priority before the reservoirs belonging to another group. The production within each group is determined as explained above for the first-order fixed-weight strategies. A production strategy of this form is referred to as a *k-th order fixed-weight strategy*.

In the second paper it was proved that the performance of an admissible strategy is uniquely characterized by the state of the reservoirs at the end of the plateau phase. Thus, it follows that an optimal production strategy can be found within a given class of admissible strategies provided that all points in the admissible plateau boundary can be reached by members of this class. It turns out that all *interior* points of the admissible plateau boundary can be reached by first-order fixed-weight strategies. However, to reach the boundary points in the admissible plateau boundary as well, higher-order strategies must be included. Fortunately, it can be shown that by considering the combined class of fixed-weight strategies of *all* orders, it is possible to reach all points in the admissible plateau boundary. Hence, an optimal production strategy can always be found within the union of all the k-th order fixed-weight strategies, where $k = 1, \ldots, n$.

Given that the value of the objective function, ϕ, is a continuous function of $\boldsymbol{Q}(T_K(\boldsymbol{b}))$, it is easy to see that for each point $\boldsymbol{Q}^*$ at the boundary of the admissible plateau boundary and $\epsilon > 0$, there exists another point, $\tilde{\boldsymbol{Q}}$ in the interior of of the admissible plateau boundary such that $|\phi(\boldsymbol{Q}^*) - \phi(\tilde{\boldsymbol{Q}})| < \epsilon$. Hence, even if the search for an optimal strategy is restricted to the first-order fixed-weight strategies, it is possible to find a strategy which is approximately optimal, at least in principle. In order to approximate a higher order fixed-weight strategy by a first-order strategy, one can assign very high weights to the reservoirs in the set with highest priority, and then use significantly smaller weights for the reservoirs in the sets with lower priorities. However, if the optimal strategy is a higher order strategy, better numerical results are obtained by searching among the fixed-weight strategies with the correct order.

A numerical algorithm using standard numerical optimization techniques is developed to search for optimal production strategies. The framework is demonstrated in some examples. Uncertainty is included in the model to enable robustness and sensitivity analysis. The purpose is to discover how vulnerable the optimal strategy is when exposed to uncertainty. If the optimal strategy is very vulnerable to uncertainty, perhaps a more robust production strategy should be selected.

Paper IV. In the final paper the model framework of the first three papers is extended and adapted to realistic production conditions. The extended model framework facilitates production profile modelling and optimization of oil and gas fields. A main field and satellite fields consist of several separate reservoirs with gas cap and/or oil rim. A process facility on the main field receives and processes the oil, gas and water from all the reservoirs.

The framework of the first three papers is extended to *two-phase* production with varying primary hydrocarbon phase, i.e. the simultaneous production of oil and associated gas from oil wells and gas and condensate from gas wells. Furthermore, the processing capacities are *time-dependent*, as will become evident from the description below. To optimize the two-phase production the parametric class of production strategies of the third paper is extended.

The available capacity is shared among several field owners with different commercial interests. The satellite field owners negotiate process capacities on the main field facility aligned with their interest. This introduces additional process capacity constraints (booking constraints) for the owners of the main field. Thus, the interests of the community may not be optimized by the individual field owners. If the total wealth created by all owners represents the economic interests of the community, it is of interest to investigate whether the total wealth may be increased by lifting the booking constraints. If all reservoirs may be produced more optimally by removing the booking constraints, all owners may benefit from this if appropriate commercial arrangements are in place. We will compare two production strategies. The first production strategy optimizes locally, at distinct time intervals. At given intervals the production is prioritized so that the maximum amount of oil is produced. The second production strategy is the extended version of the first-order fixed-weight strategy introduced in the third paper.

The main focus of the paper is the modelling approach and the basic principles for a modelling tool for general use in examination of production strategy effects on multi-reservoir fields, with different and varying hydrocarbon phases, with individual production constraints and priorities, different owners and with the functionality to extend and cover multi fields integration in a regional / processing hub evaluation.

The article also highlights the importance of being aware of local and global production optimization effects and the importance booking constraints may have.

As an illustration a case study based on real data is presented.

1.5 Topics for future research - and a brief note on real options

We now turn to describe some important topics we have not covered in this PhD project. The framework often referred to as *real options* relies on that uncertainty can be added into the model framework. The adding of uncertainty in a physical model is complex, as will be briefly discussed in Section 1.6. In order to do this we need to construct a stochastic model that accounts for the information we receive as the reservoirs are produced. We denote such a model $\mathcal{I}(t),\ t \geq 0$. It is important that $\mathcal{I}(t)$ mirrors realistically how the uncertainty changes over time in light of what we observe as the reservoirs are produced.

To be able to solve the stochastic optimization problem would be a groundbreaking accomplishment. First, we would need a stochastic model $\mathcal{I}(t),\ t \geq 0$ that accounts for the information we receive as the reservoirs are produced. Such a stochastic model could be constructed designing a statistical framework for model calibration and uncertainty estimation for complex deterministic models inspired by Larssen et al. (2006) and Kennedy et al. (2006), as explained in Section 1.6. Having an appropriate stochastic model $\mathcal{I}(t),\ t \geq 0$ we would turn to the problem of solving the stochastic optimization problem.

Having developed this generic framework in this PhD project it is also of great interest to use this framework in total value chain analysis. Examples of possible applications could include:

- A field is being prospected for additional development. How should the infrastructure be designed? (I.e, pipe line dimensions etc) How should investments be timed? How should the segment be produced, i.e., phased in with the remaining production?

- Capital expenditure decisions. Should a company upgrade the infrastructure (i.e. pump equipment, production capacities, infrastructure flexibility, technology etc.) on a platform? Decisions can be made analyzing the trade off between the cost and the improved functionality.

Although the case study in Paper IV is an example of an application of the framework, it is of interest to study more examples and case studies. This would give a more complete picture of the business relevance of the framework.

A model for $\mathcal{I}(t)$ accounts for the information we receive as the reservoirs are produced. This information gives the operator flexibility, which in turn generates options. Such options are often referred to as real options. The framework of real options relies on that the optimization problem may be decomposed into sequential optimization problems that are solved recursively, as the framework of *stochastic dynamic programming*, see Section 1.7, does. Since we have not constructed a model for the information process $\mathcal{I}(t)$, as will be discussed in Section 1.6, this section is mostly relevant for future research.

Real options have been studied extensively over the years but have not become nearly as widespread in use outside the academic community as financial options. Option pricing theory and contingent claims analysis offers an efficient framework for the valuation of corporate assets and liabilities. Although the option pricing models developed by Black & Scholes (1973) are founded on many simplifying and unrealistic assumptions, this paper avalanched a myriad of papers discussing option pricing problems. Black & Scholes (1973) designed their option pricing models for the valuation of tradable assets. The real option community has applied the framework of Black & Scholes (1973) on real investments treating them as real assets, although these assets are not tradable. The lack of a liquid market for these real assets may to some extent explain the lukewarm response to real options in the business community.

Black & Scholes (1973) constructed a portfolio of tradable assets that replicated the risk profile of the instrument of interest. In this way they constructed a portfolio that was riskless for a short period of time. Dixit & Pindyck (1994) adopted this framework and assumed the existence of spanning assets; i.e. they assumed that there exits a complete market where all project cash flows may be replicated by trading securities in the market. More formally, the securities market is complete if, for every project c there exists a *replicating strategy* β, that generates cash flows which exactly match the project's future cash flows at all times and in all states. If this assumption is realistic, it is thus possible to hedge all project cash flows for a short period of time by purchasing tradable assets in the market. Then the portfolio of interest also becomes risk-less for a short period of time, analogous to the risk-less portfolio proposed by Black & Scholes (1973).

The crucial question then becomes whether such spanning assets can be found or not. The answer to this question is closely related to the level of analysis. Bøhren & Ekern (1985) presents six levels for the analysis of an oil field development project - *The project, The company, The government income from trade, The trade, A community portfolio at national level, A community portfolio at international level.* The project level possesses the greatest multitude of non-diversifiable risks, while the international community level contains the smallest amount of non-diversifiable risks in this framework. At project level, the uncertainty associated with the reservoir volume, investment costs and maintenance costs represent risky cash flows that cannot be mirrored by tradable assets in the market. At national community level, however, it may be reasonable to assume that a well diversified portfolio of projects and companies and trades only contains market risk. Since we are mainly concerned with the project level, the assumption of spanning assets is not realistic for our purpose. The cash flows of a reservoir cannot be replicated, since securitization of a reservoir is not done in the finance industry today. But an even more important provision for using the real options framework is an appropriate model for $\mathcal{I}(t)$. In this PhD project we have not developed such a model, as discussed in Section 1.6.

Some contributions related to oil field development projects are Bjerksund & Ekern (1990), Ekern (1988), McDonald & Siegel (1986), Pickles & Smith (1993) and

Pindyck (1980). PhD projects related to oil field development projects are Lund (1997) and Jonsbraaten (1998).

Although option theory in its purest form may not be a good idea due to lack of spanning assets, hybrid techniques exist. Such techniques combine Monte Carlo simulation and the principles of dynamic programming, see Glasserman (2004) or Longstaff & Schwartz (2001) for details. For an introduction to real options in the energy industry, see Ronn (2002).

1.6 The modelling of uncertainty in a physical model

In this PhD thesis ordinary differential equations represent the potential production of each reservoir using *hybrid systems*, see Liberzon (2003) for an introduction. In the field of hybrid systems theory about dynamic systems, applied in engineering and physics, is combined with stochastic modelling. For applications of hybrid systems see Bernadsky et al. (2004), Glover & Lygeros (2004) or Koo et al. (1997). In our application we have deterministic models of every reservoir in the large oil or gas field. In reality there is considerable uncertainty associated, as the producible volume of a reservoir is unknown. There will also be random factors affecting the production, such as bad weather, need for work-overs etc.

As mentioned in Section 1.5, uncertainty can be added into the model framework introduced in Section 1.3.1 specifying a stochastic model $\mathcal{I}(t),\ t \geq 0$ that accounts for the information we receive as the reservoirs are produced. If our stochastic process were a Markov process, the framework of *stochastic differential equations*, see Øksendal (2003), could be used to add uncertainty to deterministic differential equations. The Markov property states that at any times $s > t > 0$, the conditional probability distribution of the process at time s given the whole history of the process up to and including time t, depends only on the state of the process at time t. In effect, the state of the process at time s is conditionally independent of the history of the process before time t, given the state of the process at time t. See Taylor & Karlin (1994) for details on Markov processes.

Unfortunately most of the process models we have been working with will rarely be Markov processes. To understand this intuitively, consider a simple example considering production from a single reservoir. First we assume that the only stochastic variable is the producible volume V, and that the production rate is deterministic and constant and equal to q. With this model we have produced qt after t units of time have passed. Assume that our prior knowledge of V, i.e. the knowledge of the geophysicists and other experts before production begins, can be quantified in the probability distribution G so that

$$G(v) = Pr(V > v), \tag{1.6.1}$$

heuristically referred to as the survival distribution of V. When we have produced in t units of time without interruption we know that $V > qt$. The conditional distribution for V at time t can be found by

$$Pr(V > v|V > qt) = \frac{G(v)}{G(qt)}, \text{ for } v > qt. \tag{1.6.2}$$

Then we consider the the process $\{Q(t) : t > 0\}$. Note that due to the special properties of the process we will always have that

$$0 \leq Q(t) \leq qt, \quad t \geq 0. \tag{1.6.3}$$

We also have that

$$\begin{aligned} &Pr(Q(t) = qt) = Pr(V > qt) = G(v) \\ &Pr(Q(t) > S) = Pr(V > S) = G(S), \text{ for } 0 < S < qt. \end{aligned} \tag{1.6.4}$$

In this simple, unrealistic and artificially stylized example $\{Q(t) : t > 0\}$ is in fact a Markov process. To examine this closer let $0 < u < s$ be two points in time. Then we have that

$$\begin{aligned} &Pr(Q(s) = qs|Q(u) = qu) = \frac{G(qs)}{G(qu)}, \\ &Pr(Q(s) > S|Q(u) = qu) = \frac{G(S)}{G(qu)}, \quad \text{for} \quad qu < S < qs, \\ &Pr(Q(s) > S|Q(u) = qu) = 1, \quad \text{for} \quad 0 < S <= qu, \\ &Pr(Q(s) = S|Q(u) = S) = 1, \quad \text{for} \quad 0 < S < qu. \end{aligned} \tag{1.6.5}$$

In this case we know the whole history of the process if we know $Q(u)$, which makes life very simple. If $Q(u) = qu$ we know that the production is still running, but if $Q(u) = S < qu$ we know that the production has stopped, and that it happened at time $t = \frac{S}{q}$.

What if q were stochastic ? In that case knowledge of $Q(u)$ would not be enough to decide whether the production is running at time u. If we had additional knowledge about $Q(v)$ for a given $v < u$, this would clearly change the conditional distribution of $Q(s)$. For instance, if $Q(u) = Q(v)$, this would imply that the production actually has stopped, which again leads to $Q(u) = Q(s)$ with probability 1. If we were to assume realistically that q and V were stochastically dependent, the complexity of this example increases even further. This example shows that we cannot expect that $\{Q(t) : t > 0\}$ is a Markov process in the general situation. Already when q is stochastic this breaks down.

In the framework of stochastic differential equations, a consequence of the model framework is that changes in the stochastic process are viewed as stochastic variables. Thus we get a collection of infinitely many stochastic variables when we observe such a process over a time interval. A typical application of this framework is in mathematical finance, where the underlying process for example may be a share or another asset. A physical process like the depletion of a natural resource reservoir can not be modelled using this framework. In our application production is modelled using ordinary differential equations. We do not regard infinitely many stochastic variables, but only a few variables describing the physical properties of the reservoir. As time passes and the reservoir is produced, we get more and more information about these variables.

Although we do not have a Markov process, it is possible to design a statistical framework for model calibration and uncertainty estimation for complex deterministic models, see Larssen et al. (2006) and Kennedy et al. (2006). In Larssen et al. (2006) a deterministic model is combined with a stochastic model for the observed output data. Then Markov Chain Monte Carlo (MCMC) techniques, see Gilks et al. (1996), are used to estimate all unknown parameters using Bayesian computations. For an introduction to Bayesian statistics, see Berger (1985). Prior probability distributions then have to be specified for the input parameters and for the likelihood functions for the output data. The posterior distributions are calculated by running the deterministic model repeatedly with parameters and data and allow the impact of the uncertainty on model results to be explicitly shown.

1.7 Canonical optimization methodology

In the early phases of this PhD project we considered using some of the canonical optimization techniques available in the literature. When experimenting with these techniques we encountered problems, and we found that they were not so well-suited for our application.

The *calculus of variations*, see Lebedev & Cloud (2003) or Wan (1995), is a classic optimization technique that is used for Lagrange's equations of motion. Using the calculus of variation representation leads to a set of differential equations. From paper II we recall that the performance of an admissible strategy is uniquely characterized by the state of the reservoirs at the end of the plateau phase. Thus, finding an optimal admissible production strategy, is essentially equivalent to finding the optimal state $\boldsymbol{Q}^*$ at the end of the plateau phase. When the optimal $\boldsymbol{Q}^*$ lies in the interior of the admissible plateau boundary, there is typically no unique production strategy that reaches the optimal state of the reservoirs at the end of the plateau phase. Typically there will be infinitely many admissible paths through the plateau region from $\mathbf{0}$ to $\boldsymbol{Q}^*$. In fact, it is only when $\boldsymbol{Q}^*$ belongs to the extreme points of the boundary of the admissible boundary that the admissible path through the plateau region from $\mathbf{0}$ to $\boldsymbol{Q}^*$ will be unique. Thus, if the set of differential equations produced by the calculus of variation representation exists, the solution will not be unique when the optimal $\boldsymbol{Q}^*$ does not belong to the extreme points of the boundary of the admissible boundary, i.e., in most cases. There may be infinitely many solutions to the set of differential equations in this case. This is problematic. Thus we believe that it could be problematic to use calculus of variation to solve this optimization problem.

Bellman (1972) applied the calculus of variation to *optimal control theory*, see

Zabczyk (1992), and hence developed the theory of *dynamic programming* and *stochastic dynamic programming*. An *optimal control* is a set of differential equations describing the paths of the control variables that minimize the cost functional. Dynamic programming, see Bellman (1972) or Bertsekas (2005), deals with sequential problems that are solved recursively. Formulating the problem using the *Bellman equation*, see Bellman (1972), *the method of backwards induction* can be used to solve the problem. Backwards induction can be solved analytically or numerically on a computer. We conjecture that the discovery of an analytical solution to the Bellman equation would be impeded by the same difficulty as we experienced with calculus of variation, i.e , that the solution will not be unique when the optimal $\boldsymbol{Q}^*$ does not belong to the extreme points of the admissible plateau boundary. Therefore we would have to resort to numerical solutions to the backwards induction. Experiments have indicated that this is unfeasible when the number of reservoirs exceeds 3, due to the *curse of dimensionality*, see Miranda & Fackler (2002) or Meyn (2007). Furthermore, this simulation time will increase dramatically as we increase the number of reservoirs, n. Clearly, this computer performance is not satisfactory when we want to analyze examples with n being equal to 10, 20 or even more.

Stochastic dynamic programming represents the extreme form of changing the course as new, relevant information arrives. To make this framework work, we need a stochastic model for the information process $\mathcal{I}(t),\ t > 0$. To our knowledge it is problematic to use stochastic dynamic programming if the information process $\mathcal{I}(t)$ cannot be modelled using a Markov process. As explained in Section 1.6 we do not believe that $\mathcal{I}(t)$ can be modelled with a Markov process.

1.8 A brief description of the prototype

A prototype for numerical simulation and analysis has been developed. Our ambition has been to develop a prototype that may be used as a simulation laboratory. The prototype is written in Java, using the freely available workbench Eclipse as a development tool.

Object-oriented programming has been used, and Figure 1.4 shows an overview of the most important features of the object structure. In panel 1 we see the object structure of the ordinary differential equations, introduced as PPR-functions in Section 1.3.1 in (1.3.2). The *single phase* case is modelled through the object 'ProdDiffEq', which is a subclass of the superclass 'AbstractDiffEq', which in turn implements the interface 'DiffEq'. Every production profile, denoted 'Profiles' in panel 1, can consist of any finite number of segments, i.e. , 'Segments', see paper I for details. The capacities and regularities of every reservoir also have an object-oriented structure. The *multi-phase* case is modelled through the object 'MultiProdDiffEq'. Note that the object structure is different in the single phase and multi-phase case. In the single-phase case we operate with an array of profiles, capacities and regularities. In the multi-phase case every reservoir is an object inside 'MultiProdDiffEq'.

The production strategy $\boldsymbol{b} \in \mathcal{B}$ defined in Section 1.3.1 is evaluated through the use of modifiers, denoted 'PreModifier' in panel 1. Any modifier is itself an object,

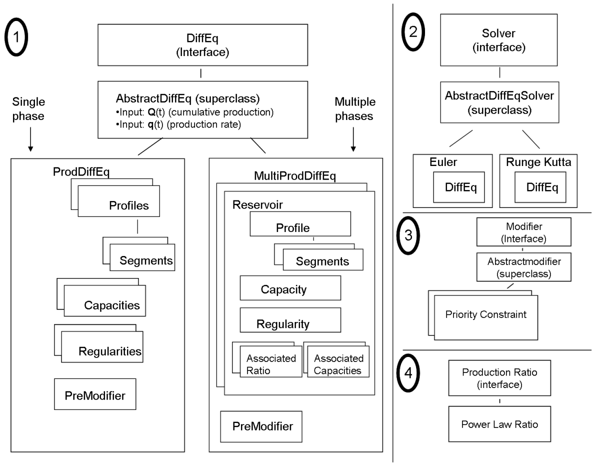

Figure 1.4: *An overview of important features of the object-oriented structure in the prototype.*

visualized in panel 3. In particular, a specific production strategy is defined through a constraint, which is a child of the superclass 'AbstractModifier'. In panel 3 this is exemplified by the modifier 'Priority Constraint' corresponding to the class of production strategies referred to as *priority strategies*, see paper II for a definition.

At any given point of time, $t \geq 0$, the vector of cumulative production $\boldsymbol{Q}(t)$ is updated using a specific solver. We have implemented two standard numerical differential equation solvers, i.e., Euler's method and Runge Kutta's 4-th order method. This is visualized in panel 2. Thus the object 'DiffEq' in panel 1 is linked to the solver by being defined as an object inside the particular solver, see panel 2.

The analysis of specific numerical examples is facilitated through the use of XML-files. The structure of the XML-files follows the object structure of the prototype. All relevant objects, i.e., field information, choice of solver, modifier and the number of phases are specified in the XML-files. The objects are then created in the program when the XML-files are read.

Chapter 2

Multisegment production profile models, a hybrid systems approach

Abstract

In the development phase of an oil or gas field, it is crucial to have a satisfactory model for the production. Since the first attempts in the 1940's, many different models have been developed for this purpose. Such a model typically incorporates knowledge about the properties of the reservoir. When used in a total value chain analysis, however, also economic and strategic factors need to be taken into account. In order to do this, more flexible modelling tools are needed. In this paper we demonstrate how this can be done using hybrid system models. In such models the production is modelled using ordinary differential equations representing both the reservoir dynamics as well as strategic control variables. The approach also allows us to break the production model into a sequence of segments. Thus, it is possible to represent various discrete events affecting the production in different ways. The flexibility of the modelling framework makes it possible to obtain realistic approximations to real-life production profiles. As the calculations can be done very efficiently, uncertainty may be added to the framework using Monte Carlo simulation. The proposed framework constitutes an important building block in total value chain analysis, that may be incorporated in a full scale analysis of a project. In such an analysis revenues, costs and investments are modelled to obtain assessments of project profitability and different strategies. As the focus of the present paper is on production profile modelling, such a full scale analysis will not be done here.

Keywords

Production profile models, Decline curve analysis, Total value chain analysis, Hybrid systems, Monte Carlo simulation, Ordinary differential equations

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2.1 Introduction

Estimating reserves and predicting production in oil and gas reservoirs has been studied extensively over many years. Many different models and methods have been suggested. A popular technique is the decline curve analysis approach. This dates back to the pioneer paper by Arps (1945) where the *exponential*, *hyperbolic* and *harmonic* curves were introduced. More recent papers considers other types of decline curves, and attempt to model the relation between the parameters of the curves and geological quantities, see e.g., Li & Horne (2003), Li & Horne (2005), Li & Horne (2006), and Marhaendrajana & Blasingame (2001).

The purpose of the present paper is to develop production models that can be used in the broader context of a total value chain analysis. In a total value chain analysis the reservoir geology may be described by structure models, sedimentary models and saturation models. Stochastic models, combined with reservoir simulation, are applied to estimate the quantitative measures Stock Tank Original Oil In Place, Original Gas In Place and the Recovery Factor of the reservoir. The geological models and the quantitative measures are crucial to assess the quality, nature and prevalence of the oil resources in the reservoir.

The knowledge gathered from sample drilling, seismic surveys and other analyses is assembled and quantified in a full-scale reservoir model. The performance of the reservoir is assessed using some reservoir simulation software. These tasks are executed by geologists, geophysicists, petroleum engineers and other specialists. The output from a full-scale reservoir simulation includes oil production profiles and important performance measures such as Gas-Oil Ratio and Water Cut. Production profiles for gas- and water production may be derived from these performance measures.

In the present paper, we will assume that the properties of the reservoirs described above have already been assessed. We will also assume that a full-scale reservoir simulation has been performed and that output from such a simulation is given. A flexible modelling environment will be developed where these assessments can be utilized. Thus, we will focus more on the mathematical and numerical aspects of the modelling. This environment is intended to be used in the context of a total value chain analysis where economic and strategic factors are taken into account as well. We will focus on one-phase production, i.e., oil production only. Multi-phase production, where oil production, gas production and water production are modelled simultaneously, is left for future work.

Execution time performance is an important issue in total value chain analysis. Usually, a simulation in a reservoir simulator takes hours, or even days. In the context of a total value chain analysis, where uncertainty should be incorporated, such execution times are unacceptable. Optimization problems, which also require low simulation execution time, are also relevant in total value chain analysis. For a discussion of total value chain analysis, see Huseby & Brækken (2000).

The perspective of the present paper is early phase analysis, done before production data are present. In the presence of production data many related problems may be studied using history matching of production data, see e.g., Liu & Olivier

(2004), Floris & Peersmann (2000) or Nævdal et al. (2005). These related problems will not be studied here. For a literature review of history matching, see e.g., Liu (2001).

The present paper presents three contributions to production profile modelling:

- Production profiles are modelled using a multisegment approach where the production profile is decomposed into a combination of decline curves, one for each segment. This enables a satisfactory fit to the output from the reservoir simulation.
- The production is modelled using ordinary differential equations representing both the reservoir dynamics as well as strategic and economic control variables. This representation enables us to incorporate various types of delay in production in the framework. At the same time the calculations can be done very efficiently, so that the execution time remains acceptable even when uncertainty is incorporated into the framework.
- In the proposed framework multiple production profiles are easily analyzed, and many optimization problems may be studied. In this paper we will study subfields that share a central process facility. Due to capacity constraints of the central process facility, the operator needs to prioritize the production of the subfields.

The paper starts out in Section 2.2 by reviewing the classical decline function approach developed by Arps (1945). We then extend the basic production profile functions to multisegment production functions in Section 2.3. In Section 2.5 production profiles are modelled as differential equations. Finally, the modelling of multiple production profiles is the topic of Section 2.6 before the conclusion is reached in Section 2.7.

2.2 Basic production profile functions

In the following models we assume that the production starts at a given time t_0. We denote the production rate at time $t \geq t_0$ by $q(t)$, and the corresponding cumulative production function at time $t \geq t_0$, by $Q(t) = \int_{t_0}^{t} q(u)du$.

The classical decline functions introduced by Arps (1945) are characterized by parameters describing the initial production rate, $r_0 > 0$, scale, $D > 0$, and shape, b. Key functions and properties of the classical decline functions are summarized in Table 2.1, see Huseby & Haavardsson (2007) for more details. Note that the exponential decline curve ($b \to 0$) and the harmonic decline curve ($b = 1$) are special cases of the hyperbolic decline curve. If the integral $Q(t) = \int_{t_0}^{t} q(u)du$ converges as $t \to \infty$, it is possible to compute ultimate recovery volume, i.e. cumulative production if the production is allowed to continue forever. The expression $\lim_{t\to\infty} Q(t)$ denotes ultimate recovery volume. Normally, however, the production is stopped when the production rate reaches some suitable cut-off level, say $r_c < r_0$. We can find the time point when this happens by solving the equation $q(t) = r_c$ with respect to t.

	Hyperbolic decline curve	Exponential decline curve	Harmonic decline curve
b	$b \in (0,1)$	$b = 0$	$b = 1$
$q(t)$	$r_0[1 + bD(t - t_0)]^{-1/b}$	$r_0 \exp(-D(t - t_o))$	$r_0[1 + D(t - t_0)]^{-1}$
$Q(t)$	$\frac{r_0}{D(1-b)}[1 - (1 + bD(t - t_0))^{1-1/b}]$	$\frac{r_0}{D}(1 - \exp(-D(t - t_0)))$	$\frac{r_0}{D}\ln(1 + D(t - t_0))$
$\lim_{t\to\infty} Q(t)$	$\frac{r_0}{D(1-b)}$	$\frac{r_0}{D}$	Not defined
t_c	$t_0 + \frac{1}{Db}[(\frac{r_0}{r_c})^b - 1]$	$t_0 + \frac{1}{D}\ln(\frac{r_0}{r_c})$	$t_0 + \frac{1}{D}[\frac{r_0}{r_c} - 1]$
V_r	$\frac{r_0}{D(b-1)}[(\frac{r_0}{r_c})^{b-1} - 1]$	$\frac{r_0 - r_c}{D}$	$\frac{r_0}{D}\ln(\frac{r_0}{r_c})$

Table 2.1: *Key functions and properties for Arps decline curves.*

The solution is denoted t_c. By inserting t_c into the expression for $Q(t) = \int_{t_0}^{t} q(u)du$ we get what we refer to as the *technical recoverable volume*, denoted by V_r. Due to economic considerations, the actual recovered volume may be smaller than this number. In this setting, however, we focus only on technical recoverable volumes. By changing the scale parameter D we obtain course changes in the steepness of the decline in the production rate function. Higher (lower) values of D yield steeper (flatter) decline in the production rate function. A fine tuning of the shape of the production rate function is obtained by changing the shape parameter b.

2.3 Multi-segmented production functions

2.3.1 Model framework

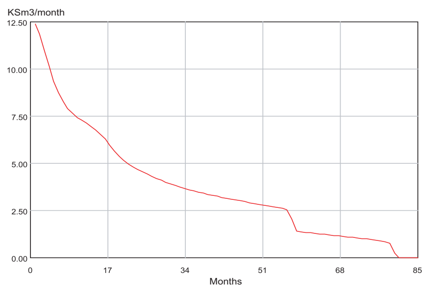

Figure 2.1: *Monthly production profile*

While the Arps decline curves covers a broad range of cases, these curves should be used with considerable caution. Camacho & Raghavan (1989) emphasize that in many cases one single Arps curve is not sufficient to obtain a satisfactory fit, as the production data span the entire set of curves. Thus, in many situations it may be necessary to use a combination of curves to get a satisfactory fit. Figure 2.1 shows an oil production profile from a real-life field. The curve was produced using a full-scale reservoir model. As one can see, the curve consists of several clearly separated segments with quite different characteristics. In order to get an acceptable fit to this curve, the curve was divided into as many as 7 segments.

To proceed it is sensible to formalize our concept of fit. Introduce $Q_{sim}(t)$ and $Q_{app}(t)$, where $Q_{sim}(t)$ denotes cumulative production from the production curve generated by the reservoir simulator and $Q_{app}(t)$ represents cumulative production from our approximation of the production curve from the reservoir simulator. We require that the distance between the simulated cumulative production and the approximated cumulative production should not exceed some threshold, i.e.,

$$|Q_{sim}(t) - Q_{app}(t)| < \epsilon, \tag{2.3.1}$$

for $t \in \{t_0, t_0 + \delta, \dots, T_c - \delta, T_c\}$, for some chosen threshold $\epsilon > 0$, and where t_0 denotes the production start, T_c denotes the time point for technical cut-off and δ represents a chosen resolution parameter. Note that the resolution parameter δ and the threshold ϵ are positively correlated, so hence a courser resolution yields a higher threshold. A sensible specification of the threshold ϵ and the resolution parameter δ depends on the context in which the framework is applied. In general δ and ϵ should not be selected too small for two reasons. First, the segmentation of the production curve is approximating the output from a reservoir simulator before production starts. The output we try to approximate is consequently a *prediction* of the production. This prediction will most likely change as our knowledge of the reservoir increases. A high degree of scientific accuracy in the segmentation of the production curve will not remove the uncertainty associated with the output from a reservoir simulator before the production commences. Second, in many applications too small values of ϵ and δ would not be sensible. In total value chain analysis a course resolution, such as $\delta = 1$ year, is sufficient.

For a given threshold ϵ and a given resolution parameter δ we then assess the number of segments, as described in detail in Huseby & Haavardsson (2007). If our fit criterion stated in (2.3.1) were defined as the difference in instantaneous production instead of cumulative production, it could lead to an erroneous recoverable volume. To see why, the absolute difference in instantaneous production could be less than ϵ for all $t \in \{t_0, t_0 + \delta, \dots, T_c - \delta, T_c\}$, but at the same time the approximated production curve could be systematically too high, systematically too low or both.

2.3.2 Stochastic simulation

A Monte Carlo simulation approach is used in the stochastic simulation. Uncertainty is added to the production model by modelling some of the key parameters as stochastic variables. A large sample, N, of the key parameters is generated, and by following Algorithm 2.3.1 described below, every simulated vector of key parameters produces one simulated production profile. Using this approach, we obtain a sample of N simulated production profiles. Thus we only need one simulation from the reservoir simulation software. Input for the uncertainty distributions of the stochastic variables are provided by industry experts.

When uncertainty is added to a production model, using a Monte Carlo simulation, the generated production functions may differ considerably from the one used in the curve fitting. In principle, the simulation could describe the uncertainty of all the segment parameters, including the number of segments and the shape parameters for the segments. It is important that the simulated production functions behave essentially like real-life functions. To obtain this we assume that the number of segments, s and the shape parameters for the segments, $b_1, \ldots, b_s$, are kept constant during the simulations in the following discussion.

The main source of uncertainty is typically the segment volumes. Thus, to assess the production uncertainty, we start out by specifying a suitable joint distribution $p(V_1, \ldots, V_s)$ for $V_1, \ldots, V_s$. We proceed by specifying a conditional joint distribution $p(r_0, r_1, \ldots, r_s | V_1, \ldots, V_s)$ for the rates at the segmentation points $r_0, r_1, \ldots, r_s$, given the segment volumes. Given all these quantities, the scale parameters $D_1, \ldots, D_s$ can be found. The point in time when the production starts, t_0, may be subject to uncertainty related to the progress of the development project, drilling activities etc. Thus, one will typically assess a separate uncertainty distribution for this quantity. The remaining segmentation points, $t_1, \ldots, t_s$, are found by exploiting relationships between $t_{i-1}, D_i, b_i, r_{i-1}$ and r_i, see Huseby & Haavardsson (2007) for details. A Monte Carlo simulation of the production can then be done using the following procedure:

Algorithm 2.3.1. STEP 1. *Generate $V_1, \ldots, V_s$ using the specified joint distribution.*

STEP 2. *Generate $r_0, r_1, \ldots, r_s$ using the specified conditional joint distribution.*

STEP 3. *Calculate $D_1, \ldots, D_s$.*

STEP 4. *Generate t_0, and calculate $t_1, \ldots, t_s$.*

STEP 5. *Calculate $q(t)$ and $Q(t)$.*

2.4 A joint distribution framework for Monte Carlo simulation

We now want to explain in detail how production uncertainty can be modelled using the multi-phased production functions. In particular, the algorithm constructed in Section 2.3.2 is based on the approach explained below. We begin by considering volume uncertainty. To obtain consistent and reasonable results it is important to maintain control over the total reservoir volume, denoted by V. Thus, it is often convenient to start out by assessing an uncertainty distribution for this volume. The segment volumes are then obtained as fractions of V. That is, we have:

$$V_i = K_i \cdot V, \qquad i = 1, \ldots, s, \tag{2.4.1}$$

where $K_i = V_i/V$ denote the fraction associated with the ith segment, $i = 1, \ldots, s$. The joint distribution of $K_1, \ldots, K_s$ must be chosen so that $\Pr(\sum_{i=1}^s K_i = 1) = 1$. A simple way of constructing such a distribution is to start out with s independent nonnegative random variables, $\tilde{K}_1, \ldots, \tilde{K}_s$, and then obtain $K_1, \ldots, K_s$ by normalizing the $\tilde{K}_i$s so that their sum becomes 1. That is, we define:

$$K_i = \frac{\tilde{K}_i}{\sum_{j=1}^s \tilde{K}_j}, \qquad i = 1, \ldots, s. \tag{2.4.2}$$

This construction ensures that $\Pr(\sum_{i=1}^s K_i = 1) = 1$ regardless of the distributions of the $\tilde{K}_i$s. Thus, one has a large variety of distributions to choose from. In particular, it is well-known that if $\tilde{K}_i$ is Gamma distributed with shape parameter α_i and scale parameter β, for $i = 1, \ldots, s$, the resulting joint distribution of $K_1, \ldots, K_s$ is a Dirichlet distribution with parameters $\alpha_1, \ldots, \alpha_s$. For more on this see e.g., Gelman et al. (1995).

