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Computation of Life Insurance Reserves under Fractional Hull-White Interest Rates

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Abstract

Hull-White interest rate models driven by fractional Brownian motion with Hurst parameter $H \neq 1/2$ is applied to life insurance policies. The theory of life insurance policies under stochastic interest rates is thus generalized to a wider class of interest rate models. Utilizing the theory of markets with small proportional transaction costs, where it is possible to avoid arbitrage even when the market noise is driven by fractional Brownian motion, we derive formulas for the reserves of life insurance policies under fractional Hull-White interest rates. Single premiums for a theoretical pension policy under a fractional Vasicek model is computed and a sensitivity analysis is carried out. The results of the analysis suggests that persistence in the interest rates might increase the single premiums substantially and thus prose a threat to a insurance company's solvency.

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Chapter 1

Introduction

The business of the insurance industry is the future. And, at risk of stating the obvious, the future is, if not unknowable, at least uncertain and therefore one will have to resort to more or less educated guessing when talking about it. This is the raison d'être of actuarial science.

The field of life insurance deals with facets of risk that often span decades into the future and so a good picture of these risks is important to have from the outset, for instance when the premium of a life insurance policy is calculated. In life insurance and finance, models based on Brownian motion is well known and widely used to get such a picture. However, these may fail to capture important features of real world phenomena such as reported persistency in interest rates ([McC+04]) or roughness in stock price volatility ([GJR18].

Fractional Brownian motion is a stochastic process that can model these features, but as usual, more realistic models comes at a cost. In the case of fractional interest rates, the mathematical machinery used in connection with the Markovian non-fractional Brownian motion breaks down faced with the non-Markovianity of fractional Brownian motion.

Some of this machinery might be salvaged however, by modifying the market model usually applied in the Markovian case and incorporate transaction costs into our model. These changes, luckily, is not a simplification and so the resulting market model might be more realistic. This thesis applies the works of [Oha09] and [FKZ13] regarding bond markets with transaction costs and we use them to price life insurance policies under fractional Hull-White interest rates. Along the way, we review material on Brownian motion, mathematical finance, life insurance and fractional Brownian motion. The rest of the text is organised as follows.

Chapter 2 : This chapter is devoted to a review of stochastic analysis and mathematical finance with respect to non-fractional Brownian motion. Gaussian stochastic processes and the multidimensional Itô integral are discussed. We also study some stochastic differential equations and review how these concepts are utilized in the field of mathematical finance.

Chapter 3 : We review the mathematical basis of life insurance. We discuss Markov chains and price a pension policy with deterministic interest rates. We also review some stochastic interest rate models based on Brownian motion and discuss life insurance reserves under stochastic interest rates.

Chapter 4 : Stochastic analysis with respect to fractional Brownian motion is reviewed and we show how fractional Brownian motion is not a martingale, unless in the special case of "standard" Brownian motion. We also discuss fractional calculus and multidimensional integrals with fractional Brownian motion as integrators.

Chapter 5 : Following [Bia+10], we demonstrate how one can make an arbitrage in a Black-Scholes market if fractional Brownian motion is the driving noise. We then review the Heath-Jarrow-Morton forward curve model and discuss the Musiela parametrization of said model. Then, following [Oha09], we discuss a forward curve based on fractional Brownian motion and mathematical finance under proportional transaction costs. Using these concepts, we follow [FKZ13] in modelling fractional Hull-White short rates under proportional transaction costs and show how these can be made arbitrage free under an average risk-neutral-measure.

Chapter 6 : We continue with fractional interest rates and follow[Fin11] in discussing the conditional distribution of some processes related to fractional Brownian motion and deriving a closed price for zero-coupon bonds for fractional Hull-White short rates.

Chapter 7 : Utilizing the above theory, life insurance reserves under the fractional Hull-White model is derived and we provide an example of a life insurance policy under fractional Vasicek interest rates. We then discuss some numerical results regarding the distribution of the fractional Vasicek short rates and provide a sensitivity analysis with respect to the single premiums of the life insurance reserves.

Chapter 2

Stochastic calculus and mathematical finance with respect to Brownian motion

This chapter aims at reviewing the fundamentals of stochastic analysis with respect to Brownian motion as well as an introduction to financial mathematics. The material is mainly collected from [LL96], [Wal12] an to some extent [Kol12] and interested readers may consult these works for further details.

2.1 Preliminaries, stochastic processes and martingales

We start (almost) at the bottom with stochastic processes. Throughout this thesis we will work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, unless otherwise stated.

Definition 2.1.1 (Stochastic process). *Let $\mathcal{I}$ be a non-empty index set. A stochastic process X is a collection of random variables $X = \{X(i), i \in \mathcal{I}\}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.*

As we work in a finance- and insurance setting the index should be thought of as time and we will usually write $X = \{X(t), t \in [0, T]\}$ or sometimes $X = \{X_t, t \in [0, T]\}$

We shall need the notion of a filtration to model how the information generated by a stochastic process is growing (heuristically; for every new step we know more about the movement of the process). In the context of finance this can be thought of as the market information flow.

Definition 2.1.2 (Filtration). *Let $\{\mathcal{F}_t, t \in [0, T]\}$ be a family of σ-algebras. If $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ for all $0 \leq t_1 \leq t_2 \leq T$ then $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration.*

If the stochastic process knows its path up until now we say that is it adapted.

Definition 2.1.3 (Adapted stochastic process). *A stochastic process X is adapted to the filtration $\mathcal{F}_t$ if, for each $t \geq 0$, $X(t)$ is an $\mathcal{F}_t$-measurable random variable.*

A special class of stochastic processes is that of Gaussian processes. They will come in to play later, but we define them here.

Definition 2.1.4 (Gaussian process). *An $\mathbb{R}$-valued stochastic process X is called Gaussian if, for any integer $k \geq 1$ and real numbers $0 \leq t_1 \leq t_2 \leq ... \leq t_k < \infty$, the random vector $(X(t_1), X(t_2), X(t_3), ..., X(t_k))$ has a joint normal distribution.*

In insurance, finance and economics a very important and useful class of stochastic processes is given by ("standard", sub- and super) martingales

Definition 2.1.5 (Martingale). *Let X be a stochastic process adapted to a filtration $\mathcal{F}_t$. X is a martingale if*

- *for each t, $X(t)$ is integrable;*
- *for each $s < t$, $E[X(t)|\mathcal{F}_s] = X(s)$*

The stochastic process is a sub- or supermartingale if the last condition is replaced by $X_s \geq E[X(t)|\mathcal{F}_s]$ (for submartingales), or $X(s) \leq E[X_t|\mathcal{F}_s]$ (for supermartingales). We see that a stochastic process is a martingale if and only if it is both a sub- and a supermartingale.

An extension of the class of martingales, which proves to be "good" stochastic integrators, are semimartingales. They are in fact the largest class of integrators for which the Itô integral (as a local martingale itself) can be defined and as such, it will prove challenging when the process one deals with is not a semimartingale (see the discussions later with respect to non-semimartingale integrators). We need a couple more concepts to define them.

First, a stopping time is a time where something interesting happens and where we know if it has happened or not. One can assign a stopping rule that determines the stopping time. For instance an investor can decide that if a stock drops below a certain value she will sell. The time τ that the stock drop below the value becomes a stopping time.

Definition 2.1.6 (Stopping time). *A random variable τ with values in $[0, \infty]$ is a stopping time if*

$$\{\tau \leq t\} \in \mathcal{F}_t, \quad \textit{for all } t \geq 0$$

If we are only interested in the process' history up until the stopping time τ we only need the stopping time σ-algebra.

Definition 2.1.7 (Stopping σ-algebra $\mathcal{F}_\tau$). *For a stopping time τ we define the stopping σ-algebra as*

$$\mathcal{F}_\tau := \{A \in \mathcal{A} : A \cap \{\tau \leq t\} \textit{ for all } t \geq 0\}$$

We are now ready to define a local martingale:

Definition 2.1.8 (Local martingale). *An $\mathcal{F}$-adapted càdlàg process M is a local martingale if there are increasing stopping times $\tau_n, n \geq 1$ with $\tau_n \to \infty$ as $n \to \infty$ with probability 1 such that*

- *the stopped process*

  $$M^{\tau_n}(t) := M_{\min(t,\tau_n)} \mathbf{1}_{\{\tau_n > 0\}}$$

  is a martingale for each n, that is

  $$\mathbb{E}[M^{\tau_n}(t)|\mathcal{F}_s] = M^{\tau_n}(s), \quad t \geq s$$

  for each $n \geq 1$, and

- *$M^{\tau_n}(t)$ is uniformly integrable. That is*

$$\sup_{t \geq 0} \mathbb{E}[|M^{\tau_n}(t)|\mathbf{1}_{\{|M^{\tau_n}(t)| \geq m\}}] \to 0, \quad \textit{as } m \to \infty$$

A semimartingale is the sum of a local martingale and a càdlàg process. More precisely:

Definition 2.1.9 (Semimartingale). *An $\mathcal{F}$-adapted càdlàg process X is a semimartingale if*

$$X(t) = X(0) + A(t) + M(t), \quad t \geq 0,$$

where A and M are càdlàg adapted processes such that A is of bounded variation (with probability one) and M is a local martingale.

It can be shown that if $A(0) = M(0) = 0$ and A is continuous then the decomposition of X in the above is unique.

2.2 Brownian motion

We now turn to a particularly useful class of Gaussian processes, the Brownian motion, used extensively in many fields, especially insurance and finance. We start with the definition:

Definition 2.2.1 (Standard Brownian Motion). *A standard Brownian motion is a stochastic process $\{W_t, t \geq 0\}$ which satisfies the following properties:*

- $W_0 = 0$
- *$W_{t+s} - W_s$ is $N(0,t)$ distributed*
- *If $0 \leq t_1 < t_2 < ... < t_n$, then the increments $W_{t_1}, W_{t_2} - W_{t_1}, ..., W_{t_n} - W_{t_{n-1}}$ are independent.*

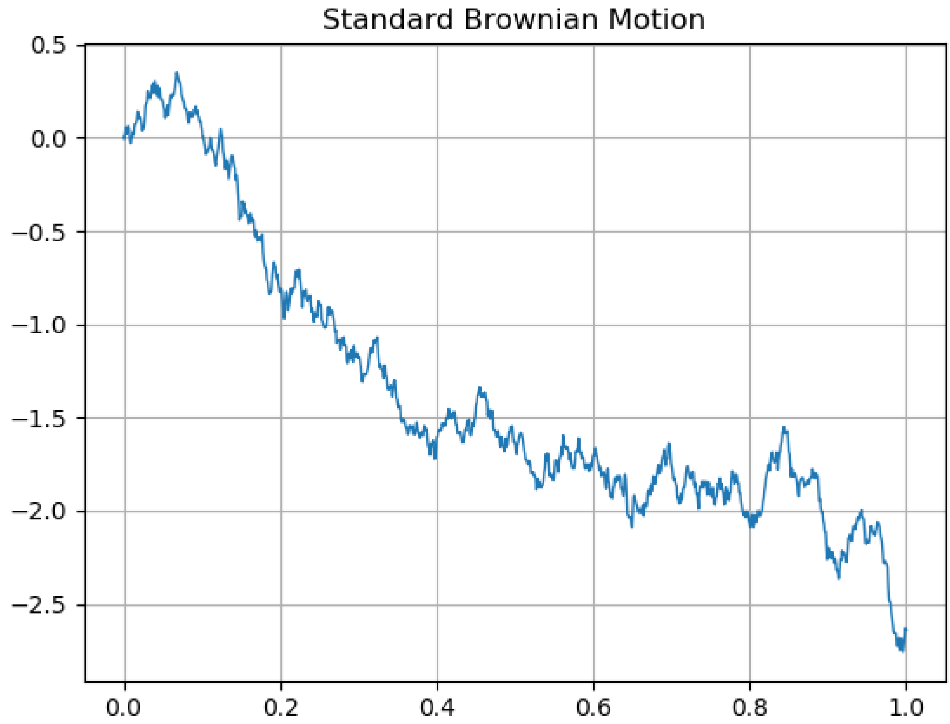


Figure 2.1: Realizations of sample paths for standard Brownian motion

We shall also need the notion of a Brownian motion with respect to a filtration $\mathcal{F}_t$.

Definition 2.2.2 ($\mathcal{F}_t$-Brownian motion). *A real-valued continuous stochastic process* $\{W_t, t \geq 0\}$ *is an* $\mathcal{F}_t$*-Brownian motion if it satisfies*

- W_t *is* $\mathcal{F}_t$*-measurable for* $t \geq 0$
- $W_t - W_s$ *is independent of the* σ*-algebra* $\mathcal{F}_s$ *for* $s \leq t$.
- $W_t - W_s$ *and* $W_{t-s} - W_0$ *have the same distribution*

So $\{W_t, t \geq 0\}$ is a Gaussian process with mean zero and covariance $E[B_s B_t] = \min(s,t)$ as well as quadratic variation $E[W_t^2] = t$. More importantly, it is a martingale and therefore also a semimartingale and we can apply Itò-calculus.

The Lèvy characterization theorem states says that in fact Brownian motion is the only local martingale with quadratic variation being identical to time t;

Theorem 2.2.3 (Lèvy's characterization theorem). *Let* $W = \{W_t, t \in [0,T]\}$ *be a stochastic process on* $(\Omega, \mathcal{A}), \mathbb{P}$ *and* $\mathcal{F}$ *its natural filtration. Then the following are equivalent:*

- W *is a Brownian motion*
- W *is a* $(\mathcal{F}, \mathbb{P})$*-martingale with* $W_0 = 0$ $\mathbb{P} - a.s$ *and quadratic variation* $[W,W]_t = t$ $\mathbb{P} - a.s$

Proof. A variant of this theorem along with the proof is provided in [KS91] (Theorem 3.16) □

2.3 Stochastic integration with respect to Brownian motion

The paths of Brownian motion are almost surely not differentiable at any point. This poses a problem as the non-differentiability makes it difficult to give meaning to integrals of the type

$$\int_0^t f(s)dW_s = \int_+^t f(s)\frac{dW_s}{ds}ds,$$

where W is a Brownian motion, as $\frac{dW_s}{ds}$ does not exist as a classical process. We want to define an integral with respect to W_s, $s \geq 0$. The overall strategy will be to approximate a process via simpler processes, very much like the Riemann integral in classical calculus. We start out by defining the stochastic integral for a small class of stochastic processes and then extend it to Brownian motion. We start with simple processes:

Definition 2.3.1 (Simple process). *A (uniformly bounded) process $\{H(t), t \in [0,T]\}$ is called a simple process if it can be written as*

$$H(t,\omega) := \sum_{i=1}^{n-1} f_i(\omega)\mathbf{1}_{(t_i,t_{i+1}]}(t)$$

where $0 = t_1 \leq t_2 \leq ... \leq t_n = T$ and f_i is $\mathcal{F}_{t_{i-1}}$-measurable.

We can now construct an integral with simple processes as integrands:

$$\int_0^t H(s)dW_s := \sum_{i=1}^{n} f_i(W_{\min(t,t_i)} - W_{\min(t,t_{i-1})})$$

this "simple" integral will be denoted by $I(H)$. It has the following properties:

Proposition 2.3.2. *If H is a simple process then*

- *The process $\{\int_0^t H(s)\,dW_s, t \in [0,T]\}$ is a continuous $\mathcal{F}_t$-martingale*
- $\mathbb{E}\left(\left(\int_0^t H(s)\,dW_s\right)^2\right) = \mathbb{E}\left(\int_0^t H(s)^2\,ds\right)$
- $\mathbb{E}\left(\sup_{t\leq T}\left|\int_0^t H(s)\,dW_s\right|^2\right) \leq 4\mathbb{E}\left(\int_0^T H(s)^2\,ds\right)$

Proof. See [LL96], Proposition 3.4.2 □

We will extend this integral to also encompass a larger class of adapted processes H as integrands, which we will denote by $\mathcal{H}$ and which is defined as

$$\mathcal{H} = \left\{\{H(t), t \in [0,T]\}, \text{ measurable and } \mathcal{F}_t\text{-adapted process, where } \mathbb{E}\left[\int_0^T (H(s))^2 ds\right] < \infty\right\}$$

where measurable is in the sense of

$$H : \Omega \times [0,T] \to \mathbb{R} \text{ is } \mathcal{F} \otimes \mathcal{B}([0,T]) \text{ -measurable}$$

where $\mathcal{F}$ is a σ-algebra on Ω and $\mathcal{B}$ is the Borel sigma algebra.

Proposition 2.3.3. *Consider a $\mathcal{F}_t$ - Brownian motion W. There exists a unique linear mapping J from $\mathcal{H}$ to the space of continuous $\mathcal{F}_t$-martingales defined on $[0,T]$ s.t:*

- *If H is a simple process, then $J(H)(t) = I(H)(t)$ a.s*
- *If $t \leq T$, then $\mathbb{E}[J(H)(t)^2)] = \mathbb{E}\Big[\int_0^t H(s)^2\, ds\Big]$*

the linear mapping is unique in the following sense: if both mappings J and J' satisfy the properties above, then we have for all $t \in [0, T]$

$$J(H)(t) = J'(H)(t) \; a.s$$

Proof. See [LL96], Proposition 3.4.4 □

The integral

$$\int_0^t H(s)\, dW_s$$

will be denoted by $J(H)(t)$ for $H \in \mathcal{H}$. The stochastic integral $J(H)(t)$ satisfies the following properties:

Proposition 2.3.4. *Let $H \in \mathcal{H}$, then*

-
$$\mathbb{E}\Big(\sup_{t\leq T}\Big|\int_0^t H(s)\, dW_s\Big|\Big) \leq 4\,\mathbb{E}\Big(\int_0^T H(s)^2\, ds\Big)$$
- *If τ is a $\mathcal{F}_t$-stopping time, then*

$$\int_0^\tau H(s)\, dW_s = \int_0^T \mathbf{1}_{s\leq\tau} H(s)\, dW_s, \quad a.s \tag{2.1}$$

Proof. See [LL96], Proposition 3.4.5. □

The integrability condition on the processes in $\mathcal{H}$ is quite strong, and so we define a class of integrand processes $\widetilde{\mathcal{H}}$ for which the requirements are a bit weaker:

$$\widetilde{\mathcal{H}} = \Big\{\{H(t), t \in [0, T]\},\, \mathcal{F}_t\text{-adapted, where } \int_0^T H(s)^2 ds < \infty \text{ a.e}\Big\}$$

The next propositions allows us to extend the integral from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$:

Proposition 2.3.5. *There exists a unique linear mapping $\tilde{J}$ from $\widetilde{\mathcal{H}}$ into the vector space of continuous processes defined on the interval $[0, T]$ s.t*

- *If H is a simple process then for all $t \in [0, T]$ we have*

$$\tilde{J}(H)(t) = I(H)(t), \quad a.s \tag{2.2}$$

- *If $\{H^n, n \geq 0\}$ is a sequence of processes in $\widetilde{\mathcal{H}}$ s.t the integral $\int_0^T (H^n(s))^2\, ds$ converges to 0 in probability, then $\sup_{t\leq T} |\tilde{J}(H^n)(t)|$ converges to 0 in probability.*

the process $\{\tilde{J}(H)(t), t \in [0, T]\}$ is not necessarily a martingale

Proof. See [LL96], proposition 3.4.6 and remark 3.4.7. □

We will from now on use the notation

$$\int_0^t H(s)\, dW_s = \widetilde{J}(H)(t)$$

With these assumptions regarding integrability in mind, we turn to Itô calculus. We start by defining an Itô process, which is a sum of an integral with respect to a Brownian motion and an integral with respect to time.

Definition 2.3.6 (Itô process). *A stochastic process $X = \{X_t, t \in [0,T]\}$ is an $\mathbb{R}$-valued Itô process if it can be written as*

$$X(t) = X(0) + \int_0^t G(s)ds + \int_0^t H(s)\, dW_s, \tag{2.3}$$

where

- *$X(0)$ is $\mathcal{F}_t$-measurable*
- *G and H are $\mathcal{F}_t$-adapted processes.*
- *$\int_0^T |G(s)|\, ds < \infty$ a.s*
- *$\int_0^T |H(s)|^2\, ds < \infty$ a.s*

The process defined in Equation (2.3) is unique, as the following proposition shows:

Proposition 2.3.7. *Let M be a continuous martingale s.t*

$$M(t) = \int_0^t G(s)\, ds,$$

where the following holds a.s:

$$\int_0^T |G(s)|\, ds < \infty.$$

Then for all $t \leq T$, $M(t) = 0$, a.e.

