Insider Trading Equilibrium in a Market with Memory

Francesca Biagini[1),3)], Yaozhong Hu[2),3)], Thilo Meyer-Brandis[1)]
and Bernt Øksendal[3),4)]

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Abstract

We consider the Kyle-Back model for insider trading, with the difference that the classical Brownian motion noise of the noise traders is replaced by the noise of a fractional Brownian motion B^H with Hurst parameter $H > \frac{1}{2}$ (when $H = \frac{1}{2}$, B^H coincides with the classical Brownian motion). Heuristically, for $H > \frac{1}{2}$ this means that the noise traders has some "memory", in the sense that any increment from time t on has a positive correlation with its value at t. (In other words, the noise trading is a *persistent* stochastic process). It also means that the paths of the noise trading process are *more regular* than in the classical Brownian motion case.

We obtain an equation for the optimal (relative) trading intensity for the insider in this setting, and we show that when $H \to \frac{1}{2}$ the solution converges to the solution in the classical case. Finally, we discuss how the size of the Hurst coefficient H influences the optimal performance and portfolio of the insider.

1   Introduction

In their seminal papers Kyle [6] and subsequently Back [2] formulate and study an equilibrium model for insider trading. There are many papers followed Kyle-Back inspired models that should be cited. The paper most closely related to ours in setup and method is [1], where a (classical) Brownian motion model is studied. Here we review then briefly the Kyle-Back model, based on the presentation in [1]. We assume the financial market has three agents:

- (i) *The insider*, who already from the initial time $t = 0$ knows the value $\tilde{v}$ at the terminal time $t = T$ of a given stock. The *portfolio* of the insider, measured in terms of the number of stocks held at time t, is denoted by x_t, $t \in [0, T]$. It is assumed that $\tilde{v}$ is a centered Gaussian random variable of known variance.

- (ii) *The noise traders*, who trade randomly without any information about the market. The portfolio z_t of the noise traders is assumed to have the form

(1.1)
$$dz_t = \sigma_t dB_t, \quad t \in [0, T],$$

where σ_t is a given continuous deterministic function and $B_t = B_t(\omega), (t, \omega) \in [0, T] \times \Omega$, is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$. It is assumed that $\tilde{v}$ is independent of the Brownian motion $B_t, t \in [0, T]$.

- (ii) *The market makers*, who at any time t can observe the *total* traded volume

(1.2)
$$y_t = x_t + z_t,$$

but not the separate trades x_t, z_t. Based on the information (filtration) $\mathcal{F}_t^y, t \in [0, T]$, generated by the observations $y_s, s \le t$, the market makers set the price of the stock at time t equal to

(1.3)
$$p_t := \mathbb{E}\left[\tilde{v} | \mathcal{F}_t^y\right], \quad 0 \le t \le T.$$

The wealth w_t at time t of the insider can be expressed as

(1.4)
$$w_t = w_0 + \int_0^t x_s dp_s, \quad 0 \le t \le T.$$

A priori this is an anticipative stochastic integral, which needs further explanation. If we assume, as Kyle and Back, that the strategy of the insider has the form

(1.5)
$$dx_t = (\tilde{v} - p_t)\beta_t dt$$

for some deterministic continuous function $\beta_t > 0$, called the *insider trading intensity*, then a natural interpretation of (1.4) is obtained by using *integration by parts*, as follows:

$$w_t = w_0 + x_t p_t - \int_0^t p_s dx_s$$

$$= w_0 + p_t \int_0^t (\tilde{v} - p_s)\beta_s ds - \int_0^t p_s(\tilde{v} - p_s)\beta_s ds$$

(1.6)
$$= w_0 + \int_0^t (\tilde{v} - p_s)^2 \beta_s ds - \int_0^t (\tilde{v} - p_t)(\tilde{v} - p_s)\beta_s ds.$$

Alternatively, one might obtain (1.6) by interpreting the stochastic integral in (1.4) as a *forward integral*. See [8] for definitions and [1] for applications of forward integrals to finance. The insider tries to find the trading intensity β_t which maximizes the expected terminal wealth

$$(1.7) \qquad \mathbb{E}\left[w_T\right] = \mathbb{E}\left[w_T^{(\beta)}\right] = w_0 + \int_0^T \mathbb{E}\left[(\tilde{v} - p_s)^2\right] \beta_s ds - \int_0^T \mathbb{E}\left[(\tilde{v} - p_T)(\tilde{v} - p_s)\right] \beta_s ds.$$

The dilemma for the insider is that an increased trading intensity at some time t will reveal more information about the value of $\tilde{v}$ to the market makers and hence induce a price p_t closer to $\tilde{v}$, which in turn implies a reduced insider advantage. The optimal insider trading strategy is proved to be

$$(1.8) \qquad \beta_t = \frac{\sigma_t^2 (\int_0^T \sigma_s^2 ds)^{\frac{1}{2}}}{S_0^{\frac{1}{2}} \int_t^T \sigma_s^2 ds}; \quad S_0 = \mathbb{E}\left[(\tilde{v} - \mathbb{E}\left[\tilde{v}\right])^2\right],$$

which gives the optimal mean square error

$$(1.9) \qquad S_t = \mathbb{E}\left[(\tilde{v} - p_t)^2\right] = \frac{S_0 \int_t^T \sigma_s^2 ds}{\int_0^T \sigma_s^2 ds},$$

and the optimal insider performance

$$(1.10) \qquad J(\beta) := \mathbb{E}\left[w_T^{(\beta)}\right] = w_0 + \left(S_0 \int_0^T \sigma_s^2 ds\right)^{\frac{1}{2}}.$$

In particular, this implies $p_T = \tilde{v}$ and $p_t = \mathbb{E}\left[\tilde{v}\right] + \lambda y_t$, where $\lambda = \left(\frac{S_0}{\int_0^T \sigma_s^2 ds}\right)^{\frac{1}{2}}$ is called the *price sensitivity*. See [1] for details.

