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OPTIMAL CONTROL FOR STOCHASTIC DELAY SYSTEM UNDER MODEL UNCERTAINTY: A STOCHASTIC DIFFERENTIAL GAME APPROACH

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Abstract. In this paper, we study a robust recursive utility maximization problem for time-delayed stochastic differential equation with jumps. This problem can be written as a stochastic delayed differential game. We suggest a maximum principle of this problem and obtain necessary and sufficient condition of optimality. We apply the result to study a problem of consumption choice optimization under model uncertainty.

1. Introduction

A common problem in mathematical finance consists of an agent who invests and want to maximize the expected utility of her instantaneous consumption and/or terminal wealth. Recently, there has been an increased interest in problems of utility maximization under model uncertainty (see e.g., [11, 15, 24, 26] and references therein.) In fact, unlike in the standard expected utility maximization, where it is assumed that the investor knows the "original" probability measure P that describes the dynamics of the wealth process; in these papers it is supposed that the investor does not know this probability. In order to take into account this uncertainty, the authors introduced a family $\mathcal{Q}$ of probability measures Q which are equivalent (or absolutely continuous with respect) to the original measure P and then choose the worst case criteria in the optimization problem. The problem is solved by dynamic programming or stochastic maximum principle or duality arguments. There is already a vast literature on the dynamic programing and the stochastic maximum principle. The reader is e.g. referred to [1, 12, 22, 35, 36] and the references therein.

The problem of optimal control for delayed systems has also received a lot of attention recently. (see for e.g., [9, 14, 19, 21] and references therein.) One of the reasons of looking at this problem is that many phenomena have memory dependence i.e., their dynamics at a present time t does not only depend on the situation at time t but also on a finite part of

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their past history. Such model may be regarded as a stochastic differential delay equations (SDDEs).

As a generalization of classical utility utility, Duffie and Epstein [5] introduced the notion of stochastic differential utility (SDU)(or recursive utility). The cost function of such utility is given in terms of an intermediate consumption rate and a future utility, therefore it can be represented as a solution of a backward stochastic differential equation (BSDE). The notion of backward stochastic differential equations (BSDEs) was introduced by Bismut [2] in the linear case to study the adjoint equation associated with the stochastic maximum principle in stochastic optimal control problem. Pardoux and Peng [28] further developed BSDEs in the nonlinear setting and since then the theory has become a useful tool for formulating many problems in mathematical finance and control theory (see [7]). They are many papers dealing with SDU maximization (see e.g., [8, 10, 23, 30] and references therein.)

In the stochastic delayed systems, let us mention that, the appearance of *time-delayed* in the coefficients of the controlled process, leads to *time-advanced* in the drift of the associated adjoint equations. Note that, time-advanced (or anticipated) BSDEs were studied by Peng and Yang [29] in the continuous case, the results were then applied to study a linear stochastic delay system when there is no delay in the noise coefficient. Øksendal et al. [27] generalized the latter results to the jumps case. Their application also extend the one by Peng and Yang [29] to a nonlinear control problem for stochastic delayed systems and with possible delay in the noise and the jumps coefficients. In the delayed case, the problem of optimal control of recursive utility can be seen as a optimal control for forward-backward stochastic differential delayed system. In the jumps case this problem was studied in [32, 33].

The problem of optimal control of recursive utility under model uncertainty was studied by Bordigoni et al. [3] in the continuous case and by Jeanblanc et al. [18] in the discontinuous case via a robust utility maximization technique. In these papers, the penalization function is given by the entropy. On the other hand, assuming that the probability measure $Q \in \mathcal{Q}$ is a market scenario controlled by the market, this problem can be interpreted as a zero-sum stochastic differential game between the agent who optimizes her instantaneous consumption and/or portfolio, and the market choosing the scenario Q. In a general non-Markovian case, this problem was solved by Øksendal and Sulem [25], using stochastic maximum principle.

In the present paper, we consider a problem of optimal control for stochastic delay system under model uncertainty, in a general non-Markovian setting. In this regard, the problem cannot be solved by a dynamic programming argument. We shall therefore study the problem using a stochastic maximum principle approach. Our problem can be regarded as a stochastic differential game of a system of forward-backward stochastic differential delay equations. We derive sufficient and necessary conditions of optimality.

This paper can be seen as a generalization of [32] to model uncertainty and with delay of moving average time in the coefficients (but without delay in the control). We also extend the work in [3, 18] by considering delay in the coefficients of the state process, and more general SDU and penalization functions. Moreover, our paper can be consider as a dynamic time delayed version of [26].

We apply the results to find the optimal consumption rate from a cash flow with delay under model uncertainty and general recursive utility. This is a generalization to the stochastic differential utility under model uncertainty of [4].

The paper is organized as follows: In Section 2, we motivate and formulate our control problem. In Section 3, we obtain a stochastic maximum principle for delayed stochastic differential games for this general non-Markovian stochastic control problem under model uncertainty. We apply our result to study a problem of consumption choice optimization under model uncertainty and delay.

2. Problem formulation

In this section, we briefly present the model in [10] and then formulate the optimization problem.

2.1. A motivating example.

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}, P)$ be a complete filtered probability space that satisfies the usual conditions with T being a finite horizon. For any probability measure $Q \ll P$ on $\mathcal{F}_T$, the density process of Q with respect to P is the RCLL P-martingale $Z^Q = (Z^Q(t))_{0\leq t\leq T}$ with

$$Z^Q(t) = \frac{dQ}{dP}\Big|_{\mathcal{F}_t} = E\Big[\frac{dQ}{dP}\Big|\mathcal{F}_t\Big]$$

The following model by Faidi et al. [10] illustrates the situation. Suppose the financial market has two investments opportunities: a bond S_0 and a risky asset S. Without loss of generality, we assume that the price of the bond in constant otherwise we consider the bond as a numeraire. We assume that S is a continuous semimartingale with canonical decomposition:

$$S(t) = S(0) + N(t) + D(t), \ \ t \in [0, T].$$

Here $< N >$ denotes the quadratic variation of the continuous martingale N. We shall assume that $< N >$ is absolutely continuous with respect to the Lebesgue measure on $[0, T]$ and we define the positive predictable process $\sigma = (\sigma(t))_{0\leq t\leq T}$ by

$$< N >_t = \int_0^t \sigma(s)ds, \ \ t \in [0, T].$$

Assume that there exists a predictable process $\lambda = (\lambda(t))_{0\leq t\leq T}$ such that

$$D(t) = \int_0^t \sigma(s)\lambda(s)ds, \ \ t \in [0, T].$$

Assume that

$$K(T) =< \lambda dN >_T = \int_0^T \sigma(s)\lambda^2(s)ds, \ \ t \in [0, T] \text{ is bounded a.s.}$$

Let us consider an investor who can consume between time 0 and time T and denote by $c = (c(t))_{0\leq t\leq T}$ her consumption rate. If she chooses a portfolio $H = (H(t))_{0\leq t\leq T}$ representing the number of risky assets invested in the portfolio and S-integrable, the corresponding wealth process $A(t), \ t \in [0, T]$, will have the dynamics

$$dA(t) = H(t)dS(t) - c(t)dt, \ \ A(0) = a_0 > 0. \tag{2.1}$$

In the case of a continuous filtration, Bordigoni et al. [3] study stochastic control problem arising in the context of utility maximization under model uncertainty. Their goal is to find $Q \in \mathcal{Q}_f$ that minimizes the following functional

$$E_Q\Big[\int_0^T \alpha S^\kappa(s)U_1(s)ds + \bar{\alpha}S^\kappa(T)U_2(T)\Big] + \beta E_Q\Big[\mathcal{R}^\kappa(0, T)\Big]$$

where

$$\mathcal{Q}_f = \Big\{Q|Q \ll P, \ Q = P \text{ on } \mathcal{F}_0 \text{ and } H(Q|P) := E_Q\Big[\log\frac{dQ}{dP}\Big]\Big\},$$

α and $\bar{\alpha}$ are non negative constants, $\beta \in (0,\infty)$, $\kappa = (\kappa(t))_{0\le t\le T}$ and $U_1 = (U_1(t))_{0\le t\le T}$ are progressively measurable processes, $U_2(T)$ is a $\mathcal{F}_T$-measurable random variable, $S^\kappa(t) = e^{\int_0^t \kappa(s)ds}$ is the the discounting factor and $\mathcal{R}^\kappa(t,T)$ is the penalization term which is the sum of the entropy rate and the terminal entropy:

$$\mathcal{R}^\kappa(t,T) = \frac{1}{S^\kappa(t)}\int_t^T \kappa(s)S^\kappa(s)\log\frac{Z^Q(s)}{Z^Q(t)}ds + \frac{S^\kappa(T)}{S^\kappa(t)}\log\frac{Z^Q(T)}{Z^Q(t)}. \tag{2.2}$$

The authors prove that in general there exists a unique optimal measure Q^* and show that Q^* is equivalent to P. In the case of a dynamic value process i.e.,

$$V(t) = \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_f} Y^Q(t), \tag{2.3}$$

where

$$Y^Q(t) = \Big(\frac{1}{S^\kappa(t)}E_Q\Big[\int_t^T \alpha S^\kappa(s)U_1(s)ds + \bar{\alpha}S^\kappa(T)U_2(T)\Big|\mathcal{F}_t\Big] + \beta E_Q\Big[\mathcal{R}^\kappa(t,T)\Big|\mathcal{F}_t\Big]\Big). \tag{2.4}$$

They also show that, if $\mathbb{F}$ is a continuous filtration, then the dynamics of $(Y(t))_{0\le t\le T}$ is given by the following BSDE

$$\begin{cases} dY(t) &=& \Big(\kappa(t)Y(t) - \alpha U_1(t)\Big)dt + \dfrac{1}{\beta}d\langle M^Y\rangle_t + dM^Y(t);\;\; t\in[0,T] \\ Y(T) &=& \bar{\alpha}U_2(T). \end{cases} \tag{2.5}$$

Faidi et al. [10] study the problem of utility maximization over a terminal wealth and consumption in complete market when the value function is given by (2.5). The existence and uniqueness of an optimal strategy is proved.

Jeanblanc et al. [18] generalize these results to model with jump and in the case of a discontinuous filtration. They prove that the robust optimization problem is the solution of a quadratic BSDE. Note that their work also extends the result of Duffie and Skiadas [6] and El Karoui et al. [8] to the robust case and including jumps.

In this paper we generalize for $\kappa = 0$ the later situation in many directions

- We study more general utility and convex penalty functions.
- We include delay in our wealth process.

2.2. Problem formulation.

Let $\{B(t)\}_{0\le t\le T}$ be a Brownian motion and $\widetilde{N}(d\zeta,ds) = N(d\zeta,ds) - \nu(d\zeta)\,ds$ be a compensated Poisson random measure associated with a Lévy process with Lévy measure ν on the (complete) filtered probability space $(\Omega,\mathcal{F},\mathbb{F}=\{\mathcal{F}_t\}_{0\le t\le T},P)$, with $T>0$, a fixed time horizon. In the sequel, we assume that the Lévy measure ν fulfills

$$\int_{\mathbb{R}_0}\zeta^2\,\nu(d\zeta) < \infty,$$

where $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$.

We also point out that the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$ is generated by the Brownian motion and the Poisson random measure.

Suppose that the state process (or wealth process) $A(t) = A^{(v)}(t,\omega);\ 0 \le t \le T,\ \omega \in \Omega$ is a controlled stochastic delay equation of the form:

$$\begin{cases} dA(t) &= b(t, A(t), A_1(t), A_2(t), v(t), \omega)\, dt \,+\, \sigma(t, A(t), A_1(t), A_2(t), v(t), \omega)\, dB(t) \\ & \quad + \int_{\mathbb{R}_0} \gamma(t, A(t), A_1(t), A_2(t), v(t), \zeta, \omega)\, \widetilde{N}(d\zeta, dt);\ \ t \in [0, T] \\ A(t) &= a_0(t);\ \ t \in [-\delta, 0], \end{cases} \tag{2.6}$$

where

$$A_1(t) = A(t-\delta),\ \ A_2(t) = \int_{t-\delta}^{t} e^{-\rho(t-r)} X(r) dr, \tag{2.7}$$

and $\delta > 0,\ \rho \ge 0$ and $T > 0$ are given constants. $v(\cdot)$ is the control process.

The functions $b : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{V} \times \Omega \to \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{V} \times \Omega \to \mathbb{R}$ and $\gamma : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{V} \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$ are given such that for all t, $b(t, a, a_1, a_2, v, \cdot)$, $\sigma(t, a, a_1, a_2, v, \cdot)$ and $\gamma(t, a, a_1, a_2, v, z, \cdot)$ are $\mathcal{F}_t$-measurable for all $a \in \mathbb{R},\ a_1 \in \mathbb{R},\ a_2 \in \mathbb{R},\ v \in \mathcal{V}$ and $z \in \mathbb{R}_0$. We assume that the function $a_0(t)$ is continuous and deterministic.

