SOME INEQUALITIES FOR CHARACTERISTIC FUNCTIONS OF DENSITIES WITH BOUNDED VARIATION*

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Abstract. Some new inequalities for characteristic functions of absolutely continuous distributions, whose densities have bounded variation, are obtained. The inequalities concern behaviour of a characteristic function both in a neighbourhood of the origin and for large values of the argument. They can be used in stability problems, local limit theorems, and statistics.

Keywords: Characteristic functions, functions of bounded variation, inequalities

1. Introduction and results

Let $f(t)$ be the characteristic function of an absolutely continuous unimodal distribution function $F(x)$ (a distribution function $F(x)$ is called unimodal if there exists x_0 such that $F(x)$ is convex on $(-\infty, x_0)$ and concave on (x_0, ∞)). Prokhorov (1962) obtained the following inequalities for $f(t)$: if $F(x)$ is symmetric about x_0 and $\sup_x F'(x) \le a$, then

$$|f(t)| \le \frac{\sin(t/2a)}{t/2a} \tag{1}$$

for $|t| \le \pi a$ and

$$|f(t)| \le \frac{2a}{|t|} \tag{2}$$

for all real t. Later Ushakov (1981) proved that (1) and (2) hold without the symmetry condition. The unimodality condition proved to be more essential: without it (1) and (2) are not true (generally speaking). However, at least for large t, the same order of decreasing ($\sim 1/|t|$) holds for characteristic functions of a wide class of non-unimodal distributions. Kent (1975) proved that if the density function $p(x)$ of $F(x)$ is a function of bounded variation, then

$$f(t) = O\left(\frac{1}{|t|}\right), \quad |t| \to \infty. \tag{3}$$

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In this work, we obtain inequalities of the form (1) and (2) for characteristic functions of non-unimodal distributions which, in particular, include (1)–(3) as partial cases.

Let $f(x)$ be a real-valued function defined on an interval $[a,b]$ of the real line. The total variation of $f(x)$ on $[a,b]$ is defined as

$$\overset{b}{\underset{a}{\mathbf{V}}}(f) = \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

where sup is taken over all n and all collections $x_0, x_1, ..., x_n$ such that $a = x_0 < x_1 < ... < x_n = b$. The total variation on the whole real line is defined as

$$\overset{\infty}{\underset{-\infty}{\mathbf{V}}}(f) = \lim_{x\to\infty} \overset{x}{\underset{-x}{\mathbf{V}}}(f).$$

We also denote

$$\overset{a}{\underset{-\infty}{\mathbf{V}}}(f) = \lim_{x\to-\infty} \overset{a}{\underset{x}{\mathbf{V}}}(f) \text{ and } \overset{\infty}{\underset{a}{\mathbf{V}}}(f) = \lim_{x\to\infty} \overset{x}{\underset{a}{\mathbf{V}}}(f).$$

For $\mathbf{V}_{-\infty}^{\infty}(f)$ we will omit limits and write $\mathbf{V}(f)$.

A function $f(x)$ is said to be a function of bounded total variation if $\mathbf{V}(f) < \infty$ ($\mathbf{V}_a^b(f) < \infty$ if it is considered on an interval $[a,b]$).

Theorem 1. *Let $p(x)$ be a probability density of bounded variation with characteristic function $f(t)$. Then*

$$|f(t)| \le \frac{\sin(t/\mathbf{V}(p))}{t/\mathbf{V}(p)} \tag{4}$$

for $|t| \le \pi\mathbf{V}(p)/2$, and

$$|f(t)| \le \frac{\mathbf{V}(p)}{|t|} \tag{5}$$

for all real t.

If $p(x)$ is unimodal, then $2\sup_x p(x) = \mathbf{V}(p)$, therefore (1) and (2) are partial cases of Theorem 1.

Estimates (4) and (5) are sharp: for an arbitrary $v > 0$ and any fixed t_0 such that $|t_0| \le \pi v/2$, there exists a probability density $p(x)$ such that $\mathbf{V}(p) = v$ and

$$|f(t_0)| = \frac{\sin t_0/v}{t_0/v},$$

where $f(t)$ is the characteristic function corresponding to $p(x)$, and a similar fact holds for inequality (5).