The purpose of this paper is to study the above model in the case when the Brownian motion B in (1.1) is generalized to a *fractional Brownian motion* B^H with *Hurst parameter* $H \in (0,1)$. By definition B_t^H, $0 \le t \le T$, is a continuous and centered Gaussian process with covariance function

$$(1.11) \qquad \mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad 0 \le t \le T.$$

If $H = \frac{1}{2}$ then B^H is the classical Brownian motion B. If $H \neq \frac{1}{2}$ the increments of B^H are not independent. For $H > \frac{1}{2}$ the two increments

$$(1.12) \qquad B_{t+h}^H - B_t^H \quad \text{and} \quad B_{s+h}^H - B_s^H$$

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are *positively correlated*, while they are *negatively correlated* for $H < \frac{1}{2}$. Thus the case for $H > \frac{1}{2}$ corresponds to systems with *memory* and *persistence*, while the case of $H < \frac{1}{2}$ corresponds to systems with *turbulence* and *anti-persistence*. If $H > \frac{1}{2}$ then the paths of B^H are more regular than for classical Brownian motion, while if $H < \frac{1}{2}$ the paths are less regular. More precisely, for any $\alpha < H$ the paths of B^H are Hölder continuous with exponent α almost surely, i.e.

$$(1.13) \qquad |B_t^H - B_s^H| \le c|t - s|^\alpha, \quad 0 \le t \le T,$$

for some (random) constant $c > 0$. For more information about fractional Brownian motion and its applications, we refer to [3], [7] and the references therein.

In this paper we restrict ourselves to the case $H > \frac{1}{2}$. In other words, we study how the introduction of *persistence* or *memory* among the noise traders influences the Kyle-Back model, in particular what effect it has on the optimal insider portfolio and maximal expected insider wealth. As in the Kyle-Back setting, we assume that $\tilde{v}$ is independent of $B_t^H, t \in [0, T]$. We prove that if an optimal smooth insider trading intensity β exists, then it is the solution of a non-linear integro-differential equation. Moreover, we show that $p_T = \tilde{v}$ in Theorem 2.4.

In the formulation adopted in this paper, we have encountered a new stochastic differential equations

$$dy_t = (\tilde{v} - \mathbb{E}(\tilde{v}|\mathcal{F}_t^y))dt + \sigma_t dB_t^H \,.$$

The existence and uniqueness of the solution to the above equation have not been studied yet. In Section 2 we shall formulate our problem, obtain an existence result for the above equation by using the innovation technique, and find an equation that the maximum trading intensity must satisfy. In Section 3, we study the uniqueness of the above equation. In Section 4, we discuss the impact of long memory on the insider trader. The Appendix provide some technical results.

2 The main result

We use the same setup as in Section 1, except that B_t, $0 \le t \le T$, is replaced with a fractional Brownian motion B_t^H, $0 \le t \le T$, with Hurst parameter $H > \frac{1}{2}$. Thus the portfolio of the noise traders gets the form

$$(2.1) \qquad dz_t = \sigma_t dB_t^H \,, \quad t \in [0, T],$$

the portfolio of the insider is as before

$$(2.2) \qquad dx_t = (\tilde{v} - p_t)\beta_t dt \,,$$

where p_t is the market price at time t set by the market makers, which will be made more precise in next lines (see equation (2.4) below). The total traded volume is hence

$$(2.3) \qquad dy_t = (\tilde{v} - p_t)\beta_t dt + \sigma_t dB_t^H \,.$$

If we let $\mathcal{F}_t^y$, $t \in [0, T]$, be the filtration generated by y_s, $s \le t$, then it is assumed that

$$(2.4) \qquad p_t := \mathbb{E}\left[\tilde{v}|\mathcal{F}_t^y\right], \quad 0 \le t \le T.$$

Substituting this into (2.3) we get that the total traded volume process must satisfy the equation

$$(2.5) \qquad dy_t = \left(\tilde{v} - \mathbb{E}\left[\tilde{v}|\mathcal{F}_t^y\right]\right)\beta_t dt + \sigma_t dB_t^H \,, \quad t \in [0, T]\,.$$

As in [1] we will prove that it is possible to find a solution of (2.5) by regarding y_t, $0 \le t \le T$, as the *innovation process* $\tilde{y}_t$, $0 \le t \le T$, of an auxiliary linear filtering problem, where the *signal process* is

$$(2.6) \qquad \xi_t = \tilde{v}; \quad t \in [0, T],$$

and the *observation process* is

$$(2.7) \qquad d\hat{y}_t = \tilde{v}\beta_t dt + \sigma_t dB_t^H \,; \quad t \in [0, T],\ \hat{y}_0 = 0\,.$$

The innovation process for this problem is, by definition,

$$
\begin{aligned}
(2.8) \qquad d\tilde{y}_t &= \left(\tilde{v} - \mathbb{E}\left[\tilde{v}|\mathcal{F}_t^{\hat{y}}\right]\right)\beta_t dt + \sigma_t dB_t^H \\
&= d\hat{y}_t - \mathbb{E}\left[\tilde{v}|\mathcal{F}_t^{\hat{y}}\right]\beta_t dt \,,
\end{aligned}
$$

where $\mathcal{F}_t^{\hat{y}} = \sigma(\hat{y}_s\,,\ 0 \le s \le t)$ is the information filtration generated by $\hat{y}$. It is obvious that we can assume that

$$\mathbb{E}(\tilde{v}) = 0 \quad \text{and} \quad \mathbb{E}(\tilde{v}^2) = 1\,.$$

We shall show that $\tilde{y}$ solves (2.5). This follows from the following lemma.

Lemma 2.1. *Assume that*

$$(2.9) \qquad s \to \frac{\beta_s}{\sigma_s} \in \mathcal{C}^2[0, t]$$

for all $t < T$. Then $\mathcal{F}_t^{\tilde{y}} = \mathcal{F}_t^{\hat{y}}$ for all $t \in [0, T]$.

