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Maximum principles for jump diffusion processes with infinite horizon

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Abstract

We prove maximum principles for the problem of optimal control for a jump diffusion with infinite horizon and partial information. The results are applied to partial information optimal consumption and portfolio problems in infinite horizon.

1 Introduction

In this paper we consider a control problem for a performance functional

$$J(u) = E\left[\int_0^\infty f(t, X(t), u(t), \omega)dt\right],$$

where $X(t)$ is a controlled jump diffusion and $u(t)$ is the control process. We allow for the case where the controller only has access to partial-information. Thus, we have a infinite horizon problem with partial information. Infinite-horizon optimal control problems arise in many fields of economics, in particular in models of economic growth. Note that because of the general nature of the partial information filtration $\mathcal{E}_t$, we cannot use dynamic programming and Hamilton-Jacobi-Bellman (HJB) equations to solve the optimization problem. Thus our problem is different from partial observation control problems.

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In the deterministic case the maximum principle by Pontryagin (1962) has been extended to infinite-horizon problems, but transversality conditions have not been given in gerneral. The 'natural' transversality condition in the infinite case would be a zero limit condition, meaning in the economic sense that one more unit of good at the limit gives no additional value. But this property is not necessarily verified. In fact [4] provides a counterexample for a 'natural' extension of the finite-horizon transversality conditions. Thus some care is needed in the infinite horizon case.

There have been a variety of articles on infinite-horizon problems. E.g. in [6] it is stated a 'natural' extension to infinite horizon discounted control problems.

We refer to [13] for more information about stochastic control in jump diffusion markets, to [8] for a background on infinite-horizon backward stochastic differential equations and [11] for a general introduction to infinite-horizon control problems in a deterministic environment.

In this paper we prove several maximum principles for an infinite horizon optimal control problem with partial information. The paper is structured as follows: In Section 4 we prove a maximum principle version of sufficient type (a verification theorem). In section 5 we give some examples, before we prove a (weak) version of a necessary type of the maximum principle in section 6.

In a forthcomming paper [1], the case of infinite horizon for delay equations is treated.

2 Preliminaries

Let $B(t) = B(t,\omega) = (B_1(t,\omega), \ldots, B_n(t,\omega))$, $t \geq 0$, $\omega \in \Omega$ and $\tilde{N}(dz,dt) = N(dz,dt) - \nu(dz)dt = (\tilde{N}_1(dz,dt), \ldots, \tilde{N}_n(dz,dt))$ be a n-dimensional Brownian motion and n independent compensated Poisson random measures, respectively, on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. Let $X(t) = X^u(t)$ be a controlled jump diffusion, described by the stochastic differential equation

$$\begin{aligned} dX(t) &= b(t, X(t), u(t), \omega)dt + \sigma(t, X(t), u(t), \omega)dB(t) \\ &\quad + \int_{\mathbb{R}_0^n} \theta(t, X(t), u(t), z, \omega)\tilde{N}(dz, dt); 0 \leq t < \infty \\ X(0) &= x \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where $b : [0,\infty] \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R}^n$ is adapted, $\sigma : [0,\infty] \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R}^{n\times n}$ is adapted and $\theta : [0,\infty] \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R}^{n\times n}$ is predictable (see [9]). See e.g. [2], [13] for notation and more information. Let

$$\mathcal{E}_t \subset \mathcal{F}_t,$$

be a given subfiltration, representing the information available to the controller at time $t; t \geq 0$. The process $u(t)$ is our control, assumed to be $\{\mathcal{E}_t\}_{t\geq 0}$ predicatble and with values in a set $U \subset \mathbb{R}^n$. Let $\mathcal{A}_{\mathcal{E}}$ be our family of $\mathcal{E}_t$-predicatble controls. Let $\mathcal{R}$ denote the set of functions $r : [0,\infty] \times \mathbb{R}_0^n \to \mathbb{R}^{n\times n}$ such that

$$\int_{\mathbb{R}_0^n} |\theta_{i,j}(t,x,u,z) r_{i,j}(t,z)| \nu_j(dz) < \infty \text{ for all } i,j,t,x.$$

Let $f : [0,\infty] \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R}^n$ be adapted and assume that

$$E\left[\int_0^\infty |f(t,X(t),u(t),\omega)| dt\right] < \infty \text{ for all } u \in \mathcal{A}_{\mathcal{E}}.$$

Then we define

$$J(u) = E\left[\int_0^\infty f(t,X(t),u(t),\omega) dt\right]$$

to be our performance functional. We study the problem to find $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ such that

$$J(\hat{u}) = \sup_{u\in\mathcal{A}_{\mathcal{E}}} J(u). \tag{2}$$

Let us define the Hamiltonian $H : [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \times \mathcal{R} \to \mathbb{R}$, by

$$\begin{aligned} H(t,x,u,p,q,r) &= f(t,x,u,\omega) + b^T(t,x,u,\omega)p + tr(\sigma^T(t,x,u,\omega)q) \\ &\quad + \sum_{i,j=1}^n \int_{\mathbb{R}_0^n} \theta_{i,j}(t,x,u,z,\omega) r_{i,j}(t,z) \nu_j(dz). \end{aligned} \tag{3}$$

For notational convenience we will in the rest of the paper suppress any ω from the notation. The adjoint equation in the unknown $\mathcal{F}_t$-predictable processes $(p(t), q(t), r(t,z))$ is the following

$$\begin{aligned} dp(t) &= -\nabla_x H(t,X(t),\hat{u}(t),p(t),q(t),r(t,\cdot))dt + q(t)dB(t) \\ &\quad + \int_{\mathbb{R}_0^n} r(t,z)\tilde{N}(dz,dt). \end{aligned} \tag{4}$$

3 Existence and Uniqueness

In this section we prove a result about existence and uniqueness of the solution $(Y(t), Z(t), K(t,\zeta))$ of infinite horizon BSDEs of the form;

$$\begin{aligned} dY(t) &= -g(t,Y(t),Z(t),K(t,\cdot))dt + Z(t)dB(t) \\ &\quad + \int_{\mathbb{R}_0^n} K(t,\zeta)\tilde{N}(d\zeta,dt); 0 \le t \le \tau, \end{aligned} \tag{5}$$

$$\lim_{t\to\tau} Y(t) = \xi(\tau)\mathbf{1}_{[0,\infty)}(\tau), \tag{6}$$

where $\tau \le \infty$ is a given $\mathcal{F}_t$-stopping time, possibly infinite. Our result is an extension to jumps of Theorem 4.1 in [7], Theorem 4 in [8] and Theorem 3.1 in [15]. It is also an extension to infinite horizon of Theorem Lemma 2.1 in [5]. See also [14], [10], [3] and [12]. We assume the following:

1. The function $g : \Omega \times \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathcal{R} \to \mathbb{R}^k$ is such that there exist real numbers μ, λ, K_1 and K_2, such that $K_1, K_2 > 0$ and

$$\lambda > 2\mu + K_1^2 + K_2^2. \tag{7}$$

We assume that the function g satisfies the following requirements:

(a) $g(\cdot, y, z, k)$ is progessively measurable for all y, z, k, and

$$|g(t,y,z,k(\cdot)) - g(t,y,z',k'(\cdot))| \leq K_1\|z - z'\| + K_2\|k(\cdot) - k'(\cdot)\|_R, \tag{8}$$

where

$$\|k(\cdot)\|_R^2 = \int_{\mathbb{R}_0^n} k^2(\zeta)\nu(d\zeta),$$

and $\|z\| = [Tr(zz^*)]^{\frac{1}{2}}$.

(b)

$$\langle y - y', g(t,y,z,k) - g(t,y',z,k)\rangle \leq \mu|y - y'|^2 \tag{9}$$

for all y, y', z, k a.s.

(c)

$$E\int_0^\tau e^{\lambda t}|g(t,0,0,0)|^2 dt < \infty. \tag{10}$$

(d) Finaly we require that

$$y \mapsto g(t,y,z,k), \tag{11}$$

is continuous for all t, z, k a.s.

2. We have a final condition ξ, which is $\mathcal{F}_\tau$-measurable such that $E(e^{\lambda\tau}|\xi|^2) < \infty$ and

$$E\int_0^\tau e^{\lambda t}|g(t,\xi_t,\eta_t,\psi_t)|^2 dt < \infty, \tag{12}$$

where $\xi_t = E(\xi|\mathcal{F}_t)$ and η,ψ are s.t.

$$\xi = E\xi + \int_0^t \eta(s)dB_s + \int_0^t\int_{\mathbb{R}_0^n} \psi(s,\zeta)\tilde{N}(d\zeta,ds). \tag{13}$$

A solution of the BSDE (5)-(6), is a trippel (Y_t, Z_t, K_t) of progressively measurable processes with values in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ s.t. Z_t, $K_t = 0$ when $t > \tau$,

1. $E[\sup_{t\geq 0} e^{\lambda t}|Y_t|^2 + \int_0^\tau e^{\lambda s}|Z_s|^2 ds + \int_0^\tau\int_{\mathbb{R}_0^n} e^{\lambda s}K^2(s,\zeta)\nu(d\zeta)ds] < \infty$,
2. $Y_t = Y_{T\wedge\tau} + \int_{t\wedge\tau}^{T\wedge\tau} g_s ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dB_s - \int_{t\wedge\tau}^{T\wedge\tau}\int_{\mathbb{R}_0^n} K(s,\zeta)\tilde{N}(d\zeta,ds)$ for all deterministic $T < \infty$ and
3. $Y_t = \xi$ on the set $\{t \geq \tau\}$.