Theorem 1 implies in particular the following estimate.

Corollary 1. *Let the conditions of Theorem 1 be satisfied. Then*

$$|f(t)| \le 1 - \frac{4t^2}{3\pi^2\mathbf{V}^2(p)}$$

for $|t| \leq \pi\,\mathbf{V}(p)/2$.

To obtain the corollary it suffices to imply the elementary inequality

$$\frac{\sin x}{x} \leq 1 - \frac{4}{3\pi^2}x^2$$

which holds for $|x| \leq \pi/2$ (see for example Ushakov, 1997).

Inequality (5) can be improved if a density function $p(x)$ is one or several times differentiable. More exactly, the following inequality holds.

Theorem 2. *Let* $p(x)$ *be a probability density and* $f(t)$ *be the corresponding characteristic function. If* $p(x)$ *is* $n-1$ *times differentiable, and* $p^{(n-1)}(x)$ *is a function of bounded variation, then*

$$|f(t)| \leq \frac{\mathbf{V}(p^{(n-1)})}{|t|^n} \tag{6}$$

for all real t.

Some examples of applications of inequalities given by Theorems 1 and 2 are contained in Glad, Hjort and Ushakov (1999).

2. Auxiliary results

A set of characteristic functions $\mathcal{B}$ is said to be closed with respect to translation if the condition $f(t) \in \mathcal{B}$ implies that $f(t)e^{itb} \in \mathcal{B}$ for any real b.

Lemma 1. (Ushakov, 1997) *Let* $\mathcal{B}$ *be a class of characteristic functions closed with respect to translation,* B *be an arbitrary set of the real line,* $g(t)$ *be a real valued function defined on* B*. If for any* $f \in \mathcal{B}$ *and any* $t \in B$,

$$|\operatorname{Re} f(t)| \leq g(t),$$

then

$$|f(t)| \leq g(t), \quad t \in B.$$

Lemma 2. *Let* $f_1(x)$, $f_2(x)$ *and* $g(x)$ *be integrable functions defined on the interval* $[a,b]$*. Suppose that* $g(x)$ *is non-increasing on* $[a,b]$,

$$\int_a^b f_1(x)dx = \int_a^b f_2(x)dx,$$

and there exists $c \in (a,b)$ *such that* $f_1(x) \geq f_2(x)$ *for* $x \in [a,c)$, *and* $f_1(x) \leq f_2(x)$ *for* $x \in [c,b]$*. Then*

$$\int_a^b f_1(x)g(x)dx \geq \int_a^b f_2(x)g(x)dx.$$

Proof.

$$\int_a^b f_1(x)g(x)dx - \int_a^b f_2(x)g(x)dx = \int_a^b [f_1(x) - f_2(x)]g(x)dx$$

$$= \int_a^c [f_1(x) - f_2(x)]g(x)dx - \int_c^b [f_2(x) - f_1(x)]g(x)dx$$

$$\geq g(c)\int_a^c [f_1(x) - f_2(x)]dx - g(c)\int_c^b [f_2(x) - f_1(x)]dx$$

$$= g(c)\int_a^b [f_1(x) - f_2(x)]dx = 0.$$

□

Lemma 3. *Let $p(x)$ and $q(x)$ be two probability density functions, and $r(x)$ be their convolution:*

$$r(x) = \int_{-\infty}^{\infty} p(x-u)q(u)du = \int_{-\infty}^{\infty} p(u)q(x-u)du.$$

Then

$$\mathbf{V}(r) \leq \min\{\mathbf{V}(p), \mathbf{V}(q)\}.$$

Proof. Let $x_0 < x_1 < ... < x_n$ be arbitrary points of the real line. We have

$$\sum_{i=1}^{n} |r(x_i) - r(x_{i-1})|$$

$$= \sum_{i=1}^{n} \left| \int_{-\infty}^{\infty} p(x_i - u)q(u)du - \int_{-\infty}^{\infty} p(x_{i-1} - u)q(u)du \right|$$

$$= \sum_{i=1}^{n} \left| \int_{-\infty}^{\infty} [p(x_i - u) - p(x_{i-1} - u)]q(u)du \right|$$

$$\leq \int_{-\infty}^{\infty} \sum_{i=1}^{n} |p(x_i - u) - p(x_{i-1} - u)|q(u)du \leq \mathbf{V}(p)\int_{-\infty}^{\infty} q(u)du = \mathbf{V}(p).$$