The implication of this is that if

$$\begin{aligned} X(t) &= X(0) + \int_0^t G(s)\, ds + \int_0^t H(s)\, dW_s \\ &= X'(0) + \int_0^t G'(s)\, ds + \int_0^t H'(s)\, dW_s, \end{aligned}$$

then

$$X(0) = X'(0)\, \mathbb{P}\text{-a.s}, \quad H(s) = H'(s)\, ds \times d\mathbb{P} \text{ a.e}, \quad G(s) = G'(s)\, ds \times d\mathbb{P} \text{ a.e}$$

and also that if X is a martingale of the form $X(0) + \int_0^t G(s)\, ds + \int_0^t H(s)\, dW_s$ then $G(t) = 0$ $dt \times d\mathbb{P}$ a.e. Remember that $[W,W]_t = t$ $\mathbb{P} - a.s$ implies $d[W,W]_t = dt$. We state the Itô formula for Brownian motion:

Theorem 2.3.8 (Itò's formula w.r.t Brownian motion). *Let $f \in C^{1,2}([0,T] \times \mathbb{R})$ and W a $(\mathcal{F}_t)$-Brownian motion. Then*

$$f(t, W_t) = f(0, B_0) + \int_0^t \Big(\frac{\partial}{\partial s} f(s, W_s) + \frac{1}{2} \frac{\partial^2}{\partial x^2} f(s, W_s) \Big) ds + \int_0^t \frac{\partial}{\partial x} f(s, W_s) dW_s$$

or, on differential form:

$$df(t, X_t) = \frac{\partial}{\partial t} f(t, X_t) + \frac{\partial}{\partial x} f(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} f(t, X_t) d[X, X]_t$$

A useful integration by parts formula is given by

Proposition 2.3.9. *Let X and Y be two Itô processes. Then*

$$X(t)Y(t) = X(0)Y(0) + \int_0^t X(s) dY_s + \int_0^t Y(s) dX_s + [X, Y]_t$$

2.3.1 Multidimensional and infinite dimensional Itò calculus

Interest rate modelling often relies on the use of a multidimensional or even infinite dimensional Brownian motions, and so an introduction to multidimensional Itô calculus is useful. The following material on the subject is inspired by [GM11].

Definition 2.3.10 (Cylindrical Gaussian random variable). *Let U be a separable Hilbert space with inner product $\langle \cdot \rangle$ and norm $||\cdot||^2$. X is a cylindrical standard Gaussian random variable on U if $X : U \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ if*

- *The mapping X is linear.*
- *For a $u \in U$, $X(u)$ is a Gaussian random variable with mean zero and variance $||u||^2$*
- *if $u, u' \in U$ are orthogonal, i.e, $\langle u, u' \rangle = 0$, then the random variables $X(u)$ and $X(u')$ are independent.*

Given an orthonormal basis on U, $(e_n, n \in \mathbb{N})$ then $\{X(e_n)\}_{i=1}^{\infty}$ is a sequence of independent Gaussian random variables with mean zero and variance one. We can then represent X as the $\mathbb{P}$-a.s convergent sum

$$X(u) = \sum_{n=1}^{\infty} \langle u, e_n \rangle X(e_n) \tag{2.4}$$

The cylindrical Brownian motion can be defined in almost the same way;

Definition 2.3.11 (Cylindrical Brownian motion). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and U be a separable Hilbert space with inner product $\langle \cdot \rangle$ and norm $|| \cdot ||^2$. The process W is a cylindrical Brownian motion if*

- *For a $t \geq 0$ we have that the mapping $W_t : U \to L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ is linear*
- *For an $u \in U$, $||u||^{-1} \cdot W_t(u)$ is a $\mathcal{F}_t$-Brownian motion*
- *For $u, u' \in U$ and $t \geq 0$ we have*

$$\mathbb{E}[W_t(u) W_t(u')] = t \langle u, u' \rangle$$

As we can scale this to be a standard cylindrical Gaussian random variable by setting $W_t/\sqrt{t}$, this also has a $\mathbb{P}$-a.s convergent sum representation akin to Equation (2.4) given by

$$W_t(u) = \sum_{n=1}^{\infty} \langle u, e_n \rangle \beta_t(e_n), \tag{2.5}$$

where $\beta_t(e_n) := W_t(e_n)$.

The stochastic integral with respect to cylindrical Brownian motion can be defined as:

Lemma 2.3.12 (Stochastic integral wrt cylindrical Brownian motion). *Let W_t be a cylindrical Brownian motion and $G : [0,T] \times \Omega \to \mathcal{L}_2(U,V)$ a square-integrable $\mathcal{F}_t$-adapted process, where $\mathcal{L}_2(U,V)$ is the family of Hilbert-Schmidt linear operators from U to V. Further, let $(e_n, n \in \mathbb{N})$ be a orthonormal basis on U. Then the stochastic integral with respect to cylindrical Brownian motion is defined as*

$$\int_0^T G(s)dW_s = \sum_{n=1}^{\infty} \int_0^T G(s)e_n d\beta_t(e_n) \tag{2.6}$$

in $\mathcal{L}^2(\Omega; V)$.

Proof. See [GM11], Lemma 2.8 □

2.4 Stochastic differential equations

Consider the stochastic differential equation (SDE):

$$S(t) = x_0 + \int_0^t S_s(\mu\, ds + \sigma\, dW_s), \quad 0 \leq t < T, T \in \mathbb{R}, \tag{2.7}$$

where $\sigma, \mu \in \mathbb{R}$, $\{W_t, t \geq 0\}$. The unique process satisfying Equation (2.7) is given by

$$S(t) = x_0 \exp((\mu - \sigma^2/2)t + \sigma W_t). \tag{2.8}$$

This can be deduced using the integration by parts formula in Itô calculus. However, we do want to study more general SDE's driven by Brownian motion. We will consider equations of the type:

$$X_t = Z + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

for Borel-measurable functions b and σ.

Definition 2.4.1 (Conditions on the solution). *Consider the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t, 0 \leq t \leq T\}$. A solution to Equation (2.8) is an $\mathcal{F}_t$-adapted stochastic process $\{X_t, t \geq 0\}$ that satisfies*

- *$b(s, X_s)$ and $\sigma(s, X_s)$ are integrable:*

$$\int_0^t |b(s, X_s)|ds < +\infty, \quad \mathbb{P} - a.s$$
$$\int_0^t |\sigma(s, X_s)|^2 ds < +\infty, \quad \mathbb{P} - a.s$$

- *$\{X_t, t \geq 0\}$ satisfies;*

$$X_t = Z + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad \forall t > 0, \mathbb{P} - a.s$$

where $b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Borel-measurable functions, Z a $\mathcal{F}_0$-measurable random variable and $\{W_t, t \geq 0\}$ an $\mathcal{F}_t$-Brownian motion.

We need some further constraints on b and σ to ensure the existence and uniqueness of a solution.

Theorem 2.4.2 (Existence and uniqueness). *Let* $b, \sigma \in C$ *and* $K < +\infty \in \mathbb{R}$. *Then if*

- $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|$ *(Lipschitz continuity)*
- $|b(t,x)| + |\sigma(t,x)| \leq K(1+|x|)$ *(Linear growth)*
- $\mathbb{E}[Z^2] < \infty$

Then the Equation (2.8) *has a unique (pathwise) solution in the interval* $[0,T]$. *The solution* $\{X_s, 0 \leq s \leq T\}$ *satisfies*

$$\mathbb{E}[\sup_{s\in[0,T]} |X_s|^2] < +\infty$$

Proof. See Theorem 2.9 in [KS91] ☐

In the following we will denote the solution starting x at time t by $\{X_s^{t,x}, s \geq t\}$ and X^x the solution starting from x at time 0. So $\{X_s^{t,x}, s \geq t\}$ satisfies

$$X_s^{t,x} = x + \int_t^s b(u, X_u^{t,x})du + \int_t^s \sigma(u, X_u^{t,x})dW_u$$

For financial applications a very important property of the solutions to equation Equation (2.8) is the Markov property. In plainspeak it says that what happens after time t only depends on the state at time t and not the processes' past. In finance the property can be utilized in, for instance, options pricing; if the asset price process is Markovian, the option price on that asset only depends on the assets price today. This is true in the Black-Scholes market model which we will encounter later, but false in markets where the noise is driven by fractional Brownian motion, which we will also encounter later. We begin with a more precise definition of Markovianity:

Definition 2.4.3 (Markov property). *An* $\mathcal{F}_t$*-process* $\{X(t), t \geq 0\}$ *satisfies the Markov property if, for any bounded Borel function* f *and for any* s *and* t *such that* $s \leq t$ *we have*

$$\mathbb{E}[f(X(t))|\mathcal{F}_s] = \mathbb{E}[f(X(t))|X(s)]$$

Lemma 2.4.4. *If* $\{X^{t,x}(s), s \geq t\}$ *exists and is unique and* $s \geq t$ *we have ,* $\mathbb{P}$*-a.s*

$$X^{0,x}(s) = X^{t,X(t)^x}(s)$$

Proof. [LL96], lemma 3.5.6. ☐

This is the *flow* property of the solution. If the above lemma is satisfied, which it is for solutions of Equation (2.8), we can state the Markov property as

Theorem 2.4.5. *Let* $\{X(t), t \geq 0\}$ *be a solution to Equation* (2.8). *It is a Markov process with respect to the Brownian filtration* $\mathcal{F}_t$. *Furthermore, for any bounded Borel function* f *we have,* $\mathbb{P}$*-a.s*

$$\mathbb{E}[f(X(t))|\mathcal{F}_s] = \psi(X(s)) \tag{2.9}$$

with $\psi(x) = \mathbb{E}[f(X^{s,x}(t))]$.

Proof. [LL96], theorem 3.5.7. ☐

2.5 Equivalent probability measures

It will later be useful to manipulate stochastic processes while retaining certain properties. If a process is a Brownian motion, the Girsanov theorem says that we can add a drift term and the new process still remains a Brownian motion, albeit under a different (but equivalent) probability measure. We define the latter:

Definition 2.5.1 (Equivalent measures). *Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures. We say that they are equivalent if they operate on the same sample space Ω and if we let $A \in \mathcal{F}$ on Ω we have*

$$\mathbb{P}(A) > 0 \iff \mathbb{Q}(A) > 0$$

Theorem 2.5.2 (Girsanov's theorem). *Let Y be an Itô-process of the form*

$$dY(t) = g(t)dt + dW_t, \quad Y_0 = 0, \quad t \in [0, T]$$

and set

$$M(t) = \exp(-\int_0^t g(s)dW_s - \frac{1}{2}\int_0^t g(s)^2 ds), \quad 0 \in [0, T] \tag{2.10}$$

Assume that M is a martingale with respect to $\mathbb{P}$ and define the measure $\mathbb{Q}$ on $\mathcal{F}_T$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = M(T)$. Then $\mathbb{Q}$ is a probability measure on $\mathcal{F}_T$ and Y is a Brownian motion under $\mathbb{Q}$.

Proof. See e.g Theorem 5.1 in [KS91] . □

As we will see later, this is a very important result in view of applications to mathematical finance. In danger of jumping too far ahead, it makes it possible to obtain discounted stock prices (modelled by an Itô-process) as martingales which in turn makes it possible to find a fair price for an option (given certain conditions).

To verify that M in fact is a martingale we can use the sufficient condition in the following lemma:

Lemma 2.5.3 (Novikov condition). *Let M be as in Girsanov's theorem. If*

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^T |g(t)|^2 dt\Big)\Big] < \infty,$$

then M is a martingale and $\mathbb{E}[M(t)] = \mathbb{E}[M(0)] = 1$

Proof. See [KS91], Corollary 5.14 □

2.6 Mathematical finance based on Brownian motion

In this section we want to use the preceding material and highlight the principles behind no-arbitrage pricing, which will be heavily utilized later in this thesis. We will provide some important definitions and results used to price bonds and financial derivatives.

Portfolios, and their value

Our market will consist of the following two assets:

- A risky asset, that is a stock or a fund with price $S(t)$ at time t. With dynamics given by

$$S(t) = x + \int_0^t \mu(s, S(s))S(0)\, ds + \int_0^t \sigma(s, S(s))s(s)\, dW_s$$

- A riskless fixed-income asset given by $B(t)$ with dynamics

$$\frac{dB(t)}{B(t)} = r(t)dt,$$

where $r(t)$ is a (for now) deterministic interest rate process.

Note that

$$B(t) = \exp\left(\int_0^t r(s)ds\right), \quad t \in [0, T]$$

is the solution of the latter.

We need a notion of the units invested in each asset at each time t. Let ϕ_t^0 be the units invested in the riskless asset at time t and analogously let ϕ_t^1 be the units invested in the risky asset. Both are adapted stochastic processes in $(\Omega, \mathcal{F}_T, \mathbb{P})$ and together they define a portfolio $\phi = (\phi^0, \phi^1)$.

Definition 2.6.1. *The couple ϕ is said to be a portfolio if it is $\mathcal{F}$-adapted and*

$$\mathbb{E}\left[\int_0^T |\phi_t^0 r(t)B(t)|dt\right] < \infty,$$
$$\mathbb{E}\left[\int_0^T |\phi_t^1 \mu(t, S(t))S(t)|dt\right] < \infty,$$
$$\mathbb{E}\left[\int_0^T |\phi_t^1 \mu(t, S(t))S(t)|^2 dt\right] < \infty$$

If we follow the trading strategy represented by the portfolio ϕ we will gain (or lose) some wealth described by the value process V_t^ϕ:

Definition 2.6.2 (Value process). *The value of the portfolio ϕ is given by*

$$V_t^\phi = \phi_t^0 B_t + \phi_t^1 S(t), \quad t \in [0, T]$$

It is also reasonable to put a limit on cash to ensure that the investor cannot lose an infinite amount of money;

Definition 2.6.3 (Admissible portfolio). *If there exists some $K \leq 0 \in \mathbb{R}$ such that*

$$V_t^\phi \geq K \quad a.s$$

we say that the portfolio is admissible

We would also like that the only gain (or loss) in wealth stems from the change of value of the assets and so in our model we are not allowed to deposit (or withdraw) more money into the process once it has started. Mathematically:

Definition 2.6.4 (Self Financing Portfolio). *We say that ϕ is a self-financing portfolio if*

$$dV_t^\phi = \phi_t^0 dB_t + \phi_t^1 dS(t), \quad t \in [0, T]$$

We discount with respect to the risk-less asset $B(t)$ to find the relative value of assets compared to each other. The discounted price process $\tilde{S}_t$ and the discounted value of the portfolio $\tilde{V}_t^\phi$ is defined by

$$\tilde{S}_t = \frac{S(t)}{B(t)}, \quad t \in [0, T]$$
$$\tilde{V}_t^\phi = \frac{V_t^\phi}{B(t)}, \quad t \in [0, T]$$

Fundamental theorems of asset pricing

If there exist a portfolio that allows us to make sure profit without taking risk we have an arbitrage opportunity in our hands.

Definition 2.6.5 (Arbitrage opportunity). *An arbitrage opportunity is a self-financing portfolio ϕ with*

$$V_0^\phi = 0$$
$$V_T^\phi \geq 0$$
$$\mathbb{P}[V_T^\phi > 0] > 0$$

The whole point of non-arbitrage pricing is to make the price so that we can rule out arbitrage opportunities. The fundamental theorems of asset pricing gives us conditions for when arbitrage pricing is fullfilled. The most important concept in this regard is that of an equivalent martingale measure:

Definition 2.6.6 (Equivalent Martingale Measure). *A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is a equivalent martingale measure (EMM) if the following holds:*

- *$\mathbb{Q}$ is equivalent to $\mathbb{P}$*
- *The discounted price process $\widetilde{S}_t$ is a $(\mathbb{Q}, \mathcal{F})$-martingale,*

that is, under the equivalent martingale measure we force the discounted price process of the asset to be a martingale and so "betting" on the discounted price process is a fair game. We now have the concepts needed to state the first fundamental theorem of asset pricing:

Theorem 2.6.7 (First Fundamental Theorem of Asset Pricing). *There are no arbitrage opportunities if and only if there exists an equivalent martingale measure $\mathbb{Q}$*

Proof. A general version of this theorem is discussed and proved in [DS06] (Theorem 2.2.7). □

This allows us to find a fair price for claims as long as we are able to find such an equivalent martingale measure. The Girsanov theorem from the chapter on stochastic analysis states the conditions which can be used to construct the martingale measure.

Definition 2.6.8 (Attainable contigent claim). *A contigent claim is a financial contract that pays the holder a nonnegative random amount H at time T (the exercise time). The random variable H is assumed to be $\mathcal{F}_T$-measurable with $\mathbb{E}[H^{2+\epsilon}] < \infty$ for some $\epsilon > 0$. The claim H is attainable if there exists a self-financing and admissible portfolio ϕ such that $V_T^{\phi} = H$, $\mathbb{P} - a.s.$ We call ϕ the replicating or hedging portfolio of H.*

The above definitions imply that two portfolios that lead to the same cash flow have the same value.

Definition 2.6.9 (Completeness). *A market is complete if all contingent claims are attainable.*

Heuristically we can say that if all the sources of noise can be traded, then the market is complete. The next theorem give a condition for when a market is complete:

Theorem 2.6.10 (Second Fundamental Theorem of Asset Pricing). *An arbitrage-free market is complete if and only if there exists a unique equivalent martingale measure.*

Proof. See e.g Theorem 1.3.4 in [LL96]. □

As mentioned above, pricing the claim $H = \max(P, S(t))$ by its expectation leads to arbitrage if we use the physical measure $\mathbb{P}$. However, the price of the claim can be given by the expectation under the equivalent martingale measure $\mathbb{Q}$. The fair price C of the claim H above is thus given by

$$C(t) = \mathbb{E}_{\mathbb{Q}}[\max(P, S(t))|\mathcal{F}_t].$$

where $\mathcal{F}_t$ is the market information up to time t (the filtration generated by the risky asset S) .

Chapter 3

Life insurance with stochastic interest rates

3.1 The life insurance setting

When someone buys a life insurance policy, the insured does not need to pay the full amount one will receive. There are two reasons for this: 1) is the time value of money and 2) not everyone who pays premiums will receive the money, as one may for instance outlive the terms of the contract (endowment) or die before the payments starts (pension). To make up for this the mathematical reserve, the amount of money the insurance company has to set aside can be computed as a conditional expectation conditioned on the state of the insured:

$$V_j(t, A_g) = \mathbb{E}\Big[\frac{1}{v(t)} \int_I v(s) dA_g(s) | X_t = j\Big], \tag{3.1}$$

where X_t is a Markov chain describing the state of the insured over the interval I. Here $X_t = j$ at the time of the calculation, I is the time interval, $A_g(t)$ is the sum of payments the insured is to receive when in state g and $v(s)$ is the discount factor. The aim of this chapter is to define these concepts more thoroughly and introduce policies with stochastic interest rates. The material is mainly gathered from [Bãn22], [Kol12] and [BM06]

3.1.1 Markov Chains and transition probabilities

The state of the insured is modelled by Markov chains. We start out by defining these and the concept of transition probabilities:

Definition 3.1.1 (Markov Chain). *Let $X_t \in S, t \in J \subset \mathbb{R}$ be a stochastic process on $(\Omega, \mathcal{A}, \mathbb{P})$. Then $X_t, t \in J$ is a Markov chain if*

$$\mathbb{P}(X_{t_{n+1}} = X_{t_1} = i_1, ..., X_{t_n} = i_n|) = \mathbb{P}(X_{t_{n+1}} = i_{n+1}|X_{t_n} = i_n)$$

for all $t_1 < t_2 < ... < t_{n+1} \in J, i_1, ..., i_{n+1} \in S$ with $\mathbb{P}(X_{t_1} = i_1, ..., X_{t_n} = i_n) \neq 0$.

The transition probabilities determine the probability that an insured switches from state i to j.