Remark 3.1 *(Infinite Horizon).* This incorperates the case where $\tau(\omega) = \infty$ on some set A with $P(A) > 0$, possibly $P(A) = 1$.

Theorem 3.1 (Existence and uniqueness). *Under the above conditions there exists a unique solution* (Y_t, Z_t, K_t) *of the BSDE* (5)-(6)*, which satisfies the condition;*

$$E[\sup_{0\le t\le\tau} e^{\lambda t}|Y_t|^2 + \int_0^\tau e^{\lambda s}(|Y_s|^2 + \parallel Z_s \parallel^2)ds + \int_0^\tau e^{\lambda s}\int_{\mathbb{R}_0^n} K^2(s,\zeta)\nu(d\zeta)ds]$$
$$\le cE[e^{\lambda\tau}|\xi|^2 + \int_0^\tau e^{\lambda s}|g(s,0,0,0)|^2 ds], \tag{14}$$

for some positive number c.

Proof. First, let us show uniqueness. Let (Y, Z, K) and (Y', Z', K') be two solutions satisfying (14) and let $(\bar{Y}, \bar{Z}, \bar{K}) = (Y - Y', Z - Z', K - K')$. From Itō's Lemma we have that

$$e^{\lambda t\wedge\tau}|\bar{Y}_{t\wedge\tau}|^2 + \int_{t\wedge\tau}^{T\wedge\tau}\left[e^{\lambda s}(\lambda|\bar{Y}_s|^2 + \|\bar{Z}_s\|^2) + e^{\lambda s}\int_{\mathbb{R}_0^n}\bar{K}^2(s,\zeta)\nu(d\zeta)\right]ds$$
$$\le e^{\lambda s}|\bar{Y}_T|^2 + 2\int_{t\wedge\tau}^{T\wedge\tau}\Bigg[e^{\lambda s}(\mu|\bar{Y}_s|^2 + K_1|\bar{Y}_s| \times \|\bar{Z}_s\|)$$
$$+ K_2|\bar{Y}_s|e^{\lambda s}(\int_{\mathbb{R}_0^n}\bar{K}^2(s,\zeta)\nu(d\zeta))^{\frac{1}{2}}\Bigg]ds$$
$$- 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\lambda s}\langle\bar{Y}_s, \bar{Z}_s dB_s\rangle$$
$$- \int_{t\wedge\tau}^{T\wedge\tau} e^{\lambda s}\int_{\mathbb{R}_0^n}\left[\bar{K}^2(s,\zeta) + 2\bar{K}(s,\zeta)\bar{Y}(s)\right]\tilde{N}(d\zeta, ds).$$

Combining the above with the fact that $2ab \le a^2 + b^2$ we deduce since $\lambda > 2\mu + K_1^2 + K_2^2$, that for $t < T$

$$E[e^{\lambda t\wedge\tau}|\bar{Y}_{t\wedge\tau}|^2] \le E[e^{\lambda T\wedge\tau}|\bar{Y}_T|^2]$$

the same holds with λ replaced by λ', with $\lambda > \lambda' > 2\mu + K_1^2 + K_2^2$

$$E\left[e^{\lambda t\wedge\tau}|\bar{Y}_{t\wedge\tau}|^2\right] \le e^{(\lambda-\lambda')T}E\left[e^{\lambda T\wedge\tau}|\bar{Y}_T|^2\mathbf{1}_{\{T<\tau\}}\right]$$

Condition (14) implies that the second factor on the right hand side remains bounded as $T \to \infty$, while the first factor tends to 0. This proves uniqueness.

Proof of existence. For each $n \in$ N we construct a solution (Y_t^n, Z_t^n, K_t^n) of the BSDE

$$Y_t^n = \xi + \int_{t\wedge\tau}^{n\wedge\tau} g(s, Y_s^n, Z_s^n, K_s^n)ds - \int_{t\wedge\tau}^{\tau} Z_s^n dB_s - \int_{t\wedge\tau}^{\tau}\int_{\mathbb{R}_0^n} K^n(s,\zeta)\tilde{N}(d\zeta, ds)$$

by letting $\{(Y_t^n, Z_t^n, K_t^n); 0 \le t \le n\}$ be defined as a solution of the following BSDE:

$$Y_t^n = E[\xi|\mathcal{F}_n] + \int_t^n \mathbf{1}_{[0,\tau]}(s)g(s, Y_s^n, Z_s^n, K_s^n)ds - \int_t^n Z_s^n dB_s - \int_t^n\int_{\mathbb{R}_0^n} K^n(s,\zeta)\tilde{N}(d\zeta, ds)$$

for $0 \leq t \leq n$ and $\{(Y_t^n, Z_t^n, K_t^n); t \geq n\}$ defined by

$$Y_t^n = \xi_t,$$

$$Z_t^n = \eta_t,$$

and

$$K_t^n = \psi_t,$$

for $t > n$. Next, we find some a priori estimates for the sequence (Y^n, Z^n, K^n). For any $\epsilon > 0$, $\rho < 1$ and α we have for all $t \geq 0, y \in \mathbb{R}^k$, $z \in \mathbb{R}^{k\times d}$, $k \in \mathcal{R}$ with $c = \frac{1}{\epsilon}$,

$$\begin{aligned}
&2\langle y, g(t,y,z,k)\rangle = 2\langle y, g(t,y,z,k) - g(t,0,z,k)\rangle \\
&+ 2\langle y, g(t,0,z,k) - g(t,0,0,0)\rangle + 2\langle y, g(t,0,0,0)\rangle \\
&\leq (2\mu + \frac{1}{\rho}K_1^2 + \frac{1}{\alpha}K_2^2 + \epsilon)|y|^2 + \rho \parallel z \parallel^2 + \alpha \int_{\mathbb{R}_0^n} k^2(\zeta)\nu(d\zeta) \\
&+ c|g(t,0,0,0)|^2.
\end{aligned}$$

From It$\bar{o}$'s Lemma we have

$$\begin{aligned}
e^{\lambda t\wedge\tau}|Y_{t\wedge\tau}^n|^2 &+ \int_{t\wedge\tau}^{\tau} \left[e^{\lambda s}(\bar{\lambda}|Y_s^n|^2 + \bar{\rho} \parallel Z_s^n \parallel^2) + \bar{\alpha} \int_{t\wedge\tau}^{\tau} e^{\lambda s} \int_{\mathbb{R}_0^n} (K^n)^2(s,\zeta)\nu(d\zeta) \right] ds \\
&\leq e^{\lambda s}|\eta|^2 + c \int_{t\wedge\tau}^{\tau} e^{\lambda s}|g(s,0,0,0)|^2 ds \\
&- 2\int_{t\wedge\tau}^{\tau} e^{\lambda s} < Y_s^n, Z_s^n dB_s > \\
&- \int_{t\wedge\tau}^{\tau} e^{\lambda s} \int_{\mathbb{R}_0^n} \left[(K^n)^2(s,\zeta) + 2K^n(s,\zeta)Y^n(s) \right] \tilde{N}(d\zeta, ds),
\end{aligned}$$

with $\bar{\lambda} = \lambda - 2\mu - \frac{1}{\rho}K_1^2 - \frac{1}{\alpha}K_2^2 - \epsilon > 0$, $\bar{\rho} = 1 - \rho > 0$ and $\bar{\alpha} = 1 - \alpha$. From this and the matingale inequality it follows that

$$\begin{aligned}
&E\left[\sup_{t\geq s} e^{\lambda t\wedge\tau}|Y_{t\wedge\tau}^n|^2 + \int_{s\wedge\tau}^{\tau} \left[e^{\lambda r}(|Y_r^n|^2 + \parallel Z_r^n \parallel^2) + e^{\lambda r} \int_{\mathbb{R}_0^n} (K^n)^2(r,\zeta)\nu(d\zeta) \right] dr \right] \\
&\leq 4E\left[e^{\lambda\tau}|\xi|^2 + \int_{s\wedge\tau}^{\tau} e^{\lambda r}|g(r,0,0,0)|^2 dr \right].
\end{aligned}$$