Let consider the preceding model uncertainty setup and assume that the law of the controlled process belong to a family of equivalent measures whose densities are

$$\begin{cases} dG^\theta(t) &= G^\theta(t^-)(\theta_0(t)\, dB(t) + \int_{\mathbb{R}_0} \theta_1(t,\zeta)\, \widetilde{N}(d\zeta, dt);\ \ t \in [0, T+\delta] \\ G^\theta(0) &= 1, \\ G^\theta(t) &= 0,\ \ t \in [-\delta, 0). \end{cases} \tag{2.8}$$

- $\theta = (\theta_0, \theta_1)$ may be regarded as a scenario control,
- $\mathcal{V}$ is the set of admissible controls v,
- $\mathcal{A}$ is the set admissible scenario controls θ assumed to be $\mathcal{F}_t$-predictable and such that $E\left[\int_0^T \left\{\theta_0^2(t) + \int_{\mathbb{R}_0} \theta_1^2(t,\zeta)\, \nu(d\zeta)\right\} dt\right] < \infty$ and $\theta_1(t,z) \ge -1 + \varepsilon$ for some $\varepsilon > 0$

Assume the following in Equation (2.4):

$$\begin{aligned} \alpha = \bar{\alpha} = \beta = 1,\ \kappa = 0,\ U_1(t) = f(t, A(t), A_1(t), A_2(t), v(t)) \\ U_2(T) = g(A(T)), \mathcal{R}^\kappa(t,T) = \mathcal{R}(t,T) = \int_t^T h(\theta(s)) ds \end{aligned} \tag{2.9}$$

where f, g are given concave functions, increasing with a strictly decreasing derivative, and h is a convex function.

The robust optimization problem we consider is therefore:

Problem P1. *Find* $(\widehat{v}, \widehat{\theta}) \in \mathcal{V} \times \mathcal{A}$ *such that*

$$\underset{v\in\mathcal{V}}{\text{ess sup}}\ \underset{\theta\in\mathcal{A}}{\text{ess inf}} E_{Q^\theta}[W_t(v,\theta)\Big|\mathcal{F}_t] = E_{Q^{\theta^*}} W_t[(\widehat{v},\widehat{\theta})\Big|\mathcal{F}_t] = \underset{\theta\in\mathcal{A}}{\text{ess inf}}\ \underset{v\in\mathcal{V}}{\text{ess sup}} E_{Q^\theta}[W_t(v,\theta)\Big|\mathcal{F}_t] \tag{2.10}$$

where

$$W_t(\widehat{v},\widehat{\theta}) = \int_t^T f(s, A(s), A_1(s), A_2(s), v(s), \omega)\, ds + g(A(T), \omega) + \int_t^T h(\theta(s))\, ds.$$

This problem can be seen as a stochastic differential game problem.

Let $\{\mathcal{E}_t^1\}_{0\le t\le T}$ and $\{\mathcal{E}_t^2\}_{0\le t\le T}$ be given subfiltration of $\{\mathcal{F}_t\}_{0\le t\le T}$ representing the amount of information available to the controllers at time t. We assume that $v \in \mathcal{V}$ is $\mathcal{E}_t^1$-predictable and $\theta \in \mathcal{A}$ is $\mathcal{E}_t^2$-predictable. We define

$$f_1(t,a,a_1,a_2,u) := f(t,a,a_1,a_2,v) + h(\theta); \;\; u = (v,\theta).$$

Then

$$E_{Q^\theta}[W(v,\theta)] = E\Big[G^\theta(T)g(A^v(T)) + \int_0^T G^\theta(s) f_1(t,A^v(t),A_1^v(t),A_2^v(t),u(t))\,ds\Big]$$

Put

$$Y(t) = E\Big[\frac{G^\theta(T)}{G^\theta(t)} g(A^v(T)) + \int_t^T \frac{G^\theta(s)}{G^\theta(t)} f_1(t,A^v(t),A_1^v(t),A_2^v(t),u(t))\,ds \Big| \mathcal{F}_t\Big] \tag{2.11}$$

If follows from Lemma A1 that $Y(t)$ is the solution of the following linear BSDE

$$\begin{cases} dY(t) &= -\Big(f_1(t,A^v(t),A_1^v(t),A_2^v(t),u(t)) + \theta_0(t)Z(t) + \int_{\mathbb{R}_0} \theta_1(t,\zeta)K(t,\zeta)\,\nu(d\zeta)\Big)dt \\ & \quad +Z(t)dB(t) + \int_{\mathbb{R}_0} K(t,\zeta)\,\widetilde{N}(d\zeta,dt); \;\; t\in[0,T] \\ Y(T) &= g(A^v(T)). \end{cases} \tag{2.12}$$

Note that

$$Y(0) = Y^{v,\theta}(0) = E_{Q^\theta}[W(v,\theta)].$$

Thus the Problem P1 becomes

Problem P2. *Find* $(\widehat{v},\widehat{\theta}) \in \mathcal{V}\times\mathcal{A}$ *such that*

$$\operatorname*{ess\,sup}_{v\in\mathcal{V}} \operatorname*{ess\,inf}_{\theta\in\mathcal{A}} Y^{v,\theta}(t) = Y^{\widehat{v},\widehat{\theta}}(t) = \operatorname*{ess\,inf}_{\theta\in\mathcal{A}} \operatorname*{ess\,sup}_{v\in\mathcal{V}} Y^{v,\theta}(t), \tag{2.13}$$

where $Y^{v,\theta}(t)$ *is given by the forward-backward delayed system (2.6) & (2.12).*

In the next section, we shall solve Problem P2 under more general coefficients using stochastic maximum principle for delayed differential games.

3. A stochastic maximum principle for delayed stochastic differential games

In this Section, we study Problem P2 with more general driver in the BSDE (2.12). We prove a necessary and sufficient stochastic maximum principle for stochastic differential games of forward-backward SDEs with delayed.

Suppose that the state process $A(t) = A^{(u)}(t,\omega);\; 0\le t\le T,\, \omega\in\Omega$ is a controlled stochastic delay equation of the form:

$$\begin{cases} dA(t) &= b(t,A(t),A_1(t),A_2(t),u(t),\omega)\,dt \,+\, \sigma(t,A(t),A_1(t),A_2(t),u(t),\omega)\,dB(t) \\ & \quad +\int_{\mathbb{R}_0} \gamma(t,A(t),A_1(t),A_2(t),u(t),\zeta,\omega)\,\widetilde{N}(d\zeta,dt); \;\; t\in[0,T] \\ A(t) &= a_0(t); \;\; t\in[-\delta,0], \end{cases} \tag{3.1}$$

where

$$A_1(t) = A(t-\delta), \;\; A_2(t) = \int_{t-\delta}^{t} e^{-\rho(t-r)} X(r)dr, \tag{3.2}$$

and $\delta > 0,\; \rho \geq 0$ and $T > 0$ are given constants. $u(\cdot)$ is the control process.

The functions $b : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \to \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \to \mathbb{R}$ and $\gamma : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$ are given such that for all t, $b(t,a,a_1,a_2,u,\cdot)$, $\sigma(t,a,a_1,a_2,u,\cdot)$ and $\gamma(t,a,a_1,a_2,u,z,\cdot)$ are $\mathcal{F}_t$-measurable for all $a \in \mathbb{R}$, $a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}$, $u \in \mathcal{U}$ and $\zeta \in \mathbb{R}_0$. We assume that the function $a_0(t)$ is continuous and deterministic.

Here $u = (u_1,u_2)$, where $u_i(t)$ is the control of player i; $i = 1,2$. We suppose that we are given two subfiltrations

$$\mathcal{E}_t^{(i)} \subset \mathcal{F}_t\,; \;\; t \in [0,T], \tag{3.3}$$

representing the information available to player i at time t; $i = 1,2$. We let $\mathcal{A}_i$ denotes the set of admissible control processes of player i, contained in the set of $\mathcal{E}_t^{(i)}$-predictable processes, $i = 1,2$.

We consider the associated BSDE's in the unknowns $Y_i(t), Z_i(t), K_i(t\zeta)$ have the form

$$\begin{cases} dY_i(t) &= g_i(t,A(t),A_1(t),A_2(t),Y_i(t),Z_i(t),K_i(t,\zeta),u(t))\,dt \,+\, Z_i(t)\,dB(t) \\ &\quad + \displaystyle\int_{\mathbb{R}_0} K_i(t,\zeta)\,\widetilde{N}(d\zeta,dt); \;\; t \in [0,T] \\ Y_i(T) &= h_i(A(T))\,; \;\; i = 1,2, \end{cases} \tag{3.4}$$

where $g_i(t,a,a_1,a_2,y,z,k,u) : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}_0\times\mathcal{U}\times\Omega \to \mathbb{R}$ and $h_i(a) : \mathbb{R} \to \mathbb{R}$, $i = 1,2$ are such that the BSDE (3.4) has a unique solution.

Let $f_i(t,a,a_1,a_2,u) : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \to \mathbb{R}$, $\varphi_i(a) : \mathbb{R} \to \mathbb{R}$ and $\psi_i(a) : \mathbb{R} \to \mathbb{R}$ $i = 1,2$ be given C^1 functions with respect to (t,a,a_1,a_2,u) such that

$$E\Big[\int_0^T \Big\{|f_i(t,A(t),A_1(t),A_2(t),u(t))| \,+\, \Big|\frac{\partial f_i}{\partial a_i} f_i(t,A(t),A_1(t),A_2(t),u(t))\Big|^2\Big\}dt$$
$$\varphi_i(A(T)) + |\varphi_i'(A(T))|^2 + |\psi_i(Y_i(0))| + |\psi_i'(Y_i(0))|^2\Big] < \infty \text{ for } a_i = a, a_1, a_2 \text{ and } u.$$

Assume that the performance functional of each player i has the following form

$$J_i(t,u) = E\Big[\int_t^T f_i(s,A(s),A_1(s),A_2(s),u(s))ds \,+\, \varphi_i(A(T)) \,+\, \psi_i(Y_i(t))\Big|\mathcal{F}_t\Big]\,; \;\; i = 1,2. \tag{3.5}$$

Here, f_i, φ_i and ψ_i can be seeing as profit rates, bequest functions and "risk evaluations" respectively, of player i ; $i = 1,2$.

We shall first consider the *non-zero-sum* stochastic differential game problem that is, we analyze the following:

Problem P3. *Find* $(u_1^*,u_2^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ *(if it exists) such that*

(1) $J_1(t,u_1,u_2^*) \leq J_1(t,u_1^*,u_2^*)$ *for all* $u_1 \in \mathcal{A}_1$

(2) $J_2(t,u_1^*,u_2) \leq J_2(t,u_1^*,u_2^*)$ *for all* $u_2 \in \mathcal{A}_2$

The pair (u_1^*, u_2^*) is called a Nash Equilibrium (if it exists). The intuitive idea is that there are two players, Player I and Player II. While Player I controls u_1, Player II controls u_2. Each player is assumed to know the equilibrium strategies of the other player, and no player has anything to gain by changing only his or her own strategy (i.e., by changing unilaterally). Player I and Player II are in Nash Equilibrium if each player is making the best decision she can, taking into account the other player's decision.

Let mention once more that in this case, u_2 is not known to the trader, but subject to uncertainty. We may regard u_2 as a market scenario or a stochastic control of the market, which is playing against the trader.

We shall first solve Problem P3 for $t = 0$ and then obtain the result for each $t \in [0, T]$ as a corollary. For $t = 0$ we put

$$J_i(u) = J_i(0,u) = E\Big[\int_0^T f_i(s, A(s), A_1(s), A_2(s), u(s))ds \,+\, \varphi_i(A(T)) \,+\, \psi_i(Y_i(0))\Big],\ \ i = 1,2 \tag{3.6}$$

Define the Hamiltonians

$$H_i : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \mathbb{U}_1 \times \mathbb{U}_2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \longrightarrow \mathbb{R},\ i = 1,2$$

by

$$\begin{aligned} H_i\,(t,a,a_1,a_2,y,z,k,u_1,u_2,\lambda,p,q,r) :=& f_i(t,a,a_1,a_2,u_1,u_2) \,+\, \lambda g_i(t,a,a_1,a_2,y,z,k,u_1,u_2) \\ &+\, p_i b(t,a,a_1,a_2,u_1,u_2) \,+\, q_i \sigma(t,a,a_1,a_2,u_1,u_2) \\ &+\, \int_{\mathbb{R}_0} r(\zeta)\gamma(t,a,a_1,a_2,u_1,u_2,\zeta)\,\widetilde{N}(d\zeta,dt) \end{aligned} \tag{3.7}$$

where $\mathcal{R}$ is the set of functions such that the last term in (3.7) converges.