Proof. Since
$$d\tilde{y}_t = d\hat{y}_t - \mathbb{E}\left[\tilde{v}|\mathcal{F}_t^{\hat{y}}\right]\beta_t dt,$$

we see that $\mathcal{F}_t^{\tilde{y}} \subset \mathcal{F}_t^{\hat{y}}$. We need to prove the other inclusion $\mathcal{F}_t^{\hat{y}} \subset \mathcal{F}_t^{\tilde{y}}$. First we shall compute $p_t := \mathbb{E}\left[\tilde{v}|\mathcal{F}_t^{\hat{y}}\right]$ by using the result obtained in [5]. Define

(2.10)
$$K_H(t,s) = \kappa_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H},$$

where $\kappa_H = 2H\Gamma(3/2-H)\Gamma(1/2+H)$. Let

(2.11)
$$y_t^* = \int_0^t K_H(t,s)\sigma_s^{-1}d\hat{y}_s.$$

Then from Theorem 1 of [5], we know that $y_t^*, 0 \le t \le T$, is a semimartingale and the information filtrations generated by y^* and $\hat{y}$ are the same:

(2.12)
$$\mathcal{F}_t^{y^*} = \mathcal{F}_t^{\hat{y}}, \quad \forall t \in [0,T].$$

We also have

(2.13)
$$d\langle y^*, y^*\rangle_s = d\ell(s), \qquad d\ell(s) := (2-2H)\kappa_H^{-1}s^{1-2H}ds.$$

Put

(2.14)
$$\gamma_t = \mathbb{E}\left[\left(\tilde{v} - \mathbb{E}\left[\tilde{v}|\mathcal{F}_t^{\hat{y}}\right]\right)^2\right], \quad t \in [0,T],$$

and define (which is $p(s,0)$ of (13) in [5])

(2.15)
$$\rho_s = \rho_s(\beta) = \frac{d}{d\ell(s)}\int_0^s K_H(s,r)\frac{\beta_r}{\sigma_r}dr \quad s \in [0,T].$$

Then by Section 5.1 of [5] we have

(2.16)
$$\gamma_t = \gamma_t(\beta) = \left(\gamma_0^{-1} + \int_0^t \rho_s^2 d\ell(s)\right)^{-1}, \quad t \in [0,T],$$

for some constant $\gamma_0 \ne 0$ and

$$p_t = \gamma_t\gamma_0^{-1}\left(p_0 + \gamma_0\int_0^t \rho_s dy_s^*\right), \quad \forall t \in [0,T].$$

From the definition of p_t we have $p_0 = \mathbb{E}\left[\tilde{v}|\mathcal{F}_0\right] = \mathbb{E}(\tilde{v}) = 0$ since we assume $\mathbb{E}(\tilde{v}) = 0$. Thus we have

(2.17)
$$p_t = \gamma_t\int_0^t \rho_s dy_s^*, \quad \forall t \in [0,T].$$

From Lemma 5.3, we have

$$(2.18) \qquad p_t = \int_0^t g(t,s) d\hat{y}_s\,,$$

where

$$(2.19) \qquad g(t,s) = \gamma_t \left[K_H(t,s)\rho_t - \int_s^t K_H(r,s)\rho_r' dr \right] \sigma_s^{-1}.$$

For any smooth deterministic function f_t, $t \in [0,T)$, we now consider

$$
\begin{aligned}
\int_0^t f_s d\tilde{y}_s &= \int_0^t f_s (d\hat{y}_s - \mathbb{E}\left[\tilde{v}|\mathcal{F}_s^{\hat{y}}\right]\beta_s ds) \\
&= \int_0^t f_s (d\hat{y}_s - p_s \beta_s ds) \\
&= \int_0^t f_s d\hat{y}_s - \int_0^t f_s \beta_s p_s ds \\
&= \int_0^t f_s d\hat{y}_s - \int_0^t f_s \beta_s (\int_0^s g(s,u) d\hat{y}_u) ds \\
(2.20) \qquad &= \int_0^t (f_u - \int_u^t g(s,u) f_s \beta_s ds) d\hat{y}_u\,.
\end{aligned}
$$

where we have used the Fubini type theorem in (2.20). We want to find a representation of $\hat{y}$ in terms of $\tilde{y}$. This is equivalent to find a solution of the equation

$$(2.21) \qquad f_u - \int_u^t g(s,u) f_s \beta_s ds = \chi_{[0,t]}(u)\,.$$

By classical results on Volterra equations, see e.g. [4], Lemma 4.3.3 on page 125, this equation has a solution if

$$(2.22) \qquad \int_0^t \int_s^t \beta_s^2 g^2(s,r) dr ds < \infty \quad \text{for all } t < T,$$

where $g(s,r)$ is given by (2.19). By the Lemma 5.2 in the Appendix we obtain that if (2.9) holds, then (2.22) is satisfied. Therefore we see that $\mathcal{F}_t^{\hat{y}} \subset \mathcal{F}_t^{\tilde{y}}$. This concludes the proof of the lemma. ∎

Corollary 2.2. *Assume that* (2.9) *holds. Then* $\tilde{y}_t$, $0 \le t \le T$, *defined by* (2.8) *is a solution of* (2.5).

Remark 2.3. *In view of Corollary 2.2 we choose to represent the total traded volume process y by $\tilde{y}$, and we write y instead of $\tilde{y}$ from now on. Note however, that we have* not *proved that the solution of* (2.5) *is unique, so this choice is not totally justified from a mathematical point*

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of view, since there might be solutions y of (2.5) which are not representable as innovation processes of linear filtering problems.

On the other hand, since the market makers are assumed to know $\mathcal{F}_t^y$ and also of course the price p_t at any time $t \in [0, T]$, then by (2.3) and (2.7) they know

$$(2.23) \qquad \hat{y}_t = y_t + \int_0^t p_s \beta_s ds \, .$$

This implies that

$$(2.24) \qquad \mathcal{F}_t^{\hat{y}} = \mathcal{F}_t^y$$

and hence $dy_t = d\tilde{y}_t$, even without hypothesis (2.9). So from a modeling point of view the assumption that $y_t = \tilde{y}_t$ is natural, and we will base our study on this.

As shown in the introduction the expected terminal wealth of the insider can be expressed as follows:

$$(2.25) \qquad \mathbb{E}(w_T) = w_0 + \int_0^T \mathbb{E}\left[(\tilde{v} - p_t)^2\right] \beta_t dt - \int_0^T \mathbb{E}\left[(\tilde{v} - p_T)(\tilde{v} - p_t)\right] \beta_t dt \, .$$

We need to compute $\mathbb{E}\left[(\tilde{v} - p_T)(\tilde{v} - p_t)\right]$. We have

$$\begin{aligned}
\mathbb{E}\left[(\tilde{v} - p_T)(\tilde{v} - p_t)\right] &= \mathbb{E}(\tilde{v}^2) - \mathbb{E}(\tilde{v}p_t) - \mathbb{E}(\tilde{v}p_T) + \mathbb{E}(p_T p_t) \\
&= \mathbb{E}(\tilde{v}^2) - \mathbb{E}(p_t^2) - \mathbb{E}(p_T^2) + \mathbb{E}(p_T p_t) \, .
\end{aligned}$$