Let $m > n$ and define $\Delta Y_t := Y_t^m - Y_t^n$, $\Delta Z_t := Z_t^m - Z_t^n$ and $\Delta K_t := K_t^m - K_t^n$, so that for $n \leq t \leq m$,

$$\Delta Y_t = \int_{t\wedge\tau}^{m\wedge\tau} g(s, Y_s^m, Z_s^m, K_s^m)ds - \int_{t\wedge\tau}^{m\wedge\tau} \Delta Z_s dB_s - \int_{t\wedge\tau}^{m\wedge\tau} \int_{\mathbb{R}_0^n} \Delta K(s,\zeta)\tilde{N}(d\zeta, ds).$$

It then follows that

$$
\begin{aligned}
e^{\lambda t\wedge\tau}|\Delta Y_{t\wedge\tau}|^2 &+ \int_{t\wedge\tau}^{m\wedge\tau}\Big\{e^{\lambda s}(\lambda|\Delta Y_s|^2+\parallel \Delta Z_s\parallel^2)+e^{\lambda s}\int_{\mathbb{R}_0^n}(\Delta K)^2(s,\zeta)\nu(d\zeta)\Big\}ds\\
&= \int_{t\wedge\tau}^{m\wedge\tau} e^{\lambda s}\langle \Delta Y_s, g(s,Y_s^m,Z_s^m,K_s^m)\rangle ds\\
&- 2\int_{t\wedge\tau}^{m\wedge\tau} e^{\lambda s}\langle \Delta Y_s,\Delta Z_s dB_s\rangle\\
&- \int_{t\wedge\tau}^{m\wedge\tau} e^{\lambda s}\int_{\mathbb{R}_0^n}\left[(\Delta K)^2(s,\zeta)+2\Delta K(s,\zeta)\Delta Y(s)\right]\tilde{N}(d\zeta,ds)\\
&2\le e^{\lambda s}|\eta|^2 c\int_{t\wedge\tau}^{m\wedge\tau} e^{\lambda s}|g(s,0,0,0)|^2 ds-2\int_{t\wedge\tau}^{m\wedge\tau} e^{\lambda s}\langle \Delta Y_s,\Delta Z_s dB_s\rangle\\
&- \int_{t\wedge\tau}^{m\wedge\tau} e^{\lambda s}\int_{\mathbb{R}_0^n}\left[(\Delta K)^2(s,\zeta)+2\Delta K(s,\zeta)\Delta Y(s)\right]\bar{N}(d\zeta,ds).
\end{aligned}
$$

From the same arguments as above

$$
\begin{aligned}
&E\Bigg[\sup_{n\le t\le m} e^{\lambda t\wedge\tau}|\Delta Y_{t\wedge\tau}|^2\\
&+\int_{n\wedge\tau}^{m\wedge\tau}\Big\{e^{\lambda s}(|\Delta Y_s|^2+\parallel \Delta Z_s\parallel^2)+e^{\lambda s}\int_{\mathbb{R}_0^n}(\Delta K)^2(s,\zeta)\nu(d\zeta)\Big\}ds\Bigg]\\
&\le 4E\left[\int_{n\wedge\tau}^{\tau} e^{\lambda s}|g(s,\xi,\eta,\psi)|^2 ds\right].
\end{aligned}
$$

The last term in the above equation goes to zero as $n\to\infty$. Now, for $t\le n$

$$
\begin{aligned}
\Delta Y_t &= \Delta Y_n + \int_{t\wedge\tau}^{n\wedge\tau}\Big\{g(s,Y_s^m,Z_s^m,K_s^m)-g(s,Y_s^n,Z_s^n,K_s^n)\Big\}ds-\int_{t\wedge\tau}^{n\wedge\tau}\Delta Z_s dB_s\\
&- \int_{t\wedge\tau}^{n\wedge\tau}\int_{\mathbb{R}_0^n}\Delta K(s,\zeta)\tilde{N}(d\zeta,ds).
\end{aligned}
$$

Using the same argument as in the case of uniqueness, we have that

$$
E[e^{\lambda t\wedge\tau}|\Delta Y_{t\wedge\tau}|^2]\le E[e^{\lambda t\wedge\tau}|\Delta Y_n|^2]\le cE\left[\int_{n\wedge\tau}^{\tau} e^{\lambda s}|g(s,\xi_s,\eta_s,\psi_s)|^2 ds\right].
$$

It now follows that the sequence (Y^n,Z^n,K^n) is Cauchy in the norm

$$
\begin{aligned}
\|(Y,Z,K)\| := E[\sup_{0\le t\le\tau} e^{\lambda t}|Y_t|^2 &+ \int_0^\tau e^{\lambda s}(|Y_s|^2+\parallel Z_s\parallel^2)ds\\
&+\int_0^\tau e^{\lambda s}\int_{\mathbb{R}_0^n}K^2(s,\zeta)\nu(d\zeta)ds].
\end{aligned}
$$

So, we have that there is an unique solution to the BSDE (5)-(6), which satisfies for all $\lambda > 2\mu + K_1^2 + K_2^2$, the condition

$$
\begin{aligned}
&E\left[\sup_{0\le t\le\tau} e^{\lambda t}|Y_t|^2+\int_0^\tau e^{\lambda s}(|Y_s|^2+\parallel Z_s\parallel^2)ds+\int_0^\tau e^{\lambda s}\int_{\mathbb{R}_0^n}K^2(s,\zeta)\nu(d\zeta)ds\right]\\
&\le cE\left[e^{\lambda\tau}|\xi|^2+\int_0^\tau e^{\lambda s}|g(s,0,0,0)|^2 ds\right].
\end{aligned}
$$

$\square$

4 Optimal control with partial information and infinite horizon

Now, let us get back to the problem of maximizing the performance functional

$$J(u) = E\left[\int_0^\infty f(t, X(t), u(t))dt\right],$$

where $X(t)$ is of the form (1). Our aim is to find a $\hat{u} \in \mathcal{A}_\mathcal{E}$ such that

$$J(\hat{u}) = \sup_{u \in \mathcal{A}_\mathcal{E}} J(u),$$

where $u(t)$ is our previsible control adapted to a subfiltration

$$\mathcal{E}_t \subset \mathcal{F}_t,$$

with values in a set $U \subset \mathbb{R}^n$. Let H be the Hamiltonian defined by (3) and p the solution to the adjoint equation (4). Then we have the following maximum principle;

Theorem 4.1 (Sufficient Infinite Horizon Maximum Principle). *Let $\hat{u} \in \mathcal{A}_\mathcal{E}$ and let $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z))$ be an associated solution to the equation (4). Assume that for all $u \in \mathcal{A}_\mathcal{E}$ the following terminal condition holds:*

$$0 \leq E\left[\overline{\lim_{t\to\infty}}[\hat{p}(t)^T(X(t) - \hat{X}(t))]\right] < \infty. \tag{15}$$

Moreover, assume that $H(t, x, u, \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))$ is concave in x and u and

$$E\left[H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))|\mathcal{E}_t\right]$$
$$= \max_{u\in U} E\left[H(t, \hat{X}(t), u, \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))|\mathcal{E}_t\right]. \tag{16}$$

In addition we assume that

$$E\left[\int_0^\infty (\hat{X}(t) - X^u(t))^T[\hat{q}\hat{q}^T + \int_{\mathbb{R}_0^n} \hat{r}\hat{r}^T(t,z)\nu(dz)](\hat{X}(t) - X^u(t))dt\right] < \infty, \tag{17}$$

$$E\left[\int_0^\infty \hat{p}(t)^T[\sigma\sigma^T(t, X(t), u(t)) + \int_{\mathbb{R}_0^n} \theta\theta^T(t, X(t), u(t))\nu(dz)]p(t)dt\right] < \infty, \tag{18}$$

$$E\left[|\nabla_u H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))|^2\right] < \infty, \tag{19}$$

and that

$$E\left[\int_0^\infty |H(s, X(s), u(s), \hat{p}(s), \hat{q}(s), \hat{r}(s,\cdot))|\right] < \infty \tag{20}$$

for all u.
Then we have that $\hat{u}(t)$ is optimal.

Remark 4.1. Note that, since $p(t)$ has the economic interpretation as the marginal value of the resource (alternativly the shadow price if representing an outside resource), the requirement

$$0 \leq E\left[\overline{\lim_{t\to\infty}}[\hat{p}(t)^T(X(t) - \hat{X}(t))]\right] < \infty,$$

has the economic interpretation that if the marginal value is positive at infinity we want to have as little resources left as possible.

Remark 4.2. The requirement in the finite horizon case that $p(T) = 0$ does not translate into $\lim_{T\to\infty} p(T) = 0$ as was shown in the deterministic case in [4].

