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On chaos representation and orthogonal polynomials for the doubly stochastic Poisson process

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Abstract. In an L_2-framework, we study various aspects of stochastic calculus with respect to the centered doubly stochastic Poisson process. We introduce an orthogonal basis via multilinear forms of the value of the random measure and we analyze the chaos representation property. We revise the structure of non-anticipating integration for martingale random fields and in this framework we study non-anticipating differentiation. We present integral representation theorems where the integrand is explicitely given by the non-anticipating derivative.

Stochastic derivatives of anticipative nature are also considered: The Malliavin type derivative is put in relationship with another anticipative derivative operator here introduced. This gives a new structural representation of the Malliavin derivative based on simple functions. Finally we exploit these results to provide a Clark-Ocone type formula for the computation of the non-anticipating derivative.

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1. Introduction

The doubly stochastic Poisson process (DSPP) also known as the Cox process, was introduced in [7] as a generalization of the Poisson process in the sense that the intensity is stochastic. Models based on DSPP's are used in risk theory, in the study of ruin probabilities in insurance and insurance-linked securities pricing, and for stochastic volatility see e.g. [1, 6, 17, 22].

For a given doubly stochastic Poisson process H with intensity α, we investigate some elements of stochastic calculus for $\tilde{H} := H - \alpha$, i.e. the *centered Doubly stochastic Poisson process* (CDSPP) on a quite general Hausdorff topological space X. The stochastic intensity α is assumed non-atomic. The paper is dedicated to the study of the structure of L_2-spaces generated by the noise and the non-anticipating integration and differentiation schemes with stochastic integral representation in view. The foreseen applications of such integral representations is in the study of backward stochastic differential equation and it is, at present, work in progress, see [16].

First we show that the observations of $\tilde{H} = H - \alpha$ give complete information on both H and α. Specifically, the σ-algebra generated by $\tilde{H}$ coincides with the one generated by H and α. With respect to the space $L_2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the CDSPP, we suggest an orthogonal system of polynomials which lead to a chaos expansion type of result. This orthogonal system is based on what we call α-multilinear forms. These prove to be key constructive elements in our proofs.

After this analysis on a general X, we specify the study to the time-space $X = (0, T] \times Z$ with the total ordering induced by time. Here we introduce an information structure associated to the CDSPP. We consider the filtration $\mathbb{G}$ generated by the CDSPP augmented by the knowledge of the whole intensity α. Note that, with respect to $\mathbb{G}$, the CDSPP is a stochastic measure with conditionally independent values. In this setup we study elements of stochastic integration and differentiation. We find a stochastic integral representation for all elements in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ by interpreting the CDSPP as a martingale random field (see e.g. [15]) and applying the corresponding Itô stochastic integration scheme. The representation of $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ is explicit in the sense that the integrand is uniquely determined as the non-anticipating stochastic derivative $\mathscr{D}^{\mathcal{G}}\xi$ with respect to the CDSPP. The non-anticipating derivative, introduced in [11] and later developed to include Lévy type random measures (see [12, 14]) and martingale random fields as integrators (see [15]) is defined by the linear operator adjoint to the Itô stochastic integral. A general formula for the calculus is here given in terms of limit of specific simple stochastic functions.

When discussing non-anticipating differentiation, the connections with the well-known correspondent of the the Clark-Ocone formulae have to be taken into account. A first study of chaos expansions in terms of iterated integrals for processes with conditionally independent increments can be found in [28]. Starting form this set up, a Malliavin derivative operator is defined.

In the present paper we discuss explicitly the relationship between the orthogonal polynomials here suggested and the Itô type iterated integrals and we retrace the relationships between the non-anticipating derivative and the Clark-Ocone formula based on the Malliavin derivative operator given in the literature. Our study however takes a different approach to Malliavin calculus. In fact we introduce a new anticipative derivative operator D^c as a limit of specific simple stochastic functions. Because of its particular structure, it is immediate to see that the non-anticipative derivative $\mathscr{D}^{\mathcal{G}}\xi$ at time t is the projection of D^c on the information $\mathcal{G}_t$. On the other side we prove that this operator coincides with the Malliavin derivative $D\xi$ as introduced in [28]. These arguments provide a new structural approach to the Malliavin derivative that is useful also for other type of random measures as integrators.

We have partially considered also integration with respect to the smaller filtration generated by the CDSPP only. Based on the martingale structure the non-anticipative differentiation can be carried through. However, we remark that there is no structure of conditional independence in this case and the study of anticipative differentiation is rather different. The study of stochastic differentiation in this setting will be developed separately.

To conclude we remark that stochastic integral representations have been investigated in [3, 9, 18, 19] for general point processes. Our contribution differs because we consider the filtration generated by the CDSPP, which is larger than the filtration generated by the DSPP alone.

The paper is organized as follows. Section 2 provides the basic information on DSPP and CDSPP on a general X. Multilinear forms and chaos expansions are studied in section 3. For $X = (0,T] \times Z$, stochastic non-anticipating integration and martingale random fields are discussed in section 4. Section 5 presents the non-anticipating derivative $\mathscr{D}^{\mathcal{G}}$. A review on iterated integrals and their connection with multilinear forms is detailed in section 6. Finally section 7 presents the anticipative derivatives D^c and D, their computation, and their relationship with the non-anticipting derivative $\mathscr{D}^{\mathcal{G}}$ via a Clark-Ocone type formula.

2. The doubly stochastic Poisson process

Let X be a locally compact, second countable Hausdorff topological space. Under these conditions, there exists a complete, seperable metric μ generating the topology on X. In particular this implies that X is σ-compact, i.e. that it admits representation as a countable union of compact sets, and that the topology on X has a countable basis consisting of precompact sets, i.e. sets with compact closure. We denote $\mathcal{B}_X$ the Borel σ-algebra of X and $\mathcal{B}_X^c$ the precompacts of $\mathcal{B}_X$. The stochastic elements considered in the paper are related to the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let α be a (positive) random measure on X. We assume that α is non-atomic, meaning that $\mathbb{P}\big(\alpha(\{x\}) = 0 \text{ for all } x \in X\big) = 1$. We further assume that α has moments of all orders on the precompact sets, i.e. that

$$\mathbb{E}\big[\alpha(\Delta)^k\big] < \infty \quad \text{for all } \Delta \in \mathcal{B}_X^c,\ k = 1, 2, \dots \tag{2.1}$$

Let us define

$$V(\Delta) := \mathbb{E}[\alpha(\Delta)], \quad \Delta \in \mathcal{B}_X.$$

We note that V is a non-atomic σ-finite measure (see e.g. [20, chapter 1.2]), which is finite at least on all precompact sets. The σ-algebra generated by α will be denoted $\mathcal{F}^\alpha$.

Let H be a random measure on X and let $\mathcal{F}_\Delta^H$ denote the σ-algebra generated by $H(\Delta')$, $\Delta' \in \mathcal{B}_X : \Delta' \subset \Delta$ (with $\Delta \in \mathcal{B}_X$). Set $\mathcal{F}^H$ to be the σ-algebra generated by all the values of H.

Definition 2.1. The random measure H is a doubly stochastic Poisson process (DSPP) if

A1) $\mathbb{P}\Big(H(\Delta) = k \,\Big|\, \alpha(\Delta)\Big) = \frac{\alpha(\Delta)^k}{k!} e^{-\alpha(\Delta)}$,

A2) $\mathcal{F}_{\Delta_1}^H$ and $\mathcal{F}_{\Delta_2}^H$ are conditionally independent given $\mathcal{F}^\alpha$ whenever Δ_1 and Δ_2 are disjoint sets.

In particular, the conditional independence A2) implies that

$$\mathbb{E}\Big[f\big(H(\Delta_1)\big) \,\Big|\, \mathcal{F}_{\Delta_2}^H \vee \mathcal{F}^\alpha\Big] = \mathbb{E}\Big[f\big(H(\Delta_1)\big) \,\Big|\, \mathcal{F}^\alpha\Big],$$

whenever $\Delta_1, \Delta_2 \in \mathcal{B}_X$, $\Delta_1 \cap \Delta_2 = \emptyset$ and for $f : \mathbb{R} \to \mathbb{R}$ such that the conditional expectation is well defined. From A1) we have

$$\mathbb{E}\Big[f\big(H(\Delta)\big) \,\Big|\, \mathcal{F}^\alpha\Big] = \sum_{k=0}^{\infty} f(k) \frac{\alpha(\Delta)^k}{k!} e^{-\alpha(\Delta)}, \quad \Delta \in \mathcal{B}_X, \tag{2.2}$$

and in particular

$$\mathbb{E}\Big[H(\Delta) \,\Big|\, \mathcal{F}^\alpha\Big] = \alpha(\Delta), \quad \Delta \in \mathcal{B}_X. \tag{2.3}$$

From the above formulae the following ones are obtained [18, Lemma 3a p23]:

$$\begin{aligned}
\mathbb{E}\Big[H\big(\Delta\big)\Big] &= \mathbb{E}\big[\alpha(\Delta)\big] = V(\Delta) \\
\mathrm{Var}\Big(H(\Delta)\Big) &= \mathbb{E}\big[\alpha(\Delta)\big] + \mathrm{Var}\big(\alpha(\Delta)\big) \\
\mathbb{E}\Big[H(\Delta)^2\Big] &= \mathbb{E}\big[\alpha(\Delta)\big] + \mathbb{E}\big[\alpha(\Delta)^2\big]
\end{aligned}$$

for $\Delta \in \mathcal{B}_X$ with $V(\Delta) < \infty$. In the case $V(\Delta) = \infty$, the above relationships hold but clearly $\mathrm{Var}(H(\Delta)) = \mathbb{E}[H(\Delta)^2] = \mathbb{E}[H(\Delta)] = \infty$.

Definition 2.2. The centered doubly stochastic Poisson process (CDSPP) is the signed random measure $\tilde{H} := H - \alpha$, ie

$$\tilde{H}(\Delta) := H(\Delta) - \alpha(\Delta), \quad \Delta \in \mathcal{B}_X. \tag{2.4}$$

We denote $\mathcal{F}^{\tilde{H}}$ the filtration generated by $\tilde{H}$. For any $\Delta \in \mathcal{B}_X$ with $V(\Delta) < \infty$, the conditional first moment is

$$\mathbb{E}\Big[\tilde{H}(\Delta)\Big|\mathcal{F}^{\alpha}\Big] = 0.$$

and the conditional second moment is

$$\mathbb{E}\Big[\tilde{H}(\Delta)^2\Big|\mathcal{F}^{\alpha}\Big] = \alpha(\Delta).$$

and thus

$$\mathbb{E}\big[\tilde{H}(\Delta)^2\big] = \mathrm{Var}\big(\tilde{H}(\Delta)\big) = \mathbb{E}\big[\alpha(\Delta)\big] = V(\Delta). \tag{2.5}$$

For the remaining conditional moments, the following recurrence formula holds:

Proposition 2.3.

$$\mathbb{E}\big[\tilde{H}(\Delta)^3\big|\mathcal{F}^{\alpha}\big] = \alpha(\Delta)$$

$$\mathbb{E}\big[\tilde{H}(\Delta)^n\big|\mathcal{F}^{\alpha}\big] = \alpha(\Delta) + \alpha(\Delta)\sum_{k=2}^{n-2}\binom{n}{k}\mathbb{E}\Big[\tilde{H}(\Delta)^k\Big|\mathcal{F}^{\alpha}\Big], \quad n \geq 4. \tag{2.6}$$

Proof. The formulae are obtained by induction for the Poisson distribution in [25, Section 3]. Those computations can easily be adapted to our case using (2.2). □

Corollary 2.4. *For $n \geq 3$, we have that*

$$\mathbb{E}\Big[\tilde{H}(\Delta)^n\Big] < \infty$$

if and only if

$$\mathbb{E}\Big[\alpha(\Delta)^{n-2}\Big] < \infty.$$

Proof. The statement holds for $n = 3$ from Proposition 2.3. Moreover, we also obtain

$$\mathbb{E}\Big[\tilde{H}(\Delta)^n\Big] = \mathbb{E}\Big[\alpha(\Delta) + \alpha(\Delta)\sum_{k=2}^{n-2}\binom{n-1}{k}\mathbb{E}\big[\tilde{H}(\Delta)^k\big|\mathcal{F}^{\alpha}\big]\Big], \quad n \geq 4. \tag{2.7}$$

Using induction we can prove that (2.7) is the expectation of a polynomial of $\alpha(\Delta)$ of degree $n-2$. □

Remark 2.5. We remark that, in view of Corollary 2.4, the assumption (2.1) is necessary and sufficient to ensure that $\tilde{H}(\Delta)$ has finite moments of all orders for $\Delta \in \mathcal{B}_X^c$.

For the arguments presented in the sequel it is crucial to investigate the relationship between the σ-algebras $\mathcal{F}^{\tilde{H}}$ and $\mathcal{F}^H \vee \mathcal{F}^{\alpha}$. While it is immediate to see that $\mathcal{F}^{\tilde{H}} \subseteq \mathcal{F}^H \vee \mathcal{F}^{\alpha}$, the opposite relationship is more delicate. Here after we introduce a dissecting system on X which is instrumental in the study of the considered random measures and associated structures. Recall that $\mathcal{B}_X^c$ is a ring generating the topology on X and that X is a Hausdorff topological space such that $X = \bigcup_{n=1}^{\infty} X_n$, where X_n, $n = 1, 2, \dots$ is a growing sequence of compacts. Hence $V(X_n) < \infty$. Denote $|\Delta| := \sup_{x,y\in\Delta} \mu(x, y)$, $\Delta \subset X$, where μ is the metric in X. Then $|X_n| < \infty$ for all n.