We first compute $\mathbb{E}(p_T p_t)$. By (1.3) we have that p_t, $0 \leq t \leq T$, is a square-integrable martingale. Hence

$$\mathbb{E}\left[p_t p_T\right] = \mathbb{E}\left[p_t^2\right] \, ,$$

and consequently

$$\begin{aligned}
\mathbb{E}\left[(\tilde{v} - p_T)(\tilde{v} - p_t)\right] &= \mathbb{E}(\tilde{v}^2) - \mathbb{E}(p_t^2) - \mathbb{E}(p_T^2) + \mathbb{E}(p_T p_t) \\
&= \mathbb{E}(\tilde{v}^2) - \mathbb{E}(p_t^2) - \mathbb{E}(p_T^2) + \mathbb{E}(p_t^2) \\
&= \mathbb{E}(\tilde{v}^2) - \mathbb{E}(p_T^2) \, .
\end{aligned}$$

But

$$\mathbb{E}(p_T^2) = \mathbb{E}(\tilde{v}^2) - \mathbb{E}(\tilde{v} - p_T)^2 = \mathbb{E}(\tilde{v}^2) - \gamma_T \, ,$$

and

$$(2.26) \qquad \mathbb{E}\left[(\tilde{v} - p_T)(\tilde{v} - p_t)\right] = \gamma_T \, .$$

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Hence by (2.25) and by (2.26) we want to maximize

$$(2.27) \qquad J(\beta(\cdot)) = w_0 + \int_0^T (\gamma_t(\beta) - \gamma_T(\beta))\beta_t dt \, .$$

First let us maximize

$$(2.28) \qquad J_0(\beta(\cdot)) = \int_0^T \gamma_t(\beta)\beta_t dt \, .$$

We do this by using a perturbation argument, as in [1]. Let ε be an arbitrary small number and $\xi_t, 0 \le t \le T$, be an arbitrary smooth function. We want to compute $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} J_0(\beta + \varepsilon\xi)$. In the following we assume that all functions involved are smooth enough to exchange the order of derivation and integration. We first note that by the definition (2.15) of ρ we obtain

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \rho_s(\beta + \varepsilon\xi) = \frac{d}{d\ell(s)} \int_0^s K_H(s,r)\frac{\xi_r}{\sigma_r} dr \, .$$

Thus

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_0^t [\rho_s(\beta + \varepsilon\xi)]^2 \, d\ell(s) = 2\int_0^t \rho_s \frac{d}{d\ell(s)} \int_0^s K_H(s,r)\frac{\xi_r}{\sigma_r} dr d\ell(s)$$

$$(2.29) \qquad\qquad\qquad = 2\int_0^t \rho_s \frac{d}{ds} \int_0^s K_H(s,r)\frac{\xi_r}{\sigma_r} dr ds \, .$$

We apply this result to compute $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \gamma_t(\beta + \varepsilon\xi)$. By (2.16) and (2.29) we have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \gamma_t(\beta + \varepsilon\xi) = -2\gamma_t^2 \int_0^t \rho_s \frac{d}{ds} \int_0^s K_H(s,r)\frac{\xi_r}{\sigma_r} dr ds$$

$$= -2\gamma_t^2 \rho_t \int_0^t K_H(t,r)\frac{\xi_r}{\sigma_r} dr + 2\gamma_t^2 \int_0^t \rho_s' \int_0^s K_H(s,r)\frac{\xi_r}{\sigma_r} dr ds$$

$$= 2\gamma_t^2 \int_0^t \sigma_r^{-1} \left[\int_r^t \rho_s' K_H(s,r) ds - \rho_t K_H(t,r) \right] \xi_r dr \, .$$

Putting everything together we obtain

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} J_0(\beta + \varepsilon\xi) = \int_0^T \gamma_t \xi_t dt + 2\int_0^T \beta_t \gamma_t^2 \left\{ \int_0^t \sigma_r^{-1} \left[\int_r^t \rho_s' K_H(s,r) ds - \rho_t K_H(t,r) \right] \xi_r dr \right\} dt$$

$$= \int_0^T \gamma_r \xi_r dr + 2\int_0^T \sigma_r^{-1} \left\{ \int_r^T \beta_t \gamma_t^2 \left[\int_r^t \rho_s' K_H(s,r) ds - \rho_t K_H(t,r) \right] dt \right\} \xi_r dr \, .$$

Since ξ_r is arbitrary, we have

$$(2.30) \qquad \gamma_r = -2\sigma_r^{-1} \int_r^T \beta_t \gamma_t^2 \left[\int_r^t \rho_s' K_H(s,r) ds - \rho_t K_H(t,r) \right] dt \, ,$$

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or equivalently

(2.31) $$\sigma_r \gamma_r(\beta) = -2 \int_r^T \beta_t \gamma_t^2(\beta) \left[\int_r^t \rho_s'(\beta) K_H(s,r) ds - \rho_t(\beta) K_H(t,r) \right] dt \,,$$

for $0 \le t \le T$. Thus we have proved that if

(2.32) $$\beta \to J_0(\beta(\cdot)) = \int_0^T \gamma_t(\beta) \beta_t dt$$

is maximal, then $\gamma_t(\beta)$ satisfies (2.31). In particular, $\gamma_T(\beta) = 0$. But this implies that $\gamma_t(\beta)$ is also optimal for

(2.33) $$J(\beta(\cdot)) = w_0 + J_0(\beta(\cdot)) = w_0 + \int_0^T (\gamma_t(\beta) - \gamma_T(\beta)) \beta_t dt$$

since we always have $\gamma_T(\beta) \ge 0$ and $\beta \ge 0$. We have proved

Theorem 2.4. *Suppose β is an optimal insider portfolio for the problem*

(2.34) $$\sup_{\beta} \mathbb{E}\left[w_T(\beta) \right] = \sup_{\beta} \left\{ w_0 + \int_0^T (\gamma_t(\beta) - \gamma_T(\beta)) \beta_t dt \right\}.$$

Then $\gamma_T(\beta) = 0$ and $\gamma_t(\beta)$ satisfies equation (2.31). In particular, by (2.26)

(2.35) $$p_T = \tilde{v}.$$

Proposition 2.5. *The process y defined by*

(2.36) $$dy_t = \left(\tilde{v} - \mathbb{E}\left[\tilde{v} | \mathcal{F}_t^{\hat{y}} \right] \right) \beta_t dt + \sigma_t dB_t^H \,, \quad y_0 = 0$$

is an $\mathcal{H}_t := \sigma \left(\tilde{v}, B_s^H ; s \le t \right)$-adapted solution of the equation

(2.37) $$d\tilde{y}_t = \left(\tilde{v} - \mathbb{E}\left[\tilde{v} | \mathcal{F}_t^{\tilde{y}} \right] \right) \beta_t dt + \sigma_t dB_t^H \,, \quad \tilde{y}_0 = 0 \,.$$

Proof. That y defined by (2.36) is a solution of equation (2.37) follows from Lemma 2.1. ■

We now let $H \to \frac{1}{2}$ in equation (2.30) and show how it converges to the equation for the optimal γ in the case $H = \frac{1}{2}$.