Proof. Let $I^\infty := E[\int_0^\infty (f(t, X(t), u(t)) - f(t, \hat{X}(t), \hat{u}(t)))dt] = J(u) - J(\hat{u})$. Then $I^\infty = I_1^\infty - I_2^\infty - I_3^\infty - I_4^\infty$, where

$$I_1^\infty := E\Bigg[\int_0^\infty (H(s, X(s), u(s), \hat{p}(s), \hat{q}(s), \hat{r}(s,\cdot)) - H(t, \hat{X}(s), \hat{u}(t), \hat{p}(s), \hat{q}(s), \hat{r}(s,\cdot)))ds\Bigg],$$

$$I_2^\infty := E\left[\int_0^\infty \hat{p}(s)^T(b(s, X(s), u(s)) - \hat{b}(s, \hat{X}(s), \hat{u}(s)))ds\right],$$

$$I_3^\infty := E\left[\int_0^\infty \mathrm{tr}[q(s)^T(\sigma(s, X(s), u(s)) - \hat{\sigma}(s, \hat{X}(s), \hat{u}(s)))]ds\right],$$

and

$$I_4^\infty := E\Bigg[\int_0^\infty \sum_{i,j} \int_{\mathbb{R}_0^n} (\theta(s, X(s), u(s), z) - \hat{\theta}(s, \hat{X}(s), \hat{u}(s), z))^T \hat{r}_{i,j}(s,z)\nu_j(dz)ds\Bigg].$$

We have from concavity that

$$H(t, X(t), u(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) - H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) \tag{21}$$

$$\leq \nabla_x H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))^T(X(t) - \hat{X}(t)) + \nabla_u H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))^T(u(t) - \hat{u}(t)). \tag{22}$$

Then we have from (16),(19) and that $u(t)$ is adapted to $\mathcal{E}_t$,

$$\begin{aligned} 0 &\geq \nabla_u E\left[H(t, \hat{X}(t), u, \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))|\mathcal{E}_t\right]^T_{u=\hat{u}(t)} (u(t) - \hat{u}(t)) \\ &= E\left[\nabla_u H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))^T(u(t) - \hat{u}(t))|\mathcal{E}_t\right]. \end{aligned} \tag{23}$$

Combining (4), (17), (21), (22) and (23)

$$\begin{aligned} I_1^\infty &\leq E\left[\int_0^\infty \nabla_x H(t, \hat{X}(s), \hat{u}(s), \hat{p}(s), \hat{q}(s), \hat{r}(s,\cdot))^T(X(s) - \hat{X}(s))ds\right] \\ &= E\left[\int_0^\infty (X(s) - \hat{X}(s))^T d\hat{p}(s)\right] =: -J_1. \end{aligned}$$

Now, using (15) and It$\bar{o}$'s formula

$$
\begin{aligned}
0 &\le E\left[\overline{\lim_{t\to\infty}}[\hat{p}(t)^T(X(t)-\hat{X}(t))]\right] \\
&= E\Bigg[\overline{\lim_{t\to\infty}}\Big[\int_0^t \hat{p}(s)^T(b(s,X(s),u(s))-\hat{b}(s,\hat{X}(s),\hat{u}(s)))ds \\
&+ \int_0^t \hat{p}(s)^T(\sigma(s,X(s),u(s))-\hat{\sigma}(s,\hat{X}(s),\hat{u}(s)))dB(s) \\
&+ \int_0^t \int_{\mathbb{R}_0^n} \hat{p}(s)^T(\theta(s,X(s),u(s),z)-\hat{\theta}(s,\hat{X}(s),\hat{u}(s),z))\tilde{N}(dz,ds) \\
&+ \int_0^t (X(s)-\hat{X}(s))^T(-\nabla_x \hat{H}(s,\hat{X}(s),\hat{u}(s),\hat{p}(s),\hat{q}(s),\hat{r}(s,\cdot)))ds \\
&+ \int_0^t \hat{q}(s)^T(X(s)-\hat{X}(s))dB(s) \\
&+ \int_0^t \int_{\mathbb{R}_0^n} \hat{r}(s,z)(X(s)-\hat{X}(s))\tilde{N}(dz,ds) \\
&+ \int_0^t \operatorname{tr}\left[\hat{q}(s)^T(\sigma(s,X(s),u(s))-\hat{\sigma}(s,\hat{X}(s),\hat{u}(s)))\right] ds \\
&+ \int_0^t \sum_{i,j} \int_{\mathbb{R}_0^n} (\theta(s,X(s),u(s),z)-\hat{\theta}(s,\hat{X}(s),\hat{u}(s),z))^T \hat{r}_{i,j}(s,z)\nu_j(dz)ds \\
&+ \int_0^t \int_{\mathbb{R}_0^n} (\theta(s,X(s),u(s),z)-\hat{\theta}(s,\hat{X}(s),\hat{u}(s),z))^T \hat{r}(s,z)\tilde{N}(dz,ds)\Big]\Bigg]
\end{aligned}
$$

From (17), (18), we have that

$$
\begin{aligned}
0 \leq E\Bigg[& \overline{\lim_{t\to\infty}} \Big[\int_0^t \hat{p}(s)^T (b(s,X(s),u(s)) - \hat{b}(s,\hat{X}(s),\hat{u}(s)))ds \\
& + \int_0^t (X(s)-\hat{X}(s))^T(-\nabla_x \hat{H}(s,\hat{X}(s),\hat{u}(s),\hat{p}(s),\hat{q}(s),\hat{r}(s,\cdot)))ds \\
& + \int_0^t \operatorname{tr}\left[\hat{q}(s)^T(\sigma(s,X(s),u(s)) - \hat{\sigma}(s,\hat{X}(s),\hat{u}(s)))\right] ds \\
& + \int_0^t \sum_{i,j} \int_{\mathbb{R}_0^n} \Big(\theta(s,X(s),u(s),z) - \hat{\theta}(s,\hat{X}(s),\hat{u}(s),z)\Big)^T \hat{r}_{i,j}(s,z)\nu_j(dz)ds\Big]\Bigg] \\
= E\Bigg[& \int_0^\infty \hat{p}(s)^T(b(s,X(s),u(s)) - \hat{b}(s,X(s),u(s)))ds \\
& + \int_0^\infty (X(s)-\hat{X}(s))^T(-\nabla_x \hat{H}(s,X(s),u(s),p(s),q(s),r(s,\cdot)))ds \\
& + \int_0^\infty \operatorname{tr}\left[\hat{q}(s)^T(\sigma(s,X(s),u(s)) - \hat{\sigma}(s,\hat{X}(s),\hat{u}(s)))\right] ds \\
& + \int_0^\infty \sum_{i,j} \int_{\mathbb{R}_0^n} (\theta(s,X(s),u(s),z) - \hat{\theta}(s,\hat{X}(s),\hat{u}(s),z))^T \hat{r}_{i,j}(s,z)\nu_j(dz)ds\Bigg] \\
= I_{1,2}^\infty & + J_1^\infty + I_{1,3}^\infty + I_{1,4}^\infty.
\end{aligned}
$$

Finally, combining the above we get

$$
\begin{aligned}
J(u) - J(\hat{u}) &\leq I_1^\infty - I_2^\infty - I_3^\infty - I_4^\infty \\
&\leq -J_1^\infty - I_2^\infty - I_3^\infty - I_4^\infty \\
&\leq 0.
\end{aligned}
$$

This holds for all $u \in \mathcal{A}_{\mathcal{E}}$, so the proof is complete. □

5 Examples

Example 5.1 (Optimal Consumption Rate Part I). *Let*

$$
J(u) = E\left[\int_0^\infty e^{-\rho t} \ln\big(u(t)X(t)\big)dt\right],
$$

where

$$
\begin{aligned}
dX(t) &= X(t)(\mu(t) - u(t))dt + X(t)\sigma(t)dB(t), \\
X(0) &= x_0,
\end{aligned}
$$

and $\rho \geq 0$. *We have that*

$$
X(t) = X_0 \exp\left[\int_0^t [(\mu(s) - u(s)) - \frac{1}{2}\sigma^2(s)]ds + \int_0^t \sigma(s)dB(s)\right].
$$