Since n and $x_0, x_1, ..., x_n$ are arbitrary, this implies that

$$\mathbf{V}(r) \leq \mathbf{V}(p).$$

Analogously we obtain

$$\mathbf{V}(r) \leq \mathbf{V}(q).$$

□

Lemma 4. *Let $p(x)$ and $q(x)$ be two probability density functions, and $r(x)$ be their convolution. If $p(x)$ is n times differentiable, then*

$$\mathbf{V}(r^{(n)}) \le \mathbf{V}(p^{(n)}).$$

Proof of the lemma is analogous to that of Lemma 3: taking n derivatives of both sides of the equality

$$r(x) = \int_{-\infty}^{\infty} p(x-u)q(u)du$$

we obtain

$$r^{(n)}(x) = \int_{-\infty}^{\infty} p^{(n)}(x-u)q(u)du$$

and now we can repeat the proof of Lemma 3 replacing $p(x)$ by $p^{(n)}(x)$.

3. Proofs of the theorems

Proof of Theorem 1. Prove the first inequality. Since the set of densities, having a given total variation, is closed with respect to translation, it suffices, due to Lemma 1, to prove that

$$|\operatorname{Re} f(t)| \le \frac{\sin(t/\mathbf{V}(p))}{t/\mathbf{V}(p)} \tag{2}$$

for $|t| \le \pi \mathbf{V}(p)/2$. Let us fix an arbitrary t_0 such that $|t_0| \le \pi \mathbf{V}(p)/2$. Without loss of generality assume that $t_0 > 0$. The cases $\operatorname{Re} f(t_0) \ge 0$ and $\operatorname{Re} f(t_0) < 0$ should be considered separately. We consider only the first one: it will be seen that the second case can be treated in a similar way.

Thus, suppose that $\operatorname{Re} f(t_0) \ge 0$. Denote

$$B_n = \left\{ x : \frac{\pi n}{t_0} \le x \le \frac{\pi(n+1)}{t_0} \right\},$$

$$M_n = \sup_{x \in B_n} p(x), \; m_n = \inf_{x \in B_n} p(x), \; I_n = \int_{B_n} p(x)dx, \; n = 0, \pm 1, \pm 2, \ldots .$$

We have

$$\operatorname{Re} f(t_0) = \int_{-\infty}^{\infty} \cos(t_0 x) p(x) dx = \sum_{n=-\infty}^{\infty} \int_{B_n} \cos(t_0 x) p(x) dx. \tag{3}$$

Prove that

$$\int_{B_n} \cos(t_0 x) p(x) dx \le \int_{B_n} \cos(t_0 x) r_n(x) dx, \tag{4}$$

where

$$r_n(x) = \begin{cases} M_n - m_n & \text{for } x \in [\pi n/t_0, \pi n/t_0 + z_n], \\ 0 & \text{otherwise} \end{cases}$$

for even n and

$$r_n(x) = \begin{cases} M_n - m_n & \text{for } x \in [\pi(n+1)/t_0 - z_n, \pi(n+1)/t_0], \\ 0 & \text{otherwise} \end{cases}$$

for odd n, where

$$z_n = \min\left\{ \frac{\pi}{2t_0}, \frac{1}{M_n - m_n \left(I - \frac{\pi}{t_0} m_n\right)} \right\}.$$

Suppose that n is even (for odd n the proof is analogous). We have

$$\int_{B_n} \cos(t_0 x) p(x) dx = \int_{B_n} \cos(t_0 x)[p(x) - m_n] dx. \tag{5}$$

Consider separately two cases:

$$1) \ \frac{1}{M_n - m_n}\left(I - \frac{\pi}{t_0} m_n\right) \leq \frac{\pi}{2t_0}$$

and

$$2) \ \frac{1}{M_n - m_n}\left(I - \frac{\pi}{t_0} m_n\right) > \frac{\pi}{2t_0}.$$