Alternatively, one can use a recursive approach where each segment volume is defined as a fraction of the remaining volume. That is, let $B_1, \ldots, B_{s-1}$ be $s-1$ independent random fractions, i.e., random variables with values in $[0, 1]$. For convenience we also introduce $B_s = 1$ reflecting that the last segment volume should always be equal to the remaining volume when all the other segment volumes are subtracted from V. We can then define the segment volumes recursively as follows:

$$V_i = B_i \cdot (V - \sum_{j=1}^{i-1} V_j), \qquad i = 1, \ldots, s. \tag{2.4.3}$$

By expanding the recursive formula the following alternative nonrecursive formula can be obtained:

$$V_i = B_i \prod_{j=1}^{i-1} (1 - B_j) \cdot V, \qquad i = 1, \ldots, s. \tag{2.4.4}$$

By comparing (2.4.1) and (2.4.4) it follows that the relationship between the K_is and the B_is can be expressed as:

$$K_i = B_i \prod_{j=1}^{i-1} (1 - B_j), \qquad i = 1, \ldots, s. \tag{2.4.5}$$

Alternatively, by inserting (2.4.1) into (2.4.3) and solving for B_i we get the following inverse relation:

$$B_i = \frac{K_i}{1 - \sum_{j=1}^{i-1} K_j}, \qquad i = 1, \ldots, s. \tag{2.4.6}$$

Note that since $B_s = 1$, it follows that:

$$K_s = 1 - \sum_{j=1}^{s-1} K_j, \tag{2.4.7}$$

or equivalently that $\sum_{j=1}^{s} K_j = 1$ as before.

Assuming that B_i has a density f_i on $[0,1]$, for $i = 1, \ldots, s-1$, the resulting joint density of $K_1, \ldots, K_{s-1}$, denoted $g(K_1, \ldots, K_{s-1})$, can be found using the standard change-of-variable formula. Noting that the Jacobian of the inverse transformation (2.4.6) is triangular, it follows that g is given by:

$$g(K_1, \ldots, K_{s-1}) = \prod_{i=1}^{s-1} f_i \left(\frac{K_i}{1 - \sum_{j=1}^{i-1} K_j} \right) \cdot \frac{1}{1 - \sum_{j=1}^{i-1} K_j}. \tag{2.4.8}$$

Assume in particular that B_i is Beta distributed with parameters α_i and β_i, $i = 1, \ldots, s-1$. Then it follows that:

$$g(K_1, \ldots, K_{s-1}) \propto \prod_{i=1}^{s-1} \left(\frac{K_i}{1 - \sum_{j=1}^{i-1} K_j} \right)^{\alpha_i - 1} \times$$
$$\left(\frac{1 - \sum_{j=1}^{i} K_j}{1 - \sum_{j=1}^{i-1} K_j} \right)^{\beta_i - 1} \cdot \frac{1}{1 - \sum_{j=1}^{i-1} K_j}, \tag{2.4.9}$$

where the normalizing constant is the product of the standard Beta distribution normalizing constants. By rearranging the terms it is easy to see that (2.4.9) can be written as:

$$g(K_1, \ldots, K_{s-1}) \propto \left[\prod_{i=1}^{s-1} K_i^{\alpha_i - 1} \right] \times$$
$$\left[\prod_{i=1}^{s-2} (1 - \sum_{j=1}^{i} K_j)^{\beta_i - (\alpha_{i+1} + \beta_{i+1})} \right] \cdot \left[1 - \sum_{j=1}^{s-1} K_j \right]^{\beta_{s-1} - 1}. \tag{2.4.10}$$

Inserting $K_s = 1 - \sum_{j=1}^{s-1} K_j$ and $\alpha_s = \beta_{s-1}$, the density becomes:

$$g(K_1, \ldots, K_s) \propto \left[\prod_{i=1}^{s} K_i^{\alpha_i - 1} \right] \times$$
$$\left[\prod_{i=1}^{s-2} (1 - \sum_{j=1}^{i} K_j)^{\beta_i - (\alpha_{i+1} + \beta_{i+1})} \right]. \tag{2.4.11}$$

In particular if $\beta_i = \alpha_{i+1} + \beta_{i+1}, i = 1, \ldots, s-2$, we see that $K_1, \ldots, K_s$ once again becomes Dirichlet distributed with parameters $\alpha_1, \ldots, \alpha_s$. This special case is a well-known result as well. See Gelman et al. (1995) for more details. In the following, however, we will not impose these restrictions on the parameters of the Beta distributions and instead use the more general distribution given in (2.4.11).

We then turn to modelling production rate uncertainty, i.e., the joint distribution of $r_0, r_1, \ldots, r_s$. Since the production function is assumed to be strictly decreasing, this distribution must be chosen so that $r_0 > r_1 > \cdots > r_s$ with probability one. A simple way to accomplish this is to introduce random fractions $C_1, \ldots, C_s$ with values in $[0,1]$, and use the following multiplicative model:

$$r_i = C_i \cdot r_{i-1}, \qquad i = 1, \ldots, s. \tag{2.4.12}$$

If $C_1, \ldots, C_s$ are independent, the r_is will form a discrete time Markov chain. In addition to this we need to specify a suitable distribution for r_0. In general it may also be of interest to incorporate some sort of (typically positive) dependence between the r_is and the segment volumes. In many cases, however, it may be sufficient to include dependence between V and r_0.

2.5 Modelling production delays and random effects

2.5.1 Basic production profiles

In the previous sections we started out by fitting the production rate function, and then used this as the basis for calculating the cumulative production function. Since the Arps functions are easy to handle analytically, all the necessary calculations can be done very efficiently using explicit integration formulas. This is especially convenient when the production model is part of a Monte Carlo simulation model. As demonstrated in the previous section, it is easy to incorporate reservoir uncertainty into the model, and still keep the overall structure.

In many cases, however, this approach becomes too static. Having a production model with a fixed number of segments, makes it difficult to incorporate various types of production constraints, random irregularities in the flow, maintenance operations, etc. An important observation is that such external factors tend to affect the short term production rate, but not the ultimate recoverable volume. Thus, if a well has to be shut down temporarily for maintenance, the total volume produced from this well may still be the same. The consequence of the operation is that the production is *delayed*. In order to incorporate external factors into the model, we need to include both the short term effect (the reduced production rate due to the maintenance operation), as well as the long term effect (the delayed production). In principle it is possible to model such effects by modifying the production rate function. However, it turns out to be much easier to do this by establishing a feedback loop between the production rate and the cumulative production. That is, instead of modelling a simple one-way causal relation from the production rate to the cumulative production, we model the dynamic two-way relation between these two functions in terms of a differential equation. We will do this in two steps. In the first step we model the *internal* relation between the production rate and the cumulative production assuming no external factors, while in the second step we modify the internal relation by including external effects.

In general one can argue that the internal relation between the production rate and the cumulative production is essentially *time independent*, if we disregard the need to specify the production start. Thus, if t_0 denotes the point in time when the production starts, the relation between the production rate function, q, and the cumulative production function, Q, should be of the following form:

$$q(t) = f(Q(t)), \qquad \text{for all } t \geq t_0, \tag{2.5.1}$$

with $Q(t_0) = 0$ as a boundary condition. Provided that the function f satisfies the well-known *Lipschitz condition*, there exists a unique solution to (2.5.1). See

Dettman (1986) for details about this. A differential equation of the form given in (2.5.1), where the function f depends only on Q, is called a first order *autonomous* differential equation. Using standard numerical differential equation solvers, like Runge-Kutta's 4th order method, it is very easy to calculate the resulting production rate and cumulative production regardless of the function f. For details see Kloeden et al. (2003). Thus, this formulation allows us to work with many different functions. However, since the reservoir pressure falls as the reservoir is emptied, the function f will typically be a decreasing function. The results of the detailed deductions stated in Huseby & Haavardsson (2007) are summarized in Table 2.2.

Type of decline curve	Production function	Relation function $f(Q(t))$
Hyperbolic	$r_0[1+bD(t-t_0)]^{-1/b}$	$r_0\{1-\frac{D(1-b)}{r_0}Q(t)\}^{1/(1-b)}$
Exponential	$r_0\exp(-D(t-t_o))$	$r_0\{1-\frac{D}{r_0}Q(t)\}$
Harmonic	$r_0[1+D(t-t_0)]^{-1}$	$r_0\exp\{-\frac{DQ(t)}{r_0}\}$

Table 2.2: *Key functions and properties for Arps decline curves.*

2.5.2 Multi-segmented production profiles

The differential equation approach can also be extended to the more general situation where the production rate function consists of s segments. For each segment we assume that we have fitted a model in terms of a differential equation. As before we denote the production rate function and the cumulative production function for the ith segment by q_i and Q_i respectively. Moreover, we assume that the relations between these functions are given by:

$$q_i(t) = f_i(Q_i(t)), \text{ for } i = 1, \ldots, s. \tag{2.5.2}$$

In order to connect these segment models, we need to specify a *switching rule* describing when to switch from one segment model to the next one. Such a switched system is a special case of so-called hybrid systems. For an introduction to switched systems, see Liberzon (2003). One possible way of specifying a switching rule is to calculate the switching points, $t_1, \ldots, t_s$ as we did in the previous section, and then switch from model to model as we pass these points. This is called a *time-dependent* switching rule. In our case it turns out to be more convenient to use the *state* of the process, represented by the cumulative volume as a basis for the switching rule. This is called a *state-dependent* switching rule. As in the previous sections we denote the volumes produced in the s segments by $V_1, \ldots, V_s$ respectively. Moreover, the production rate function and the cumulative production function of the combined profile are denoted by q and Q respectively. The switching rule, $\sigma : \mathbb{R}^+ \to \{1, \ldots, s\}$, maps the state value, Q, onto the corresponding segment index, and is given by:

$$\sigma(Q) = \begin{cases} 1 & \text{if } 0 \le Q \le V_1 \\ \vdots & \\ k & \text{if } \sum_{i=1}^{k-1} V_i < Q \le \sum_{i=1}^{k} V_i \\ \vdots & \\ s & \text{if } \sum_{i=1}^{s-1} V_i < Q \le \sum_{i=1}^{s} V_i. \end{cases} \tag{2.5.3}$$

By using this switching rule, the combined differential equation can be written as:

$$q = f_{\sigma(Q)}(Q - \sum_{i<\sigma(Q)} V_i). \tag{2.5.4}$$

To summarize this section, we have shown that all the models introduced in the previous sections can be described equivalently in terms of differential equations. In itself this reformulation does not simplify anything. In fact, it does not make much sense to use this approach at all whenever exact, explicit solutions, like those presented in the previous sections, are available. The main advantage of the differential equation approach is that it allows us to model complex interactions between the pure production models and other models affecting the production in various ways, and still keep the physical relation between the production rate and the cumulative production intact. We will illustrate how this can be done, by considering an example.

2.5.3 An example: Deterministic calculations and stochastic simulations

Assume that we have fitted a production model in terms of a differential equation in the form (2.5.1), as done above. In the following we will refer to the function f as the *state-dependent* part of the production model. In a real-life situation the production rate may be affected by other factors as well. One such factor may be the ability of each well or well cluster to produce oil. In this paper we refer to this ability as the deliverability function and it represents the subfield's aggregated ability to produce oil. The deliverability function depends on parameters such as pipe dimensions from the wells to the central process facility. The deliverability function is typically not a function of the production state, but rather a function of time. We let $d(t)$ denote the deliverability function at time $t \ge t_0$. A model combining both the state-dependent part and the deliverability can then be written as:

$$q(t) = \min(d(t), f(Q(t))), \text{ for all } t \ge t_0. \tag{2.5.5}$$

In principle any nonnegative, measurable function can represent the deliverability function. As long as f satisfies the Lipschitz condition, so will the combined function.

Normally, a typical deliverability function would be a step function representing the combined production capacity from the available wells at a given point in time. Each time a well is put in production, the deliverability increases, while each time a well drops out, the deliverability decreases.

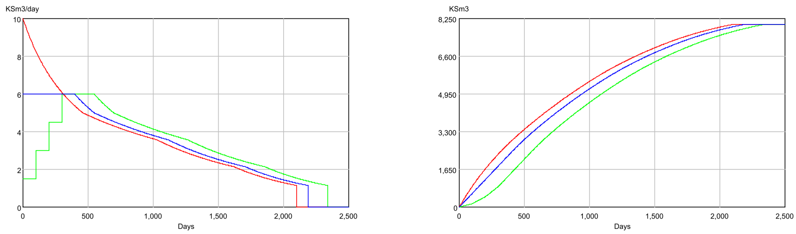

Figure 2.2: *The production rate (left panel) and the cumulative production function (right panel) for different choices of the deliverability function. In both panels the green curve represents the situation when the wells are put in production sequentially, the blue curve represents the situation when all the wells are put in at the production start, and the red curve represents the case of the unrestricted production profile.*

In the simplest cases the deliverability function can be modelled as a deterministic step function. As an example consider the deterministic production function with parameters as listed in Table 2.3. We then combine this state-dependent function with a deterministic deliverability function. The deliverability function is partitioned into four segments. We consider a situation where the wells are put in production sequentially, and never drops out. The beginning of each segment corresponds to the point in time when a new well is put in production. Thus, the deliverability values should form an increasing sequence. Moreover, the deliverability values are assumed to be constant within each segment. The segment capacities (in KSm3) and durations (in days) are listed in Table 2.4. As the last segment of the d function is assumed to last throughout the remaining lifetime of the field, the duration of this segment is set to ∞. As a contrast to the situation where the wells are put in production sequentially, we also consider the situation where all the wells are put in production at the production start. We also consider the case of an unrestricted production profile, i.e., only the state dependent part $f(Q(t))$. An unrestricted production profile refers to a production profile where the production is not limited by the deliverability function, the process capacity of the platform, transport capacity or other forms of capacity constraints. The unrestricted production of a reservoir refers to the potential production of that reservoir. Consequentially, the unrestricted profile of a depletion field has no plateau phase. In the case of a field with gas or water injection, a plateau phase may occur.

Segment (i)	1	2	3	4
Volume (V_i)	3600.0	2200.0	1500.0	700.0
Initial rate (r_{i-1})	10.0	3.5	2.5	1.5
End rate (r_i)	3.5	2.5	1.5	0.8
Shape parameter (b_i)	0.0	0.5	0.5	1.0

Table 2.3: *Parameter values for the deterministic production function*

Segment (i)	1	2	3	4
Capacities	1.5	3.0	4.5	6.0
Durations	100	100	100	∞

Table 2.4: *Segment capacities and durations*

We now want to study the production rates and the cumulative production of these three models. We calculated the production rates and the cumulative production for the three models using a standard Runge-Kutta's 4th order differential equation solver. The results are shown in Figure 2.2. We observe that the three curves are quite different in the beginning when the d function is the dominating term of (2.5.5). As the state-dependent part becomes dominating, the three curves become identical except for a time lag. This lag occurs since the deliverability constraints slows the production down in the early stages. This lag is seen in the cumulative curves as well. Still the three cumulative curves end up at the same level corresponding to the total volume of the field.

More generally, both the deliverability part and the state-dependent part can of course be stochastic. In order to focus on the effect of a stochastic deliverability, we keep the state-dependent part deterministic for now, and include only uncertainty about the deliverability. More specifically we consider two cases. In the first case we let the durations of each segment be stochastic, while the deliverability values are kept constant. For simplicity the durations are assumed to be independent and lognormally distributed with means 100 and standard deviations 50.

In the second case we also include uncertainty about the deliverability values. To get an increasing sequence of deliverability values for the four segments, we used a simple additive model, where the deliverability at a given time point is equal to a sum of independent stochastic variables, and where the number of terms in the sum is equal to the number of wells in production at that given point in time. All the terms where assumed to be lognormally distributed with means 1.5 and standard deviations 0.5.

Using Runge-Kutta's 4th order method we ran 5000 simulations on the two uncertainty models, and calculated means and standard deviations for each point on the production rate functions and cumulative production functions. Using these numbers we calculated upper and lower bounds for the rates and the cumulative values. As in the previous section we used the mean value plus and minus the standard deviation as upper and lower bounds respectively.

In Figure 2.3 we have plotted the deterministic production rate and the cumulative production together with the two corresponding sets of upper and lower bounds. In particular, the cumulative curves all reach the fixed total volume.

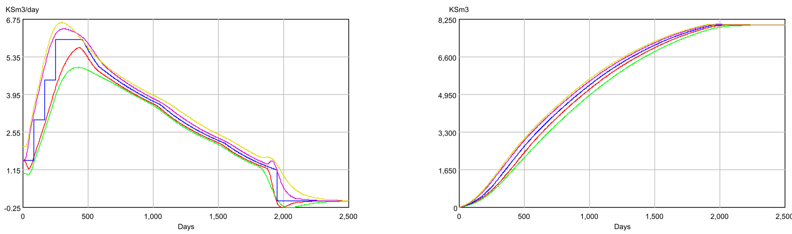

Figure 2.3: *The left panel illustrates deterministic production rate (blue curve), upper and lower bounds corresponding to the first stochastic model (purple and red curves) and the second stochastic model(yellow and green curves). The right panel illustrates deterministic cumulative production (blue curve), upper and lower bounds corresponding to the first stochastic model (purple and red curves) and the second stochastic model (yellow and green curves).*

2.5.4 Introducing regularity

As we have seen, the deliverability function allows us to incorporate variable production constraints related to major changes in the operational settings, corresponding to wells being put in or out of production. However, when operating an oil field, the production rate is also affected by short term irregularities. Such irregularities may be caused by random irregularities in the flow or for example bad weather. To include such phenomenons into the production model, it is more natural to use a continuous time stochastic process. More specifically, we introduce a stochastic process $\{r(t)\}$ such that for any given point in time t, $r(t)$ represents the fraction of the production rate that is actually produced at this point in time. This process is referred to as the *regularity process*. When this process is incorporated into the model, the resulting differential equation becomes:

$$q(t) = r(t) \cdot \min(d(t), f(Q(t))), \text{ for all } t \geq t_0. \tag{2.5.6}$$

As long as $r(t)$ is measurable almost surely, we can easily extend our framework to cover this situation. A suitable model for $r(t)$ and examples are presented in Huseby & Haavardsson (2007).

2.6 Multiple production profiles

2.6.1 Introduction

In many cases it is of interest to model the production from several subfields simultaneously. Common capacity constraints are often imposed on subfields belonging to a larger field or group of fields. Such constraints may for example be the total capacity at a processing facility. We will now show how such a situation can be managed within our framework. More specifically, we assume that we have a field consisting of n subfields. The production from these subfields are processed on a common processing facility with a given processing capacity per unit of time.

2.6.2 Common constraint models

We start out by modelling the *potential production* from each the subfields given no common capacity constraints. The potential production rate function for subfield i at time t is denoted by $q_i^{pot}(t)$ and is expressed as:

$$q_i^{pot}(t) = g_i(t, Q_i(t)), \; i = 1, \ldots, n, \tag{2.6.1}$$

where $Q_i(t)$ represents the cumulative production function of subfield i at time t, and g_i represents both the time and state dependent parts of the model for subfield i. Thus, g_i may include segments, time dependent subfield constraints and regularity as explained in the previous section.

The processing capacity at time t is denoted by $K(t)$. To ensure that the total production does not exceed $K(t)$, the production from each subfield has to be restricted in some way. This can be done by introducing *choke factors* for each of the subfields. Thus, let $a_i(t) \in [0, 1]$ denote the choke factor for the ith subfield at time t, $i = 1, \ldots, n$. The resulting production from the ith subfield at time t is then given by $q_i(t) = a_i(t)q_i^{pot}(t)$, $i = 1, \ldots n$. In order to satisfy the processing constraint, the choke factors must be chosen so that:

$$\sum_{i=1}^{n} a_i(t)q_i^{pot}(t) \leq K(t) \text{ for all } t \geq 0. \tag{2.6.2}$$

Clearly there are an infinite number choices for the choke factors. Typically, one would be interested in finding choke factors that are optimal with respect to some suitable criterion. E.g., one could try to maximize the total discounted production from all the subfields. Solving such optimization problems can be difficult, especially when uncertainty is included in the model, so this is beyond the scope of the present paper. Instead we will present some possible ad. hoc. strategies for the choke factors.

The symmetric strategy. When using the symmetric strategy, the available process capacity is shared proportionally between the different fields. If the current production exceeds the process capacity of the central processing facility, the production of every field is scaled down with a common choke factor. That is, we let

$$a_i(t) = \frac{K(t)}{\sum_{j=1}^{n} q_j^{pot}(t)},\ i = 1, \ldots, n, \tag{2.6.3}$$

where $K(t)$ denotes process capacity of the central processing facility.

The priority strategy. For this strategy the fields are prioritized initially according to some suitable criterion. Let $\boldsymbol{\pi} = (\pi(1), \ldots, \pi(n))$ be the permutation vector representing the order at which the fields are prioritized. That is, field number $\pi(1)$ is given the highest priority, field number $\pi(2)$ is given the second highest priority, etc. The choke factors are then defined as:

$$\begin{aligned}
a_{\pi(1)}(t) &= \min\{1, \frac{K(t)}{q_{\pi(1)}^{pot}(t)}\}, \qquad (2.6.4)\\
a_{\pi(2)}(t) &= \min\{1, \frac{K(t) - a_{\pi(1)}(t) q_{\pi(1)}^{pot}(t)}{q_{\pi(2)}^{pot}(t)}\}\},\\
&\cdots\\
a_{\pi(n)}(t) &= \min\{1, \frac{K(t) - \sum_{j=1}^{n-1} a_{\pi(j)}(t) q_{\pi(j)}^{pot}(t)}{q_{\pi(n)}^{pot}(t)}\}\}.
\end{aligned}$$

Note that if $q_{\pi(i)}^{pot}(t) = 0$ for some i and t, the corresponding choke factor is undefined in (2.6.4). To avoid this problem, we simply replace the corresponding choke factor by 1 in such cases.

The fixed maximal quota strategy. Sometimes it may not be possible to adjust the choke factors continuously. Instead one have to choose fixed maximal production rates for the subfields initially, and use the same values throughout the lifeime of the field. More specifically, assume that the ith subfield is given a fixed maximal quota κ_i for the entire production period, $i = 1, \ldots, n$. The resulting production from the ith subfield at time t is then given by $q_i(t) = \min\{\kappa_i, q_i^{pot}(t)\}$, $i = 1, \ldots n$. In order to satisfy the processing constraint, the quotas must be chosen so that:

$$\sum_{i=1}^{n} \kappa_i \leq K(t), \text{ for all } t \geq 0. \tag{2.6.5}$$

In particular, if $K(t) = K$ for all t the quotas should be chosen so that their sum is equal to K.

Regardless of which strategy one chooses for the choke factors, it is easy to run simulations on all the subfields simultaneously. The fact that we now have a set of n ordinary differential equations instead of just one, does not cause any problems. All the calculations can still be done using e.g., a standard Runge-Kutta's 4th order differential equation solver.

2.6.3 An example with different choices of production quota constraints

In this example we will study both continuous control constraints and quota model constraints. This case study is based on real reservoir simulation data from two actual fields. We assume that two fields share a central processing facility. In one of the fields water is injected into the reservoir to maintain reservoir pressure. We will refer to this field as the *maintenance field*. In the other field the reservoir is depleted without water being injected. This field is referred to as the *depletion field*. In most cases water or gas will be injected into the reservoir to maintain reservoir pressure if it is economically viable to do so. However, some times there are good reasons not to inject water or gas into the reservoir, for example if the field of interest is too small.

Table 2.5 summarizes the characteristics of the production rates of the two fields. The depletion field is described by a production rate curve with two segments, while the maintenance field has a production rate curve with four segments. Table 2.6 states the characteristics of the field capacities of the two fields. The capacity of the central processing facility is assumed to be 2500 Sm3/d in this example. Production is assumed to commence immediately in both fields. Finally, Table 2.7 states the regularity parameters selected in this example. Figure 2.4 illustrates the simulation profile and the matched profile comparison of the production rate for the depletion field and the maintenance field.

Figure 2.5 show the results of implementing three different choices of constraints in our prototype. Symmetry constraints are illustrated in the upper left panel in Figure 2.5 and priority constraints are illustrated in the upper right panel. The maintenance field is prioritized throughout the entire production period. As a contrast to these continuous control constraints we have also implemented a constant quota model. In such a model the quotas of each field is decided once and for all before the production commences. The lower panel in Figure 2.5 shows the results of an implementation of a constant quota model where the maintenance field receives a constant quota of 1800 Sm3/d while the depletion field receives a constant quota of 700 Sm3/d. As we can see from these three figures the production rates of the two fields are quite different depending on what constraints are being implemented.

An important issue is how our choice of constraints impacts the total cumulative production. The left panel in Figure 2.6 shows total cumulative production for the three choices of constraints. Since the fields will be emptied regardless of the choice of constraints, we are ultimately interested in selecting the constraints that maximizes the discounted production and the net present value. From Figure 2.6 we can see that the priority constraints yields the highest discounted production value. The constant quota model yields the poorest discounted production value, while the discounted production of the symmetry constraints is a little lower than the discounted production of the priority constraints.

The initial choice of quotas in the constant quota clearly has a tremendous impact on the discounted production. To optimize discounted production with respect to initial quotas we let the constant quotas of the maintenance field and the depletion field vary between 0 and 2500 which is the capacity of the central processing facility. The right panel in Figure 2.6 illustrates discounted production as a function of the constant quota assigned to the maintenance field. We can see from the right panel in Figure 2.6 that by assigning a quota of 2058 Sm3/d to the maintenance field and 442 Sm3/d to the depletion field the discounted production is maximized. In Table 2.8 the discounted production values are summarized for the different strategies. We can see that when the constant quota model is optimized with respect to constant quota, its discounted production is almost as high as the discounted production of the symmetry constraints. The priority constraints yields the highest discounted production regardless of whether the constant quota model is optimized.

To give some intuition on the economic values of the different choices, Table 2.9 summarizes the net present value of the different choices of constraints. The net present value is calculated using an oil price of 55 USD per barrel of oil, which reflects the spot price level of Brent at the time the article is written. The oil price has fluctuated considerably in the past. Table 2.10 shows the difference in net present value in USD between the priority constraints and the quota constraints for some different choices of oil price.

Field	Segm. i	Start rate r_{i-1} (Sm3/d)	Stop rate r_i (Sm3/d)	Segm. vol. V_i (KSm3)	Shape param. b_i
Depletion	1	3995	1395	315.3	0.0
Depletion	2	1395	100	1182.5	0.8
Maintenance	1	9995	5525	993.2	0.0
Maintenance	2	5525	2919	2437.8	0.7
Maintenance	3	2919	1693	1062.3	1.0
Maintenance	4	1693	45	2681.8	0.5

Table 2.5: *Parameter values for the production rates of the depletion field and the maintenance field.*

Field	Capacity (Sm3/d)	End of epoch (Days)
Depletion	1395	230
Maintenance	5525	180

Table 2.6: *Parameter values for the field capacities of the depletion field and the maintenance field. In both cases the epoch considered starts at production start.*

Field	Minimum regularity	Maximum regularity	Correlation factor ρ
Depletion	0.95	1.0	0.8
Maintenance	0.95	1.0	0.8

Table 2.7: *Parameter values for the regularity parameters of the depletion field and the maintenance field.*

	Disc. Prod.
Symmetry constraints	6,656.7
Priority constraints	6,678.9
Constant (1800, 700 Sm3/d)	6,594.3
Constant (2058, 442 Sm3/d)	6,657.5

Table 2.8: *Discounted production values (KSm3) for the different strategies.*

	Net Present Value
Symmetry constraints	2,303
Priority constraints	2,310
Constant (1800, 700 Sm3/d)	2,281
Constant (2058, 442 Sm3/d)	2,303

Table 2.9: *Net present values in* $1,000,000$ *USD for the different strategies.*

Oil price in USD per barrel						
20	30	40	50	60	70	80
2,692	4,038	5,384	6,730	8,075	9,421	10,767

Table 2.10: *Difference in net present value in* $1,000$ *USD between priority constraints and optimized quota constraints for different choices of oil price.*

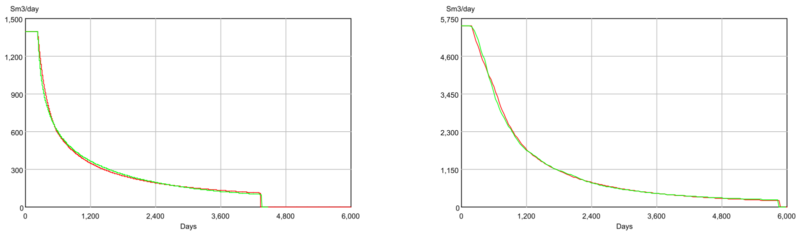

Figure 2.4: *The simulation profile and the matched profile comparison for the depletion field (left panel) and the maintenance field (right panel). In both panels the green curve represents the production rate from the simulation software, while the red curve represents the production rate from our model.*

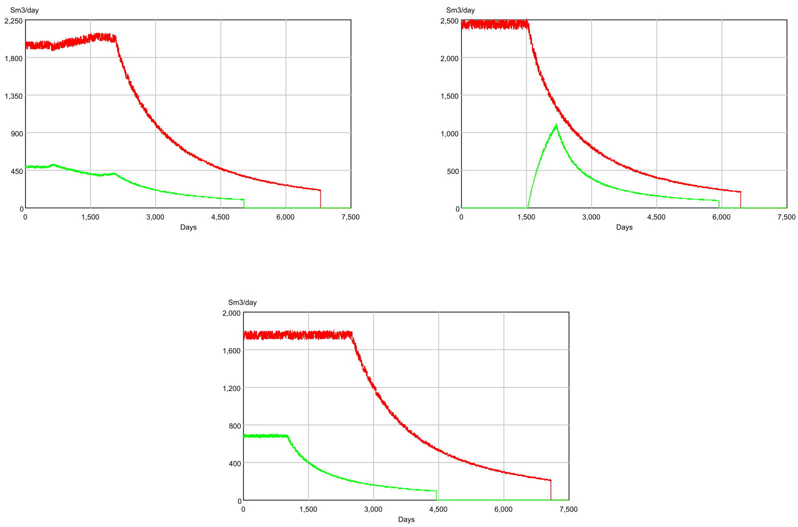

Figure 2.5: *The production rate of the depletion field (green graph) and the maintenance field (red graph) for different choices of constraints. The upper left panel illustrates continuous control with symmetry constraints. The upper right panel illustrates continuous control with priority constraints, where the maintenance field is being prioritized throughout the entire production period. The lower panel illustrates quota model with constant constraints. The maintenance field receives a constant quota of* $1800 Sm3/day$ *while the depletion field receives a constant quota of* $700 Sm3/day$*. In all panels the x-axis represents days and the y-axis represents* $Sm3/day$*.*

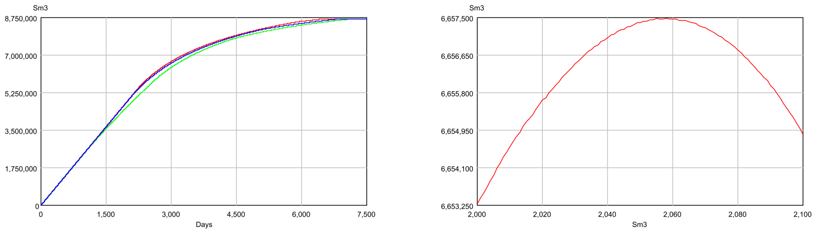

Figure 2.6: *The left panel illustrates total cumulative production for the three choices of constraints. Total cumulative production curves for priority constraints and symmetry constraints are illustrated by the red and blue graph, respectively. The green graph illustrates total cumulative production for the quota constraints model where the quota is* $1800Sm3/d$ *for the maintenance field and* $700Sm3/d$ *for the depletion field. The right panel illustrates discounted production as a function of the constant quota assigned to the maintenance field.*

2.7 Conclusions

We have demonstrated that the proposed hybrid system approach for production profile modelling is flexible and generic enough to be useful in the context of a total value chain analysis. We have seen that our framework handles multisegment production profiles, the foreseen and unforeseen delays in production and multiple production profile modelling. Further, this paper has studied different allocation strategies for subfields with common capacity constraints.

A natural next step is to optimize these allocation strategies. The inclusion of uncertainty in the analysis of multiple production profiles with common capacity constraints is also an important issue that needs to be treated and studied at a later stage. Further, it is also important to incorporate gas and water production into the model framework. As a consequence the complexity of the model will increase and the transparency may decrease. Bearing in mind that parsimonious modelling is a virtue these consequences are not desired. All the same we believe it is of great value to be able to incorporate gas and water production into the framework since these aspects are essential in the context of a total value chain analysis. Further, it is of interest to try to optimize multiple production profiles in higher dimensions than $n = 2$. Finally, we want to apply the framework to analyze other problems that arise in the context of total value chain analysis. Such problems could include crucial investment decisions regarding infrastructure on the platform, i.e. capital expenditure decisions. Another problem that could be of interest to analyze is the economic effect of different production plan schedules, such as the issue of phasing in and phasing out different subfields of a major field.

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Chapter 3

A framework for multi-reservoir production optimization

Abstract

When a large oil or gas field is produced, several reservoirs often share the same processing facility. This facility is typically capable of processing only a limited amount of oil, gas and water per unit of time. In order to satisfy these processing limitations, the production needs to be *choked*, i.e., scaled down by a suitable *choke factor*. A *production strategy* is defined as a vector valued function defined for all points of time representing the choke factors applied to reservoirs at any given time. In the present paper we consider the problem of optimizing such production strategies with respect to various types of objective functions. A general framework for handling this problem is developed. A crucial assumption in our approach is that the *potential production rate* from a reservoir can be expressed as a function of the remaining producible volume. The solution to the optimization problem depends on certain key properties, e.g., convexity or concavity, of the objective function and of the potential production rate functions. Using these properties several important special cases can be solved. An *admissible production strategy* is a strategy where the total processing capacity is fully utilized throughout a plateau phase. This phase lasts until the total potential production rate falls below the processing capacity, and after this all the reservoirs are produced without any choking. Under mild restrictions on the objective function the performance of an admissible strategy is uniquely characterized by the state of the reservoirs at the end of the plateau phase. Thus, finding an optimal admissible production strategy, is essentially equivalent to finding the optimal state at the end of the plateau phase. Given the optimal state a backtracking algorithm can then used to derive an optimal production strategy. We will demonstrate this on a specific example.