Suppose that H_i is Fréchet differentiable in the variables $a, a_1, a_2, y, z, k, u_i$ and that $\nabla_k H_i(t,\zeta)$ as a random measure which is absolutely continuous with respect to ν; $i = 1,2$. Define the adjoint processes $\lambda_i(t)$, $p_i(t)$, $q_i(t)$ and $r_i(t,\zeta)$, $\ t \in [0,T],\ \zeta \in \mathbb{R}_0$ associated to these Hamiltonians by the following system of advanced forward-backward stochastic differential equation (AFBSDEs)

(1) Forward SDE in $\lambda_i(t)$

$$\begin{cases} d\lambda_i(t) &=& \dfrac{\partial H_i}{\partial y}(t)dt + \dfrac{\partial H_i}{\partial z}(t)dB(t) \,+\, \displaystyle\int_{\mathbb{R}_0} \dfrac{d\nabla_k H_i}{d\nu(\zeta)}(t,\zeta)\,\widetilde{N}(d\zeta,dt),\ \ t \in [0,T] \\ \lambda_i(0) &=& \psi_i'(Y(0))\,;\ i = 1,2,. \end{cases} \tag{3.8}$$

Here and in what follows, we use the notation

$$\frac{\partial H_i}{\partial y}(t) = \frac{\partial H_i}{\partial y}(t, A(t), A_1(t), A_2(t), u_1(t), u_2(t), Y_i(t), Z_i(t), K_i(t,\cdot), \lambda_i(t),\, p_i(t),\, q_i(t), r_i(t,\cdot)),$$

etc and $\dfrac{d\nabla_k H_i}{d\nu(\zeta)}(t,\zeta)$ is the Radon-Nikodyn derivative of $\nabla_k H_i(t,\zeta)$ with respect to $\nu(t,\zeta)$.

(2) Anticipative BSDE in $p_i(t)$, $q_i(t)$, $r_i(t,\zeta)$

$$\begin{cases} dp_i(t) &=& E[\mu_i(t)\big|\mathcal{F}_t] \,+\, q_i(t)\,dB(t) \,+\, \displaystyle\int_{\mathbb{R}_0} r_i(t,\zeta)\,\widetilde{N}(d\zeta,dt),\ \ t \in [0,T] \\ p_i(T) &=& \varphi_i'(A(T)) \,+\, h_i'(A(T)), q(T) = r(T,\cdot) = 0 \\ p(t) &=& q(t) = r(t,\cdot) = 0\,;\ t \in (T, T+\delta],\ \ i = 1,2,, \end{cases} \tag{3.9}$$

where

$$\mu(t) = -\frac{\partial H_i}{\partial a}(t) - \frac{\partial H_i}{\partial a_1}(t+\delta)\chi_{[0,T-\delta]}(t) - e^{\rho t}\int_t^{t+\delta} \frac{\partial H_i}{\partial a_2}(s) e^{-\rho s}\chi_{[0,T]}(s)\, ds, \tag{3.10}$$

and

$$\frac{\partial H_i}{\partial a}(t) = \frac{\partial H_i}{\partial a}(t, A(t), A_1(t), A_2(t), u_1(t), u_2(t), Y_i(t), Z_i(t), K_i(t,\cdot), \lambda_i(t),\, p_i(t),\, q_i(t), r_i(t,\cdot)),$$

$$\frac{\partial H_i}{\partial a_1}(t+\delta)$$
$$= \frac{\partial H_i}{\partial a_1}(t+\delta, A^\delta(t), A_1^\delta(t), A_2^\delta(t), u_1^\delta(t), u_2^\delta(t), Y_i^\delta(t), Z_i^\delta(t), K_i^\delta(t,\cdot), \lambda_i^\delta(t),\, p_i^\delta(t),\, q_i^\delta(t), r_i^\delta(t,\cdot)),$$

with $x_i^\delta = x_i(t+\delta)$, $\ x_i = a, a_1, a_2, u_1, u_2, y, z, k, \lambda, p, q, r$.

Note that $\mu(t)$ contains future values of $A(s), A_1(s), A_2(s), u_1(s), u_2(s), Y_i(s), Z_i(s), K_i(s,\cdot), \lambda_i(s),\, p_i(s),\, q_i(s), r_i(s,\cdot);\ \ s \le t+\delta$

Remark 3.1. *Let V be an open subset of a Banach space $\mathcal{X}$ and let $F : V \to \mathbb{R}$.*

- *We say that F has a directional derivative (or Gateaux derivative) at $x \in V$ in the direction $y \in \mathcal{X}$ if*
$$D_y F(x) := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(F(x+\varepsilon y) - F(x)) \text{ exists.}$$
- *We say that F is Fréchet differentiable at $x \in V$ if there exists a linear map*
$$L : \mathcal{X} \to \mathbb{R}$$
such that
$$\lim_{\substack{h\to 0\\ h\in\mathcal{X}}} \frac{1}{\|h\|}|F(x+h) - F(x) - L(h)| = 0.$$
In this case we call L the Fréchet derivative of F at x, and we write
$$L = \nabla_x F$$
- *If F is Fréchet differentiable, then F has a directional derivative in all directions $y \in \mathcal{X}$ and*
$$D_y F(x) = \nabla_x F(y)$$

3.1. A sufficient maximum principle for FBSDDE games.

In the following result, we give a sufficient maximum principle for FBSDDE games. In fact, we prove that, under some assumptions, maximizing the Hamiltonians leads to an optimal control.

Theorem 3.2. *[**Sufficient maximum principle for FBSDDE games**] Let $(\widehat{u}_1, \widehat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ with corresponding solutions $\widehat{A}(t), \widehat{Y}_i(t), \widehat{Z}_i(t), \widehat{K}_i(t,\zeta), \widehat{\lambda}_i(t), \widehat{p}_i(t), \widehat{q}_i(t), \widehat{r}_i(t,\zeta)$ of equations (3.1), (3.4), (3.8) and (3.9) for $i = 1, 2$. Suppose that the following are true:*

- *The functions*

$$a \mapsto h_i(a),\ a \mapsto \varphi_i(a),\ y \mapsto \psi_i(y), \tag{3.11}$$

$$(a, a_1, a_2, y, z, k, v_1) \mapsto H_1(a, a_1, a_2, y, z, k, v_1, \widehat{u}_2, \widehat{\lambda}_i(t), \widehat{p}_i(t), \widehat{q}_i(t), \widehat{r}_i(t, \cdot)) \tag{3.12}$$

and

$$(a, a_1, a_2, y, z, k, v_2) \mapsto H_2(a, a_1, a_2, y, z, k, \widehat{u}_1, v_2, \widehat{\lambda}_i(t), \widehat{p}_i(t), \widehat{q}_i(t), \widehat{r}_i(t, \cdot)) \tag{3.13}$$

are concave, when $u_i(t) = v(t), u_{3-i}(t) = \widehat{u}_{3-i}(t);\ \ i = 1, 2.$

- $$\begin{aligned}&\max_{v\in\mathbb{U}_i}\Big\{E\Big[H_i(\widehat{A}(t), \widehat{A}_1(t), \widehat{A}_2(t), \widehat{Y}_i(t), \widehat{Z}_i(t), \widehat{K}_i(t, \cdot), u_1(t), u_2(t), \widehat{\lambda}_i(t), \widehat{p}_i(t), \widehat{q}_i(t), \widehat{r}_i(t, \cdot))\Big|\mathcal{E}_t^{(i)}\Big];\\ &\quad u_i(t) = v(t), u_{3-i}(t) = \widehat{u}_{3-i}(t)\Big\}\\ &\quad = E\Big[H_i(t, \widehat{A}(t), \widehat{A}_1(t), \widehat{A}_2(t), \widehat{Y}_i(t), \widehat{Z}_i(t), \widehat{K}_i(t, \cdot), \widehat{u}_1(t), \widehat{u}_2(t), \widehat{\lambda}_i(t), \widehat{p}_i(t), \widehat{q}_i(t), \widehat{r}_i(t, \cdot))\Big|\mathcal{E}_t^{(i)}\Big]\\ &\quad \text{for } i = 1, 2.\end{aligned} \tag{3.14}$$

  for all $t \in [0, T]$*, a.s.*
- *In addition, assume the following growth conditions*

$$\begin{aligned}E\Big[\int_0^T \Big\{\widehat{p}_i^2(t)\Big((\sigma(t) - \widehat{\sigma}(t))^2 + \int_{\mathbb{R}_0} (\gamma_i(t, \zeta) - \widehat{\gamma}_i(t, \zeta))^2\, \nu(d\zeta)\Big)&\\ (A(t) - \widehat{A}(t))^2\Big(\widehat{q}_i^2(t) + \int_{\mathbb{R}_0} \widehat{r}_i^2(t, \zeta)\, \nu(d\zeta)\Big)&\\ (Y(t) - \widehat{Y}(t))^2\Big((\frac{\partial \widehat{H}_i}{\partial z})^2(t) + \int_{\mathbb{R}_0} \Big\|\nabla_k \widehat{H}_i(t, \zeta)\Big\|^2 \nu(d\zeta)\Big)&\\ \widehat{\lambda}_i^2(t)\Big((Z_i(t) - \widehat{Z}_i(t))^2 + \int_{\mathbb{R}_0} (K_i(t, \zeta) - \widehat{K}_i(t, \zeta))^2\, \nu(d\zeta)\Big)\Big\}\Big] < \infty \text{ for } i = 1, 2.&\end{aligned} \tag{3.15}$$

Then $\widehat{u}(t) = (\widehat{u}_1(t), \widehat{u}_2(t))$ *is a Nash equilibrium for (3.1)-(3.4) and (3.6).*

Remark 3.3. *In the Theorem and in the following, we are using the subsequent notation: If* $i = 1$, $A(t) = A^{(u_1, \widehat{u}_2)}(t)$ *and* $Y_1(t) = Y_1^{(u_1, \widehat{u}_2)}(t)$ *are the processes associated to the control* $u(t) = (u_1(t), \widehat{u}_2(t))$*, while* $\widehat{A}(t) = A^{(\widehat{u})}(t)$ *and* $\widehat{Y}_1(t) = Y_1^{(\widehat{u})}(t)$ *are those associated to the control* $\widehat{u}(t) = (\widehat{u}_1(t), \widehat{u}_2(t))$*.*

Furthermore, we put

$$\frac{\partial \widehat{H}_i}{\partial a}(t) = \frac{\partial H_i}{\partial a}(t, \widehat{A}(t), \widehat{A}_1(t), \widehat{A}_2(t), \widehat{Y}_i(t), \widehat{Z}_i(t), \widehat{K}_i(t, \cdot), \widehat{u}, \widehat{\lambda}_i(t), \widehat{p}_i(t), \widehat{q}_i(t), \widehat{r}_i(t, \cdot))$$

and similarly with $\frac{\partial \widehat{H}_i}{\partial a_1}(t), \frac{\partial \widehat{H}_i}{\partial a_2}(t), \frac{\partial \widehat{H}_i}{\partial y}(t), \frac{\partial \widehat{H}_i}{\partial z}(t), \frac{\partial \widehat{H}_i}{\partial u_1}(t), \frac{\partial \widehat{H}_i}{\partial u_2}(t)$ *and* $\nabla_{k_i} \widehat{H}_i(t, \zeta),\ \ i = 1, 2.$

Proof. We shall first prove that $J_1(u_1, \widehat{u}_2) \leq J_1(\widehat{u}_1, \widehat{u}_2)$ for all $u \in \mathcal{A}_1$.