1) In this case,

$$\int_{B_n} r_n(x) dx = I - \frac{\pi}{t_0} m_n = \int_{B_n} [p(x) - m_n] dx,$$

and, evidently,

$$r_n(x) \geq p(x) - m_n$$

for

$$\frac{\pi n}{t_0} \leq x \leq \frac{\pi n}{t_0} + z_n$$

and

$$r_n(x) = 0 \leq p(x) - m_n$$

for

$$\frac{\pi n}{t_0} + z_n < x \leq \frac{\pi(n+1)}{t_0}.$$

Therefore, due to Lemma 2 ($\cos(t_0 x)$ decreases on the interval B_n),

$$\int_{B_n} \cos(t_0 x) r_n(x) dx \geq \int_{B_n} \cos(t_0 x)[p(x) - m_n] dx.$$

Taking into account (5), we obtain (4).

2) In this case, since $\cos(t_0 x)$ is negative on the interval

$$\left(\frac{\pi}{2t_0} + \frac{\pi n}{t_0}, \frac{\pi(n+1)}{t_0}\right)$$

and positive on the interval

$$\left(\frac{\pi n}{t_0}, \frac{\pi}{2t_0} + \frac{\pi n}{t_0}\right),$$

we have

$$\int_{B_n} \cos(t_0 x)[p(x) - m_n]dx \le \int_{\pi n/t_0}^{\pi/2t_0+\pi n/t_0} \cos(t_0 x)[p(x) - m_n]dx$$

$$\le \int_{\pi n/t_0}^{\pi/2t_0+\pi n/t_0} \cos(t_0 x)[M_n - m_n]dx = \int_{B_n} \cos(t_0 x) r_n(x)dx.$$

Again, taking into account (5), we obtain (4).

Now, define functions $p_n(x)$ as follows. If n is even, then

$$p_n(x) = \begin{cases} r_n(x + \pi n/t_0) & \text{for } 0 < x \le \pi/2t_0, \\ 0 & \text{otherwise.} \end{cases}$$

If n is odd, then

$$p_n(x) = \begin{cases} r_n(x + \pi(n+1)/t_0) & \text{for } -\pi/2t_0 \le x < 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\int_{B_n} \cos(t_0 x) r_n(x)dx = \int_{-\infty}^{\infty} \cos(t_0 x) p_n(x)dx,$$

or, since cos is an even function,

$$\int_{B_n} \cos(t_0 x) r_n(x)dx = \int_{-\infty}^{\infty} \cos(t_0 x) \left(\frac{p_n(x) + p_n(-x)}{2}\right) dx. \tag{6}$$

Consider the function

$$q_n(x) = \left(\frac{p_n(x) + p_n(-x)}{2}\right).$$

It has bounded support, the interval $[-\pi/2t_0, \pi/2t_0]$, and

$$\max_x q_n(x) \le \frac{M_n - m_n}{2}. \tag{7}$$

Indeed, $p_n(x)$ and $p_n(-x)$ have non-intersecting supports, therefore

$$\max_x(p_n(x)+p_n(-x)) = \max\left\{\max_x p_n(x), \max_x p_n(-x)\right\} = M_n - m_n.$$

In addition, evidently,

$$\int_{-\infty}^{\infty} q_n(x)dx \le I_n. \tag{8}$$

From (3), (4) and (6) we obtain

$$\operatorname{Re} f(t_0) \le \int_{-\infty}^{\infty} \cos(t_0 x)\left(\sum_{n=-\infty}^{\infty} q_n(x)\right) dx. \tag{9}$$

Define

$$q(x) = \left(\sum_{n=-\infty}^{\infty} q_n(x)\right). \tag{10}$$

Then, due to (8),

$$I = \int_{-\infty}^{\infty} q(x)dx \le 1,$$

and, due to (7),

$$\sup_x q(x) \le \frac{1}{2}\sum_{n=-\infty}^{\infty}(M_n - m_n) \le \frac{1}{2}\mathbf{V}(p).$$