Definition 3.1.2 (Transition probabilities). *We call the functions*

$$p_{ij}(s,t) = \mathbb{P}(X_t = j | X_s = i), \quad s \geq t, \quad i,j \in S$$

transition probabilities. Here, $p_{ij}(s,t)$ *denotes the probability that* X *will be in state* j *at time* t *given that* X *was in state* i *at a previous time* s.

Transition probabilities put together in a matrix is aptly named a transition matrix:

Definition 3.1.3 (Transition matrix). *Let* $P(s,t) = \{p_{ij}(s,t)\}_{i,j \in S}$ *be a matrix with entries* $p_{ij}(s,t)$. *Then* P *is called a tranition matrix if*

- $p_{ij}(s,t) \geq 0$
- $\sum_{j \in S} p_{ij}(s,t) = 1$ *for all* $i \in S$
- $p_{ij}(s,s) = \mathbf{1}_{i=j}$ *provided that* $\mathbb{P}(X_s = i) \neq 0$

In our model, as the state of the insured is defined by a Markov chain, it is natural that this process has the Markov property. That is; we do not care if the insured used to be disabled yesterday as long as they are healthy today. The state tomorrow is only determined by the state today.

Definition 3.1.4 (Markov property). *Consider a Markov chain* $X = \{X_t\}_{t \in J}$. *Let* $t_1 < t_2 < ... < t_n < t_{n+1} < ... < t_{n+m}, i \in S, A \subset S^{n-1}, B \subset S^m$. *Assume that*

$$\mathbb{P}((X_{t_1}, X_{t_2}, ..., X_{t_{n-1}}) \in A, X_{t_n} = i) \neq 0,$$

then the Markov property holds. That is

$$\begin{aligned}&\mathbb{P}((X_{t_{n+1}}, X_{t_{n+2}}, ..., X_{t_{n+m}}) \in B | X_{t_1}, X_{t_2}, ..., X_{t_{n-1}}) \in A, X_{t_n} = i) \\ &= \mathbb{P}((X_{t_{n+1}}, X_{t_{n+2}}, ..., X_{t_{n+m}}) \in B | X_{t_n} = i)\end{aligned}$$

Definition 3.1.5 (Transition rates). *Let* $X = \{X_t, t \in J\}$ *be a Markov process with finite state space* S. *The transition rates* $\mu_i, \mu_{ij}, i,j \in S, j \neq i$ *are the functions defined by*

$$\mu_i(t) = \lim_{h \to 0, h > 0} \frac{1 - p_{ii}(t, t+h)}{h}, \quad t \in J, \quad i \in S$$

and

$$\mu_{ij}(t) = \lim_{h \to 0, h > 0} \frac{p_{ij}(t, t+h)}{h}, \quad t \in J, \quad i,j \in S, \quad i \neq j$$

Definition 3.1.6. *Let* $X = \{X_t, t \in J\}$ *be a Markov process with finite state space* S. *We say that* X *is regular if the transition rates* $\mu_i, \mu_{ij}, i,j \in S, j \neq i$ *exist and are continuous as functions of* t.

Often only the transition rates are avaliable. However, the Kolmogorov equations provide a way to retrieve the transition probabilities from the transition rates;

Theorem 3.1.7 (Kolmogorov equations).

- ***Backward Kolmogorov equation:***

$$\frac{d}{ds}p_{ij}(s,t) = \mu_i(s)p_{ij}(s,t) - \sum_{k\in S, k\neq i} \mu_{ik}(s)p_{kj}(s,t)$$

- ***Forward Kolmogorov equation:***

$$\frac{d}{ds}p_{ij}(s,t) = -p_{ij}(s,t)\mu_i(t) + \sum_{k\in S, k\neq i} p_{kj}(s,t)\mu_{ik}(t)$$

These are pretty unwieldy (sometimes even impossible) to solve analytically, so we usually need to resort to numerical methods such as the RK4-scheme, Euler-scheme etc (see e.g [Bãn22] for a discussion on this).

Finally we do have a closed formula for the probability that the insured remains in state j:

Theorem 3.1.8 (Calculation of $\bar{p}_{jj}(s,t)$). *If $X = \{X_t, t \in J\}$ is regular then*

$$\bar{p}_{jj}(s,t) = \exp\Big(-\sum_{k\neq j}\int_s^t \mu_{jk}(u)du\Big)$$

3.1.2 Mathematical reserves

We are now ready to define mathematical reserves in greater detail. The ingredients we need to get a closed formula for mathematical reserves are the stochastic cash flow, policy functions and the discount factor. The mathematical reserves amount to the expected present value at time $t \geq 0$, of all payments given $X_t = i$.

Definition 3.1.9 (Stochastic cash flow). *A stochastic cash flow is a stochastic process $A = \{A(t), t \in [0,T]\}$ with almost all sample paths of bounded variation.*

Definition 3.1.10 (Policy functions). *Let $a_i, a_{ij} : [0,\infty) \to \mathbb{R}, i,j \in S, i \neq j$ be functions of bounded variation. We call them policy functions if they model the following quantities:*

- *$a_i(t)$ = the accumulated payments from the insurer to the insured up to time t, given that we know that insured has always been in state i*

- *$a_{ij}(t)$ = the punctual payments which are due when the insured switches from state to j at time t.*

The "given that we know that insured has always been in state i"-part can be somewhat confusing. Remember that we are in a Markovian world where we only care about the state of today to predict the state tomorrow. This means that "Always been in state i" means that the insured is in state i at the time of the computation.

Definition 3.1.11 (Policy cash flow). *Let $a_i(t)$ and $a_{ij}(t), t \geq 0, i,j \in S, j \neq i$ be policy functions. The (stochastic) cash flow associated to this insurance is defined by*

$$A(t) = \sum_{i\in S} A_i(t) + \sum_{i,j\in S, j\neq i} A_{ij}(t),$$

where

$$A_i(t) = \int_0^t I_i^X(s)da_i(s), \quad A_{ij}(t) = \int_0^t a_{ij}(s)dN_{ij}^X(s),$$

where both integrals are given a.s in the Riemann-Stieltjes sense. The quantity A_i corresponds to the accumulated liabilities while the insured is in state i and A_{ij} for the case when the insured switches from i to j

As we deal with time-value of money we need a discount factor:

Definition 3.1.12 (Discount factor). *The following function $v : [0, \infty) \to [0, \infty)$ will be called a discount factor,*

$$v(t) = e^{-\int_0^t r(u)du}, \quad t \geq 0$$

where $r : [0, \infty)$ is an interest rate process.

The interest rate process $r(t)$ can be stochastic, but we will deal with a deterministic one for now and introduce the stochastic version later.

We are now ready to put together the stochastic value of the cash flow:

Definition 3.1.13 (Stochastic prospective value of a cash flow). *The prospective value of a stochastic cash flow A at time t will be denoted by $V^+(t, A)$ and is defined, via discounting, as*

$$V^+(t, A) = \frac{1}{v(t)} \int_t^\infty v(s)dA(s), \quad t \geq 0$$

where v is the discount factor given above.

The process $V^+(t, A)$ can be adapted to the Markov process modelling the states of the insured, to the performance of a fund or to an interest rate process, which we will do later. Combining the above yields an explicit formula for the prospective reserves. Note that there also exist formulas for discrete time. See [Kol12] and [Bãn22].

Theorem 3.1.14 (Mathematical reserves). *Let x be the age of the insured at the start of the contract. The value at time t of the liability A associated with to policy functions a_i and a_{ij}, $i, j \in \mathcal{S}$, $j \neq i$, given that the insured is in state i at time t is given by*

$$\begin{aligned} V_i^+(t, S(t)) = & \frac{1}{v(t)} \sum_{j \in \mathcal{S}} v(s) p_{ij}^x(t, T) \Delta a_i(T) \\ & + \frac{1}{v(t)} \sum_{j \in \mathcal{S}} \int_t^T v(s) p_{ij}^x(t, s) \dot{a}_i(t) \\ & + \frac{1}{v(t)} \sum_{j \in \mathcal{S}, k \neq j} \int_t^T v(s) p_{ij}^x(t, s) \mu_{jk}^x(s) a_{ij}(t) ds \end{aligned}$$

Example 3.1.15 (Pension policy with deterministic interest rate). We illustrate this with an example which we will use for benchmarking purposes later. Consider a pension policy which pays out a yearly benefit B from T_0 until T. We do not consider premiums. The policy functions are given by

$$a_*(t) = \begin{cases} 0, & t \in [0, T_0) \\ B(t - T_0), & t \in [T_0, T] \end{cases}$$

which gives us

$$\dot{a}_*(t) = \begin{cases} 0, & t \in [0, T_0) \\ B, & t \in [T_0, T] \end{cases}$$

We get

$$V_*^+(t, S(t)) = B \int_{\max(t,T_0)}^{T} \frac{v(s)}{v(t)} p_{**}^x(t, s) ds$$

Let the insured be a male aged 30 at the start of the contract with the pension policy paying 200000 NOK a year, starting at 70 ($T_0 = 40$) and lasting until he is 110 ($T = 80$) with a deterministic interest rate given by $r_0 = 2\%$. For simplicity we will use the Gompertz-Makeham mortality model for the rest of this thesis, see e.g. [Bøl14]. The single premium (i.e the present value of the benefits at time $t = 0$) for this policy is then given by $\pi_0 \approx 1173531$.

To see how the reserve develops, one can compute the present value of the benefits for every time $t \in [0, T]$, this is shown in Figure 3.1.

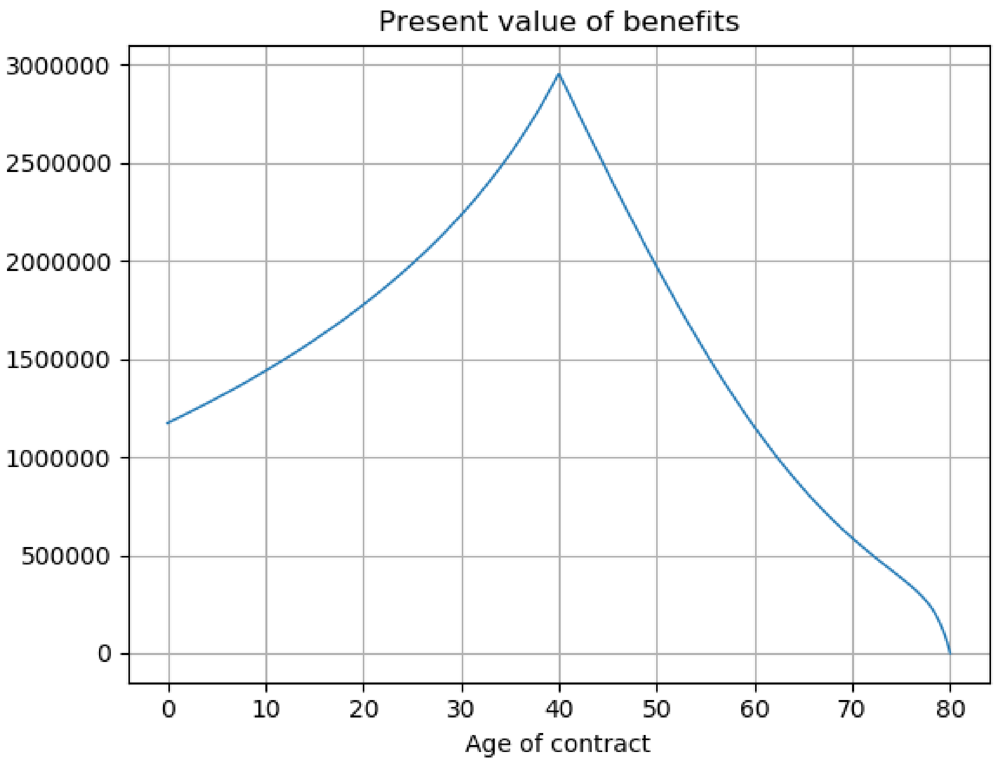


Figure 3.1: Present values for the pension policy in Example 3.1.15

3.1.3 Premiums

In the above example we did not take into account premiums. We can find the premium of a life insurance policy quite easily by introducing a new term $-\pi$ in the policy functions, denoting the premium paid from time 0 to time T_0 (usually only while the insured was in an active state). In Example 3.1.15, the policy function then becomes

$$a_*(t) = \begin{cases} -\pi t, & t \in [0, T_0) \\ -\pi T_0 + B(t - T_0), & t \in [T_0, T] \end{cases}$$

which gives us

$$\dot{a}_*(t) = \begin{cases} -\pi, & t \in [0, T_0) \\ B, & t \in [T_0, T] \end{cases}$$

The present value of the policy is thus given by

$$V_*^+(t, A = -\pi \int_t^{\max(t,T_0)} \frac{v(s)}{v(t)} p_{**}^x(t, s) ds + B \int_{\max(t,T_0)}^T \frac{v(s)}{v(t)} p_{**}^x(t, s) ds$$

Using the equivalence principle, want the premium-part of the policy to be equal to the benefit part of the policy at time $t = 0$. In Example 3.1.15 this amounts to

$$\pi \int_0^{T_0} v(s) p_{**}^x(0, s) ds = B \int_{T_0}^T v(s) p_{**}^x(0, s) ds$$

which yields

$$\pi = \frac{B \int_{T_0}^T v(s) p_{**}^x(0, s) ds}{\int_0^{T_0} v(s) p_{**}^x(0, s) ds}$$

The premium in Example 3.1.15 is $\pi \approx 43552$.

The reserve of Example 3.1.15 is plotted in Figure 3.2. We see that at $t = T_0$, the premiums is fully paid and the payouts start.

3.2 Short rate models and bond prices

We will soon turn to a more mathematically juicy (and realistic) kind of policy, where the interest rate is not determined but stochastic. For a life insurance company this is important to be able to price accurately, as the fluctuation of the interest rate market can pose a severe threat to its solvency.

As rates can not be traded per se, the market is not complete by the definition of complete markets. However, we will complete the market by introducing bonds, more specifically zero-coupon bonds. That is a security that pay out 1 unit of money at time T and nothing else. Formally:

Definition 3.2.1 (Zero-coupon bond). *A zero-coupon bond price $P(t, T)$ with maturity T is defined as the time t value of 1 unit of money at the future time $T \geq t$.*

Note that $P(T, T) = 1$ for all T and one may assume that $P(t, T)$ is differentiable in T. If the interest rate $r(u) = r$ is deterministic, the price of the zero-coupon bond is simply $P(t, T) = \exp(-\int_t^T r du)$ and can be regarded as the bank account, or the risk-less

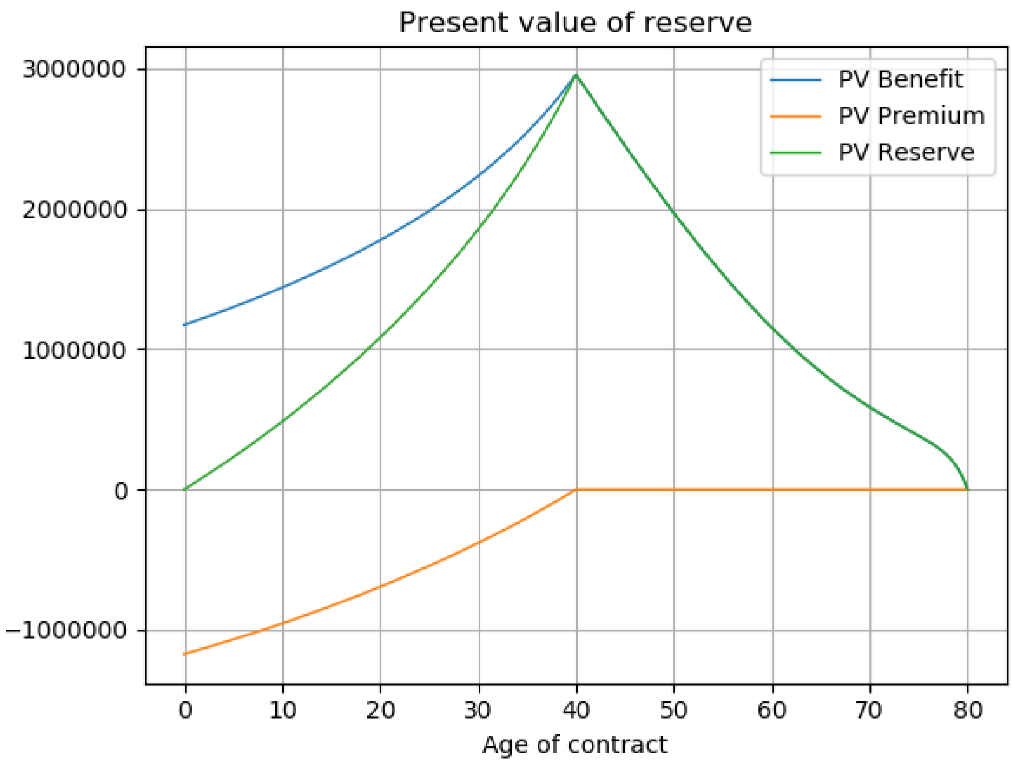


Figure 3.2: Present value of reserves for the pension policy in Example 3.1.15

fixed-income asset $v(t) = B(t)$. However, we will deal with stochastic interest rates. For now we will use interest rate models driven by a Brownian motion.

The short rate, $r(t)$, is the rate of which the interest rate grows instantaneously, whereas the forward rate $f(t,T)$ is the rate contracted at time t for a loan starting at time T and returned instantaneously. We will introduce a model for the forward rate in later chapters. In the Markovian case we assume the following:

- the short rate follows an Itô-process:

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW_t,$$

  which determines the money-market account $B(t) = e^{\int_0^t r(s)ds}$

- there exists an equivalent martingale measure $\mathbb{Q}$ such that the discounted bond price process $Z(t) = P(t,T)/B(t), t \leq T$, is a $\mathbb{Q}$-martingale.

The last assumption yields the following

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} | \mathcal{F}_t\right] \tag{3.2}$$

Take the Girsanov transformed $\mathbb{Q}$-Brownian motion $\widetilde{W}_t = W_t - \int_0^t \gamma_s ds$. Under the above assumptions, the process r satisfies under the measure $\mathbb{Q}$

$$dr(t) = \mu(t, r(t)) + \gamma_t \sigma(t, r(t))dt + \sigma(t, r(t))\widetilde{W}_t \tag{3.3}$$

under the real world measure $\mathbb{P}$ this amounts to

$$\frac{dP(t,T)}{P(t,T)} = \big(r(t) - v_t\gamma_t\big)dt + v_t dW_t \tag{3.4}$$

Here, the term γ_t represents the market price of risk as the excess of instantaneous return over $r(t)$ in units of volatility.

A class of short rate models which we will deal with variations of later is given by affine term structure models; short rate models where the bond price is of the form

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}, \tag{3.5}$$

where A and B are smooth functions.

Proposition 3.2.2. *The short rate model described by Equation* (3.3) *provides an affine term structure iff its diffusion and drift terms are of the form*

$$\sigma(t,r) = a(t) + \alpha(t)r$$
$$b(t,r) = b(t) + \beta(t)r$$

for some continuous functions a, α, b, β *and the functions* A, B *satisfies the following system of ordinary differential equations for all* $t \le T$*:*

$$\begin{aligned}
\frac{\partial}{\partial t}A(t,T) &= \frac{1}{2}a(t)B^2(t,T) - b(t)B(t,T)\\
\frac{\partial}{\partial t}B(t,T) &= \frac{1}{2}\alpha(t)B^2(t,T) - \beta(t)B(t,T) - 1\\
A(T,T) &= 0\\
B(T,T) &= 0
\end{aligned}$$

Proof. [Fil12], Proposition 5.2 □

In the following, we will concentrate on the Vasicek model, the Hull-White model and the Hull-White extension of the Vasicek model.

The Vasicek model

The $\mathbb{Q}$-dynamics of the Vasicek model is given by:

$$dr(t) = a[b - r(t)]dt + \sigma\widetilde{W}_t \tag{3.6}$$

where b is the long-term mean, a is the speed of reversion, and σ is the volatility. We require $a, b, \sigma > 0$. We can also note that as the market price of risk is assumed to be constant, the dynamics is the same under the real world measure $\mathbb{P}$ as under the measure $\mathbb{Q}$. A useful property of the Vasicek model is the mean-reversion effect. What was long thought of as a downside is that the probability that it can take on negative interest rates is not zero (albeit, very small). However, later developments have shown that this need not be a drawback as negative -or zero- interest rates have occured recently and might do so more frequently in the future (see e.g [HK20] for a discussion on this).