Keywords

Convex optimization theory, Convex sets, quasi-convex and quasi-concave functions, Simplified Production Profiles, Separating and supporting hyperplane theorems

Publication details

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3.1 Introduction

Optimization is an important element in the management of large offshore Exploration & Production (E&P) assets, since many investment decisions are irreversible and finance is committed long-term. van den Heever et al. (2001) classify decisions made in reservoir management in two main categories, design decisions and operational decisions. Design decisions comprise selecting the type of platform, the staging of compression and assessing the number of wells to be drilled in a reservoir. These decisions are discrete in nature. In operational decisions production rates from individual reservoirs and wells are assessed. In contrast to design decisions, operational decisions are continuous in nature.

Neiro & Pinto (2004) propose a framework for modelling the entire petroleum supply chain. Ivyer & Grossmann (1998) present a multi-period mixed-integer linear programming formulation for the planning and scheduling of investment and operation in offshore oilfields. In other approaches a case and scenario analysis system is constructed for evaluating uncertainties in the E&P value chain, see Narayanan et al. (2003) for details. In Floris & Peersmann (2000) a decision scenario analysis framework is presented. Here, scenario and probabilistic analysis is combined with Monte Carlo simulation. Optimization can also be performed using a simulator, where real-time decisions are made subject to production constraints. Davidson & Beckner (2003) and Wang et al. (2002) use this technique. Their decision variables include binary on/off conditions and continuous variables. Uncertainty was not considered in these works. Optimization of oil and gas recovery is also a considerable research area, see Bittencourt & Horne (1997), Horne (2002) or Merabet & Bellah (2002).

Many of the contributions listed above focus on the problem of modelling the entire hydrocarbon value chain, where the purpose is to make models for scheduling and planning of hydrocarbon field infrastructures with complex objectives. Since the entire value chain is very complex, many aspects of it needs to be simplified to be able to construct such a comprehensive model. The purpose of the present paper is to focus on the problem of optimizing production in an oil or gas field consisting of many reservoirs, which constitutes an important component in the hydrocarbon value chain. By focusing on only one important component we are able to develop a framework that provides insight into how a large oil or gas field should be produced. The optimization methods developed here can thus be used in the broader context of a total value chain analysis.

To obtain reliable and valid results, having realistic production models is very important. Key properties of the reservoirs are typically assessed by geologists, geophysicists, petroleum engineers and other specialists. This knowledge is then assembled and quantified into a reservoir model. Our analysis starts at the stage where a full-scale reservoir simulation has been performed, and the output from this simulation is given. Simplified production models can then be constructed based on this output. See Haavardsson & Huseby (2007) for details about this. The present paper will utilize such production models. For a related approach see Li & Horne (2002).

We consider a situation where several reservoirs share the same processing facility. Oil, water and gas flow from each reservoir to this facility. The processing facility is only capable of handling limited amounts of the commodities per unit of time. In order to satisfy the resulting constraints, the production needs to be *choked*. In this setting we focus on optimizing the oil production and leave the simultaneous analysis of oil, gas and water production for future work. To avoid issues of dependence between the production profiles of the reservoirs, the production from any reservoir is assumed to be independent of the production from the other reservoirs.

A fundamental model assumption is that the *potential production rate* from a reservoir, can be expressed as a function of the remaining producible volume, or equivalently as a function of the volume produced. Thus, if $Q(t)$ denotes the cumulative production at time $t \geq 0$, and $f(t)$ denotes the potential production rate at the same point of time, we assume that $f(t) = f(Q(t))$. This assumption implies that the potential production rate at a given point of time only depends on the volume produced at that time (or equivalently on the volume left in the reservoir). Thus, if we delay the production from a reservoir, we can still produce the same volume at a later time. We refer to the function f as the *potential production rate function* or *PPR-function* of the reservoir. If a reservoir is produced without any production constraint from time $t = 0$, the cumulative production function will satisfy the following autonomous differential equation:

$$\frac{dQ(t)}{dt} = f(Q(t)), \tag{3.1.1}$$

with the boundary condition $Q(0) = 0$. The function f would typically be a non-increasing function. In order to ensure a unique solution to (3.1.1), we will also assume that f is Lipschitz continuous. If $Q = Q(t)$ is the solution to (3.1.1), we assume that:

$$\lim_{t\to\infty} Q(t) = \int_0^\infty f(Q(u))du = V < \infty. \tag{3.1.2}$$

That is, the *recoverable volume* from the reservoir, denoted V, is assumed to be *finite*. Note that since f is continuous, (3.1.2) implies that:

$$\lim_{t\to\infty} f(Q(t)) = f(V) = 0, \tag{3.1.3}$$

since otherwise the integral in (3.1.2) would be divergent.

Due to various kinds of restrictions, including possible time-dependent constraints, the actual production rate will typically be less than or equal to $f(t)$. Still it turns out that the PPR-functions play an important part in the analysis.

The present paper presents the following contributions:

- Section 3.2 introduces basic concepts and results, including a discussion of objective functions and some mild restrictions we impose on them.

- In Section 3.3 we turn to the problem of finding the best production strategy. An algorithm for finding the best production strategy and two main results are presented. The first result deals with the solution to the optimization problem if the PPR-functions are convex and the extended version of objective function ϕ is quasi-convex[1], while the second result analogously treats the situation when the PPR-functions are concave and the extended version of objective function ϕ is quasi-concave. A specific type of objective function and an important class of production strategies are presented.

- In Section 3.4 we consider the case where all the PPR-functions are linear. In this case a specific production strategy is proven to be optimal for a wide class of objective functions. The framework is illustrated on a specific example.

- Section 3.5 is devoted to generate optimal production strategies using backtracking. Since the performance of an admissible strategy is uniquely characterized by the state of the reservoirs at the end of the plateau phase, the backtracking is initiated using the optimal state at the end of the plateau phase. Given the optimal state a backtracking algorithm can then be used to derive an optimal production strategy.

3.2 Basic concepts and results

We consider the oil production from n reservoirs that share a processing facility with a constant process capacity $K > 0$, expressed in some suitable unit, e.g., kSm^3 per day. Let $\boldsymbol{Q}(t) = (Q_1(t), \ldots, Q_n(t))$ denote the vector of cumulative production functions for the n reservoirs, and let $\boldsymbol{f}(t) = (f_1(t), \ldots, f_n(t))$ be the corresponding vector of PPR functions. We assume that the PPR functions can be written as:

$$f_i(t) = f_i(Q_i(t)), \qquad t \geq 0,\ i = 1, \ldots n.$$

Note that this assumption implies that the potential production rate of one reservoir does not depend on the volumes produced from the other reservoirs. We will also assume for $i = 1, \ldots n$ that f_i is non-negative and non-increasing as a function of $Q_i(t)$ for all t, and that $\lim_{t\to\infty} Q_i(t) = V_i < \infty$. As already stated, this implies that $\lim_{t\to\infty} f_i(Q_i(t)) = f_i(V_i) = 0$. These assumptions reflect the natural properties that the production rate cannot be negative, that reservoir pressure typically decreases towards zero as more and more oil is produced, and that the recoverable volume is finite. Finally, to ensure uniqueness of potential production profiles we will also assume that f_i is Lipschitz continuous in Q_i, $i = 1, \ldots, n$.

A *production strategy* is defined as a vector valued function $\boldsymbol{b} = \boldsymbol{b}(t) = (b_1(t), \ldots, b_n(t))$, defined for all $t \geq 0$, where $b_i(t)$ represents the *choke factor* applied to the ith reservoir at time t, $i = 1, \ldots, n$. We refer to the individual b_i-functions as the *choke factor functions* of the production strategy. The *actual production rates* from the reservoirs, after the production is choked is given by:

$$\boldsymbol{q}(t) = (q_1(t), \ldots, q_n(t)),$$

[1] For a definition of quasi-convex and quasi-concave functions see Appendix 3.7.2

where:

$$q_i(t) = \frac{dQ(t)}{dt} = b_i(t) f_i(Q_i(t)), \qquad i = 1, \dots, n.$$

We also introduce the total production rate function $q(t) = \sum_{i=1}^n q_i(t)$ and the total cumulative production function $Q(t) = \sum_{i=1}^n Q_i(t)$. To reflect that $\boldsymbol{q}$, q, $\boldsymbol{Q}$, and Q depend on the chosen production strategy $\boldsymbol{b}$, we sometimes indicate this by writing $\boldsymbol{q}(t) = \boldsymbol{q}(t, \boldsymbol{b})$, $q(t) = q(t, \boldsymbol{b})$, $\boldsymbol{Q}(t) = \boldsymbol{Q}(t, \boldsymbol{b})$, and $Q(t) = Q(t, \boldsymbol{b})$.

To satisfy the physical constraints of the reservoirs and the process facility, we require that:

$$0 \leq q_i(t) \leq f_i(Q_i(t)), \qquad i = 1, \dots n, \ t \geq 0, \tag{3.2.1}$$

and that

$$q(t) = \sum_{i=1}^{n} q_i(t) \leq K, \qquad t \geq 0. \tag{3.2.2}$$

Expressed in terms of the production strategy $\boldsymbol{b}$, this implies that:

$$0 \leq b_i(t) \leq 1, \qquad i = 1, \dots n, \ t \geq 0, \tag{3.2.3}$$

and that

$$\sum_{i=1}^{n} b_i(t) f_i(Q_i(t), t) \leq K, \qquad t \geq 0. \tag{3.2.4}$$

The constraint (3.2.3) implies that the actual production rate cannot be increased beyond the potential production rate at any given point of time, while the constraint (3.2.4) states that the actual, total production rate cannot exceed the capacity of the processing facility. Let $\mathcal{B}$ denote the class of production strategies that satisfy the physical constraints (3.2.3) and (3.2.4). We refer to production strategies $\boldsymbol{b} \in \mathcal{B}$ as *valid production strategies*.

Intuitively, choosing lower values for the choke factors has the effect that the volumes are produced more slowly. The following proposition, proved in Huseby & Haavardsson (2008), formalizes this.

Proposition 3.2.1. *Consider a reservoir with PPR-function $f(t) = f(Q(t))$, and let b^1 and b^2 be two choke factor functions such that $0 \leq b^1(t) \leq b^2(t) \leq 1$ for all $t \geq 0$. Let Q^1 and Q^2 denote the resulting cumulative production functions, and let $q^1(t) = b^1(t) f(Q^1(t))$ and $q^2(t) = b^2(t) f(Q^2(t))$ be the corresponding actual production rates. We assume that $Q^1(0) = Q^2(0) = 0$. Then $Q^1(t) \leq Q^2(t)$ for all $t \geq 0$.*

3.2.1 Objective functions

To evaluate production strategies we introduce an *objective function*, i.e., a mapping $\phi : \mathcal{B} \to \mathbb{R}$ representing some sort of a performance measure. If $\boldsymbol{b}^1, \boldsymbol{b}^2 \in \mathcal{B}$, we prefer $\boldsymbol{b}^2$ to $\boldsymbol{b}^1$ if $\phi(\boldsymbol{b}^2) \geq \phi(\boldsymbol{b}^1)$. Moreover, an *optimal production strategy* with respect to ϕ is a production strategy $\boldsymbol{b}^{opt} \in \mathcal{B}$ such that $\phi(\boldsymbol{b}^{opt}) \geq \phi(\boldsymbol{b})$ for all $\boldsymbol{b} \in \mathcal{B}$. In this paper we will impose some mild restrictions on the objective functions. The following section will explain the rationale behind these restrictions. Complete proofs of the results are given in Huseby & Haavardsson (2008).

If $\boldsymbol{b}^1, \boldsymbol{b}^2 \in \mathcal{B}$ are two production strategies such that $\boldsymbol{Q}(t, \boldsymbol{b}^1) \leq \boldsymbol{Q}(t, \boldsymbol{b}^2)$ for all $t \geq 0$, one would most likely prefer $\boldsymbol{b}^2$ to $\boldsymbol{b}^1$. Thus, a sensible objective function should have the property that $\phi(\boldsymbol{b}^1) \leq \phi(\boldsymbol{b}^2)$ whenever $\boldsymbol{Q}(t, \boldsymbol{b}^1) \leq \boldsymbol{Q}(t, \boldsymbol{b}^2)$ for all $t \geq 0$. Objective functions satisfying this property will be referred to as *monotone objective functions*. The following result states that monotone objective functions also satisfies a monotonicity with respect to the production strategy.

Proposition 3.2.2. *Let ϕ be a monotone objective function, and let $\boldsymbol{b}^1, \boldsymbol{b}^2 \in \mathcal{B}$ be such that $\boldsymbol{b}^1(t) \leq \boldsymbol{b}^2(t)$ for all $t \geq 0$. Then $\phi(\boldsymbol{b}^1) \leq \phi(\boldsymbol{b}^2)$.*

Monotone objective functions will encourage production strategies where the total production rate is sustained at the plateau level K as long as possible. Furthermore, when the plateau level cannot be sustained, all the reservoirs should be produced without choking.

To study this further we introduce:

$$T_K = T_K(\boldsymbol{b}) = \inf\{t \geq 0 : \sum_{i=1}^{n} f_i(Q_i(t, \boldsymbol{b})) \leq K\}. \tag{3.2.5}$$

If $\sum_{i=1}^{n} f_i(Q_i(0)) \leq K$, it follows that $T_K = 0$. In this case the optimization problem is trivial since no choking is necessary. To avoid this we henceforth assume that $\sum_{i=1}^{n} f_i(Q_i(0)) = \sum_{i=1}^{n} f_i(0) > K$. It then follows by the continuity and monotonicity of the PPR-functions that $\sum_{i=1}^{n} f_i(Q_i(t)) \geq K$ for all $t \in [0, T_K]$. The quantity $T_K(\boldsymbol{b})$ will be referred to as the *plateau length* for the production strategy $\boldsymbol{b}$.

We now define an *admissible production strategy* as a production strategy $\boldsymbol{b} \in \mathcal{B}$ satisfying the following constraints:

$$q(t) = \sum_{i=1}^{n} q_i(t) = \sum_{i=1}^{n} b_i(t) f_i(Q_i(t)) = K, \qquad 0 \leq t \leq T_K, \tag{3.2.6}$$

and

$$q_i(t) = b_i(t) f_i(Q_i(t)) = f_i(Q_i(t)), \qquad t > T_K,\ i = 1, \ldots n. \tag{3.2.7}$$

Moreover, we let $\mathcal{B}' \subseteq \mathcal{B}$ denote the class of admissible strategies.

The following results states that if the objective function is monotone, an optimal production strategy can always be found within the class of admissible production strategies. Thus, when searching for optimal strategies we can restrict the search to the class $\mathcal{B}'$.

Proposition 3.2.3. *Let ϕ be a monotone objective function, and let $\boldsymbol{b} \in \mathcal{B}$. Then there exists $\boldsymbol{b}' \in \mathcal{B}'$ such that $\phi(\boldsymbol{b}') \geq \phi(\boldsymbol{b})$.*

In general the revenue generated by the production may vary between the reservoirs. This may occur if e.g., the quality of the oil, or the average production cost per unit are different from reservoir to reservoir. Such differences should then be reflected in the chosen objective function. On the other hand, if all the reservoirs are similar, we could restrict ourselves to considering objective functions depending on the production strategy $\boldsymbol{b}$ only through the total production rate function $q(\cdot, \boldsymbol{b})$ (or equivalently through $Q(\cdot, \boldsymbol{b})$). We refer to such objective functions as *symmetric*.

Within the class of admissible production strategies any symmetric objective function can be expressed in terms of the system state at the end of the plateau phase. The following result formalizes this:

Proposition 3.2.4. *Let ϕ be a symmetric objective function, and let $\boldsymbol{b} \in \mathcal{B}'$. Then $\phi(\boldsymbol{b})$ is uniquely determined by $\boldsymbol{Q}(T_K(\boldsymbol{b}))$. Thus, we may write $\phi(\boldsymbol{b}) = \phi(\boldsymbol{Q}(T_K(\boldsymbol{b})))$.*

3.3 Optimizing production strategies

We now turn to the problem of finding the *best* production strategy, i.e., the one that maximizes the value of the objective function, ϕ. To simplify this problem, only *monotone*, *symmetric* objective functions will be discussed. As we shall see, Proposition 3.2.4 plays a key role when searching for optimal production strategies. In order to explain this, we consider the set of all possible cumulative production vectors for the given field, denoted by $\mathcal{Q}$:

$$\mathcal{Q} = [0, V_1] \times \cdots \times [0, V_n], \tag{3.3.1}$$

where $V_1, \ldots, V_n$ are the recoverable volumes from the n reservoirs. We then introduce the subsets $\mathcal{M}, \bar{\mathcal{M}} \subseteq \mathcal{Q}$ given respectively by:

$$\mathcal{M} = \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{n} f_i(Q_i) \geq K\}, \tag{3.3.2}$$

$$\bar{\mathcal{M}} = \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{n} f_i(Q_i) < K\}. \tag{3.3.3}$$

We also need the set of boundary points of $\mathcal{M}$ separating $\mathcal{M}$ from $\bar{\mathcal{M}}$, which we denote by $\partial(\mathcal{M})$. Thus, $\boldsymbol{Q} \in \partial(\mathcal{M})$ if and only if every neighborhood of $\boldsymbol{Q}$ intersects both $\mathcal{M}$ and $\bar{\mathcal{M}}$.

Since we have assumed that $\sum_{i=1}^{n} f_i(0) > K > 0$ and $\sum_{i=1}^{n} f_i(V_i) = 0$, both $\mathcal{M}$ and $\bar{\mathcal{M}}$ are non-empty. Moreover, since the PPR-functions are assumed to be continuous, it is easy to see that:

$$\partial(\mathcal{M}) \subseteq \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{n} f_i(Q_i) = K\}, \tag{3.3.4}$$

where equality holds if the PPR-functions are *strictly decreasing*.

The following key result shows how the shapes of the sets $\mathcal{M}$ and $\bar{\mathcal{M}}$ depend on the shapes of the PPR-functions.

Proposition 3.3.1. *Consider a field with n reservoirs with PPR-functions $f_1, \ldots, f_n$.*
(i) If $f_1, \ldots, f_n$ are convex, the set $\bar{\mathcal{M}}$ is convex.
(ii) If $f_1, \ldots, f_n$ are concave, the set $\mathcal{M}$ is convex.

Proof: Assume first that the PPR-functions are convex, and let $\boldsymbol{Q}^1 = (Q_1^1, \ldots, Q_n^1)$ and $\boldsymbol{Q}^2 = (Q_1^2, \ldots, Q_n^2)$ be two vectors in $\bar{\mathcal{M}}$. Thus, we have:

$$\sum_{i=1}^{n} f_i(Q_i^j) < K, \qquad j = 1, 2. \tag{3.3.5}$$

Then let $0 \leq \alpha \leq 1$, and consider the vector $\boldsymbol{Q} = (Q_1, \ldots, Q_n) = \alpha \boldsymbol{Q}^1 + (1 - \alpha)\boldsymbol{Q}^2$. Since the PPR-functions are convex, we have:

$$\begin{aligned}\sum_{i=1}^{n} f_i(Q_i) &= \sum_{i=1}^{n} f_i(\alpha Q_i^1 + (1-\alpha)Q_i^2) \\ &\leq \alpha \sum_{i=1}^{n} f_i(Q_i^1) + (1-\alpha) \sum_{i=1}^{n} f_i(Q_i^2) < K\end{aligned}$$

Thus, we conclude that $\boldsymbol{Q} \in \bar{\mathcal{M}}$ as well. Hence $\bar{\mathcal{M}}$ is convex. The second part of the proposition is proved in a similar way ■

Note that since convexity is preserved under set closure, we also have the following corollary

Corollary 3.3.2. *Consider a field with n reservoirs with convex PPR-functions $f_1, \ldots, f_n$. Then the set $\bar{\mathcal{M}} \cup \partial(\mathcal{M})$ is convex.*

Proof: The result follows by realizing that the closure of $\bar{\mathcal{M}}$ is $\bar{\mathcal{M}} \cup \partial(\mathcal{M})$ ■

By combining (3.3.3) and (3.3.4) we get that:

$$\bar{\mathcal{M}} \cup \partial(\mathcal{M}) \subseteq \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{n} f_i(Q_i) \leq K\}, \tag{3.3.6}$$

where equality holds if the PPR-functions are *strictly decreasing*.

The set $\mathcal{M}$ has the property that the total production rate can be sustained at plateau level as long as $\boldsymbol{Q}(t) \in \mathcal{M}$. More specifically, let $\boldsymbol{b}$ be any production strategy, and consider the points in $\mathcal{Q}$ generated by $\boldsymbol{Q}(t) = \boldsymbol{Q}(t, \boldsymbol{b})$ as t increases. From the boundary conditions we know that $\boldsymbol{Q}(0) = \boldsymbol{0}$. By the continuity of the PPR-functions, $\boldsymbol{Q}(t)$ will move along some path in $\mathcal{M}$ until the boundary $\partial(\mathcal{M})$ is reached.

If $\boldsymbol{b} \in \mathcal{B}$, the resulting path is said to be a *valid path*, while if $\boldsymbol{b} \in \mathcal{B}'$, the path is called an *admissible path*. In general only a subset of $\mathcal{M}$ can be reached by admissible paths. We denote this subset by $\mathcal{M}'$. Moreover, we let $\partial(\mathcal{M}') = \partial(\mathcal{M}) \cap \mathcal{M}'$. We now make the mild but important assumption that $\partial(\mathcal{M}')$ is a $(n-1)$-manifold with boundary denoted by $\partial(\partial(\mathcal{M}'))$. In particular we assume that all points in $\partial(\partial(\mathcal{M}'))$ can be reachable by admissible paths.

For an admissible path the total production rate equals K all the way until the path reaches $\partial(\mathcal{M}')$. Moreover, the plateau length $T_K(\boldsymbol{b})$ is the point of time when the path reaches $\partial(\mathcal{M}')$, implying that:

$$\partial(\mathcal{M}') = \{\boldsymbol{Q}(T_K(\boldsymbol{b})) : \boldsymbol{b} \in \mathcal{B}'\} \tag{3.3.7}$$

By Proposition 3.2.4 we know that $\phi(\boldsymbol{b}) = \phi(\boldsymbol{Q}(T_K(\boldsymbol{b})))$ given that $\boldsymbol{b} \in \mathcal{B}'$ and ϕ is symmetric. Hence, the best production strategy can, at least in principle, be found using the following two-stage process:

Algorithm 3.3.3. *Let ϕ be a monotone, symmetric objective function. Then a production strategy $\boldsymbol{b}$ which is optimal with respect to ϕ can be found as follows:*
STEP 1. *Find $\boldsymbol{Q}^{opt} \in \partial(\mathcal{M}')$ such that $\phi(\boldsymbol{Q}^{opt}) \geq \phi(\boldsymbol{Q})$ for all $\boldsymbol{Q} \in \partial(\mathcal{M}')$.*
STEP 2. *Find a production strategy $\boldsymbol{b} \in \mathcal{B}'$ such that $\boldsymbol{Q}(T_K(\boldsymbol{b})) = \boldsymbol{Q}^{opt}$.*

We observe that in the first step of Algorithm 3.3.3 the objective function ϕ is interpreted simply as a function of the vector $\boldsymbol{Q}$, while in the second step we look for a production strategy $\boldsymbol{b} \in \mathcal{B}'$ generating an admissible path in $\mathcal{M}$ from the origin to the optimal vector $\boldsymbol{Q}^{opt}$.

To solve the optimization problem given in Step 1 of Algorithm 3.3.3, we assume that it is possible to extend the definition of ϕ to all vectors $\boldsymbol{Q} \in \mathcal{Q}$. Moreover, we assume that the extended version of ϕ is non-decreasing in $\boldsymbol{Q}$. That is, if $\boldsymbol{Q}^1, \boldsymbol{Q}^2 \in \mathcal{Q}$ and $\boldsymbol{Q}^1 \leq \boldsymbol{Q}^2$, then $\phi(\boldsymbol{Q}^1) \leq \phi(\boldsymbol{Q}^2)$. Having extended ϕ in this way, the problem is now to maximize $\phi(\boldsymbol{Q})$ subject to the constraint that $\boldsymbol{Q} \in \partial(\mathcal{M}')$.

Note that since the PPR-functions are assumed to be non-decreasing, it follows that for any $\boldsymbol{Q} \in \mathcal{M}$, we can always find another vector $\boldsymbol{Q}' \in \partial(\mathcal{M})$ such that $\boldsymbol{Q} \leq \boldsymbol{Q}'$. Thus, since ϕ is assumed to be non-decreasing as well, we have $\phi(\boldsymbol{Q}) \leq \phi(\boldsymbol{Q}')$. In particular, if $\boldsymbol{Q}^* \in \partial(\mathcal{M})$ maximizes ϕ over $\partial(\mathcal{M})$, it follows that $\phi(\boldsymbol{Q}^*) \geq \phi(\boldsymbol{Q})$ for all $\boldsymbol{Q} \in \mathcal{M}$. We also introduce the set $\mathcal{N}$:

$$\mathcal{N} = \{\boldsymbol{Q} \in \mathcal{Q} : \phi(\boldsymbol{Q}) > \phi(\boldsymbol{Q}^*)\}. \tag{3.3.8}$$

Since $\phi(\boldsymbol{Q}^*) \geq \phi(\boldsymbol{Q})$ for all $\boldsymbol{Q} \in \mathcal{M}$, it follows that $\mathcal{M} \cap \mathcal{N} = \emptyset$.

If $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ as well, then obviously $\boldsymbol{Q}^*$ is a solution to the optimization problem in Step 1 of Algorithm 3.3.3. Hence, we may let $\boldsymbol{Q}^{opt} = \boldsymbol{Q}^*$. In many cases, however, it may happen that $\boldsymbol{Q}^* \notin \partial(\mathcal{M}')$. In such cases the optimal vector $\boldsymbol{Q}^{opt} \in \partial(\mathcal{M}')$ can typically be found at the boundary, $\partial(\partial(\mathcal{M}'))$.

Using results from Appendix 3.7 we are now ready to prove the two main results of this section.

Theorem 3.3.4. *Consider a field with n reservoirs with convex PPR-functions $f_1, \ldots, f_n$. Furthermore, let ϕ be a symmetric, monotone objective function. Assume also that ϕ, interpreted as a function of $\boldsymbol{Q}$, can be extended to a non-decreasing,* *quasi-convex*[2] *function defined on the set* $\mathcal{Q}$*. Then an optimal vector, denoted* $\boldsymbol{Q}^{opt}$*, i.e., a vector maximizing* $\phi(\boldsymbol{Q})$ *subject to* $\boldsymbol{Q} \in \partial(\mathcal{M}')$*, can always be found within the set* $\partial(\partial(\mathcal{M}'))$*.*

Proof: Let $\boldsymbol{Q} \in \partial(\mathcal{M}')$ be chosen arbitrarily. Then by Theorem 3.7.4 there exists m vectors $\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_m \in \partial(\partial(\mathcal{M}'))$ and non-negative numbers $\alpha_1, \ldots, \alpha_m$ such that $\sum_{i=1}^m \alpha_i \leq 1$ and such that:

$$\boldsymbol{Q} = \sum_{i=1}^{m} \alpha_i \boldsymbol{Q}_i.$$

We then introduce $\boldsymbol{Q}' = (\sum_{i=1}^m \alpha_i)^{-1}\boldsymbol{Q}$. Thus, $\boldsymbol{Q}'$ is a convex combination of $\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_m$. Moreover, since $\sum_{i=1}^m \alpha_i \leq 1$, we have $\boldsymbol{Q} \leq \boldsymbol{Q}'$.

By Corollary 3.3.2 we know that the set $\bar{\mathcal{M}} \cup \partial(\mathcal{M})$ is convex, so $\boldsymbol{Q}'$ must belong to this set. Hence, since ϕ is assumed to be non-decreasing and quasi-convex, it follows that:

$$\phi(\boldsymbol{Q}) \leq \phi(\boldsymbol{Q}') \leq \max\{\phi(\boldsymbol{Q}_1), \ldots, \phi(\boldsymbol{Q}_m)\}. \tag{3.3.9}$$

Since $\boldsymbol{Q}$ was chosen arbitrarily, we conclude that for any $\boldsymbol{Q} \in \partial(\mathcal{M}')$, there exists some boundary point $\boldsymbol{Q}^* \in \partial(\partial(\mathcal{M}'))$ such that $\phi(\boldsymbol{Q}) \leq \phi(\boldsymbol{Q}^*)$. Hence, an optimal vector, $\boldsymbol{Q}^{opt}$, can always be found within the set $\partial(\partial(\mathcal{M}'))$ ■

Note that in the proof of Theorem 3.3.4 will hold even if the definition of ϕ is extended only to the set $\bar{\mathcal{M}} \cup \partial(\mathcal{M})$, i.e., not to the entire set $\mathcal{Q}$.

Theorem 3.3.5. *Consider a field with* n *reservoirs with concave PPR-functions* $f_1, \ldots, f_n$*. Furthermore, let* ϕ *be a symmetric, monotone objective function. Assume also that* ϕ*, interpreted as a function of* $\boldsymbol{Q}$*, can be extended to a non-decreasing quasi-concave*[3] *function defined on the set* $\mathcal{Q}$*. Furthermore, assume that the vector,* $\boldsymbol{Q}^*$*, maximizes* $\phi(\boldsymbol{Q})$ *subject to* $\boldsymbol{Q} \in \partial(\mathcal{M})$*, and that the set* $\mathcal{N}$ *defined relative to* $\boldsymbol{Q}^*$ *as in* (3.3.8)*, is non-empty. Then there exists a hyperplane* $H = \{\boldsymbol{Q} : \ell(\boldsymbol{Q}) = c\}$ *separating* $\mathcal{M}$ *and* $\mathcal{N}$*. Moreover, if* ϕ *is strictly increasing at* $\boldsymbol{Q}^*$*, then* H *supports* $\mathcal{M}$ *at* $\boldsymbol{Q}^*$*. Finally, if* $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ *as well, we may let* $\boldsymbol{Q}^{opt} = \boldsymbol{Q}^*$*.*

Proof: We first note that since the PPR-functions are assumed to be concave, it follows by Proposition 3.3.1 that $\mathcal{M}$ is convex. Moreover, since ϕ interpreted as a function of $\boldsymbol{Q}$, is assumed to be quasi-concave, it follows by Proposition 3.7.8 that $\mathcal{N}$ is convex. As already pointed out we obviously have that $\mathcal{M} \cap \mathcal{N} = \emptyset$. Hence, it follows by Theorem 3.7.1 there exists a hyperplane H separating $\mathcal{M}$ and $\mathcal{N}$.

If ϕ is strictly increasing at $\boldsymbol{Q}^*$, it follows that any neighborhood of $\boldsymbol{Q}^*$ must contain a vector $\boldsymbol{Q}$ such that $\phi(\boldsymbol{Q}) > \phi(\boldsymbol{Q}^*)$. Thus, by the definition of $\mathcal{N}$ any such neighborhood must intersect $\mathcal{N}$. Hence, by Proposition 3.7.3 H supports $\mathcal{M}$ at $\boldsymbol{Q}^*$. The final statement that if $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ as well, we may let $\boldsymbol{Q}^{opt} = \boldsymbol{Q}^*$ is obvious from the previous discussion ■

The two above results indicate how to solve the optimization problem given in Step 1 of Algorithm 3.3.3 in two important cases. If the PPR-functions are convex

[2]For a definition of quasi-convex functions see Appendix 3.7.2

[3]For a definition of quasi-concave functions see Appendix 3.7.2

and the extended version of objective function ϕ is quasi-convex, the optimal $\boldsymbol{Q}^{opt}$ can be found within the set $\partial(\partial(\mathcal{M}'))$. The extreme points of this set correspond to a certain class of admissible production strategies called *priority strategies* which will be discussed in the next subsection. In certain cases it can be shown that the optimal solution can be found within this class. Since there are only a finite number of priority rules, finding the optimal one is easy, at least in principle. Moreover, given an optimal priority strategy, Step 2 of Algorithm 3.3.3 is trivial, as the corresponding production strategy $\boldsymbol{b} \in \mathcal{B}'$ is essentially uniquely defined by this rule. We will discuss this further in Section 3.3.2.

If the PPR-functions are concave and the extended version of objective function ϕ is quasi-concave, Step 1 of Algorithm 3.3.3 typically involves finding the hyperplane separating $\mathcal{M}$ and $\mathcal{N}$, and thus identify the point $\boldsymbol{Q}^*$ where the hyperplane supports $\mathcal{M}$. Assuming that $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ as well, Step 1 is completed by letting $\boldsymbol{Q}^{opt} = \boldsymbol{Q}^*$. Note that verifying that $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ may in general be a difficult task. Often the easiest way to do this, is by proceeding directly to Step 2, using the $\boldsymbol{Q}^*$ found in Step 1. If we are able to successfully complete Step 2 as well, this implies that $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$.

If the PPR-functions and the extended ϕ-function are differentiable, the standard approach to finding $\boldsymbol{Q}^*$ is by using Lagrange multipliers. An example where this method is used, is given in Section 3.5.

If the extended ϕ-function is a quasi-linear function of the form $\phi(\boldsymbol{Q}) = h(\ell(\boldsymbol{Q}))$, where h is an increasing function and ℓ is a non-zero linear form, it follows that finding the optimal $\boldsymbol{Q}^*$ is equivalent to maximizing $\ell(\boldsymbol{Q})$ subject to $\boldsymbol{Q} \in \partial(\mathcal{M}')$. If the PPR-functions are piecewise linear and concave, then finding the optimal $\boldsymbol{Q}^*$ can be formulated as a linear programming problem. We will return to this in a future paper.

When $\boldsymbol{Q}^{opt}$ lies in the interior of $\partial(\mathcal{M}')$, there is typically no unique solution to Step 2 of Algorithm 3.3.3. Typically there will be many admissible paths through $\mathcal{M}$ from $\mathbf{0}$ to $\boldsymbol{Q}^{opt}$. When searching for such a path it turns out to be easier to solve the problem backwards, i.e., by starting at $\boldsymbol{Q}^{opt}$ and finding an admissible path back to the origin. The reason for this is that the constraints (3.2.6) and (3.2.7) are much easier to satisfy close to the origin where $f_1(Q_1), \dots, f_n(Q_n)$ are large than at the boundary of M where $f_1(Q_1), \dots, f_n(Q_n)$ are small. Thus, in order to carry out Step 2 of Algorithm 3.3.3, we will use a certain *backtracking algorithm* which will be described in Section 3.5.