Then we deal with the problem of maximizing $J(u)$ over all $u(t) \geq 0$. We have the Hamiliton function takes the form

$$H(t, x, u, p, q) = e^{-\rho t} \ln(ux) + x(\mu - u)p + x\sigma q,$$

so that we get the partial derivatives

$$\nabla_x H(t, x, u, p, q) = \frac{e^{-\rho t}}{x} + (\mu - u)p + \sigma q,$$

and

$$\nabla_u H(t, x, u, p, q) = \frac{e^{-\rho t}}{u} - xp,$$

This gives us that

$$-dp(t) = \left[\frac{e^{-\rho t}}{X(t)} + (\mu(t) - u(t))p(t) + \sigma(t)q(t)\right] dt - q(t)dB(t).$$

so that

$$\hat{u}(t) = \frac{e^{-\rho t}}{\hat{X}(t)\hat{p}(t)}.$$

Let us try the infinite horizon BSDE with terminal condition $\lim_{t\to\infty} p(t) = 0$,

$$-dp(t) = \left[\frac{e^{-\rho t}}{X(t)} + (\mu(t) - u(t))p(t) + \sigma(t)q(t)\right] dt - q(t)dB(t), \tag{24}$$

$$\lim_{t\to\infty} p(t) = 0. \tag{25}$$

Lemma 5.1 (Solution of infinite horizon linear BSDE with jumps). *Let $A(t), \beta(t)$ and $\alpha(t, \zeta)$ be $\mathcal{F}_t$-predictable processes such that*

$$E\left[\int_0^\infty \{|A(t)| + \beta^2(t) + \int_{\mathbf{R}} \alpha^2(s, \zeta)\nu(d\zeta)\}dt\right] < \infty,$$

and define $\Gamma_{t,s}$ as the solution of the linear SDE

$$d\Gamma_{t,s} = \Gamma_{t^-,s}\left(A(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}_0^n} \alpha(t, \zeta)\bar{N}(d\zeta, dt)\right), s \geq t \geq 0,$$

$$\Gamma_{t,t} = 1.$$

Let $C(t)$ be a predictable process such that

$$E\left[\int_0^\infty \Gamma_{0,s}|C(s)|ds\right] < \infty.$$

Then a solution $(Y(t), Z(t), K(t, \zeta))$ of the linear BSDE

$$-dY(t) = \left[A(t)Y(t) + Z(t)\beta(t) + C(t) + \int_{\mathbb{R}_0^n} \alpha(t, \zeta)K(t, \zeta)d\nu(\zeta)\right] dt$$

$$- Z(t)dB(t) - \int_{\mathbb{R}_0^n} K(t, \zeta)\bar{N}(d\zeta, dt),$$

$$\lim Y(t) = 0, t \to \infty,$$

is given by

$$Y(t) = E\left[\int_t^\infty \Gamma_{t,s} C(s) ds | \mathcal{F}_t\right], t \geq 0.$$

If in additon

$$E\left[\int_0^\infty e^{\lambda t} |Y(t)|^2 dt\right] < \infty,$$

where λ *as in* (7), *then* $Y(t)$ *is the unique solution.*

Proof. By Itō's Lemma we have that

$$\begin{aligned} d(\Gamma_{0,t} Y_t) &= -\Gamma_{0,t} C_t dt + \Gamma_{0,t}(Z_t + Y_t \beta_t) dB_t \\ &\quad + \int_{\mathbb{R}_0^n} \Big[Y(t)\alpha(t,\zeta)\Gamma_{0,t} + K(t,\zeta)\Gamma_{0,t} + K(t,\zeta)\alpha(t,\zeta)\Gamma_{0,t} \Big] \tilde{N}(d\zeta, dt). \end{aligned}$$

So

$$\begin{aligned} \Gamma_{0,t} Y_t + \int_t^\infty \Gamma_{0,s} C_s ds &= \int_t^\infty \Gamma_{0,s}(Z_s + Y_s \beta_s) dB(s) \\ &\quad + \int_t^\infty \int_{\mathbb{R}_0^n} \Big[Y(s)\alpha(s,\zeta)\Gamma_{0,s} + K(s,\zeta)\Gamma_{0,s} + K(s,\zeta)\alpha(s,\zeta)\Gamma_{0,s} \Big] \tilde{N}(d\zeta, ds). \end{aligned}$$

By taking expectation we get the desired result. The uniqueness follows from Theorem 3.1. □

From the above lemma we see that the solution of the linear, infinite horizon BSDE (24) - (25) *is*

$$\hat{p}(t) = E\left[\int_t^\infty \frac{\hat{\Gamma}_s}{\hat{\Gamma}_t} \frac{e^{-\rho s}}{\hat{X}_s} ds | \mathcal{F}_t\right],$$

where

$$\hat{\Gamma}_t = e^{\int_0^t [(\mu(s) - u(s)) - \frac{1}{2}\sigma^2(s)] ds + \int_0^t \sigma(s) dB(s)} = \frac{\hat{X}(t)}{x_0}.$$

Hence

$$\hat{p}(t) = \frac{1}{\rho} e^{-\rho t} \frac{1}{\hat{X}(t)}.$$

and

$$\overline{\lim_{t\to\infty}} \hat{p}(t)(X(t) - \hat{X}(t)) \geq \overline{\lim_{t\to\infty}} \hat{p}(t) X(t) \geq 0.$$

So

$$\hat{u}(t) = \rho,$$

is an optimal control.

Example 5.2 (Optimal Consumption Rate - part II). *Let*

$$J(u) = E\left[\int_0^\infty e^{-\rho t} \ln\big(u(t)X(t)\big) dt\right],$$

where

$$dX(t) = X(t)\mu(t)(1-u(t))dt + X(t)\sigma(t)(1-u(t))dB(t),$$
$$X(0) = x_0,$$

and $\rho \geq 0$. *We have that*

$$X(t) = X_0 \exp\left[\int_0^t [\mu(s)(1-u(s)) - \frac{1}{2}\sigma^2(s)(1-u(t))^2]ds + \int_0^t \sigma(s)(1-u(s))dB(s)\right].$$

Then we deal with the problem of maximizing $J(u)$ *over all* $u(t) \geq 0$. *We have the Hamiliton function takes the form*

$$H(t,x,u,p,q) = e^{-\rho t}\ln(ux) + x\mu(1-u)p + x\sigma(1-u)q,$$

so that we get the partial derivatives

$$\nabla_x H(t,x,u,p,q) = \frac{e^{-\rho t}}{x} + \mu(1-u)p + \sigma(1-u)q,$$

and

$$\nabla_u H(t,x,u,p,q) = \frac{e^{-\rho t}}{u} - x\mu p - x\sigma q.$$

This gives us that

$$-dp(t) = \left[\frac{e^{-\rho t}}{X(t)} + \mu(t)(1-u(t))p(t) + \sigma(t)(1-u(s))q(t)\right]dt - q(t)dB(t).$$

So that

$$\hat{u}(t) = \frac{e^{-\rho t}}{\hat{X}(t)(\mu\hat{p}(t) + \sigma\hat{q}(t))}.$$

Let us try the infinite horizon BSDE with terminal condition $\lim_{t\to\infty} p(t) = 0$, *so that*

$$-dp(t) = \left[\frac{e^{-\rho t}}{X(t)} + \mu(t)(1-u(t))p(t) + \sigma(t)(1-u(s))q(t)\right]dt - q(t)dB(t), \tag{26}$$
$$\lim_{t\to\infty} p(t) = 0. \tag{27}$$

From the above lemma we see that the solution of the linear, infinite horizon BSDE (26) - (27) *is*

$$\hat{p}(t) = E\left[\int_t^\infty \frac{\hat{\Gamma}_{0,s}}{\hat{\Gamma}_{0,t}} \frac{e^{-\rho s}}{\hat{X}_s} ds|\mathcal{F}_t\right],$$

where

$$\hat{\Gamma}_t = \exp\left[\int_0^t \left[\mu(s)(1-u(s)) - \frac{1}{2}\sigma^2(s)(1-u(s))^2\right] ds \right.$$
$$\left. + \int_0^t \sigma(s)(1-u(s))dB(s)\right]$$
$$= \frac{\hat{X}(t)}{x_0}.$$

Hence

$$\hat{p}(t) = \frac{1}{\rho} e^{-\rho t} \frac{1}{\hat{X}(t)}.$$

and

$$\overline{\lim_{t\to\infty}} \hat{p}(t)(X(t) - \hat{X}(t)) \geq \overline{\lim_{t\to\infty}} \hat{p}(t)X(t) \geq 0.$$

Since

$$d(e^{-\rho t}\frac{1}{X(t)}) = e^{-\rho t}\frac{1}{X}(t)dt - e^{-\rho t}\frac{1}{X(t)}(\mu(t) - u(t))dt$$
$$+ e^{-\rho t}\frac{1}{X(t)}\sigma^2(t)dt + e^{-\rho t}\frac{1}{X(t)}\sigma(t)dB(t),$$

we must have that

$$\hat{q}(t) = \frac{1}{\rho} e^{-\rho t} \frac{1}{\hat{X}(t)}\sigma(t).$$

So

$$\hat{u}(t) = \frac{\rho}{\mu + \sigma},$$

is an optimal control.