In addition, since each $p_n(x)$ vanishes outside the interval $[-\pi/2t_0, \pi/2t_0]$, the support of $q(x)$ belongs to this interval as well. Applying Theorem 2 of Ushakov (1997), we obtain

$$\int_{-\infty}^{\infty} \cos(tx)q(x)dx \le \frac{\mathbf{V}(p)}{t}\sin\frac{It}{\mathbf{V}(p)} \le \frac{\mathbf{V}(p)}{t}\sin\frac{t}{\mathbf{V}(p)}$$

for all $|t| \le \frac{\pi}{2(\pi/2t_0)} = t_0$, in particular,

$$\int_{-\infty}^{\infty} \cos(t_0 x)q(x)dx \le \frac{\mathbf{V}(p)}{t_0}\sin\frac{t_0}{\mathbf{V}(p)}. \tag{11}$$

From (9), (10) and (11) we finally obtain (2).

Now let us prove inequality (5). First we prove it in the case when $p(x)$ is differentiable. We have

$$f(t) = \int_{-\infty}^{\infty} e^{itx}p(x)dx = \frac{1}{it}\int_{-\infty}^{\infty} p(x)de^{itx}$$

$$= -\frac{1}{it}\int_{-\infty}^{\infty} e^{itx}dp(x) = -\frac{1}{it}\int_{-\infty}^{\infty} e^{itx}p'(x)dx$$

which implies

$$|f(t)| \leq \frac{1}{|t|} \int_{-\infty}^{\infty} |p'(x)| dx.$$

Now it suffices to observe that

$$\mathbf{V}(p) = \int_{-\infty}^{\infty} |p'(x)| dx.$$

Now let us consider the general case: $p(x)$ is not obligatory differentiable. Consider the convolution

$$p_\varepsilon(x) = \int_{-\infty}^{\infty} p(x-u) n_\varepsilon(u) du$$

where

$$n_\varepsilon(x) = \frac{1}{\sqrt{2\pi}\varepsilon} \exp\left\{-\frac{x^2}{2\varepsilon^2}\right\}$$

is the normal density function with zero mean and variance ε^2. The function $p_\varepsilon(x)$ is differentiable because $n_\varepsilon(x)$ is differentiable hence

$$\left| f(t) e^{-\varepsilon^2 t^2/2} \right| \leq \frac{\mathbf{V}(p_\varepsilon)}{|t|}$$

or, taking into account Lemma 3,

$$\left| f(t) e^{-\varepsilon^2 t^2/2} \right| \leq \frac{\mathbf{V}(p)}{|t|}.$$

Let $\varepsilon \to 0$, then we finally obtain

$$|f(t)| \leq \frac{\mathbf{V}(p)}{|t|}.$$

□

Proof of Theorem 2. The proof is analogous to that of inequality (5). First, suppose that $p^{(n-1)}(x)$ is differentiable (i.e. $p(x)$ is n times differentiable). The procedure, which was used in the proof of inequality (5) can be repeated as many times as many derivatives of $p(x)$ exist. More exactly, if $p(x)$ is n times differentiable, and its first $n-1$ derivatives satisfy the condition

$$\lim_{|x| \to \infty} p^{(k)}(x) = 0, \quad k = 1, 2, ..., n-1,$$

then

$$f(t) = -\frac{1}{it} \int_{-\infty}^{\infty} e^{itx} p'(x) dx = -\frac{1}{(it)^2} \int_{-\infty}^{\infty} p'(x) de^{itx} = \frac{1}{(it)^2} \int_{-\infty}^{\infty} e^{itx} dp'(x)$$

$$= \frac{1}{(it)^2} \int_{-\infty}^{\infty} e^{itx} p''(x) dx = ... = \left(-\frac{1}{it}\right)^n \int_{-\infty}^{\infty} e^{itx} p^{(n)}(x) dx.$$

This implies that

$$|f(t)| \le \frac{\mathbf{V}(p^{(n-1)})}{|t|^n}.$$

Transition to the case when $p^{(n-1)}(x)$ is not differentiable can be performed in exactly the same way as in the proof of inequality (5).

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