3.3.1 Truncated discounted production

In order to exemplify the results given in the previous subsection, we now consider a more specific type of symmetric monotone objective function, referred to as *truncated discounted production*, and given by the following expression:

$$\phi_{C,R}(\boldsymbol{b}) = \int_0^\infty I\{q(u) \geq C\} q(u) e^{-Ru} du, \qquad 0 \leq C \leq K,\ R \geq 0. \tag{3.3.10}$$

The parameter R is interpreted as a *discount rate*, while C defines the *level of truncation*, typically reflecting the minimal acceptable production rate, e.g., the lowest production rate resulting in a non-negative cash-flow.

Since $\phi_{C,R}$ only depends on the production strategy through the total production rate q, it follows that $\phi_{C,R}$ is symmetric. Moreover, the truncation factor $I\{q(u) \geq C\}$ and the discounting factor e^{-Ru} ensure that it is monotone as well.

Different choices of C and R yield different types of objective functions. If we e.g., let $C = 0$ and $R > 0$, the integrand of the objective function is not truncated at any level, so we simply get the *total discounted production*.

On the other hand if we let $C = K$, the production is truncated as soon as it leaves the plateau level. In this case the integrand is positive only when $q(u) = K$. In particular if $\boldsymbol{b} \in \mathcal{B}'$, we know that $q(u) = K$ if and only if $0 \leq u \leq T_K(\boldsymbol{b})$, so in this case (3.3.10) is reduced to:

$$\phi_{C,R}(\boldsymbol{b}) = \phi_{K,R}(\boldsymbol{b}) = K \int_0^{T_K(\boldsymbol{b})} e^{-Ru} du = KR^{-1}(1 - e^{-RT_K(\boldsymbol{b})}), \tag{3.3.11}$$

when $R > 0$, while $\phi_{C,0}(\boldsymbol{b}) = \phi_{K,0}(\boldsymbol{b}) = KT_K(\boldsymbol{b})$. Moreover, when $\boldsymbol{b} \in \mathcal{B}'$, we have $q(u) = K$ for all $0 \leq u \leq T_K(\boldsymbol{b})$, so:

$$KT_K(\boldsymbol{b}) = \sum_{i=1}^{n} Q_i(T_K(\boldsymbol{b})).$$

From this it follows that $\phi_{K,R}$, interpreted as a function of $\boldsymbol{Q}$, can be extended to $\mathcal{Q}$ by letting:

$$\phi_{K,R}(\boldsymbol{Q}) = \begin{cases} KR^{-1}[1 - \exp(-K^{-1}R\ell(\boldsymbol{Q}))] & \text{if } R > 0, \\ \ell(\boldsymbol{Q}) & \text{if } R = 0, \end{cases} \tag{3.3.12}$$

where we have introduced $\ell(\boldsymbol{Q}) = \sum_{i=1}^{n} Q_i$. Thus, it follows by Proposition 3.7.9 that $\phi_{K,R}$ is quasi-linear. Moreover, $\boldsymbol{Q}^*$ can be found by maximizing $\ell(\boldsymbol{Q})$ subject to $\boldsymbol{Q} \in \partial(\mathcal{M})$.

Maximizing the plateau production $\ell(\boldsymbol{Q})$ or equivalently the plateau length T_K is often easier than maximizing a general objective function of the form $\phi_{C,R}$. Still the special case where $C = K$ and $R = 0$ and the general case are closely related, and an optimal solution to one of them will often be at least a good approximation to an optimal solution of the others. In Section 3.4 we shall prove that this in fact holds exactly when the PPR-functions are linear.

3.3.2 Priority strategies

In this subsection we introduce a specific class of production strategies referred to as *priority strategies*. A priority strategy is characterized by prioritizing the reservoirs according to some suitable criterion. More specifically, we define a priority strategy as follows:

Definition 3.3.6. *Consider a field with n reservoirs with PPR-functions $f_1, \dots, f_n$, and let $\boldsymbol{\pi} = (\pi_1, \dots, \pi_n)$ be a permutation vector representing the prioritization order of the reservoirs. Then the priority strategy relative to $\boldsymbol{\pi}$ is defined by letting the production rates at time t, $q_1(t), \dots, q_n(t)$, be given by:*

$$q_{\pi_i}(t) = \min[f_{\pi_i}(Q_{\pi_i}(t)), K - \sum_{j<i} q_{\pi_j}(t)], \qquad i = 1, \dots, n. \tag{3.3.13}$$

We observe that when assigning the production rate $q_{\pi_i}(t)$ to reservoir π_i, this is limited by $K - \sum_{j<i} q_{\pi_j}(t)$, i.e., the remaining processing capacity after assigning production rates to all the reservoirs with higher priority. If $f_{\pi_i}(Q_{\pi_i}(t)) \leq K - \sum_{j<i} q_{\pi_j}(t)$, reservoir π_i can be produced without any choking, and the remaining processing capacity is passed on to the reservoirs with lower priorities. If on the other hand $f_{\pi_i}(Q_{\pi_i}(t)) > K - \sum_{j<i} q_{\pi_j}(t)$, the production at reservoir π_i is choked so that $q_{\pi_i}(t) = K - \sum_{j<i} q_{\pi_j}(t)$. Thus, all the remaining processing capacity is used on this reservoir, and nothing is passed on to the reservoirs with lower priorities.

The priority strategy can also be expressed in terms of the choke factors at time t, see Huseby & Haavardsson (2008) for details. The production strategy corresponding to the priority strategy relative to the permutation $\boldsymbol{\pi}$ is denoted by $\boldsymbol{b}^{\boldsymbol{\pi}}$. Moreover, the class of all priority strategies is denoted by $\mathcal{B}^{PR}$.

To further explore the properties of priority strategies, we introduce:

$$T_i = T_i(\boldsymbol{b}^{\boldsymbol{\pi}}) = \inf\{t \geq 0 : \sum_{j=1}^{i} f_{\pi_j}(Q_{\pi_j}(t, \boldsymbol{b}^{\boldsymbol{\pi}})) \leq K\}, \qquad i = 1, \dots, n. \tag{3.3.14}$$

We also let $T_0 = 0$, and note that we obviously have: $0 = T_0 \leq T_1 \leq \cdots \leq T_n = T_K(\boldsymbol{b}^{\boldsymbol{\pi}})$. Thus, $T_1, \dots, T_n$ defines an increasing sequence of *subplateau sets*, $[0, T_1], \dots, [0, T_n]$, where the last one is equal to the plateau set Π_K. We will refer to $T_1, \dots, T_n$ as the *subplateau lengths* for the given priority strategy.

We now let $i \in \{1, \dots, n\}$, and assume that $T_{i-1} < t < T_i$. Then the reservoirs $\pi_1, \dots, \pi_{i-1}$ are produced without choking, i.e.:

$$q_{\pi_j}(t) = f_{\pi_j}(Q_{\pi_j}(t)), \qquad j = 1, \dots, i-1. \tag{3.3.15}$$

Furthermore, the reservoir π_i is produced *with choking* so that:

$$q_{\pi_i}(t) = K - \sum_{j<i} q_{\pi_j}(t) = K - \sum_{j<i} f_{\pi_j}(Q_{\pi_j}(t)). \tag{3.3.16}$$

Finally the reservoirs $\pi_{i+1}, \dots, \pi_n$ are not produced at all. Note also that $t = T_i$ is the smallest t where:

$$f_{\pi_i}(Q_{\pi_i}(t)) \leq K - \sum_{j<i} q_{\pi_j}(t) = K - \sum_{j<i} f_{\pi_j}(Q_{\pi_j}(t)). \tag{3.3.17}$$

Thus, from this point of time the reservoir π_i can be produced without any choking.

Summarizing this we see that for $i = 1, \ldots, n$, the production rate, $q_i(t)$ is given by:

$$q_i(t) = \begin{cases} 0 & \text{if } t < T_{i-1}, \\ K - \sum_{j<i} f_{\pi_j}(Q_{\pi_j}(t)) & \text{if } T_{i-1} \leq t < T_i, \\ f_{\pi_i}(Q_{\pi_i}(t)) & \text{if } t \geq T_i. \end{cases} \tag{3.3.18}$$

The priority strategies have the important property that they generate admissible paths through $\mathcal{M}'$ such that $\boldsymbol{Q}(T_K(\boldsymbol{b}^{\boldsymbol{\pi}}), \boldsymbol{b}^{\boldsymbol{\pi}}) \in \partial(\partial(\mathcal{M}'))$. In order to study this further we introduce the set $\mathcal{A} \subseteq \mathcal{Q}$ consisting of the union of all admissible paths. Thus, we have:

$$\mathcal{A} = \{\boldsymbol{Q}(t, \boldsymbol{b}) : t \geq 0, \boldsymbol{b} \in \mathcal{B}'\}.$$

The following lemma, proven in Huseby & Haavardsson (2008), shows that the path of a priority strategy follows the boundary of $\mathcal{A}$.

Lemma 3.3.7. *Consider a field with n reservoirs with PPR-functions $f_1, \ldots, f_n$. Moreover, let $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ be a permutation vector, and let $\boldsymbol{b}^{\boldsymbol{\pi}}$ be the corresponding priority strategy. Then we have:*

$$\boldsymbol{Q}(t, \boldsymbol{b}^{\boldsymbol{\pi}}) \in \partial(\mathcal{A}) \text{ for all } t \geq 0.$$

Using Lemma 3.3.7 we can now show:

Theorem 3.3.8. *Consider a field with n reservoirs, and let $\boldsymbol{b}^{\boldsymbol{\pi}}$ be a priority strategy. Then $\boldsymbol{Q}(T_K(\boldsymbol{b}^{\boldsymbol{\pi}}), \boldsymbol{b}^{\boldsymbol{\pi}}) \in \partial(\partial(\mathcal{M}'))$.*

Proof: By Lemma 3.3.7 we have that $\boldsymbol{Q}(T_K(\boldsymbol{b}^{\boldsymbol{\pi}}), \boldsymbol{b}^{\boldsymbol{\pi}}) \in \partial(\mathcal{A})$. Moreover, by definition of $T_K(\boldsymbol{b}^{\boldsymbol{\pi}})$ it follows that $\boldsymbol{Q}(T_K(\boldsymbol{b}^{\boldsymbol{\pi}}), \boldsymbol{b}^{\boldsymbol{\pi}}) \in \partial(\mathcal{M})$. Hence, since obviously $\partial(\partial(\mathcal{M}')) = \partial(\mathcal{A}) \cap \partial(\mathcal{M})$, we must have $\boldsymbol{Q}(T_K(\boldsymbol{b}^{\boldsymbol{\pi}}), \boldsymbol{b}^{\boldsymbol{\pi}}) \in \partial(\mathcal{A}) \cap \partial(\mathcal{M}) = \partial(\partial(\mathcal{M}'))$ ■

When the PPR-functions are convex, and the objective function, ϕ, interpreted as a function of $\boldsymbol{Q}$, is quasi-convex, we know by Theorem 3.3.4 that an optimal production strategy $\boldsymbol{b}^*$ should be chosen so that $\boldsymbol{Q}(T_K(\boldsymbol{b}^*), \boldsymbol{b}^*) \in \partial(\partial(\mathcal{M}'))$. By Theorem 3.3.8 we see that priority strategies always satisfies this condition. Thus, priority strategies provide a good starting point for the optimal strategy. We close this section by a result providing a sufficient criterion for when the optimal strategy can be found within the class of priority rules.

Theorem 3.3.9. *Consider a field with n reservoirs with convex PPR-functions $f_1, \ldots, f_n$. Furthermore, let ϕ be a symmetric, monotone objective function. Assume also that ϕ, interpreted as a function of $\boldsymbol{Q}$, can be extended to a non-decreasing, quasi-convex function defined on the set $\mathcal{Q}$. Finally assume that $\partial(\mathcal{M}')$ is contained in the convex hull of the points $\{\boldsymbol{Q}(T_K(\boldsymbol{b}), \boldsymbol{b}) : \boldsymbol{b} \in \mathcal{B}^{PR}\}$. Then an optimal production strategy can be found within the class $\mathcal{B}^{PR}$.*

Proof: Let $\boldsymbol{Q} \in \partial(\mathcal{M}')$ be chosen arbitrarily. Then by the assumption there exists non-negative numbers $\{\alpha_{\boldsymbol{b}} : \boldsymbol{b} \in \mathcal{B}^{PR}\}$ such that $\sum_{\boldsymbol{b} \in \mathcal{B}^{PR}} \alpha_{\boldsymbol{b}} \leq 1$ and such that:

$$\boldsymbol{Q} = \sum_{\boldsymbol{b} \in \mathcal{B}^{PR}} \alpha_{\boldsymbol{b}} \boldsymbol{Q}(T_K(\boldsymbol{b}), \boldsymbol{b}).$$

From this the result follows by arguments similar to the proof of Theorem 3.3.4 ■

3.4 Optimization with linear PPR-functions

In this section we consider the case where all the PPR-functions are linear. That is, we consider a field with n reservoirs with PPR-functions $f_1, \ldots, f_n$, such that:

$$f_i(Q_i(t)) = D_i(V_i - Q_i(t)), \qquad i = 1, \ldots, n, \tag{3.4.1}$$

where $V_1, \ldots, V_n$ denotes the recoverable volumes from the n reservoirs, and where we assume that the reservoirs have been indexed so that $0 < D_1 \leq D_2 \leq \cdots \leq D_n$.

We then consider the ith reservoir, and let $T \geq 0$. If this reservoir is produced without any choking, i.e., with a choking factor function $b_i(t) = 1$ for all $t \geq T$, we can solve the differential equation (3.4.1) for $t \geq T$ given that the cumulative production at time T is $Q_i(T)$, and get:

$$q_i(t) = D_i(V_i - Q_i(T)) \exp(-D_i(t - T)), \qquad t \geq T. \tag{3.4.2}$$

Moreover, by integrating $q_i(t)$ from T to t we also get:

$$Q_i(t) = V_i(1 - e^{-D_i(t-T)}) + Q_i(T)e^{-D_i(t-T)}, \qquad t \geq T. \tag{3.4.3}$$

If on the other hand, the reservoir is produced with a choking factor function $b_i(t) \leq 1$ for $t \geq T$ it follows by Proposition 3.2.1 that $Q_i(t)$ will be less than or equal to the right-hand side of (3.4.3). These relations will be used in order to state the following result, which is proved in Appendix 3.8:

Theorem 3.4.1. *Consider a field with n reservoirs with linear PPR-functions $f_1, \ldots, f_n$ given by* (3.4.1)*. Then let $\boldsymbol{b}^1$ denote the priority strategy corresponding to the permutation $\boldsymbol{\pi} = (1, 2, \ldots, n)$, and let $\boldsymbol{b}^2$ be any other valid production strategy. Then $Q(t, \boldsymbol{b}^1) \geq Q(t, \boldsymbol{b}^2)$ for all $t \geq 0$. Thus, $\boldsymbol{b}^1$ is optimal with respect to any monotone, symmetric objective function.*

Having identified the optimal production strategy in the case of linear PPR-function, we proceed to calculating the resulting production rates and cumulative production functions. Since the optimal solution is a priority strategy, it turns out that it is fairly easy to solve this. We consider once again a field with n reservoirs with PPR-functions $f_1, \ldots, f_n$, of the form given in (3.4.1). The formulas we are about to present, are valid for any priority strategy, not just the optimal one. Thus, we consider an arbitrary priority strategy $\boldsymbol{b}^{\boldsymbol{\pi}}$ where the permutation vector is $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$.

In order to find the production rates and cumulative production functions, we start out by assuming that the subplateau lengths, $T_1, \ldots, T_n$, are *known*. As in Section 3.3.2 we also let $T_0 = 0$. Then by combining (3.3.18) and (3.4.2) it is easy to see that for $i = 1, \ldots, n$, the production rate, $q_{\pi_i}(t)$ is given by:

$$q_{\pi_i}(t) = \begin{cases} 0 & \text{if } t < T_{i-1}, \\ K - \sum_{j<i} D_{\pi_j}(V_{\pi_j} - Q_{\pi_j}(T_j))e^{-D_{\pi_j}(t-T_j)} & \text{if } t \in [T_{i-1}, T_i), \\ D_{\pi_i}(V_{\pi_i} - Q_{\pi_i}(T_i))e^{-D_{\pi_i}(t-T_i)} & \text{if } t \geq T_i. \end{cases} \tag{3.4.4}$$

Moreover, by integrating these production rates we get the following cumulative production functions:

$$Q_{\pi_i}(t) = \begin{cases} 0 & \text{if } t < T_{i-1}, \\ K[t - T_{i-1}] - \sum_{j<i}(V_{\pi_j} - Q_{\pi_j}(T_j))[e^{-D_{\pi_j}(T_{i-1}-T_j)} - e^{-D_{\pi_j}(t-T_j)}] & \text{if } t \in [T_{i-1}, T_i], \\ V_{\pi_i}(1 - e^{-D_{\pi_i}(t-T_i)}) + Q_{\pi_i}(T_i)e^{-D_{\pi_i}(t-T_i)} & \text{if } t > T_i. \end{cases} \tag{3.4.5}$$

In order to complete these formulas we need to explain how to determine the subplateau lengths, $T_1, \ldots, T_n$. This will be done as a sequential process where T_1 is determined first. Then T_1 is used to determine T_2, T_1 and T_2 are used to determine T_3, and so on until all the subplateau lengths have been found.

To determine T_1 we first consider the case where $f_{\pi_1}(Q_{\pi_1}(T_0)) \leq K$. In this case it follows by (3.3.14) that $T_1 = T_0 = 0$, i.e., the first subplateau has zero length. On the other hand, if $f_{\pi_1}(Q_{\pi_1}(T_0)) > K$, T_1 is found as the solution to the equation:

$$f_{\pi_1}(Q_{\pi_1}(t)) = D_{\pi_1}(V_{\pi_1} - Q_{\pi_1}(t)) = K. \tag{3.4.6}$$

Since obviously $Q_{\pi_1}(t) = Kt$ for all $t \leq T_1$, we get that $T_1 = V_{\pi_1}K^{-1} - D_{\pi_1}^{-1}$ in this case.

We then assume that we have determined $T_1, \ldots, T_{i-1}$, and consider the problem of how to determine T_i. As for T_1 we first consider the case where $f_{\pi_i}(Q_{\pi_i}(T_{i-1})) \leq K - \sum_{j<i} f_{\pi_j}(Q_{\pi_j}(T_{i-1}))$. In this case it follows by (3.3.14) that $T_i = T_{i-1}$, i.e., the ith subplateau has the same length as the $(i-1)$th subplateau. On the other hand, if $f_{\pi_i}(Q_{\pi_i}(T_{i-1})) > K - \sum_{j<i} f_{\pi_j}(Q_{\pi_j}(T_{i-1}))$, T_i is found as the solution to the equation:

$$f_{\pi_i}(Q_{\pi_i}(t)) = D_{\pi_i}(V_{\pi_i} - Q_{\pi_i}(t)) = K - \sum_{j<i} f_{\pi_j}(Q_{\pi_j}(t)), \tag{3.4.7}$$

where $Q_{\pi_i}(t)$ for all $t \in [T_{i-1}, T_i]$ is given by (3.4.5). In general this equation is easily solvable using standard numerical methods.

3.4.1 An example with linear PPR-functions

We consider a field with $n = 3$ reservoirs with linear PPR functions, f_1, f_2, f_3 of the form given in (3.4.1). Moreover, we assume, as above, that the reservoirs are indexed so that $0 < D_1 < D_2 < D_3$. More specifically, we let $D_1 = 0.0003$ $D_2 = 0.0006$, and $D_3 = 0.0010$.

According to Theorem 3.4.1 the optimal production strategy with respect to any symmetric monotone objective function is the priority strategy corresponding to the permutation $\boldsymbol{\pi} = (1, 2, 3)$. In this example we focus on the objective function $\phi_{K,0}$ defined by letting $C = K$ and $R = 0$ in (3.3.10). As explained in Section 3.3.1, the optimal solution maximizes the plateau volume, $\ell(\boldsymbol{Q}) = Q_1 + Q_2 + Q_3$ subject to $\boldsymbol{Q} \in \partial(\mathcal{M}')$.

We observe that the optimal priority strategy does not depend on the producible volumes V_1, V_2, V_3. However, the volumes may still have an impact on the ranking of the different priority rules as well as the differences in performance. To see this we consider two different cases. In the first case we let $V_1 = 15.0$ MSm^3, $V_2 = 10.0$ MSm^3 and $V_3 = 5.0$ MSm^3, while in the second example we let $V_1 = 5.0$ MSm^3, $V_2 = 10.0$ MSm^3 and $V_3 = 15.0$ MSm^3.

	Case 1			Case 2		
Res.	Producible volume V_i (MSm^3)	Scale parameter D_i	Max rate D_iV_i (kSm^3/d)	Producible volume V_i (MSm^3)	Scale parameter D_i	Max rate D_iV_i (kSm^3/d)
1	15.0	0.0003	4.5	5.0	0.0003	1.5
2	10.0	0.0006	6.0	10.0	0.0006	6.0
3	5.0	0.0010	5.0	15.0	0.0010	15.0

Table 3.1: *Parameter values for the three reservoirs.*

In Table 3.1 we have listed the parameter values for the two cases. We have also included columns showing the maximum value of the PPR-functions, i.e., $f_i(0) = D_iV_i$, $i = 1, 2, 3$. In both cases we let $K = 3.0$ kSm^3 per day. We note that in the second case the maximum value of f_1 is just 1.5 kSm^3 per day. Thus, in this case the first reservoir can never reach the plateau level K alone. Hence, if this reservoir is given the highest priority, the subplateau length T_1 is zero.

By using the formulas (3.4.4) and (3.4.5), we may calculate the plateau length $T_K(\boldsymbol{b}^{\boldsymbol{\pi}})$ for each of the six possible priority strategies. Moreover, we may calculate cumulative production for each of the reservoirs as well as the total cumulative production $KT_K(\boldsymbol{b}^{\boldsymbol{\pi}})$ at this point of time. The results are shown in Table 3.2.

From the table we see that the priority strategy corresponding to the permutation $(1, 2, 3)$ is indeed optimal in both cases. The second and third best priority strategies correspond to the permutations $(2, 1, 3)$ and $(1, 3, 2)$ respectively. Both these permutations are “neighbors” of the optimal permutation in the sense that they can be obtained from $(1, 2, 3)$ by switching two consecutive entries in the vector. That is, $(2, 1, 3)$ is obtained from $(1, 2, 3)$ by switching the two first entries, while $(1, 3, 2)$ is obtained from $(1, 2, 3)$ by switching the two last entries. We observe, however, that in the first case the the total cumulative productions using the permutations $(2, 1, 3)$

and $(1,3,2)$ are very close to each other, while in the second case the permutation $(2,1,3)$ produces a result which is closer to the result of the optimal strategy.

Another observation is that the results using the two worst permutations, i.e., $(2,3,1)$ and $(3,2,1)$ switch places in the two cases. In the first case $(2,3,1)$ produces the worst results, while in the second case $(3,2,1)$ comes in last.

Summarizing the example, we see that the results confirm that the optimal priority strategy indeed corresponds to the permutation $(1,2,3)$ and thus agree with Theorem 3.4.1. Still we see that the producible volumes also affect the results significantly.

Priority strategy $\boldsymbol{\pi}$	Plateau prod. res. 1 $Q_1(T_K)$ MSm^3	Plateau prod. res. 2 $Q_2(T_K)$ MSm^3	Plateau prod. res. 3 $Q_3(T_K)$ MSm^3	Tot. plateau production $\ell(\boldsymbol{Q}(T_K))$ MSm^3	Rank
Case 1					
(1, 2, 3)	13.745	9.083	2.927	25.755	1
(2, 1, 3)	11.352	9.897	3.156	24.405	2
(1, 3, 2)	13.551	5.828	4.938	24.317	3
(3, 1, 2)	12.525	6.241	4.998	23.764	4
(3, 2, 1)	6.173	9.424	4.994	20.591	5
(2, 3, 1)	5.810	9.774	4.893	20.477	6
Case 2					
(1, 2, 3)	4.654	9.885	12.173	26.712	1
(2, 1, 3)	4.331	9.932	12.241	26.504	2
(1, 3, 2)	4.585	5.466	14.845	24.896	3
(3, 1, 2)	3.396	5.887	14.949	24.232	4
(2, 3, 1)	0.461	9.883	13.432	23.776	5
(3, 2, 1)	0.655	7.306	14.920	22.880	6

Table 3.2: *Plateau production for the six priority strategies in the two cases.*

We recall that for any admissible strategy the vector $\boldsymbol{Q}(T_K)$ always belongs to the set $\partial(\mathcal{M}')$. In the linear case $\partial(\mathcal{M}')$ is a part of the hyperplane with equation:

$$\sum_{i=1}^{n} f_i(Q_i) = \sum_{i=1}^{n} D_i(V_i - Q_i) = K.$$

Thus, in particular, $\boldsymbol{Q}(T_K, \boldsymbol{b}^{\boldsymbol{\pi}})$ belongs to this hyperplane for any priority strategy $\boldsymbol{b}^{\boldsymbol{\pi}}$. In Figure 3.1 and Figure 3.2 we have illustrated the resulting hyperplanes for Case 1 and 2 respectively. Moreover, the plots also show the locations of $\boldsymbol{Q}(T_K, \boldsymbol{b}^{\boldsymbol{\pi}})$ for each of the six priority strategies. In both cases these six points forms a hexagon. However, as we see, the shapes of these hexagons are quite different. Obviously if the points of two priority strategies are close to each other, then so are their respective plateau productions as well. Thus, in particular the points corresponding to the two best permutations $(1,2,3)$ and $(2,1,3)$ are much closer together in Case 2 than in Case 1. At the same time their respective plateau productions are closer in Case 2 than Case 1.

A similar but opposite effect holds for the two worst permutations, i.e., $(2,3,1)$ and $(3,2,1)$. In Case 1 both the points representing these two strategies and the plateau productions are very close to each other. In Case 2 the points are much further apart, and so are the plateau productions.

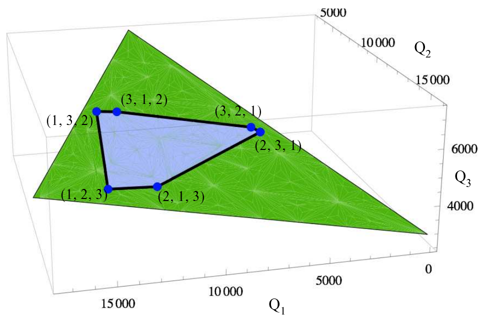

Figure 3.1: *The hyperplane containing* $\partial(\mathcal{M}')$ *in Case 1*

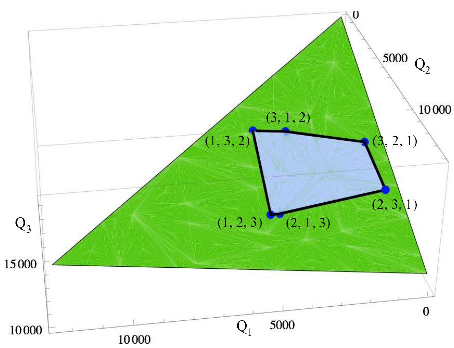

Figure 3.2: *The hyperplane containing* $\partial(\mathcal{M}')$ *in Case 2*

3.5 Generating optimal strategies using backtracking

In this section we present a methodology for Step 2 of Algorithm 3.3.3. Thus, we consider a field with n reservoirs with PPR-functions $f_1, \dots, f_n$, and assume that Step 1 of this algorithm is completed, where $\boldsymbol{Q}^* = (Q_1^*, \dots, Q_n^*)$ is the vector maximizing $\phi(\boldsymbol{Q})$ subject to $\boldsymbol{Q} \in \partial(\mathcal{M})$. Moreover, we assume that $T_K^* = K^{-1} \sum_{i=1}^n Q_i^*$ is the point of time when $\boldsymbol{Q}^*$ is reached. The idea is to construct an admissible production strategy generating a path $\{\boldsymbol{Q}(t) : 0 \leq t \leq T_K^*\}$, where $\boldsymbol{Q}(0) = \boldsymbol{0}$ and $\boldsymbol{Q}(T_K^*) = \boldsymbol{Q}^*$. As pointed out in the discussion following Theorem 3.3.5, finding such a production strategy implies that $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ as well. Thus, the constructed admissible production strategy is indeed an optimal strategy.

Except in special cases, like e.g., when $\boldsymbol{Q}^*$ corresponds to a priority strategy, there will typically be an infinite number of admissible paths from $\boldsymbol{0}$ to $\boldsymbol{Q}^*$. In order to find *one* such path, we search within the class of piecewise linear paths. More specifically, let $0 = t_0 < t_1 < \cdots < t_N = T_K^*$, and let $\boldsymbol{q}_j = (q_{1,j}, \dots, q_{n,j})$, $j = 1, \dots, N$. We then assume that the reservoirs are produced using the following rates:

$$q_i(t) = q_{i,j}, \quad t \in (t_{j-1}, t_j],\ i = 1, \dots, n,\ j = 1, \dots, N.$$

Thus, the production rates are constant within each of the N intervals $(t_0, t_1], \dots, (t_{N-1}, t_N]$. Hence, the cumulative production functions are given by:

$$Q_i(t) = Q_i(t_{j-1}) + q_{i,j}(t - t_{j-1}), \quad t \in (t_{j-1}, t_j],\ i = 1, \dots, n,\ j = 1, \dots, N,$$

where we of course assume that $Q_i(t_0) = Q_i(0) = 0$, $i = 1, \dots, n$. In order to ensure that we have an admissible path, we must have:

$$\sum_{i=1}^n q_{i,j} = K, \quad j = 1, \dots, N, \tag{3.5.1}$$

and that:

$$0 \leq q_{i,j} \leq f_i(Q_i(t)), \quad t \in (t_{j-1}, t_j],\ i = 1, \dots, n,\ j = 1, \dots, N. \tag{3.5.2}$$

Since the PPR-functions are assumed to be non-increasing, it follows that the last condition is satisfied if and only if

$$0 \leq q_{i,j} \leq f_i(Q_i(t_j)), \quad i = 1, \dots, n,\ j = 1, \dots, N. \tag{3.5.3}$$

Finally, we want the path to end up at the optimal point, i.e., we must have $\boldsymbol{Q}(T_K^*) = \boldsymbol{Q}(t_N) = \boldsymbol{Q}^*$. Expressed in terms of $\boldsymbol{q}_1, \dots, \boldsymbol{q}_N$ we get the following condition:

$$\sum_{j=1}^N \boldsymbol{q}_j (t_j - t_{j-1}) = \boldsymbol{Q}^*. \tag{3.5.4}$$

Thus, the problem is reduced to choosing the intervals $(t_0, t_1], \dots, (t_{N-1}, t_N]$, in particular, the number of intervals N, and finding the vectors $\boldsymbol{q}_1, \dots, \boldsymbol{q}_N$ subject to the conditions (3.5.1), (3.5.3) and (3.5.4).

From a practical point of view it is of interest to keep the number of intervals as small as possible, since this means that the reservoirs can be produced with stable rates. However, if N is too small, it may not be possible to find a piecewise linear admissible path from $\mathbf{0}$ to $\boldsymbol{Q}^*$. In order to find a suitable N, we start out by letting N be small, e.g., $N = 1$. If it is possible to find an admissible path from $\mathbf{0}$ to $\boldsymbol{Q}^*$ with this N, we are done. If not, we increase N and try once more. This process is repeated until we eventually find an admissible path from $\mathbf{0}$ to $\boldsymbol{Q}^*$.

For a given N we also need to choose the numbers $t_1, \dots, t_N$. The easiest choice here would be to distribute these partition points uniformly over the interval $[0, T_K^*]$. Since, however, the condition (3.5.2) is stricter when $f_i(Q_i(t))$ is small, i.e., when t is close to T_K^*, it may be a good idea to distribute the partition points so that we have shorter intervals when t is close to T_K^*, and longer intervals when t is close to 0. In Huseby & Haavardsson (2008) we show how this can be done in a systematic way.

Having chosen N the partition points $t_1, \dots, t_N$, we now turn to the problem of choosing the vectors $\boldsymbol{q}_1, \dots, \boldsymbol{q}_N$ subject to the conditions (3.5.1), (3.5.3) and (3.5.4). Rather than finding all these vectors at once, it turns out to be easier to determine them one by one, starting backwards with $\boldsymbol{q}_N$. Thus, let:

$$\boldsymbol{Q}^{(k)} = \boldsymbol{Q}^* - \sum_{j>k} \boldsymbol{q}_j (t_j - t_{j-1}), \qquad k = 0, 1, \dots, N.$$

Thus, in particular $\boldsymbol{Q}^{(0)} = \mathbf{0}$, while $\boldsymbol{Q}^{(N)} = \boldsymbol{Q}^*$. As we move backwards from $\boldsymbol{Q}^*$ to $\mathbf{0}$, we follow a piecewise linear path through the points $\boldsymbol{Q}^{(N)}, \boldsymbol{Q}^{(N-1)}, \dots, \boldsymbol{Q}^{(0)}$. At each of these points we are allowed to change direction by choosing the next vector in the set $\{\boldsymbol{q}_N, \boldsymbol{q}_{N-1}, \dots, \boldsymbol{q}_1\}$. Thus, assume that we have chosen the directions $\boldsymbol{q}_N, \dots, \boldsymbol{q}_{k+1}$, and that we want to choose $\boldsymbol{q}_k$. At this stage we have constructed an admissible path backwards from $\boldsymbol{Q}^*$ to the point $\boldsymbol{Q}^{(k)}$. Since our goal is to find an admissible path back to $\mathbf{0}$, the ideal direction from the point $\boldsymbol{Q}^{(k)}$ is a vector $\boldsymbol{q}_k$ that is parallel to $\boldsymbol{Q}^{(k)}$. If we can find such a vector which at the same time satisfies the conditions (3.5.1), (3.5.3) and (3.5.4), we would be right on track back to $\mathbf{0}$. In general, however, this may not be possible. Thus, we instead look for a vector $\boldsymbol{q}_k$ satisfying the conditions (3.5.1), (3.5.3) and (3.5.4), and such that the angle between $\boldsymbol{q}_k$ and $\boldsymbol{Q}^{(k)}$ is as small as possible. That is, we choose $\boldsymbol{q}_k$ by maximizing the scalar product $\boldsymbol{Q}^{(k)}\boldsymbol{q}_k$ subject to (3.5.1), (3.5.3) and (3.5.4). This optimization problem is a standard linear programming problem which can easily be solved using the well-known Simplex algorithm.