Proposition 2.6. *For $H \to \frac{1}{2}$ equation (2.30) becomes*

$$\gamma_r = 2 \frac{\beta_r}{\sigma_r^2} \int_r^T \beta_t \gamma_t^2 dt \,, \quad 0 \le r \le T,$$

that is equivalent to the equation (4.28) of [1]:

(2.38) $$1 = 2 \frac{\beta_r \gamma_r}{\sigma_r^2} \int_r^T \beta_t \exp\left(-2 \int_r^t \frac{\gamma_u \beta_u^2}{\sigma_u^2} du \right) dt, \quad 0 \le r \le T,$$

for the optimal γ in the case $H = \frac{1}{2}$.

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Proof. First of all we note that by (2.10) $K_H(s, r)$ converges to 1 for $H \to \frac{1}{2}$. Furthermore by taking the limit in (2.15) we obtain that ρ_t goes to $\frac{\beta_t}{\sigma_t}$ for all $t \in [0, T]$. Hence (2.30) becomes

$$(2.39) \qquad \gamma_r = 2\sigma_r^{-1}\rho_r \int_r^T \beta_t \gamma_t^2 dt = 2\frac{\beta_r}{\sigma_r^2} \int_r^T \beta_t \gamma_t^2 dt , \ 0 \le r \le T,$$

if $H \to \frac{1}{2}$ by uniform integrability. When $H \to 1/2$, the equation (2.16) becomes

$$\gamma_t = \left(\frac{1}{\gamma_0} + \int_0^t \left(\frac{\beta_s}{\sigma_s} \right)^2 ds \right)^{-1} , \ 0 \le r \le T.$$

Therefore, γ satisfies

$$(2.40) \qquad \frac{d\gamma_t}{dt} = -\frac{\beta_t^2}{\sigma_t^2} \gamma_t^2 .$$

Hence for $t > r$ we have that

$$(2.41) \qquad \gamma_t = \gamma_r \exp \left(-\int_r^t \frac{\beta_u^2}{\sigma_u^2} \gamma_u du \right) .$$

Substituting (2.41) into (2.39), we obtain

$$(2.42) \qquad \gamma_r = 2\frac{\beta_r}{\sigma_r^2} \int_r^T \gamma_r^2 \beta_t \exp \left(-2 \int_r^t \frac{\gamma_u \beta_u^2}{\sigma_u^2} du \right) dt, \quad r \in [0, T].$$

This is equation (2.38). ∎

3 Uniqueness of the equation

The equation (2.5) which we reproduce here

$$(3.1) \qquad dy_t = (\tilde{v} - \mathbb{E}[\tilde{v}|\mathcal{F}_t^y]) \beta_t dt + \sigma_t dB_t^H , \quad t \in [0, T],$$

with $y_0 = 0$ is a new type of equation even in the case B^H is replaced by a Brownian motion, where B^H is a fractional Brownian motion of Hurst parameter H, β_t and σ_t are deterministic functions and $\tilde{v}$ is a standard normal random variable independent of the fractional Brownian motion B^H.

Lemma 2.1 yields the existence of a solution. In Remark 2.3, we explain from economic point of view the rationale of uniqueness. However, mathematically the uniqueness is still an open problem mathematically. It is our conjecture that the uniqueness holds as well.

Here we give an attempt to this problem. We restrict the solution to the form

$$(3.2) \qquad y_t = h_1(t)\tilde{v} + \int_0^t h_2(t, s) dB_s^H$$

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for some unknown functions $h_1(t)$ and $h_2(t,s)$, $0 \le s \le t \le T$. Since $\tilde{v}$ and y_t, $0 \le t \le T$, are jointly Gaussian, there is a $g(t,s)$, $0 \le s \le t \le T$, such that (because $\mathbb{E}(\tilde{v}) = 0$ and $\mathbb{E}(\tilde{v})^2 = 1$)

$$
\begin{aligned}
\mathbb{E}\left[\tilde{v}|\mathcal{F}_t^y\right] &= \int_0^t g(t,s)dy_s \\
&= \int_0^t g(t,s)h_1'(s)ds\ \tilde{v} + \int_0^t g(t,s)h_2(s,s)dB_s^H + \int_0^t g(t,s)\int_0^s \frac{\partial}{\partial s}h_2(s,r)dB_r^H ds \\
(3.3)\quad &= \int_0^t g(t,s)h_1'(s)ds\ \tilde{v} + \int_0^t \left[g(t,s)h_2(s,s) + \int_s^t g(t,r)\frac{\partial}{\partial r}h_2(r,s)dr\right]dB_s^H\,.
\end{aligned}
$$

By the property of conditional expectation we have

$$
(3.4)\qquad\qquad \mathbb{E}\left(y_s\tilde{v}\right) = \mathbb{E}\left(y_s\mathbb{E}\left[\tilde{v}|\mathcal{F}_t^y\right]\right),\quad \forall\, 0 \le s \le t\,.
$$