Being V non-atomic, for every n and $\epsilon_n > 0$, there exists a partition of X_n, i.e. a finite family of pairwise disjoint sets:

$$\Delta_{n,1}, \dots, \Delta_{n,K_n} \in \mathcal{B}_X^c : \quad X_n = \bigcup_{k=1}^{K_n} \Delta_{n,k} \tag{2.8}$$

such that $\sup_{k=1,\dots,K_n} V(\Delta_{n,k}) \le \epsilon_n$ and $\sup_{k=1,\dots,K_n} |\Delta_{n,k}| \le \epsilon_n$.

Let us consider a decreasing sequence $\epsilon_n \searrow 0$, $n \to \infty$. Then, based on (2.8), we give the following definition.

Definition 2.6. A *dissecting system* of X is the sequence of partitions of X:

$$\Delta_{n,1}, \dots, \Delta_{n,K_n+1}, \quad n = 1, 2, \dots \tag{2.9}$$

with $\bigcup_{k=1}^{K_n} \Delta_{n,k} = X_n$ from (2.8) and $\Delta_{n,K_n+1} := X \setminus X_n$, satisfying the *nesting property*:

$$\Delta_{n,k} \cap \Delta_{n+1,j} = \Delta_{n+1,j} \text{ or } \emptyset \tag{2.10}$$

for all $k = 1, \dots, K_n + 1$ and $j = 1, \dots, K_{n+1} + 1$.

We remark that, from (2.8) and (2.10), we have

$$\sup_{k=1,\dots,K_n} V(\Delta_{n,k}) \le \epsilon_n \to 0, \text{ and } \sup_{k=1,\dots,K_n} |\Delta_{n,k}| \le \epsilon_n \to 0 \;\; n \to \infty \tag{2.11}$$

We can refer to e.g. [20] and [8] for more on disscting systems and partitions.

Lemma 2.7. *For any $\Delta \in \mathcal{B}_X$ such that $\alpha(\Delta) < \infty$ $\mathbb{P}$-a.s. we have that*

$$\sup_{k=1,\dots,K_n+1} \alpha(\Delta \cap \Delta_{n,k}) \longrightarrow 0, \; n \to \infty \; \mathbb{P}\text{-}a.s. \tag{2.12}$$

Proof. The sets

$$\tilde{\Delta}_{n,k} := \Delta \cap \Delta_{n,k}, \quad k = 1, \dots K_n, \; n = 1, 2, \dots$$

constitute a dissecting system of Δ. Note that $\alpha(\Delta_{k,n}) < \infty$ $\mathbb{P}$-a.s. for all k and n. Let $\tilde{\Omega}$ be the event where α is non-atomic and $\alpha(\Delta) < \infty$. Then $\mathbb{P}(\tilde{\Omega}) = 1$. From (2.9) we have

$$\alpha(\tilde{\Delta}_{n+1,j}, \omega) \le \sup_{k=1,\dots,K_n+1} \alpha(\tilde{\Delta}_{n,k}, \omega) \le \alpha(\Delta, \omega), \quad \omega \in \tilde{\Omega}$$

for all $j = 1, \dots, K_n + 1$. Hence, for every $n = 1, 2, \dots,$ we have

$$\sup_{j=1,\dots,K_{n+1}+1} \alpha(\tilde{\Delta}_{n+1,j}, \omega) \le \sup_{k=1,\dots,K_{n+1}+1} \alpha(\tilde{\Delta}_{n,j}, \omega), \quad \omega \in \tilde{\Omega}.$$

We denote $A(\omega) := \lim_{n\to\infty} \sup_{k=1,\dots K_{n+1}+1} \alpha(\tilde{\Delta}_{n,k}, \omega)$, $\omega \in \tilde{\Omega}$. Naturally $A(\omega) \ge 0$, but we need to prove $A(\omega) = 0$. We proceed by contradiction. Set $\tilde{\Omega}_0 := \{\omega \in \tilde{\Omega} | A(\omega) > 0\}$ and suppose $\mathbb{P}(\tilde{\Omega}_0) > 0$. For each n there exists a set $\tilde{\Delta}_{n,\delta(n)}$ such that $\alpha(\tilde{\Delta}_{n,\delta(n)}, \omega) \ge A(\omega) > 0$, $\omega \in \tilde{\Omega}_0$. Comparing $\tilde{\Delta}_{n,\delta(n)}$ with the sets $\tilde{\Delta}_{n-1,j}$, $j = 1, \dots K_{n-1} + 1$, we see that there is a set $\tilde{\Delta}_{n-1,\delta(n-1)}$ such that $\tilde{\Delta}_{n-1,\delta(n-1)} \supseteq \tilde{\Delta}_{n,\delta(n)}$.

Hence there exists a decreasing sequence of sets

$$\tilde{\Delta}_{n,\delta(n)}, \quad n = 1, 2, \dots$$

such that for every n, $\tilde{\Delta}_{n,\delta(n)}$ is an element of the dissecting system of Δ and $0 < A(\omega) \leq \alpha(\tilde{\Delta}_{n,\delta(n)}, \omega)$ ($\omega \in \tilde{\Omega}_0$). On the other side, from the property (2.11) of the dissecting system on X, and hence on Δ, the limit of a decreasing sequence of sets is either empty or a singleton. Thus we have

$$\lim_{n\to\infty} \alpha(\tilde{\Delta}_{n,\delta(n)}, \omega) = 0, \quad \omega \in \tilde{\Omega}_0$$

since α is a non-atomic measure for $\omega \in \tilde{\Omega}_0$. This is a contradiction, and hence $A(\omega) = 0$ for all $\omega \in \tilde{\Omega}_0$. □

Theorem 2.8. *The following equality holds:*

$$\mathcal{F}^{\tilde{H}} = \mathcal{F}^{H} \vee \mathcal{F}^{\alpha}.$$

Proof. It is sufficient to show that $H(\Delta)$ and $\alpha(\Delta)$ are $\mathcal{F}^{\tilde{H}}$-measurable for any $\Delta \in \mathcal{B}^c_X$. Let $\Delta \in \mathcal{B}^c_X$ and recall its representation

$$\Delta = \bigcup_{k}^{K_n+1} \tilde{\Delta}_{n,k} = \bigcup_{k}^{K_n+1} \Big(\Delta \cap \Delta_{n,k}\Big), \quad n = 1, 2, \dots,$$

as a pairwise disjoint union of sets obtained from the dissecting system (2.9) of X. Consider

$$g_n(\Delta) := \sum_{k=1}^{K_n+1} \operatorname{ceil}\Big(\tilde{H}(\tilde{\Delta}_{n,k})\Big) = \sum_{k=1}^{K_n+1} \operatorname{ceil}\Big(H(\tilde{\Delta}_{n,k}) - \alpha(\tilde{\Delta}_{n,k})\Big),$$

where $\operatorname{ceil}(y)$ is the smallest integer greater than y. The random variables $g_n(\Delta)$, $n = 1, \dots,$ are clearly $\mathcal{F}^{\tilde{H}}$-measurable. From Lemma 2.7 there exists for $\mathbb{P}$-a.a. ω, a $N(\omega) \in \mathbb{N}$ such that $\sup_{k=1,\dots,K_n+1} \alpha(\tilde{\Delta}_{n,j}, \omega) < 1$ for $n > N(\omega)$. Then we have

$$\lim_{n\to\infty} \operatorname{ceil}\Big(H(\tilde{\Delta}_{n,k}) - \alpha(\tilde{\Delta}_{n,k})\Big) = H(\tilde{\Delta}_{n,k}) \quad \mathbb{P}\text{-a.s.}$$

Thus

$$\lim_{n\to\infty} g_n(\Delta) = \lim_{n\to\infty} \sum_{k=1}^{K_n+1} \operatorname{ceil}\Big(\tilde{H}(\tilde{\Delta}_{n,k})\Big) = H(\Delta) \quad \mathbb{P}\text{-a.s.}$$

and $H(\Delta)$ is a pointwise limit of $\mathcal{F}^{\tilde{H}}$-measurable functions. Since $\alpha(\Delta) = H(\Delta) - \tilde{H}(\Delta)$, we also have that $\alpha(\Delta)$ is $\mathcal{F}^{\tilde{H}}$-measurable. □

Note that the initial assumption that α is $\mathbb{P}$-a.s. non-atomic is crucial for this result. On the other side we remark that the assumption (2.1) is here not required.

Theorem 2.8 can be regarded as an extension of a result proved for a time-changed Lévy processes with independent time-change in [26].

3. Multilinear forms, polynomials, and chaos expansions

In this section we construct a system of multilinear forms and show how they describe the intrinsic orthogonal structures in $L_2(\Omega, \mathcal{F}, \mathbb{P})$. Here and in the sequel we set $\mathcal{F} = \mathcal{F}^{\tilde{H}} = \mathcal{F}^H \vee \mathcal{F}^\alpha$, see Theorem 2.8.

Definition 3.1. For any group of pairwise disjoint sets $\Delta_1, \dots \Delta_p \in \mathcal{B}_X^c$, an *$\alpha$-multilinear form of order p* is a random variable of type

$$\beta \prod_{j=1}^{p} \tilde{H}(\Delta_j), \quad p \geq 1, \tag{3.1}$$

where β is an $\mathcal{F}^\alpha$-measurable random variables with finite moments of all orders. The 0-order α-multilinear forms are the $\mathcal{F}^\alpha$-measurable random variable with finite moments of all orders.

This definition is a generalization of the one given in [13, page 7]: A *p-order multilinear form* of the values $\tilde{H}(\Delta_j)$, $j = 1, \dots p$, is a random variable of type

$$\prod_{j=1}^{p} \tilde{H}(\Delta_j), \quad p \geq 1. \tag{3.2}$$

The 0-order multilinear forms are the constants.

Note that any α-multilinear form is an element of $L_2(\Omega, \mathcal{F}, \mathbb{P})$. In fact, by assumption (2.1), the following holds:

$$\mathbb{E}\Big[\xi^2\Big] = \mathbb{E}\Big[\beta^2 \prod_{j=1}^{p} \mathbb{E}\big[\tilde{H}(\Delta_j)^2 \big| \mathcal{F}^\alpha\big]\Big] = \mathbb{E}\Big[\beta^2 \prod_{j=1}^{p} \alpha(\Delta_j)\Big] < \infty. \tag{3.3}$$

In the sequel we will consider multilinear forms on the sets (2.8)-(2.9) of the dissecting system of X.

The present section completes and extends to the CDSPP the results presented in [13] in which measure based multilinear forms were introduced for the study of stochastic calculus for Lévy stochastic measures. In that case the structure of independence of the random measure values was heavily exploited. In particular we stress that the space $\mathbb{H}^p$ via (3.4) here below is a substantial element of novelty and it is crucial for the forthcoming analysis.

Definition 3.2. For $p \geq 1$ we write $\mathbb{H}^p$ for the subspace in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the finite linear combinations of p-order α-multilinear form:

$$\sum_i \beta_i \prod_{j=1}^{p} \tilde{H}(\Delta_j^i). \tag{3.4}$$

Here above the sets Δ_j^i, $j = 1, \dots, p$, are pairwise disjoint and belong to the dissecting system (2.8)-(2.9) on X. The subspace $\mathbb{H}^0$ is the $\mathcal{F}^\alpha$-measurable random variables with finite variance.

Remark 3.3. We may consider the multipliers β in Definition 3.2 to be finite products of the form $\prod_{i=1}^{n} \alpha(\Delta_i)$ with Δ_i, $i = 1, \dots n$ pairwise disjoint sets from the dissecting system (2.8)-(2.9).

Remark 3.4. Let $p \geq 1$. By definition, for any $\xi \in \mathbb{H}^p$ there exists a sequence $\{\xi_m\}_m$ such that $\xi_m \to \xi$, $m \to \infty$ in $L_2(\Omega, \mathcal{F}, \mathbb{P})$, with $\xi_m = \sum_{l=1}^{L_m} \xi_{ml}$ and ξ_{ml} p-order α-multilinear forms. Indeed we can always consider the finite sums above with orthogonal terms. In fact two p-order α-multilinear forms are not orthogonal if and only if they are written via the product of the values of $\tilde{H}$ on the same sets.