As for the zero-coupon bond price, the Vasicek model has affine term structure and is given by the following

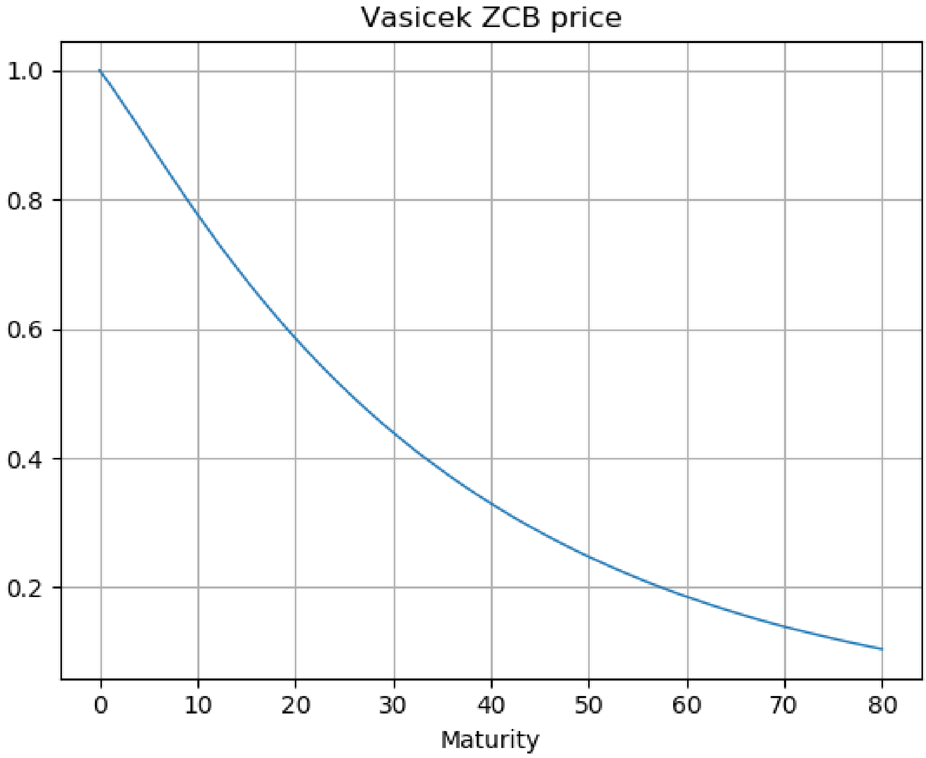


Figure 3.3: Vasicek zero-coupon bond price as a function of maturities with parameters $r_0 = 2\%, a = 20\%, b = 3\%$ and $\sigma = 1\%$. A long maturity is chosen as this is relevant for our purpose.

Theorem 3.2.3 (Vasicek Zero-Coupon bond price). *A Zero coupon bond under the Vasicek model issued at time t with maturity T is given by*

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where A and B are given by the following:

$$\begin{aligned} B(t,T) &= \frac{(1-\exp(-a(T-t))}{a} \\ A(t,T) &= \exp\left\{\left(b - \frac{\sigma^2}{2a^2}\right)(B(t,T) - T + t) - \frac{\sigma^2}{4a}B(t,T)^2\right\} \end{aligned}$$

Proof. [BM06], pp.59 and references therein □

The Hull-White model

Its $\mathbb{Q}$-dynamics is, quite similarly to the Vasicek model, given by

$$dr(t) = [k(t) - a(t)r(t)]dt + \sigma(t)\widetilde{W}_t \tag{3.7}$$

where $k(t), a(t)$ and $\sigma(t)$ are deterministic functions of time and so allows for fitting to the yield- and volatility curve. This model can generally not be handled analytically and, as noted by [BM06](pp.72-73), it is also prone to treacherous volatility models when fitted to the forward curve. Therefore, the following model is rather used:

The Extended Vasicek model

Taking the Hull-White model and choosing the coefficient functions $a(t)$ and $\sigma(t)$ as constants and fitting only the term $k(t)$ to the initial forward curve yields the Extended Vasicek model. The function $k(t)$ is given by

$$k(t) = \frac{\partial}{\partial T} f(0,t) + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}), \tag{3.8}$$

where $f(0,t)$ is the instantaneous forward rate at time 0. The extended Vasicek model is still affine, so that we can use Equation (3.5) to price the bond. The closed formula is given by;

Theorem 3.2.4 (Extended Vasicek Zero-Coupon bond price). *A Zero coupon bond under the Extended Vasicek model issued at time t with maturity T is given by*

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where A and B are given by the following:

$$\begin{aligned} B(t,T) =& \frac{(1 - \exp(-a(T-t)}{a} \\ A(t,T) =& \frac{P(0,T)}{P(0,t)} \exp\left\{ B(t,T)f(0,t) - \frac{\sigma^2}{4a}(1 - e^{-2a(T-t)})B(t,T)^2) \right\} \end{aligned}$$

Proof. [BM06], pp.75 and references therein. □

3.3 Policies with stochastic interest rates

We return to life insurance. Using the machinery above we can price policies where the interest rate is stochastic. Note that we can set

$$B(t) = \exp\left(\int_0^t r(s)ds \right) = \frac{1}{v(t)}, \quad t \in [0,T]$$

Which, together with Equation (3.1), yields

$$V_j(t, A_g) = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{v(t)} \int_I v(s)dA_g(s) \middle| \mathcal{F}_t \right], \tag{3.9}$$

Where the (completed) filtration $\mathcal{F}_t$ is generated by the bond price process and the state process $\{X_t, t \geq 0\}$, which are assumed to be independent. We can, in the similar vein as the deterministic case expand this to be

$$\begin{aligned} V_i^+(t, r(t)) =& \sum_{j \in \mathcal{S}} \mathbb{E}_{\mathbb{Q}}\left[\frac{v(s)}{v(t)} | \mathcal{F}_t \right] p_{ij}^x(t,T)\Delta a_i(T) \\ &+ \sum_{j \in \mathcal{S}} \int_t^T \mathbb{E}_{\mathbb{Q}}\left[\frac{v(s)}{v(t)} | \mathcal{F}_t \right] p_{ij}^x(t,s)\dot{a}_i(t) \\ &+ \sum_{j \in \mathcal{S}, k \neq j} \int_t^T \mathbb{E}_{\mathbb{Q}}\left[\frac{v(s)}{v(t)} | \mathcal{F}_t \right] p_{ij}^x(t,s)\mu_{jk}^x(s)a_{ij}(t)ds \end{aligned}$$

The idea is to substitute the term $\mathbb{E}_{\mathbb{Q}}[\frac{v(s)}{v(t)} | \mathcal{F}_t]$ with the price of the bond.

Definition 3.3.1 (Mathematical reserves with stochastic interest rates). *Let x be the age of the insured at the start of the contract. Let $P(t,s)$ be the price of a zero-coupon bond. The value at time t of the liability A associated with to policy functions a_i and a_{ij}, $i,j \in \mathcal{S}$, $j \neq i$, given that the insured is in state i at time t is given by*

$$\begin{aligned} V_i^+(t,r(t)) = &\sum_{j\in\mathcal{S}} P(t,T)p_{ij}^x(t,T)\Delta a_i(T) \\ &+ \sum_{j\in\mathcal{S}} \int_t^T P(t,s)p_{ij}^x(t,s)\dot{a}_i(t) \\ &+ \sum_{j\in\mathcal{S},k\neq j} \int_t^T P(t,s)p_{ij}^x(t,s)\mu_{jk}^x(s)a_{ij}(t)ds \end{aligned}$$

Example 3.3.2 (Pension policy under the Vasicek model). Consider the same policy as in Example 3.1.15. The policy functions are the same, however we consider stochastic interest rates. The mathematical reserves are then expressed by

$$V_*^+(t,r(t)) = B\int_{\max(t,T_0)}^T P(t,s)p_{**}^x(t,s)ds$$

where $P(t,s)$ is given as in Theorem 3.2.3.

As in the deterministic case, let the insured be a male aged 30 at the start of the contract with the pension paying 200000 NOK, starting at 70 ($T_0 = 40$) and lasting until he is 110 ($T = 80$) with Vasicek parameters given by $a = 20\%, b = 3\%$ and $\sigma = 1\%$.

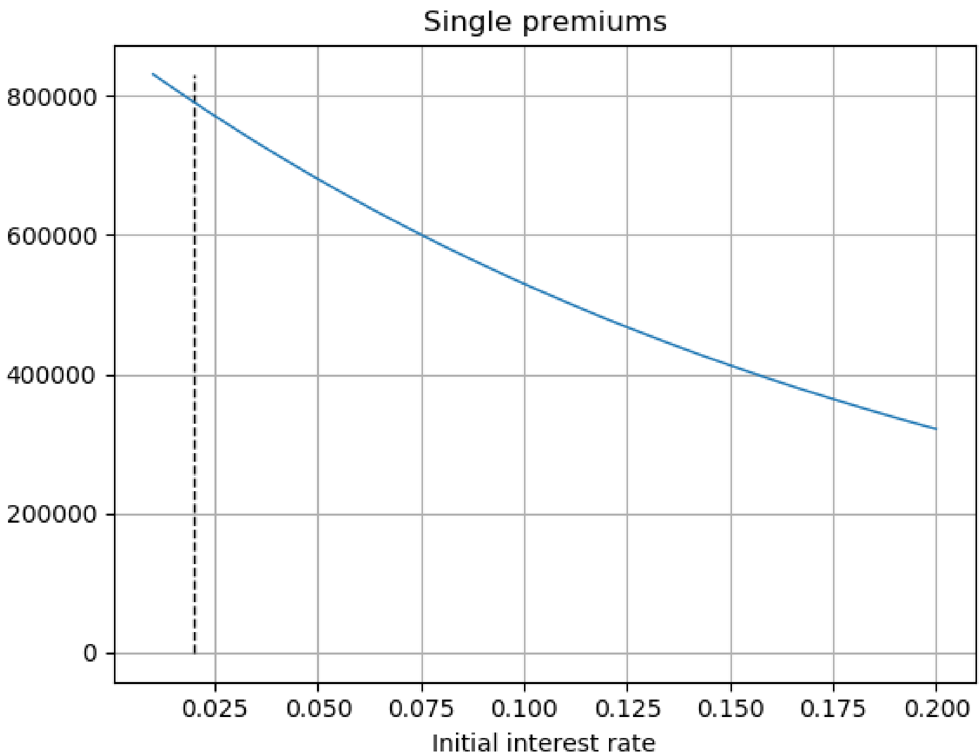


Figure 3.4: Single premiums of the policy in Example 3.3.2 for some values of r_0

For $r_0 = 2\%$ this amounts to $\pi_0 \approx 790225$.

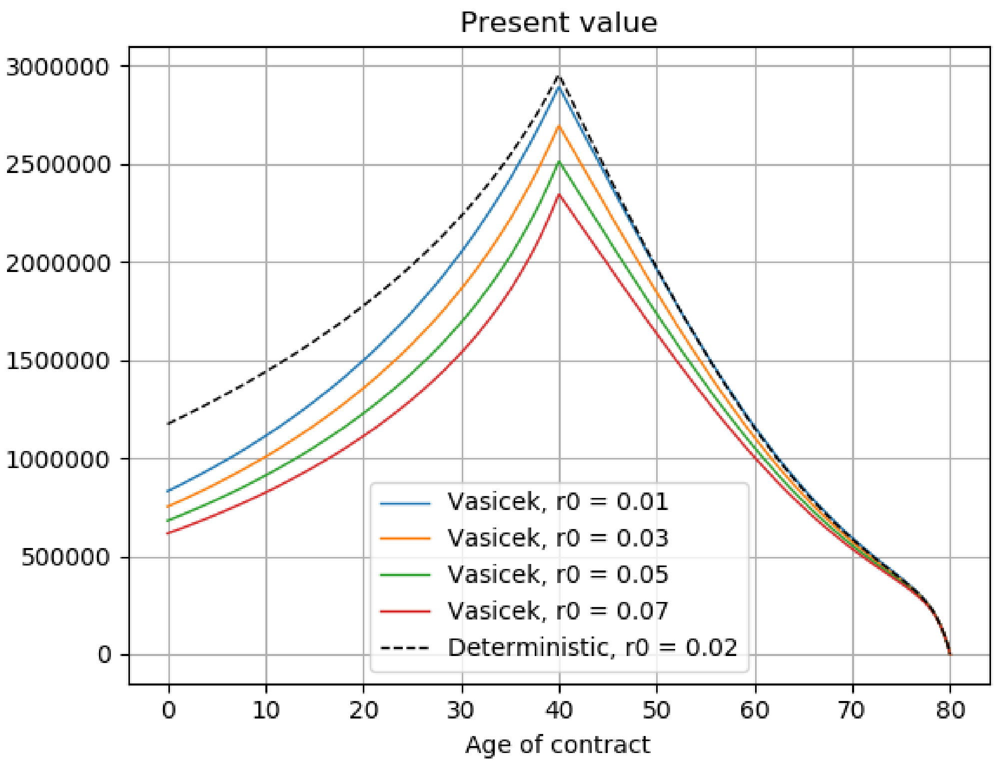


Figure 3.5: Present value of the policy in Example 3.3.2 for some values of r_0

Chapter 4

Stochastic calculus with respect to fractional Brownian motion

Up until now we have mainly dealt with standard Brownian motion and semimartingales in a Markovian setting. This is useful for computational purposes, but empirical evidence suggests that interest rates does not behave that way (see [McC+04] and references therein). A candidate for a more realistic model might be the fractional Brownian motion (fBm). For reasons that will become apparent, the Itô calculus presented earlier is not able to capture fBm (if the Hurst parameter $H \neq 1/2$). We need a different theory. In the following we first present fBm and its properties, we show that is not a semimartingale and that it does not have the Markov property. We then review some concepts from fractional calculus and end with the derivation of a stochastic integral with respect to fractional Brownian motion.

4.1 Definitions and properties of fBm

The following material is gathered from [Bia+10], chapter 1.

Definition 4.1.1 (Fractional Brownian motion). *(Standard) Fractional Brownin motion with Hurst-parameter $H \in (0, 1)$ is a continuous and centered Gaussian stochastic process $(W_t^H, t \geq 0)$ with covariance function*

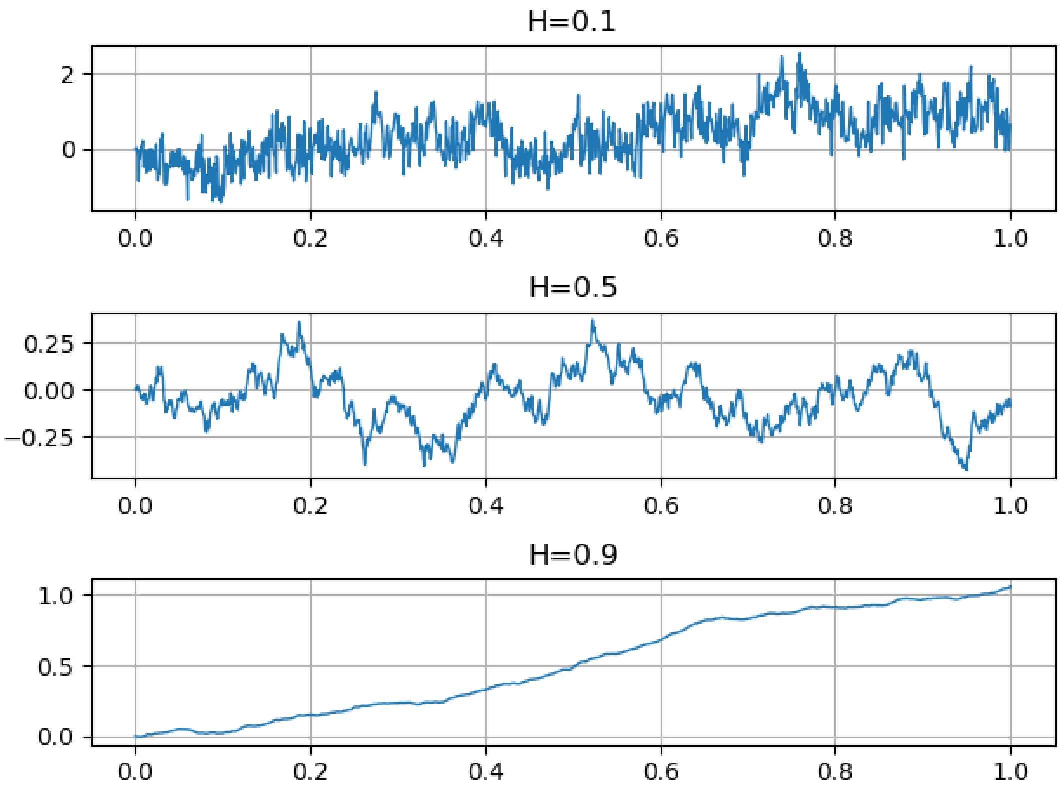
$$\mathbb{E}(W_t^H W_s^H) = \frac{1}{2}(|t|^{2H} - |t-s|^{2H} + |s|^{2H})$$

for all $s, t \geq 0$.

From the definition it can be shown that the standard fBm has the following properties:

1. $W_0^H = 0$
2. $\mathbb{E}[W_t^H] = 0$ for all $t \geq 0$
3. $W_{t+s}^H - W_s^H$ has the same distribution as W_t^H for $s, t \geq 0$ (i.e it has homogeneous increments)
4. $\mathbb{E}[(W_t^H)^2] = t^{2H}, t \geq 0$ for all $H \in (0, 1)$.
5. W^H has continuous trajectories.

When $H = 1/2$ the fBm corresponds to a standard Brownian motion. It is important to note that fBm with Hurst parameters $H \in (0, 1/2), H = 1/2$ and $H \in (1/2, 1)$ exhibit very different behaviour and so it is often natural to treat them separately.



We will now present the stochastic integral representation of the fBm over finite intervals. We need to treat each Hurst parameter family separately.

Theorem 4.1.2 (Stochastic integral representation over finite intervals). *Let W_t be a standard Brownian motion. Then*

$$Z_t := \int_0^t K_H(t,s) dW_s$$

is an fBm for Hurst parameter H, where;

- $H > 1/2$:

$$K_H(t,s) = c_H s^{1/2-H} \int_s^t |u-s|^{H-3/2} u^{H-1/2} du, \quad t > s$$
$$\text{where } c_H = [H(2H-1)/\beta(2-2H, H-1/2)]^{1/2}$$

- $H < 1/2$:

$$K_H(t,s) = b_H [\frac{t}{s}^{H-1/2} (t-s)^{H-1/2} - (H - \frac{1}{2}) s^{1/2-H} \int_s^t (u-s)^{H-1/2} u^{H-3/2} du],$$
$$\text{where } b_H = [2H/((1-2H)\beta(1-2H, H+1/2))]^{1/2}$$

in the above $\beta(a,b) := \Gamma(a+b)/(\Gamma(a)\Gamma(b))$ *and* $t > s$

As a fBm with Hurst parameter $H = 1/2$ is a standard Brownian motion, $W^{1/2}$ has independent increments. This is not the case for $H \neq 1/2$. In fact if one has two increments $W^H_{t+h} - W^H_t$ and $W^H_{t+2h} - W^H_{t+h}$, they are negatively correlated for $H < 1/2$ and positively correlated for $H > 1/2$.

For Hurst-parameters $H \in (1/2, 1)$ fBm displays the property of long-range dependence. We state it here in two ways. First:

Definition 4.1.3 (Long-range dependence I). *A stationary sequence* $\{X_n, n \in \mathbb{N}\}$ *exhibits long-range dependence if the autocovariance function (defined as* $\rho(n) = \mathrm{Cov}(X_k, X_{k+n})$*) satisfy*

$$\lim_{n\to\infty} \frac{\rho(n)}{cn^{-\alpha}}$$

for some constant c *and* $\alpha \in (0,1)$. *In this case the dependence between* X_k *and* X_{k+n} *decays slowly as* n *tends to infinity and*

$$\sum_{n=1}^{\infty} \rho(n) = \infty$$

Take the increments X_k and X_{k+n} of W^H, that is $X_k := W^H_k - W^H_{k-1}$ and $X_{k+n} := W^H_{k+n} - W^H_{k+n-1}$. If we look at the case $H > 1/2$ we see that

$$\rho_H(n) = \frac{1}{2}[(n+1)^{2H} + (n+1)^{2H} - 2n^{2H}] \sim H(2H-1)n^{2H-2}$$

as $n \to \infty$ and it is clear that

$$\lim_{n\to\infty} \frac{\rho_H(n)}{H(2H-1)n^{2H-2}} = 1$$

Hence we obtain that X_k and X_{k+n} exhibits the long-range dependence property.