Example 5.3 (Optimal consumption rate - part III). *As above, let*

$$J(u) = E\left[\int_0^\infty e^{-\rho t} \ln\big(u(t)X(t)\big)dt\right].$$

But add a jump part

$$dX(t) = X(t)(\mu(t) - u(t))dt + X(t)\sigma(t)dB(t) + X(t)\int_{\mathbb{R}_0} \theta(t)z\tilde{N}(dz, dt)$$
$$X(0) = x_0,$$

and we also add the assumption that we only know a subset of the information given by the market available at time t, represented by $\mathcal{E}_t \subset \mathcal{F}_t$. Let $\rho \geq 0$, be a random variable adapted to $\mathcal{F}_t$. Then we deal with the problem of maximizing $J(u)$ over all $u(t) \geq 0$. We have

$$H(t, x, u, p, q, r) = e^{-\rho t}\ln(ux) + x(\mu - u)p + x\sigma q + x\int_{\mathbb{R}_0} \theta(t)zr(t, z)\nu(dz)$$

$$\nabla_x H(t, x, u, p, q, r) = \frac{e^{-\rho t}}{x} + (\mu - u)p + \sigma q + \int_{\mathbb{R}_0} \theta(t)zr(t, z)\nu(dz),$$

$$\nabla_u H(t,x,u,p,q,r) = \frac{e^{-\rho t}}{u} - xp$$

and

$$\begin{aligned} -dp(t) &= [\frac{e^{-\rho t}}{X(t)} + (\mu(t) - u(t))p(t) + \sigma(t)q(t) + \int_{\mathbb{R}_0} \theta(t) z r(t,z)\nu(dz)]dt \\ &- q(t)dB(t) - \int_{\mathbb{R}_0} \theta(t) z \tilde{N}(dz,dt), \\ \lim_{t\to\infty} p(t) &= 0. \end{aligned}$$

If we maximice

$$E[H(t, \hat{X}(t), u, \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))|\mathcal{E}_t],$$

we get that

$$\begin{aligned} \nabla_u E[H(t, \hat{X}(t), u, \hat{p}(t), \hat{q}(t))|\mathcal{E}_t] &= E[\nabla_u H(t, \hat{X}(t), u, \hat{p}(t), \hat{q}(t))|\mathcal{E}_t] \\ &= E[\frac{e^{-\rho t}}{u} - \hat{X}(t)\hat{p}(t)|\mathcal{E}_t]. \end{aligned}$$

So that

$$\hat{u}(t) = E[\frac{e^{-\rho t}}{\hat{X}(t)\hat{p}(t)}|\mathcal{E}_t].$$

The solution of the linear, infinite horizon BSDE (24) - (25) *is (see [10])*

$$\hat{p}(t) = E\left[\int_t^\infty \frac{\hat{\Gamma}_s}{\hat{\Gamma}_t}\frac{e^{-\rho s}}{\hat{X}_s} ds|\mathcal{F}_t\right],$$

where

$$\begin{aligned} d\hat{\Gamma}_t &= X(t)(\mu(t) - u(t))dt + X(t)\sigma(t)dB(t) + X(t^-)\int_{\mathcal{R}_0} \theta(t) z \tilde{N}(dz,dt), \\ X(0) &= 1. \end{aligned}$$

So

$$\hat{\Gamma}_t = \frac{\hat{X}(t)}{x_0}.$$

Hence

$$\hat{p}(t) = \frac{1}{x_0 \hat{\Gamma}_t}\frac{1}{\rho}e^{-\rho t} = \frac{1}{\rho}e^{-\rho t}\frac{1}{\hat{X}(t)}.$$

Therefore we have that

$$\overline{\lim_{t\to\infty}}\, \hat{p}(t)(X(t) - \hat{X}(t)) = \overline{\lim_{t\to\infty}}\, \hat{p}(t)X(t) \geq 0.$$

So

$$\hat{u}(t) = E[\rho, |\mathcal{E}_t]$$

is an optimal control.

Example 5.4 (Optimal Portfolio Selection With Consumption). *For this example let us look at a market with two investment possibilities:*

1. *A bond or bank account*

$$dZ_0(t) = \rho Z_0(t)dt.$$

2. *A stock*

$$dZ_1(t) = \mu Z_1(t)dt + \sigma Z_1(t)dB(t).$$

Let (Y_0, Y_1) *denote the amount the agent has invested in the bonds and stocks repectively at time* t*. Consider then* $u(t,\omega) = u(t)$*, the fraction of the wealth invested in the stocks, e.g.*

$$u(t) = \frac{Z_1(t)}{Z_0(t) + Z_1(t)}.$$

Further let $\lambda(t,\omega) = \lambda(t)$ *be the consumption rate relative to the wealth so that the investor controls*

$$c(t) = (u(t), \lambda(t)).$$

Then let

$$J^{\lambda,u}(s,z) = E^{s,z}\left[\int_0^\infty e^{-\delta(s+t)}\frac{(\lambda(t)X(t))^\gamma}{\gamma}\right],$$

be a performance functional, where

$$dX(t) = X(t)\left[(\rho + u(t)(\mu - \rho) - \lambda(t))dt + \sigma u(t)dB(t)\right],$$

and $\rho \geq 0$*. We have that*

$$X(t) = x_0 \exp\left[\int_0^t [\rho + u(s)(\mu - \rho) - \lambda(s) - \frac{1}{2}\sigma^2 u^2]ds + \int_0^t \sigma u(s)dB(s)\right].$$

Then we want to maximize $J^{u,\lambda}(s,t)$ *over all* $l = (u(t), \lambda(t))$*,* $\lambda \geq 0$*. We have that*

$$H(t,x,l,p,q) = e^{-\delta(s+t)}\frac{(\lambda(t)X(t))^\gamma}{\gamma} + x(\rho + u(s)(\mu - \rho) - \lambda)p + x\sigma uq,$$

so that

$$\nabla_x H(t,x,l,p,q) = e^{-\delta(s+t)}\lambda^\gamma x^{\gamma-1} + (\rho + u(\mu - \rho) - \lambda)p + \sigma uq.$$

Further, we also have

$$-dp(t) = [e^{-\delta(s+t)}\lambda^\gamma(t)X^{\gamma-1}(t) + (\rho + u(t)(\mu - \rho) - \lambda(t))p + \sigma u(t)q]dt - qdB(t).$$

and

$$\nabla_u H(t,x,l,p,q) = (\mu - \rho)xp + x\sigma q,$$
$$\nabla_\lambda H(t,x,l,p,q) = e^{-\delta(s+t)}(\lambda(t))^{\gamma-1}X^\gamma - xp.$$

So that

$$q(t) = -\frac{(\mu - \rho)}{\sigma}p(t),$$

and

$$\hat{\lambda} = \frac{1}{x}p^{\frac{1}{\gamma-1}}e^{\frac{\delta(s+t)}{\gamma-1}}.$$

Then

$$dp(t) = -e^{\frac{\delta(s+t)}{\gamma-1}} \frac{1}{X} p^{\frac{\gamma}{\gamma-1}}(t)dt - [\rho + u(t)(\mu-\rho) - \frac{1}{X}(t) p^{\frac{1}{\gamma-1}} e^{\frac{\delta(s+t)}{\gamma-1}}]p(t)dt$$
$$+ \sigma u(t)\frac{(\mu-\rho)}{\sigma}p(t)dt - \frac{(\mu-\rho)}{\sigma}p(t)dB(t)$$
$$= -\rho p(t)dt - \frac{(\mu-\rho)}{\sigma}p(t)dB(t).$$

So to ensure that the requirement

$$E[\overline{\lim_{t\to\infty}} \hat{p}(t)(X(t) - \hat{X}(t))] \geq 0,$$

is satisfied we need that

$$E[\overline{\lim_{t\to\infty}} - \hat{p}(t)\hat{X}(t)] \geq 0.$$

Since

$$\hat{p}(t) = (\hat{\lambda}(t)\hat{X}(t))^{(\gamma-1)} e^{\delta(s+t)},$$

we see that

$$-\hat{p}(t)\hat{X}(t) = \hat{\lambda}^{(\gamma-1)}(t)\hat{X}^{\gamma}(t)e^{\delta(s+t)}.$$

So, by considering

$$\hat{\lambda} = \frac{1}{x} p^{\frac{1}{\gamma-1}} e^{\frac{\delta(s+t)}{\gamma-1}},$$

we try to let

$$p^{\frac{1}{\gamma-1}}(t) = X(t)Ke^{Bt},$$

for some constants K and B.It is now clear that

$$d(p^{\frac{1}{\gamma-1}}(t)) = p^{\frac{1}{\gamma-1}}(t)\frac{1}{\gamma-1}(-\rho dt - \frac{(\mu-\rho)}{\sigma}dB(t))$$
$$+ p^{\frac{1}{\gamma-1}} \frac{1}{2}\frac{1}{\gamma-1}\frac{2-\gamma}{\gamma-1}\frac{(\mu-\rho)^2}{\sigma^2}dt.$$

On the other hand we have that

$$d(X(t)Ke^{Bt}) = BX(t)Ke^{Bt}dt + X(t)Ke^{Bt}[\rho + u(t)(\mu-\rho) - Ke^{Bt}e^{\frac{\delta(s+t)}{\gamma-1}}]dt$$
$$+ X(t)Ke^{Bt}\sigma u(t)dB(t).$$