Lemma 3.5. *For $p' \neq p''$, the subspaces $\mathbb{H}^{p'}$ and $\mathbb{H}^{p''}$ are orthogonal in $L_2(\Omega, \mathcal{F}, \mathbb{P})$.*

Proof. We assume that $p'' > p'$. It is sufficient to prove the statement for $\xi' \in \mathbb{H}^{p'}$, $\xi'' \in \mathbb{H}^{p''}$ of type

$$\xi' = \beta' \prod_{i=1}^{p'} \tilde{H}(\Delta^i), \quad \xi'' = \beta'' \prod_{j=1}^{p''} \tilde{H}(\Delta^j)$$

where Δ^i, $i = 1, \dots, p'$ and Δ^j, $j = 1 \dots p'$ are two groups of pairwise disjoint sets (2.9) of the dissecting system of X. Note that, in view of the nesting property (2.10), there exists $n \in \mathbb{N}$ such that all the sets above can be represented in terms of finite disjoint unions of elements from the same n'th partition (2.9)-(2.10). Thus we can represent ξ' and ξ'' by finite sums of p'-order and p''-order α-multilinear forms respectively over sets (2.9) in the same n'th partition (2.10):

$$\xi' = \beta' \sum_k \prod_{i=1}^{p'} \tilde{H}(\Delta^i_{n,k})$$

$$\xi'' = \beta'' \sum_l \prod_{j=1}^{p''} \tilde{H}(\Delta^j_{n,l}).$$

To prove the statement it is then enough to verify that for all k, l,

$$\mathbb{E}\Big[\beta' \prod_{i=1}^{p'} \tilde{H}(\Delta^i_{n,k}) \beta'' \prod_{j=1}^{p''} \tilde{H}(\Delta^j_{n,l})\Big] = 0.$$

We remark that being $p'' > p'$, there is at least one set among $\Delta^j_{n,l}$, $j = 1, \dots, p''$ that is different from $\Delta^i_{n,k}$, $i = 1, \dots, p'$. Denote such a set by $\Delta^{\hat{j}}_{n,l}$.

We have

$$\mathbb{E}\Big[\beta' \prod_{i=1}^{p'} \tilde{H}(\Delta^i_{n,k})\beta''\tilde{H}(\Delta^{\hat{j}}_{n,l}) \prod_{\substack{j=1\\ j\neq\hat{j}}}^{p''} \tilde{H}(\Delta^j_{n,l})\Big]$$

$$= \mathbb{E}\Big[\beta'\beta''\mathbb{E}\Big[\prod_{i=1}^{p'} \tilde{H}(\Delta^i_{n,k}) \prod_{\substack{j=1\\ j\neq\hat{j}}}^{p''} \tilde{H}(\Delta^j_{n,l})\Big|\mathcal{F}^\alpha\Big]\mathbb{E}\Big[\tilde{H}(\Delta^{\hat{j}}_{n,l})\Big|\mathcal{F}^\alpha\Big]\Big] = 0$$

By this we end the proof. □

Definition 3.6. We write $\mathbb{H}_p$ for the subspaces of $L_2(\Omega, \mathcal{F}, \mathbb{P})$ defined by:

$$\mathbb{H}_p := \sum_{q=0}^{p} \oplus \mathbb{H}^q.$$

Namely the subspaces generated by the linear combinations of α-multilinear forms:

$$\sum_i \beta_i \prod_{j=1}^{p_i} \tilde{H}(\Delta^i_j), \quad p_i \leq p. \tag{3.5}$$

We set

$$\mathbb{H} := \sum_{q=0}^{\infty} \oplus \mathbb{H}^q.$$

Lemma 3.7. *Let $\Delta', \Delta'' \in \mathcal{B}_X : \Delta' \cap \Delta'' = \emptyset$. Consider $\mathcal{F}_{\Delta'}$ and $\mathcal{F}_{\Delta''}$ as the σ-algebras generated by $\tilde{H}(\Delta)$, $\Delta \in \mathcal{B}_X : \Delta \subset \Delta'$ and $\Delta \subset \Delta''$, respectively. Let $\xi' \in \mathbb{H}_{p'}$ be $\mathcal{F}_{\Delta'}$-measurable and $\xi'' \in \mathbb{H}_{p''}$ be $\mathcal{F}_{\Delta''}$-measurable. The product $\xi'\xi''$ is measurable with respect to $\mathcal{F}_{\Delta' \cup \Delta''}$ and belongs to $\mathbb{H}_{p'+p''}$.*

Proof. If ξ' and ξ'' are of type (3.4), then clearly the product $\xi'\xi'' \in \mathbb{H}_{p'+p''}$ and it is $\mathcal{F}_{\Delta'\cup\Delta''}$-measurable. In the general case, ξ' and ξ'' are approximated in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ by sequences of elements ξ'_n and ξ''_n, $n = 1, 2, \dots$ of type (3.4):

$$\xi' = \lim_{n\to\infty} \xi'_n = \lim_{n\to\infty} \sum_i \beta'_{ni} \prod_{j=1}^{p'} \tilde{H}(\Delta^j_{n,i})$$

$$\xi'' = \lim_{n\to\infty} \xi''_n = \lim_{n\to\infty} \sum_k \beta'_{nk} \prod_{l=1}^{p''} \tilde{H}(\Delta^l_{n,k}).$$

Note that in view of the measurability assumptions we have $\Delta^j_{n,i} \subset \Delta'$, $j = 1, \dots, p'$ and $\Delta^j_{n,i} \subset \Delta''$, $j = 1, \dots, p''$ and also β'_{ni} are $\mathcal{F}^\alpha_{\Delta'}$-measurable, while β''_{nk} are $\mathcal{F}^\alpha_{\Delta''}$-measurable. Then it is easy to see that the statement holds. □

We remark that the result still holds true if we consider the σ-algebras $\mathcal{F}^H_{\Delta'} \vee \mathcal{F}^\alpha$ and $\mathcal{F}^H_{\Delta''} \vee \mathcal{F}^\alpha$, for $\Delta' \cap \Delta'' = \emptyset$.

The *polynomials* of the values of $\tilde{H}$ of degree p are the random variables ξ admitting representation as

$$\xi = \sum_{m=1}^{M} c_m \prod_{j=1}^{J_m} \tilde{H}(\Delta_j^m)^{p_j}, \quad c_m \in \mathbb{R},\ M, J_m, p_j \in \mathbb{N} \tag{3.6}$$

such that $\sum_{j=1}^{J_m} p_j \leq p$, $m = 1, 2 \ldots, M$ and $\Delta_j^m \in \mathcal{B}_X^c$, $j = 1, \ldots, J_m$ are pairwise disjoint.

Theorem 3.8. *All the polynomials of values of* $\tilde{H}$ *of degree less or equal to* p *belong to the subspace* $\mathbb{H}_p$.

Proof. Let ξ be a polynomial of degree p as in (3.6). We proceed by induction. If $p = 0$ then $\xi \in \mathbb{H}_0$ and if $p = 1$ then $\xi \in \mathbb{H}_1$. Suppose the statement holds for $q < p$, we verify this for p. For each m, let us consider elements

$$\xi_m := \prod_{j=1}^{J_m} \tilde{H}(\Delta_j^m)^{p_j}, \quad \sum_{j=1}^{J_m} p_j \leq p.$$

If $p_j < p$ for $j = 1, \ldots, J_m$

i) and $\sum_{j=1}^{J_m} p_j < p$ then the induction hypothesis holds and $\xi_m \in \mathbb{H}_p$.

ii) and $\sum_{j=1}^{J_m} p_j = p$ with $J_m > 1$, then for any j we have $\tilde{H}(\Delta_j^m)^{p_j} \in \mathbb{H}_{p_j}$ by the induction hypothesis. Furthermore, being the sets disjoint, from Lemma 3.7 we have that $\prod_{j=1}^{m} \tilde{H}(\Delta_j^{J_m}) \in \mathbb{H}_{\sum p_j}$.

Hence we only have to verify the case $J_m = 1$. Namely

$$\xi = \tilde{H}(\Delta_j^m)^p \in \mathbb{H}_p, \quad \text{for } p > 1.$$

Set $\Delta := \Delta_j^m$. For all n, we can represent Δ in terms of the sets (2.9) of the dissecting system of X

$$\Delta = \bigcup_{k=1}^{K_n+1} \big(\Delta \cap \Delta_{n,k}\big) := \bigcup_{k=1}^{K_n+1} \tilde{\Delta}_{n,k}$$

thus

$$\xi = \tilde{H}(\Delta)^p = \Big(\sum_{k=1}^{K_n+1} \tilde{H}(\tilde{\Delta}_{n,k})\Big)^p.$$

Let

$$\xi_n^{(1)} := \xi - \sum_{k=1}^{K_n+1} \tilde{H}(\tilde{\Delta}_{n,k})^p = \Big(\sum_{k=1}^{K_n+1} \tilde{H}(\tilde{\Delta}_{n,k})\Big)^p - \sum_{k=1}^{K_n+1} \tilde{H}(\tilde{\Delta}_{n,k})^p$$

and

$$\xi_n^{(2)} := \sum_{k=1}^{K_n+1} \tilde{H}(\tilde{\Delta}_{n,k})^p.$$

For all n we have $\xi = \xi_n^{(1)} + \xi_n^{(2)}$ and thus $\xi = \lim_{n\to\infty} \xi_n^{(1)} + \xi_n^{(2)}$ in $L_2(\Omega, \mathcal{F}, \mathbb{P})$. Let us investigate $\xi_n^{(1)}$ and $\xi_n^{(2)}$ seperately. First of all we note that $\xi_n^{(1)}$ is a polynomial, as in (3.6), with $p_j < p$ for all $j = 1, \dots, K_n + 1$. Thus $\xi_n^{(1)} \in \mathbb{H}_p$. Hence we have $\xi^{(1)} := \lim_{n\to\infty} \xi_n^{(1)} \in \mathbb{H}_p$ since $\mathbb{H}_p$ is closed in $L_2(\Omega, \mathcal{F}, \mathbb{P})$.

Let us consider the following limit in $L_2(\Omega, \mathcal{F}, \mathbb{P})$:

$$\begin{aligned}
\xi^{(2)} &:= \lim_{n\to\infty} \xi_n^{(2)} \\
&= \lim_{n\to\infty} \sum_{k=1}^{K_n+1} \Big(H(\tilde{\Delta}_{n,k}) - \alpha(\tilde{\Delta}_{n,k}) \Big)^p \\
&= \lim_{n\to\infty} \sum_{k=1}^{K_n+1} \sum_{j=0}^{p} \binom{p}{j} H(\tilde{\Delta}_{n,k})^{p-j} (-1)^j \alpha(\tilde{\Delta}_{n,k})^j \\
&= \sum_{j=0}^{p} \lim_{n\to\infty} \sum_{k=1}^{K_n+1} \binom{p}{j} H(\tilde{\Delta}_{n,k})^{p-j} (-1)^j \alpha(\tilde{\Delta}_{n,k})^j
\end{aligned} \tag{3.7}$$

Since $\mathbb{P}\big(H(\{x\}) \neq 0, 1 \text{ for some } x \in X\big) = 0$ see [18, Theorem 1.3 page 19], we will ultimately have $H(\tilde{\Delta}_{n,k}) = 0$ or 1 $\mathbb{P}$-a.s. as $n \to \infty$. Thus for the first term ($j = 0$) in (3.7), using dominated convergence, we have:

$$\lim_{n\to\infty} \sum_{k=1}^{K_n+1} (H(\tilde{\Delta}_{n,k})^p = \sum_{k=1}^{H(\Delta)} 1^p = H(\Delta) = \tilde{H}(\Delta) + \alpha(\Delta).$$

For the remaining terms ($j > 0$) in (3.7) the following estimate applies

$$\begin{aligned}
\Big| \sum_{k=1}^{K_n+1} (-1)^j H(\tilde{\Delta}_{n,k})^p \alpha(\tilde{\Delta}_{n,k})^{p-j} \Big| &\le \sum_{k=1}^{K_n+1} \mathbf{1}_{\{H(\tilde{\Delta}_{n,k})>0\}} H(\tilde{\Delta}_{n,k})^p \alpha(\tilde{\Delta}_{n,k})^{p-j} \\
&\le \sup_k \alpha(\tilde{\Delta}_{n,k})^{p-j} \sum_{k=1}^{K_n} \mathbf{1}_{\{H(\tilde{\Delta}_{n,k})>0\}} H(\tilde{\Delta}_{n,k})^p \\
&\le H(\tilde{\Delta})^{p+1} \sup_k \alpha(\tilde{\Delta}_{n,k})^{p-j} \longrightarrow 0, \ n \to \infty,
\end{aligned}$$

by Lemma 2.7. Thus $\xi^{(2)} = \tilde{H}(\Delta) + \alpha(\Delta) \in \mathbb{H}_1 \subseteq \mathbb{H}_p$. Hence $\xi = \xi^{(1)} + \xi^{(2)} \in \mathbb{H}_p$. □

The following statement is a direct consequence of the theorem above.

Corollary 3.9. All the polynomials of all degrees *of the values of* $\tilde{H}$ *belong to* $\mathbb{H}$.

Remark 3.10. We note that if the sets in (3.6) were not disjoint, then one could always represent the same polynomials via disjoint sets by applying the additivity of the measure $\tilde{H}$, but the degree would naturally change.