By solving a linear programming problem at each of the points $\boldsymbol{Q}^{(N)}, \boldsymbol{Q}^{(N-1)}, \dots, \boldsymbol{Q}^{(1)}$, we may be able to construct an admissible path from $\boldsymbol{Q}^*$ back to $\mathbf{0}$. If the procedure fails, we increase N, and run the procedure once again, and so forth. In order to avoid an infinite number of runs, however, one would typically specify some suitable maximum number of intervals, denoted by N_{max}. Ideally the process produces an admissible path with $N \leq N_{max}$ intervals. Still it may happen that no such path is found even for a very large value of N_{max}. This obviously happens if $\boldsymbol{Q}^* \notin \partial(\mathcal{M}')$ since by definition no admissible path from $\mathbf{0}$ to $\boldsymbol{Q}^*$ exists in this case. Unfortunately, since N_{max} is finite, the process may also fail when $\boldsymbol{Q}^*$ is a point in $\partial(\mathcal{M}')$ very close to or at the boundary of this set. Thus, the algorithm is not guaranteed to work even though there may exist an admissible path from $\mathbf{0}$ to $\boldsymbol{Q}^*$. Still in cases where $\boldsymbol{Q}^*$ is located in the central parts of $\partial(\mathcal{M}')$, the algorithm tends to work very well.

3.5.1 An example with concave PPR-functions

We consider a field with concave PPR-functions $f_1, \ldots, f_{10}$ given by

$$f_i(Q_i(t)) = \sqrt{D_i(V_i - Q_i(t))},\ i = 1, \ldots, 10, \tag{3.5.5}$$

where $V_1, \ldots, V_n$ denote the producible volumes from the n reservoirs. Table 3.3 shows the parameter values for the 10 reservoirs. The process capacity constraint $K = 7.5$ kSm^3 per day is used. The max rate for the i-th reservoir is given by $\sqrt{D_i V_i}$ and is obtained by inserting $Q_i(0) = 0$ in (3.5.5). In this example we use the objective function $\phi_{K,0}$ defined by letting $C = K$ and $R = 0$ in (3.3.10). By (3.3.12) it follows that $\phi_{K,0}$, interpreted as a function defined for all $\boldsymbol{Q} \in \mathcal{Q}$ is given by $\phi_{K,0}(\boldsymbol{Q}) = \ell(\boldsymbol{Q}) = \sum_{i=1}^{10} Q_i$. Since the PPR-functions and the extended objective function $\phi_{K,0}(\boldsymbol{Q})$ are differentiable, we may apply Lagrange multipliers in order to find $\boldsymbol{Q}^*$. Hence, it is easy to show that $\boldsymbol{Q}^*$ is given by

$$\boldsymbol{Q}^* = (Q_1^*, \ldots, Q_n^*) = (V_1 - \frac{D_1}{2}\{\frac{K}{\sum_{i=1}^n D_i}\}^2, \ldots, V_n - \frac{D_n}{2}\{\frac{K}{\sum_{i=1}^n D_i}\}^2).$$

Reservoir	Producible volume V_i (MSm^3)	Scale parameter D_i	Max rate $\sqrt{D_i V_i}$ (kSm^3/d)	Plateau production Q_i^* (MSm^3)
1	4.5	1.174	3.25	4.204
2	6.5	1.356	4.20	6.158
3	7.0	0.643	3.00	6.838
4	10.0	0.313	2.50	9.921
5	5.0	0.625	2.50	4.842
6	4.0	1.125	3.00	3.716
7	6.0	1.333	4.00	5.664
8	8.0	1.000	4.00	7.748
9	9.0	0.500	3.00	8.874
10	5.0	2.500	5.00	4.370

Table 3.3: *Parameter values for the 10 reservoirs used in the backtracking example.*

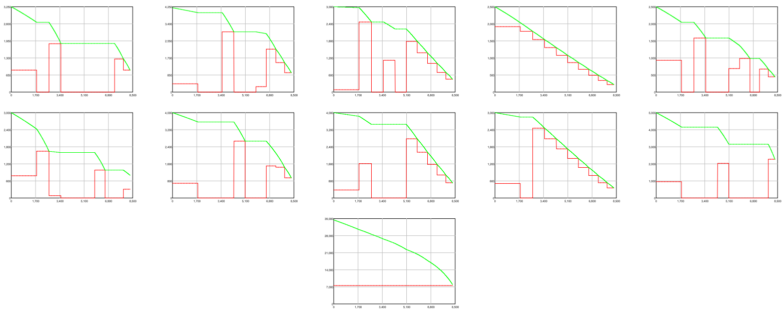

Figure 3.3: *Actual production rates* $(q_1(t), \ldots, q_{12}(t))$ *(red curves) satisfying the conditions* (3.5.1), (3.5.3) *and* (3.5.4) *and PPR-functions* $(f_1(Q_1(t)), \ldots, f_{10}(Q_{10}(t)))$ *(green curves) for the backtracking example. The last panel shows the total production rate* $q(t)$ *(red curve) and the sum of the PPR-functions* $\sum_{i=1}^{10} f_i(Q_i(t))$ *(green curve).*

We then proceed to Step 2 of Algorithm 3.3.3. To generate a production strategy reaching $\boldsymbol{Q}^*$ we use the approach described in Section 5 where we search for intervals $(t_0, t_1], \ldots, (t_{N-1}, t_N]$ so that the condition expressed in (3.5.2) is satisfied. For simplicity we distributed the partition points uniformly over the interval $[0, T_K^*]$. Starting out with $N = 1$ and increasing N until an admissible path from $\mathbf{0}$ to $\boldsymbol{Q}^*$ was found, it turned out that $N = 12$ periods were needed. Figure 3.3 shows the actual production rates and the PPR-rates of the 10 reservoirs. The total actual production rate and the total PPR-rate are also displayed in Figure 3.3. From Figure 3.3 we see that the conditions (3.5.1) and (3.5.2) are satisfied for all $t \geq 0$.

3.6 Conclusions

In the present paper we have focused on the problem of optimizing the production of an oil or gas field consisting of many reservoirs. We have shown how to construct an optimal production strategy using a procedure described in Algorithm 3.3.3. The first step of the algorithm involves finding the optimal state of the reservoirs at the end of the plateau phase, i.e, when the path defined by the vector of cumulative productions reaches the set $\partial(\mathcal{M}')$. The second step involves finding an admissible production strategy such that the optimal state is reached.

The key results given in Theorem 3.3.4 and Theorem 3.3.5 indicate how to solve the optimization problem given in Step 1 of Algorithm 3.3.3 in two important cases characterized by the convexity or concavity of the PPR-functions and the quasi-convexity or quasi-concavity of the objective function.

If the optimal state is located at the *boundary* of $\partial(\mathcal{M}')$, the priority strategies play an important part, since these strategies correspond to the extreme points of the boundary of $\partial(\mathcal{M}')$. Searching for an optimal rule within this class is, at least in principle, easy, since there are only a finite number of such strategies. Moreover, having found the best priority strategy, the second step of the algorithm is trivial, since any priority strategy is uniquely defined by the permutation vector representing the ordering of the reservoirs. While there of course are infinitely many other production strategies with cumulative production paths reaching the boundary of $\partial(\mathcal{M}')$, we believe that the priority strategies at least provide a very good approximation to the optimal solution.

In the special case where all the PPR-functions are *linear*, a specific priority strategy is shown to be optimal with respect to any monotone, symmetric objective function.

If the optimal state is located in the *interior* of the set $\partial(\mathcal{M}')$, a backtracking algorithm is proposed for handling Step 2 of Algorithm 3.3.3. Unless the optimal state is too close to the boundary of $\partial(\mathcal{M}')$ this method produces an admissible production strategy such that the optimal state is reached.

We believe that the general framework developed in this theoretical paper is of fundamental importance in order to gain insight into the general production optimization problem. Still there are many unsolved problems left. In particular, by running Step 1 of Algorithm 3.3.3 as proposed in the present paper, we only get a *candidate* for the optimal state in the set $\partial(\mathcal{M}')$. Thus, having a precise and easy condition for when this candidate actually belongs to $\partial(\mathcal{M}')$, would be very convenient. Using this we could e.g., avoid running Step 2 of the algorithm in cases where the candidate state does not belong to $\partial(\mathcal{M}')$, in which case we know that no admissible strategy reaching this state can be found. Given such a condition we could also be able to handle combinations of convex and concave PPR-functions.

Furthermore, in order to analyze the robustness of the derived production strategies, it is of interest to incorporate uncertainty into the framework. These issues will addressed in a forth-coming paper, where a certain parametric class of production strategies will be proposed.

In this paper we have focused on single-phase production optimization, i.e., either oil or gas. In real life typically oil, gas and water are produced simultaneously. Thus, extending the framework so that *multi-phase* production optimization can be handled, is of great interest. We will return to this problem in a future research project.

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3.7 Some results on convexity

3.7.1 Separating and supporting hyperplanes

Many results in convex optimization theory rest upon the well-known *separating and supporting hyperplane theorems*. For more details see Boyd & Vandeberghe (2004*b*). In the space $\mathbb{R}^n$ a *hyperplane* $H = \{\boldsymbol{x} \in \mathbb{R}^n : \ell(\boldsymbol{x}) = c\}$, where ℓ is a non-zero linear form, divides the space into two closed half-spaces, $H^+ = \{\boldsymbol{x} \in \mathbb{R}^n : \ell(\boldsymbol{x}) \geq c\}$ and $H^- = \{\boldsymbol{x} \in \mathbb{R}^n : \ell(\boldsymbol{x}) \leq c\}$. A hyperplane H is said to *separate* the sets S and T if one of the sets is contained in H^+ while the other is contained in H^-. A hyperplane, H, is said to *support* a set S, if either $S \subseteq H^+$ or $S \subseteq H^-$, and $S \cap H \neq \emptyset$. The separating and supporting hyperplane theorems can now be stated as follows:

Theorem 3.7.1. *Let $S, T \subset \mathbb{R}^n$ be two disjoint convex sets. Then there exists a hyperplane H separating S and T.*

Theorem 3.7.2. *Let $S \subset \mathbb{R}^n$ be a closed convex set, and let $\boldsymbol{x}_0 \in S$ be a point on the boundary of S. Then there exists a hyperplane, H, supporting S such that $\boldsymbol{x}_0 \in H$.*

The following proposition, proved in Huseby & Haavardsson (2008), combines Theorem 3.7.1 and Theorem 3.7.2:

Proposition 3.7.3. *Let $S, T \subset \mathbb{R}^n$ be two disjoint convex sets. Moreover, assume that there exists a $\boldsymbol{x}_0 \in S$ such that any neighborhood of $\boldsymbol{x}_0$ intersects T. Then there exists a hyperplane H separating S and T such that H supports S at $\boldsymbol{x}_0$.*

Using the various sets and notation introduced in Section 3.3, we can now formulate the following important result:

Theorem 3.7.4. *Consider a field with n reservoirs with convex PPR-functions $f_1, \ldots, f_n$. Moreover, let $\boldsymbol{Q} \in \partial(\mathcal{M}')$. Then there exists m (m suitably chosen) vectors $\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_m \in \partial(\partial(\mathcal{M}'))$ and non-negative numbers $\alpha_1, \ldots, \alpha_m$ such that $\sum_{i=1}^m \alpha_i \leq 1$ and such that:*

$$\boldsymbol{Q} = \sum_{i=1}^{m} \alpha_i \boldsymbol{Q}_i. \tag{3.7.1}$$

Proof: See Huseby & Haavardsson (2008).

Note that in the above argument we embed the set $\lambda[\partial(\mathcal{M}')]$ in a convex polytope P with m extreme points, where m is a suitably chosen integer. If $n = 2$, the set $\lambda[\partial(\mathcal{M}')]$ can be embedded within an interval, i.e., a polytope with two extreme points. Similarly, if $n = 3$, the set can be embedded withing a triangle, which is a polytope with three extreme points. In general the set $\lambda[\partial(\mathcal{M}')]$ can always be embedded within an n-dimensional simplex which is a polytope with n extreme points. Thus, we may always choose the polytope P such that $m = n$.

3.7.2 Quasi-convex functions

In Section 3.3 we needed the concept of *quasi-convexity*. This is defined as follows (see Boyd & Vandeberghe (2004*b*)):

Definition 3.7.5. *Let $S \subseteq \mathbb{R}^n$ be a convex set. We say that a function $g : S \to \mathbb{R}$ is* quasi-convex *if for any pair of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2 \in S$ and $\lambda \in [0,1]$ we have:*

$$g(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) \leq \max\{g(\boldsymbol{x}_1), g(\boldsymbol{x}_2)\}.$$

Furthermore, the function g is said to be quasi-concave *if $-g$ is quasi-convex. Finally, if g is both quasi-convex and quasi-concave, we say that g is* quasi-linear.

More generally quasi-convexity implies the following result which is proved in Huseby & Haavardsson (2008):

Proposition 3.7.6. *Let $S \subseteq \mathbb{R}^n$ be a convex set, and let $g : S \to \mathbb{R}$ be a quasi-convex function. Moreover, let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in S$, and let $\lambda_1, \ldots, \lambda_n \in [0,1]$, be such that $\sum_{i=1}^n \lambda_i = 1$. Then:*

$$g(\sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i) \leq \max\{g(\boldsymbol{x}_1), \ldots, g(\boldsymbol{x}_n)\}. \tag{3.7.2}$$

The following result provides alternative definitions of quasi-convexity and quasi-concavity (see Boyd & Vandeberghe (2004*b*)):

Proposition 3.7.7. *Let $S \subseteq \mathbb{R}^n$ be a convex set, and let $g : S \to \mathbb{R}$. Then g is quasi-convex if and only if the sets $L_y = \{\boldsymbol{x} \in S : g(\boldsymbol{x}) \leq y\}$ are convex for all y. Similarly, g is quasi-concave if and only if the sets $U_y = \{\boldsymbol{x} \in S : g(\boldsymbol{x}) \geq y\}$ are convex for all y. Finally, g is quasi-linear if and only if L_y and U_y are convex for all y.*

Note that for some y L_y or U_y may be empty. In this setting, however, $\emptyset$ is defined to be convex, so in order to verify quasi-convexity or quasi-concavity, only non-empty sets need to be considered.

Using Proposition 3.7.7 we can also state the following characterizations; the proof is given in Huseby & Haavardsson (2008):

Proposition 3.7.8. *Let $S \subseteq \mathbb{R}^n$ be a convex set, and let $g : S \to \mathbb{R}$. Then g is quasi-convex if and only if the sets $L_y^o = \{\boldsymbol{x} \in S : g(\boldsymbol{x}) < y\}$ are convex for all y. Similarly, g is quasi-concave if and only if the sets $U_y^o = \{\boldsymbol{x} \in S : g(\boldsymbol{x}) > y\}$ are convex for all y. Finally, g is quasi-linear if and only if L_y^o and U_y^o are convex for all y.*

By combining the above results, we see that a function $g : S \to \mathbb{R}$ is quasi-linear if and only if L_y and its complement are both convex for all y. It is easy to see that this implies that for all y, $\partial(L_y) = S \cap H_y$ where H_y is a hyperplane. The following result provides a sufficient condition for quasi-linearity. The proof is given in Huseby & Haavardsson (2008)

Proposition 3.7.9. *Let $S \subseteq \mathbb{R}^n$ be a convex set, and let $g : S \to \mathbb{R}$. Moreover assume that there exists a non-zero linear form ℓ such that $g(\boldsymbol{x}) = h(\ell(\boldsymbol{x}))$ for all $\boldsymbol{x} \in S$, where $h : \mathbb{R} \to \mathbb{R}$ is either non-decreasing or non-increasing. Then g is quasi-linear.*

3.8 Proof of Theorem 3.4.1

In order to prove Theorem 3.4.1, we need the following lemma, proved in Huseby & Haavardsson (2008).

Lemma 3.8.1. *Assume that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ are such that:*

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i, \qquad k = 1, \ldots, n. \tag{3.8.1}$$

Then for any $\boldsymbol{a} \in \mathbb{R}^n$ such that:

$$a_1 \geq a_2 \geq \ldots \geq a_n \geq 0, \tag{3.8.2}$$

we also have:

$$\sum_{i=1}^{k} x_i a_i \geq \sum_{i=1}^{k} y_i a_i, \qquad k = 1, \ldots, n.$$

We now turn to the proof of the theorem and start by introducing the plateau lengths $T_1, \ldots, T_n$ as defined in (3.3.14). When the priority strategy $\boldsymbol{b}^1$ is used, reservoir 1 is produced at the rate K throughout the interval $[0, T_1]$, the reservoirs 1 and 2 are produced at a total rate K throughout the interval $[0, T_2]$, etc. Moreover, reservoir 1 will be produced without any choking for $t \geq T_1$, reservoir 1 and 2 will be produced without any choking for $t \geq T_2$, etc.

We shall now prove by induction that:

$$\sum_{j=1}^{i} Q_j(t, \boldsymbol{b}^1) \geq \sum_{j=1}^{i} Q_j(t, \boldsymbol{b}^2), \qquad t \geq 0,\ i = 1, \ldots, n. \tag{3.8.3}$$

Thus, we start out by considering the case where $i = 1$. If $0 \leq t \leq T_1$, then obviously:

$$Q_1(t, \boldsymbol{b}^1) = Kt \geq Q_1(t, \boldsymbol{b}^2).$$

If $t > T_1$, we know that reservoir 1 is produced without any choking when $\boldsymbol{b}^1$ is used. Thus, we have:

$$Q_1(t, \boldsymbol{b}^1) = V_1(1 - e^{-D_1(t-T_1)}) + Q_1(T_1, \boldsymbol{b}^1)e^{-D_1(t-T_1)}.$$

If, on the other hand, $\boldsymbol{b}^2$ is used, we get:

$$Q_1(t, \boldsymbol{b}^2) \leq V_1(1 - e^{-D_1(t-T_1)}) + Q_1(T_1, \boldsymbol{b}^2)e^{-D_1(t-T_1)}.$$

Thus, since $Q_1(T_1, \boldsymbol{b}^1) \geq Q_1(T_1, \boldsymbol{b}^2)$, it follows that $Q_1(t, \boldsymbol{b}^1) \geq Q_1(t, \boldsymbol{b}^2)$ for all $t > T_1$. Hence, we conclude that $Q_1(t, \boldsymbol{b}^1) \geq Q_1(t, \boldsymbol{b}^2)$ for all $t \geq 0$, i.e., (3.8.3) is proved for $i = 1$.

We then assume that (3.8.3) is proved for $i = 1, \ldots, (k-1)$, and consider the case where $i = k$. If $0 \leq t \leq T_k$, we have:

$$\sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^1) = Kt \geq \sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^2). \tag{3.8.4}$$

We then consider the case where $t > T_k$. If $\boldsymbol{b}^1$ is used, the reservoirs $1, 2, \ldots, k$ are produced without any choking, thus:

$$\sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^1) = \sum_{j=1}^{k} V_j(1 - e^{-D_j(t-T_k)}) + \sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^1) e^{-D_j(t-T_k)}. \tag{3.8.5}$$

If, on the other hand, $\boldsymbol{b}^2$ is used, we get:

$$\sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^2) \leq \sum_{j=1}^{k} V_j(1 - e^{-D_j(t-T_k)}) + \sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^2) e^{-D_j(t-T_k)}. \tag{3.8.6}$$

By the induction hypothesis and (3.8.4) we have that:

$$\sum_{j=1}^{i} Q_j(T_k, \boldsymbol{b}^1) \geq \sum_{j=1}^{i} Q_j(T_k, \boldsymbol{b}^2), \qquad i = 1, \ldots, k.$$

Moreover, since $D_1 \leq D_2 \leq \cdots \leq D_k$, we have:

$$e^{-D_1(t-T_k)} \geq \cdots \geq e^{-D_k(t-T_k)}, \text{ for all } t \geq T_k.$$

Then it follows by Lemma 3.8.1 that:

$$\sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^1) e^{-D_j(t-T_k)} \geq \sum_{j=1}^{k} Q_j(T_k, \boldsymbol{b}^2) e^{-D_j(t-T_k)} \tag{3.8.7}$$

By combining (3.8.5), (3.8.6) and (3.8.7), for all $t > T_k$ and (3.8.4) for $0 \leq t \leq T_k$, we get for $t \geq 0$:

$$\sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^1) \geq \sum_{j=1}^{k} Q_j(t, \boldsymbol{b}^2).$$

Thus, (3.8.3) is proved for $i = k$ as well. Hence, the result is proved by induction ■

Chapter 4

A parametric class of production strategies for multi-reservoir production optimization

Abstract

When a large oil or gas field is produced, several reservoirs often share the same processing facility. This facility is typically capable of processing only a limited amount of oil, gas and water per unit of time. In the present paper only single phase production, e.g., oil production, is considered. In order to satisfy the processing limitations, the production needs to be choked. That is, for each reservoir the production is scaled down by suitable *choke factors* between zero and one, chosen so that the total production does not exceed the processing capacity. Huseby & Haavardsson (2008) introduced the concept of a *production strategy*, a vector valued function defined for all points of time $t \geq 0$ representing the choke factors applied to the reservoirs at time t. As long as the total potential production rate is greater than the processing capacity, the choke factors should be chosen so that the processing capacity is fully utilized. When the production reaches a state where this is not possible, the production should be left unchoked. A production strategy satisfying these constraints is said to be *admissible*. Huseby & Haavardsson (2008) developed a general framework for optimizing production strategies with respect to various types of objective functions. In the present paper we present a parametric class of admissible production strategies. Using the framework of Huseby & Haavardsson (2008) it can be shown that under mild restrictions on the objective function an optimal strategy can be found within this class. The number of parameters needed to span the class is bounded by the number of reservoirs. Thus, an optimal strategy within this class can be found using a standard numerical optimization algorithm. This makes it possible to handle complex, high-dimensional cases. Furthermore, uncertainty may be included, enabling robustness and sensitivity analysis.

Keywords

Convex optimization theory, Numerical optimization methods, Conjugate Gradient Method, Nelder-Mead Method, Risk Analysis, Total Value Chain Analysis, Simplified Production Profiles

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4.1 Introduction

Optimization is an important element in the management of multiple-field oil and gas assets, since many investment decisions are irreversible and finance is committed for the long term. Optimization of oil and gas recovery in petroleum engineering is a considerable research field, see Bittencourt & Horne (1997), Horne (2002) or Merabet & Bellah (2002). Another important research tradition focuses on the problem of modelling the entire hydrocarbon value chain, where the purpose is to make models for scheduling and planning of hydrocarbon field infrastructures with complex objectives, see van den Heever et al. (2001), Ivyer & Grossmann (1998) or Neiro & Pinto (2004). Since the entire value chain is very complex, many aspects of it need to be simplified to be able to construct such a comprehensive model.

Huseby & Haavardsson (2008) considered the more limited problem of hydrocarbon production optimization in an oil or gas field consisting of many reservoirs sharing the same processing facility. In order to satisfy the processing limitations of the facility, the production needs to be choked. That is, at any given point of time the production from each of the reservoirs are scaled down by suitable *choke factors* between zero and one, chosen so that the total production does not exceed the processing capacity. This situation was handled by introducing the concept of a *production strategy*. A production strategy is a vector valued function defined for all points of time $t \geq 0$ representing the choke factors applied to the reservoirs at time t. The problem is then reduced to finding a production strategy which is optimal with respect to a suitable objective function. Huseby & Haavardsson (2008) developed a general framework for solving such optimization problems, and provided solutions to the problem in several important special cases.

In the present paper we present a parametric class of production strategies. Using the framework of Huseby & Haavardsson (2008) it can be shown that under mild restrictions on the objective function an optimal strategy can be found within this class. The number of parameters needed to span the class is bounded by the number of reservoirs. Thus, an optimal strategy within this class can be found using a standard numerical optimization algorithm. This makes it possible to handle complex, high-dimensional cases. Furthermore, uncertainty may be included, enabling robustness and sensitivity analysis.

As in Huseby & Haavardsson (2008), we assume that the field has been analyzed using state-of-the-art reservoir simulation methods. Based on the output from these simulations simplified *production profile models* for each of the reservoirs can be constructed and used as input to the optimization procedure. How to construct such profile models is described in Haavardsson & Huseby (2007). We also follow the approach of Huseby & Haavardsson (2008) by focussing on optimizing oil production, and leave simultaneous analysis of oil, gas and water production for future work. Still the optimization methods developed here can be used in a broader context of a total value chain analysis.

4.2 Some basic concepts and established results

4.2.1 Model and notation

We consider the oil production from n reservoirs that share a processing facility with a constant process capacity $K > 0$, expressed in some suitable unit, e.g., kSm^3 per day. Let $\boldsymbol{Q}(t) = (Q_1(t), \ldots, Q_n(t))$ denote the vector of cumulative production functions for the n reservoirs, and let $\boldsymbol{f}(t) = (f_1(t), \ldots, f_n(t))$ be the corresponding vector of *potential production rate functions* or *PPR-functions*, where each PPR-function can be written as:

$$f_i(t) = f_i(Q_i(t)), \qquad t \geq 0,\ i = 1, \ldots n. \tag{4.2.1}$$

This assumption implies that the potential production rate can be expressed as a function of the volume produced. As a consequence the total producible volume from a reservoir does not depend on the production schedule. The assumption expressed in (4.2.1) also implies that the potential production rate of one reservoir does not depend on the volumes produced from the other reservoirs. We will also assume for $i = 1, \ldots n$ that f_i is non-negative and non-increasing as a function of $Q_i(t)$ for all t and that the recoverable volume of each reservoir is finite. Finally, to ensure uniqueness of potential production profiles we will also assume that f_i is Lipschitz continuous in Q_i, $i = 1, \ldots, n$.

A *production strategy* is defined by a vector valued function $\boldsymbol{b} = \boldsymbol{b}(t) = (b_1(t), \ldots, b_n(t))$, defined for all $t \geq 0$, where $b_i(t)$ represents the *choke factor* applied to the ith reservoir at time t, $i = 1, \ldots, n$. The *actual production rates* from the reservoirs, after the production is choked is given by

$$\boldsymbol{q}(t) = (q_1(t), \ldots, q_n(t)),$$

where

$$q_i(t) = \frac{dQ_i(t)}{dt} = b_i(t) f_i(Q_i(t)), \quad i = 1, \ldots, n. \tag{4.2.2}$$

We also introduce the total production rate function $q(t) = \sum_{i=1}^{n} q_i(t)$ and the total cumulative production function $Q(t) = \sum_{i=1}^{n} Q_i(t)$. To reflect that $\boldsymbol{q}$ and $\boldsymbol{Q}$ depend on the chosen productions strategy $\boldsymbol{b}$, we sometimes indicate this by writing $\boldsymbol{q}(t) = \boldsymbol{q}(t, \boldsymbol{b})$ etc.

To satisfy the physical constraints of the reservoirs and the process facility, we require that the actual production rate cannot exceed its potential production rate. Moreover, the total production rate cannot exceed the production capacity. Let $\mathcal{B}$ denote the class of production strategies that satisfy these physical constraints. We refer to production strategies $\boldsymbol{b} \in \mathcal{B}$ as *valid production strategies*.

For a given production strategy $\boldsymbol{b} \in \mathcal{B}$ the *plateau length* is defined as

$$T_K = T_K(\boldsymbol{b}) = \sup\{t \geq 0 : \sum_{i=1}^{n} f_i(Q_i(t)) \geq K\}. \tag{4.2.3}$$

An *admissible production strategy* is defined as a production strategy $\boldsymbol{b} \in \mathcal{B}$ satisfying the following constraint:

$$q(t) = \sum_{i=1}^{n} q_i(t) = \sum_{i=1}^{n} b_i(t) f_i(Q_i(t)) = \min\{K, \sum_{i=1}^{n} f_i(Q_i(t))\}. \tag{4.2.4}$$

Moreover, we let $\mathcal{B}' \subseteq \mathcal{B}$ denote the class of admissible strategies.

4.2.2 Objective functions

To evaluate production strategies we introduce an *objective function*, i.e., a mapping $\phi : \mathcal{B} \to \mathbb{R}$ representing some sort of a performance measure. If $\boldsymbol{b}^1, \boldsymbol{b}^2 \in \mathcal{B}$, we prefer $\boldsymbol{b}^2$ to $\boldsymbol{b}^1$ if $\phi(\boldsymbol{b}^2) \geq \phi(\boldsymbol{b}^1)$. Moreover, an *optimal production strategy* with respect to ϕ is a production strategy $\boldsymbol{b}^{opt} \in \mathcal{B}$ such that $\phi(\boldsymbol{b}^{opt}) \geq \phi(\boldsymbol{b})$ for all $\boldsymbol{b} \in \mathcal{B}$.

If $\boldsymbol{b}^1, \boldsymbol{b}^2 \in \mathcal{B}$ are two production strategies such that $\boldsymbol{Q}(t, \boldsymbol{b}^1) \leq \boldsymbol{Q}(t, \boldsymbol{b}^2)$ for all $t \geq 0$, one would most likely prefer $\boldsymbol{b}^2$ to $\boldsymbol{b}^1$. Thus, a sensible objective function should have the property that $\phi(\boldsymbol{b}^1) \leq \phi(\boldsymbol{b}^2)$ whenever $\boldsymbol{Q}(t, \boldsymbol{b}^1) \leq \boldsymbol{Q}(t, \boldsymbol{b}^2)$ for all $t \geq 0$. Objective functions satisfying this property will be referred to as *monotone objective functions*.

In general the revenue generated by the production may vary between the reservoirs. This may occur if e.g., the quality of the oil, or the average production cost per unit are different from reservoir to reservoir. Such differences should then be reflected in the chosen objective function. On the other hand, if all the reservoirs are similar, we could restrict ourselves to considering objective functions depending on the production strategy $\boldsymbol{b}$ only through the total production rate function $q(\cdot, \boldsymbol{b})$ (or equivalently through $Q(\cdot, \boldsymbol{b})$). We refer to such objective functions as *symmetric*.

In this paper we will consider the following monotone, symmetric objective function:

$$\phi_{C,R}(\boldsymbol{b}) = \int_0^\infty I\{q(u) \geq C\} q(u) e^{-Ru} du, \quad 0 \leq C \leq K, \quad R \geq 0. \tag{4.2.5}$$

The parameter R may be interpreted as a discount factor, while C is a threshold value reflecting the minimum acceptable production rate. If we insert $C = 0$ and $R > 0$ in (4.2.5), the resulting value of the objective function is simply the *discounted production*. On the other hand if we insert $C = K$ in (4.2.5), the integrand is positive only when $q(u) = K$. When $R = 0$ we obtain that $\phi_{C,0}(\boldsymbol{b}) = \phi_{K,0}(\boldsymbol{b}) = KT_K(\boldsymbol{b})$. It also follows from the definition of $\phi_{C,R}$ in (4.2.5) and T_K in (4.2.3) that $\phi_{K,0}(\boldsymbol{b}) = KT_K(\boldsymbol{b}) = \sum_{i=1}^n Q_i(T_K(\boldsymbol{b}))$.

4.2.3 Principles for optimizing production strategies

We now turn to the problem of finding the best production strategy. Consider the set

$$\mathcal{Q} = [0, V_1] \times \cdots \times [0, V_n], \tag{4.2.6}$$

where $V_1, \ldots, V_n$ are the recoverable volumes from the n reservoirs. We then introduce the subset $\mathcal{M} \subseteq \mathcal{Q}$ given by:

$$\mathcal{M} = \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{n} f_i(Q_i) \geq K\}.$$

The subset $\bar{\mathcal{M}} \subseteq \mathcal{Q}$ is defined as $\bar{\mathcal{M}} = \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{n} f_i(Q_i) < K\}$. We also need the set of boundary points of $\mathcal{M}$ separating $\mathcal{M}$ from $\bar{\mathcal{M}}$, which we denote by $\partial(\mathcal{M})$. Thus, $\boldsymbol{Q} \in \partial(\mathcal{M})$ if and only if every neighborhood of $\boldsymbol{Q}$ intersects both $\mathcal{M}$ and $\bar{\mathcal{M}}$.

The set $\mathcal{M}$ has the property that the total production rate can be sustained at plateau level as long as $\boldsymbol{Q} \in \mathcal{M}$. More specifically, let $\boldsymbol{b}$ be any production strategy, and consider the points in $\mathcal{Q}$ generated by $\boldsymbol{Q}(t) = \boldsymbol{Q}(t, \boldsymbol{b})$ as t increases. From the boundary conditions we know that $\boldsymbol{Q}(0) = \boldsymbol{0}$. By the continuity of the PPR-functions, $\boldsymbol{Q}(t)$ will move along some path in $\mathcal{M}$ until the boundary $\partial(\mathcal{M})$ is reached.

If $\boldsymbol{b} \in \mathcal{B}$, the resulting path is said to be a *valid path*, while if $\boldsymbol{b} \in \mathcal{B}^{'}$, the path is called an *admissible path*. in general only a subset of $\mathcal{M}$ can be reached by admissible paths. We denote this subset by $\mathcal{M}^{'}$. Moreover, we let $\partial(\mathcal{M}^{'}) = \partial(\mathcal{M}) \cap \mathcal{M}^{'}$.

For an admissible path the total production rate equals K all the way until the path reaches $\partial(\mathcal{M}^{'})$. Moreover, the plateau length $T_K(\boldsymbol{b})$ is the point in time when the path reaches $\partial(\mathcal{M}^{'})$, implying that $\partial(\mathcal{M}^{'}) = \{\boldsymbol{Q}(T_K(\boldsymbol{b})) : \boldsymbol{b} \in \mathcal{B}^{'}\}$.