Remark 4.1.4. *Long range dependence implies that fBm does not have the Markov property for Hurst parameter* $H \in (1/2, 1)$

In the second definition the notion of a slowly varying function is needed, so we start with that.

Definition 4.1.5 (Slowly varying functions). *Let* $a > 0$ *and* L *be a measurable function on a finite interval. Then*

- *L is slowly varying at zero, if*

$$\lim_{x\to 0} \frac{L(ax)}{L(x)} = 1$$

- *L is slowly varying at infinity, if*

$$\lim_{x\to\infty} \frac{L(ax)}{L(x)} = 1$$

We will also need the spectral density of autocovariance functions $\rho(k)$:

Definition 4.1.6 (Spectral density). *Let $\lambda \in [-\pi, \pi]$. Then the spectral density of the autocovariance function $\rho(k)$ is given by*

$$f(\lambda) := \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \exp(-i\lambda k)\rho(k)$$

The long-range dependence property can also be stated as

Definition 4.1.7 (Long-range dependence II). *For stationary sequences $\{X_n, n \in \mathbb{N}\}$ with finite variance, we say that $\{X_n, n \in \mathbb{N}\}$ exhibits long-range dependence if one of the following properties holds:*

- $$\lim_{n\to\infty} (\sum_{k=-n}^{n} \rho(k))/cn^{\beta} L_1(n) = 1 \textit{ for } c, \beta \in (0,1)$$

- $$\lim_{k\to\infty} \rho(k)/ck^{\gamma} L_2(k) = 1 \textit{ for } c, \gamma \in (0,1)$$

- $$\lim_{\lambda\to\infty} f(\lambda)/c|\lambda|^{-}\delta L_3(|\lambda|) = 1 \textit{ for } c, \delta \in (0,1)$$

for slowly varying functions L_1, L_2 (at infinity) and L_3 (at zero).

It is worth noting that for a Hurst parameter H between $1/2$ and 1 the definitions above are equivalent.

Definition 4.1.8 (Self-similarity). *We say that an $\mathbb{R}^d$-valued random process $X = \{X_t, t \geq 0\}$ is self-similar if for every $a > 0$ there exists $b > 0$ such that $\{X_{at}, t \geq 0\}$ and $\{bX_t, t \geq 0\}$ has the same distribution. That is*

$$\{X_{at}, t \geq 0\} \stackrel{d}{=} \{bX_t, t \geq 0\}$$

This means that the two processes X_{at} and bX_t have the same finite-dimensional distribution functions. Fractional Brownian motion also displays (local) Hölder continuity and is not differentiable, as the following results state.

Theorem 4.1.9 (Hölder continuity). *Let $H \in (0,1)$. W^H admits a version whose sample paths are a.s (locally) Hölder continuous of order strictly less than H.*

Proof. See [Bia+10], Theorem 1.6.1 □

Proposition 4.1.10. *The fBm sample path $W^H_\cdot$ is not differentiable. For every $t_0 \in [0, \infty)$ we have with probability one that*

$$\limsup_{t\to t_0} |\frac{W^H_t - W^H_{t_0}}{t - t_o}| = \infty$$

Proof. See [Bia+10], Proposition 1.7.1. □

Definition 4.1.11 (Self-similarity with Hurst index H). *If $b = a^{-H}$ in the above definition we say that $X = \{X_t, t \geq 0\}$ is a self-similar process with Hurst index H or that it satisfies the property of statistical self-similarity with Hurst index H. The quantity $D = 1/H$ is called the statistical fractal dimension of X.*

As the covariance function of the fBm is homogeneous of order $2H$ we obtain that W^H is a self-similar process with Hurst index H. That is; for any $a > 0$ the processes W^H_{at} and $a^{-H}W^H_t$ have the same distribution.

In previous chapters we introduced an integral and a calculus for semimartingales. fBm is, however, not a semimartingale. The main idea behind the proof for this is to show that fractional Brownian motion does not have quadratic or bounded variation for $H \neq 1/2$, opposed to semimartingales. To show this we need some definitions;

Definition 4.1.12 (p-variation). *Let $\{X_t, t \geq 0\}$ be a stochastic process and consider a partition $\pi = \{0 = t_0 < t_1 < ... < t_n = T\}$. Set*

$$\mathcal{S}_p(X, \pi) = \sum_{i=1}^{n} |X_{t_k} - X_{t_{k-1}}|^p$$

The p-variation of X over the interval [0,T] is defined as

$$V_p(X, [0, T]) = \sup_{\pi} \mathcal{S}_p(X, \pi)$$

where π is the finite partition of $[0, T]$.

Definition 4.1.13 (Index of p-variation). *The index of p-variation of a process is defined as*

$$I(X, [0, T]) := \inf\{p > 0; V_p(X, [0, T]) < \infty\}$$

We are now ready to prove the following proposition (gathered from [Bia+10]).

Proposition 4.1.14 (fBm is not a semimartingale). *The fBm is not a semimartingale for Hurst parameter $H \neq 1/2$*

Proof. As every semimartingale X consists of a local martingale and a process of locally bounded variation, it has quadratic variation (2-variation) or, if the martingale part is zero, bounded variation (1-variation). The index $I(X, [0, T])$ of a semimartingale must then belong to $\{2\}$ or $[0, 1]$. To show that this is not the case for fBm with $H \neq 1/2$ we need to compute the p-variation of an fBm W^H with $H \neq 1/2$. We want to show that

$$I(W^H, [0, T]) = \frac{1}{H}$$

as this would put $I(W^H, [0, T])$ in $[0, 1] \cup \{2\}$ if and only if $H = 1/2$.

Define for $p > 0$

$$Y_{n,p} = n^{pH-1} \sum_{i=1}^{n} |W^H_{i/n} - W^H_{(i-1)/n}|^p$$

Because W^H is self-similar, $\{Y_{n,p}, n \in \mathbb{N}\}$ has the same distribution as

$$\widetilde{Y}_{n,p} = n^{-1} \sum_{i=1}^{n} |W_i^H - W_{i-1}^H|^p$$

Now, by the Ergodic theorem (see e.g [Wei]), the sequence $\{\widetilde{Y}_{n,p}, n \in \mathbb{N}\}$ converges almost surely and in L^1 to $\mathbb{E}[|W_1^H|^p]$ as $n \to \infty$. As convergence in L^1 implies convergence in probability we get that $\{\widetilde{Y}_{n,p}, n \in \mathbb{N}\} \xrightarrow{p} \mathbb{E}[|W_1^H|^p$. Thus we obtain that

$$V_{n,p} = \sum_{i=1}^{n} |W_{i/n}^H - W_{(i-1)/n}^H|^p$$

converges in probability to 0 if $pH > 1$ and to ∞ if $pH < 1$ as $n \to \infty$. Hence we conclude that

$$I(W^H, [0,T]) = \frac{1}{H}$$

□

4.2 Fractional calculus

We will need some basic concepts from fractional calculus and this section introduces the Riemann-Liouville integral as well as the fractional derivative. The material is gathered from [Bia+10] Appendix B as well as [FKZ13].

There exists separate definitions for the right- and left-hand sided Riemann-Liouville integrals and derivatives. We will mainly use the right-hand side version, but we state both for completeness. We will however only need the integral defined for finite time e.g on an interval.

Definition 4.2.1 (Riemann-Liouville fractional integral of order κ). *We have for almost all $x \in [0,T]$ that*

- *Left hand side:*

$$I_{a+}^{\kappa} f(x) := \frac{1}{\Gamma(\kappa)} \int_a^x f(y)(x-y)^{\kappa-1} dy$$

- *Right hand side:*

$$I_{b-}^{\kappa} f(x) := \frac{1}{\Gamma(\kappa)} \int_x^b f(y)(y-x)^{\kappa-1} dy$$

where $f \in L^1([0,T])$, $\kappa > 0$, and Γ is the gamma-function.

We also state an integration by parts formula:

Definition 4.2.2 (Integration by parts for Riemann-Liouville integrals).

$$\int_a^b I_{a+}^{\kappa} f(x) g(x) dx = \int_a^b f(x) I_{b-}^{\alpha} g(x) dx,$$

where $f \in L^p$, $g \in L^q(a,b)$, and the following holds: $1/p + 1/q \leq 1 + \kappa$ and $p, q > 1$ if $1/p + 1/q = 1 + \kappa$.

The fractional integrals satisfy the following compositions

1. $I^{\kappa}_{a+}(I^{\beta}_{a+}f) = I^{\kappa+\beta}_{a+}f$
2. $I^{\kappa}_{b-}(I^{\beta}_{b-}f) = I^{\kappa+\beta}_{b-}f$

We define the Liouville derivatives as the opposite operation:

Definition 4.2.3 (Liouville fractional derivative for $\kappa < 1$). *We have*

$$D^{\kappa}_{a+}f := \frac{d}{dx}I^{1-\kappa}_{a+}f \quad \text{and} \quad D^{\kappa}_{b-}f := \tfrac{d}{dx}I^{1-\kappa}_{b-}f,$$

if the right-hand sides are well-defined.

It can be shown that the following holds for almost every $x \in (a,b)$:

Definition 4.2.4 (Weyl-representation of Liouville derivatives). *Let $f \in L^p(a,b)$ then for almost every $x \in (a,b)$*

$$D^{\kappa}_{a+}f(x) = \frac{1}{\Gamma(1-\kappa)}\left[\frac{f(x)}{(x-a)^{\kappa}} + \kappa\int_a^x \frac{f(x)-f(y)}{(x-y)^{\kappa-1}}dy\right]$$

and

$$D^{\kappa}_{b-}f(x) = \frac{1}{\Gamma(1-\kappa)}\left[\frac{f(x)}{(b-x)^{\kappa}} + \kappa\int_x^b \frac{f(x)-f(y)}{(y-x)^{\kappa-1}}dy\right]$$

for the left- and right side respectively.

As with its integral counterparts the following relations hold:

1. $D^{\kappa}_{a+}(D^{\beta}_{a+}f) = D^{\kappa+\beta}_{a+}f$
2. $D^{\kappa}_{b-}(D^{\beta}_{b-}f) = D^{\kappa+\beta}_{b-}f$

The following relations between the Riemann-Liouville integrals and Liouville derivatives hold:

1. Let $f \in L^1(a,b)$. Then $D^{\alpha}_{a+}I^{\alpha}_{a+}f = f$ and $D^{\alpha}_{b-}I^{\alpha}_{b-}f = f$
2. Let $f \in I^{\kappa}_{a+}(L^p(a,b))$ (that is; functions that can be represented as an I^{κ}_{a+}-integral). Then $I^{\kappa}_{a+}D^{\kappa}_{a+}f = f$

We can now define a second integration by parts formula:

Definition 4.2.5 (Integration by parts for Riemann-Liouville integrals II).

$$\int_a^b D^{\kappa}_{a+}f(x)g(x)dx = \int_a^b f(x)D^{\kappa}_{b-}g(x)dx$$

where $f \in I^{\kappa}_{a+}(L^p(a,b))$, $g \in I^{\kappa}_{b-}(L^q(a,b))$, and the following holds: $1/p + 1/q \leq 1 + \kappa$ and $\kappa \in [0,1]$.

4.3 Integration with respect to cylindrical fBm

With later applications in mind, we want to define integration with respect to fractional Brownian motion. As we have seen, fBm with $H \neq 1/2$ is not a martingale and so classical Itô calculus can not be used. In the sequel, we do not however, need to integrate processes driven by fBm, only deterministic functions. The integral utilized in [Oha09], where fundamental results used below is proved, is defined in [DPB02].

Definition 4.3.1 (Cylindrical fractional Brownian motion). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and U be a separable Hilbert space with inner product $\langle \cdot \rangle$ and norm $|| \cdot ||$. A cylindrical process $\langle W^H, \cdot \rangle : \Omega \times \mathbb{R}_+ \times U \to \mathbb{R}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a standard cylindrical fractional Brownian motion with Hurst parameter $H \in (0,1)$ if*

- *For each $x \in U - \{0\}$, $\frac{1}{||x||}\langle W^H(\cdot), x\rangle$ is a standard scalar fractional Brownian motion with Hurst parameter H.*
- *For $a, b \in \mathbb{R}$ and $x, y \in U$:*

$$\langle W^H(t), ax + by\rangle = a\langle W^H(t), x\rangle + b\langle W^H(t), y\rangle \quad a.s\ \mathbb{P}$$

Let $(e_n, n \in \mathbb{N})$ be a complete orthonormal basis of U and set

$$\beta_n^H(t) = \langle W^H(t), e_n\rangle \text{ for } n \in \mathbb{N}$$

The sequence of scalar processes $(\beta_n^H, n \in \mathbb{N})$ is independent and with the same Hurst parameter. Then W^H can be represented by the series

$$W^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t) e_n \tag{4.1}$$

this series does not converge in $L^2(\mathbb{P})$ and is as such not well-defined as a U-valued random variable. One can, however, show that if U_1 is a Hilbert space such that U is included in U_1, and the linear embedding is a Hilbert-Schmidt operator, then Equation (4.1) is a U_1-valued fBm ([DPB02], pp.228).

We turn to the construction of a integral with respect to (cylindrical) fBm. We want to make sense of the expression

$$\int_0^T G(s)\, dW^H(s) \tag{4.2}$$

where G is a deterministic function.

Lemma 4.3.2. *If $p > 1/H$, then for a $\varphi \in L^p(0,T;\mathbb{R})$ the following inequality is satisfied*

$$\int_0^T \int_0^T \varphi(u)\varphi(v)\eta_H(u-v)\, du\, ds \leq C_T |\varphi|^2_{L^p(0,T;\mathbb{R})} \tag{4.3}$$

for some $C_T > 0$ that only depends on T where

$$\eta_H(u) = H(2H-1)|u|^{2H-2} \tag{4.4}$$

Proof. See [DPB02], Lemma 2.1 $\square$

Assume that the integral

$$\int_0^T f(s)\,dW^H(s) \tag{4.5}$$

is defined for $f \in L^p(0,T;V)$. Let $\mathcal{H}_*$ be the family of V-valued step-functions H defined by

Definition 4.3.3 (Step function).

$$H(t) = \sum_{i=1}^{n-1} f_i \mathbf{1}_{[t_i,t_{i+1})}(t)$$

where $0 = t_1 < t_2 < ... < t_n = T$ *and* $f_i \in V$.

For $f \in \mathcal{H}_*$ define the stochastic integral given by Equation (4.5) as

$$\int_0^T f(s)\,dW^H(s) := \sum_{i=1}^{n-1} f_i(\beta^H(t_{i+1}) - \beta^H(t_i)) \tag{4.6}$$

This random variable constituted by the integral has mean zero and second moment given by

$$\mathbb{E}\Big[\,\Big|\int_0^T f(s)\,dW^H(s)\Big|^2\,\Big] = \int_0^T\int_0^T \langle f(u), f(v)\rangle \phi(u-v)\,du\,dv \tag{4.7}$$

By Lemma 4.3.2 it follows that

$$\begin{aligned}\Big|\Big|\int_0^T f(s)\,d\beta^H(s)\Big|\Big|^2_{L^2(\mathbb{P})} &= \int_0^T\int_0^T \langle f(u), f(v)\rangle \phi(u-v)\,du\,dv\\ &\le c_{T,p}\Big(\int_0^T |f(s)|^p\,ds\Big)^{2/p}\\ &= c_{T,p}|f|^2_{L^p(0,T;V)},\end{aligned}$$

where $c_{T,p}$ is some constant that depends only on p and T. As $\mathcal{H}_*$ is dense in $L^p(0,T;V)$, the stochastic integral can thus be extended a.s from $\mathcal{H}_*$ to $L^p(0,T;V)$.

The integral given by Equation (4.2) is now defined for a U-valued standard fractional Brownian motion and for $G : [0,T] \to \mathcal{L}_2(U,V)$ where $\mathcal{L}_2(U,V)$ is the family of Hilbert-Schmidt linear operators from U to V. We assume the following about G:

- For each $x \in U$ we have $G(\cdot)x \in L^p(0,T;V)$
- $$\int_0^T\int_0^T |G(s)|_{\mathcal{L}_2(U,V)}|G(r)|_{\mathcal{L}_2(U,V)}\phi(s-r)\,ds\,dr < \infty$$

and further define the integral Equation (4.2) for cylindrical fBm as

$$\int_o^T G(s)\,dW^H(s) := \sum_{n=1}^{\infty}\int_0^T G(s)e_n\,d\beta_n^H(s) \tag{4.8}$$

We shall need a notion for the space of integrands we deal with and so we introduce the following spaces of deterministic integrands, as done in [FKZ13] and references therein:

Definition 4.3.4 (Possible spaces of integrands for fBm).

$$\tilde{\Lambda}_T^\kappa := \begin{cases} \{f : [0,T] \to \mathbb{R} | \int_0^T [s^\kappa I_{T-}^H((\cdot)^\kappa f(\cdot))(s)]^2 ds < \infty\} & \kappa \in (0,1/2) \\ \{f : [0,T] \to \mathbb{R} | \exists \phi_f \in L^2[0,T] \ s.t \ f(s) = s^{-\kappa} I_{T-}^{-\kappa}((\cdot)^\kappa \phi_f(\cdot))(s)\} & \kappa \in (0,1/2) \end{cases}$$

closed under multiplication with the index function we get

$$\Lambda_T^\kappa := \{f : [0,T] \to \mathbb{R} | \forall [s,t] \subset [0,T] : f \mathbf{1}_{[s,t]} \in \tilde{\Lambda}_T^\kappa\}, \quad \kappa \in (-1/2, 1/2)$$

We define the scalar product of Λ_T^κ for $f, g \in \Lambda_T^\kappa$ by

$$\langle f, g \rangle_{\kappa,T} = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)} \int_0^T s^{-2\kappa} [I_{T-}^\kappa((x)^\kappa f(x))(s)][I_{T-}^\kappa((x)^\kappa g(x))(s)] \, ds \tag{4.9}$$

and its corresponding norm by

$$||f||_{\kappa,T}^2 = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)} \int_0^T s^{-2\kappa} [I_{T-}^\kappa((x)^\kappa f(x))(s)]^2 \, ds \tag{4.10}$$

It can be shown that when $\kappa = 0$ (remember $\kappa = 0 \implies H = 1/2$) the space Λ_T^κ is equal to $\tilde{\Lambda}_T^\kappa$ which is again equal to $L^2([0,T])$, the norms and inner products will align accordingly.

Chapter 5

Bond markets with proportional transaction costs

We have seen how we can price insurance contracts with stochastic interest rates driven by Brownian motion. As mentioned above, empirical evidence suggests that interest rates does not behave Markovian and so we want to be able to price insurance contracts where the stochastic interest rate is non-Markovian. In a first step, we discuss bond markets where the noise is driven by fBm.