Following classical arguments via Fourier transforms (see e.g. [24, Lemma 4.3.1 and Lemma 4.3.2]) one can see that the random variables

$$\exp\Big\{ \sum_{j=1}^{J} x_j \tilde{H}(\Delta_j) \Big\}, \quad j = 1, 2 \dots, J; \ x = (x_1, \dots, x_J) \in \mathbb{R}^J,$$

with Δ_j, $j = 1, \dots, J$ pairwise disjoint sets in $\mathcal{B}_X^c$, constitue a complete system in $L_2(\Omega, \mathcal{F}, \mathbb{P})$. By Taylor approximations of the analytic extension on $\mathbb{C}^J$ we have that

$$\mathbb{E}\Big[\Big| \exp\Big\{\sum_{j=1}^{J} x_j \tilde{H}(\Delta_j)\Big\} - \sum_{p=0}^{q} \frac{\sum_{j=1}^{J} i x_j \tilde{H}(\Delta_j)^p}{p!}\Big|^2\Big] \longrightarrow 0, \; q \to \infty$$

(see e.g. [2, Eq. (26.4)] for an estimate of the quantity here above justifying the convergence.) Hence we can conclude:

Lemma 3.11. *The polynomials of the values of $\tilde{H}(\Delta)$, $\Delta \in \mathcal{B}_X^c$ are dense in $L^2\big(\Omega, \mathcal{F}, \mathbb{P}\big)$.*

Theorem 3.12 (Chaos expansion). *The following equality holds:*

$$\mathbb{H} = L_2(\Omega, \mathcal{F}, \mathbb{P}).$$

Namely, any $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ can be written as

$$\xi = \sum_{p=0}^{\infty} \xi_p, \quad \textit{where } \xi_p \in \mathbb{H}^p \textit{ for } p = 1, 2 \dots$$

Proof. The polynomials of the values of $\tilde{H}(\Delta)$ are dense in $L_2(\Omega, \mathcal{F}, \mathbb{P})$, see Lemma 3.11. By Theorem 3.8 and Corollary 3.9 all the polynomials are in $\mathbb{H}$. Since $\mathbb{H}$ is closed, we must have $L_2(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathbb{H}$. On the other side we recall that by construction $\mathbb{H} \subseteq L_2(\Omega, \mathcal{F}, \mathbb{P})$, see (3.3) and Definitions 3.2 and 3.6. □

Remark 3.13. Definitions 3.2 and 3.6 describe the spaces generated by α-multilinear forms. We can also consider analogous spaces generated only by the multilinear forms as in (3.2). However we have to stress that in this case the multilinear forms are not dense in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ with the only exception made when H is a Poisson random measure, i.e. if α is deterministic. Indeed write $\widetilde{\mathbb{H}}^p$ for the subspace in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the finite linear combinations of p-order multilinear forms:

$$\sum_i c_i \prod_{j=1}^{p} \tilde{H}(\Delta_j^i). \tag{3.8}$$

The sets Δ_j^i, $j = 1, \dots, p$, are pairwise disjoint and the c_i are constants. Set $\widetilde{\mathbb{H}}^0 = \mathbb{R}$ and $\widetilde{\mathbb{H}} := \sum_{p=0}^{\infty} \oplus \widetilde{\mathbb{H}}_p$. It is easily seen that $(\beta - \mathbb{E}[\beta]) \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ is orthogonal to $\widetilde{\mathbb{H}}$ whenever β is $\mathcal{F}^\alpha$-measurable. There are also α-multilinear forms of higher orders that are orthogonal to $\widetilde{\mathbb{H}}$, one example is $(\mathbb{E}[\beta | \mathcal{F}_\Delta^\alpha] - \mathbb{E}[\beta])\tilde{H}(\Delta)$ $(\Delta \in \mathcal{B}_X^c)$. Thus, in general, $\widetilde{\mathbb{H}} \neq L_2(\Omega, \mathcal{F}, \mathbb{P})$. The case if the Poisson random measure is a particular Lévy random measure and it falls in the study of [13]. In that case $\widetilde{\mathbb{H}} = L_2(\Omega, \mathcal{F}, \mathbb{P})$.

4. Non-anticipating stochastic integration

In the sequel we consider $X = [0,T] \times Z$ for $T < \infty$ and Z a locally compact, second countable Hausdorff topological space. Being interested in integration, without loss of generality we assume that

$$\alpha\big(\{0\} \times Z\big) = 0 \ \mathbb{P}\text{-a.s.}$$

Hence we can restrict the attention to $X = (0,T] \times Z$.

We chose a dissecting system of X to be given by partitions (2.8)-(2.9) of the form

$$\Delta_{n,k} = (s_{n,k}, u_{n,k}] \times B_{n,k}, \quad s_{n,k} < u_{n,k},\ B_{n,k} \in \mathcal{B}_Z^c \tag{4.1}$$

for $n = 1, 2, \dots$ and $k = 1, 2, \dots, K_n$, see Definition 2.6. Here $\mathcal{B}_Z$ denotes the Borel σ-algebra on Z and $\mathcal{B}_Z^c$ the family of precompacts for the topology in Z. The set X is ordered with the natural ordering given by time in $[0,T]$. Two filtrations naturally appear in the present setting:

- $\mathbb{F} := \big\{\mathcal{F}_t, t \in [0,T]\big\}$ where $\mathcal{F}_t$ is generated by $\{\tilde{H}(\Delta) : \Delta \in \mathcal{B}_{[0,t]\times Z}\}$
- $\mathbb{G} := \{\mathcal{G}_t, t \in [0,T]\}$ with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}^\alpha$.

Clearly we have that $\mathcal{F}_t \subseteq \mathcal{G}_t$, $\mathcal{F}_0$ is trivial but $\mathcal{G}_0 = \mathcal{F}^\alpha$ and $\mathcal{F}_T = \mathcal{G}_T = \mathcal{F}$.

We define a martingale random field as in [15], see in particular Remark [15, Remark 2.3] for historical notes. We can also refer to the work of [27] and [5] as pioneering in the use of martingale random fields in stochastic calculus, though mostly related to Brownian sheet.

Hence we can see that the stochastic set function $\tilde{H}(\Delta)$, $\Delta \in \mathcal{B}_X$ is a martingale random field (with square integrable values) with respect to $\mathbb{F}$ and $\mathbb{G}$ as it satisfies the following properties:

i) $\tilde{H}$ has a σ-finite variance measure $V(\Delta) = E\big[\tilde{H}(\Delta)^2\big]$, $\Delta \in \mathcal{B}_X$, recall (2.5).

ii) $\tilde{H}$ is additive, i.e. for pairwise disjoint sets $\Delta_1, \dots, \Delta_K$: $V(\Delta_k) < \infty$

$$\tilde{H}\Big(\bigcup_{k=1}^{K} \Delta_k\Big) = \sum_{k=1}^{K} \tilde{H}(\Delta_k)$$

and σ-additive in L_2, i.e. for pairwise disjoint sets $\Delta_1, \Delta_2, \dots$: $V(\Delta_k) < \infty$

$$\tilde{H}\Big(\bigcup_{k=1}^{\infty} \Delta_k\Big) = \sum_{k=1}^{\infty} \tilde{H}(\Delta_k)$$

with convergence in $L_2(\Omega, \mathcal{F}, \mathbb{P})$.

iii) $\tilde{H}$ is adapted to $\mathbb{F}$ and $\mathbb{G}$.

iv) $\tilde{H}$ has the *martingale property*. Consider $\Delta \subseteq (t,T] \times Z$. Then, from (2.3) we have:

$$\mathbb{E}\Big[\tilde{H}(\Delta)\,\Big|\,\mathcal{F}_t\Big] = \mathbb{E}\Big[\mathbb{E}\big[\tilde{H}(\Delta)\,|\,\mathcal{G}_t\big]\,\Big|\,\mathcal{F}_t\Big] = \mathbb{E}\Big[\mathbb{E}\big[\tilde{H}(\Delta)\,|\,\mathcal{F}^\alpha\big]\,\Big|\,\mathcal{F}_t\Big] = 0.$$

v) $\tilde{H}$ has conditionally orthogonal values. For any $\Delta_1, \Delta_2 \subseteq (t,T]\times Z$ such that $\Delta_1 \cap \Delta_2 = \emptyset$ and. Then, from A2), we have:

$$\begin{aligned}\mathbb{E}\Big[\tilde{H}(\Delta_1)\tilde{H}(\Delta_2)\,\Big|\,\mathcal{F}_t\Big] &= \mathbb{E}\Big[\mathbb{E}\big[\tilde{H}(\Delta_1)\tilde{H}(\Delta_2)\,\big|\,\mathcal{G}_t\big]\,\Big|\,\mathcal{F}_t\Big]\\ &= \mathbb{E}\Big[\mathbb{E}\big[\tilde{H}(\Delta_1)\big|\mathcal{F}^\alpha\big]\mathbb{E}\big[\tilde{H}(\Delta_2)\big|\mathcal{F}^\alpha\big]\Big|\mathcal{F}_t\Big] = 0.\end{aligned}$$

Given the martingale structure of the CDSPP $\tilde{H}$ with respect to the filtrations $\mathbb{G}$ and $\mathbb{F}$, we can construct a stochastic integration of Itô type according to the classical scheme, as retraced in [15]. Recall that α is a positive random measure.

We define the $\mathbb{G}$-predictable σ-algebra $\mathcal{P}_\mathbb{G}$ as the σ-algebra generated by $\{F\times(s,u]\times B :\ F\in\mathcal{G}_s, s<u, B\in\mathcal{B}_Z\}$ and, as usual, we will say that a stochastic process ϕ is $\mathbb{G}$-*predictable* if the mapping $\phi=\phi(\omega,t,z)$, $\omega\in\Omega$, $(t,z)\in X$, is $\mathcal{P}_\mathbb{G}$-measurable. Hence we define

$$\|\phi\|_\Phi := \Big(\mathbb{E}\Big[\int_0^T\int_Z \phi^2(t,z)\,\alpha(dt,dz)\Big]\Big)^{1/2}$$

and set Φ to be the L_2-subspace of stochastic processes ϕ admitting a $\mathbb{G}$-predictable modification and such that $\|\phi\|_\Phi<\infty$.

Lemma 4.1. *$\mathcal{F}^\alpha\times\mathcal{B}_X\subset\mathcal{P}_\mathcal{G}$, and α is the $\mathbb{G}$-predictable compensator of H.*

We take the $\mathbb{G}$-predictable compensator to be as in [21], a predictable, locally integrable random measure such that $\mathbb{E}[H(\Delta)]=\mathbb{E}[\alpha(\Delta)]$.

Proof. For the first claim it is sufficient to show that $A\times(a,b]\times B$ with $A\in\mathcal{F}^\alpha$, $a<b$ and $B\in\mathcal{B}_Z^c$ is an element of $\mathcal{P}_\mathbb{G}$. Recall that $A\in\mathcal{G}_s$ for all s and the claim follows. Since $E[H(\Delta)]=\mathbb{E}[\alpha(\Delta)]$ for all $\Delta\in\mathcal{B}_X$, and α is $\mathbb{G}$-predictable, it is the $\mathbb{G}$-predictable compensator of H. □

The *non-anticipating stochastic integral* with respect to $\tilde{H}$ under $\mathbb{G}$ is the isometric operator I mapping:

$$I : domI \Longrightarrow L_2(\Omega,\mathcal{F},\mathbb{P})$$

such that

$$I(\phi) := \int_0^T\int_Z \phi(t,z)\,\tilde{H}(dt,dz) := \sum_{k=1}^K \phi_k\tilde{H}\big(\Delta_k\big) \tag{4.2}$$

for any

$$\phi(t,z) = \sum_{k=1}^K \phi_k\mathbf{1}_{\Delta_k}(t,z) \tag{4.3}$$

with $\Delta_k=(s_k,u_k]\times B_k\in\mathcal{B}_X^c$ and ϕ_k a $\mathcal{G}_{s_k}$-measurable random variable such that $\|\phi\|_\Phi<\infty$. In fact,

$$\mathbb{E}\big[I(\phi)^2\big] = \mathbb{E}\Big[\big(\sum_{k=1}^K\phi_k\tilde{H}\big(\Delta_k\big)\big)^2\Big] = \mathbb{E}\Big[\sum_{k=1}^K\phi_k^2\alpha\big(\Delta_k\big)\Big] = \|\phi\|_\Phi^2. \tag{4.4}$$

Naturally the integrands are given by $domI \subseteq L_2(\Omega\times X)$, with $L_2(\Omega\times X) := L_2(\Omega\times X, \mathcal{F}\times\mathcal{B}_X, \mathbb{P}\times\alpha)$, which is the linear closure of the stochastic processes (4.3) and the integral is characterized in a standard manner exploiting the isometry (4.4).

Actually $domI = \Phi$. In fact the following result holds true.

Lemma 4.2. *For any $\phi \in \Phi$ there exists an approximating sequence of stochastic processes ϕ_n, $n = 1, 2, ...$, of type* (4.3) *having the form:*

$$\phi_n(t,z) = \sum_{k=1}^{K_n} \mathbb{E}\Big[\frac{1}{\alpha(\Delta_{n,k})}\int_{\Delta_{n,k}} \phi(\tau,\zeta)\,\alpha(d\tau,d\zeta)\,|\mathcal{G}_{s_{n,k}}\Big]\mathbf{1}_{\Delta_{n,k}}(t,z)$$

where $\Delta_k = (s_{n,k}, u_{n,k}] \times B_{n,k}$ are the sets (4.1) *of the dissecting system of $X = (0,T] \times Z$.*

Proof. The arguments of [15, Lemma 3.1] can be easily adapted to the present framework. □

Then, by isometry, it is clear that for any $\phi \in \Phi$, $I(\phi) = \lim_{n\to\infty} I(\phi_n)$ in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ where $\phi_n \in \Phi$ are processes of type (4.3) approximating ϕ in $L_2(\Omega \times X)$.