Consider

$$\hat{u}(t) = -\frac{(\mu-\rho)}{\sigma^2(\gamma-1)},$$

and

$$K = e^{-Bt}e^{-\frac{\delta(s+t)}{\gamma-1}}[B + \frac{\gamma\rho}{\gamma-1} - \frac{1}{2}\gamma\frac{(\mu-\rho)^2}{\sigma^2(\gamma-1)^2}].$$

For K to be independent of t, we must have $B = -\frac{\delta}{\gamma-1}$, which gives us

$$K = \left[-\frac{\delta}{\gamma-1} + \frac{\gamma\rho}{\gamma-1} - \frac{1}{2}\gamma\frac{(\mu-\rho)^2}{\sigma^2(\gamma-1)^2}\right].$$

With this K and

$$\hat{u}(t) = -\frac{(\mu - \rho)}{\sigma^2(\gamma - 1)}$$

we can conclude that we have

$$p^{\frac{1}{\gamma-1}}(t) = X(t)Ke^{Bt}.$$

It is now clear that

$$\hat{\lambda}(t) = K = \hat{\lambda}.$$

which gives us that

$$\begin{aligned}\hat{p}(t)\hat{X}(t) &= X^{\gamma}(t)K^{\gamma}(t)e^{\gamma Bt}\\ &= X^{\gamma}(t)\hat{\lambda}^{\gamma},\end{aligned}$$

so that

$$\begin{aligned}-\hat{p}(t)\hat{X}(t) &= -e^{-\frac{\gamma\delta(s+t)}{\gamma-1}}\hat{\lambda}^{\gamma}x_0^{\gamma}e^{\gamma\int_0^t \rho-\frac{(\mu-\rho)^2}{\sigma^2(\gamma-1)}-\hat{\lambda}-\frac{(\mu-\rho)^2}{\sigma^2(\gamma-1)^2}]ds-\gamma\int_0^t \frac{(\mu-\rho)}{\sigma(\gamma-1)}dB(s)}\\ &= -e_0^{-\frac{\gamma\delta(s+t)}{\gamma-1}}\hat{\lambda}^{\gamma}e^{\gamma\rho t-\gamma\frac{(\mu-\rho)^2}{\sigma^2(\gamma-1)^2}t-\gamma\hat{\lambda}t-\gamma\frac{(\mu-\rho)^2}{\sigma^2(\gamma-1)^2}t-\gamma\frac{(\mu-\rho)}{\sigma(\gamma-1)}B(t)}\\ &\geq -e^{-\frac{\gamma\delta(s+t)}{\gamma-1}}\hat{\lambda}^{\gamma}x_0^{\gamma}e^{-\gamma^2\frac{(\mu-\rho)^2}{\sigma^2(\gamma-1)^2}-\gamma\frac{(\mu-\rho)}{\sigma(\gamma-1)}B(t)}.\end{aligned}$$

If δ, γ, ρ deterministic, then

$$\begin{aligned}E[\overline{\lim_{t\to\infty}}\hat{p}(t)(X(t)-\hat{X}(t))] &\geq -\lim e^{-\delta(s+t)}\hat{\gamma}x_0^{\gamma}e^{-\gamma^2\frac{(\mu-\rho)^2}{\sigma^2(\gamma-1)^2}}E[e^{-\frac{(\mu-\rho)}{\sigma(\gamma-1)}B(t)}]\\ &= 0.\end{aligned}$$

So we have that $E[\overline{\lim_{t\to\infty}}\hat{p}(t)(X(t)-\hat{X}(t))] = 0$, which gives us that $(\hat{\lambda}, \hat{u})$, where

$$\hat{\lambda} = \left[-\frac{\delta}{\gamma-1} + \frac{\gamma\rho}{\gamma-1} - \frac{1}{2}\gamma\frac{(\mu-\rho)^2}{\sigma^2(\gamma-1)^2}\right],$$

and

$$\hat{u} = -\frac{(\mu-\rho)}{\sigma^2(\gamma-1)}$$

is an optimal control.

6 Necessary Maximum Principle

To answer the question: if $\hat{u}$ is optimal does it satisfy

$$\begin{aligned}&E\left[H(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))|\mathcal{E}_t\right]\\ &= \max_{u\in U} E\left[H(t,\hat{X}(t),u,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))|\mathcal{E}_t\right], \end{aligned} \tag{28}$$

we assume the following two requirements:

A1 For all t, h such that $0 \le t < t+h \le T$, all $i = 1, \dots, k$ and for all bounded $\mathcal{E}_t$-measurable $\alpha = \alpha(\omega)$, the control $\beta(s) := (0, \dots, \beta_i(s), 0, \dots, 0) \in U \subset \mathbb{R}^k$ with

$$\beta(s) := \alpha_i \mathbf{1}_{[t,t+h]}(s),$$

belongs to $\mathcal{A}_{\mathcal{E}}$.

A2 For all $u, \beta \in \mathcal{A}_{\mathcal{E}}$ with β bounded, there exists $\delta > 0$ such that $\hat{u}+\epsilon\beta \in \mathcal{A}_{\mathcal{E}}$ for all $\epsilon \in (-\delta, \delta)$.

Given $u, \beta \in \mathcal{A}_{\mathcal{E}}$ with β bounded, define the process $Y(t) = Y^{(u,\beta)}(t)$ by

$$Y(t) = \frac{d}{d\epsilon} X^{\hat{u}+\epsilon\beta}(t)|_{\epsilon=0} = (Y_1(t), ..., Y_n(t))^T.$$

Notice that $Y(0) = 0$ and

$$dY_i(t) = \lambda_i(t)dt + \sum_{j=1}^{n} \xi_{ij}(t)dB_j(t) + \sum_{j=1}^{n} \int_{\mathbb{R}_0^n} \zeta_{ij}(t,z)\tilde{N}_j(dz,dt),$$

where

$$\begin{aligned} \lambda_i(t) &= \nabla_x b_i(t, X(t), u(t))^T Y(t) + \nabla_u b_i(t, X(t), u(t))^T \beta(t), \\ \xi_{ij}(t) &= \nabla_x \sigma_{ij}(t, X(t), u(t))^T Y(t) + \nabla_u \sigma_{ij}(t, X(t), u(t))^T \beta(t), \\ \zeta_{ij}(t,z) &= \nabla_x \theta_{ij}(t, X(t), u(t))^T Y(t) + \nabla_u \theta_{ij}(t, X(t), u(t))^T \beta(t). \end{aligned}$$

We can then give a answer to the question.

Theorem 6.1 (Partial Information Necessary Maximum Principle). *Suppose $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ is a local maximum for $J(u)$, meaning that for all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$ there exists a $\delta > 0$ such that $\hat{u} + \epsilon\beta \in \mathcal{A}_{\mathcal{E}}$ for all $\epsilon \in (-\delta, \delta)$ and*

$$h(\epsilon) := J(\hat{u} + \epsilon\beta), \epsilon \in (-\delta, \delta)$$

is maximal at $\epsilon = 0$. Suppose there exists a solution $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z))$ to the adjoint equation

$$\begin{aligned} d\hat{p}(t) = &-\nabla_x H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))dt + \hat{q}(t)dB(t) \\ &+ \int_{\mathbb{R}_0^n} \hat{r}(z,t)\tilde{N}(dz,dt), \end{aligned}$$

and

$$0 \le E\left[\overline{\lim_{t\to\infty}}[\hat{p}(t)^T(X(t) - \hat{X}(t))]\right] < \infty,$$

for all $u \in \mathcal{A}_{\mathcal{E}}$ and $p(t)Y(t,\epsilon)$ converges as $t \to \infty$, uniformly in ϵ, where $Y(t,\epsilon) := \frac{\partial}{\partial\epsilon} X^{\hat{u}+\epsilon\beta}$. Moreover assume that if $\hat{Y}(t) = Y^{(\hat{u},\beta)}(t)$, with corresponding coefficients $\hat{\lambda}_i$, $\hat{\xi_{ij}}$, $\hat{\zeta_{ij}}$, we have

$$E\left[\hat{Y}(t)^T[\hat{q}\hat{q}^T(t) + \int_{\mathbb{R}_0^n} \hat{r}\hat{r}^T(t,z)\nu(dx)]\hat{Y}(t)dt\right] < \infty,$$

and

$$E\left[\int_0^\infty \hat{p}^T(t)[\hat{\xi}\hat{\xi}^T(t,\hat{X}(t),\hat{u}(t)) + \int_{\mathbb{R}_0^n} \hat{\zeta}\hat{\zeta}^T(t,\hat{X}(t),\hat{u}(t),z)\nu(dz)]\hat{p}(t)dt\right] < \infty.$$

Then $\hat{u}$ is a stationary point for $E[H|\mathcal{E}]$ in the sense that for all $t \geq 0$,

$$E[\nabla_u H(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))|\mathcal{E}_t] = 0.$$