Choose $u_1 \in \mathcal{A}_1$ and consider

$$J_1(u_1, \widehat{u}_2) - J_1(\widehat{u}_1, \widehat{u}_2) = I_1 + I_2 + I_3 \tag{3.16}$$

where

$$I_1 = E\Big[\int_0^T \Big\{f_1(t, A(t), A_1(t), A_2(t), u(t)) \;-\; f_1(t, \widehat{A}(t), \widehat{A}_1(t), \widehat{A}_2(t), \widehat{u}(t))dt\Big\}\Big] \tag{3.17}$$

$$I_2 = E\Big[\varphi_1(A(T)) - \varphi_1(\widehat{A}(T))\Big] \tag{3.18}$$

$$I_3 = E\Big[\psi_1(Y_1(0))\Big] \;-\; \psi_1(\widehat{Y}_1(0))\Big] \tag{3.19}$$

By the definition of H_1 and concavity, we get

$$\begin{aligned}
I_1 = E\Big[\int_0^T \Big\{&H_1(t) - \widehat{H}_1(t) - \widehat{\lambda}_1(g_1(t) - \widehat{g}_1(t)) - \widehat{p}_1(t)(b(t) - \widehat{b}(t)) \\
&- \widehat{q}_1(t)(\sigma(t) - \widehat{\sigma}(t)) - \int_{\mathbb{R}_0} \widehat{r}_1(t,\zeta)(\gamma_1(t,\zeta) - \widehat{\gamma}_1(t,\zeta))\nu(d\zeta)\Big\}dt\Big] \\
\leq E\Big[\int_0^T \Big\{&\frac{\partial \widehat{H}_1}{\partial a}(t)(A(t) - \widehat{A}(t)) + \frac{\partial \widehat{H}_1}{\partial a_1}(t)(A_1(t) - \widehat{A}_1(t)) + \frac{\partial \widehat{H}_1}{\partial a_2}(t)(A_2(t) - \widehat{A}_2(t)) \\
&+ \frac{\partial \widehat{H}_1}{\partial y}(t)(Y_1(t) - \widehat{Y}_1(t)) + \frac{\partial \widehat{H}_1}{\partial z}(t)(Z_1(t) - \widehat{Z}_1(t)) + \int_{\mathbb{R}_0} \nabla_k \widehat{H}_i(t,\zeta)(K_1(t,\zeta) - \widehat{K}_1(t,\zeta))\nu(d\zeta) \\
&\frac{\partial \widehat{H}_1}{\partial u_1}(t)(u_1(t) - \widehat{u}_1(t)) - \widehat{\lambda}_1(g_1(t) - \widehat{g}_1(t)) - \widehat{p}_1(t)(b(t) - \widehat{b}(t)) \\
&- \widehat{q}_1(t)(\sigma(t) - \widehat{\sigma}(t)) - \int_{\mathbb{R}_0} \widehat{r}_1(t,\zeta)(\gamma_1(t,\zeta) - \widehat{\gamma}_1(t,\zeta))\nu(d\zeta)\Big\}dt\Big]
\end{aligned} \tag{3.20}$$

By concavity of φ_1, Itô formula, (3.1) and (3.9), we get

$$\begin{aligned}
I_2 =&\leq E\Big[\varphi_1'(A(T))(A(T) - \widehat{A}(T))\Big] \\
&= E\Big[\widehat{p}_1(T)(A(T) - \widehat{A}(T))\Big] - E\Big[\widehat{\lambda}_1(T)(A(T) - \widehat{A}(T))\Big] \\
&= E\Big[\int_0^T \widehat{p}_1(t)(dA(t) - d\widehat{A}(t)) \;+\; \int_0^T (A(t^-) - \widehat{A}(t^-))d\widehat{p}_1(t) \\
&+ \int_0^T (\sigma(t) - \widehat{\sigma}(t))\widehat{q}_1(t)dt \;+\; \int_0^T \int_{\mathbb{R}_0} (\gamma(t) - \widehat{\gamma}(t))\widehat{r}_1(t,\zeta)\nu(d\zeta)\, dt\Big] \\
&- E\Big[\widehat{\lambda}_1(T)(A(T) - \widehat{A}(T))\Big] \\
&= E\Big[\int_0^T \widehat{p}_1(t)(b(t) - \widehat{b}(t))\, dt \;+\; \int_0^T (A(t^-) - \widehat{A}(t^-))E[\mu(t)|\mathcal{F}_t]dt \\
&+ \int_0^T (\sigma(t) - \widehat{\sigma}(t))\widehat{q}_1(t)dt \;+\; \int_0^T \int_{\mathbb{R}_0} (\gamma(t,\zeta) - \widehat{\gamma}(t,\zeta))\widehat{r}_1(t,\zeta)\nu(d\zeta)\, dt\Big] \\
&- E\Big[\widehat{\lambda}_1(T)(A(T) - \widehat{A}(T))\Big]
\end{aligned} \tag{3.21}$$

By concavity of $\psi_1,\ h_1$, Itô formula, (3.4) and (3.8), we get

$$
\begin{aligned}
I_2 =& \leq E\Big[\psi_1(Y_1(0))\Big] \;-\; \psi_1(\widehat{Y}_1(0))\Big]\\
&\leq E\Big[\psi_1'(\widehat{Y}_1(0))(Y_1(0)-\widehat{Y}_1(0))\Big]\\
&= E\Big[\widehat{\lambda}_1(0)(Y_1(0)-\widehat{Y}_1(0))\Big]\\
&= E\Big[\widehat{\lambda}_1(T)(Y_1(T)-\widehat{Y}_1(T))\Big]\\
&- E\Big[\int_0^T \widehat{\lambda}_1(t)(dY_1(t)-d\widehat{Y}_1(t)) \;+\; \int_0^T (Y_1(t^-)-\widehat{Y}_1(t^-))d\widehat{\lambda}_1(t)\\
&+ \int_0^T (Z_1(t)-\widehat{Z}_1(t))\frac{\partial \widehat{H}_1}{\partial z}(t)dt \;+\; \int_0^T\int_{\mathbb{R}_0}(K_1(t,\zeta)-\widehat{K}_1(t,\zeta))\nabla_k \widehat{H}_1(t,\zeta)\nu(d\zeta)\,dt\Big]\\
&= E\Big[\widehat{\lambda}_1(T)(h_1(A(T))-h_1(\widehat{A}(T)))\Big]\\
&- E\Big[\int_0^T \frac{\partial \widehat{H}_1}{\partial y}(t)(Y_1(t)-\widehat{Y}_1(t))dt \;+\; \int_0^T \widehat{\lambda}_1(t)(-g_1(t)+\widehat{g}_1(t))dt\\
&+ \int_0^T (Z_1(t)-\widehat{Z}_1(t))\frac{\partial \widehat{H}_1}{\partial z}(t)dt \;+\; \int_0^T\int_{\mathbb{R}_0}(K_1(t,\zeta)-\widehat{K}_1(t,\zeta))\nabla_k \widehat{H}_1(t,\zeta)\nu(d\zeta)\,dt\Big]\\
&= E\Big[\widehat{\lambda}_1(T)h_1'(\widehat{A}(T))(A(T)-\widehat{A}(T))\Big]\\
&- E\Big[\int_0^T \frac{\partial \widehat{H}_1}{\partial y}(t)(Y_1(t)-\widehat{Y}_1(t))dt \;+\; \int_0^T \widehat{\lambda}_1(t)(-g_1(t)+\widehat{g}_1(t))dt\\
&+ \int_0^T (Z_1(t)-\widehat{Z}_1(t))\frac{\partial \widehat{H}_1}{\partial z}(t)dt \;+\; \int_0^T\int_{\mathbb{R}_0}(K_1(t,\zeta)-\widehat{K}_1(t,\zeta))\nabla_k \widehat{H}_1(t,\zeta)\nu(d\zeta)\,dt\Big]
\end{aligned}
\tag{3.22}
$$

229 Summing (3.20), (3.21) and (3.22), we have

$$
\begin{aligned}
I_1 \;+\; I_2 \;+\; I_3 \leq E\Big[\int_0^T &\Big\{\frac{\partial \widehat{H}_1}{\partial a}(t)(A(t)-\widehat{A}(t)) + \frac{\partial \widehat{H}_1}{\partial a_1}(t)(A_1(t)-\widehat{A}_1(t))\\
&+ \frac{\partial \widehat{H}_1}{\partial a_2}(t)(A_2(t)-\widehat{A}_2(t)) + \frac{\partial \widehat{H}_1}{\partial u_1}(t)(u_1(t)-\widehat{u}_1(t)) + \mu_1(t)(A_1(t)-\widehat{A}_1(t))\Big\}dt\Big]\\
&= E\Big[\int_\delta^{T+\delta}\Big\{\frac{\partial \widehat{H}_1}{\partial a}(t-\delta) + \frac{\partial \widehat{H}_1}{\partial a_1}(t)\chi_{[0,T]}(t) + \mu_1(t-\delta)\Big\}(A_1(t)-\widehat{A}_1(t))\,dt\\
&+ \int_0^T \frac{\partial \widehat{H}_1}{\partial a_2}(t)(A_2(t)-\widehat{A}_2(t))\,dt + \int_0^T \frac{\partial \widehat{H}_1}{\partial u_1}(t)(u_1(t)-\widehat{u}_1(t))\,dt\Big]
\end{aligned}
\tag{3.23}
$$

230 Using integration by parts and substituting $r = t-\delta$, we get

$$
\begin{aligned}
&\int_0^T \frac{\partial \widehat{H}_1}{\partial a_2}(s)(A_2(s) - \widehat{A}_2(s))\,ds \\
&= \int_0^T \frac{\partial \widehat{H}_1}{\partial a_2}(s) \int_{s-\delta}^{s} e^{\rho(s-r)}(A(r) - \widehat{A}(r))dr\,ds \\
&= \int_0^T \left(\int_r^{r+\delta} \frac{\partial \widehat{H}_1}{\partial a_2}(s) e^{-\rho s}\chi_{[0,T]}(s)\,ds \right) e^{\rho r}(A(r) - \widehat{A}(r))dr \\
&= \int_\delta^{T+\delta} \left(\int_{t-\delta}^{t} \frac{\partial \widehat{H}_1}{\partial a_2}(s) e^{-\rho s}\chi_{[0,T]}(s)\,ds \right) e^{\rho(t-\delta)}(A(t-\delta) - \widehat{A}(t-\delta))dt.
\end{aligned} \tag{3.24}
$$

Combining this with (3.10) and using (3.23), we obtain

$$
\begin{aligned}
&J_1(u_1, \widehat{u}_2) - J_1(\widehat{u}_1, \widehat{u}_2) \\
&\le E\Big[\int_\delta^{T+\delta} \Big\{ \frac{\partial \widehat{H}_1}{\partial a}(t-\delta) + \frac{\partial \widehat{H}_1}{\partial a_1}(t)\chi_{[0,T]}(t) \\
&+ \left(\int_{t-\delta}^{t} \frac{\partial \widehat{H}_1}{\partial a_2}(s) e^{-\rho s}\chi_{[0,T]}(s)\,ds \right) e^{\rho(t-\delta)} + \mu_1(t-\delta) \Big\} (A_1(t) - \widehat{A}_1(t))\,dt \\
&+ \int_0^T \frac{\partial \widehat{H}_1}{\partial u_1}(t)(u_1(t) - \widehat{u}_1(t))\,dt \Big] \\
&= E\Big[\int_0^T \frac{\partial \widehat{H}_1}{\partial u_1}(t)(u_1(t) - \widehat{u}_1(t))\,dt \Big] \\
&= E\Big[\int_0^T E\Big[\frac{\partial \widehat{H}_1}{\partial u_1}(t)(u_1(t) - \widehat{u}_1(t)) | \mathcal{E}_t^{(1)} \Big]\,dt \Big] \le 0.
\end{aligned} \tag{3.25}
$$

The last inequality follows from condition (3.14) for $i = 1$. Hence

$$J_1(u_1, \widehat{u}_2) \le J_1(\widehat{u}_1, \widehat{u}_2) \text{ for all } u_1 \in \mathcal{A}_1$$

The inequality

$$J_2(\widehat{u}_1, u_2) \le J_1(\widehat{u}_1, \widehat{u}_2) \text{ for all } u_2 \in \mathcal{A}_2$$

is proved in the same way.

This completed the proof.

□

If we now start from $t \in [0, T]$, the it can be easily derived that the following result holds

Corollary 3.4. *Let* $(\widehat{u}_1, \widehat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ *with corresponding solutions* $\widehat{A}(t), \widehat{Y}_i(t), \widehat{Z}_i(t), \widehat{K}_i(t,\zeta), \widehat{\lambda}_i(t), \widehat{p}_i(t), \widehat{q}_i(t), \widehat{r}_i(t,\zeta)$ *of equations (3.1), (3.4), (3.8) and (3.9) for* $i = 1, 2$. *If the other conditions of Theorem 3.2 hold. Then* $\widehat{u}(t) = (\widehat{u}_1(t), \widehat{u}_2(t))$ *is a Nash equilibrium for (3.1)-(3.5).*

Proof. It easily follows from the proof of Theorem 3.2 with the starting value being t instead of 0 and using the fact that $\mathcal{E}_t^i \subset \mathcal{F}_t, \ i = 1, 2.$ □

3.2. **A necessary maximum principle for FBSDDE games.**

One of the drawback with the sufficient maximum principle is the concavity condition (3.12), which may fail in some interesting applications. In particular for a zero-sum game, the concavity condition implies that φ_1, ψ_1 and h_1 are affine functions, which is too strong.