The following proposition, proved in Huseby & Haavardsson (2008), plays a key role when searching for optimal production strategies:

Proposition 4.2.1. *Let ϕ be a symmetric, monotone objective function and let $\boldsymbol{b} \in \mathcal{B}^{'}$. Then ϕ is uniquely determined by $\boldsymbol{Q}(T_K(\boldsymbol{b}))$. Thus, we may write $\phi(\boldsymbol{b}) = \phi(\boldsymbol{Q}(T_K(\boldsymbol{b})))$.*

As a consequence of Proposition 4.2.1 the following corollary can be stated:

Corollary 4.2.2. *Let ϕ be a symmetric, monotone objective function and let $\boldsymbol{b} \in \mathcal{B}'$ and let $\boldsymbol{Q}^* \in \partial\mathcal{M}'$ denote the point with the property that $\phi(\boldsymbol{Q})$ is maximized for $\boldsymbol{Q} = \boldsymbol{Q}^*$. Assume that $\boldsymbol{Q}(T_K(\boldsymbol{b})) = \boldsymbol{Q}^*$. Then $\boldsymbol{b}$ is optimal with respect to ϕ.*

Corollary 4.2.2 states that any admissible production strategy which path reaches the optimal $\boldsymbol{Q}^*$ is optimal. The following corollary will be useful in the present paper:

Corollary 4.2.3. *Let ϕ be a symmetric, monotone objective function and let $\mathcal{C} \subseteq \mathcal{B}'$ be a class of admissible production strategies such that for all $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ there exists a $\boldsymbol{b}^* \in \mathcal{C}$ such that $\boldsymbol{Q}(T_K(\boldsymbol{b}^*)) = \boldsymbol{Q}^*$. Then an optimal production strategy with respect to ϕ can always be found within $\mathcal{C}$.*

Motivated by Corollary 4.2.2 Huseby & Haavardsson (2008) proposed a two-step process for finding an optimal production strategy. The first step consisted of finding $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ such that $\phi(\boldsymbol{Q}^*) \geq \phi(\boldsymbol{Q})$ for all $\boldsymbol{Q} \in \partial(\mathcal{M}')$. In the second step a *backtracking algorithm* was used to derive a production strategy $\boldsymbol{b}^* \in \mathcal{B}'$ such that $\boldsymbol{Q}(T_K(\boldsymbol{b})) = \boldsymbol{Q}^*$ which by Corollary 4.2.2 is optimal.

If all the PPR-functions are differentiable, the first step can often be solved very efficiently using e.g., the method of Lagrange multipliers. Using such a method one can at least find a $\boldsymbol{Q}' \in \partial(\mathcal{M})$ such that $\phi(\boldsymbol{Q}') \geq \phi(\boldsymbol{Q})$ for all $\boldsymbol{Q} \in \partial(\mathcal{M})$. If $\boldsymbol{Q}' \in \partial(\mathcal{M}')$ as well, we let $\boldsymbol{Q}^* = \boldsymbol{Q}'$. To verify that $\boldsymbol{Q}' \in \partial(\mathcal{M}')$, Huseby & Haavardsson (2008) uses the backtracking algorithm. If this algorithm successfully produces an admissible path, this shows that $\boldsymbol{Q}' \in \partial(\mathcal{M}')$. Thus, an optimal production strategy is found. On the other hand, if $\boldsymbol{Q}' \in \partial(\mathcal{M} \setminus \mathcal{M}')$, no such admissible path exists. Thus, the backtracking algorithm cannot succeed. Note, however, that even if $\boldsymbol{Q}' \in \partial(\mathcal{M}')$, the backtracking algorithm may sometimes fail. This occurs when $\boldsymbol{Q}'$ is very close to or at the border of $\partial(\mathcal{M}')$.

In the next section we propose an alternative approach to the optimization problem. We introduce a parametric class of admissible production strategies and restrict our search for optimal strategies within this class. If all $\boldsymbol{Q}^* \in \partial(\mathcal{M}')$ can be reached using production strategies from this parametric class, Corollary 4.2.3 then states that an optimal production strategy can always be found within this class.

4.3 A parametric class of production strategies

A simple production strategy can always be constructed by using the same choke factor for all the reservoirs. That is, we let $b_i(t) = c(t)$, $i = 1, \ldots, n$. For such a production strategy to be admissible $c(t)$ must satisfy the following:

$$\sum_{i=1}^{n} c(t) f_i(Q_i(t)) = \min\{K, \sum_{i=1}^{n} f_i(Q_i(t))\}. \tag{4.3.1}$$

Thus, for $0 \leq t \leq T_K$, we have:

$$c(t) = \frac{K}{\sum_{i=1}^{n} f_i(Q_i(t))}, \tag{4.3.2}$$

while $c(t) = 1$ for all $t > T_K$. Note that since $\sum_{i=1}^{n} f_i(Q_i(t)) \geq K$ for $0 \leq t \leq T_K$, the common choke factor, $c(t)$ will always be less than or equal to 1. A production strategy defined in this way, will be referred to as a *symmetry strategy.* We observe that when a symmetry strategy is used, the available production capacity is shared among the reservoirs such that none of the reservoirs are given any kind of priority. The idea now is to expand this class by allowing the production capacity to be shared asymmetrically. To facilitate this we start out by considering production strategies where for $0 \leq t \leq T_K$ the choke factors are given by:

$$b_i(t) = w_i c(t), \qquad i = 1, \ldots, n, \tag{4.3.3}$$

where $w_1, \ldots, w_n$ are positive real numbers representing the relative priorities assigned to the n reservoirs, and where $c(t)$ is chosen so that the strategy is admissible. For $t > T_K$, we of course define $b_i(t) = 1$, $i = 1, \ldots, n$. Note that if $w_1 = \cdots = w_n$ we get a symmetry strategy.

In order to ensure admissibility, $c(t)$ must be chosen so that:

$$\sum_{i=1}^{n} w_i c(t) f_i(Q_i(t)) = \min\{K, \sum_{i=1}^{n} f_i(Q_i(t))\}. \tag{4.3.4}$$

Thus, for $0 \leq t \leq T_K$ the choke factors are given by:

$$b_i(t) = w_i c(t) = \frac{w_i K}{\sum_{j=1}^{n} w_j f_j(Q_j(t))}, \qquad i = 1, \ldots, n. \tag{4.3.5}$$

Unfortunately, this definition does not guarantee that the choke factors are less than or equal to 1. To fix this problem, we instead let:

$$b_i(t) = \min\{1, w_i c(t)\}, \qquad i = 1, \ldots, n. \tag{4.3.6}$$

While this ensures that the resulting production strategy is valid, it makes the calculation of $c(t)$ slightly more complicated. To ensure admissibility, $c(t)$ must now be chosen so that:

$$\sum_{i=1}^{n} \min\{1, w_i c(t)\} f_i(Q_i(t)) = \min\{K, \sum_{i=1}^{n} f_i(Q_i(t))\}. \tag{4.3.7}$$

When $t > T_K$, $c(t)$ must be chosen large enough so that $\min\{1, w_i c(t)\} = 1$, $i = 1, \ldots, n$. One obvious possibility is to let $c(t) = \max_i\{w_i^{-1}\}$. When $0 \leq t \leq T_K$, there is always a unique value of $c(t)$ satisfying (4.3.7). To see this we first note that if we let $c(t) = 0$, the left-hand side of (4.3.7) is zero which is less than K. On the other hand, letting $c(t) = \max\{w_i^{-1}\}$, we get that $\min\{1, w_i c(t)\} = 1$, $i = 1, \ldots, n$. Inserting this, the left-hand side of (4.3.7) becomes $\sum_{i=1}^{n} f_i(Q_i(t))$, which is greater than or equal to K for $0 \leq t \leq T_K$. Between these two extremes the left-hand side of (4.3.7) is a continuous function of $c(t)$. Thus, the existence of a $c(t)$ satisfying (4.3.7) is guaranteed by the intermediate value theorem. Moreover, since the left-hand side of (4.3.7) is a strictly increasing function of $c(t)$, this $c(t)$-value is unique.

In order to take a closer look at the calculation of $c(t)$ for the case where $0 \leq t \leq T_K$, it is convenient to sort the weights in decreasing order. This can always be done by identifying a permutation π of the index set so that $w_{\pi(1)} \geq w_{\pi(2)} \geq \cdots \geq w_{\pi(n)}$. In order to simplify the notation, however, we instead assume that the reservoirs are indexed so that $w_1 \geq w_2 \geq \cdots \geq w_n$. We then introduce the following sets:

$$\mathcal{M}_k = \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i=1}^{k} f_i(Q_i) + \sum_{i=k+1}^{n} \frac{w_i}{w_k} f_i(Q_i) \geq K\}, \qquad k = 1, \ldots, n. \tag{4.3.8}$$

We also define $\mathcal{M}_0 = \emptyset$. We observe that the left-hand side of the inequality defining the $\mathcal{M}_k$, is a weighted sum of the PPR-functions, where the weight associated to f_i is 1 for $i = 1, \ldots, k$ and $\frac{w_i}{w_k}$ for $i = k+1, \ldots, n$. Moreover, since $w_1 \geq w_2 \geq \cdots \geq w_n$, it follows that $\frac{w_i}{w_k} \leq 1$ for $i = k+1, \ldots, n$. As k increases, the number of PPR-functions with weight 1 increases. At the same time the weights of form $\frac{w_i}{w_k}$ increases as well. Thus, the left-hand side of the inequality defining the set $\mathcal{M}_k$ increases with k. Hence, it follows that $\mathcal{M}_1 \subseteq \ldots \subseteq \mathcal{M}_n$. When $k = n$, all the PPR-functions have weight 1, implying that $\mathcal{M}_n = \mathcal{M}$. From this it follows that the difference sets $(\mathcal{M}_k \setminus \mathcal{M}_{k-1})$, $k = 1, \ldots, n$, form a partition of the set $\mathcal{M}$. Note, however, that if $w_k = w_{k-1}$, then $\mathcal{M}_k = \mathcal{M}_{k-1}$. Thus, some of the difference sets may be empty.

In order to find the $c(t)$-function satisfying (4.3.7) when $0 \leq t \leq T_K$, i.e., when $\boldsymbol{Q}(t) \in \mathcal{M}$, it is convenient to solve this problem separately for each of the difference sets. Thus, we let $\boldsymbol{Q}(t) \in \mathcal{M}_k \setminus \mathcal{M}_{k-1}$, and claim that in this case $c(t)$ is given by:

$$c(t) = \frac{K - \sum_{i<k} f_i(Q_i(t))}{\sum_{i=k}^{n} w_i f_i(Q_i(t))}. \tag{4.3.9}$$

To prove this we note that since $\boldsymbol{Q}(t) \in \mathcal{M}_k$, it follows that:

$$\sum_{i=1}^{k} f_i(Q_i(t)) + \sum_{i=k+1}^{n} \frac{w_i}{w_k} f_i(Q_i(t)) \geq K. \tag{4.3.10}$$

By multiplying both sides by w_k and rearranging the terms we obtain that:

$$\frac{w_k(K - \sum_{i<k} f_i(Q_i(t)))}{\sum_{i=k}^{n} w_i f_i(Q_i(t))} \leq 1. \tag{4.3.11}$$

Combining this with (4.3.9) we get that $w_k c(t) \leq 1$, and since $w_k \geq w_{k+1} \geq \cdots \geq w_n$, it follows that:

$$w_i c(t) \leq 1, \qquad i = k, \ldots, n. \tag{4.3.12}$$

On the other hand we have that $\boldsymbol{Q}(t) \notin \mathcal{M}_{k-1}$. Thus, by using a similar argument as above, we get that:

$$w_i c(t) \geq 1, \qquad i = 1, \ldots, k-1. \tag{4.3.13}$$

Hence, it follows that:

$$b_i(t) = \min\{1, w_i c(t)\} = \begin{cases} 1 & i = 1, \ldots, k-1 \\ w_i c(t) & i = k, \ldots, n \end{cases} \tag{4.3.14}$$

By inserting (4.3.9) and (4.3.14) into the left-hand side of (4.3.7) we get that:

$$\begin{aligned} \sum_{i=1}^{n} \min\{1, w_i c(t)\} f_i(Q_i(t)) &= \sum_{i<k} f_i(Q_i(t)) + \sum_{i=k}^{n} w_i c(t) f_i(Q_i(t)) \\ &= \sum_{i<k} f_i(Q_i(t)) + \frac{[K - \sum_{i<k} f_i(Q_i(t))] \sum_{i=k}^{n} w_i f_i(Q_i(t))}{\sum_{i=k}^{n} w_i f_i(Q_i(t))} \\ &= K. \end{aligned}$$

That is, $c(t)$ as given in (4.3.9), satisfies (4.3.7) when $\boldsymbol{Q}(t) \in \mathcal{M}_k \setminus \mathcal{M}_{k-1}$. Since the same argument holds for all $k = 1, \ldots, n$, it follows that $c(t)$ satisfies (4.3.7) for all $\boldsymbol{Q}(t) \in \mathcal{M}$, i.e., whenever $0 \leq t \leq T_K$ as claimed.

By varying the weights $w_1, \ldots, w_n$ in $\mathbb{R}^n_+$ a whole range of admissible production strategies is obtained. We will refer to such production strategies as *first-order fixed-weight strategies*, and denote the class of all such strategies by $\mathcal{B}^w_1$. We always assume that the corresponding $c(t)$ is determined by (4.3.9) ensuring that the resulting production strategy is admissible. Thus, $\mathcal{B}^w_1 \subseteq \mathcal{B}'$. If $\boldsymbol{b} \in \mathcal{B}^w_1$ is a fixed-weight strategy with weight vector $\boldsymbol{w} = (w_1, \ldots, w_n)$, we sometimes indicate this by writing $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{w})$.

From the formula (4.3.9) it is easy to see that if we replace the weight vector $\boldsymbol{w}$ by $\tilde{\boldsymbol{w}} = \lambda \boldsymbol{w}$ where $\lambda > 0$, then $c(t)$ is replaced by $\tilde{c}(t) = \lambda^{-1} c(t)$. As a result the choke factors are not affected by this change of weights. Thus, we have shown that:

$$\boldsymbol{b}(\boldsymbol{w}) = \boldsymbol{b}(\lambda \boldsymbol{w}). \tag{4.3.15}$$

That is, the production strategy is invariant with respect to scale transformations of the weight vector $\boldsymbol{w}$. This means that one can reduce the dimension of the space of possible weight vectors to $(n-1)$ without changing the class $\mathcal{B}^w_1$. There are several ways of doing this. One possibility is to consider only $\boldsymbol{w}$ of length 1. Another possibility is to restrict the search to $\boldsymbol{w}$ normalized so that the sum of weights is 1. Here, however, we have chosen a third option, where the dimension is reduced by fixing the value of one of the weights, e.g., by letting $w_n = 1$. All the remaining weights may be chosen as arbitrary positive real numbers. Thus, the resulting search area is the unbounded $(n-1)$-dimensional set $\mathbb{R}^{n-1}_+$. Sometimes, however, it is easier to carry out the search on a bounded set. This can be achieved by using the following reparametrization:

$$w_i = \frac{v_i}{1 - v_i}, \qquad i = 1, \ldots, (n-1). \tag{4.3.16}$$

By letting v_i run through all values in the interval $(0, 1)$, the resulting values of w_i will run through all positive real numbers. Thus, searching for the optimal values of $(w_1, \ldots, w_{n-1})$ within the unbounded set $\mathbb{R}^{n-1}_+$ is reduced to searching for the optimal values of $(v_1, \ldots, v_{n-1})$ within the bounded set $(0, 1)^{n-1}$. This reparametrization is used in the numerical examples discussed in Section 4.

4.3.1 Higher order fixed-weight strategies

A weakness with the class $\mathcal{B}^w_1$ is that it does not allow strict priorities between the reservoirs. In order to study this further we introduce the concept of a *priority strategy*. A *kth order priority strategy* is an admissible production strategy defined relative to an ordered partition $\{A_j\}_{j=1}^k$ of the index set $\{1, \ldots, n\}$ of the reservoirs. The available processing capacity K is divided between the n reservoirs so that the reservoirs in A_1 are given the highest priority, the reservoirs in A_2 are given the second highest priorities, and so on. More specifically, at any given point of time t we let $K_j(t)$ denote the processing capacity available to the reservoirs in A_j,

$j = 1, \dots, k$. Then:

$$\begin{aligned} K_1(t) &= K, \\ K_2(t) &= \max\{0, K_1(t) - \sum_{i \in A_1} f_i(Q_i(t))\}, \\ K_3(t) &= \max\{0, K_2(t) - \sum_{i \in A_2} f_i(Q_i(t))\}, \\ &\dots \\ K_k(t) &= \max\{0, K_{k-1}(t) - \sum_{i \in A_{k-1}} f_i(Q_i(t))\}. \end{aligned} \tag{4.3.17}$$

In order to ensure admissibility it is assumed that the reservoirs in all groups use as much as possible of the available processing capacity. Thus, the choke factors $b_1(t), \dots, b_n(t)$ are chosen so that:

$$\sum_{i \in A_j} b_i(t) f_i(Q_i(t)) = \min\{K_j(t), \sum_{i \in A_j} f_i(Q_i(t))\}, \qquad j = 1, \dots, k. \tag{4.3.18}$$

Note that that (4.3.18) implies that the production strategy is admissible since adding up all k equalities yields:

$$\sum_{i=1}^{n} b_i(t) f_i(Q_i(t)) = \min\{K, \sum_{i=1}^{n} f_i(Q_i(t))\}. \tag{4.3.19}$$

The most extreme type of a priority strategy is an nth order priority strategy. For such strategies $|A_j| = 1$ for $j = 1, \dots, n$. Thus, the ordered partition, $\{A_j\}_{j=1}^n$, simply represents a permutation of the reservoirs. In this case the production strategy is uniquely defined by this permutation. Thus, if $A_j = \{i_j\}$, $j = 1, \dots, n$, then:

$$K_j(t) = \max\{0, K_{j-1}(t) - f_{i_{j-1}}(Q_{i_{j-1}}(t))\}, \qquad j = 2, \dots n, \tag{4.3.20}$$

while the choking factors, $b_1(t), \dots, b_n(t)$, satisfies:

$$b_{i_j}(t) = \min\{1, \frac{K_j(t)}{f_{i_j}(Q_{i_j}(t))}\}, \qquad j = 1, \dots n. \tag{4.3.21}$$

We now proceed by combining fixed-weight strategies and priority strategies. Thus, we let $\{A_j\}_{j=1}^k$ be a partition of the index set, and let $\boldsymbol{w} = (w_1, \dots, w_n)$ be a vector of positive real numbers. We then consider choke factor functions of the form:

$$b_i(t) = \min\{1, w_i c_j(t)\}, \qquad i \in A_j, \ j = 1, \dots, k, \tag{4.3.22}$$

where $c_1(t), \dots, c_k(t)$ are determined for each t so that the resulting production strategy is admissible., i.e., so that:

$$\sum_{i \in A_j} \min\{1, w_i c_j(t)\} f_i(Q_i(t)) = \min\{K_j(t), \sum_{i \in A_j} f_i(Q_i(t))\}, \qquad j = 1, \dots, k. \tag{4.3.23}$$

A production strategy of this form will be referred to as a *kth order fixed-weight strategy*, and we denote the class of all such strategies by $\mathcal{B}_k^w$. Computing $c_1(t), \ldots, c_k(t)$ can be done in exactly the same way as for first-order fixed-weight strategies, so we skip the details here.

If $\boldsymbol{b} \in \mathcal{B}_k^w$ is a kth order fixed-weight strategy relative to the partition $\{A_j\}_{j=1}^k$ and with weight vector $\boldsymbol{w} = (w_1, \ldots, w_n)$, we introduce the following vectors $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k$, where $\boldsymbol{w}_j$ is obtained from $\boldsymbol{w}$ by replacing w_i by 0 for all $i \notin A_j$, $j = 1, \ldots, k$. Thus, since $A_1, \ldots, A_k$ are pairwise disjoint and $A_1 \cup \cdots \cup A_k = \{1, \ldots, n\}$, it follows that $\boldsymbol{w} = \boldsymbol{w}_1 + \cdots + \boldsymbol{w}_k$. Now, let $\lambda_1, \ldots, \lambda_k$ be positive numbers, and assume that the vector of weights, $\boldsymbol{w}$, is replaced by $\tilde{\boldsymbol{w}} = \sum_{j=1}^k \lambda_j \boldsymbol{w}_j$. Then, using the same argument as in the first-order case, it follows that $c_1(t), \ldots, c_k(t)$ are replaced by $\tilde{c}_1(t), \ldots, \tilde{c}_k(t)$, where $\tilde{c}_j(t) = \lambda_j^{-1} c(t)$. As a result the choke factors are not affected by this change of weights. Thus, we have shown that:

$$\boldsymbol{b}(\boldsymbol{w}) = \boldsymbol{b}(\sum_{j=1}^{n} \lambda_j \boldsymbol{w}_j). \tag{4.3.24}$$

This implies that we in the kth order case may reduce the dimension of the space of possible weight vectors to $(n-k)$ without reducing the class $\mathcal{B}_k^w$. We have chosen to do this by fixing the value of one weight for each of the sets $A_1, \ldots, A_k$. In the case where $k = n$, we know that the priority strategy is uniquely determined by the permutation given by the single element sets $A_1, \ldots, A_n$. Thus, in this case the weight vector does not affect the production strategy, which is reflected by the fact that the dimension of the space of possible weight vectors can be reduced to zero.

We recall that by Corollary 4.2.3 an optimal production strategy can be found within a given class of admissible strategies provided that all points in $\partial(\mathcal{M}')$ can be reached by members of this class. It turns out that all *interior* points of $\partial(\mathcal{M}')$ can be reached by first-order fixed-weight strategies. However, to reach the boundary points in $\partial(\mathcal{M}')$ as well, higher-order strategies must be included. In a forthcoming paper it will be proved that by considering the combined class of fixed-weight strategies of *all* orders, it is possible to reach all points in $\partial(\mathcal{M}')$. Hence, an optimal production strategy can always be found within $\mathcal{B}_1^w \cup \cdots \cup \mathcal{B}_n^w$.

Assuming that the value of the objective function, ϕ, interpreted as a function of $\boldsymbol{Q}(T_K(\boldsymbol{b}))$, is a continuous function of this vector, it follows that for each point $\boldsymbol{Q}^* \in \partial(\partial(\mathcal{M}'))$ and $\epsilon > 0$, there exists another point, $\tilde{\boldsymbol{Q}}$ in the interior of $\partial(\mathcal{M}')$ such that $|\phi(\boldsymbol{Q}^*) - \phi(\tilde{\boldsymbol{Q}})| < \epsilon$. Hence, even if the search for an optimal strategy is restricted to $\mathcal{B}_1^w$, it is possible to find a strategy which is approximately optimal. In order to approximate a higher order fixed-weight strategy by a first-order strategy, one can assign very high weights to the reservoirs in the set with highest priority, and then use significantly smaller weights for the reservoirs in the sets with lower priorities. As we shall see, however, if the optimal strategy is a higher order strategy, better numerical results are obtained by searching among the fixed-weight strategies with the correct order.

4.4 Numerical optimization

We will now describe how the objective function $\phi_{C,R}(\boldsymbol{b})$ defined in (4.2.5) can be maximized for the parametric class defined in Section 4.3 using numerical optimization techniques. The numerical Java library Java Tools for Experimental Mathematics (JTEM)[1] is used for the optimization.

4.4.1 Initialization

Case studies have suggested that the objective function $\phi_{C,R}(\boldsymbol{b}(\boldsymbol{w}))$ is multi-modal as a function of $\boldsymbol{w}$. To ensure that the global maximum is found, several initialization techniques may be used, see Liu (2001). We have chosen to initialize the search by sampling N n-dimensional vectors at random. Then the objective function is evaluated in all N vectors. The vector where the object function attains its maximum, constitutes the initialization vector for the numerical search.

4.4.2 A sequential approach for numerical optimization

To find the optimal strategy within $\mathcal{B}_1^w \cup \cdots \mathcal{B}_1^w$ we need to identify the correct order of the optimal strategy. Moreover, for a given order, say k, we need an algorithm for finding the optimal strategy within $\mathcal{B}_k^w$. We start out by presenting the last algorithm:

Algorithm 4.4.1. *Let ϕ be a monotone, symmetric objective function. Assume that an ordered partition $\{A_j\}_{j=1}^k$ is given. Denote the highest element in each A_j with i_{A_j}, $j = 1, \ldots, k$. Then a production strategy $\boldsymbol{b}^* \in \mathcal{B}_k^w$ which maximizes ϕ numerically can be found as follows:*

STEP 1. *Find N random samples of $\boldsymbol{w}_k$ using the techniques described in Section 4.4.1. We set $w_{i_{A_j}} = 1.0$ for $j = 1, \ldots, k$ to avoid over-parametrization, as explained in Section 4.3.1. Denote these samples $\boldsymbol{w}_k^1, \ldots, \boldsymbol{w}_k^N$. Among these we select a vector $\boldsymbol{w}_k^j$ such that*

$$\phi(\boldsymbol{b}(\boldsymbol{w}_k^j)) \geq \phi(\boldsymbol{b}(\boldsymbol{w}_k^i)), \text{ for all } i \in \{1, \ldots, N\}.$$

STEP 2. *Maximize ϕ numerically with respect to $\boldsymbol{w}$ using $\boldsymbol{w}_k^j$ as initialization vector. In the maximization we always keep $w_{i_{A_j}} = 1.0$ for $j = 1, \ldots, k$. Denote the resulting vector of weights $\boldsymbol{w}_k^*$.*

To find the correct order of the optimal strategy we start by searching among the first-order fixed-weight strategies, and denote by $\boldsymbol{w}_1^*$ the resulting candidate obtained from Algorithm 4.4.1. Assuming that ϕ is continuous, it follows, as explained in Section 4.3.1, that $\boldsymbol{w}_1^*$ will be approximately optimal. We then proceed by inspecting $\boldsymbol{w}_1^*$. If the ratio between the smallest and largest element of this vector is large, this indicates that the optimal strategy may be a higher order strategy. Thus, the natural

[1] For documentation see `http://www.jtem.de/`.

next step is to consider second order fixed-weight strategies. Prior to this we sort the elements of $\boldsymbol{w}_1^*$ so that
$$w_{1,i_1}^* \geq \cdots \geq w_{1,i_n}^*.$$
Since this ordering indicates a prioritization order of the reservoirs, we consider only second order fixed-weight strategies such that the weights corresponding to the indices in A_1 are larger than the weights corresponding to the indices in A_2. Thus, only the following $n-1$ partitions need to be considered:

$$A_1 = \{i_1\}, \quad A_2 = \{i_2, \ldots, i_n\}, \tag{4.4.1}$$
$$A_1 = \{i_1, i_2\}, \quad A_2 = \{i_3, \ldots, i_n\}, \tag{4.4.2}$$
$$\ldots$$
$$A_1 = \{i_1, \ldots, i_{n-1}\}, \quad A_2 = \{i_n\}. \tag{4.4.3}$$
$$\tag{4.4.4}$$

We then run Algorithm 4.4.1 for all these partitions and denote by $\boldsymbol{w}_2^*$ the best-performing weight vector. If $\phi(\boldsymbol{b}(\boldsymbol{w}_2)) < \phi(\boldsymbol{b}(\boldsymbol{w}_1^*))$ we use $\boldsymbol{w}_1^*$ and conclude that the optimal strategy is a first-order strategy. Otherwise we proceed using $\boldsymbol{w}_2^*$ and the corresponding partition instead of $\boldsymbol{w}_1^*$. We then inspect the two sub-vectors of $\boldsymbol{w}_2^*$ corresponding to A_1 and A_2. If the ratio between the smallest and largest element of any of these two sub-vectors is large, this indicates that the optimal strategy may be an even higher order strategy. We then proceed by considering third order strategies. Now, however, only refinements of the previous partitions are considered. This implies that only $n-2$ partitions need to be examined at this stage. The process is repeated until no further improvement can be obtained.

By only considering successive refinements of the previous partitions, and taking into account the ordering of the weights, the number of times we need to run Algorithm 4.4.1 is reduced to a minimum. Thus, the total order of the sequential optimization process is dominated by the order of this algorithm.

4.5 Examples

4.5.1 The fixed-weight strategy as an alternative to backtracking

In this first example we will assume that both steps of the optimization algorithm developed in Huseby & Haavardsson (2008) can be executed. We start with finding the optimal state $\boldsymbol{Q}^*$ of the reservoirs at the end of the reservoirs. Then we use backtracking to derive an admissible production strategy to reach $\boldsymbol{Q}^*$. Corollary 4.2.2 states that any admissible production strategy which path reaches the optimal $\boldsymbol{Q}^*$ is optimal. Thus, as an alternative to backtracking we use the proposed parametric class to find another admissible production strategy to reach $\boldsymbol{Q}^*$.

To find $\boldsymbol{Q}^*$ we use the theory of Huseby & Haavardsson (2008), which states that if all PPR-functions are concave, the optimal $\boldsymbol{Q}^*$ may typically be located in central parts of $\partial(\mathcal{M}')$. The PPR-functions $f_1, \ldots, f_n$ are given by

$$f_i(Q_i(t)) = \sqrt{D_i(V_i - Q_i(t))},\ i = 1, \ldots, n, \tag{4.5.1}$$

where $V_1, \ldots, V_n$ denote the recoverable volumes of the n reservoirs. The chosen parameter values of the example are listed in Table 4.5. The objective function $\phi_{K,0}$ defined by letting $C = K$ and $R = 0$ in (4.2.5) is used. As explained in Section 4.2.2 the optimal solution maximizes the plateau volume, $\phi_{K,0}(\boldsymbol{Q}) = \sum_{i=1}^{n} Q_i$ subject to $\boldsymbol{Q} \in \partial(\mathcal{M}')$. When all PPR-functions are concave, an optimal solution to the first step of the optimization algorithm developed in Huseby & Haavardsson (2008) typically involves finding the separating hyperplane supporting $\mathcal{M}$ at the optimal $\boldsymbol{Q}^*$. Further, we realize that the PPR-functions on the form given by (4.5.1) and the extended objective function $\phi_{K,0} = \sum_{i=1}^{n} Q_i$ are differentiable, so Lagrange multipliers may be used. Using Lagrange multipliers it is straight-forward to show that the optimal $\boldsymbol{Q}^*$, denoted $\boldsymbol{Q}_L^*$, is given by

$$\boldsymbol{Q}^* = (Q_1^*, \ldots, Q_n^*) = (V_1 - \frac{D_1}{2}\{\frac{K}{\sum_{i=1}^{n} D_i}\}^2, \ldots, V_n - \frac{D_n}{2}\{\frac{K}{\sum_{i=1}^{n} D_i}\}^2). \tag{4.5.2}$$

To compare $\boldsymbol{Q}_L^*$ with the boundary point $\boldsymbol{Q}_{\boldsymbol{w}_1^*}^*$ obtained using the best first-order fixed-weight strategy we calculate $\boldsymbol{b}^* \in \mathcal{B}_1^w$, as explained in Section 4.4.2. Table 4.1 lists the coordinates of $\boldsymbol{Q}_L^*$ and $\boldsymbol{Q}_{\boldsymbol{w}_1^*}^*$ for the example of this section and the example of Section 4.5.2. Table 4.2 correspondingly lists the objective function values for $\boldsymbol{Q}_L^*$ and $\boldsymbol{Q}_P^*$ and the Euclidian distance between these two points. From Table 4.2 we observe that the distance between $\boldsymbol{Q}_L^*$ and $\boldsymbol{Q}_P^*$ is small, as expected. Table 4.3 lists the weights of the best numerical first-order fixed-weight strategy $\boldsymbol{w}_1^*$. As we can see from Table 4.3 none of the weights are significantly larger than the others, indicating that the optimum $\boldsymbol{Q}_{\boldsymbol{w}_1^*}^*$ is an interior point of $\partial(\mathcal{M})$. When we execute Algorithm 4.4.1 with second-order fixed-weight strategies this is confirmed; we are not able to find any second-order fixed-weight strategy with the property that ϕ_K is increased. Thus we conclude $\boldsymbol{Q}_{\boldsymbol{w}_1^*}^*$ is an interior point of $\partial(\mathcal{M}')$. The fact that the backtracking algorithm manages to propose an admissible production strategy to reach $\boldsymbol{Q}_L^*$ also indicates that $\boldsymbol{Q}_L^*$ is an interior point, see Huseby & Haavardsson (2008) for a discussion.

Proceeding to the backtracking, Figure 4.1 shows the production rates of this example when backtracking is used. The backtracking algorithm follows a piecewise linear path from the optimal $\boldsymbol{Q}^*$ to $\boldsymbol{0}$. At distinct time points the actual production is found using the well-known Simplex algorithm, see Huseby & Haavardsson (2008) for details. Due to the extreme nature of this algorithm, the production rates of the individual reservoirs tend to oscillate in periods. The oscillation occurs when it is equally beneficial to produce from two or more reservoirs, so that when the reservoirs compete for capacities they will alternate between being produced in one period and choked the next. If the primary purpose of the production optimization is to give decision support to project teams, the oscillations are not critical. The focus can for example be the assessment of different infrastructure investment alternatives. Hence, we are interested in the resulting cash flows of these different alternatives so

Example	Boundary point	Reservoir					
		1	2	3	4	5	6
Section 4.5.1	$\boldsymbol{Q}_L^*$	3,619.6	4,459.1	5,454.4	n.a.	n.a.	n.a.
	$\boldsymbol{Q}_{\boldsymbol{w}_1^*}^*$	3,619.2	4,458.4	5,453.9	n.a.	n.a.	n.a.
Section 4.5.2	$\boldsymbol{Q}_L^*$	3,879.2	4,731.7	5,896.0	5,275.6	7,832.3	8,141.2
	$\boldsymbol{Q}_{\boldsymbol{w}_1^*}^*$	3,881.7	4,747.0	5,968.2	5,273.6	7,673.3	8,190.4

Table 4.1: *Coordinates of the points* $\boldsymbol{Q}_L^*, \boldsymbol{Q}_{\boldsymbol{w}_1^*}^* \in \partial(\mathcal{M}')$ *in the examples in the sections 4.5.1 and 4.5.2.*

Example	Process constraint (in kSm^3 per sd)	$\phi_K(\boldsymbol{Q}_L^*)$ (in kSm^3)	$\phi_K(\boldsymbol{Q}_P^*)$ (in kSm^3)	Distance between $\boldsymbol{Q}_L^*$ and $\boldsymbol{Q}_P^*$ (in kSm^3)
Section 4.5.1	3.0	13,531.5	13,533.1	1.3
Section 4.5.2	7.0	35,756.2	35,737.3	182.1

Table 4.2: *Comparison of* $\phi_K(\boldsymbol{Q}_L^*)$ *and* $\phi_K(\boldsymbol{Q}_{\boldsymbol{w}_1^*}^*)$ *and the distance between* $\boldsymbol{Q}_L^*$ *and* $\boldsymbol{Q}_P^*$ *in the examples of the sections 4.5.1 and 4.5.2.*

Example Section	Best first-order fixed-weight strategy, $\boldsymbol{w}_1^*$	$\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_1^*))$ (kSm^3)	Best higher-order fixed-weight strategy, $\{A_j\}_{j=1}^k$ and $\boldsymbol{w}_k^*$	$\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_k^*))$ (kSm^3)	k
4.5.1	$(2.28, 2.0, 1.0)$	13,533.1	n.a.	n.a	n.a.
4.5.2	$(2.74, 1.56, 0.67, 0.82, 44.73, 1.0)$	35,737.3	$\{5\}$, $\{1,2,3,4,6\}$ $(2.75, 1.56, 0.67, 0.82, 1.0, 1.0)$	35,737.3	2
4.5.3	$(0.047, 1.141, 0.036, 0.028, 16.34, 12.28, 1.41, 1.0)$	31,856.2	$\{5\}, \{6\}, \{7\}, \{8\}, \{1,2,3,4\}$ $(1.54, 1.48, 1.19, 1.0, 1.0, 1.0, 1.0, 1.0)$	32,230.3	5

Table 4.3: *The best first-order fixed-weight strategies, and the best higher-order fixed-weight strategies if the optimum is in the boundary of* $\partial(\mathcal{M}')$ *in the examples of the sections 4.5.1, 4.5.2 and 4.5.3.*

that we can ultimately select and recommend one of the alternatives. The purpose is not to give the obtained production strategy as an input for long-term production

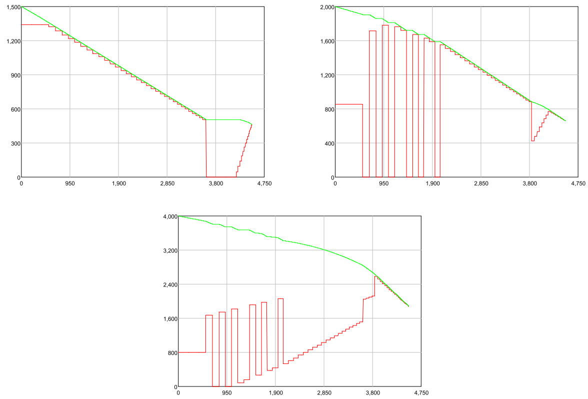

Figure 4.1: *The red graphs display the actual production rates when the backtracking algorithm proposed in Huseby & Haavardsson (2008) is used. The green graphs display the potential production rate functions.*

planning to a field manager. In a real production setting it would not be advisable to produce the reservoirs as prescribed by the backtracking algorithm, due to the oscillations of the individual production rates.