First we have

$$
\mathbb{E}\left(y_s\tilde{v}\right) = h_1(s)\,.
$$

On the other hand, we have

$$
\begin{aligned}
\mathbb{E}\left(y_s\mathbb{E}\left[\tilde{v}|\mathcal{F}_t^y\right]\right) &= h_1(s)\int_0^t g(t,u)h_1'(u)du \\
(3.5)\quad &+ \int_0^r \int_0^t h_2(r,s_1)\left[g(t,s_2)h_2(s_2,s_2) + \int_{s_2}^t g(t,r)h_2(r,s_2)dr\right]\phi(s_1-s_2)ds_1 ds_2\,,
\end{aligned}
$$

where

$$
\phi(u) = H(2H-1)|u|^{2H-2}\,.
$$

Thus equation (3.4) becomes

$$
\begin{aligned}
h_1(s) &= h_1(s)\int_0^t g(t,u)h_1'(u)du) \\
(3.6)\quad &+ \int_0^r \int_0^t h_2(r,s_1)\left[g(t,s_2)h_2(s_2,s_2) + \int_{s_2}^t g(t,r)h_2(r,s_2)dr\right]\phi(s_1-s_2)ds_1 ds_2\,,
\end{aligned}
$$

Substituting (3.3) into (3.1), we have

$$
\begin{aligned}
y_t &= \int_0^t \left(\tilde{v} - \mathbb{E}\left[\tilde{v}|\mathcal{F}_r^y\right]\right)\beta_r dr + \int_0^t \sigma_r dB_r^H \\
&= \int_0^t \left\{\tilde{v} - \tilde{v}\int_0^r g(r,s)h_1'(s)ds\right. \\
&\quad \left.+ \int_0^r \left[g(r,s)h_2(s,s) + \int_s^r g(r,u)\frac{\partial}{\partial u}h_2(u,s)du\right]dB_s^H\right\}\beta_r dr + \int_0^t \sigma_r dB_r^H \\
&= \int_0^t \left[1 - \int_0^r g(r,s)h_1'(s)ds\right]\beta_r dr\ \tilde{v} \\
(3.7)\quad &- \int_0^t \left\{\int_s^t \left[g(r,s)h_2(s,s) + \int_s^r g(r,u)\frac{\partial}{\partial u}h_2(u,s)du\right]\beta_r dr\right\}dB_s^H + \int_0^t \sigma_r dB_r^H\,.
\end{aligned}
$$

Comparing (3.7) with (3.2) and using the fact that $\tilde{v}$ and B_t^H are independent, we have

$$(3.8) \qquad h_1(t) \;=\; \int_0^t \left[1 - \int_0^r g(r,s)h_1'(s)ds \right] \beta_r dr$$

$$(3.9) \qquad h_2(t,s) \;=\; \sigma_s - \int_0^t \left\{ \int_s^t \left[g(r,s)h_2(s,s) + \int_s^r g(r,u)\frac{\partial}{\partial u}h_2(u,s)du \right] \beta_r dr \right\} .$$

Thus we obtain

Proposition 3.1. *The equation (3.1) has a unique solution of the form (3.2) if the following system of equations has a unique solution* $(h_1(t), h_2(t,s), g(t,s), 0 \le s \le t \le T)$:

$$
\begin{cases}
h_1(t) & = \int_0^t \left[1 - \int_0^r g(r,s)h_1'(s)ds \right] \beta_r dr \\
h_2(t,s) & = \sigma_s - \int_0^t \left\{ \int_s^t \left[g(r,s)h_2(s,s) + \int_s^r g(r,u)\frac{\partial}{\partial u}h_2(u,s)du \right] \beta_r dr \right\} \\
h_1(s) & = h_1(s)\int_0^t g(t,u)h_1'(u)du \\
& \quad + \int_0^r \int_0^t h_2(r,s_1) \left[g(t,s_2)h_2(s_2,s_2) + \int_{s_2}^t g(t,r)h_2(r,s_2)dr \right] \phi(s_1 - s_2)ds_1 ds_2 .
\end{cases}
$$

The existence of the above system was obtained in Section 2 through the technique of filtering.

4 The impact of memory (persistence) in the noise trades

One of the motivations of this paper is to investigate how the memory (persistence) and regularity of the noise process of the noise traders, represented by the Hurst coefficient $H > \frac{1}{2}$, influence the performance of the insider.

Unfortunately, we are not able to solve our general equation (2.31) to obtain the optimal $\beta_t = \beta_t(H)$, $t \in [0,T]$, explicitly, and thus we are unable to make any conclusion about this influence in general. However, if we restrict ourselves to *constant* insider trading intensity $\beta = \beta(H) > 0$, our equations simplify as follows.

Consider σ constant. By (5.1) in Lemma 5.1 we obtain that

$$
\rho_t = s^{2H-2}\frac{\beta}{\sigma}\int_0^s r^{\frac{1}{2}-H}(s-r)^{\frac{1}{2}-H}dr
$$

$$
= \frac{\beta}{\sigma}\int_0^1 u^{\frac{1}{2}-H}(1-u)^{\frac{1}{2}-H}du
$$

$$(4.1) \qquad\qquad = \frac{\Gamma(\frac{3}{2}-H)^2}{\Gamma(3-2H)}\frac{\beta}{\sigma}.$$

Therefore equation (2.16) becomes

$$\gamma_t = \left(\gamma_0^{-1} + \int_0^t \rho_s^2 d\ell(s)\right)^{-1}$$

(4.2)
$$= \left(\gamma_0^{-1} + a_H t^{2-2H} \frac{\beta^2}{\sigma^2}\right)^{-1},$$

where $a_H := \frac{\Gamma(\frac{3}{2}-H)^3}{2H\Gamma(3-2H)^2\Gamma(\frac{1}{2}+H)}$. Hence, we can write down the performance functional as

$$J(\beta) = \beta \int_0^T (\gamma_t - \gamma_T)dt$$

(4.3)
$$= \beta \int_0^T \frac{1}{\gamma_0^{-1} + a_H t^{2-2H}\frac{\beta^2}{\sigma^2}} dt - \frac{\beta T}{\gamma_0^{-1} + a_H T^{2-2H}\frac{\beta^2}{\sigma^2}}.$$