In what follows, we shall prove a version of the maximum principle which does not need concavity. In fact we shall show the equivalence between a critical point for the conditional Hamiltonian and a directional derivative point for the performance functional. To this end, we make the following assumptions:

Assumption A1. *For all $t_0 \in [0,T]$ and all bounded $\mathcal{E}_t^{(i)}$-measurable random variable $\alpha_i(\omega)$, the control process $\beta_i(t)$ defined by*

$$\beta_i(t) := \chi_{(t_0,T)}(t)\alpha_i(\omega)\,;\ t \in [0,T] \tag{3.26}$$

belong to $\mathcal{A}_i$, $\ i = 1, 2$

Assumption A2. *For all $u_i \in \mathcal{A}_i$ and all bounded $\beta_i \in \mathcal{A}_i$, there exists $\varepsilon > 0$ such that*

$$\widetilde{u}_i(t) := u_i(t) + s\beta_i(t) \in \mathcal{A}_i \text{ for all } s \in (-\varepsilon, \varepsilon)$$

Assumption A3. *For all bounded $\beta_i \in \mathcal{A}_i$, the derivatives processes*

$$\begin{aligned}
X_1(t) &= \frac{d}{ds}A^{(u_1+s\beta_1,\widehat{u}_2)}(t)\Big|_{s=0};\ \ X_2(t) = \frac{d}{ds}A^{(\widehat{u}_1,u_2+s\beta_2)}(t)\Big|_{s=0}\\
y_1(t) &= \frac{d}{ds}Y_1^{(u_1+s\beta_1,\widehat{u}_2)}(t)\Big|_{s=0};\ \ y_2(t) = \frac{d}{ds}Y_2^{(\widehat{u}_1,u_2+s\beta_2)}(t)\Big|_{s=0}\\
z_1(t) &= \frac{d}{ds}Z_1^{(u_1+s\beta_1,\widehat{u}_2)}(t)\Big|_{s=0};\ \ z_2(t) = \frac{d}{ds}Z_2^{(\widehat{u}_1,u_2+s\beta_2)}(t)\Big|_{s=0}\\
k_1(t,\zeta) &= \frac{d}{ds}K_1^{(u_1+s\beta_1,\widehat{u}_2)}(t,\zeta)\Big|_{s=0}\ ;k_2(t,\zeta) = \frac{d}{ds}K_2^{(\widehat{u}_1,u_2+s\beta_2)}(t,\zeta)\Big|_{s=0}
\end{aligned}$$

exist and belong to $L^2(\lambda \times P)$.

It follows from that (3.1) that

$$\begin{aligned}
dX_1(t) &= \Big\{\frac{\partial b}{\partial a}(t)X_1(t) + \frac{\partial b}{\partial a_1}(t)X_1(t-\delta) + \frac{\partial b}{\partial a_2}(t)\int_{t-\delta}^t e^{\rho(t-r)}X_1(r)\,dr + \frac{\partial b}{\partial u_1}(t)\beta_1(t)\Big\}dt\\
&\quad + \Big\{\frac{\partial \sigma}{\partial a}(t)X_1(t) + \frac{\partial \sigma}{\partial a_1}(t)X_1(t-\delta) + \frac{\partial \sigma}{\partial a_2}(t)\int_{t-\delta}^t e^{\rho(t-r)}X_1(r)\,dr + \frac{\partial \sigma}{\partial u_1}(t)\beta_1(t)\Big\}dB(t)\\
&\quad + \int_{\mathbb{R}_0}\Big\{\frac{\partial \gamma}{\partial a}(t,\zeta)X_1(t) + \frac{\partial \gamma}{\partial a_1}(t,\zeta)X_1(t-\delta) + \frac{\partial \gamma}{\partial a_2}(t,\zeta)\int_{t-\delta}^t e^{\rho(t-r)}X_1(r)\,dr\\
&\quad + \frac{\partial \gamma}{\partial u_1}(t,\zeta)\beta_1(t)\Big\}\widetilde{N}(dt,d\zeta),\ t \in [0,T] \qquad (3.27)\\
X_1(t) &= 0,\ \ t \in [-\delta, 0].
\end{aligned}$$

Here we have used for notational simplicity

$$\begin{aligned}
\frac{d}{ds}A_1^{(u_1+s\beta_1,\widehat{u}_2)}(t)\Big|_{s=0} &= \frac{d}{ds}A^{(u_1+s\beta_1,\widehat{u}_2)}(t-\delta)\Big|_{s=0} = X_1(t-\delta)\\
\frac{d}{ds}A_2^{(u_1+s\beta_1,\widehat{u}_2)}(t)\Big|_{s=0} &= \frac{d}{ds}\Big(\int_{t-\delta}^{t} e^{\rho(t-r)}A^{(u_1+s\beta_1,\widehat{u}_2)}(r)dr\Big)\Big|_{s=0}\\
&= \int_{t-\delta}^{t} e^{\rho(t-r)}\frac{d}{ds}A^{(u_1+s\beta_1,\widehat{u}_2)}(r)\Big|_{s=0}dr = \int_{t-\delta}^{t} e^{\rho(t-r)}X_1(r)\,dr
\end{aligned}$$

It follows from (3.4) that

$$\begin{aligned}
dy_1(t) = &\Big\{\frac{\partial g_1}{\partial a}(t)X_1(t) + \frac{\partial g_1}{\partial a_1}(t)X_1(t-\delta) + \frac{\partial g_1}{\partial a_2}(t)\int_{t-\delta}^{t} e^{\rho(t-r)}X_1(r)\,dr\\
&+ \frac{\partial g_1}{\partial u_1}(t)\beta_1(t) + \frac{\partial g_1}{\partial y}(t)y_1(t) + \frac{\partial g_1}{\partial z}(t)z_1(t) + \int_{\mathbb{R}_0}\nabla_k g_1(t,\zeta)k_1(t,\zeta)\nu(d\zeta)\Big\}dt\\
&+ z_1(t)dB(t) + \int_{\mathbb{R}_0}k_1(t,\zeta)\widetilde{N}(dt,d\zeta),\ \ t\in[0,T] \qquad (3.28)\\
y_1(T) = &\ h_1'(A^{(u_1,\widehat{u}_2)}(T))X_1(T),
\end{aligned}$$

and similarly, we obtain $dx_2(t)$, $dy_2(t)$.

Theorem 3.5. *[**Necessary maximum principle for FBSDDE games**] Let $u \in \mathcal{A}$ with corresponding solutions $A(t)$ of (3.1), $(Y_i(t), Z_i(t), K_i(t\,zeta))$ of (3.4), $\lambda_i(t)$ of (3.8), $(p_i(t), q_i(t), r_i(t,\zeta))$ of (3.9) and corresponding derivative processes $X_i(t)$ and $(y_i(t), z_i(t), k_i(t,\zeta))$ given by (3.27) and (3.28) respectively. Assume that Assumption A1, A2 and A3 hold. Moreover assume the following conditions*

$$\begin{aligned}
E\Big[&\int_0^T p_i^2(t)\Big\{(\frac{\partial\sigma}{\partial a})^2(t)X_i^2(t) + (\frac{\partial\sigma}{\partial a_1})^2(t)X_i^2(t-\delta) + (\frac{\partial\sigma}{\partial a_2})^2(t)\Big(\int_{t-\delta}^{t} e^{\rho(t-r)}X_i(r)\,dr\Big)^2\\
&+ (\frac{\partial\sigma}{\partial u_i})^2(t)\beta_i^2(t) + \int_{\mathbb{R}_0}\Big((\frac{\partial\gamma}{\partial a})^2(t,\zeta)X_i^2(t) + (\frac{\partial\gamma}{\partial a_1})^2(t,\zeta)X_i^2(t-\delta)\\
&+ (\frac{\partial\gamma}{\partial a_2})^2(t,\zeta)\Big(\int_{t-\delta}^{t} e^{\rho(t-r)}X_i^2(r)\,dr\Big)^2 + (\frac{\partial\gamma}{\partial u_i})^2(t,\zeta)\beta_i^2(t)\Big)\nu(d\zeta)\Big\}dt\\
&+ \int_0^T X_i^2(t)\Big\{q_i^2(t) + \int_{\mathbb{R}_0} r_i^2(t,\zeta)\nu(d\zeta)\Big\}dt < \infty,\ \ i=1,2. \qquad (3.29)
\end{aligned}$$

and

$$\begin{aligned}
E\Big[&\int_0^T y_i^2(t)\Big\{(\frac{\partial H_i}{\partial z})^2(t) \ + \int_{\mathbb{R}_0}(\nabla_k H_i)^2(t,\zeta)\nu(d\zeta)\Big\}dt\\
&+ \int_0^T \lambda_i^2(t)\Big\{z_i^2(t) + \int_{\mathbb{R}_0} k_i^2(t,\zeta)\nu(d\zeta)\Big\}dt < \infty,\ \ i=1,2. \qquad (3.30)
\end{aligned}$$

Then the following are equivalent:

(1)

$$\frac{d}{ds}J_1^{(u_1+s\beta_1,u_2)}(t)\Big|_{s=0} = \frac{d}{ds}J_2^{(u_1,u_2+s\beta_2)}(t)\Big|_{s=0} = 0 \tag{3.31}$$

for all bounded $\beta_1 \in \mathcal{A}_1,\ \ \beta_2 \in \mathcal{A}_2$

(2)

$$\begin{aligned} &E\Big[\frac{\partial H_1}{\partial v_1}(t, A(t), A_1(t), A_2(t), v_1(t), u_2(t), Y_1(t), Z_1(t), K_1(t,\cdot), \lambda_1(t),\ p_1(t),\ q_1(t), r_1(t,\cdot))\Big|\mathcal{E}_t^{(1)}\Big]_{v_1=u_1} \\ &= E\Big[\frac{\partial H_2}{\partial v_2}(t, A(t), A_1(t), A_2(t), u_1(t), v_2(t), Y_2(t), Z_2(t), K_2(t,\cdot), \lambda_2(t),\ p_2(t),\ q_2(t), r_2(t,\cdot))\Big|\mathcal{E}_t^{(2)}\Big]_{v_2=u_2} \end{aligned} \tag{3.32}$$

Proof. Put

$$\begin{aligned} \Delta_1 &= \frac{d}{ds}J_1^{(u_1+s\beta_1,u_2)}(t)\Big|_{s=0} \\ &= E\Big[\int_0^T \Big\{\frac{\partial f_1}{\partial a}(t)X_1(t) + \frac{\partial \sigma}{\partial a_1}(t)X_1(t-\delta) + \frac{\partial f_1}{\partial a_2}(t)\int_{t-\delta}^t e^{\rho(t-r)}X_1(r)\,dr + \frac{\partial f_1}{\partial u_1}(t)\beta_1(t)\Big\}dt \\ &\quad + \varphi'(A^{(u_1,\widehat{u}_2)}(T))X_1(T) + \psi_1'(Y_1(0))y_1(0)\Big] \\ &= I_1' + I_2' + I_3' \end{aligned} \tag{3.33}$$

with

$$\begin{aligned} I_1' &= E\Big[\int_0^T \Big\{\frac{\partial f_1}{\partial a}(t)X_1(t) + \frac{\partial \sigma}{\partial a_1}(t)X_1(t-\delta) + \frac{\partial f_1}{\partial a_2}(t)\int_{t-\delta}^t e^{\rho(t-r)}X_1(r)\,dr + \frac{\partial f_1}{\partial u_1}(t)\beta_1(t)\Big\}dt\Big] \\ I_2' &= E\Big[\varphi'(A^{(u_1,\widehat{u}_2)}(T))X_1(T)\Big] \\ I_3' &= E\Big[\psi_1'(Y_1(0))y_1(0)\Big] \end{aligned}$$