From the construction of the stochastic integral, it follows that for any $\phi \in \Phi$, the stochastic set function

$$\mu(\phi,\Delta) := \int\limits_{\Delta} \phi(t,z)\,\tilde{H}(dt,dz), \quad \Delta \in \mathcal{B}_X \tag{4.5}$$

is again a martingale random field [15, Remark 3.2] under $\mathbb{G}$ with variance measure

$$m(\phi,\Delta) := \mathbb{E}\Big[\int\limits_{\Delta} \phi^2(t,z)\,\alpha(dt,dz)\Big], \quad \Delta \in \mathcal{B}_X.$$

Proposition 4.3. *Consider the $\mathcal{F}^\alpha$-measurable $\beta \in L^2\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ and $\phi \in \Phi$. Then*

$$\beta I(\phi) = I(\beta\phi)$$

if either side of the equality exists as an element of $L^2\big(\Omega, \mathcal{F}, \mathbb{P}\big)$.

Proof. Assume $\phi \in \Phi$ is of type (4.3) and β is bounded. Then, for every k, $\beta\phi_k$ is $\mathcal{G}_{s_k}$-measurable and

$$\beta I(\phi) = \sum_{k=1}^{K} \beta\phi_k \tilde{H}\big((s_k,u_k] \times B_k\big) = I(\beta\phi).$$

The general case follows by taking limits. □

The classical calculus rules hold: for any $\phi \in \Phi$ we have

$$\mathbb{E}\Big[\int_{\Delta} \phi(t,z)\,\tilde{H}(dt,dz)\,|\mathcal{G}_t\Big] = 0, \quad \Delta \in \mathcal{B}_{(s,T]\times Z}$$

and

$$\begin{aligned}
&\mathbb{E}\Big[\int_\Delta \phi_1(t,z)\tilde{H}(dt,dz)\int_\Delta \phi_2(t,z)\,\tilde{H}(dt,dz)\,|\mathcal{G}_s\Big]\\
&\quad= \mathbb{E}\Big[\int_\Delta \phi_1(t,z)\phi_2(t,z)\,\alpha(dt,dz)\,|\mathcal{G}_s\Big]\\
&\quad= \int_\Delta \mathbb{E}\big[\phi_1(t,z)\phi_2(t,z)\ \,|\mathcal{G}_s\big]\alpha(dt,dz), \quad \Delta\in\mathcal{B}_{(s,T]\times Z}
\end{aligned}$$

and in particular we have

$$\mathbb{E}\Big[I(\phi)^2\Big|\mathcal{F}^\alpha\Big] = \int_0^T\int_Z \mathbb{E}\big[\phi^2(t,z)\big|\mathcal{F}^\alpha\big]\,\alpha(dt,dz). \tag{4.6}$$

Remark 4.4. In the same way as for the case of information flow $\mathbb{G}$, we can define the $\mathbb{F}$-predictable σ-algebra $\mathcal{P}_\mathbb{F}$ and consider the associated $\mathbb{F}$-predictable random fields. Being any $\mathbb{F}$-predictable stochastic process also $\mathbb{G}$-predictable, the integration can be done in the same setting as above with the result that the corresponding stochastic functions of type (4.5) will be martingale random fields under $\mathbb{F}$. Clearly results as in Proposition 4.3 fail in general in this context. In fact β is $\mathcal{G}_0$-measurable, but not $\mathcal{F}_0$-measurable in general.

Due to the dependence structure of α and the way it is taken into account by the integrator $\tilde{H}$, integral representations for $\mathbb{F}$-martingale random fields are not obvious and will not be further investigated here.

5. Non-anticipating stochastic derivative and representation theorem

In this section we discuss stochastic differentiation in the context of non-anticipative calculus. We will use the terminology non-anticipating derivative to emphasize the fact that the operator introduced embeds the information flow associated with the framework as time evolves. This differs from other concepts of stochastic differentiation, as the Malliavin type derivative. We consider the relationships with anticipative derivatives in section 7.

The *non-anticipating stochastic derivative* is the adjoint linear operator $\mathscr{D}^\mathbb{G} = I^*$ of the stochastic integral:

$$\mathscr{D}^\mathcal{G} : L_2(\Omega,\mathcal{F},\mathbb{P}) \Longrightarrow \Phi.$$

We can see that the non-anticipating derivative can be computed as the limit

$$\mathscr{D}^\mathcal{G}\xi = \lim_{n\to\infty}\phi_n \tag{5.1}$$

with convergence in Φ of the stochastic functions of type (4.3) given by

$$\phi_n(t,z) := \sum_{k=1}^{K_n}\mathbb{E}\Big[\xi\frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{s_{n,k}}\Big]\mathbf{1}_{\Delta_{n,k}}(t,z) \tag{5.2}$$

where $\Delta_{n,k} = (s_{n,k}, u_{n,k}] \times B_{n,k}$ refers to the n'th partition sets (4.1) in the dissecting system of $X = (0,T] \times Z$ (as per Definition 2.6). In fact we have the following result:

Theorem 5.1. *All* $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ *have representation*

$$\xi = \xi_0 + \int_0^T \int_Z \mathscr{D}^{\mathcal{G}}_{t,z}\xi \, \tilde{H}(dt, dz). \tag{5.3}$$

Moreover $\mathscr{D}^{\mathcal{G}}\xi_0 = 0$. *In particular we have* $\xi_0 = \mathbb{E}[\xi | \mathcal{F}^\alpha]$.

Proof. The proof follows similar arguments as the proof of [11, Theorem 2.1]. Set $\phi_{n,k} := \mathbb{E}\big[\xi \frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})} \,|\mathcal{G}_{s_{n,k}}\big]$. First note that

$$\mathbb{E}\big[\big|\phi_{n,k}\tilde{H}(\Delta_{n,k})\big|^2 \,|\mathcal{G}_{s_{n,k}}\big] \le \mathbb{E}\big[\xi^2 \big|\mathcal{G}_{s_{n,k}}\big]$$

by application of the conditional Hölder inequality. Thus $\mathbb{E}\big[|\phi_{n,k}\tilde{H}(\Delta_{n,k})|^2\big] < \infty$. Moreover, we have that

$$\mathbb{E}\Big[\big(\xi - \phi_{n,k}\tilde{H}(\Delta_{n,k})\big)\psi\tilde{H}(\Delta_{n,k})\Big] = 0 \tag{5.4}$$

for all $\mathcal{G}_{s_{n,k}}$-measurable $\psi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$. In fact, we have

$$\begin{aligned}\mathbb{E}\Big[\Big(\xi - \phi_{n,k}\tilde{H}(\Delta_{n,k})\Big)\psi\tilde{H}(\Delta_{n,k})\Big|\mathcal{G}_{s_{n,k}}\Big]& \\ = \psi\mathbb{E}\big[\xi\tilde{H}(\Delta_{n,k})\big|\mathcal{G}_{s_{n,k}}\big] - \psi\phi_{n,k}\alpha(\Delta_{n,k}) = 0.&\end{aligned}$$

Then, from (5.4), we conclude that

$$\hat{\xi}_n := \sum_{k=1}^{K_n} \phi_{n,k}\tilde{H}(\Delta_{n,k}) = \int_0^T \int_Z \phi_n(s,z)\, \tilde{H}(ds, dz)$$

is the projection of ξ onto the subspace of the stochastic integrals of type (4.3). Moreover, $\hat{\xi} := \lim_{n\to\infty} \hat{\xi}_n$ in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ is the projection of ξ onto the subspace of all the stochastic integrals with respect to $\tilde{H}$. Indeed, for any integral $I(\psi)$ with $\psi \in \Phi$, and $\psi := \lim_{n\to\infty} \sum_{k=1}^{K_n} \psi_{n,k}\mathbf{1}_{\Delta_{n,k}} \in \Phi$, we have

$$\mathbb{E}\Big[\Big(\xi - \hat{\xi}\Big)I(\psi)\Big] = \lim_{n\to\infty} \sum_{k=1}^{K_n} \mathbb{E}\Big[\big(\xi - \phi_{n,k}\tilde{H}(\Delta_{n,k})\big)\psi_{n,k}\tilde{H}(\Delta_{n,k})\Big] = 0$$

(with convergence in $L_2(\Omega, \mathcal{F}, \mathbb{P})$). Denote by $\hat{\phi}$ the integrand representing $\hat{\xi}$, i.e.

$$\hat{\xi} = \int_0^T \int_Z \hat{\phi}(s,z)\, \tilde{H}(ds, dz).$$

Then, by isometry, we have

$$\big\|\hat{\phi} - \phi_n\big\|_\Phi^2 = \|\hat{\xi} - \hat{\xi}_n\big\|^2_{L_2(\Omega,\mathcal{F},\mathbb{P})} \to 0, \; n \to \infty.$$

Hence $\hat{\phi} = \mathscr{D}^{\mathcal{G}}\xi$. Moreover, being the difference $\xi^0 := \xi - \hat{\xi}$ orthogonal to all stochastic integrals, we have $\mathscr{D}^{\mathcal{G}}\xi = 0$. In addition we also have that

$$\xi_0 = \mathbb{E}\big[\xi\big|\mathcal{G}_0\big] = \mathbb{E}\big[\xi\big|\mathcal{F}^\alpha\big].$$

By this we end the proof. □

Remark 5.2. Note that the non-anticipating derivative is continuous in L_2. Namely, if $\xi = \lim_{n\to\infty}\xi_n$ in $L_2(\Omega,\mathcal{F},\mathbb{P})$, then

$$\mathscr{D}^{\mathcal{G}}\xi = \lim_{n\to\infty}\mathscr{D}^{\mathcal{G}}\xi_n \quad \text{in } \Phi.$$

In fact $\|\mathscr{D}^{\mathcal{G}}\xi - \mathscr{D}^{\mathcal{G}}\xi_n\|_\Phi^2 \le \mathbb{E}\Big[(\xi-\xi_n)^2\Big] \longrightarrow 0$, $n \to \infty$.

Example 5.3. Let $\xi \in \mathbb{H}^p$ be a α-multilinear form $\xi = \beta\prod_{j=1}^p \tilde{H}(\Delta_j)$ with $\Delta_1 = (s_1,u_1]\times B_1, \Delta_2 = (s_2,u_2]\times B_2,\dots,\Delta_p = (s_p,u_p]\times B_p$ and $0 \le s_1 < u_1 \le s_2 < u_2 < \cdots < u_p \le T$. Then

$$\mathscr{D}^{\mathcal{G}}_{t,z}\xi = \beta\prod_{j=1}^{p-1}\tilde{H}(\Delta_j)\mathbf{1}_{\Delta_p}(t,z)$$

and

$$\xi = \int\limits_{\Delta_p}\beta\prod_{j=1}^{p-1}\tilde{H}\big(\Delta_j\big)\,\tilde{H}(dt,dz).$$

Example 5.4. If $\beta \in L_2(\Omega,\mathcal{F},\mathbb{P})$ is $\mathcal{F}^\alpha$-measurable, then $\mathscr{D}^{\mathcal{G}}\beta = 0$. In fact $\mathbb{E}\Big[\beta\frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{t_{n,k}}\Big] = 0$ for all $\Delta_{n,k}$.

In general we have the following formula:

Proposition 5.5. *Let ξ be an α-multilinear form, $\xi = \beta\prod_{j=1}^p\tilde{H}(\Delta_j)$. Then*

$$\xi = \int\limits_0^T\int\limits_Z \mathscr{D}^{\mathcal{G}}_{s,z}\xi\,\tilde{H}(ds,dz)$$

with

$$\mathscr{D}^{\mathcal{G}}_{s,z}\xi = \beta\sum_{\substack{1\le i\le p\\ \Delta_i\subseteq\Delta'}}\mathbf{1}_{\Delta_i}(s,z)\prod_{j\neq i}^{p}\tilde{H}(\Delta_j\cap[0,s)\times Z) \tag{5.5}$$

Here the set Δ' is given by

$$\Delta' = \bigcup_{j\notin\mathcal{I}}\Delta_j \tag{5.6}$$

where $\mathcal{I} = \Big\{1 \le i \le p\,\Big|\Delta_i \subset [0,t)\times Z$ and $\Delta_j \subset [t,T]\times Z$ for some $1 \le j \le p$ and $t \in [0,T]\Big\}$.

To explain the set $\mathcal{I}$ in Proposition 5.5, in Example 5.3 we would have $\mathcal{I} = \{1,\dots,p-1\}$, corresponding to the sets $\Delta_1,\dots,\Delta_{p-1}$, i.e. the elements of the multilinear form that “are entirely before the last set”.