Figure 4.2 shows the production rates of this example when the proposed parametric class is used. The production rates from the proposed parametric class yield smooth, interpretable production rates that do not fluctuate. These production rates can also be used in decision support, as explained above. In addition the production rates can be used in long-term planning of the actual production of a field consisting of many reservoirs. In particular the production strategy can be used to assist production managers when they want to coordinate the production of many reservoirs. Furthermore, the proposed parametric class is better suited than the backtracking algorithm for feedback to the reservoir simulation team on possible modifications of the simulations. Hence, the proposed parametric class serves multiple purposes.

4.5.2 A case where backtracking fails

In the second example we consider an example where we are able to execute the first step but not the second step of the two step optimization algorithm developed in Huseby & Haavardsson (2008). Thus, we find the optimal state $\boldsymbol{Q}^*$ of the reservoirs at the end of the reservoirs, but the backtracking algorithm fails to propose an admissible production strategy to reach $\boldsymbol{Q}^*$. As explained in Huseby & Haavardsson (2008), this indicates that $\boldsymbol{Q}^* \notin \partial(\mathcal{M}')$, i.e. , $\boldsymbol{Q}^*$ cannot be reached by an admissible path. Alternatively, $\boldsymbol{Q}^*$ may be a point in $\partial(\partial(\mathcal{M}'))$ or a point very close to this set.

As in Section 4.5.1, the PPR-functions are given by (4.5.1). The chosen parameter

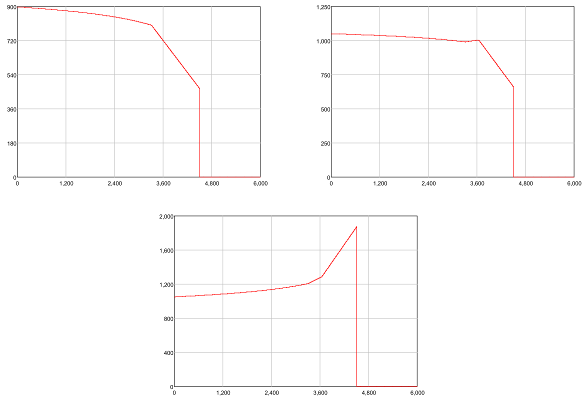

Figure 4.2: *The actual production rates when the proposed parametric class is used.*

values of the example are listed in Table 4.5. The objective function $\phi_{K,0}$ defined by letting $C = K$ and $R = 0$ in (4.2.5) is used. Thus, we use Lagrange multipliers to find $\boldsymbol{Q}^*$, which can be found using (4.5.2). As in Section 4.5.1 we are interested in comparing $\boldsymbol{Q}_L^*$ with the boundary point $\boldsymbol{Q}_{\boldsymbol{w}_1^*}^*$ obtained using Algorithm 4.4.1, so we calculate $\boldsymbol{b}^* \in \mathcal{B}_1^w$. We consider one example, and Table 4.1 lists the coordinates of $\boldsymbol{Q}_L^*$ and $\boldsymbol{Q}_{\boldsymbol{w}_1^*}^*$. Table 4.2 correspondingly lists the objective function values for $\boldsymbol{Q}_L^*$ and $\boldsymbol{Q}_{\boldsymbol{w}_1^*}^*$ and the Euclidian distance between these two points. In this case we see that the distance between $\boldsymbol{Q}_L^*$ and $\boldsymbol{Q}_{\boldsymbol{w}_1^*}^*$ is greater than in the previous example. Table 4.3 lists the weights of the best numerical first-order fixed-weight strategy $\boldsymbol{w}_1^*$.

As we can see from Table 4.3, the weight of reservoir 5, w_5^*, is significantly larger than the other weights. reservoirs. When we use Algorithm 4.4.1 to calculate the second-order fixed-weight strategy where reservoir 5 is given strict priority, denoted $\boldsymbol{w}_2^*$, we find that $\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_2^*)) = \phi_K(\boldsymbol{b}^*(\boldsymbol{w}_1^*))$, as can be read from Table 4.3. For all other higher-order fixed-weight strategies we obtain that $\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_k^*)) < \phi_K(\boldsymbol{b}^*(\boldsymbol{w}_1^*))$. Thus, the optimum is a boundary point of $\partial(\mathcal{M}')$ and Algorithm 4.4.1 managed to find it among the first-order fixed-weight strategies. This result is also consistent with the failure of the backtracking algorithm to find an admissible production strategy to reach $\boldsymbol{Q}_L^*$. Thus $\boldsymbol{Q}_L^*$ represents an inadmissible boundary point, i.e., $\boldsymbol{Q}_L^* \in \partial(\mathcal{M} \setminus \mathcal{M}')$.

4.5.3 A case where the optimal state is hard to find

In the final example neither steps of the two step optimization algorithm developed in Huseby & Haavardsson (2008) can be executed. The optimal state $\boldsymbol{Q}^*$ of the reservoirs at the end of the reservoirs is hard to find. Thus, the execution of the second step becomes difficult, since it assumes that Step 1 is done.

The theory in Huseby & Haavardsson (2008) puts restrictions on the PPR-functions of a specific field. In particular all the PPR-functions are assumed to be either convex or concave. In many examples the PPR-functions of some reservoirs are concave, and others are convex. Furthermore, the concave PPR-functions may be described by different classes of functions. Finally, some PPR-functions may be convex for some values of Q and concave for other values of Q. All these eventualities may be handled by the proposed parametric class.

Consider an examples where a field consists of 8 reservoirs. Four of the reservoirs may be described by linear PPR-functions, while the remaining four can be described by concave PPR-functions. For the first four reservoirs the PPR-functions are given by

$$f_i(Q_i(t)) = D_i(V_i - Q_i(t)),\ i = 1, \ldots, 4, \tag{4.5.3}$$

where $V_1, \ldots, V_4$ denote the recoverable volumes from the 4 reservoirs and D_i is the *scale parameter* of reservoir i. For the four remaining reservoirs the PPR-functions are given by (4.5.1). The chosen parameter values of the example are listed in Table 4.5.

The theory in Huseby & Haavardsson (2008) states that if all the PPR-functions are linear a specific $n-$th priority strategy is optimal with respect to a wide class of objective functions. In this case we may not apply this theory directly, since some of the PPR-functions are linear and other concave. The method of Lagrange multipliers may be used numerically or analytically if the optimum is an interior point of $\partial(\mathcal{M}')$. If the optimum is a point of the boundary of $\partial(\mathcal{M}')$, Lagrange multipliers may not be used.

As described in Section 4.4.2 we start out using Algorithm 4.4.1 to find the best numerical first-order fixed-weight strategy $\boldsymbol{w}_1^*$, which is displayed in Table 4.3. An inspection of $\boldsymbol{w}_1^*$ indicates that reservoir $5, 6, 7$ and 8 receive far higher priorities than the other reservoirs. This might indicate that the optimal production strategy reaches the boundary of $\partial(\mathcal{M}')$. Consequently we are interested in examining higher-order strategies, as described in Section 4.4.2. Examination among the $n - 1 = 8 - 1 = 7$ relevant partitions of second order strategies assigning the highest priority to reservoir 5 we find that the objective function ϕ_K indeed increases when we search among second order strategies. Carrying on we obtain improvements until we reach fifth order strategies denoting the resulting optimum candidate $\boldsymbol{w}_5^*$. Table 4.3 lists the weights of $\boldsymbol{w}_5^*$ and the corresponding ordered partition $\{A_j\}_j^5$. From Table 4.3 we see that ϕ_K is increased by 1.2% compared with $\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_1^*))$. In this example we did not obtain any further improvement in ϕ_K by searching among even higher order strategies, i.e. , among sixth, seventh and eighth order strategies. Thus we conclude that fifth order strategies represent the correct order and that $\boldsymbol{w}_5^*$ is optimal. To illustrate that further improvement could not be obtained by searching among higher order strategies we compare $\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_5^*))$ with the best eighth order priority strategy, denoted $\boldsymbol{\pi}^*$. This strategy is a strict priority strategy and we find that $\{A_j\}_j^8 = \{5\}, \{6\}, \{7\}, \{8\}, \{2\}, \{0\}, \{1\}, \{3\}$. Furthermore, we find that $\phi_K(\boldsymbol{b}^*(\boldsymbol{\pi}^*)) = 30,926$ kSm^3, so $\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_5^*))$ is 4.2 % larger than $\phi_K(\boldsymbol{b}^*(\boldsymbol{\pi}^*))$. Note also that the performance of $\boldsymbol{\pi}^*$ is significantly inferior to the performance of $\boldsymbol{w}_1^*$.

When the number of reservoirs is fairly small, say $n \leq 6$, we have seen that

Algorithm 4.4.1 manages to find good solutions among the first-order fixed weight production strategies even when the optimal $\boldsymbol{Q}^*$ belongs to the boundary of $\partial(\mathcal{M}')$. This occurred in the example of Section 4.5.2. However, this breaks down in higher dimensional examples, as demonstrated in the present example.

4.6 The modelling of uncertainty

4.6.1 Including uncertainty in the model

We will now describe how robustness and sensitivity analyzes of an optimal production strategy $\boldsymbol{b}^*$ can be carried out, where $\boldsymbol{b}^*$ is found using the approach explained in Section 4.4.2. The robustness and sensitivity analysis is typically run *before* any production starts. The purpose is to discover how vulnerable the optimal strategy is when exposed to uncertainty. If $\boldsymbol{b}^*$ is very vulnerable to uncertainty, perhaps a more robust production strategy should be selected.

In this paper we will add uncertainty to the framework using the approach developed in Haavardsson & Huseby (2007), where a Monte Carlo simulation approach is used in the stochastic simulation. Uncertainty is added to the production model by modelling some of the key parameters as stochastic variables. A large sample, N, of the key parameters is generated, and every simulated vector of key parameters produces one simulated production profile. Using this approach, we obtain a sample of N simulated production profiles. A Monte Carlo simulation of the production can be done using Algorithm 4.8.1 stated in Appendix 4.8.

If we were to add uncertainty to the framework utilizing the framework developed in Huseby & Haavardsson (2008), we would use the constructed two-step optimization algorithm, where the second step is solved using a backtracking algorithm. Then we would use the Monte Carlo approach described in Algorithm 4.8.1. A natural approach would be to apply the backtracking algorithm on a base case, i.e., a case that expressed the expected values of the stochastic variables. Denote the optimum for the base case $\boldsymbol{Q}^e \in \partial(\mathcal{M}')$. Then we would obtain an admissible production strategy $\boldsymbol{b}^e \in \mathcal{B}'$, i.e., an admissible path from $\boldsymbol{Q}^e$ back to $\mathbf{0}$, assuming that the second step involving backtracking may be successfully solved. A natural next step would be to use the Monte Carlo sampling technique described in Algorithm 4.8.1 to create N simulated objective functions.

The backtracking works in a deterministic model where all parameters are known. For every point in time we then know how to produce every reservoir, because the proportions between the different reservoirs are known and the backtracking algorithm has found an admissible path, based on these proportions. When uncertainty is added these proportions will be distorted, and we cannot be guaranteed that $\boldsymbol{b}^e \in \mathcal{B}'$, i.e., the production strategy that yielded an admissible path from $\boldsymbol{Q}^e$ back to $\mathbf{0}$ in the base case, produces an admissible path when uncertainty is added. In fact, it is not obvious at all how the production strategy found with a deterministic model should be interpreted when uncertainty is added.

Using the proposed parametric class of the present paper we obtain an admissible production strategy for every sample of stochastic recoverable volumes and start rates as specified in Algorithm 4.8.1, which is clearly very advantageous.

4.6.2 Robustness and sensitivity analysis

The algorithm below describes how the robustness and sensibility analysis is executed.

Algorithm 4.6.1.

STEP 1. *Use ordinary differential equations and multi-segmented models as explained in Appendix 4.8 and Haavardsson & Huseby (2007) to create a vector of PPR-functions* $\boldsymbol{f}(\boldsymbol{Q}(t))$.

STEP 2. *Use Algorithm 4.4.1 to find the production strategy* $\boldsymbol{b}^* \in \mathcal{B}_1^w$ *that maximizes* $\phi_{K,0} = \phi_K$ *numerically, where* $\phi_{K,0} = \phi_K$ *is defined in* (4.2.5).

STEP 3. *Use the Monte Carlo sampling technique described in Algorithm 4.8.1 to create* N *simulated objective functions* $\phi_K^j(\boldsymbol{b}^*) = \sum_{i=1}^n Q_i^j(T_K(\boldsymbol{b}^*)) = KT_K^j(\boldsymbol{b}^*), \quad j = 1, \ldots, N$.

Note that the vector $\boldsymbol{f}(\boldsymbol{Q}(t))$ in Step 1 is a vector of simplified production profile models, i.e., a curve fit of the vector of deterministic production models generated in the reservoir simulator.

We will assume that the recoverable volumes and the start rates of the reservoirs are stochastic. Since the start rates can be predicted with a high degree of certainty from e.g. well tests, we will assume that the uncertainty associated with the recoverable volumes is far greater than the uncertainty associated with the start rates. The sensitivity analysis will give us a variation in the plateau length T_K as a function of the decline rates, since $\phi_{K,0}(\boldsymbol{b}) = KT_K(\boldsymbol{b})$. If the variability in T_K is great compared to the gain in plateau volume we achieve by using the optimizing techniques, this is not so good. The expected gain obtained using the optimization should be considerable. If the variability of T_K using $\boldsymbol{b}^*$ is great compared to the variability using other selected production strategies, it is relevant to ask whether the more robust production strategies should be selected.

Inspired by the Sharpe ratio used in portfolio analysis, see Sharpe (1994), we will propose a measure to compare production strategies. The Sharpe ratio is a measure of the mean excess return per unit of risk in an investment asset or a trading strategy and is defined as:

$$S = \frac{E[R - R_f]}{\sigma} = \frac{E[R - R_f]}{\sqrt{\{\mathrm{Var}[R - R_f]\}}},$$

where R is the asset return, R_f is the return on a benchmark asset, such as the risk free rate of return, $E[R - R_f]$ is the expected value of the excess of the asset return over the benchmark return, and σ is the standard deviation of the excess return.

In our situation selected production strategies play the roles of the assets. We will compare the performance of the n-th order priority strategies, defined in Section 4.3.1, and the production strategy obtained using Algorithm 4.4.1. The symmetry production strategy, defined in Section 4.3, will be used as a benchmark production strategy. The production strategies will then be compared to the symmetry production strategy using our version of the Sharpe ratio, referred to as the *performance ratio*:

$$P_i = \frac{E[\phi_K(\boldsymbol{b}^i) - \phi_K(\boldsymbol{b}^s)]}{\sqrt{\{\text{Var}[\phi_K(\boldsymbol{b}^i) - \phi_K(\boldsymbol{b}^s)]\}}}, \tag{4.6.1}$$

where $\phi_K(\boldsymbol{b}^i)$ is the value of the objective function using production strategy i for a selection of production strategies $i = 1, \ldots, M$, where $M = n! + 1$ in this paper. The moments $E\{\phi_K(\boldsymbol{b}^i)\}$ and $\sqrt{\{\text{Var}[\phi_K(\boldsymbol{b}^i) - \phi_K(\boldsymbol{b}^s)]\}} = \sqrt{\{\text{Var}[\phi_K(\boldsymbol{b}^i) - \phi_K(\boldsymbol{b}^M)]\}}$ are estimated empirically using the simulations. The optimal production strategy should ideally come out best most frequently in the simulations. Thus we compare the frequency at which each production strategy is best-performing during all the simulations.

The uncertainties used in the example

The framework described above will now be demonstrated in an example. Table 4.4 displays P10 and P90, i.e. the 10 percentile and the 90 percentile in the distributions of the stochastic producible volumes of the examples.

	Reservoir 1		Reservoir 2		Reservoir 3		Reservoir 4	
Example	P10	P90	P10	P90	P10	P90	P10	P90
Section 4.6.2	84 %	117 %	81 %	121 %	74 %	128 %	80 %	121 %

Table 4.4: *The P10 and P90 of the stochastic distributions of the producible volumes in the example. 100 % refers to the expected value, which is selected to be the deterministic producible volume.*

An example with concave PPR-functions

We consider an example with four reservoirs, where the *multi-segmented* PPR-functions $\{f_1, f_2, f_3, f_4\}$ are given by

$$f_{i,j}(Q_i(t)) = D_{i,j}(V_{i,j} - (Q_i(t) - \sum_{k<j} V_{i,k})), \quad i = 1,2,3,4 \quad j = 1,2,3 \tag{4.6.2}$$

where $D_{i,j}$ denotes the *scale parameter* of the j-th segment of the i-th reservoir. We assume that $D_{i,1} \leq D_{i,2} \leq D_{i,3}$ for $i = 1,2,3,4$. For an introduction to multi-segmented production profiles, see Appendix 4.8 or Haavardsson & Huseby (2007). The parameter values of the PPR-functions are given in Table 4.6. We let $K = 5.0$ kSm^3 per day.

We then proceed to Step 2 of Algorithm 4.6.1, where we use Algorithm 4.4.1 to find the candidate $\boldsymbol{b}^* \in \mathcal{B}_1^w$ that the numerical algorithm proposes as the best production strategy. Thus we obtain that $\boldsymbol{w}_1^* = (1.706, 0.575, 0.399, 1.0)$. Finally, we perform Step 3 of Algorithm 4.6.1 by simulating $N = 5,000$ objective functions for M selected production strategies. We consider the n-th order priority strategies and the strategy $\boldsymbol{b}^* = \boldsymbol{b}^*(\boldsymbol{w}_1^*)$, so we let $M = n! + 1 = 4! + 1 = 24 + 1 = 25$. The performance ratio P_i defined by (4.6.1) may now be estimated for the 25 strategies.

Figure 4.3 shows results from the simulations. In the upper panel we see the frequency at which every production strategy is best-performing, indicated with the columns in the graph. The frequencies can be read from the left axis in the graph. The blue curve in the left panel shows the performance ratio P_i of each production strategy, which can be read from the right axis of the graph in the same panel. The red curve, also relating to the right axis of the graph in the left panel, shows the performance of each production strategy in the deterministic case relative to $\phi_K(\boldsymbol{b}^*(\boldsymbol{w}_1^*))$ in the deterministic case. We see that $\boldsymbol{b}^*$ is optimal in the deterministic case and remains the best-performing also when uncertainty is introduced in this example. The best-performing frequency of each production strategy is reconcilable with the magnitude of each performance ratio P_i.

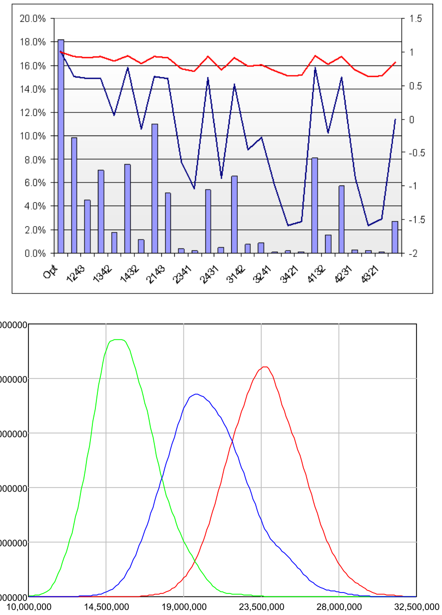

Figure 4.3: *The upper panel shows every production strategy's best-performance frequency, performance ratio and* ϕ_K *in the deterministic case relative to* $\phi_K(\boldsymbol{b}^*)$*. The lower panel shows the estimated densities of the plateau production* ϕ_K *of the strategy that is optimal in the deterministic situation (red graph), the symmetry strategy (blue graph), and the n-th order priority strategy* $\boldsymbol{\pi} = \{3, 4, 1, 2\}$ *(green graph).*

4.7 Conclusions

In the present paper we have introduced a parametric class of admissible production strategies referred to as *fixed-weight strategies*. Such strategies are stable, robust solutions that are easy to interpret. Thus, the production rates can be used in long-term planning of the actual production of a field consisting of many reservoirs. Compared to the strategies obtained using the two-step algorithm proposed in Huseby & Haavardsson (2008), fixed-weight strategies are also better suited for feedback to the reservoir simulation team on possible modifications of the simulations.

In cases where the first step of the algorithm proposed in Huseby & Haavardsson (2008) can be handled analytically, this method is extremely fast having a simulation time which grows linearly in the number of reservoirs. Compared to this, finding the optimal fixed- weight strategy is not as numerically efficient. Since, however, the number of parameters needed to define a fixed-weight strategy, is bounded by the number of reservoirs, complex, high-dimensional examples can easily be handled.

Hence, the efficiency of this method is sufficient for most applications.

We have also demonstrated how uncertainty can be added into the proposed framework. This enables robustness and sensitivity studies of different production strategies. The performance criterion gives an indication of how robust every strategy is when exposed to uncertainty.

Under mild restrictions on the objective function it can be shown that an optimal production strategy can be found within the class of fixed- weight strategies. We will return to this important issue in a forthcoming paper.

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4.8 A brief introduction to multi-segmented production profiles using ordinary differential equations

Single Arps curves, introduced by Arps (1945) model the production rate function and the cumulative production function mathematically through a one-way, causal relation. In Haavardsson & Huseby (2007) this approach is extended to multiple segments so that a combination of Arps curves may be used to get a satisfactory fit to a specific set of production data.

To also take into account various production delays, the dynamic two-way relation between the production rate function and the cumulative production is modelled in terms of a differential equation. The relation between the production rate function, q, and the cumulative production function, Q, should be of the following form:

$$q(t) = f(Q(t)), \qquad \text{for all } t \geq 0, \tag{4.8.1}$$

with $Q(t_0) = 0$ as a boundary condition.

The differential equation approach can also be extended to the more general situation where the production rate function consists of s segments. For each segment we assume that we have fitted a model in terms of a differential equation on the form given in (4.8.1). In order to connect these segment models, we need to specify a *switching rule* describing when to switch from one segment model to the next one. We define a switching rule based on the produced volume. By using this switching rule, we obtain a model for the combined differential equation.

[2]Mr. Wickmann graduated from University of Oslo in 2007.

A Monte Carlo simulation of the production can be done using the following procedure:

Algorithm 4.8.1.

STEP 1. *Assume that a production profile is divided in* s *segments. Generate* $V_1, \ldots, V_s$ *using the specified joint distribution* $p(V_1, \ldots, V_s)$ *for* $V_1, \ldots, V_s$*, where* V_i *denotes the producible volume of segment* i, $\quad i = 1, \ldots, s$.

STEP 2. *Generate* $r_0, r_1, \ldots, r_s$ *using the specified conditional joint distribution.* $p(r_0, r_1, \ldots, r_s | V_1, \ldots, V_s)$ *for the rates at the segmentation points* $r_0, r_1, \ldots, r_s$*, given the segment volumes.*

STEP 3. *Calculate* $D_1, \ldots, D_s$.

STEP 4. *Generate* t_0*, which may be subject to uncertainty related to the progress of the development project, drilling activities etc. Thus, one will typically assess a separate uncertainty distribution for this quantity. Then calculate* $t_1, \ldots, t_s$.

STEP 5. *Calculate* $q(t)$ *and* $Q(t)$.

4.9 Descriptions of reservoirs used in examples

Example	Reservoir	Producible volume V_i (MSm^3)	Scale parameter D_i	Max rate $\sqrt{D_i V_i}$ (kSm^3/d)
Section 4.5.1	1	4.0	0.28	1.5
	2	5.0	0.40	2.0
	3	7.0	1.14	4.0
Section 4.5.2	1	4.0	0.28	1.5
	2	5.0	0.62	2.5
	3	7.0	2.57	6.0
	4	6.0	1.69	4.5
	5	8.0	0.39	2.5
	6	9.0	1.99	6.0
Section 4.5.3	1	4.0	0.0781	4.6
	2	4.0	0.0782	4.8
	3	4.4	0.0776	6.0
	4	4.4	0.0892	7.0
	5	4.0	0.0311	0.8
	6	5.0	0.0512	2.5
	7	3.0	0.0731	3.0
	8	7.0	0.0913	12.5

Table 4.5: *Parameter values for the examples in Section 4.5.*

		Segment 1		Segment 2		Segment 3		
Reservoir	Total reserves (kSm^3)	Producible volume (kSm^3)	Start rate (kSm^3)	Producible volume (kSm^3)	Start rate (kSm^3)	Producible volume (kSm^3)	Start rate (kSm^3)	Stop rate (kSm^3)
1	10,000	7,000	3.0	1,800	1.9	1,200	1.3	0.01
2	6,000	3,600	2.6	1,320	2.1	1,080	1.1	0.01
3	7,000	5,460	5.0	1,001	3.1	539	2.2	0.02
4	4,000	3,080	3.0	570	1.9	350	1.3	0.01

Table 4.6: *Parameter values for the four reservoirs used in Section 4.6.2.*

Chapter 5

Hydrocarbon production optimization in fields with different ownership and commercial interests

Abstract

A main field and satellite fields consist of several separate reservoirs with gas cap and/or oil rim. A processing facility on the main field receives and processes the oil, gas and water from all the reservoirs. This facility is typically capable of processing only a limited amount of oil, gas and water per unit of time. In order to satisfy these processing limitations, the production needs to be choked. The available capacity is shared among several field owners with different commercial interests. In this paper we focus on how total oil and gas production from all the fields could be optimized. The satellite field owners negotiate processing capacities on the main field facility. This introduces additional processing capacity constraints (booking constraints) for the owners of the main field. If the total wealth created by all owners represents the economic interests of the community, it is of interest to investigate whether the total wealth may be increased by lifting the booking constraints. If all reservoirs may be produced more optimally by removing the booking constraints, all owners may benefit from this when appropriate commercial arrangements are in place. We will compare two production strategies. The first production strategy optimizes locally, at distinct time intervals. At given intervals the production is prioritized so that the maximum amount of oil is produced. In the second production strategy a fixed weight is assigned to each reservoir. The reservoirs with the highest weights receive the highest priority.

Keywords

Production profile models, Total value chain analysis, Two-phase production optimization, Numerical optimization methods, Conjugate Gradient Method, Nelder-Mead Method, Risk Analysis

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5.1 Introduction

Optimization is an important element in the management of multiple-field oil and gas assets, since many investment decisions are irreversible and finance is committed for the long term. Optimization of oil and gas recovery in petroleum engineering is a considerable research field, see Bittencourt & Horne (1997), Horne (2002) or Merabet & Bellah (2002). Another important research tradition focuses on the problem of modelling the entire hydrocarbon value chain, where the purpose is to make models for scheduling and planning of hydrocarbon field infrastructures with complex objectives, see van den Heever et al. (2001), Ivyer & Grossmann (1998) or Neiro & Pinto (2004). Since the entire value chain is very complex, many aspects of it needs to be simplified to be able to construct such a comprehensive model.

The purpose of the present paper is to focus on the problem of optimizing production in an oil or gas field consisting of many reservoirs, which constitutes an important component in the hydrocarbon value chain. By focusing on only one important component we are able to develop a framework that provides insight into how an oil or gas field should be produced. The optimization methods developed here can thus be used in the broader context of a total value chain analysis. The present paper applies an already developed model framework for hydrocarbon production optimization of an oil and gas field development project. More specifically, the methodology developed in Haavardsson & Huseby (2007), Huseby & Haavardsson (2008) and Haavardsson et al. (2008) will be used.

We assume that state-of-the-art production profile models based on reservoir simulation models exist for every reservoir. Simplified *production profile models* can then be constructed, as described in Haavardsson & Huseby (2007). In the present paper we will utilize such production profile models in production optimization where several reservoirs share the same processing facilities. These facilities are only capable of processing limited amounts of oil, gas and water per unit of time. In order to satisfy these processing limitations, the production needs to be choked according to a production strategy. Each reservoir produces a primary hydrocarbon phase - oil or gas. In addition to the primary phases, most reservoirs also produce associated phases; gas in oil reservoirs, condensate in gas reservoirs and water.

Huseby & Haavardsson (2008) is a theoretical paper, where the problem of optimizing production strategies with respect to various types of objective functions is considered. It is shown that the solution to the optimization problem depends on certain key properties, e.g., convexity or concavity, of the objective function and of the potential production rate functions. An algorithm for finding the best production strategy and two main analytical results are presented.

Haavardsson et al. (2008) focuses on applied multi-reservoir production optimization, and an alternative approach to production optimization is proposed. By introducing a parametric class of production strategies the best production strategy is found using standard numerical optimization techniques.

We close this section listing the main interests of the present paper:

- The main focus of the paper is the modelling approach and the basic principles for a modelling tool for general use in examination of production strategy effects on multi-reservoir fields, with different and varying hydrocarbon phases, with individual production constraints and priorities, different owners and with the functionality to extend and cover multi fields integration in a regional / processing hub evaluation.

- The article also highlights the importance of being aware of local and global production optimization effects and the importance booking constraints may have. To study this two different production strategies are presented.
- As an illustration a case study based on real data[1] will be presented. Thus, the case study serves as a tool for the investigation of the general issues listed above. We seek knowledge that is valid beyond the numerical results obtained in the case study.

5.2 Model framework

5.2.1 Production profile model framework

The reservoir simulation output available from the reservoir simulator Eclipse [2] is used to construct simplified production profile models for each well. See Appendix 5.6 for a broad-brush introduction to simplified production profile models, or Haavardsson & Huseby (2007) for details.

To model multiple phases of production we assume that the production of each associated hydrocarbon phase can be expressed as a function of the cumulative production of the primary hydrocarbon phase. If the primary hydrocarbon phase is oil, we denote the cumulative production $Q(t)$, while $G(t)$ is used analogously for gas.

A fundamental model assumption is that the *potential production rate* of the primary hydrocarbon phase from a reservoir, can be expressed as a function of the remaining producible volume, or equivalently as a function of the volume produced. Thus, if $Q(t)$ denotes the cumulative production of the primary hydrocarbon phase at time $t \geq 0$, and $f(t)$ denotes the potential production rate at the same point in time, we assume that $f(t) = f(Q(t))$. This assumption implies that the total producible volume from a reservoir does not depend on the production schedule. In particular, if we delay the production from a reservoir we can still produce the same volume at a later time. We refer to the function f as the *potential production rate function* or *PPR-function* of the reservoir. If a reservoir is produced without any production constraints from time $t = 0$, the cumulative production function will satisfy the following autonomous differential equation:

$$\frac{dQ(t)}{dt} = f(Q(t)),$$

[1] In the present paper the case study is un-named and the data are made anonymous to reduce the ability to derive commercial values.

[2] For details on Schlumberger's Eclipse Reservoir Engineering Software, see `www.slb.com`.

with the boundary condition $Q(0) = 0$.

A single production well

We assume that we are given a ratio expressing the units of the associated hydrocarbon phase that is produced depending on the units produced of the primary hydrocarbon phase. We refer to this function as the *associated ratio* and denote it $\psi(Q(t))$ or $\gamma(G(t))$ depending on whether oil or gas is the primary hydrocarbon phase. Although we can handle any finite number of associated phases we will assume in this paper that there is only one associated phase. Thus, we are not concerned with water production in this application. If the primary hydrocarbon phase is oil, the associated ratio expresses the Gas-Oil-Ratio (GOR). If the primary hydrocarbon phase is gas, the associated ratio expresses the Condensate-Gas-Ratio (CGR).

To model $\psi(Q(t))$ we choose to use the following representation

$$\psi(Q(t)) = \psi(0) + (\psi(V) - \psi(0)) \cdot R(t)^{P_\psi}, \tag{5.2.1}$$

where $R(t) = \frac{Q(t)}{V}$ denotes the fraction produced, $R(t) \in [0, 1]$, where V denotes producible volume of the primary hydrocarbon phase. The parameter P_ψ is assumed to be positive. The parameters $\psi(0), \psi(V)$ and P_ψ are estimated using the output from the reservoir simulator. Typically $\psi(Q(t))$ is increasing in $Q(t)$, reflecting the increasing quantity of gas produced per unit produced oil as the reservoir is produced.

For $\gamma(G(t))$ we use the same representation, i.e.,

$$\gamma(G(t)) = \gamma(0) + (\gamma(V) - \gamma(0)) \cdot R(t)^{P_\gamma}, \tag{5.2.2}$$

where $P_\gamma > 0$. Typically $\gamma(G(t))$ is decreasing in $G(t)$, so that typically $\gamma(0) > \gamma(V)$. This reflects the decreasing quantity of condensate produced per unit produced gas as the reservoir is produced. Furthermore, we will typically choose $P_\gamma < 1$.

Multiple production wells

We consider oil and gas production from N wells that share a processing facility with a constant oil processing capacity K_o and a constant gas capacity K_g.

Let $\boldsymbol{I} = (I_1, \ldots, I_N)$ be the vector expressing the type of primary hydrocarbon phase of each well, so that

$$I_i = \begin{cases} 1 & \text{if the primary hydrocarbon phase of well } i \text{ is oil,} \\ 0 & \text{if the primary hydrocarbon phase of well } i \text{ is gas,} \end{cases} \tag{5.2.3}$$

for $i = 1, \ldots, N$. Let $\mathcal{O} = \{i \mid I_i = 1\}$ and $\mathcal{G} = \{i \mid I_i = 0\}$, so that $\mathcal{O}$ contains the indices of the oil wells and $\mathcal{G}$ contains the indices of the gas wells.