By Itô formula, (3.9), (3.27) and (3.29), we have

$$
\begin{aligned}
I_2' &= E\Big[\varphi'(A^{(u_1,\widehat{u}_2)}(T))X_1(T)\Big] \\
&= E\Big[p_1(T)X_1(T)\Big] - E\Big[h_1'(A^{(u_1,\widehat{u}_2)}(T))\lambda_1(T)X_1(T)\Big] \\
&= E\Big[\int_0^T \Big\{p_1(t)dX_1(t) + X_1(t^-)dp_1(t) + q_1(t)\Big(\frac{\partial \sigma}{\partial a}(t)X_1(t) + \frac{\partial \sigma}{\partial a_1}(t)X_1(t-\delta) \\
&+ \frac{\partial \sigma}{\partial a_2}(t)\int_{t-\delta}^t e^{\rho(t-r)}X_1(r)\,dr + \frac{\partial \sigma}{\partial u_1}(t)\beta_1(t)\Big)dt + \int_{\mathbb{R}_0} r_1(t,\zeta)\Big(\frac{\partial \gamma}{\partial a}(t,\zeta)X_1(t) \\
&+ \frac{\partial \gamma}{\partial a_1}(t,\zeta)X_1(t-\delta) + \frac{\partial \gamma}{\partial a_2}(t,\zeta)\int_{t-\delta}^t e^{\rho(t-r)}X_1(r)\,dr + \frac{\partial \gamma}{\partial u_1}(t,\zeta)\beta_1(t)\Big)\nu(d\zeta)dt\Big\}\Big] \\
&- E\Big[h_1'(A^{(u_1,\widehat{u}_2)}(T))\lambda_1(T)X_1(T)\Big] \\
&= E\Big[\int_0^T \Big\{p_1(t)\Big(\frac{\partial b}{\partial a}(t)X_1(t) + \frac{\partial b}{\partial a_1}(t)X_1(t-\delta) + \frac{\partial b}{\partial a_2}(t)\int_{t-\delta}^t e^{\rho(t-r)}X_1(r)\,dr \\
&+ \frac{\partial b}{\partial u_1}(t)\beta_1(t)\Big) + X_1(t^-)E\Big[\mu_1(t)|\mathcal{F}_t\Big] + q_1(t)\Big(\frac{\partial \sigma}{\partial a}(t)X_1(t) + \frac{\partial \sigma}{\partial a_1}(t)X_1(t-\delta) \\
&+ \frac{\partial \sigma}{\partial a_2}(t)\int_{t-\delta}^t e^{\rho(t-r)}X_1(r)\,dr + \frac{\partial \sigma}{\partial u_1}(t)\beta_1(t)\Big) + \int_{\mathbb{R}_0} r_1(t,\zeta)\Big(\frac{\partial \gamma}{\partial a}(t,\zeta)X_1(t) \\
&+ \frac{\partial \gamma}{\partial a_1}(t,\zeta)X_1(t-\delta) + \frac{\partial \gamma}{\partial a_2}(t,\zeta)\int_{t-\delta}^t e^{\rho(t-r)}X_1(r)\,dr + \frac{\partial \gamma}{\partial u_1}(t,\zeta)\beta_1(t)\Big)\nu(d\zeta)\Big\}dt\Big] \\
&- E\Big[h_1'(A^{(u_1,\widehat{u}_2)}(T))\lambda_1(T)X_1(T)\Big] \qquad (3.34)
\end{aligned}
$$

By Itô formula, (3.8), (3.28) and (3.30), we get

$$
\begin{aligned}
I_3' &= E\Big[\psi_1'(Y_1(0))y_1(0)\Big] = E\Big[\lambda_(0)y_1(0)\Big] \\
&= E\Big[\lambda_(T)y_1(T)\Big] - E\Big[\int_0^T \Big\{\lambda_1(t^-)dy_1(t) + y_1(t^-)d\lambda_1(t) + \frac{\partial H_1}{\partial z}(t)z_1(t)dt \\
&+ \int_{\mathbb{R}_0} \nabla_k H_1(t,\zeta)k_1(t,\zeta)\nu(d\zeta)dt\Big\}\Big] \\
&= E\Big[h_1'(A^{(u_1,\widehat{u}_2)}(T))\lambda_1(T)X_1(T)\Big] \\
&- E\Big[\int_0^T \Big\{\lambda_1(t^-)\Big(\frac{\partial g_1}{\partial a}(t)X_1(t) + \frac{\partial g_1}{\partial a_1}(t)X_1(t-\delta) + \frac{\partial g_1}{\partial a_2}(t)\int_{t-\delta}^t e^{\rho(t-r)}X_1(r)\,dr \\
&+ \frac{\partial g_1}{\partial u_1}(t)\beta_1(t) + \frac{\partial g_1}{\partial y}(t)y_1(t) + \frac{\partial g_1}{\partial z}(t)z_1(t) + \int_{\mathbb{R}_0} \nabla_k g_1(t,\zeta)k_1(t,\zeta)\nu(d\zeta)\Big) \\
&+ \frac{\partial H_1}{\partial y}(t)y_1(t) + \frac{\partial H_1}{\partial z}(t)z_1(t) + \int_{\mathbb{R}_0} \nabla_k H_1(t,\zeta)k_1(t,\zeta)\nu(d\zeta)\Big\}dt\Big] \qquad (3.35)
\end{aligned}
$$

By the definition of the Hamiltonian, we have

$$\begin{aligned}
I_1' &= E\Big[\int_0^T \Big\{\frac{\partial f_1}{\partial a}(t)X_1(t) + \frac{\partial f_1}{\partial a_1}(t)X_1(t-\delta) + \frac{\partial f_1}{\partial a_2}(t)\int_{t-\delta}^t e^{\rho(t-r)}X_1(r)\,dr + \frac{\partial f_1}{\partial u_1}(t)\beta_1(t)\Big\}dt\Big] \\
&= E\Big[\int_0^T \Big(\frac{\partial H_1}{\partial a}(t) - \lambda(t)\frac{\partial g_1}{\partial a}(t) - p(t)\frac{\partial b}{\partial a}(t) - q(t)\frac{\partial \sigma}{\partial a}(t) - \int_{\mathbb{R}_0} r(t,\zeta)\frac{\partial \gamma}{\partial a}(t)\nu(d\zeta)\Big)X_1(t)dt \\
&+ \int_0^T \Big(\frac{\partial H_1}{\partial a_1}(t) - \lambda(t)\frac{\partial g_1}{\partial a_1}(t) - p(t)\frac{\partial b}{\partial a_1}(t) - q(t)\frac{\partial \sigma}{\partial a_1}(t) - \int_{\mathbb{R}_0} r(t,\zeta)\frac{\partial \gamma}{\partial a_1}(t)\nu(d\zeta)\Big)X_1(t-\delta)dt \\
&+ \int_0^T \Big(\frac{\partial H_1}{\partial a_2}(t) - \lambda(t)\frac{\partial g_1}{\partial a_2}(t) - p(t)\frac{\partial b}{\partial a_2}(t) - q(t)\frac{\partial \sigma}{\partial a_2}(t) - \int_{\mathbb{R}_0} r(t,\zeta)\frac{\partial \gamma}{\partial a_2}(t)\nu(d\zeta)\Big)\Big(\int_{t-\delta}^t e^{\rho(t-r)}X_1(r)\,dr\Big)dt \\
&+ \int_0^T \Big(\frac{\partial H_1}{\partial u_1}(t) - \lambda(t)\frac{\partial g_1}{\partial u_1}(t) - p(t)\frac{\partial b}{\partial u_1}(t) - q(t)\frac{\partial \sigma}{\partial u_1}(t) - \int_{\mathbb{R}_0} r(t,\zeta)\frac{\partial \gamma}{\partial u_1}(t)\nu(d\zeta)\Big)\beta_1(t)dt\Big]
\end{aligned} \tag{3.36}$$

Summing I_1', I_2' and I_3', we get

$$\begin{aligned}
\Delta_1 &= \frac{d}{ds}J_1^{(u_1+s\beta_1,u_2)}(t)\Big|_{s=0} \\
&= E\Big[\int_0^T \Big\{\mu_1(t)X_1(t) + \frac{\partial H_1}{\partial a}(t)X_1(t) + \frac{\partial H_1}{\partial a_1}(t)X_1(t-\delta) \\
&\quad + \frac{\partial H_1}{\partial a_2}(t)\int_{t-\delta}^t e^{\rho(t-r)}X_1(r)\,dr + \frac{\partial H_1}{\partial u_1}(t)\beta_1(t)\Big\}dt\Big] \\
&= E\Big[\int_0^T X_1(t)\Big(\mu_1(t) + \frac{\partial H_1}{\partial a}(t)\Big)dt + \int_0^T \frac{\partial H_1}{\partial a_1}(t)X_1(t-\delta)\,dt \\
&+ \int_0^T \Big(\int_{t-\delta}^t e^{-\rho(t-r)}X_1(r)\,dr\Big)\frac{\partial H_1}{\partial a_2}(t)dt + \int_0^T \frac{\partial H_1}{\partial u_1}(t)\beta_1(t)\,dt\Big]
\end{aligned}$$

$$\begin{aligned}
&= E\Big[\int_0^T X_1(t)\Big\{\frac{\partial H_1}{\partial a}(t) - \frac{\partial H_1}{\partial a}(t) - \frac{\partial H_1}{\partial a_1}(t+\delta)\chi_{[0,T-\delta]}(t) \\
&- e^{\rho t}\Big(\int_t^{t+\delta}\frac{\partial H_1}{\partial a_2}(s)e^{-\rho s}\chi_{[0,T]}(s)\,ds\Big)\Big\}dt + \int_0^T \frac{\partial H_1}{\partial a_1}(t)X_1(t-\delta)\,dt \\
&+ \int_0^T \Big(\int_{s-\delta}^s e^{-\rho(s-t)}X_1(t)\,dt\Big)\frac{\partial H_1}{\partial a_2}(s)ds + \int_0^T \frac{\partial H_1}{\partial u_1}(t)\beta_1(t)\,dt\Big] \\
&= E\Big[\int_0^T X_1(t)\Big\{-\frac{\partial H_1}{\partial a_1}(t+\delta)\chi_{[0,T-\delta]}(t)\Big\}dt + \int_0^T \frac{\partial H_1}{\partial a_1}(t)X_1(t-\delta)\,dt \\
&- \int_0^T X_1(t)e^{\rho t}\Big(\int_t^{t+\delta}\frac{\partial H_1}{\partial a_2}(s)e^{-\rho s}\chi_{[0,T]}(s)\,ds\Big)dt \\
&+ \int_0^T X_1(t)e^{\rho t}\Big(\int_t^{t+\delta}\frac{\partial H_1}{\partial a_2}(s)e^{-\rho s}\chi_{[0,T]}(s)\,ds\Big)dt + \int_0^T \frac{\partial H_1}{\partial u_1}(t)\beta_1(t)\,dt\Big] \\
&= E\Big[\int_0^T \frac{\partial H_1}{\partial u_1}(t)\beta_1(t)\,dt\Big],
\end{aligned} \tag{3.37}$$

where we have used once more integration by parts.

If $\frac{d}{ds}J_1^{(u_1+s\beta_1,u_2)}(t)\Big|_{s=0} = 0$ for all bounded $\beta_1 \in \mathcal{A}_1$, then this holds in particular for β_1 of the form

$$\beta_1(t) = \alpha_1(\omega)\chi_{[s,T]}(t),$$

where $\alpha_1(\omega)$ is bounded and $\mathcal{E}_{t_0}^{(1)}$-measurable, $s \geq t_0$. Then

$$E\Big[\int_s^T \frac{\partial H_1}{\partial u_1}(t)\,dt\,\alpha_1\Big] = 0.$$

Differentiating with respect to s, we have

$$E\Big[\frac{\partial H_1}{\partial u_1}(s)\,\alpha_1\Big] = 0.$$

Since the equality is true for all $s \geq t_0$ and $\alpha_1(\omega)$ bounded and $\mathcal{E}_{t_0}^{(1)}$-measurable random variable, we conclude that

$$E\Big[\frac{\partial H_1}{\partial u_1}(t_0)|\mathcal{E}_{t_0}^{(1)}\Big] = 0 \text{ for a.a. } t_0 \in [0,T].$$

A similar argument gives that

$$E\Big[\frac{\partial H_2}{\partial u_2}(t_0)|\mathcal{E}_{t_0}^{(2)}\Big] = 0 \text{ for a.a. } t_0 \in [0,T],$$

under the condition that

$$\frac{d}{ds}J_2^{(u_1,u_2+s\beta_2)}(t)\Big|_{s=0} = 0 \text{ for all bounded } \beta_2 \in \mathcal{A}_2.$$

This shows that (i) $\Rightarrow$ (ii)

Conversely, using the fact that every bounded $\beta_i \in \mathcal{A}_i$, $i = 1, 2$ can be approximated by a linear combinations of controls β_i of the form (3.26), the above argument can be reversed to shoe that (ii) $\Rightarrow$ (i). □

Remark 3.6. *The result also easily follows for if we start from $t \geq 0$ in the performance functional.*

Zero-sum game.