Proof. Let ξ be a multilinear form of order $p \geq 1$, $\xi = \beta \prod_{j=1}^{p} \tilde{H}(\Delta_j)$. For simplicity assume $\Delta_j \cap \Delta_{n,k} = \emptyset$ or $\Delta_{n,k}$. Denote

$$\psi(n,k) = \mathbb{E}\Big[\beta \prod_{j=1}^{p} \tilde{H}(\Delta_j) \frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{t_{n,k}}\Big].$$

The computation of $\psi(n,k)$ is divided into three cases.

i) If $(\bigcup_{j=1}^{P} \Delta_j) \cap \Delta_{n,k} = \emptyset$ then $\psi(n,k) = 0$.

ii) If there exists i such that $\Delta_i \subset (t_{n,k}, T] \times Z$ and $\Delta_i \cap \Delta_{n,k} = \emptyset$ then

$$\psi(n,k) = \mathbb{E}\Big[\beta \prod_{j \neq i} \tilde{H}(\Delta_j) \frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{t_{n,k}}\Big] \mathbb{E}\Big[\tilde{H}(\Delta_i)\Big|\mathcal{F}^{\alpha}\Big] = 0.$$

iii) Neither case i) or ii) is true. This implies that $\Delta_{n,k} \subset \Delta'$. By assumption there exists $1 \leq i \leq p$ such that $\Delta_i \cap \Delta_{n,k} = \Delta_{n,k}$. We have

$$\begin{aligned}
\psi(n,k) = \mathbb{E}\Big[\beta \prod_{j \neq i} \Big(\tilde{H}\big(\Delta_j \cap [0, t_{n,k}] \times Z\big) + \tilde{H}\big(\Delta_j \cap (t_{n,k}, T] \times Z\big)\Big) \\
\Big(\tilde{H}(\Delta_i \cap \Delta_{n,k}) + \tilde{H}(\Delta_i \cap \Delta_{n,k}^c\Big) \frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{t_{n,k}}\Big] \\
= \mathbb{E}\Big[\beta \prod_{j \neq i} \tilde{H}\big(\Delta_j \cap [0, t_{n,k}] \times Z\big)\Big|\mathcal{G}_{t_{n,k}}\Big] \\
= \beta \prod_{j \neq i} \tilde{H}\big(\Delta_j \cap [0, t_{n,k}] \times Z\big).
\end{aligned}$$

Thus

$$\psi(n,k) = \mathbf{1}_{\{\Delta_{n,k} \cap \Delta' \neq \emptyset\}}(n,k)\, \beta \prod_{\substack{j \\ \Delta_j \cap \Delta_{n,k} = \emptyset}} \tilde{H}\big(\Delta_j \cap [0, t_{n,k}] \times Z\big), \tag{5.7}$$

and with Δ' as above, $\mathscr{D}^{\mathcal{G}}\xi$ is given by (5.5). Since $\mathbb{E}[\xi|\mathcal{F}^{\alpha}] = 0$ the representation is

$$\xi = \mathbb{E}\big[\xi\big|\mathcal{F}^{\alpha}\big] + I(\mathscr{D}^{\mathcal{G}}\xi) = I(\mathscr{D}^{\mathcal{G}}\xi).$$

□

The doubly stochastic Poisson process H is an example of a point process. For point processes in general, some integral representations have been developed in [4, 19, 3], see also the survey [10]. Note that the filtration of reference in these studies is $\mathbb{F}^H = \{\mathcal{F}_t^H|, t \in [0,T]\}$. As an illustration consider [4, Theorem 8.8]:

Theorem 5.6. *Let $\xi \in L_2(\Omega, \mathcal{F}_T^H, \mathbb{P})$. Let Λ be the $\mathbb{F}^H$-predictable compensator of H. Then there exists an $\mathbb{F}^H$-predictable process ϕ such that*

$$\xi = \mathbb{E}[\xi] + \int_0^T \int_Z \phi(s,z)\big(H(ds,dz - \Lambda(ds,dz)\big) \tag{5.8}$$

and $\mathbb{E}\Big[\int_0^T \int_Z \phi(s,z)^2 \Lambda(ds,dz)\Big] < \infty$.

We remark that Theorem 5.1 allows the repesentation of random variables that are $\mathcal{F}_T = \mathcal{F}_T^\alpha \vee \mathcal{F}_T^H$-measurable, which is a larger σ-algebra than $\mathcal{F}_T^H$. The function ϕ in (5.8) can be described in explicit terms depending on conditional expectations. This approach exploits the fact that the filtration $\mathbb{F}^H$ can be fully characterized by the jump times. This is not the case for the filtration $\mathbb{G}$ in which case we consider additional random noise such as the one generated by α. Theorem 5.1 provides an explicit characterization of the integrand in this setting.

6. Iterated integrals and chaos expansions

In this section, we revise the notion of Itô-type iterated integrals, with the intent to relate them with the α-multilinear forms of section 3. With this in mind, the iterated integrals are developed without any symmetrization schemes. These iterated integrals will later help us connect the α-multilinear forms with the Malliavin-type derivatives developed in [28] using symmetrization schemes and multiple integrals. In particular, Theorem 6.4 resembles [28, Corollary 14], but our construction is better suited for an analysis starting from α-multilinear forms. Let

$$S_p := \Big\{(s_1, z_1 \ldots s_p, z_p) \in ([0,T] \times Z)^p \Big| 0 \le s_1 \le s_2 \le \ldots s_p \le T\Big\}$$

Denote Φ_α^p the set of $\mathcal{F}^\alpha$-measurable functions, $\phi : \Omega \times S_p \to \mathbb{R}$, such that

$$\|\phi\|_{\Phi_\alpha^p} := \Big(\mathbb{E}\Big[\int_{S_p} \phi^2(s_1, z_1, s_2 \ldots s_p, z_p)\, \alpha(ds_1 dz_1) \ldots \alpha(ds_p dz_p)\Big]\Big)^{\frac{1}{2}} < \infty. \tag{6.1}$$

For any $\phi \in \Phi_\alpha^p$, the *p'th iterated integral* is defined as

$$J_p(T,\phi) := \int_0^T \int_Z \int_0^{s_p^-} \int_Z \cdots \int_0^{s_2^-} \int_Z \phi(s_1, z_1 \ldots s_p, z_p)\, \tilde{H}(ds_1 dz_1) \ldots \tilde{H}(ds_p dz_p), \tag{6.2}$$

and we set $J_p := \big\{J_p(T,\phi), \phi \in \Phi_\alpha^p\big\}$. From (4.4) and (4.6) we have

$$\begin{aligned}\mathbb{E}\Big[J_p^2(T,\phi)\Big] = \mathbb{E}\Big[\int_0^T \int_Z \mathbb{E}\Big[\Big(\int_0^{s_p} \int_Z \cdots \int_0^{s_2} \int_Z \phi(s_1, z_1 \ldots s_p, z_p) \\ \tilde{H}(ds_1 dz_1) \ldots \tilde{H}(ds_{p-1} dz_{p-1})\Big)^2 \Big| \mathcal{F}^\alpha\Big] \alpha(ds_p dz_p)\Big]\end{aligned}$$

$$= \mathbb{E}\Big[\int_0^T\int_Z \big(\int_0^{s_p}\int_Z \dots \int_0^{s_2}\int_Z \phi^2(s_1, z_1 \dots s_p, z_p)\,\alpha(ds_1, dz_1)\dots\alpha(ds_p dz_p)\Big]$$

$$= \|\phi\|^2_{\Phi^p_\alpha}. \tag{6.3}$$

The iterated integrals J_p are in correspondence with the space of α-multilinear forms $\mathbb{H}^p$ (see Definition 3.6). An example is instructive before considering the general case.

Example 6.1. Let ξ be a p-order α-multilinear form with pairwise disjoint time-intervals, i.e.

$$\xi = \beta\prod_{j=1}^p \tilde{H}(\Delta_j),$$

with $\Delta_1 = (s_1, u_1] \times Z_1$, $\Delta_2 = (s_2, u_2] \times Z_2, \dots$ and $0 \le s_1 < u_1 \dots < s_p \le u_1$. Then

$$\begin{aligned}
\xi &= \beta\prod_{j=1}^p \tilde{H}(\Delta_j) = \beta\prod_{j=1}^p I\big(\mathbf{1}_{\Delta_j}\big)\\
&= \int_0^T\int_Z \mathbf{1}_{\Delta_p}(s_p, z_p)\beta\prod_{j=1}^{p-1} I\big(\mathbf{1}_{\Delta_k}\big)\,\tilde{H}(ds_p, dz_p)\\
&= \int_0^T\int_Z \mathbf{1}_{\Delta_p}(s_p, z_p)\int_0^{s_p}\int_Z \mathbf{1}_{\Delta_{p-1}}(s_{p-1}, z_{p-1})\beta\\
&\qquad\qquad \prod_{j=1}^{p-2} I\big(\mathbf{1}_{\Delta_j}\big)\,\tilde{H}(ds_{p-1}, dz_{p-1})\,\tilde{H}(ds_p, dz_p)\\
&= \int_0^T\int_Z\int_0^{s_p}\int_Z \dots \int_0^{s_2}\int_Z \beta\mathbf{1}_{\Delta_p}(s_p, z_p)\dots\mathbf{1}_{\Delta_1}(s_1, z_1)\,\tilde{H}(ds_1 dz_1)\dots\tilde{H}(ds_p dz_p)
\end{aligned} \tag{6.4}$$

Next we need to expand this representation to the case when sets are "overlapping in time". It is possible to investigate this using Itô's formula or symmetric functions, but instead we exploit the explicit result from Proposition 5.5.

Theorem 6.2. *If $\xi \in \mathbb{H}^p$, $p \ge 1$, then ξ can be represented as a p'th iterated integral, ie*

$$\xi = \int_0^T\int_Z\int_0^{s_p-}\int_Z \dots \int_0^{s_2-}\int_Z \phi(s_p, z_p, \dots s_1, z_1)\tilde{H}(ds_1 dz_1)\dots\tilde{H}(ds_p dz_p), \tag{6.5}$$

where $\phi \in \Phi^p_\alpha$. Furthermore we have

$$\|\xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})} = \|\phi\|_{\Phi^p_a}. \tag{6.6}$$

Proof. First we prove this for the α-multilinear forms by induction. The result is true for α-multilinear forms of order $p = 1$. Consider $p \geq 2$. Assume, as induction hypothesis, that a representation of type (6.5) holds for all multilinear forms of order $p - 1$. Let

$$\xi' = \prod_{j=1}^{p-1} \tilde{H}(\Delta_j \cap [0,t) \times Z) \tag{6.7}$$

Being a $(p-1)$-order α-multilinear for, it has representation

$$\xi' = \int_0^t \int_Z \dots \int_0^{s_2-} \phi'_{p-1} \tilde{H}(ds_1, dz_1) \dots \tilde{H}(dz_{p-1} ds_{p-1}),$$

with means of $\phi'_{p-1} \in \Phi_\alpha^{p-1}$. Denote this integral as $J_{p-1}(t, \phi'_{p-1})$.

Let ξ be an α-multilinear form of order p, $\xi = \beta \prod_{j=1}^{p} \tilde{H}(\Delta_j)$. From Proposition 5.5, we know that $\xi = I(\mathscr{D}^{\mathcal{G}}\xi)$, with

$$\mathscr{D}^{\mathcal{G}}_{s,z}\xi = \beta \sum_{\substack{1 \leq i \leq p \\ \Delta_i \subseteq \Delta'}} \mathbf{1}_{\Delta_i}(s,z) \prod_{j \neq i}^{p} \tilde{H}(\Delta_j \cap [0,s) \times Z).$$

Hence, by (6.7) we have

$$\begin{aligned}
\xi &= I(\mathscr{D}^{\mathcal{G}}\xi) \\
&= \int_0^T \int_Z \Big[\beta \sum_{\substack{1 \leq i \leq p \\ \Delta_i \subseteq \Delta'}} \mathbf{1}_{\Delta_i}(s,z) \prod_{j \neq i}^{p} \tilde{H}(\Delta_j \cap [0,s) \times Z) \Big] \tilde{H}(ds, dz) \\
&= \int_0^T \int_Z \Big[\beta \sum_{\substack{1 \leq i \leq p \\ \Delta_i \subseteq \Delta'}} \mathbf{1}_{\Delta_i}(s,z) J_{p-1}(s, \phi_i) \Big] \tilde{H}(ds, dz) \\
&= \int_0^T \int_Z \int_0^{s_p-} \dots \int_0^{s_2-} \int_Z \Big[\beta \sum_{\substack{1 \leq i \leq p \\ \Delta_i \subseteq \Delta'}} \mathbf{1}_{\Delta_i}(s_p, z_p) \phi_i(s_1, \dots s_{p-1}, z_{p-1}) \Big] \\
&\qquad\qquad \tilde{H}(ds_1, dz_1) \dots \tilde{H}(ds_p, dz_p)
\end{aligned}$$

which is an iterated integral of order p.