5.1 Arbitrage with fBm in a Black-Scholes market

The price of a zero-coupon bond is, as in the Markovian case above, given by the conditional expectation

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r(u)du}|\mathcal{F}_t], \quad t \in [0,T]$$

where $\mathbb{Q}$ is a risk-neutral measure of sorts. The Markov property makes these sometimes relatively easy to calculate if the short rate is driven by standard Brownian motion. For fBm, however, things are more complicated and arbitrage opportunities arise if the the fBm-process is introduced naively into a market. We illustrate this with an example gathered from [Bia+10]; We need to show that there exists a portfolio which is an arbitrage opportunity. We do this by providing a arbitrage example. We assume the same market as in the classical Black-Scholes case, but with the driving factor in the risky asset being a fBm such that its dynamic is given by:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW_t^H, \quad S(0) > 0$$

with solution

$$S(t) = S(0)\exp(\sigma W_t^H + \mu t)$$

for simplicity we assume that $\mu = r$ and $\sigma = S_0 = 1$. We recall that a portfolio is given by the tuple $\phi = (\phi^0, \phi^1)$ corresponding to the proportion invested in the riskless- and risky asset respectively. Further, we define

$$\begin{aligned}\phi^0(t) &= 1 - \exp(2W_t^H)\\ \phi^1(t) &= 2(\exp(W_t^H) - 1),\end{aligned}$$

which result in the following value process:

$$\begin{aligned} V^{\phi}(t) &= \phi^0(t)B(t) + \phi^1(t)S(t) \\ &= (1 - \exp(2W_t^H)\exp(rt) + 2(\exp(W_t^H) - 1)\exp(W_t^H + rt) \\ &= \exp(rt)(\exp(W_t^H - 1)^2 \\ &> 0 \end{aligned}$$

It can be shown that the portfolio is self-financing and thus, it is an arbitrage opportunity. In fact, if the driving process is not a semimartingale, Delbaen and Schachermayer has shown that it does imply a weak form of arbitrage [DS94]. However, even though it is true in Black-Scholes-markets that fBm as noise produces arbitrage, it is not generally true in markets with (small) proportional transaction costs. This is proven in the works of Guasoni [Gua06], [GSR10].

Handwaivingly, Guasonis argument is that when we introduce transaction costs we also introduce a "hurdle" the asset price will have to jump in order to be sold with profit. There will always be a probability that the asset price will not make the jump over the hurdle and so we can not make a sure profit. Arbitrage is therefore ruled out.

This chapter is devoted to a review of Ohashi [Oha09] who utilizes the results of Guasoni to develop no-arbitrage conditions for a fractional forward curve.

The strategy going forward is to model a fractional forward rate under a quasi-martingale measure $\mathbb{Q}_*$, to be defined. Once that is done we can extract certain short rates from the forward rate using the relation $f(t,t) = r(t)$ and price bonds.

As this is a relatively new field of research, we will keep it more thorough than in the well established martingale case.

5.2 The forward curve

We begin with an introduction to the forward curve driven by Brownian motion, more precisely the Heath-Jarrow-Morton (HJM) framework. It allows us to calibrate the short-rate models better to the initial term structure by directly modelling the forward curve. This review highlights the most important features from [HJM92] for our task.

We set our trading horizon to be $[0, T^\star]$ and define a continuum of zero-coupon bonds with maturities for each trading date $T \in [0, T^\star]$. We start with the instantaneous forward rate $f(t,T)$ which is the rate contracted at time t for a loan starting at time T and returned instantaneously. Let $T > t$, then we define $f(t,T)$ by

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T), \quad \forall T \in [0, T^\star], 0 \le t \le T \tag{5.1}$$

If we solve Equation (5.1) we get a familiar expression:

$$P(t,T) = \exp\Big(-\int_t^T f(t,s)ds\Big), \quad 0 \le t \le T \le T^\star$$

the relation between the instantaneous forward rate and the short rate can be expressed as follows:

$$r(t) = f(t,t), \quad 0 \le t \le T^\star \tag{5.2}$$

the interpretation is that the short rate is the instantaneous forward rate at a time t if the bond is set to be paid back an instant later. The strength of the HJM model is in its ability to model the forward rate movements, and in extension the short rates and thus also the bond prices.

For every $T \in [0, T^\star]$, the forward rate process $f(t,T)$ is described by

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \sum_{i=1}^{d} \int_0^t \sigma^i(s,T)dW_s^i, \quad \forall 0 \leq t \leq T, \tag{5.3}$$

where $W_t = (W_t^1, ..., W_t^d)'$ is a Brownian motion under a risk-neutral measure $\mathbb{Q}$ and the following holds for all $0 \leq t \leq T$ and $i = 1, 2, ..., d$:

$$\int_0^T |\alpha(t,T)|dt < \infty, \quad \int_0^T |\sigma^i(t,T)|^2 dt < \infty$$

Invoking Equation (5.2) we can write the dynamics of the short rate process by

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)ds + \sum_{i=1}^{d} \int_0^t \sigma^i(0,t)dW_s^i \tag{5.4}$$

5.2.1 The Musiela parametrization

We conclude this section with the Musiela parametrization. Under the regular unparametrized HJM model, the state space is a function space on an interval that varies with the parameter t. The Musiela parametrization uses time *to* maturity, $x = T - t$ instead of time *of* maturity, T. As such, the state space under the Musiela parametrization does not vary with t. Another advantage is that the volatility term σ is also not dependent on the state, but rather on the whole forward curve. For a detailed discussion of the advantages, see [Fil01].

The forward rate and the coefficient functions can be written in the Musiela parametrization as

$$\begin{aligned} f_t(x) &= f(t, t+x) \\ \alpha_t :=& \alpha_t(x) = \alpha(t, t+x) \\ \sigma_t :=& \sigma_t(x) = \sigma(t, t+x) \end{aligned}$$

Defining the semigroup of right shifts $\{S(t); t \geq 0\}$ by $S(t)g(x) = g(t+x)$ for any $g : \mathbb{R}_+ \to \mathbb{R}$ allows us to write Equation (5.3) as

$$f_t(x) = S(t)f(0,x) + \int_0^t S(t-s)\alpha_s(x)ds + \sum_{i=1}^{d} \int_0^t S(t-s)\sigma_s^i(x)dW_s^i \tag{5.5}$$

As the infinitesimal generator of S is given by $A = \frac{\partial}{\partial x}$, the dynamics of the Musiela parametrization $f_t(x)$ is given by

$$df_t(x) = \Big(Af_t(x) + \alpha(t, f_t(x))\Big) + \sum_{i=1}^{d} \sigma^i(t, f_t(x))dW_t^i \tag{5.6}$$

In order for Equation (5.6) to be well defined when regarded as a Hilbert space - valued process with solution given by Equation (5.5), we need to specify some conditions on the Hilbert space. The space we seek needs to fulfill the following three criteria:

- H1: The functions h in the space are continuous and the pointwise evaluation $\mathcal{J}_x := h(x)$ is a continuous linear functional on the space for all $x \in \mathbb{R}_+$

- H2: The semigroup $\{S(t)|t \in \mathbb{R}_+\}$ is strongly continuous in the space with infinitesmal generator denoted by A.
- H3: There exists a constant K such that

$$||Sh|| \leq K||h|||^2$$

for all h in the space with Sh also in the space for a norm $||\cdot||$ to be defined.

Such a space, denoted by E, can be defined as the following weighted Sobolev space:

Definition 5.2.1 (E-space). *Let $w : \mathbb{R}_+ \to [1, \infty)$ be a non-decreasing C^1-function such that*

$$w^{\frac{1}{3}} \in L^1(\mathbb{R}_+)$$

We write

$$||h||_E^2 := |h(0)|^2 + \int_{\mathbb{R}_+} |h'(x)|^2 w(x)\, dx$$

and define

$$E := \{h \in L^1_{loc}(\mathbb{R}_+)|\ \exists h' \in L^1_{loc}(\mathbb{R}_+) \text{ and } ||h||_E < \infty\}$$

the following result ensures that the space E does indeed meet the requirements H1-H3.

Theorem 5.2.2. *The set E equipped with the norm $||\cdot||_E$ forms a separable Hilbert space meeting H1 - H3*

Proof. [Fil01], Theorem 5.1.1 □

For a detailed discussion of the requirements on the space E we refer to [Fil01].

5.3 The fractional forward curve

We turn to the fractional version of the forward curve as described by [Oha09]. Our starting point is to assume that the forward curve is described by the Hilbert-space valued linear SPDE

$$df_t(x) = \Big(Af_t(x) + \alpha_t\Big) + \sum_{i=1}^{d} \sigma_t^i\, dW_t^{H,i} \tag{5.7}$$

with W_s^H under the physical measure $\mathbb{P}$ (for the time being).

In the following, the space E is analogous to the identically named space used in the non-fractional case.

The forward rate is assumed to be given by the following process

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \sum_{i=1}^{d} \int_0^t \sigma^i(s,T)dW_s^{H,i}, \tag{5.8}$$

where the coefficients$(\sigma^i, ..., \sigma^d)$ and α are deterministic functions. To make sure Equation (5.8) is well defined we require that the following holds for all $0 < T < \infty$:

$$\int_0^T |\alpha(s,T)|ds + \int_0^T \int_0^T |\sigma^i(s,T)||\sigma^i(t,T)|\eta_H(t-s)\, ds\, dt < \infty,$$

where $\eta_H(u) = H(2H-1)|u|^{2H-2}$ and $u \in \mathbb{R}$.

Using the parametrization $T = t + x$ and the same notation as in the Musiela parametrization for the coefficient functions we write Equation (5.8) as

$$f_t(x) = S(t)f(0,x) + \int_0^t S(t-s)\alpha_s(x)ds + \sum_{i=1}^{d} \int_0^t S(t-s)\sigma_s^i(x)dW_s^{H,i} \tag{5.9}$$

The following two assumptions make sure that Equation (5.9) is the continuous mild solution to Equation (5.7);
1) We assume that α and σ are integrable:

$$\int_0^T ||\alpha_s||_E \, ds + \int_0^T ||\sigma_s||^2 \, ds < \infty \tag{5.10}$$

and
2) we assume that there exists an $\gamma \in (0, 1/2)$ such that

$$\int_0^T \int_0^T u^{-\gamma} v^{-\gamma} ||S(u)\sigma_u)|| \, ||S(v)\sigma_v|| \, \eta_H(u-v) \, du \, dv < \infty \tag{5.11}$$

By Proposition 3.1 and 3.2 of [DPB02], we see that Equation (5.7) is indeed the continuous mild solution to Equation (5.9).

Returning to the task at hand, we assume that for every $0 < T < \infty$ the following two expressions hold:

$$\int_{[0,T]^4} ||\sigma_u(s)||_{\mathbb{R}^d} \, ||\sigma_v(r)||_{\mathbb{R}^d} \, \eta_H(u-v) \, du \, dv \, ds \, dr < \infty \tag{5.12}$$

$$\int_{[0,T]^3} ||\sigma_u(t)||_{\mathbb{R}^d} \, ||\sigma_v(t)||_{\mathbb{R}^d} \, \eta_H(u-v) \, du \, dv \, ds < \infty \tag{5.13}$$

For the following, it is useful to define

$$\mathcal{I}_v(s,T) = \int_0^{T-s} v_s(x) \, dx,$$

where $v : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ is locally integrable in $\mathbb{R}_+$. We also define $\Delta^2 = \{(t,T) \in \mathbb{R}^2 | 0 \leq t \leq T < \infty\}$.

Lemma 5.3.1. *Assume that α and σ satisfy Equation* (5.10)*, Equation* (5.11)*, Equation* (5.12) *and Equation* (5.13)*. Then $f(t, t+x)$ is given by Equation* (5.9) *is the continuous mild solution of Equation* (5.7) *and the term structure of bond prices is given by*

$$P(t,T) = P(0,T) \times \exp\Big(\int_0^t [r(s) - \mathcal{I}_\alpha(s,T)] \, ds + \sum_{i=1}^{d} \int_0^t -\mathcal{I}_{\sigma_i}(s,T) \, dW_{s,i}^H\Big) \tag{5.14}$$

for $(t,T) \in \Delta^2$.

Proof. See [Oha09], Lemma 2.1. □

5.4 Arbitrage-free pricing under transaction costs

In the transaction cost market, defining the usual finance concepts such as portfolio, value process and arbitrage opportunity is more involved than in the standard case.

We start out by defining $\mathcal{M}_{T^\star}$ as the space of finite signed measures on $[0, T^\star]$ endowed with the total norm denoted $||x||_{T\overline{S}}$. Set $m_i \in \mathcal{M}_{T^\star}$, $F_i \in \mathcal{F}_{t_i}$ and $0 = t_0 < ... < T_N \leq T^\star$. We define the elementary process ϕ by

$$\phi(t)(\omega, x) = \sum_{i=0}^{N-1} \chi_{F_i \times (t_i, t_{i+1}]}(\omega, t) m_i(x) \tag{5.15}$$

and $\mathcal{S}$ the set of elementary processes of the form Equation (5.15) with the norm given by

$$||\phi||^2_{\overline{S}} = \mathbb{E}\Big[\sup_{0 \leq t \leq T^\star} ||\phi_t||^2_{TV}\Big] \tag{5.16}$$

and the completion with respect to Equation (5.16) given by $\overline{\mathcal{S}}$.

We let the relative bond price be given by

$$Z(t, T) := \frac{P(t, T)}{B_t} = \frac{P(t, T)}{\exp(\int_0^t r(s) ds)}, \tag{5.17}$$

where B_t is the numeraire and $r(t)$ is a short rate. As our trading horizon is $[0, T^\star]$ we set $Z(t, T) = 0$ if $(t, T) \notin [0, T^\star]^2$. Assuming Equation (5.10), Equation (5.11), Equation (5.12) and Equation (5.13), the discounted bond price process $Z(t, T)$ satisfies the following two conditions.

- Condition I: $\{Z(t, T); (t, T) \in [0, T^\star]^2\}$ is a jointly continuous real-valued stochastic process s.t

$$\mathbb{E}\Big[\sup_{(t,T) \in [0, T^\star]^2} |Z(t, T)|^2\Big] < \infty \tag{5.18}$$

We define the integral representing the value gain of the portfolio by

$$\int_0^t \phi(s) dZ_s = \sum_{i=0}^{N-1} \chi_{F_i}(Z_{t_{i+1}} - Z_{t_i}) m_i \tag{5.19}$$

it is well defined for every $\phi \in \overline{S}$ and it can be shown that the following relation hold:

$$\mathbb{E}\Big[\sup_{0 \leq t \leq T^\star} \Big| \int_0^t \phi_s dZ_s \Big|\Big] \leq ||\phi||_{\overline{S}} \mathbb{E}^{1/2}\Big[\sup_{0 \leq s, t \leq T^\star} ||Z_s - Z_t||^2_\infty\Big] < \infty \tag{5.20}$$

Under proportional transaction costs one has also to take into account the transaction costs with respect to the portfolio value needed to sell the whole portfolio as well as the cost of selling parts of it. It is therefore given by:

$$\begin{aligned} V_t^k(\phi) = &\sum_{i=0}^{N-1} \chi_{F_i}(Z_{t_{i+1}} - Z_{t_i}) m_i \\ &- k \sum_{i=0}^{N-1} Z_{t_i} |\phi_{t_{i+1}} - \phi_{t_i}| - k Z_t |\phi_t|, \end{aligned} \tag{5.21}$$

where $k > 0$ is an arbitrary number.

After introducing Condition II, we can extend the process to continuous trading;

- Condition II: Let $\mathcal{P}_{T^\star}$ denote the set of all partitions of $[0, T^\star]$ and assume that $\Pi_{T^\star}(\phi) = \sup_{\pi \in \mathcal{P}_{T^\star}} \sum_{t_i \in \pi} ||\phi_{t_{i+1}} - \phi_{t_i}||_{TV}$ is square integrable.

Leading to the following lemma:

Lemma 5.4.1. *Let $\phi \in \overline{S}$ satisfy Condition II. Then Equation* (5.21) *converges to*

$$V_t^k(\phi) = \int_0^t \phi_s dZ_s - k \int_0^t Z_s d|\phi_s| - kZ_t|\phi_t|$$

Proof. [Oha09], pp.9 and references therein. □

We are ready to define some familiar concepts;

Definition 5.4.2 (Admissible portfolio). *For proportional transaction costs with proportionality factor $k > 0$ a portfolio $\phi \in \overline{\mathcal{S}}$ is called admissible if it is adapted, it satisfies Condition II, and there exists a $M > 0$ such that $V_t^k(\phi) \geq -M$ a.s for every $0 \leq t \leq T^\star$.*

The intuition is as in the classical case; the investor is not allowed to "disturb" the process by depositing or withdrawing cash once it has started and it is not possible to lose an infinite amount of money. We follow up with the familiar notions of an arbitrage opportunity and arbitrage free market:

Definition 5.4.3 (Arbitrage opportunity). *An admissible portfolio $\phi \in \overline{\mathcal{S}}$ is called an arbitrage opportunity with transaction costs $k > 0$ if $V_{T^\star}^k(\phi) \geq 0$ a.s and $\mathbb{P}(V_{T^\star}^k(\phi) > 0) > 0$*

Definition 5.4.4 (k-arbitrage free market). *The bond market is called k-arbitrage free with transaction costs $k > 0$ if for every admissible portfolio ϕ, $V_{T^\star}^k(\phi) \geq 0$ a.s implies $V_{T^\star}^k(\phi) = 0$*

As before, the process should not be able to give a sure profit, and as always in mathematical finance our most important task is to avoid the possibility of arbitrage. The following proposition is a general criteria for no-arbitrage;

Proposition 5.4.5. *Fix $k > 0$. If for every $\{\mathcal{F}_t, t \geq 0\}$-stopping time τ s.t $\mathbb{P}(\tau < T^\star) > 0$ we have*

$$\mathbb{P}\left(\sup_{\tau \leq t \leq T \leq T^\star} \left| \frac{Z_\tau(\tau)}{Z(t,T)} - 1 \right| < k, \tau < T^\star \right) > 0, \tag{5.22}$$

then the bond market is arbitrage free on $[0, T^\star]$ with transaction costs k

Proof. See [Oha09], Proposition 3.1. □

The reasoning behind the proposition is as follows: we choose τ as our first trading point in time. The trade generates the transaction costs k. To make an arbitrage the price will have to be larger than k in a future time. If we choose an event A as the event that the price will not go above k and find that the probability of A happening is positive, we can not make a risk less profit and hence arbitrage is impossible.

We would like to find more specific criteria for k-arbitrage free bond markets. In the following denote $\mathcal{C}(\Delta^2_{T^\star})$ as the space of all real-valued continuous functions on $\Delta^2_{T^\star}$.

The following lemma gives a condition for when we can use Proposition 5.4.5.

Lemma 5.4.6. *Let $\mathbf{Y} : \Omega \to \mathcal{C}(\Delta^2_{T^\star})$ be a measurable map s.t $\mathbf{X} := \log \mathbf{Y}$ has full $\mathbb{P}$-support. Then $\mathbf{Y}$ satisfies the assumption in Proposition 5.4.5*

Proof. See [Oha09], Lemma 3.1 □

We remember that the paths of fBm is Hölder-continuous. The following lemma states volatility conditions that need to be fulfilled in order for the market to be k-arbitrage free.

Lemma 5.4.7. *Assume that $\mathcal{I}_{\sigma_i}(t,T)$ is λ-Hölder continuous on $\Delta^2_{T^\star}$ for every $i \geq 1$ where $1/2 < \lambda < 1$. Then the process*

$$\sum_{i=1}^{d} \int_0^t \mathcal{I}_{\sigma_i}(s,T) dW^H_{s,i}$$

has $\mathbb{P}$-full support on $\mathcal{C}(\Delta^2_{T^\star})$.

Proof. See Ohashi [Oha09], Lemma 3.2. □

If we put together Lemma 5.4.6 and Proposition 5.4.5 we see that the bond market is k-arbitrage free if the process $\log(Z)$ has $\mathbb{P}$-full support. By Lemma 5.4.7 we constrain the volatility σ by imposing conditions such that $\log(Z)$ has the full support property. We would however, like to find conditions on α such that a) the full-support property is ensured, and b) there exist a quasi-martingale measure. We start with the definition of the latter:

Definition 5.4.8 (Quasi-martingale measure). *We say that an equivalent probability measure $\mathbb{Q}_* \sim \mathbb{P}$ is a quasi-martingale measure if the discounted bond price process $Z(t,T)$ has $\mathbb{Q}_*$-constant expectation. That is, for every $0 < T < \infty$*

$$\mathbb{E}_{\mathbb{Q}*}[Z(t,T)] = P(0,T) \tag{5.23}$$

holds for all $0 \leq t \leq T$.

We start by stating a condition on α under the measure $\mathbb{P}$. We will soon come up with a change of measure which let us state the fractional forward curve under a quasi-martingale measure $\mathbb{Q}_*$.