In this section, we solve the *zero-sum* delayed stochastic differential game problem (or *worst case scenario optimal problem*) that is, we suppose that the given performance functional for Player I is the negative of that for Player II, i.e.,

$$\begin{aligned} J(t,u_1,u_1) &= J_1(t,u_1,u_2) \\ &:= E\Big[\int_t^T f(s,A(s),A_1(s),A_2(s),u_1(s),u_2(s))ds \,+\, \varphi(A(T)) \,+\, \psi(Y(t))\Big|\mathcal{F}_t\Big] \\ &= -J_2(t,u_1,u_2). \end{aligned} \tag{3.38}$$

In this case, we see that (u_1^*, u_2^*) is a *Nash equilibrium* if and only if

$$\operatorname*{ess\,sup}_{u_1\in\mathcal{A}_1} J(t,u_1,u_2^*) = J(t,u_1^*,u_2^*) = \operatorname*{ess\,inf}_{u_2\in\mathcal{A}_2} J(t,u_1^*,u_2). \tag{3.39}$$

This implies that

$$\begin{aligned}\underset{u_2\in\mathcal{A}_2}{\text{ess inf}}(\underset{u_1\in\mathcal{A}_1}{\text{ess sup}}\, J(t,u_1,u_2)) &\leq \underset{u_1\in\mathcal{A}_1}{\text{ess sup}}\, J(t,u_1,u_2^*)\\ &= J(t,u_1^*,u_2^*) = \underset{u_2\in\mathcal{A}_2}{\text{ess inf}}\, J(t,u_1^*,u_2)\\ &\leq \underset{u_1\in\mathcal{A}_1}{\text{ess sup}}(\underset{u_2\in\mathcal{A}_2}{\text{ess inf}}\, J(t,u_1,u_2)).\end{aligned}$$

On the other hand, we always have ess inf(ess sup) $\geq$ ess sup(ess inf). This means that if (u_1^*,u_2^*) is a *Nash equilibrium* then

$$\underset{u_2\in\mathcal{A}_2}{\text{ess inf}}(\underset{u_1\in\mathcal{A}_1}{\text{ess sup}}\, J(t,u_1,u_2)) = \underset{u_1\in\mathcal{A}_1}{\text{ess sup}}(\underset{u_2\in\mathcal{A}_2}{\text{ess inf}}\, J(t,u_1,u_2)).$$

The zero-sum delayed stochastic differential game problem is therefore the following:

Problem P4. *Find $u_1^*\in\mathcal{A}_1$ and $u_2^*\in\mathcal{A}_2$ (if it exists) such that*

$$\underset{u_2\in\mathcal{A}_2}{\text{ess inf}}(\underset{u_1\in\mathcal{A}_1}{\text{ess sup}}\, J(t,u_1,u_2)) = J(t,u_1^*,u_2^*) = \underset{u_1\in\mathcal{A}_1}{\text{ess sup}}(\underset{u_2\in\mathcal{A}_2}{\text{ess inf}}\, J(t,u_1,u_2)). \tag{3.40}$$

Such a control (u_1^*,u_2^*) is called an *optimal control* (if it exists). The intuitive idea is that while Player I controls u_1, Player II controls u_2. The actions of the players are antagonistic, which means that between player I and II there is a payoff $J(t,u_1,u_2)$ and it is a reward for Player I and cost for Player II.

Remark 3.7. *The above Problem P4 can be seen as a generalization of Problem P2 in Section 2. We shall as in the non-zero sum case give the result for $t=0$ and conclude for $t\in[0,T]$. The results obtained in this Section generalize the ones in* [3, 10, 18] *and* [26].

In the case of a zero-sume game, we only have one value function for the players and therefore, Theorem 3.5 becomes

Theorem 3.8. *[**Necessary maximum principle for zero-sum FBSDDE games**]*
Let $u\in\mathcal{A}$ with corresponding solutions $A(t)$ of (3.1), $(Y(t),Z(t),K(t,\zeta))$ of (3.4), $\lambda(t)$ of (3.8), $(p(t),q(t),r(t,\zeta))$ of (3.9) and corresponding derivative processes $X(t)$ and $(y(t),z(t),k(t,\zeta))$ given by (3.27) and (3.28) respectively. Assume that conditions of Theorem 3.5 are satisfied. Then the following are equivalent:

(1)

$$\frac{d}{ds}J^{(u_1+s\beta_1,u_2)}(t)\Big|_{s=0} = \frac{d}{ds}J^{(u_1,u_2+s\beta_2)}(t)\Big|_{s=0} = 0 \tag{3.41}$$

for all bounded $\beta_1\in\mathcal{A}_1,\ \ \beta_2\in\mathcal{A}_2$

(2)

$$\begin{aligned}0 &= E\Big[\frac{\partial H}{\partial v_1}(t,A(t),A_1(t),A_2(t),v_1(t),u_2(t),Y_1(t),Z_1(t),K_1(t,\cdot),\lambda_1(t),\ p_1(t),\ q_1(t),r_1(t,\cdot))\Big|\mathcal{E}_t^{(1)}\Big]_{v_1=u_1}\\ &= E\Big[\frac{\partial H}{\partial v_2}(t,A(t),A_1(t),A_2(t),u_1(t),v_2(t),Y_2(t),Z_2(t),K_2(t,\cdot),\lambda_2(t),\ p_2(t),\ q_2(t),r_2(t,\cdot))\Big|\mathcal{E}_t^{(2)}\Big]_{v_2=u_2}\end{aligned} \tag{3.42}$$

Proof. It follows directly from Theorem 3.5. □

Remark 3.9. *This result extends the one obtained in* [3] *and* [10] .

Corollary 3.10. *If $u = (u_1, u_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ is a Nash equilibrium for the zero-sum game in Theorem 3.8, then equalities (3.42) holds.*

Proof. If $u = (u_1, u_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ is a Nash equilibrium, then, it follows from Theorem 3.8 that (3.41) holds by (3.39). □

4. Application to optimal consumption from a cash flow with delay under model uncertainty and general recursive utility

In this section, we apply our maximum principle for stochastic delayed differential game to study a problem of consumption choice optimization under model uncertainty.

The model of this problem is a modification of the one in [4, 17]. Assume that the investor can invest his cash flow to generate some production, and get profit. Let $A(t)$ and $\alpha(t)$ denote the capital (cash flow) of the investor and the labor at time t. We assume that at time $t \in [0, T]$, the investor consumes at the rate $c(t) \geq 0$, a *càdlàg* adapted process. The rate of change of capital and labor was described in [31] as follows:

$$\frac{dA(t)}{dt} = f(A(t), \alpha(t)) - c(t), \tag{4.1}$$

where f is some function.

Assuming that the production rate is subject to random perturbations, 4.1 becomes

$$dA(t) = \Big[f(A(t), \alpha(t)) - c(t)\Big]dt + \sigma(A(t))dB(t); \;\; t \in [0, T]. \tag{4.2}$$

Here, B is a 1-dimensional standard Brownian motion and σ is the volatility. The key assumption in the previous model is that there is instant transformation of investments. However, this assumption does not reflects the reality. In fact there is a non negligible time delay in the production, such as length of the production cycle. This leads to the following modified model obtain in [17].

$$\begin{cases} dA(t) &= \Big[f(A(t-\delta), \beta(t)) - c(t)\Big]dt + \sigma(A(t-\delta))dB(t); \;\; t \in [0, T] \\ A(t) &= a_0(t) > 0; \;\; t \in [-\delta, 0], \end{cases} \tag{4.3}$$

where $\delta > 0$, $a_0(t)$ is a given bounded deterministic function which represents the initial capital and β is a deterministic and bounded.

In our model, we shall assume moreover that the function f in (4.3) do not only depend on the investment made at time $t - \delta$ but also on the investment at time t. We will also assume that the production rate is subject to jumps. Our model is then given by

$$\begin{cases} dA(t) &= \Big[f(A(t), A(t-\delta), \alpha(t), \beta(t)) - c(t)\Big]dt + \sigma(A(t-\delta))dB(t) \\ & \quad + \int_{\mathbb{R}_0} \gamma(A(t-\delta), \zeta)\, \widetilde{N}(d\zeta, dt); \;\; t \in [0, T] \\ A(t) &= a_0(t) > 0; \;\; t \in [-\delta, 0]. \end{cases} \tag{4.4}$$

For simplicity, we assume that $f(a, a_1, \alpha, \beta) = L_1 a^{r_1} \alpha^{r_1'} + L_2 a^{r_2} \alpha^{r_2'}$ where $L_1, r_1, r_1', L_2, r_2, r_2'$ are some constants (note that for $L_2 = 0$ this model can be reduced to the one in [13].) We set

$$\begin{aligned}
&L_1 = r_1 = r_1' = L_2 = r_1 = r_2' = 1,\\
&\sigma(A(t-\delta)) = \sigma(t)A(t-\delta),\\
&\gamma(A(t-\delta),\zeta) = \gamma(t,\zeta)A(t-\delta),
\end{aligned}$$

where $\sigma(t)$ and $\gamma(t,\zeta)$ are bounded adapted processes and $\int_{\mathbb{R}_0} \gamma^2(t,\zeta)\,\nu(d\zeta) < \infty$.

The dynamic of the cash flow $A(t) = A^c(t)$ is therefore given by

$$\left\{\begin{array}{lcl}
dA(t) &=& \Big[A(t)\alpha(t) + A(t-\delta)\beta(t) - c(t)\Big]dt + A(t-\delta)\sigma(t)\,dB(t)\\
&& +A(t-\delta)\displaystyle\int_{\mathbb{R}_0} \gamma(t,\zeta)\,\widetilde{N}(d\zeta,dt);\;\; t\in[0,T]\\
A(t) &=& a_0(t) > 0;\;\; t\in[-\delta,0],
\end{array}\right. \tag{4.5}$$

Recall that our objective is to solve an optimal consumption problem for recursive utility under model uncertainty. To this end, let $U_1(t,c,\omega) : [0,T]\times\mathbb{R}^+\times\Omega \to \mathbb{R}$ be a stochastic utility function satisfying:

$$\begin{aligned}
t \to U_1(t,c,\omega) &\text{ is } \mathcal{F}_t-\text{adapted for each } c\geq 0,\\
c \to U_1(t,c,\omega) &\text{ is } C^1,\;\; \frac{\partial U_1}{\partial c}(t,c,\omega) > 0,\\
c \to \frac{\partial U_1}{\partial c}(t,c,\omega) &\text{ is strictly decreasing}\\
\lim_{c\to\infty}\frac{\partial U_1}{\partial c}(t,c,\omega) = 0 &\text{ for all } (t,\omega)\in[0,T]\times\Omega
\end{aligned}$$

and $\frac{\partial U_1}{\partial c}$ has an inverse in the sense that

$$I_1(t,c,\omega) = \left\{\begin{array}{lll}
0 & \text{if} & v \geq v_0(t,\omega)\\
\Big(\frac{\partial U_1}{\partial c}(t,\cdot,\omega)\Big)^{-1}(v) & \text{if} & 0\leq v < v_0(t,\omega)),
\end{array}\right. \tag{4.6}$$

where $v_0(t,\omega) = \lim_{c\to 0^+}\frac{\partial U_1}{\partial c}(t,c,\omega)$

Let $U_2(x,\omega) : \mathbb{R}^+\times\Omega\to\mathbb{R}$ be another stochastic utility function. We assume that U_2 satisfies similar conditions as U_1 and define I_2 as the inverse of its dervivative. Let $h(x,y) : \mathbb{R}^+\times\mathbb{R}_0\to\mathbb{R}$ be a convex C^1 function such that h' has an inverse.

Choose the functions of Problem P2 in Section 2 as follow:

$$\begin{aligned}
&f_1(t,a,a_1,a_2,c,\theta_1,\theta_2) = U_1(t,c) + h(\theta_1,\theta_2),\\
&g(A(T)) = U_2(A(T).
\end{aligned}$$

Therefore, the stochastic differential utility given by (2.12) becomes

$$\left\{\begin{array}{lcl}
dY(t) &=& -\Big(U_1(t,c) + h(\theta) + \theta_0(t)Z(t) + \displaystyle\int_{\mathbb{R}_0}\theta_1(t,z)K(t,z)\,\nu(dz)\Big)dt\\
&& +Z(t)dB(t) + \displaystyle\int_{\mathbb{R}_0} K(t,z)\,\widetilde{N}(dz,dt);\;\; t\in[0,T]\\
Y(T) &=& U_2(A(T)),
\end{array}\right. \tag{4.7}$$

and our problem is to find $(\widehat{v}, \widehat{\theta}) \in \mathcal{V} \times \mathcal{A}$ such that

$$\sup_{v \in \mathcal{V}} \inf_{\theta \in \mathcal{A}} Y^{v,\theta}(0) = Y^{\widehat{v},\widehat{\theta}}(0) = \inf_{\theta \in \mathcal{A}} \sup_{v \in \mathcal{V}} Y^{v,\theta}(0), \tag{4.8}$$

where $Y^{v,\theta}(0)$ is given by the forward-backward delayed system (4.5) & (4.7).