Any ξ in $\mathbb{H}^p$ is the limit in $L_2(\Omega, \mathcal{F}, \mathbb{P})$ of a finite sums of multilinear forms of order p. Let ξ_n be such a sequence. Any finite sum of multilinear forms can be written as a p'th iterated integral, let

$$\xi_n = J_p(\phi_n), \quad \phi_n \in \Phi_\alpha^p.$$

Since ξ_n is a Cauchy sequence in $L_2(\Omega, \mathcal{F}, \mathbb{P})$, ϕ_n is a Cauchy sequence in Φ_α^p by the isometry in (6.3). Hence there exists a $\phi \in \Phi_\alpha^p$ such that $\phi_n \to \phi$ as

$n \to \infty$ and we must have $\xi = J_p(\phi)$. Finally, equation (6.6) follows directly from (6.3). $\square$

Remark 6.3. From (6.4), we can see that if $\xi = \beta \prod_{j=1}^{p} \tilde{H}(\Delta_j)$ is an α-multilinear form, with $\beta \in \mathbb{R}$ then $\xi = J_p(\phi)$ with ϕ deterministic. For general $\xi \in \widetilde{\mathbb{H}}^p$ (Remark 3.13), we can use the same arguments as in Theorem 6.2 to conclude that $\xi = J_p(\phi)$, where $\phi \in \Phi_\alpha^p$ is deterministic. Thus $\widetilde{\mathbb{H}}^p$ is the space spanned by iterated integrals of order p with deterministic integrands.

Theorem 6.4 (Chaos expansion). *For any $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$, there is unique sequence of integrands $\phi_p \in \Phi_\alpha^p$, $p = 1, 2, \ldots$ such that the following representation holds:*

$$\xi = \mathbb{E}[\xi | \mathcal{F}^\alpha] + \sum_{p=1}^{\infty} J_p(\phi_p).$$

Proof. Theorem 3.12 shows that any $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ has orthogonal decomposition

$$\xi = \sum_{p=0}^{\infty} \xi_p$$

with $\xi_p \in \mathbb{H}^p$, $p = 0, 1, \ldots$. Any ξ_p, $p \geq 1$ can be written as a p'th iterated integral by Theorem 6.2 and $\xi_0 = \mathbb{E}[\xi | \mathcal{G}_0] = \mathbb{E}[\xi | \mathcal{F}^\alpha]$ is the projection of ξ on $\mathbb{H}^0$. $\square$

Directly from Theorem 6.4 we can formulate an integral representation theorem.

Corollary 6.5. *For any $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ there exists a unique $\phi \in \Phi$ such that*

$$\xi = \mathbb{E}[\xi | \mathcal{F}^\alpha] + \int_0^T \int_Z \phi(s, z) \, \tilde{H}(ds, dz)$$

Note that this corollary is in line with classical stochastic integral representation theorems with respect to square integrable martingales as integrators. Corollary 6.5 offers no immediate way of computing the integrand ϕ since only the existence of the representation via the kernel functions of the iterated integrals is given. Corollary 6.5 deeply differs from Theorem 5.1 and the following Theorem 7.9. The last ones characterize the integrand ϕ in terms of derivative operators.

7. Anticipative stochastic derivatives and integral representations

Motivated by Clark-Ocone type formulae we study ways to compute the non-anticipating derivative and to have stochastic integral representations. We introduce a new anticipative derivative operator D^c, partially inspired by

[14]. We study this operator in relation with a Malliavin-type derivative for processes with conditionally independent increments as developed in [28].

Let $\mathcal{G}_{\Delta^c}$ be the minimal complete σ-algebra containing $\mathcal{F}^\alpha$ and the sets $\{\tilde{H}(\Delta')|\Delta' \subset \Delta^c\}$, where Δ^c is the complement of Δ.

Definition 7.1. The operator $D^c : \mathbb{D}^c \to L_2(\Omega \times X)$, where $\mathbb{D}^c$ is the subset of $L_2(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\|\xi\|_{\mathbb{D}^c} = \Big(\mathbb{E}\Big[\int_0^T \int_Z (D^c_{s,z}\xi)^2 \alpha(ds, dz)\Big]\Big)^{\frac{1}{2}} < \infty,$$

is defined as the limit in $L_2(\Omega \times X)$ over the partitions in the dissecting system of X given by

$$D^c\xi = \lim_{n\to\infty} D^c\xi(n) \tag{7.1}$$

where

$$D^c_{s,z}\xi(n) := \sum_{k=1}^{K_n} \mathbb{E}\Big[\xi \frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{\Delta^c_{n,k}}\Big] \mathbf{1}_{\Delta_{n,k}}(s, z). \tag{7.2}$$

We remark that for any $\beta \in \mathbb{H}^0$ (recall Definition 3.2), $\beta \in \mathbb{D}^c$ and $D^c\beta = 0$.

Lemma 7.2. *For $p \geq 1$, let ξ be a p-order α-multilinear form, i.e. we have $\xi = \beta \prod_{j=1}^p \tilde{H}(\Delta_j)$. Then*

$$D^c_{s,z}\xi = \beta \sum_{i=1}^p \mathbf{1}_{\Delta_i}(s, z) \prod_{j\neq i} \tilde{H}(\Delta_j), \tag{7.3}$$

and

$$D^c\xi(n) = \sum_{i=1}^p \sum_{k=1}^{K_n} \frac{\alpha(\Delta_i \cap \Delta_{n,k})}{\alpha(\Delta_{n,k})} \prod_{j\neq i} \tilde{H}(\Delta_j)\mathbf{1}_{\{\Delta_{n,k}\cap\Delta_j=\emptyset\}}(k, j)\ \mathbf{1}_{\Delta_{n,k}}(s, z). \tag{7.4}$$

Furthermore

$$\|\xi\|_{\mathbb{D}^c} = \|D^c\xi\|_{L_2(\Omega\times X)} = \sqrt{p}\|\xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})}.$$

Proof. For any n and $k = 1, \ldots K_n$, denote

$$\psi_{n,k} = \mathbb{E}\Big[\beta \prod_{j=1}^p \tilde{H}(\Delta_j) \frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{\Delta^c_{n,k}}\Big].$$

If $\Delta_{n,k} \cap \big(\bigcup_{j=1}^p \Delta_j\big) = \emptyset$ or if $\Delta_{n,k}$ intersects with *more than one* of the sets Δ_j's, then $\psi_{n,k}$ is equal to zero by direct computation. If $\Delta_{n,k} \subset \Delta_i$ for some

i, then

$$\begin{aligned}\psi(n,k) &= \mathbb{E}\Big[\beta\prod_{j=1}^{p}\tilde H(\Delta_j)\frac{\tilde H(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{\Delta^c_{n,k}}\Big]\\ &= \beta\prod_{j\neq i}\tilde H(\Delta_j)\mathbb{E}\Big[\Big(\tilde H(\Delta_i\cap\Delta_{n,k})+\tilde H(\Delta_i\setminus\Delta_{n,k})\Big)\frac{\tilde H(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{\Delta^c_{n,k}}\Big]\\ &= \beta\prod_{j\neq i}\tilde H(\Delta_j)\end{aligned}$$

If $\Delta_i\subsetneq\Delta_{n,k}$ for some i and $\Delta_{n,k}\cap\Delta_j=\emptyset$ for all $j\neq i$, then

$$\psi_{n,k}=\beta\frac{\alpha(\Delta_i)}{\alpha(\Delta_{n,k})}\prod_{j\neq i}^{p}\tilde H(\Delta_j).$$

Combining the above cases we conclude that

$$\psi(n,k)=\sum_{i=1}^{p}\frac{\alpha(\Delta_i\cap\Delta_{n,k})}{\alpha(\Delta_{n,k})}\prod_{j\neq i}\tilde H(\Delta_j)\mathbf{1}_{\{\Delta_{n,k}\cap\Delta_j=\emptyset\}}(k,j)\mathbf{1}_{\Delta_{n,k}}(s,z).$$

and (7.4) follows. Passing to the limit in $L_2(\Omega\times X)$ we have

$$D^c\xi=\lim_{n\to\infty}D^c\xi(n)=\beta\sum_{i=1}^{p}\mathbf{1}_{\Delta_i}(s,z)\prod_{j\neq i}\tilde H(\Delta_j).$$

Moreover

$$\begin{aligned}\|\xi\|^2_{\mathbb{D}^c} &= \mathbb{E}\Big[\int_0^T\int_Z\Big(\beta\sum_{i=1}^{p}\mathbf{1}_{\Delta_i}(s,z)\prod_{j\neq i}\tilde H(\Delta_j)\Big)^2\alpha(ds,dz)\Big]\\ &= \mathbb{E}\Big[\beta^2\sum_{i=1}^{p}\alpha(\Delta_i)\prod_{j\neq i}\tilde H(\Delta_j)^2\Big]\\ &= \mathbb{E}\Big[\beta^2 p\prod_{j=1}^{p}\alpha(\Delta_j)\Big]\\ &= p\|\xi\|^2_{L_2(\Omega,\mathcal{F},\mathbb{P})}.\end{aligned}$$

□

Comparing (7.3) and (7.4) we can see that for the p-order α-multilinear form ξ, the following estimate holds for all n:

$$\|D^c\xi(n)\|_{L_2(\Omega\times X)}\le\|D^c\xi\|_{L_2(\Omega\times X)}=\sqrt{p}\|\xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})}. \tag{7.5}$$

The following statements are an immediate consequence in Lemma 7.2.

Corollary 7.3. *Let $p\ge 1$. Let ξ_1 and ξ_2 be orthogonal p-order α-multilinear forms. Then, for all n, $D^c\xi_1(n)$ and $D^c\xi_2(n)$ are orthogonal in $L_2(\Omega\times X)$. The same holds for $D^c\xi_1$ and $D^c\xi_2$.*

Corollary 7.4. *For $p_1 > p_2 \geq 1$, let $\xi_1 \in \mathbb{H}^{p_1}$ and $\xi_2 \in \mathbb{H}^{p_2}$ be α-multilinear forms. Then $D^c\xi_1$ and $D^c\xi_2$ are orthogonal in $L_2(\Omega \times X)$.*

Finally we have the following result.

Proposition 7.5. *For $p \geq 1$, if $\xi \in \mathbb{H}^p$ then $\xi \in \mathbb{D}^c$ with*

$$\|\xi\|_{\mathbb{D}^c} = \sqrt{p}\|\xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})} < \infty. \tag{7.6}$$

Proof. Any $\xi \in \mathbb{H}^p \subset L_2(\Omega,\mathcal{F},\mathbb{P})$ can be approximated by a sequence ξ_m, $m = 1, 2, ...,$ of finite sums of α-multilinear forms of order p: $\lim_{m\to\infty} \|\xi_m - \xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})} = 0$. First of all we observe that, from Remark 3.4, and ξ_m above can be represented as finite sums of orthogonal p-order α-multilinear forms. From Lemma 7.2 and Corollary 7.3 we can see that $D^c\xi_m$ is a Cauchy sequence in $L_2(\Omega \times X)$ with limit ϕ such that $\|\phi\|_{L_2(\Omega\times X)} = \sqrt{p}\|\xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})}$.

We show that indeed $\phi = D^c\xi := \lim_{n\to\infty} D^c\xi(n)$ in $L_2(\Omega \times X)$. By application of Corollary 7.3 and (7.5) we have

$$\|D^c\xi_m(n)\|_{L_2(\Omega\times X)} \leq \sqrt{p}\|\xi_m\|_{L_2(\Omega,\mathcal{F},\mathbb{P})}. \tag{7.7}$$

Moreover we note that

$$\begin{aligned}
&\|D^c\xi_m(n) - D^c\xi(n)\|^2_{L_2(\Omega\times X)} \\
&= \mathbb{E}\Big[\int_0^T \int_Z \sum_{k=1}^{K_n} \Big(\mathbb{E}\Big[(\xi_m - \xi)\frac{\tilde{H}(\Delta_{n,k})}{\alpha(\Delta_{n,k})}\Big|\mathcal{G}_{\Delta^c_{n,k}}\Big]\Big)^2 \mathbf{1}_{\Delta_{n,k}}(s,z)\,\alpha(ds,dz)\Big] \\
&\leq \mathbb{E}\Big[\sum_{k=1}^{K_n} \mathbb{E}\Big[(\xi_m - \xi)^2\Big|\mathcal{G}_{\Delta^c_{n,k}}\Big]\mathbb{E}\big[\tilde{H}(\Delta_{n,k})^2\big|\mathcal{G}_{\Delta^c_{n,k}}\big]\frac{1}{\alpha(\Delta_{n,k})}\Big] \\
&= K_n\|\xi_m - \xi\|^2.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\lim_{n\to\infty} \|\phi - D^c\xi(n)\|_{L_2(\Omega\times X)} \\
&\qquad \leq \lim_{n\to\infty}\lim_{m\to\infty} \Big\{\|\phi - D^c\xi_m\|_{L_2(\Omega\times X)} + \|D^c\xi_m - D^c\xi_m(n)\|_{L_2(\Omega\times X)} \\
&\qquad\qquad + \|D^c\xi_m(n) - D^c\xi(n)\|_{L_2(\Omega\times X)}\Big\} = 0
\end{aligned}$$

In fact

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty} \|D^c\xi_m(n) - D^c\xi(n)\|_{L_2(\Omega\times X)} \\
&\qquad\qquad \leq \lim_{n\to\infty} \Big\{\sqrt{K_n}\lim_{m\to\infty} \|\xi_m - \xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})}\Big\} = 0
\end{aligned}$$

and by (7.7)

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\|D^c\xi_m - D^c\xi_m(n)\|_{L_2(\Omega\times X)}\\
&\le \lim_{q\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\Big\{\|D^c\xi_m - D^c\xi_q(n)\|_{L_2(\Omega\times X)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad + \|D^c\xi_q(n) + D^c\xi_m(n)\|_{L_2(\Omega\times X)}\Big\}\\
&\le \lim_{q\to\infty}\|\phi - D^c\xi_q\|_{L_2(\Omega\times X)} + \sqrt{p}\lim_{q\to\infty}\lim_{m\to\infty}\|\xi_q - \xi_n\|_{L_2(\Omega,\mathcal{F},\mathbb{P})} = 0.
\end{aligned}
$$

□

The Malliavin calculus for processes with conditionally independent increments was developed in [28], this include the CDSPP. The results and developments therein are close to those of [23, Chapter 1]. We summarize some of those results with the aim of showing how these operators relate to the operator D^c and the non-anticipating derivative $\mathscr{D}^{\mathcal{G}}$.