It is easy to see that for a given $H \in (\frac{1}{2}, 1)$ and $T > 0$, $J(\beta)$ is a continuous function of β, $J(0) = 0$ and $\lim_{\beta \to \infty} J(\beta) = 0$. Thus $J(\beta)$ attains its maximum values over all $\beta > 0$. We illustrate the relation between β and $J(\beta)$ numerically as plots. We choose $T = 2$, $\sigma = 1$ and plot the function $J(\beta)$ for the Hurst parameters $H = 0.5$ (top curve), $H = 0.6$ (second from top), $H = 0.75$ (third from top), $H = 0.9$ (fourth from top), and $H = 1$ (bottom straight line).
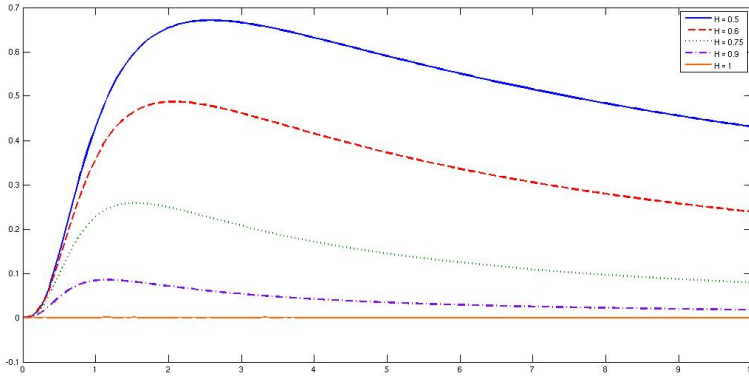

Figure 1: Plot of the functions $J(\beta)$ for 5 different values of H
x-axis is β and y-axis is $J(\beta)$

The graphs show that the performance of an insider *decreases* with increasing $H \in [\frac{1}{2}, 1)$. They also show that the optimal insider trading intensity $\beta^*(H)$ *decreases* with increasing $H \in [\frac{1}{2}, 1)$. These results can perhaps be understood as follows. Increasing the Hurst coefficient H of the noise trading reduces the "complexity" of the noise in two ways: (i) the noise process becomes more persistent, and

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(ii) the paths of the noise process become more regular.

Both these effects contribute to the decrease of the information advantage of the insider, because with reduced noise the actions of the insider become more apparent to the market makers. Thus increasing H might have the same effect on the insider performance as reducing the noise level $|\sigma|$ in the classical Brownian motion model ($H = \frac{1}{2}$).

Similarly, the *decrease* of the optimal trading intensity $\beta^*(H)$ with increasing H, is also in line with what happens when $|\sigma|$ decreases in the classical setting. (See Section 1).

It is not clear, though, what the effects of increasing H would be if the portfolios β were allowed to vary with time. Then the insider might be able to take advantage of the increased "predictability" of the noise traders to increase her performance, and this might outweigh the disadvantage coming from reduced noise complexity mentioned above.

In either case, we have not been able to give rigorous proofs of any of these statements regarding the effects of increasing H, and we leave the task of doing so as an open problem.

5 Appendix

In this appendix we provided some technical computations needed in the previous sections.

Lemma 5.1. *If* β_r, σ_r *are twice differentiable and* $\sigma_r > 0$ *on* $[0, T]$, *then*

$$(5.1) \qquad \rho_s = s^{2H-2} \int_0^s r^{\frac{1}{2}-H}(s-r)^{\frac{1}{2}-H} \frac{\beta_r}{\sigma_r} dr$$

$$+ (2-2H)^{-1} s^{2H-2} \int_0^s r^{\frac{3}{2}-H}(s-r)^{\frac{1}{2}-H} \frac{d}{dr}\left[\frac{\beta_r}{\sigma_r}\right] dr,$$

$$(5.2) \qquad \rho_s' = -\left[1 + (2-2H)^{-1}\right] s^{2H-3} \int_0^s r^{\frac{3}{2}-H}(s-r)^{\frac{1}{2}-H} \frac{d}{dr}\left[\frac{\beta_r}{\sigma_r}\right] dr$$

$$- (2-2H)^{-1} s^{2H-3} \int_0^s r^{\frac{5}{2}-H}(s-r)^{\frac{1}{2}-H} \frac{d^2}{dr^2}\left[\frac{\beta_r}{\sigma_r}\right] dr,$$

for $0 \leq t \leq T$.

Proof. By definition (2.15) for ρ, we obtain

$$
\begin{aligned}
\rho_s &= \frac{d}{d\ell(s)} \int_0^s K_H(s,r) \frac{\beta_r}{\sigma_r} dr \\
&= (2-2H)^{-1} \kappa_H s^{2H-1} \frac{d}{ds} \left[\int_0^s K_H(s,r) \frac{\beta_r}{\sigma_r} dr \right] \\
&= (2-2H)^{-1} s^{2H-1} \frac{d}{ds} \left[s^{2-2H} \int_0^1 u^{\frac{1}{2}-H}(1-u)^{\frac{1}{2}-H} \frac{\beta_{su}}{\sigma_{su}} du \right] \\
&= (2-2H)^{-1} s^{2H-1} \left[(2-2H) s^{1-2H} \int_0^1 u^{\frac{1}{2}-H}(1-u)^{\frac{1}{2}-H} \frac{\beta_{su}}{\sigma_{su}} du \right. \\
&\qquad \left. + s^{2-2H} \int_0^1 u^{\frac{1}{2}-H}(1-u)^{\frac{1}{2}-H} \frac{d}{ds}\left[\frac{\beta_{su}}{\sigma_{su}}\right] du \right] \\
&= \int_0^1 u^{\frac{1}{2}-H}(1-u)^{\frac{1}{2}-H} \frac{\beta_{su}}{\sigma_{su}} du + (2-2H)^{-1} s \int_0^1 u^{\frac{1}{2}-H}(1-u)^{\frac{1}{2}-H} \frac{d}{ds}\left[\frac{\beta_{su}}{\sigma_{su}}\right] du \\
&= s^{2H-2} \int_0^s r^{\frac{1}{2}-H}(s-r)^{\frac{1}{2}-H} \frac{\beta_r}{\sigma_r} dr \\
&\qquad + (2-2H)^{-1} s^{2H-2} \int_0^s r^{\frac{3}{2}-H}(s-r)^{\frac{1}{2}-H} \frac{d}{dr}\left[\frac{\beta_r}{\sigma_r}\right] dr \, .
\end{aligned}
$$

Taking the derivative again we have

$$
\begin{aligned}
\rho_s' &= \left[1 + (2-2H)^{-1} \right] \int_0^1 u^{\frac{1}{2}-H}(1-u)^{\frac{1}{2}-H} \frac{d}{ds}\left[\frac{\beta_{su}}{\sigma_{su}}\right] du \\
&\qquad + (2-2H)^{-1} s \int_0^1 u^{\frac{1}{2}-H}(1-u)^{\frac{1}{2}-H} \frac{d^2}{ds^2}\left[\frac{\beta_{su}}{\sigma_{su}}\right] du
\end{aligned}
$$