The Hamiltonian is by (3.7) simplified to:

$$\begin{aligned} H(t, a, a_1, a_2, y, z, k, c, \theta, \lambda, p, q, r) = \lambda\Big[U_1(t, c) + h(\theta) + \theta_0(t)z + \int_{\mathbb{R}_0} \theta_1(\zeta)k(\zeta)\,\nu(d\zeta)\Big] \quad (4.9) \\ + p\Big[a\alpha(t) + a_1\beta(t) - c(t)\Big] + a_1\sigma(t)q + a_1 \int_{\mathbb{R}_0} r(\zeta)\gamma(t,\zeta)\,\nu(d\zeta) \end{aligned}$$

Maximizing H with respect to c gives the following first order condition for an optimal $\widehat{c}$

$$\frac{\partial U_1}{\partial c}(t, \widehat{c}, \omega) = E[p(t)|\mathcal{E}_t^{(1)}]. \tag{4.10}$$

Minimizing H with respect to $\theta = (\theta_0, \theta_1)$ gives the following first order condition for an optimal $\widehat{\theta}$

$$\frac{\partial h}{\partial \theta_0}(\widehat{\theta}) = -E[Z(t)|\mathcal{E}_t^{(2)}], \tag{4.11}$$

$$\nabla_{\theta_1} h(\widehat{\theta}) = -E\Big[\int_{\mathbb{R}_0} K(t,\zeta)\,\nu(d\zeta)|\mathcal{E}_t^{(2)}\Big]. \tag{4.12}$$

The time-advanced BSDE for $p(t), q(t), r(t,\zeta)$ becomes

$$\begin{cases} dp(t) &= -E\Big[\alpha(t)p(t) + \Big\{\beta(t)p(t+\delta) + \sigma(t)q(t+\delta) + \displaystyle\int_{\mathbb{R}_0} \gamma(t,\zeta)r(t+\delta,\zeta)(d\zeta)\Big\}\chi_{[0,T-\delta]}(t)\Big|\mathcal{F}_t\Big]dt \\ &\quad +q(t)\,dB(t) + \displaystyle\int_{\mathbb{R}_0} r(t,\zeta)\,\widetilde{N}(d\zeta, dt), \ \ t \in [0,T] \\ p(T) &= \lambda(T)U_2'(A(T)), \end{cases} \tag{4.13}$$

and the forward SDE for λ becomes

$$\begin{cases} d\lambda(t) &= \lambda(t)\Big[\theta_0(t)dB(t) + \displaystyle\int_{\mathbb{R}_0} \theta_1(t,\zeta)\,\widetilde{N}(d\zeta, dt), \ \ t \in [0,T] \\ \lambda(0) &= 1. \end{cases} \tag{4.14}$$

For simplicity, we will restrict ourselves to the case with no jumps, that is $K = \theta_1 = 0$. It is possible to solve the time-advanced ABSDE (4.13) recursively. A similar proof (for $\beta = 0$) can be found in [27]. For completeness, we give the proof here. We will solve the ABSDE (4.13) recursively. To this end, we will use a n steps scheme.

(1) If $t \in [T-\delta, T]$, the BSDE has the form

$$\begin{cases} dp(t) &= \alpha(t)p(t)dt + q(t)\,dB(t), \ \ t \in [T-\delta, T] \\ p(T) &= \lambda(T)U_2'(A(T)), \end{cases} \tag{4.15}$$

which has the solution

$$p(t) = E\Big[\lambda(T)U_2'(A(T))e^{-\int_t^T \alpha(s)ds}\Big|\mathcal{F}_t\Big], \ \ t \in [T-\delta, T],$$

and using variational smoothness of solutions of time-advanced BSDEs, we get

$$q(t) = e^{-\int_t^T \alpha(s)ds} E\Big[D_t(\lambda(T)U_2'(A(T)))\Big|\mathcal{F}_t\Big].$$

Let us mention that Malliavin differentiability of time-advanced BSDEs is proved in [20].

(2) If $t \in [T-2\delta, T-\delta]$, and $T-2\delta > 0$, we get by (1) the following BSDE

$$\begin{cases} dp(t) &=& -E\Big[\alpha(t)p(t) + \Big\{\alpha(t)p(t+\delta) + \sigma(t)q(t+\delta)\Big\}\Big|\mathcal{F}_t\Big]dt \\ & & +q(t)\,dB(t), \;\; t \in [T-2\delta, T-\delta] \end{cases} \tag{4.16}$$

with $p(t-\delta)$ known from step 1. Note that $p(t+\delta)$ and $q(t+\delta)$ are also known from step 1. Therefore, this is a simple BSDE which can be solved for $p(t)$, $q(t)$; $t \in [T-2\delta, T-\delta]$. Applying the same procedure by induction up to and including step j, where j is such that $T - j\delta \le 0 < T-(j-1)\delta$. We then end up with with a solution $p(t)$ of (4.13) which depends on the (optimal) terminal value $A(T)$ (given by (4.22)) and the terminal value $\lambda(T)$ of the FSDE (4.14).

If

$$0 \le p(t) \le v_0(t,\omega) \text{ for all } t \in [0,T]. \tag{4.17}$$

Then, the optimal consumption rate $\widehat{c}(t)$ is by (4.10) given by

$$\widehat{c}(t) = \widehat{c}_{\widehat{A}(T)}(t) = I_1(t, \widehat{p}(t), \omega), \;\; t \in [0,T], \tag{4.18}$$

and the optimal scenario parameter is by (4.11) given by

$$\widehat{\theta}_0(t) = (h')^{-1}(-\widehat{Z}(t)), \;\; t \in [0,T], \tag{4.19}$$

where $(\widehat{Y}(t), \widehat{Z}(t))$ is the solution of the corresponding BSDE (4.7) i.e.,

$$\begin{cases} d\widehat{Y}(t) &=& -\Big(U_1(t,\widehat{c}(t)) + h(\widehat{\theta}_0) + \widehat{\theta}_0(t)\widehat{Z}(t)\Big)dt + \widehat{Z}(t)dB(t) \\ \widehat{Y}(T) &=& U_2(\widehat{A}(T)). \end{cases} \tag{4.20}$$

Substituting the expression of $\widehat{c}(t)$ into (4.5) we get the SDE for the optimal wealth process $A(t)$. Solving this, we find $A(T)$ and hence $\widehat{c}(t)$. More precisely, we shall write the forward SDE cash equation (4.5) as a BSDE in $(A(t), \widetilde{Z}(t))$ as follows

$$\begin{cases} dA(t) &=& -\Big[I_1(t,\widehat{p}(t)) - \alpha(t)A(t) - \widetilde{Z}(t)\dfrac{\beta(t)}{\sigma(t)}\Big]dt \,+\, \widetilde{Z}(t)\,dB(t), \;\; t \in [0,T] \\ A(T) &=& I_2\Big(\dfrac{p(T)}{\lambda(T)}\Big), \end{cases} \tag{4.21}$$

where we have $\widetilde{Z}(t) = A(t-\delta)\sigma(t)$.

It follows from Lemma A1 that, the solution of this linear BSDE is given by

$$A(t) = E\Big[I_2\Big(\frac{p(T)}{\lambda(T)}\Big)\frac{G(T)}{G(t)} + \int_0^T I_1(s,\widehat{p}(s))\frac{G(s)}{G(t)}ds\Big|\mathcal{F}_t\Big], \;\; t \in [0,T], \tag{4.22}$$

and $\widetilde{Z}(t) = D_tA(t)$ if $A(t)$ is Malliavin differentiable.

We can now summarize the above result in the following Theorem

Theorem 4.1. *Let $A^c(t)$ be a cash flow with delay given by (4.5) with $\gamma = 0$. Consider the optimization problem to find $\widehat{c} \in \mathcal{A}_1$ and $\widehat{\theta} \in \mathcal{A}_2$ such that (4.8) holds, with*

$$Y^{c,\theta}(0) = E_{Q^\theta}\Big[U_2(A^c(T)) + \int_0^T \Big\{U_1(t, c(t)) + h(\theta(t))\Big\}dt\Big].$$

Let $\lambda(t)$ be the solution of the FSDE (4.14) and $p(t)$, $q(t)$ be the solution of the BSDDE (4.13). Suppose that (4.17) holds. Then the optimal consumption rate $\widehat{c}(t)$ and the optimal scenario measure of the market $\widehat{\theta}(t)$ are given by (4.18) and (4.19) respectively, with $\widehat{A}^c(t)$ and $Z(t)$ given by (4.22) and (4.20) respectively.

Remark 4.2. *This result is a generalization of* [4, Proposition 4.1], *where the same conclusion was obtained for classical utility with*

$$U_1(t, c) = \frac{c^\gamma}{\gamma},\ \gamma \in (0, 1),\ U_2(X(T)) = X(T),\ h(\theta) = 0,\ Z(t) = 0,\ \textit{for all } t.$$

Appendix

Lemma A1. *Suppose that $\delta > 0$ is a given constant, $\beta, \theta_0 \in L^2_{\mathcal{F}}(-\delta, T + \delta)$, $\ell \in L^2_{\mathcal{F}}(0, T)$, $\theta_1 \in H^2(-\delta, T + \delta)$, $\theta_1(t, z) > -1 + \varepsilon$ and $\beta, \theta_0, \theta_1$ are uniformly bounded, Q is such that $Q \in S^2_{\mathcal{F}}(T, T + \delta)$ and $E\Big[\sup_{0\le t\le T} |Q^2(t)|\Big] < \infty$.*

Then the linear anticipated BSDE

$$\begin{cases} dY(t) &= \Big(\ell(t) + \beta(t)Y(t) + \theta_0(t)Z(t) + \int_{\mathbb{R}_0} \theta_1(t, z)K(t, z)\,\nu(dz)\Big)dt \\ &\quad +Z(t)dB(t) + \int_{\mathbb{R}_0} K(t, z)\,\widetilde{N}(dz, dt);\ \ t \in [0, T] \\ Y(t) &= Q(t);\ \ t \in [T, T + \delta], \\ Z(t) &= 0,\ \ t \in [T, T + \delta], \\ K(t, z) &= 0,\ \ t \in [T, T + \delta]. \end{cases} \tag{4.23}$$

has the unique solution

$$Y(t) = E\Big[Q(T)G(t, T) + \int_0^T G(t, s)l(s)\,ds\Big|\mathcal{F}_t\Big] \tag{4.24}$$

where $G(t, s)$ is defined by

$$\begin{cases} dG^\theta(t, s) &= G^\theta(t, s^-)(\beta(s)ds + \theta_0(s)\,dB(s) + \int_{\mathbb{R}_0} \theta_1(s, z)\,\widetilde{N}(dz, ds);\ \ s \in [t, T + \delta] \\ G^\theta(t, t) &= 1, \\ G^\theta(t, s) &= 0,\ \ s \in [t - \delta, t). \end{cases} \tag{4.25}$$

Proof. The existence and uniqueness results follows by general theorem for advanced BSDEs (see [27]).

Equation (4.25) has a unique solution. In fact, if $s \in [t, t + \delta]$, then (4.25) becomes

$$\begin{cases} dG^\theta(t, s) &= G^\theta(t, s^-)(\beta(s)ds + \theta_0(s)\,dB(s) + \int_{\mathbb{R}_0} \theta_1(s, z)\,\widetilde{N}(dz, ds);\ \ s \in [t, t + \delta] \\ G^\theta(t, t) &= 1. \end{cases} \tag{4.26}$$

We can then get a unique solution $\xi(t,\cdot)$ for (4.26). When $s \in [t+\delta, T+\delta]$, (4.25) can be written has

$$\begin{cases} dG^{\theta}(t,s) &= G^{\theta}(t,s^-)(\beta(s)ds + \theta_0(s)\,dB(s) + \int_{\mathbb{R}_0} \theta_1(s,z)\,\widetilde{N}(dz,ds); \;\; s \in [t+\delta, T+\delta] \\ G^{\theta}(t,s) &= \xi(t,s), \;\; s \in [t, t+\delta]. \end{cases} \tag{4.27}$$

(4.27) is a classical SDDE and therefore has a unique solution. It only remains to proove that if $Y(t)$ is defined to be solution of (4.23), then (4.24) holds.

By Itô formula, we have

$$\begin{aligned} d(G(t,s)Y(s)) &= G(t,s^-)dY(s) + Y(s)dG(t,s) + d(GY)(s) \\ &= G(t,s^-)\Big\{ - \Big(\ell(t) + \beta(t)Y(t) + \theta_0(t)Z(t) \; + \int_{\mathbb{R}_0} \theta_1(t,z)K(t,z)\,\nu(dz)\Big)dt \\ &\quad + Z(t)dB(t) + \int_{\mathbb{R}_0} K(t,z)\,\widetilde{N}(dz,dt)\Big\} + Y(s)G(t,s^-)\Big[\beta(s)ds + \theta_0(s)\,dB(s) \\ &\quad + \int_{\mathbb{R}_0} \theta_1(s,z)\,\widetilde{N}(dz,ds)\Big] + G(t,s^-)\Big[\theta_0(s)Z(s)\,ds + \int_{\mathbb{R}_0} \theta_1(s,z)K(t,z)\,\nu(dz)\,ds\Big] \end{aligned}$$

Taking the conditional expectation under $\mathcal{F}_t$, we have

$$E\Big[G(t,T)Y(T)\Big|\mathcal{F}_t\Big] - G(t,t)Y(t) = E\Big[\int_t^T G(t,s^-)\ell(s)\,ds\Big|\mathcal{F}_t\Big]$$

Since $G(t,t) = 1$, we obtain

$$Y(t) = E\Big[G(t,T)Y(T) + \int_t^T G(t,s^-)\ell(s)\,ds\Big|\mathcal{F}_t\Big]$$

□

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