We introduce

$$P_i(t) = \begin{cases} Q_i(t) & \text{if } i \in \mathcal{O}, \\ G_i(t) & \text{if } i \in \mathcal{G}. \end{cases} \tag{5.2.4}$$

and assume that the PPR-functions can be written as

$$f_i(t) = \begin{cases} f_i(Q_i(t)) & \text{if } i \in \mathcal{O}, \\ f_i(G_i(t)) & \text{if } i \in \mathcal{G}, \end{cases} \tag{5.2.5}$$

for $i = 1, \ldots, N$. Then $\boldsymbol{P}(t) = (P_1(t), \ldots, P_N(t))$ represents the vector of cumulative primary hydrocarbon phase production functions for the N wells, and $\boldsymbol{f}(t) = (f_1(t), \ldots, f_N(t))$ the corresponding vector of PPR-functions. Thus, f_i represents the PPR-function of well i. Note that the formulation (5.2.5) implies that the potential production rate of one well does not depend on the volumes produced from the other wells.

A *production strategy* is defined by a vector valued function $\boldsymbol{b} = \boldsymbol{b}(t) = (b_1(t), \ldots, b_N(t))$, defined for all $t \geq 0$, where $b_i(t)$ represents the *choke factor* applied to the ith well at time t, $i = 1, \ldots, N$. We refer to the individual b_i-functions as the *choke factor functions* of the production strategy. The *actual oil production rates* from the wells, after the production is choked is given by:

$$\boldsymbol{q}(t) = (q_1(t), \ldots, q_N(t)),$$

where

$$q_i(t) = \begin{cases} b_i(t) f_i(Q_i(t)) & \text{if } i \in \mathcal{O}, \\ b_i(t)\gamma_i(G_i(t)) f_i(G_i(t)) & \text{if } i \in \mathcal{G}, \end{cases} \tag{5.2.6}$$

so that $q_i(t)$ either expresses the actual oil rate from an oil well or the actual condensate rate from a gas well. The *actual gas production rates* from the wells are similarly denoted

$$\boldsymbol{g}(t) = (g_1(t), \ldots, g_N(t)),$$

where

$$g_i(t) = \begin{cases} b_i(t) f_i(G_i(t)) & \text{if } i \in \mathcal{G}, \\ b_i(t)\psi_i(Q_i(t)) f_i(Q_i(t)) & \text{if } i \in \mathcal{O}, \end{cases} \tag{5.2.7}$$

so that $g_i(t)$ either expresses the actual gas rate from a gas well or the actual associated gas rate from an oil well.

We also introduce the total oil production rate function $q(t) = \sum_{i=1}^{N} q_i(t)$ and the total cumulative oil production function $Q(t) = \sum_{i=1}^{N} Q_i(t)$. The total gas production rate function is analogously denoted $g(t) = \sum_{i=1}^{N} g_i(t)$, while the total cumulative gas production function is denoted $G(t) = \sum_{i=1}^{N} G_i(t)$.

To satisfy the physical constraints of the wells and the processing facility, we require that for a hydrocarbon phase, the actual well production rate cannot exceed its potential production well rate. Moreover, the total well production rate cannot exceed the production capacity. These requirements imply that

$$\begin{aligned} 0 &\leq q_i(t) \leq f_i(Q_i(t)), & t \geq 0, \quad i \in \mathcal{O}, \\ 0 &\leq q_i(t) \leq \gamma_i(G_i(t)) f_i(G_i(t)), & t \geq 0, \quad i \in \mathcal{G}, \\ 0 &\leq g_i(t) \leq \psi_i(Q_i(t)) f_i(Q_i(t)), & t \geq 0, \quad i \in \mathcal{O}, \\ 0 &\leq g_i(t) \leq f_i(G_i(t)), & t \geq 0, \quad i \in \mathcal{G}, \end{aligned} \tag{5.2.8}$$

for $i = 1, \dots, N$ and that

$$\begin{aligned} q(t) &= \sum_{i=1}^{N} q_i(t) \le K_o, \quad t \ge 0, \\ g(t) &= \sum_{i=1}^{N} g_i(t) \le K_g, \quad t \ge 0. \end{aligned} \tag{5.2.9}$$

Expressed in terms of the production strategy $\boldsymbol{b}$, this implies that

$$0 \le b_i(t) \le 1, \quad t \ge 0, \quad i = 1, \dots, N, \tag{5.2.10}$$

and that

$$\begin{aligned} &\sum_{i \in \mathcal{O}} b_i(t) f_i(Q_i(t)) + \sum_{i \in \mathcal{G}} b_i(t) \gamma_i(G_i(t)) f_i(G_i(t)) \le K_o, \quad t \ge 0, \\ &\sum_{i \in \mathcal{O}} b_i(t) \psi_i(Q_i(t)) f_i(Q_i(t)) + \sum_{i \in \mathcal{G}} b_i(t) f_i(G_i(t)) \le K_g, \quad t \ge 0. \end{aligned} \tag{5.2.11}$$

The constraint (5.2.10) implies that the actual production rate cannot be increased beyond the potential production rate at any given point in time, while the constraint (5.2.11) states that the actual, total production rates cannot exceed the capacities of the processing facility. Let $\mathcal{B}$ denote the class of production strategies that satisfy the physical constraints (5.2.10) and (5.2.11). We refer to production strategies $\boldsymbol{b} \in \mathcal{B}$ as *valid production strategies.*

We need to specify how the choke factors are determined. In this paper we will determine the choke factors sequentially. A sequential approach only produces one of the phases at the plateau level. First the choke factors are determined so that the constraint of the primary hydrocarbon phase is not exceeded. Then, if the constraint of the associated hydrocarbon phase is exceeded the choke factors are modified accordingly.

Definition 5.2.1. *We say that* $\boldsymbol{x} > \boldsymbol{y}$ *if* $x_i \ge y_i \;\; \forall i$ *and* $\exists\, j \in \{1, \dots, n\}$ *such that* $x_j > y_j$. *Let* $\boldsymbol{b}, \boldsymbol{b}' \in \mathcal{B}$ *be two production strategies. If* $\boldsymbol{b}'(t) > \boldsymbol{b}(t)$ *implies that either*

$$\sum_{i \in \mathcal{O}} b_i'(t) f_i(Q_i(t)) + \sum_{i \in \mathcal{G}} b_i'(t) \gamma_i(G_i(t)) f_i(G_i(t)) > K_o$$

or

$$\sum_{i \in \mathcal{O}} b_i'(t) \psi_i(Q_i(t)) f_i(Q_i(t)) + \sum_{i \in \mathcal{G}} b_i'(t) f_i(G_i(t)) > K_g,$$

then $\boldsymbol{b}$ *is an admissible production strategy. We denote the class of admissible production strategies* $\mathcal{B}'$.

5.2.2 Production strategies and objective functions

Strategy for local production optimization

Consider a production strategy that optimizes production locally at predefined, discrete time intervals. The production is prioritized so that the oil production is accelerated following an argument that the oil has the highest value and accelerated production would be beneficial from a net present value perspective if disregarding other potentially overriding effects as e.g. gas capacity utilization and oil price assumptions. This is obtained by assigning the highest priority to the well with the highest oil to gas production ratio, meaning first sorting the oil wells after the GOR in ascending order followed by the gas wells sorted after CGR in descending order. The production at time t is thus prioritized strictly according to $\boldsymbol{\pi} = (\pi(1), \dots, \pi(N))$. At the next decision point, i.e., at time $t + \delta$, the procedure is repeated.

To be a bit more precise we start by dividing a finite time horizon $[0, T]$ into S intervals. Thus we obtain a partition $[0, \delta, 2\delta, \dots, (S-1)\delta, T]$, where $\delta = T/S$. Let ϕ^l denote the objective function of the local production strategy. At time $t = 0$ ϕ^l is initialized so that $\phi^l = 0$. At time t the following algorithm is used:

Algorithm 5.2.2. STEP 1. *Sort the wells by any predefined order given by commercial agreements or other priorities. Sort the remaining wells by first sorting the oil wells after GOR in ascending order followed by the gas wells sorted after CGR in descending order. Denote the resulting permutation vector* $\boldsymbol{\pi}$.

STEP 2. *Find the number of producing wells* $i_c = 1 + min(i_q, i_g)$ *where* i_q *and* i_g *are the largest integers that fulfill*

$$\begin{aligned} \sum_{j \le i_q, \pi(j) \in \mathcal{O}} f_{\pi(j)}(Q_{\pi(j)}(t)) + \sum_{j \le i_q, \pi(j) \in \mathcal{G}} \gamma_{\pi(j)}(G_{\pi(j)}(t)) f_{\pi(j)}(G_{\pi(j)}(t)) &\le K_o \\ \sum_{j \le i_g, \pi(j) \in \mathcal{G}} f_{\pi(j)}(G_{\pi(j)}(t)) + \sum_{j \le i_g, \pi(j) \in \mathcal{O}} \psi_{\pi(j)}(Q_{\pi(j)}(t)) f_{\pi(j)}(Q_{\pi(j)}(t)) &\le K_g \end{aligned}$$

Note that if $min(i_q, i_g) = N$ *choking is not necessary. We let* $\boldsymbol{b} = \boldsymbol{1}$ *in this case. If* $min(i_q, i_g) < N$ *the (time-dependent) choke factors are given as*

$$b_{\pi(i)} = \begin{cases} 1, & i < i_c, \\ b_c, & i = i_c, \\ 0, & i > i_c, \end{cases} \tag{5.2.12}$$

where $b_c = min(b_q, b_g)$ *and*

$$b_q = \begin{cases} \frac{K_o - \sum_{j < i_c} q_{\pi(j)}(t)}{f_{\pi(i_c)}(Q_{\pi(i_c)}(t))}, & \pi(i_c) \in \mathcal{O}, \\ \frac{K_o - \sum_{j < i_c} q_{\pi(j)}(t)}{\gamma_{\pi(i_c)}(G_{\pi(i_c)}(t)) f_{\pi(i_c)}(G_{\pi(i_c)}(t))}, & \pi(i_c) \in \mathcal{G}, \end{cases} \tag{5.2.13}$$

and

$$b_g = \begin{cases} \frac{K_g - \sum_{j<i_c} g_{\pi(j)}(t)}{\psi_{\pi(i_c)}(Q_{\pi(i_c)}(t)) f_{\pi(i_c)}(Q_{\pi(i_c)}(t))}, & \pi(i_c) \in \mathcal{O}, \\ \frac{K_g - \sum_{j<i_c} g_{\pi(j)}(t)}{f_{\pi(i_c)}(G_{\pi(i_c)}(t))}, & \pi(i_c) \in \mathcal{G}. \end{cases} \quad (5.2.14)$$

STEP 3. *Update ϕ^l, so that*

$$\phi^l = \phi^l + \int_t^{t+\delta} \sum_{i=1}^{N} \{q_i(u) + \alpha g_i(u)\} e^{-ru} du.$$

Algorithm 5.2.2 is repeated at every grid point in the partition $[0, \delta, 2\delta, \dots, (S-1)\delta]$. The parameter α converts a unit of gas into an oil unit equivalent. Thus, we are capable of comparing the energy amount in gas versus oil. In this paper we use $\alpha = 0.001$, as stated by the Norwegian Petroleum Directorate [3].

Due to the nature of the local production strategy, the production rates of some of the individual wells might fluctuate in periods. The fluctuation occurs when it is equally beneficial to produce from two or more wells, so that when the wells compete for capacities they will alternate between being produced in one period and choked the next. The primary purpose of the local production strategy is to give decision support to project teams. The focus can for example be the assessment of different infrastructure investment alternatives. Hence, we are interested in the resulting cash flows of these different alternatives so that we can ultimately select and recommend one of the alternatives. The purpose is not to give the obtained production strategy as an input for long-term production planning to a field manager. In the case of fluctuating production it would not be advisable to produce the wells exactly as prescribed by the local production strategy.

Strategy for fixed-weight production optimization

The following production strategy is introduced in Haavardsson et al. (2008). Consider the set

$$\mathcal{Q} = [0, V_1] \times \cdots \times [0, V_N], \quad (5.2.15)$$

where $V_1, \dots, V_N$ are the recoverable volumes of the primary hydrocarbon phase from the N reservoirs. We then introduce the subset $\mathcal{M}_o \subseteq \mathcal{Q}$ given by:

$$\mathcal{M}_o = \{\boldsymbol{Q} \in \mathcal{Q} : \sum_{i \in \mathcal{O}} f_i(Q_i(t)) + \sum_{i \in \mathcal{G}} \gamma_i(G_i(t)) f_i(G_i(t)) \geq K_o\}, \quad (5.2.16)$$

so that $\mathcal{M}_o$ the points in $\mathcal{Q}$ where the oil production rate can be sustained at its plateau level. Furthermore we introduce the *oil plateau length* defined as

$$T_{K,o} = T_{K,o}(\boldsymbol{b}) = \sup\{t \geq 0 : \sum_{i \in \mathcal{O}} f_i(Q_i(t)) + \sum_{i \in \mathcal{G}} \gamma_i(G_i(t)) f_i(G_i(t)) \geq K_o\}. \quad (5.2.17)$$

[3] `http://www.npd.no/English/Om+OD/Nyttig/Olje-ABC/maaleenheter_oljeoggass.htm`

First we explain intuitively how the fixed-weight strategy was constructed in single-phase production, for the moment neglecting the gas constraint expressed in (5.2.11). Then we will explain how the fixed-weight strategy can be modified to handle two-phase production.

A simple production strategy can always be constructed using the same choke factor for all the reservoirs. That is, we let $b_i(t) = c(t)$, $i = 1, \ldots, N$. For such a production strategy to be admissible $c(t)$ must satisfy the following:

$$\sum_{i\in\mathcal{O}} c(t)f_i(Q_i(t))+\sum_{i\in\mathcal{G}} c(t)a_i(G_i(t))f_i(G_i(t)) = \min\{K_o, \sum_{i\in\mathcal{O}} f_i(Q_i(t))+\sum_{i\in\mathcal{G}} a_i(G_i(t))f_i(G_i(t))\}. \tag{5.2.18}$$

Thus, for $0 \le t \le T_{K,o}$, we have:

$$c(t) = \frac{K}{\sum_{i\in\mathcal{O}} f_i(Q_i(t)) + \sum_{i\in\mathcal{G}} \gamma_i(G_i(t))f_i(G_i(t))}, \tag{5.2.19}$$

while $c(t) = 1$ for all $t > T_{K,o}$, neglecting for the moment the gas constraint expressed in (5.2.11). Note that since $\sum_{i\in\mathcal{O}} c(t)f_i(Q_i(t)) + \sum_{i\in\mathcal{G}} c(t)a_i(G_i(t))f_i(G_i(t)) \ge K_o$ for $0 \le t \le T_{K,o}$, the common choke factor, $c(t)$ will always be less than or equal to 1. A production strategy defined in this way, will be referred to as a *symmetry strategy*. We observe that when a symmetry strategy is used, the available production capacity is shared proportionally among the reservoirs such that none of the reservoirs are given any kind of priority. The idea now is to expand this class so that some reservoirs can be prioritized before others. To facilitate this we start out by considering production strategies where for $0 \le t \le T_{K,o}$ the choke factors are given by:

$$b_i(t) = w_i c(t), \qquad i = 1, \ldots, N, \tag{5.2.20}$$

where $w_1, \ldots, w_N$ are positive real numbers representing the relative priorities assigned to the N reservoirs, and where $c(t)$ is chosen so that the strategy is admissible. For $t > T_{K,o}$, we define $b_i(t) = 1$, $i = 1, \ldots, N$. Note that if $w_1 = \cdots = w_N$ we get a symmetry strategy.

In order to ensure admissibility, $c(t)$ must be chosen so that:

$$\begin{aligned} &\sum_{i\in\mathcal{O}} w_i c(t)f_i(Q_i(t)) + \sum_{i\in\mathcal{G}} w_i c(t)\gamma_i(G_i(t))f_i(G_i(t)) \\ = \;& \min\{K, \sum_{i\in\mathcal{O}} f_i(Q_i(t)) + \sum_{i\in\mathcal{G}} \gamma_i(G_i(t))f_i(G_i(t))\}. \end{aligned}$$

Thus, for $0 \le t \le T_{K,o}$ the choke factors are given by:

$$b_i(t) = w_i c(t) = \frac{w_i K}{\sum_{j\in\mathcal{O}} w_j f_j(Q_j(t)) + \sum_{j\in\mathcal{G}} w_j\gamma_j(G_j(t))f_j(G_j(t))}, \qquad i = 1, \ldots, N. \tag{5.2.21}$$

Unfortunately, this definition does not guarantee that the choke factors are less than or equal to 1. To fix this problem, we instead let:

$$b_i(t) = \min\{1, w_i c(t)\}, \qquad i = 1, \ldots, N. \tag{5.2.22}$$

While this ensures that the resulting production strategy is valid, it makes the calculation of $c(t)$ slightly more complicated. To ensure admissibility, $c(t)$ must now be chosen so that:

$$\begin{aligned} &\sum_{i \in \mathcal{O}} \min\{1, w_i c(t)\} f_i(Q_i(t)) + \sum_{i \in \mathcal{G}} \min\{1, w_i c(t)\} \gamma_i(G_i(t)) f_i(G_i(t)) \\ = \ &\min\{K, \sum_{i \in \mathcal{O}} f_i(Q_i(t)) + \sum_{i \in \mathcal{G}} \gamma_i(G_i(t)) f_i(G_i(t))\}, \end{aligned}$$

see Haavardsson et al. (2008) for details on how $c(t)$ is calculated in single-phase production. Then, if the gas constraint K_g is exceeded with this choice of $c(t)$, the choke vector $\boldsymbol{b}(t)$ is modified so that

$$\sum_{i \in \mathcal{O}} \min\{1, w_i c(t)\} \psi_i(Q_i(t)) f_i(Q_i(t)) + \sum_{i \in \mathcal{G}} \min\{1, w_i c(t)\} f_i(G_i(t)) = K_g. \quad (5.2.23)$$

By varying the weights $w_1, \ldots, w_N$ in $\mathbb{R}_+^N$ a whole range of admissible production strategies is obtained. We will refer to such production strategies as *fixed-weight strategies*. It is straight-forward to show that

$$\boldsymbol{b}(\boldsymbol{w}) = \boldsymbol{b}(\lambda \boldsymbol{w}) \quad (5.2.24)$$

for any $\lambda > 0$. Thus, to avoid over-parametrization, the dimension of the search space is reduced by fixing the value of one of the weights, e.g., by letting $w_N = 1$, see Haavardsson et al. (2008) for details.

A numerical algorithm is used to maximize the following objective function

$$\phi_{C,r}(\boldsymbol{b}) = \int_0^\infty I\{q(u) \geq C\}\{q(u) + \alpha g(u)\} e^{-ru} du, \quad r \geq 0 \quad (5.2.25)$$

with respect to the vector of weights $\boldsymbol{w} = (w_1, \ldots, w_N)$, see Haavardsson et al. (2008) for details. We denote the vector of weights that maximizes $\phi_{C,r}$ in (5.2.25) $\boldsymbol{w}^*$. The parameter r may be interpreted as a discount factor, while the parameter C represents a threshold value for total production, i.e., all wells are shut down when the total production is below this total field production rate. As in Section 5.2.2, the parameter α converts one unit of gas into one oil unit equivalent and is set equal to 0.001. $\phi_{C,r}$ in (5.2.25) expresses *discounted total production.*

The fixed-weight production strategy can be used in decision support, as the local production strategy defined in Section 5.2.2. Using the fixed-weight strategy for production planning and forecasting we avoid the fluctuations we might experience using the local strategy as discussed in Section 5.2.2, which is clearly an advantage. However, the weights assigned to each reservoir are fixed over the life of the field, which is clearly a disadvantage if the chosen fixed-weight production strategy is not optimal. If it can be proved that an optimal production strategy can always be found within the parametric class of fixed-weight strategies, this does not represent a problem. In Haavardsson et al. (2008) it is explained that in single-phase production optimization an optimal production strategy can always be found within the parametric class of fixed-weight strategies. A forth-coming paper will extend the framework to two-phase production and examine the optimality properties of the parametric class in two-phase production.

5.3 Description of the case study

In the case considered two parties referred to as the *main field* and the *satellite field* are involved in offshore[4] oil and gas production. The main field consists of separate reservoirs containing gas or gas cap with oil rim, as illustrated in Figure 5.1. In reservoirs with gas cap and oil rim, the oil must be produced before the gas cap to avoid significant loss in oil recovery due to pressure depletion. The oil and gas are processed to export specification on a central production facility.

The satellite field consists of one gas reservoir and one oil reservoir, with associated condensate and gas, respectively. The satellite field is developed with two gas production wells and one oil production well. The oil and gas of the satellite field are sent to the main field in pipelines, where it is being processed at the processing facility of the main field. The main field and the satellite field have different owners and hence different commercial interests regarding production optimization.

Relating the case study to the notation and model framework of Section 5.2 the number of wells is 16. Thus, $N = 16$ and the vector $\boldsymbol{I}$ expressing the type of hydrocarbon phase of each well is $\boldsymbol{I} = (I_1, \ldots, I_N) = (I_1, \ldots, I_{16}) = (1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0)$.

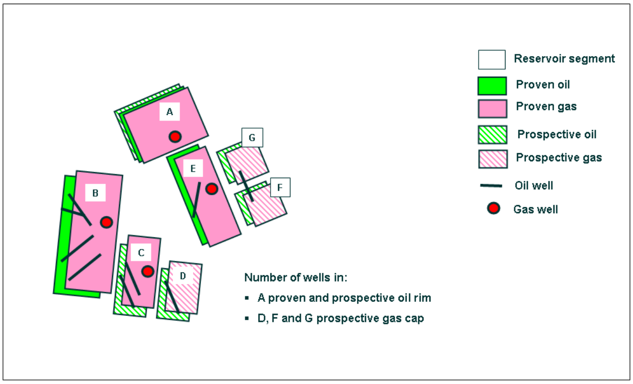

Figure 5.1: *An overview of the seven reservoirs of the main field. Oil is proven in the reservoirs A, B and E, while gas is proven in the reservoirs A, B, C and E. There are oil prospects in the reservoirs A, C, D, F and G. There are gas prospects in the reservoirs D, F and G.*

[4] In principle the problems considered also apply to onshore fields; however in this specific case offshore fields are considered.

5.4 Optimization of total production under booking constraints

The satellite field and the main field have agreed to allocate a share of the main field processing capacities to the satellite field. The allocated capacities are in the following called booking constraints. Table 5.1 lists the booking constraints of the satellite field in percent of the processing capacities of the main field. The main field will thus use the remaining capacities for its own production, as long as its processing capacities are not exceeded. [5] Note that these booking constraints necessitate modification in the capacity constraints introduced in (5.2.9), yielding different capacity constraints for each year the booking constraints apply.

	Year												
	1	2	3	4	5	6	7	8	9	10	11	12	13
Gas	21.5	23.2	21.5	21.5	21.5	19.8	18.5	14.9	11.6	9.9	8.9	7.3	6.6
Oil	19.2	14.8	12.2	10.6	8.8	6.3	5.6	3.7	3.4	2.8	3.0	2.3	1.5

Table 5.1: *Booking constraints of the satellite field stated in percent of the processing capacities at the main platform. The main field uses the remaining capacities to process its own hydrocarbons.*

Since the satellite field has booked the capacities specified in Table 5.1 it is in its self-interest to exploit this capacity. We are interested in analyzing the effect of lifting the booking constraints, since different owners with potentially conflicting commercial interests are excepted to have different preferences. If total discounted production increases when the booking constraints are lifted, both owners may benefit from this. If both owners benefit when the booking constraints are lifted, it is sensible to do so. If one owner benefits and the other suffers, it may still be beneficial to lift the constraints, if the gain of the profiting owner exceeds the loss of the suffering owner. In this case the profiting owner may buy out the suffering owner, compensating him for his loss. This way all owners benefit if total production increases. If both owners suffer when the booking constraints are lifted, or the gain of the beneficiary owner does not exceed the loss of the losing owner, it is not sensible to lift the booking constraints. However, if this is the case, there may exist booking constraints that increase total production. Hence, a new optimization problem arises, where the total discounted production of the owners is maximized. The booking constraints are the free parameters in this optimization problem. We leave this optimization problem for future research.

Lifting the booking constraints may result in a radical change in the production rates of each party. Reservoir and production engineering considerations often play

[5] In the implementation the main field uses the total capacity minus booked capacity. In reality one would expect that the total field would use total capacity minus the capacity actually used by the satellite field.

an important role in production optimization. High production rates may have negative effect on reservoir behaviour. Such effects are not addressed in this paper. In real life such reservoir considerations need to be taken into account.

The local and fixed-weight production strategy described in the sections 5.2.2 and 5.2.2 will be compared. Since we are interested in the effect of lifting the booking constraints, we calculate the production for the satellite field and the main field with and without booking constraints. The satellite field production *with* booking constraints is calculated optimizing the production strategies as described in the sections 5.2.2 and 5.2.2. The booking constraints specified in Table 5.1 are used. The production for the main field with booking constraints is calculated analogously. In the local optimization calculations, two gas wells, i.e. , well 4 and 5, have received fixed priority in the production phasing. This is done as there is an underlying assumption in the applied reservoir simulation results that there will be early gas production from the respective two reservoirs. If this assumption is not accounted for, the results will not reflect the expected physical performance of the reservoirs.

The satellite and main field production *without* booking constraints is calculated optimizing the production strategies as described in the sections 5.2.2 and 5.2.2. In the local optimization calculations, the two gas wells still received fixed priority in the production phasing. The production rates of the remaining wells are found using Algorithm 5.2.2. For the fixed-weight production strategy we use $\phi_{C,r}(\boldsymbol{b})$ specified in (5.2.25) as an objective function. We denote this production strategy $\boldsymbol{b}_C^*$, where the subscript C denotes *Combined*. The satellite and main field production without constraints is then found by aggregating the oil and gas production from all the satellite and main wells, respectively, using strategy $\boldsymbol{b}_C^*$. Having inspected individual gas well rates, the fixed-weight production strategy assigns a fair amount of gas production from day one from well 4 and 5 which is in accordance with some of the main assumptions in and results from the reservoir simulation.

Table 5.2 summarizes the results of the calculations. The results indicate that it is beneficial to lift the constraints with both strategies. With the local strategy the discounted production increases with 1.5% when the booking constraints are lifted, while the corresponding increase with the fixed-weight strategy is 1.7%. However, with the local strategy the satellite field benefits far more than the main field from lifting the constraint, while it is the other way around with the fixed-weight strategy. To understand this we take a look at the actual production rates. Figure 5.2 shows the resulting total production rates of oil and gas with and without booking constraints for the two production strategies.

For the local strategy we observe that lifting the booking constraints has a large impact on the discounted gas production of the satellite field. The gas can be produced far more efficiently when the constraints are lifted for this field. Without the constraints the main field manages to maintain its gas plateau level for approximately 3.5 years, i.e. ,from approximately 600 days until 1,800 days. Then the gas plateau level cannot be sustained anymore and the satellite field is given an increasing share of the production capacity. In fact, for a long period from approximately 1,800 days until 2,700 days, the local production strategy without constraints as-

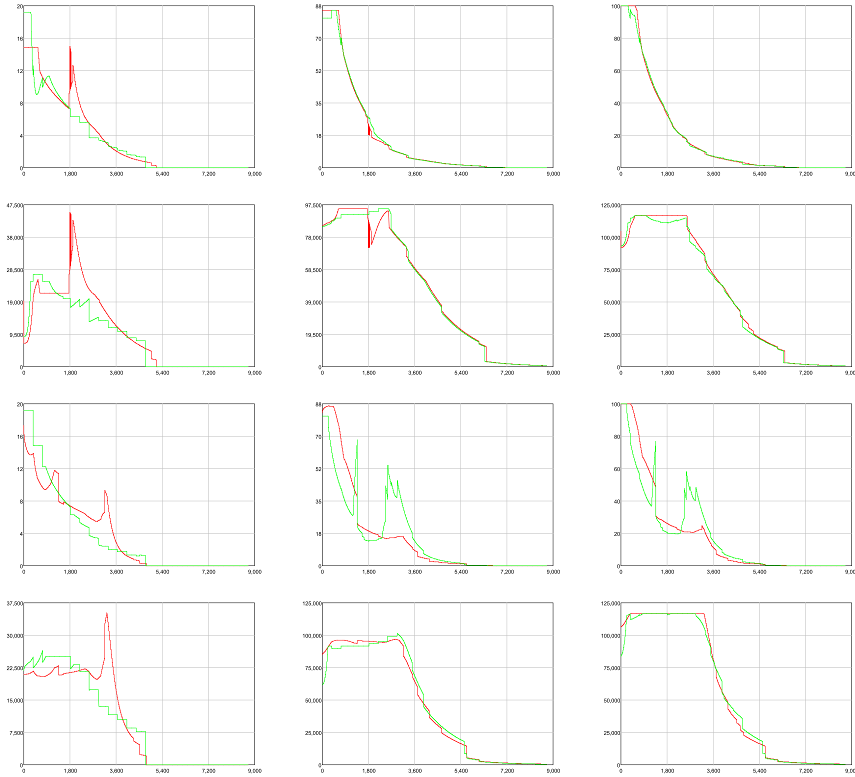

Figure 5.2: *In every row the satellite field, the main field and the total are displayed in the left, middle and right panel, respectively. The red and green graph display the production rates without and with booking constraints, respectively. The upper two rows show the production rates for the local strategy, while the lower two rows show the production rates for the fixed-weight strategy. The first and third rows display oil rates, while the second and fourth rows display gas rates. A coarser and standardized scale is used in the plots.*

signs far less gas production to the main field than it obtained with the quotas. As a result, the satellite field can now produce far more than it could with the booking constraints in place. This advantage is held for several years. This positive effect on the discounted production is reduced by heavier discounting due to the delay in time, but the advantage of the increased production by far outweighs the disadvantage represented by the delay.

With the fixed-weight strategy it is the main field that benefits from lifting the constraints. The main field is able to sustain a very high gas production for a very long time, almost 3,000 days. The satellite field suffers from this and is allocated a

Main field						
Production strategy	Oil	Gas	Total with booking	Oil	Gas	Total without booking
Local	22.6	58.0	80.6	22.7	58.0	80.7
Fixed-weight	21.3	58.5	79.8	22.5	59.1	81.7
Satellite field						
Production strategy	Oil	Gas	Total with booking	Oil	Gas	Total without booking
Local	4.9	11.8	16.8	5.2	13.1	18.3
Fixed-weight	5.2	13.2	18.5	5.1	13.2	18.3
Main field and satellite field combined						
Production strategy	Oil	Gas	Total with booking	Oil	Gas	Total without booking
Local	27.5	69.9	97.4	27.8	71.1	98.9
Fixed-weight	26.6	71.8	98.3	27.6	72.4	100.0

Table 5.2: *Results with and without booking constraints. All numbers are discounted production of oil equivalents, stated in* kSm^3*, as measured in percent using total discounted production of main field and satellite field combined without booking constraints with the fixed-weight strategy as a base case.* $\alpha = 0.001$ *has been used to convert gas into oil equivalents.*

relatively low gas production in this period. For the main field the oil is produced far more efficiently with the fixed-weight strategy when the booking constraints are lifted. Again, this efficient oil recovery is at the expense of the resulting lower share the satellite field receives. The satellite field can produce more when the main field goes into decline, as we observed with the local strategy. However, since the main field is able to maintain a high gas production rate for a very long time, the heavy discounting reduces the advantage of this unrestricted production. Furthermore, the efficient main field oil and gas production the first 3,000 days leads to relatively low satellite field production in this period. The satellite field needs a large increase in later production to balance out the loss earlier on. From Table 5.2 we see that the satellite field experiences a loss of 1% when the constraints are lifted with the fixed-weight strategy, i.e., the reduction in total discounted production from 18.5 % to 18.3 % relative to the base case.

Comparing the local strategy and the fixed-weight strategy without booking constraints, the total discounted production of the fixed-weight strategy is 1.1% larger than total discounted production of the local strategy.

5.5 Conclusions

This paper has analyzed production of oil and gas fields with different ownership and commercial interests. Satellite field booking constraints are negotiated due to different ownerships in field and an important issue is to assess the effects imposed by these constraints. Two different production strategies have been compared, with respect to performance measured in discounted production of oil equivalents.

The modelling results highlight the importance of the booking constraints. In particular the results obtained in the case study indicate that the total wealth expressed in discounted production of oil equivalents created from the satellite field and main field combined can increase when the booking constraints are lifted using both production strategies. The gain for the society as a whole thus increases. Using terminology from *game theory*, see Myerson (1991), both production strategies mimic the behaviour of a positive sum game since total discounted production increases in both cases when the booking constraints are lifted. Producing with the local strategy the satellite field receives the lion's share of the gain. Since the main field does not sustain gas plateau level for a very long time when the constraints are lifted, and the main field subsequently for a long period receives a lower share of the production capacity than it received with the booking constraints in place, the gas of the satellite field can be produced far more efficiently. Selecting the fixed-weight strategy the main field is able to sustain a high gas plateau level for a substantial amount of time. When the gas production of the satellite field is let in, it happens so late that the discounting effect outweighs the advantage of being able to produce unrestrictedly. Furthermore, the satellite field has to produce effectively and fast later in the production period to offset the loss in discounted production it suffered early in the production period. This loss is caused by the high proportion of the production capacity the main field received in this period. Thus, with the fixed-weight strategy it is the main field that benefits from lifting the constraints.

5.6 A brief introduction to multi-segmented production profiles using ordinary differential equations

Single Arps curves, introduced by Arps (1945) model the production rate function and the cumulative production function mathematically through a one-way, causal relation. In Haavardsson & Huseby (2007) this approach is extended to multiple segments so that a combination of Arps curves may be used to get a satisfactory fit to a specific set of production data.

To also take into account various production delays, the dynamic two-way relation between the production rate function and the cumulative production is modelled in terms of a differential equation. The relation between the production rate function, q, and the cumulative production function, Q, should be of the following form:

$$q(t) = f(Q(t)), \qquad \text{for all } t \geq 0, \tag{5.6.1}$$

with $Q(t_0) = 0$ as a boundary condition.

The differential equation approach can also be extended to the more general situation where the production rate function consists of s segments. For each segment we assume that we have fitted a model in terms of a differential equation on the form given in (5.6.1). In order to connect these segment models, we need to specify a *switching rule* describing when to switch from one segment model to the next one. We define a switching rule based on the produced volume. By using this switching rule, we obtain a model for the combined differential equation.

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