Denote $r = su$. Then

$$
\frac{d}{ds}\left[\frac{\beta_{su}}{\sigma_{su}}\right] = u \frac{d}{dr}\left[\frac{\beta_r}{\sigma_r}\right] , \qquad \frac{d^2}{ds^2}\left[\frac{\beta_{su}}{\sigma_{su}}\right] = u^2 \frac{d^2}{dr^2}\left[\frac{\beta_r}{\sigma_r}\right] .
$$

Therefore,

$$
\begin{aligned}
\rho'_s &= \left[1 + (2 - 2H)^{-1}\right] \int_0^1 u^{\frac{1}{2}-H}(1-u)^{\frac{1}{2}-H} \frac{d}{ds}\left[\frac{\beta_{su}}{\sigma_{su}}\right] du \\
&\quad + (2-2H)^{-1}s \int_0^1 u^{\frac{1}{2}-H}(1-u)^{\frac{1}{2}-H} \frac{d^2}{ds^2}\left[\frac{\beta_{su}}{\sigma_{su}}\right] du \\
&= \left[1 + (2-2H)^{-1}\right] \int_0^1 u^{\frac{3}{2}-H}(1-u)^{\frac{1}{2}-H} \frac{d}{dr}\left[\frac{\beta_r}{\sigma_r}\right] du \\
&\quad + (2-2H)^{-1}s \int_0^1 u^{\frac{5}{2}-H}(1-u)^{\frac{1}{2}-H} \frac{d^2}{dr^2}\left[\frac{\beta_r}{\sigma_r}\right] du \\
&= - \left[1 + (2-2H)^{-1}\right] s^{2H-3} \int_0^s r^{\frac{3}{2}-H}(s-r)^{\frac{1}{2}-H} \frac{d}{dr}\left[\frac{\beta_r}{\sigma_r}\right] dr \\
&\quad + (2-2H)^{-1}s^{2H-3} \int_0^s r^{\frac{5}{2}-H}(s-r)^{\frac{1}{2}-H} \frac{d^2}{dr^2}\left[\frac{\beta_r}{\sigma_r}\right] dr\,.
\end{aligned}
$$

∎

Lemma 5.2. *Suppose*

$$
\sup_{0 \le r \le t} \left\{ \left|\frac{\beta_r}{\sigma_r}\right| + \left|\frac{d}{dr}\left(\frac{\beta_r}{\sigma_r}\right)\right| + \left|\frac{d^2}{dr^2}\left(\frac{\beta_r}{\sigma_r}\right)\right| \right\} < \infty\,.
$$

Then

$$
\sup_{0 \le s \le t} \left[|\rho_s| + |\rho'_s|\right] < \infty\,.
$$

Proof. We use C to denote a generic constant which may have different value in different occurrences. From Lemma 5.1 and the assumption, we have

$$
\begin{aligned}
|\rho_s| &\le C \left[s^{2H-1} \int_0^s r^{\frac{1}{2}-H}(s-r)^{\frac{1}{2}-H} dr + s^{2H-2} \int_0^s r^{\frac{3}{2}-H}(s-r)^{\frac{1}{2}-H}\, dr \right] \\
&\le Cs \le C\,.
\end{aligned}
$$

$$
\begin{aligned}
|\rho'_s| &\le C \left[s^{2H-3} \int_0^s r^{\frac{3}{2}-H}(s-r)^{\frac{1}{2}-H}\, dr + s^{2H-3} \int_0^s r^{\frac{5}{2}-H}(s-r)^{\frac{1}{2}-H} dr \right] \\
&\le C + Cs \le C\,.
\end{aligned}
$$

∎

Finally we need to express $\int_0^t \rho_s dy^*_s$ in terms of $\int_0^t \tilde{\rho}_s d\hat{y}_s$, where y^* and $\hat{y}$ are defined through (2.11).

Introduce the following operator

$$
T(f)(t) = \frac{d}{dt} \int_0^t K_H(t,s) f_s ds\,, \quad f \in C^1([0,T];\mathbb{R})\,.
$$

Then up to an argument of approximation for $y_t, 0 \leq t \leq T$, by smooth functions and from

$$y_s^* = \int_0^s K_H(s,r)\sigma_r^{-1}d\hat{y}_r = \int_0^s K_H(s,r)\sigma_r^{-1}\dot{\hat{y}}_r dr\,,$$

we can write

$$\int_0^t \rho_s dy_s^* = \int_0^t \rho_s T(\sigma^{-1}\dot{\hat{y}})(s)ds\,.$$

Let T_t^* be the transpose of T on the interval $[0,t]$, more precisely, T_t^* is defined by the following identity:

$$\int_0^t g_r(T(f)(r))dr = \int_0^t (T_t^*(g)(r))f_r dr\,, \forall \text{ nice smooth functions} \quad f\,,\ g \in C^1([0,T]\,;\mathbb{R})\,.$$

Then

$$\int_0^t \rho_s dy_s^* = \int_0^t (T_t^*(\rho)(s)\sigma_s^{-1}d\hat{y}_s\,.$$

A simple computation yields that

$$T_t^*(\rho)(s) = K_H(t,s)\rho_t - \int_s^t K_H(r,s)\rho_r' dr\,.$$

Therefore, we have

Lemma 5.3. *For any continuous function $\rho : [0,T] \to \mathbb{R}$, we have*

(5.3)
$$\int_0^t \rho_s dy_s^* = \int_0^t \left[K_H(t,s)\rho_t - \int_s^t K_H(r,s)\rho_r' dr \right] \sigma_s^{-1}d\hat{y}_s\,.$$

Acknowledgment

[1]   Department of Mathematics, Ludwig-Maximilians Universität,
      D-80333 Munich, Germany,
      email: francesca.biagini@math.lmu.de, meyerbra@math.lmu.de.
[2]   Department of Mathematics, University of Kansas
      405 Snow Hall, Lawrence, Kansas 66045-2142, USA,
      email: hu@math.ku.edu.
[3]   Center of Mathematics for Applications (CMA)
      Department of Mathematics, University of Oslo
      Box 1053 Blindern, N-0316, Oslo, Norway,
      email: oksendal@math.uio.no.
[4]   Norwegian School of Economics and Business Administration,
      Helleveien 30, N-5045, Bergen, Norway.

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