Proof. Since

$$0 \leq E\left[\overline{\lim_{t\to\infty}}[\hat{p}(t)^T(X(t)-\hat{X}(t))]\right],$$

we have that

$$E\left[\overline{\lim_{t\to\infty}}[\hat{p}(t)^T X^{\hat{u}+\epsilon\beta}(t)]\right] \geq E\left[\overline{\lim_{t\to\infty}}[\hat{p}(t)^T X^{\hat{u}}(t))]\right],$$

for all $\beta \in \mathcal{A}_{\mathcal{E}}$ for some ϵ. Define

$$g(\epsilon) = \overline{\lim_{t\to\infty}}[\hat{p}(t)^T X^{\hat{u}+\epsilon\beta}(t)],$$

so that

$$Eg(\epsilon) \geq Eg(0),$$

for all $\beta \in \mathcal{A}_{\mathcal{E}}$. This means that

$$\frac{d}{d\epsilon}(Eg(\epsilon))_{\epsilon=0} = 0.$$

So

$$\begin{aligned} 0 &= \frac{\partial}{\partial\epsilon}(E\left[\overline{\lim_{t\to\infty}}[\hat{p}(t)^T X^{\hat{u}+\epsilon\beta}(t)]\right])|_{\epsilon=0} \\ &= E\left[\frac{\partial}{\partial\epsilon}(\overline{\lim_{t\to\infty}}[\hat{p}(t)^T X^{\hat{u}+\epsilon\beta}(t)])|_{\epsilon=0}\right] \\ &= E\left[\overline{\lim_{t\to\infty}}[\hat{p}(t)^T \frac{\partial}{\partial\epsilon}(X^{\hat{u}+\epsilon\beta}(t))|_{\epsilon=0}]\right]. \end{aligned}$$

the interchanging of the limit w.r.t. the derivative operator holds for uniform limits with uniform convergence of the derivative. Interchanging derivative and integration is justified if

$$\left|\frac{\partial}{\partial\epsilon}(\overline{\lim_{t\to\infty}}[\hat{p}(t)^T X^{\hat{u}+\epsilon\beta}(t,\omega)])|_{\epsilon=0}\right| \leq F(\omega),$$

for some integrable function F. Now let

$$h(\epsilon) = J(\hat{u}+\epsilon\beta),$$

so that we have

$$\begin{aligned} 0 &= h'(0) \\ &= E\Bigg[\int_0^\infty \Big\{\nabla_x f(t,\hat{X}(t),\hat{u}(t))^T \frac{d}{d\epsilon} X^{\hat{u}+\epsilon\beta}(t)|_{\epsilon=0} + \nabla_u f(t,\hat{X}(t),\hat{u}(t))^T \beta(t)\Big\} dt \\ &+ \overline{\lim_{t\to\infty}}[\hat{p}(t)^T \frac{d}{d\epsilon}(X^{\hat{u}+\epsilon\beta}(t))|_{\epsilon=0}]\Bigg]. \end{aligned}$$

Using Itō's Lemma we get

$$E\left[\overline{\lim_{t\to\infty}}[\hat{p}(t)^T \frac{d}{d\epsilon}(X^{\hat{u}+\epsilon\beta}(t))|_{\epsilon=0}]\right]$$

$$= E\left[\int_0^\infty \Big\{\hat{p}(t)\Big[\nabla_x b(t,\hat{X}(t),\hat{u}(t))^T \frac{d}{d\epsilon}X^{\hat{u}+\epsilon\beta}(t)|_{\epsilon=0} + \nabla_u b(t,\hat{X}(t),\hat{u}(t))^T\beta(t)\Big]^T\right.$$

$$+ \frac{d}{d\epsilon}X^{\hat{u}+\epsilon\beta}(t)|_{\epsilon=0}(-\nabla_x H(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))$$

$$+ q(t)(\nabla_x\sigma(t,\hat{X}(t),\hat{u}(t))^T \frac{d}{d\epsilon}X^{\hat{u}+\epsilon\beta}(t)|_{\epsilon=0} + \nabla_u\sigma(t,\hat{X}(t),\hat{u}(t))^T\beta(t)$$

$$\left. + \hat{r}(t,z)(\nabla_x\theta(t,\hat{X}(t),\hat{u}(t))^T \frac{d}{d\epsilon}X^{\hat{u}+\epsilon\beta}(t)|_{\epsilon=0} + \nabla_u\theta(t,\hat{X}(t),\hat{u}(t))^T\beta(t)\nu(dz)\Big\}dt\right].$$

Since

$$\nabla_u H(t,x,u,p,q,r) = \nabla_u f(t,x,u) + \nabla_u b(t,x,u)p(t) + \nabla_u\sigma(t,x,y)q(t) + \int_{\mathbb{R}_0^n}\nabla_u\theta(t,x,u,z)r(t,z)\nu(dz),$$

and

$$\nabla_u H(t,x,u,p,q,r) = \nabla_x f(t,x,u) + \nabla_x b(t,x,u)p(t) + \nabla_x\sigma(t,x,y)q(t) + \int_{\mathbb{R}_0^n}\nabla_x\theta(t,x,u,z)r(t,z)\nu(dz),$$

we have

$$0 = E\left[\int_0^\infty \Big\{\nabla_u f(t,\hat{X}(t),\hat{u}(t)) + \nabla_u b(t,\hat{X}(t),\hat{u}(t))\hat{p}^T + \nabla_u\sigma(t,\hat{X}(t),\hat{u}(t))\hat{q}^T\right.$$

$$\left. + \hat{r}\nabla_u\theta(t,\hat{X}(t),\hat{u}(t))\beta(t))\Big\}dt\right]$$

$$= E\left[\int_0^\infty \nabla_u H(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))^T\beta(t)dt\right].$$

Define

$$\beta(s) := \alpha\mathbf{1}_{[t,t+h]}(s).$$

Then

$$E\left[\int_t^{t+h}\nabla_u H(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))^T\alpha(t)dt\right] = 0.$$

Differentiating with respect to h at $h = 0$ gives

$$E\left[\nabla_u H(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))^T\alpha\right] = 0.$$

Since this holds for all $\mathcal{E}$ measurable α, we have that

$$E\left[\nabla_u H(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))^T\alpha|\mathcal{E}\right] = 0,$$

which proves the theorem. □

References

[1] N. Agram, S. Haadem, B. Øksendal, and F. Proske. A maximum principle for infinite horizon delay equations. *Preliminary version*, 2012.

[2] F. Baghery and B. Øksendal. A maximum principle for stochastic control with partial information. *Stochastic Analysis and Applications*, 25(3):705–717, 2007.

[3] E. Pardoux G. Barles, R. Buckdahn. Backward stochastic differential equations and integral-partial differential equations. *Stochastics and Stochastics Reports*, 60:57–83, 2009.

[4] H. Halkin. Necessary conditions for optimal control problems with infinite horizons. *Econometrica*, 42:267–272, 1974.

[5] J.Li and S. Peng. Stochastic optimization theory of backward stochastic differential equations with jumps and viscosity solutions of Hamilton-Jacobi-Bellman equations. *Nonlinear Analysis*, 70:1779–1796, 2009.

[6] B. Maslowski and P. Veverka. Infinite horizon maxmimum principle for the discounted control problem - incomplete version. *arXiv*, 2011.

[7] E. Pardoux. Bsdes', weak convergence and homogenizations of semilinear pdes. In F.H. Clark and R.J. Stern, editors, *Nonlinear Analysis, Differential Equations and Control*, pages 503–549. Kluwer Academic, Dordrecht, 1999.

[8] S. Peng and Y. Shi. infinite horizon forward-backward stochastic differential equations. *Stoch. Proc. and their Appl.*, 85:75–92, 2000.

[9] L. C. G. Rogers and D. Williams. *Diffusions, Markov Processes, and Martingales*. Cambridge University Press, second edition, 2000.

[10] M. Royer. Backward stochastic differential equations with jumps and related non-linear expectations. *Stochastic Processes and Their Applications*, 116:1358–1376, 2006.

[11] A. Seierstad, A. Strøm, and K. Sydsæter. *Matematisk Analyse*. Gyldendal Akademisk, 4 edition, 2006.

[12] R. Situ. On solutions of backward stochastic differential equations with jumps and with non-Lipschitzian coefficients in Hilbert spaces and stochastic control. *Statistics and Probability Letters*, 60:279–288, 2002.

[13] A. Sulem and B. Øksendal. *Applied Stochastic Control of Jump Diffusions*. Springer, second edition, 2007.

[14] X.Li and S. Tang. Necessary conditions for optimal control of stochastic systems with random jumps. *SIAM Journal of Control and Optimization*, 32:1447–1475, 1994.

[15] J. Yin. On solutions of a class of infinite horizon fbsdes. *Statistics and Probability Letters*, 78:2412–2419, 2008.