Lemma 5.4.9. *The $\mathbb{P}$-constant expectation $\mathbb{E}[Z(t,T)] = P(0,T)$ holds for every $0 < T < \infty$, $0 \leq t \leq T$ iff the drift α satisfies the following equality:*

$$\begin{aligned} \alpha_t(x) = \sum_{i=1}^{d} \Big[&\sigma_t(x) \int_0^t \mathcal{I}_{\sigma_i}(\theta, x+t) \eta_H(t-\theta) d\theta \\ &+ \int_0^x \sigma_{t,i}(y) dy \int_0^t \sigma_{\theta,i}(x+t-\theta)\eta_H(t-\theta) d\theta \Big] \end{aligned} \tag{5.24}$$

Proof. See [Oha09], Corollary 3.1 □

5.4.1 Change of measure

We now turn to the change of measure. Consider

$$\mathcal{K}h(t) = \int_0^t K(t,s)h(s)ds, \quad h \in L^2(0,T^\star;\mathbb{R}), \tag{5.25}$$

where K is given by

$$K(t,s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du \tag{5.26}$$

for some $c_H > 0$.

We recognize $K(t,s)$ as the stochastic integral representation of fBm over finite intervals which let us describe fBm by a standard Brownian motion $H > 1/2$. Using this we can state a version of the Girsanov theorem:

Lemma 5.4.10. *Let $\gamma(t)$ be a $\mathbb{R}$-valued measurable function s.t $\int_0^{T^\star} ||\gamma(t)||_{\mathbb{R}} dt < \infty$ and $R(x) = \int_0^x \gamma(s)ds \in \mathbf{H}$. Then $\widetilde{W}_t^H = W_t^H - \int_0^t \gamma(s)ds$ is a $\mathbb{Q}_*$-fBm on $[0, T^\star]$ s.t*

$$\frac{d\mathbb{Q}_\star}{d\mathbb{P}} = \mathcal{E}(\mathcal{K}^{-1}R \cdot W)_T^\star \tag{5.27}$$

where

$$\mathcal{E}(\mathcal{K}^{-1}R \cdot W)_{T^\star} = \exp\Big[(\mathcal{K}^{-1}R \cdot W)_{T^\star} - \frac{1}{2}\int_0^{T^\star} ||\mathcal{K}^{-1}R(t)||_{\mathbb{R}}^2 dt\Big] \tag{5.28}$$

Proof. See [Oha09], Lemma 3.3. □

Lastly, we develop a no-arbitrage drift condition. Fix the proportional transaction cost $k > 0$ and define

$$\begin{aligned} \mathcal{S}_H\sigma_t(x) = &\sum_{i=1}^d \Big[\sigma_{t,i}(x)\int_0^t \mathcal{I}_{\sigma_i}(\theta, x+t)\eta_H(t-\theta)d\theta \\ &+ \int_0^x \sigma_{t,i}(y)dy \int_0^t \sigma_{\theta,i}(x+t-\theta)\eta_H(t-\theta)d\theta\Big], \end{aligned} \tag{5.29}$$

where we require that the the volatilities are regular:

$$\int_0^T ||\mathcal{S}_H\sigma_t||_E dt < \infty \tag{5.30}$$

and we observe that Equation (5.29) corresponds to α_t in Equation (5.24).

By Lemma 5.4.9 we know that if α_t in Equation (5.24) holds, the constant expectation condition is satisfied. We also see that $\mathcal{S}_H\sigma_t(x)$ does exactly that. In the following theorem we define the process $\sigma_t\gamma_t = \mathcal{S}_H\sigma_t - \alpha_t$. Making use of Lemma 5.4.10 we construct a new measure $\mathbb{Q}_*$, under which the drift $\mathcal{S}_H\sigma_t$ satisfies Equation (5.24) by definition. This ensures the existence of a quasi-martingale measure. Condensed:

Theorem 5.4.11. *Assume that the volatility satisfies assumptions in Lemma 5.4.7 and there exists an $\mathbb{R}$-valued measurable function γ_t satisfying assumptions in Lemma 5.4.10 in a way such that*

$$\sigma_t\gamma_t = \mathcal{S}_H\sigma_t - \alpha_t, \quad t \geq 0 \tag{5.31}$$

Then there exists a quasi-martingale measure for the bond market. In addition, the market is arbitrage free on $[0, T^\star]$ with transaction costs k

Proof. See [Oha09], Theorem 3.1 □

Rearranging the last term $\sigma_t\gamma_t = \mathcal{S}_H\sigma_t - \alpha_t$ to $\sigma_t\gamma_t + \alpha_t = \mathcal{S}_H\sigma_t$ we can, as in the classic case, interpret $-\gamma$ as the market price of risk. Moreover, if we set $d\widetilde{W}^H = dW^H - \gamma_t dt$ (from Lemma 5.4.10) we can write the forward curve under the measure $\mathbb{Q}_*$ as

$$f(t,T) = f(0,t) + \int_0^t \widetilde{\alpha}(s,T)ds + \sum_{i=1}^{d}\int_0^t \sigma^i(s,T)d\widetilde{W}_s^{H,i} \tag{5.32}$$

where $\widetilde{\alpha}(s,T) = \mathcal{S}_H\sigma(s,T)$

Note that the change of measure does not affect the bonds volatility nor its rate of return. This is contrary to the real world where we would expect the volatility to be a major factor in an investors decision to buy or sell a bond. A higher volatility would be deemed to give a higher return. Under the quasi-martingale measure, all bonds have the same expected rate of return regardless of their riskiness.

5.5 Modelling fractional short rates under the average risk-neutral-measure

We retrieve the short rates:

$$r(t) = f(t,t) = f(0,t) + \int_0^t \widetilde{\alpha}(s,t)ds + \int_0^t \sigma(s,t)d\widetilde{W}_s^H$$

computing $\int_0^t \widetilde{\alpha}(s,t)ds$:

$$\begin{aligned}\int_0^t \widetilde{\alpha}(s,t)ds &= \int_0^t \mathcal{S}_H\sigma(s,T)ds\\ &= \sum_{i=1}^{d}\Bigg(\int_0^t \bigg[\sigma^i(s,t)\int_0^s\int_0^{t-\theta}\sigma^i_\theta(x)\eta_H(s-\theta)dxd\theta\\ &\quad + \int_0^{t-s}\sigma^i_s(x)dx\int_0^s \sigma(\theta,t)\eta_H(s-\theta)d\theta\bigg]ds\Bigg)\end{aligned}$$

factorizing $\sigma(t,T)$ to $\xi(t)\nu(T)$, where $\xi(x),\nu(x) > 0$, $\nu(x)$ differentiable and $\xi(x)$ of bounded p-variation for $0 < p < 1/H$ yields

$$\begin{aligned}\int_0^t \widetilde{\alpha}(s,t)ds =&\nu(t)\sum_{i=1}^{d}\Bigg(\int_0^t\bigg[\int_0^s\int_\theta^t \xi^i(s)\xi^i(\theta)\nu(x)\eta_{H,i}(s-\theta)dxd\theta\\ &+\int_s^t \xi^i(s)\nu(x)dx\int_0^s \xi^i(\theta)\eta_{H,i}(s-\theta)d\theta\bigg]ds\Bigg)\\ =&\nu(t)\sum_{i=1}^{d}\Bigg(\bigg[\int_0^t\int_0^s \xi^i(s)\xi^i(\theta)\int_\theta^t \nu(x)dx\eta_{H,i}(s-\theta)d\theta ds\\ &+\int_0^t\int_0^s \xi^i(s)\xi^i(\theta)\int_s^t \eta_{H,i}(s-\theta)d\theta ds\bigg]\Bigg).\end{aligned}$$

Furthermore, setting

$$\lambda(t) := f(0,t) + \int_0^t \tilde{\alpha}(t,s)ds.$$

We can write the fractional Vasicek short rate model under $\mathbb{Q}_*$, for $t \in [0, T^\star]$ by

$$r(t) = \lambda(t) + \nu(t) \sum_{i=1}^{d} \int_0^t \xi^i(s) d\widetilde{W}_s^{H,i}. \tag{5.33}$$

However, we are interested in the fractional short rate model for $0 \leq s \leq t \leq T^\star$.

$$\begin{aligned}
r(t) - r(s) =& \lambda(t) + \nu(t) \sum_{i=1}^{d} \int_0^t \xi^i(u) d\widetilde{W}_u^{H,i} - \lambda(s) - \nu(s) \sum_{i=1}^{d} \int_0^s \xi^i(u) d\widetilde{W}_u^{H,i} \\
=& \int_s^t \lambda'(u) du + \sum_{i=1}^{d} \Big[\int_s^t \Big(\int_0^u \xi^i(v) d\widetilde{W}_v^{H,i} \Big) d\nu(u) + \int_s^t \nu(u) d \Big(\int_0^u \xi^i(v) d\widetilde{W}_v^{H,i} \Big) \Big] \\
=& \int_s^t \lambda'(u) du + \sum_{i=1}^{d} \Big[\int_s^t \nu'(u) (\int_0^u \xi^i(v) d\widetilde{W}^{H,i}) du + \int_s^t \nu(u) \xi^i(u) d\widetilde{W}_u^{H,i} \Big] \\
=& \int_s^t \Big[\lambda'(u) + \nu'(u) \sum_{i=1}^{d} \Big(\int_0^u \xi^i(v) d\widetilde{W}_v^{H,i} \Big) \Big] du + \sum_{i=1}^{d} \int_s^t \nu(u) \xi^i(u) d\widetilde{W}_u^{H,i} \\
=& \int_s^t \Big[\lambda'(u) + \nu'(u) \frac{r(u) - \lambda(u)}{\nu(u)} \Big] du + \sum_{i=1}^{d} \int_s^t \nu(u) \xi^i(u) d\widetilde{W}_u^{H,i}.
\end{aligned}$$

Using the following (Musiela-) equations for $t \in [0, T^\star]$ and $i \in \{0, 1, 2..., d\}$

$$k(t) = \lambda'(t) - \frac{\nu'(t)}{\nu(t)} \lambda(t) \tag{5.34}$$

$$a(t) = \frac{-\nu'(t)}{-\nu(t)} \tag{5.35}$$

$$\sigma^i(t) = \xi^i(t) \nu(t), \tag{5.36}$$

we end up with the following short-rate dynamics under $\mathbb{Q}_*$:

$$r(t) = r(s) + \int_s^t \Big(k(u) - a(u) r(u) \Big) du + \sum_{i=1}^{d} \int_s^t \sigma^i(u) d\widetilde{W}_u^{H,i}, \tag{5.37}$$

where we again can factor out $a(u)$ to get the mean reversion effect and hence a fractional Vasicek model under $\mathbb{Q}_*$:

$$r(t) = r(s) + \int_s^t a(u) \Big(\frac{k(u)}{a(u)} - r(u) \Big) du + \int_s^t \sigma^i(u) d\widetilde{W}_u^{H}$$

for simplicity we again denote $b(t) = \frac{k(t)}{a(t)}$. In differential form:

$$dr(t) = dr(s) + a(t)(b(t) - r(t)) dt + \sigma(t) d\widetilde{W}_t^H. \tag{5.38}$$

We want to model directly under $\mathbb{Q}_*$ and the following lemma ensures that we can find $\xi(x), \nu(x), f(0, x)$ such that Equation (5.34), Equation (5.35) and Equation (5.36) holds:

Lemma 5.5.1. *Given* b, a, σ *continuous on* $t \in [0, T^\star]$, *take*

$$\nu(t) = \exp\Big(-\int_0^t a(s)ds\Big)$$
$$\xi(t) = \sigma(t)\exp\Big(\int_0^t a(s)ds\Big)$$

and let $f(0, x)$ *be the first order linear differential equation*

$$\frac{\partial}{\partial t} f(0,t) = -a(t)f(0,t) + \big[k(t) - a(t)\int_+^t \widetilde{a}(s,t)ds - \frac{\partial}{\partial t}\int_0^t \widetilde{\alpha}(s,t)ds\big], \quad t \in [0, T^\star]$$

then Equation (5.34)*, Equation* (5.35) *and Equation* (5.36) *is satisfied.*

Proof. [Fin11], Lemma 3.3.10 □

Thus we can model short rates directly under $\mathbb{Q}_*$ given that the coefficient functions satisfies Lemma 5.5.1.

Chapter 6

Bond prices under fBm-driven Hull-White models

6.1 Conditional distributions of fBm

We recall the space of suitable integrands for fBm Λ_T^κ defined earlier and we recall that a random variable X is normally distributed with expectation $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ iff

$$\mathbb{E}[e^{iuX}] = e^{iu\mu - \frac{u^2}{2}\sigma^2}$$

Throughout the rest of this chapter we will condition on the completion of the generated σ-algebra denoted by

$$\mathcal{F}_s := \sigma\overline{\{W_v^H, 0 \leq v \leq s\}}, \quad 0 \leq s \leq T$$

We will also rescale the Hurst parameter so that the notation align with that of fractional calculus:

$$\kappa = H - \frac{1}{2}$$

The main building block going forward in this chapter is the following lemma

Lemma 6.1.1. *Let* $0 \leq t \leq T \leq T^\star$ *and* $0 \leq \kappa \leq 1$. *Then*

$$\mathbb{E}[W_T^\kappa|\mathcal{F}_t] = W_t^\kappa + \int_0^t \Psi^\kappa(t, T, v)dW_v^\kappa,$$

where

$$\begin{aligned}\Psi^\kappa(s,t,v) =& v^{-\kappa}(I_{s-}^{-\kappa}(I_{t-}^{-\kappa}(x)^\kappa \mathbf{1}_{[s,t]}(x)))(v) \\ =& \frac{\sin(\pi\kappa)}{\pi} v^{-\kappa}(s-v)^{-\kappa}\int_s^t \frac{z^\kappa(z-s)^\kappa}{z-v}dz\end{aligned}$$

for $v \in (0, t)$ *and with* $\Psi^\kappa(s, t, v) = 0$ *for* $v \in \{0, s\}$.

Proof. [FKZ13], Lemma 2.1. □

the following auxiliary result will also be utilized. Take $c \in \Lambda_T^\kappa$. We start with the expectation of the fractional process $\int_0^t c(v)dW_v^H$:

Proposition 6.1.2. *For $0 \le s \le t \le$, let $c \in \Lambda_T^H$. Then*

$$\mathbb{E}\Big[\int_0^t c(v)dW_v^\kappa \Big| W_v^\kappa, v \in [0,s]\Big] = \int_0^s c(v)dW_v^\kappa + \int_0^s \Psi_c^\kappa(s,t,v)dW_v^\kappa$$

where

$$\begin{aligned}\Psi_c^H(s,t,v) =& v^{-\kappa}(I_{s-}^{-\kappa}(I_{t-}^{-\kappa}(x)^\kappa c(z)\mathbf{1}_{[s,t]}(z)))(v)\\ =& \frac{\sin(\pi\kappa)}{\pi} v^{-\kappa}(s-v)^{-\kappa}\int_s^t \frac{z^\kappa(z-s)^\kappa}{z-v}c(z)dz\end{aligned}$$

Proof. [Dun06], Lemma 1. □

The distribution of $\int_0^t c(v)dW_v^H$ is characterized by:

Theorem 6.1.3. *Let $c \in \Lambda_T^\kappa$ and let $0 \le t \le T \le T^\star$. Then $\int_0^t c(v)dW_v^H$ is normally distributed with expectation*

$$\mathbb{E}\Big[\int_0^t c(v)dW_v^\kappa \Big| \mathcal{F}_s\Big] = \int_0^s c(v)dW_v^\kappa + \int_0^s \Psi_c^\kappa(s,t,v)dW_v^\kappa$$

and variance

$$\mathbb{V}\Big[\int_0^t c(v)dW_v^\kappa \Big| \mathcal{F}_s\Big] = ||c(x)\mathbf{1}_{[s,t]}(x)||_{\kappa,T}^2 - ||\Psi_c^\kappa(s,t,x)\mathbf{1}_{[0,s]}(x)||_{\kappa,T}^2.$$

The conditional expectation of $\int_0^t c(v)dW_v^\kappa|\mathcal{F}_s$ is already established by Proposition 6.1.2. However, utilizing Gaussianity and an application of Lemma 5.1 in [FKZ13] the conditional variance is calculated as a limit. This is done by by partitioning the interval $[0,s]$ and approximate the covariance matrices $(\Sigma_{22}^n)^{-1}\Sigma_{21}^n$ and $\Sigma_{12}^n(\Sigma_{22}^n)^{-1}\Sigma_{21}^n$. For the full proof, see [FKZ13], Theorem 3.1.

Tying in to the overall theme, consider a fractional Hull-White processes with the dynamics

$$dX(t) = (k(t) - a(t)X(t))dt + \sigma(t)dW_t^\kappa, \quad X(0) \in \mathbb{R}, \quad t \in [0,T] \tag{6.1}$$

and solution given by

$$\begin{aligned}X(t) =& X(0)\exp\Big(-\int_0^t a(s)ds\Big) + \int_0^t \exp\Big(-\int_s^t a(u)du\Big)k(s)ds\\ &+ \int_0^t \exp\Big(-\int_s^t a(u)du\Big)\sigma(s)dW_s^\kappa\end{aligned} \tag{6.2}$$

We constrain $k(x)$ and $a(x)$ to be locally integrable and $\sigma(x) \in \Lambda_T^\kappa$. Setting $\sigma(x) \neq 0$ yields the following equality

$$\mathcal{F}_s := \sigma\overline{\{W_v^\kappa, 0 \le v \le s\}} = \sigma\overline{\{X(v), 0 \le v \le s\}}, \quad 0 \le s \le T.$$

Recall the norm on Λ_T^κ;

$$||f||_{\kappa,T}^2 = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)}\int_0^T s^{-2\kappa}[I_{T-}^\kappa((x)^\kappa f(x))(s)]^2\,ds \tag{6.3}$$

By Equation (6.2), we see that $X(t)|\mathcal{F}_s$ is Gaussian distributed. Setting $c(x) = \exp(-\int_x^t a(w)dw)\sigma(x)$ and invoking Theorem 6.1.3 yields the following result:

Theorem 6.1.4. *Let $0 \leq t \leq T \leq T^\star$ and set $c(x) = \exp(-\int_x^t a(w)dw)\sigma(x)$, $c(x) \in \Lambda_T^\kappa$. Then $X(t)|\mathcal{F}_t$ is normally distributed with expectation*

$$\begin{aligned}\mathbb{E}\big[X(t)|\mathcal{F}_t\big] =&X(s)\exp\Big(-\int_t^T a(v)dv\Big) + \int_t^T \exp\Big(-\int_v^T a(u)du\Big)k(v)dv \\ &+ \int_0^t \Psi_c^\kappa(t,T,v)dW_v^\kappa\end{aligned}$$

variance

$$\mathbb{V}\big[X(t)|\mathcal{F}_s\big] = ||c(x)\mathbf{1}_{[s,t]}(x)||_{\kappa,T}^2 - ||\Psi_c^\kappa(s,t,x)\mathbf{1}_{[0,s]}(x)||_{\kappa,T}^2$$

and characteristic function given by

$$\begin{aligned}\mathbb{E}[e^{iuX(t)}|\mathcal{F}_s] =&\exp\Big\{iu\Big[X(s)\exp\Big(-\int_s^t a(v)dv\Big) + \int_s^t \exp\Big(-\int_v^t a(w)dw\Big)k(v)dv \\ &+ \int_0^s \Psi_c^\kappa(s,t,v)dW_v^\kappa\Big]\Big\} \\ &\times \exp\Big\{-\frac{u^2}{2}[||\mathbf{1}_{[s,t]}(x)||_{\kappa,T}^2 - ||\Psi^\kappa(s,t,x)\mathbf{1}_{[0,s]}(x)||_{\kappa,T}^2]\Big\}\end{aligned}$$

Proof. For a full proof, see [FKZ13], Theorem 3.2 □

Assume that $\sigma(x)$ and $1/\sigma(x)$ are of bounded p-variation for some $0 < p < 1/\kappa$. Inverting Equation (6.1) ([FKZ13], proof of Proposition 3.1) we can write the the distribution in terms of X only.