Let $f_p : L_2\big(\Omega \times ((0,T] \times Z)^p\big) \to \mathbb{R}$ where f_p is $\mathcal{F}^\alpha \times \mathcal{B}_X$-measurable. Remark that f_p is *not* defined on Φ^p_α, which is a smaller space. We say that f_p is *simple* if

$$
f_p = \sum_{i=1}^n \beta_i(\omega)\mathbf{1}_{\Delta_1}(s_1,z_1)\dots\mathbf{1}_{\Delta_p}(s_p,z_p)
$$

where β_i, $i = 1,\dots,n$ is a bounded $\mathcal{F}^\alpha$-measurable random variable and the sets $\Delta_1,\dots,\Delta_p$ are pairwise disjoint. The multiple integrals of order p of a simple function are then

$$
I_p(T, f_p) := \sum_i^n \beta_i \prod_{j=1}^p \tilde{H}(\Delta_j),
$$

i.e. the multiple integrals of simple functions of order p are sums of α-multilinear forms of order p. These multiple integrals are extended to integrals of general $\mathcal{F}^\alpha \times \mathcal{B}_X$-measurable functions $f_p : L_2\big(\Omega \times (0,T] \times Z)^p\big) \to \mathbb{R}$ by taking limits of simple functions. We conclude that the space spanned by multiple integrals of order p on the functions above coincide with $\mathbb{H}^p$.

Any $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbb{P})$ has representation (as per Theorem 6.4 and [28, Corollary 14])

$$
\xi = \mathbb{E}\Big[\xi\Big|\mathcal{F}^\alpha\Big] + \sum_{p=1}^\infty I_p(\tilde{f}_p), \tag{7.8}
$$

by means if a sequence $\tilde{f}_p$, $p \ge 1$, of symmetric functions in $L_2\big(\Omega \times ((0,T] \times z)^p\big)$.

Denote the symmetrization of f_p by

$$
\tilde{f}_p := \frac{1}{p!}\sum_\sigma f\big(s_{\sigma(1)}, z_{\sigma(1)}, \dots, s_{\sigma(p)}, z_{\sigma(p)}\big)
$$

where σ is running over all permutations of $1, \dots, p$. Let $\phi_p \in \Phi_\alpha^p$ (see (6.1)) and $f_p = \mathbf{1}_{S_p}\phi_p$. Then the following equalities hold [28, Section 3]:

$$J_p(T, \phi_p) = I_p(T, f_p) = I_p(T, \tilde{f}_p) = p! J_p(T, \tilde{f}_p).$$

The *Malliavin derivative* $D : \mathbb{D}_{1,2} \to L_2(\Omega \times X)$ is given by

$$D_{s,z}\xi := \sum_{p=1}^{\infty} p I_{p-1}\big(\tilde{f}_p(\cdot, s, z)\big) \tag{7.9}$$

for all $\xi \in L_2(\Omega, \mathcal{F}, \mathbb{P})$ with $\xi = \mathbb{E}[\xi|\mathcal{F}^\alpha] + \sum_{p=1}^{\infty} I_p(\tilde{f}_p)$, such that

$$\|\xi\|_{\mathbb{D}_{1,2}} := \Big(\sum_{p=1}^{\infty} pp! \|\tilde{f}_p\|^2_{\Phi_\alpha^p}\Big)^{\frac{1}{2}} < \infty.$$

Indeed, $\|D\xi\|^2_{L_2(\Omega\times X)} = \sum_{p=1}^{\infty} pp! \|\tilde{f}_p\|^2_{\Phi_\alpha^p}$.

Lemma 7.6. *For $p \geq 1$, let $\xi = \beta \prod_{j=1}^{p} \tilde{H}(\Delta_j)$ be an α-multilinear form. Then*

$$D_{s,z}\xi = \beta \sum_{i=1}^{p} \mathbf{1}_{\Delta_i}(s, z) \prod_{i \neq j} \tilde{H}(\Delta_j) \tag{7.10}$$

and

$$\|\xi\|_{\mathbb{D}_{1,2}} = \sqrt{p}\|\xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})}$$

Proof. Since $\xi = J_p(T, \beta\mathbf{1}_{\Delta_1} \dots \mathbf{1}_{\Delta_p})$, $\xi = I_p(\tilde{f}_p)$ with

$$\tilde{f} = \beta \frac{1}{p!} \sum_{\sigma} \mathbf{1}_{\Delta_1}(s_{\sigma(1)}, z_{\sigma(1)}) \dots \mathbf{1}_{\Delta_p}(s_{\sigma(p)}, z_{\sigma(p)}).$$

Thus, from (7.9), we have

$$\begin{aligned} D_{s,z}\xi = {} & \beta \frac{1}{p!} p \sum_{\sigma} \mathbf{1}_{\Delta_p}(s_{\sigma(p)}, z_{\sigma(p)}) \\ & \qquad I_{p-1}\Big(\mathbf{1}_{\Delta_1}(s_{\sigma(1)}, z_{\sigma(1)}) \dots \mathbf{1}_{\Delta_p}(s_{\sigma(p-1)}, z_{\sigma(p-1)})\Big)\Big|_{\substack{s_{\sigma(p)}=s \\ z_{\sigma(p)}=z}} \\ = {} & \frac{p}{p!}\beta \sum_{i=1}^{p} \mathbf{1}_{\Delta_i}(s, z)(p-1)! \prod_{j \neq i} \tilde{H}(\Delta_j) \\ = {} & \beta \sum_{i=1}^{p} \mathbf{1}_{\Delta_i}(s, z) \prod_{j \neq i} \tilde{H}(\Delta_j) \end{aligned}$$

Let us compute the norm of ξ in $\mathbb{D}_{1,2}$. Note that

$$\|\tilde{f}_p\|^2_{\Phi_\alpha^p} = \mathbb{E}\Big[\beta^2 \frac{1}{p!} \prod_{j=1}^{p} \alpha(\Delta_j)\Big].$$

Hence

$$\|\xi\|^2_{\mathbb{D}_{1,2}} = pp!\|\tilde{f}_p\|^2_{\Phi^p_\alpha} = p\mathbb{E}\Big[\beta^2 \prod_{j=1}^{p} \alpha(\Delta_j)\Big] = p\|\xi\|^2_{L_2(\Omega,\mathcal{F},\mathbb{P})}.$$

□

We observe that if $\xi \in \mathbb{H}^p$, $p \geq 1$, then by the closability of D, [28, Lemma 21], and Lemma 7.6 it follows that $\xi \in \mathbb{D}_{1,2}$ with

$$\|\xi\|_{\mathbb{D}_{1,2}} = \sqrt{p}\|\xi\|_{L_2(\Omega,\mathcal{F},\mathbb{P})} < \infty. \tag{7.11}$$

Moreover, if β is $\mathcal{F}^\alpha$ measurable we have $D\beta = 0$ by [28, Proposition 25].

Recall that $L_2(\Omega,\mathcal{F},\mathbb{P}) = \mathbb{H} = \sum_{p=0}^{\infty} \oplus\mathbb{H}^p$ (see Theorem 3.12).

Proposition 7.7. *For any $\xi \in \mathbb{H}$, $\xi \in \mathbb{D}^c$ if and only if*

$$\|D^c\xi\|_{\mathbb{D}^c} = \|D\xi\|_{\mathbb{D}_{1,2}} = \sum_{p=1}^{\infty} \sqrt{p}\|\xi_p\|_{L_2(\Omega,\mathcal{F},\mathbb{P})} < \infty. \tag{7.12}$$

Here $\xi_0, \xi_1, \ldots$ is the orthogonal decomposition of ξ in the chaos expansion of Theorem 3.12.

Proof. This is a direct application of Lemma 7.2, Corollary 7.4 for D^c and of Theorem 6.2, Lemma 7.6, and (7.10), for D. □

We conclude that the spaces $\mathbb{D}^c$ and $\mathbb{D}_{1,2}$ coincide but are not equal to the whole of $L_2(\Omega,\mathcal{F},\mathbb{P})$, i.e.

$$\mathbb{D}^c = \mathbb{D}_{1,2} \subsetneq L_2(\Omega,\mathcal{F},\mathbb{P}).$$

Moreover, we stress that for any $\xi \in \mathbb{D}^c$, there exists a sequence ξ_m, $m = 1, 2 \ldots$, of finite sums of α-multilinear forms approximating ξ. Then $D^c\xi_m$ and $D\xi_m$ are two identical converging sequences by Lemmas 7.2 and 7.6. These two sequences must have the same limit in $L_2(\Omega \times X)$.

We conclude summarizing the arguments into the following statement:

Theorem 7.8. *The operators D^c and D coincide, i.e. $\mathbb{D}^c = \mathbb{D}_{1,2}$ and*

$$D^c\xi = D\xi \quad \text{in } L_2(\Omega \times X)$$

After the above result we can also interpret the operator D^c as and alternative approach to describe the Malliavin derivative which shows the anticipative dependence of the operator on the information in a much more structural and explicit way than the classical approach via chaos expansions of iterated integrals.

The following theorem is a Clark-Ocone type result which provides an alternative way to compute the non-anticipating derivative in the integral representation of Theorem 5.1.

Theorem 7.9. *For any* $\xi \in \mathbb{D}^c$ *we have*

$$\mathbb{E}\Big[D^c\xi(s,z)\Big|\mathcal{G}_{s-}\Big] = \mathbb{E}\Big[D\xi(s,z)\Big|\mathcal{G}_{s-}\Big] = \mathscr{D}^{\mathcal{G}}\xi(s,z) \quad \mathbb{P}\times\alpha \ a.e.$$

Proof. The first equality follows from Theorem 7.8. Assume $\xi \in \mathbb{H}^p$ is a p-order α-multilinear form, $\xi = \beta\prod_{j=1}^p \tilde{H}(\Delta_j)$. From (7.3),

$$\begin{aligned}\mathbb{E}\Big[D^c\xi\Big|\mathcal{G}_{s-}\Big] &= \beta\sum_{i=1}^p \mathbb{E}\Big[\prod_{j\neq i}\tilde{H}(\Delta_j)\Big|\mathcal{G}_{s-}\Big]\mathbf{1}_{\Delta_i}(s,z)\\ &= \beta\sum_{\substack{1\le i\le p\\ \Delta_i\subseteq\Delta'}}\mathbf{1}_{\Delta_i}(s,z)\prod_{j\neq i}^p\tilde{H}(\Delta_j\cap[0,s)\times Z)\\ &= \mathscr{D}^{\mathcal{G}}\xi(s,z)\end{aligned}$$

by comparing to (5.5). The set Δ' is as described in (5.6). By approximation we obtain the statement first for the general $\xi \in \mathbb{H}^p$ and then for $\xi \in \mathbb{H}$: $\xi = \lim_{q\to\infty}\sum_{p=0}^q \xi_p$ with $\xi_p \in \mathbb{H}^p$. □

Denote $\mathbb{E}[D^c\xi|\mathcal{G}]$ the stochastic process given by $\phi(s,z) = \mathbb{E}\big[D^c_{s,z}\xi|\mathcal{G}_{s-}\big]$.

Corollary 7.10. *For any* $\xi \in L_2(\Omega,\mathcal{F},\mathbb{P})$ *there exists a sequence* $\xi_q \in \mathbb{D}^c$, $q = 1,\dots$ *such that* $\xi_q \to \xi$ *in* $L_2(\Omega,\mathcal{F},\mathbb{P})$ *and*

$$\mathscr{D}^{\mathcal{G}}\xi_q = \mathbb{E}\big[D^c\xi_q|\mathcal{G}\big] \longrightarrow \mathscr{D}^{\mathcal{G}}\xi \quad \text{as } q\to\infty \qquad \text{in } \Phi.$$

Thus

$$\xi = \mathbb{E}[\xi|\mathcal{F}^\alpha] + \lim_{q\to\infty}\int_0^T\int_Z \mathbb{E}\big[D^c_{s,z}\xi_q|\mathcal{G}_{s-}\big]\,\tilde{H}(ds,dz)$$

with convergence in $L_2(\Omega,\mathcal{F},\mathbb{P})$.

Proof. Take ξ_q to be the projection of ξ on $\mathbb{H}_q = \sum_{p=0}^q \oplus\mathbb{H}^p$, this is $\xi_q = \mathbb{E}\big[\xi\big|\mathcal{F}^\alpha\big] + \sum_{p=1}^q \xi_p$, and apply Remark 5.2 and Theorem 7.9. □

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