Proposition 6.1.5. *Assume the same situation as in Theorem 6.1.4. Assume that $\sigma(x)$ and $1/\sigma(x)$ are of bounded p-variation for some $0 < p < 1/\kappa$. Let $0 \leq t \leq T$ and set $c(x) = \exp(-\int_x^t a(w)dw)\sigma(x)$. Then $X(t)|\mathcal{F}_s$ is normally distributed with expectation*

$$\begin{aligned}\mathbb{E}\big[X(t)|\mathcal{F}_s\big] =&X(s)\exp\Big(-\int_s^t a(v)dv\Big) + \int_s^t \exp\Big(-\int_v^t a(w)dw\Big)k(v)dv \\ &- \int_0^s \Psi_c^\kappa(s,t,b)\frac{\tilde{v}}{\sigma(v)}dv + \int_0^s \Psi_c^\kappa(s,t,v)\frac{a(v)}{\sigma(v)}X(v)dv \\ &+ \int_0^t \Psi_c^\kappa(s,t,v)\frac{1}{\sigma(v)}dX(v), \quad t \in [0,T]\end{aligned}$$

and variance

$$\mathbb{V}\big[X(t)|\mathcal{F}_s\big] = ||c(x)\mathbf{1}_{[s,t]}(x)||_{\kappa,T}^2 - ||\Psi_c^H(s,t,x)\mathbf{1}_{[0,s]}(x)||_{\kappa,T}^2$$

where $c(x) = \exp(-\int_x^t a(v)dv)\sigma(x)$. The characteristic function is given by

$$\begin{aligned}\mathbb{E}[e^{iuX(t)}|\mathcal{F}_t] =&\exp\Big\{iu\Big[X(t)\exp\Big(-\int_t^T a(v)dv\Big) + \int_t^T \exp\Big(-\int_v^T a(w)dw\Big)k(v)dv \\ &- \int_0^t \Psi_c^\kappa(t,T,v)\frac{k(v)}{\sigma(v)}dX(v) + \int_0^t \Psi_c^\kappa(t,T,v)\frac{a(v)}{\sigma(v)}dX(v) \\ &+ \int_0^t \Psi_c^\kappa(t,T,v)\frac{1}{\sigma(v)}dX(v)\Big]\Big\} \\ &\times \exp\Big(-\frac{u^2}{2}[||\mathbf{1}_{[t,T]}(x)||_{\kappa,T^\star}^2 - ||\Psi^\kappa(t,T,x)\mathbf{1}_{[0,t]}(x)||_{\kappa,T^\star}^2]\Big)\end{aligned}$$

Proof. [FKZ13], Proposition 3.1 and references therein. □

As bond prices are not given in terms of the "pure" short rate $X(t)$, but rather the integrated short rate, we need a formula for the conditional characteristic function of $\int_s^t X(v)dv$. To ease notation, define

$$D(x,t) = \int_x^t \exp\Big(-\int_x^v a(w)dw\Big)dv, \quad t \in [0,T]$$

and assume in the following that

$$c(x) = D(x,t)\sigma(x) \in \Lambda_T^\kappa \tag{6.4}$$

Proposition 6.1.6. *Let X be the process in Equation* (6.2)*. Let $0 \leq t \leq T \leq T^\star$. Assume that $\sigma(x)$ and $1/\sigma(x)$ are of bounded p-variation for some $0 < p < 1/\kappa$. Then for $u \in \mathbb{R}$:*

$$\begin{aligned}\mathbb{E}[e^{iu\int_0^t X(v)dv}|\mathcal{F}_s] =& \exp\Big(iu\Big[\int_0^t X(v)dv + D(t,T)X(t) + \int_t^T D(v,T)k(v)dv\\ &- \int_0^t \Psi_c^\kappa(t,T,v)\frac{k(v)}{\sigma(v)}dv + \int_0^t \Psi_c^\kappa(t,T,v)\frac{a(v)}{\sigma(v)}X(v)dv\\ &+ \int_0^t \Psi_c^\kappa(t,T,v)\frac{1}{\sigma(v)}dX(v)\Big]\Big)\\ &\times \exp\Big(-\frac{u^2}{2}[||c(x)\mathbf{1}_{[t,T]}(x)||^2_{\kappa,T^\star} - ||\Psi_c^\kappa(t,T,x)\mathbf{1}_[0,t](x)||_{\kappa,T^\star}]\Big)\end{aligned}$$

Proof. [FKZ13], Proposition 3.2 □

6.2 Zero-coupon bonds for fBm-driven short rates

Due to Gaussianity we can extend the characteristic function from Proposition 6.1.6 to $\mathbb{C}$. If we set $u = i$ we see that the above formula becomes the price of a zero-coupon Bond for a fractional Hull-White short rate process $r(t) = X(t)$, under the measure $\mathbb{Q}_*$. We recall r is given by

$$dr(t) = (k(t) - a(t)r(t))dt + \sigma(t)d\widetilde{W}_t^\kappa, \quad r(0) \in \mathbb{R}, \quad t \in [0,T] \tag{6.5}$$

Theorem 6.2.1 (Zero-coupon bond price for fractional Hull-White models). *Let $r(t) = X(t)$ be the process in Equation* (6.2)*. Let $0 \leq t \leq T \leq T^\star$. Assume that $\sigma(x)$ and $1/\sigma(x)$ are of bounded p-variation for some $0 < p < 1/\kappa$, $\kappa \neq 0$. Assume that $c(x) \in \Lambda_T^\kappa$ for $\kappa \in (0,1)$. Then the price of a zero-coupon bond $P_{fHW}(t,T)$ at time t with maturity T is given by:*

$$\begin{aligned}P_{fHW}(t,T) =& \mathbb{E}_{\mathbb{Q}_*}\Big[e^{-\int_0^T r(v)dv}\Big|\mathcal{F}_t\Big]\\ =& \exp\Big\{-\Big[D(t,T)r(t) + \int_t^T D(v,T)k(v)dv\\ &- \int_0^t \Psi_c^\kappa(t,T,v)\frac{k(v)}{\sigma(v)}dv + \int_0^t \Psi_c^\kappa(t,T,v)\frac{a(v)}{\sigma(v)}r(v)dv\\ &+ \int_0^t \Psi_c^\kappa(t,T,v)\frac{1}{\sigma(v)}dr(v)\Big]\Big\}\end{aligned}$$

$$\times \exp\left(-\frac{1}{2}[||c(x)\mathbf{1}_{[t,T]}(x)||^2_{\kappa,T^\star} - ||\Psi^\kappa_c(t,T,x)\mathbf{1}_[0,t](x)||_{\kappa,T^\star}]\right)$$

Remark 6.2.2. *Theorem 6.2.1 is originally given for d-dimensional short rates. We will focus on one-dimensional models and so we only state the $d = 1$ case. The original theorem and its proof can be found in [FKZ13], Theorem 4.1.*

For the reasons mentioned above with regards to the dangers of the Hull-White method, we will concentrate on a fractional Vasicek model with constant coefficients given by

$$dr(t) = a(b - r(t))dt + \sigma d\widetilde{W}^\kappa_t, \quad r(0) \in \mathbb{R}, \quad t \in [0,T] \tag{6.6}$$

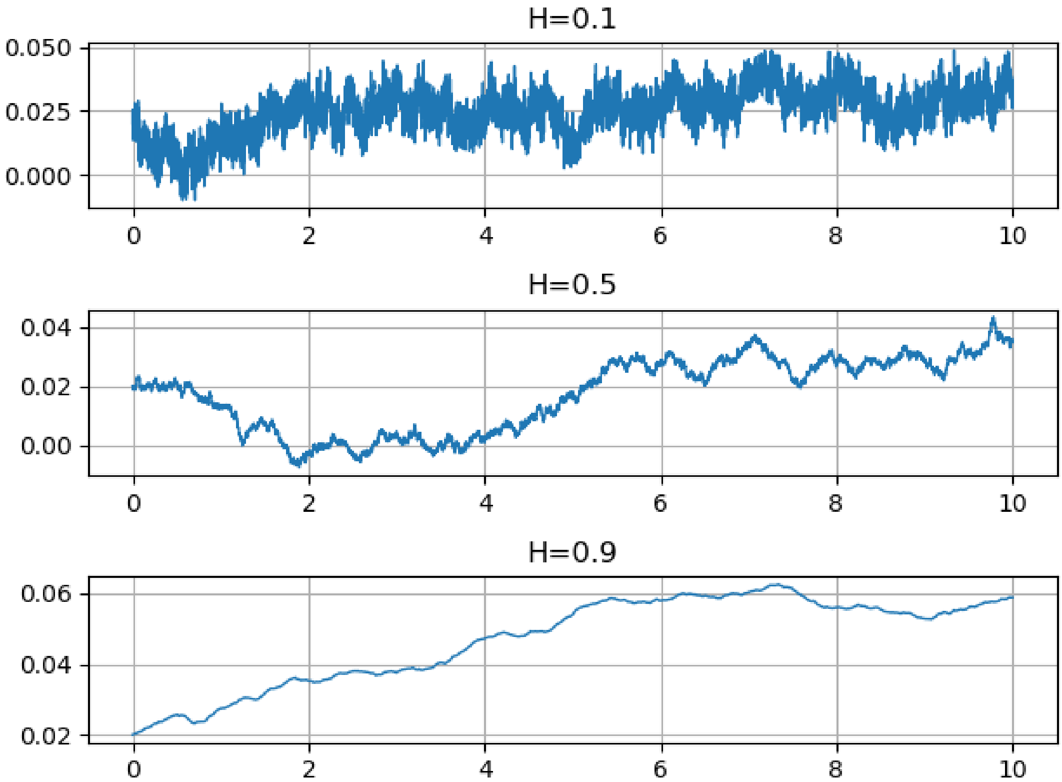


Figure 6.1: Sample paths from the fractional Vasicek model for some values of H

The price of a zero-coupon bond based on such a model can easily be derived from Theorem 6.2.1:

Corollary 6.2.3 (Zero-coupon bond price for fractional Vasicek models). *Assume the same situation as in Theorem 6.2.1 and short rates $r(t)$ given by Equation* (6.6). *The price of a zero-coupon bond P_{fV} is given by:*

$$\begin{aligned} P_{fV}(t,T) = \exp\Bigg\{-\Bigg[&D(t,T)\left(r(0)e^{-at} + ab\int_0^t e^{-a(t-s)}\,ds + \int_0^t e^{-a(t-s)}\sigma dW^\kappa_s\right) \\ &+ ab\int_t^T D(v,T)dv - \frac{ab}{\sigma}\int_0^t \Psi^\kappa_c(t,T,v)dv \\ &+ \frac{a}{\sigma}\int_0^t \Psi^\kappa_c(t,T,v)\left(r(0)e^{-av} + ab\int_0^v e^{-a(v-s)}\,ds + \int_0^v e^{-a(v-s)}\sigma dW^\kappa_s\right)dv \end{aligned}$$

$$+\frac{1}{\sigma}\int_0^t \Psi_c^\kappa(t,T,v)(a(b-1)\Big(r(0)e^{-av}+ab\int_0^v e^{-a(v-s)}\,ds+\int_0^v e^{-a(v-s)}\sigma dW_s^\kappa\Big)\,dt$$
$$+\int_0^t \Psi_c^\kappa(t,T,v)dW_t^\kappa\Big]\Big\}$$
$$\times\exp\Big(-\frac{1}{2}[||c(x)\mathbf{1}_{[t,T]}(x)||^2_{\kappa,T^\star}-||\Psi_c^\kappa(t,T,x)\mathbf{1}_{[0,t]}(x)||_{\kappa,T^\star}]\Big) \tag{6.7}$$

where $c(x)=\sigma D(x,T)$

Proof. Take the equation from Theorem 6.2.1. The result follow from setting $k(t)=ab$, $a(t)=a$ and $\sigma(t)=\sigma$ and inserting Equation (6.2) for $r(t)$-terms. □

Though this equation is still mildly abhorrent, it is now in terms of fBm and not the short rate process. It is also worth to note that the most messy parts disappear when $t=0$ as the next example shows

Example 6.2.4 (Fractional Vasicek ZCB price for $t=0$). Using $t=0$, the above equation simplifies to the more tractable

$$P_{\text{fV}}(0,T)=\exp\Big\{-\Big[D(0,T)\Big(r(0)+k\int_0^T D(v,T)dv\Big]\Big\}$$
$$\times\exp\Big(-\frac{1}{2}[||c(x)\mathbf{1}_{[0,T]}(x)||^2_{\kappa,T^\star}]\Big)$$

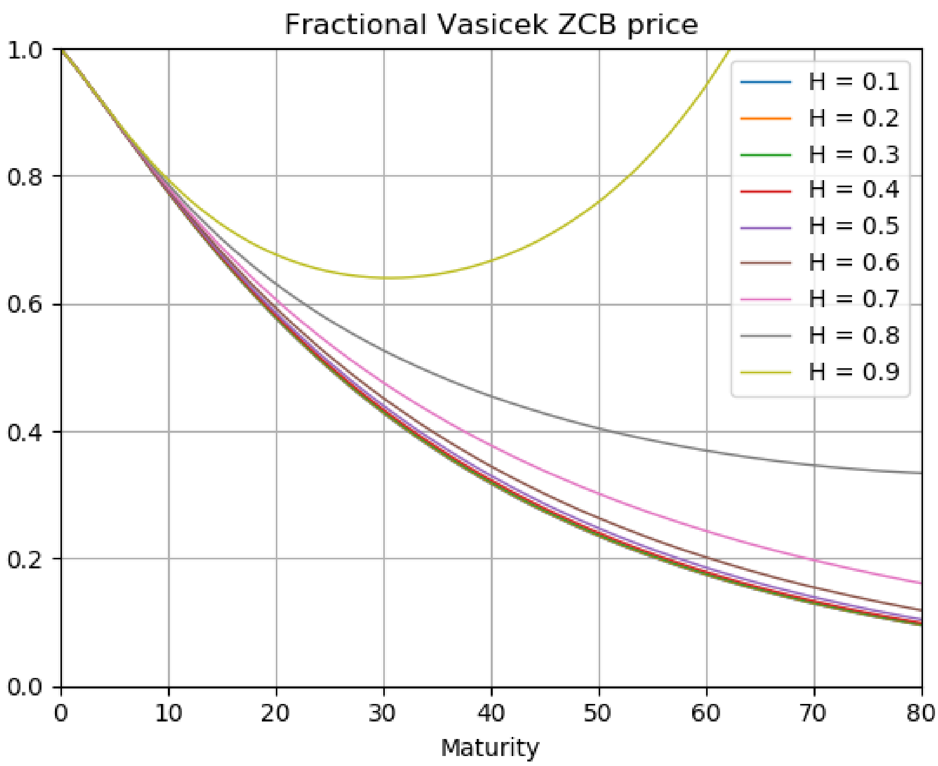


Figure 6.2: Prices of a zero-coupon bond under the fractional Vasicek model with parameters $a=20\%, b=3\%$ and $\sigma=1\%$ for some values of H

Remark 6.2.5 (What about parameter estimation?). *It is out of the scope of this thesis to do parameter estimation of the fractional Vasicek model, both with respect to the Hurst parameter, but also with respect to* a*,* b *and* σ*. We do however direct the interested reader to [TXY20] for a discussion of Maximum Likelihood Estimation with respect to the fractional Vasicek model and to [GSP08] for a discussion of the Rescaled Range (R/S)-method for the estimation of long-range dependence in financial datasets. There is also a discussion of estimation methods with respect to* H *in [Bia+10].*

Chapter 7

Life Insurance reserves under fractional interest rates

Arriving at the main result of this thesis, we apply the theory of fractional interest rates from previous chapters to life insurance reserves.

7.1 Life insurance reserves

Under the measure $\mathbb{Q}_*$, the definition of life insurance reserves with stochastic interest rates is given by;

$$V_j(t, A_g) = \mathbb{E}_{\mathbb{Q}_*}\left[\frac{1}{v(t)}\int_I v(s)dA_g(s)\Big|\mathcal{F}_t\right], \tag{7.1}$$

As in the Markovian case, we can expand this to be:

$$\begin{aligned} V_i^+(t, r(t)) = &\sum_{j\in\mathcal{S}} \mathbb{E}_{\mathbb{Q}_*}\left[\frac{v(T)}{v(t)}|\mathcal{F}_t\right] p_{ij}^x(t,T)\Delta a_i(T) \\ &+ \sum_{j\in\mathcal{S}} \int_t^T \mathbb{E}_{\mathbb{Q}_*}\left[\frac{v(s)}{v(t)}|\mathcal{F}_t\right] p_{ij}^x(t,s)\dot{a}_i(t) \\ &+ \sum_{j\in\mathcal{S},k\neq j} \int_t^T \mathbb{E}_{\mathbb{Q}_*}\left[\frac{v(s)}{v(t)}|\mathcal{F}_t\right] p_{ij}^x(t,s)\mu_{jk}^x(s)a_{ij}(t)ds \end{aligned}$$

setting

$$P_{\text{fHW}}(t,T)(t,T) = \mathbb{E}_{\mathbb{Q}_*}\left[\frac{v(T)}{v(t)}|\mathcal{F}_t\right]$$

yields, analogous to the Markovian case, the following expression

Definition 7.1.1 (Mathematical reserves with fractional stochastic interest rates). *Let x be the age of the insured at the start of the contract and let $P_{fHW}(t,T)$ be given as in Theorem 6.2.1. The value at time t of the liability A associated with to policy functions a_i and a_{ij}, $i, j \in \mathcal{S}$, $j \neq i$, given that the insured is in state i at time t is given by*

$$\begin{aligned} V_i^+(t, r(t)) = &\sum_{j\in\mathcal{S}} P_{fHW}(t,T)p_{ij}^x(t,T)\Delta a_i(T) \\ &+ \sum_{j\in\mathcal{S}} \int_t^T P_{fHW}(t,s)p_{ij}^x(t,s)\dot{a}_i(t) \end{aligned}$$

$$+ \sum_{j \in \mathcal{S}, k \neq j} \int_t^T P_{fHW}(t,s) p_{ij}^x(t,s) \mu_{jk}^x(s) a_{ij}(t) ds$$

These reserves can handle time-dependent parameters and can in principle be computed for every $t \in [0,T]$. However, the formula for the bond used, given by Theorem 6.2.1, forces the life insurance reserves to be a non-Markovian stochastic variable due to the loss of semimartingality for $t \in (0,T)$. Therefore, the derivation of a Thiele's PDE (see e.g [Bãn22]) and computing the present value for time $t \in (0,T]$ as in Figure 3.5 is impossible (at least in the classical sense). For $t = 0$, however, the computation is quite straight forward as indicated by Example 6.2.4. This is illustrated by the next example.

Example 7.1.2 (Pension policy with fractional Vasicek interest rates). Consider the same policy as in Example 3.1.15; I.e. a policy which pays out a yearly benefit B from T_0 until T. We still do not consider premiums. Recall that the policy functions are then given by

$$a_*(t) = \begin{cases} 0, & t \in [0,T_0) \\ B(t-T_0), & t \in [T_0,T] \end{cases}$$

which gives us

$$\dot{a}_*(t) = \begin{cases} 0, & t \in [0,T_0) \\ B, & t \in [T_0,T] \end{cases}$$

Our stochastic interest rates are modelled by a fractional Vasicek model. The mathematical reserves are then expressed by

$$V_*^+(t, r(t)) = B \int_{\max(t,T_0)}^T P_{\mathrm{fV}}(t,s) p_{**}^x(t,s) ds$$

where $P_{\mathrm{fV}}(t,s)$ is given as in Equation (6.7). As in the above cases, let the insured be a male aged 30 at the start of the contract with the pension payouts of 200000 NOK a year starting at 70 ($T_0 = 40$) and lasting until he is 110 ($T = 80$) with Vasicek parameters given by $a = 20\%, b = 3\%$ and $\sigma = 1\%$. We will compute the single premiums for some values of H, including the Markovian case $H = 1/2$.

In Figure 7.1 we see that the single premium for interest rates with $H = 0.9$ is significantly higher than the life insurance reserves based on "standard" Vasicek interest rates ($H = 0.5$) for all initial interest rates r_0. Seeing the results for the bond prices, this should come at no surprise.

7.2 Numerical analysis

Admittedly, the parameters in the above examples is chosen carefully so that the present values does not explode. This happens with certain combinations of values and especially over the long maturities used in connection with life insurance. It is therefore interesting to have a closer look at the parameters in the fractional Vasicek model and how they influence the single premium of a life insurance policy. Varying any of the parameters vary the price of the bond, and as the value of the policy is contingent on the price of the bond, we expect the former to be very sensitive to the latter.

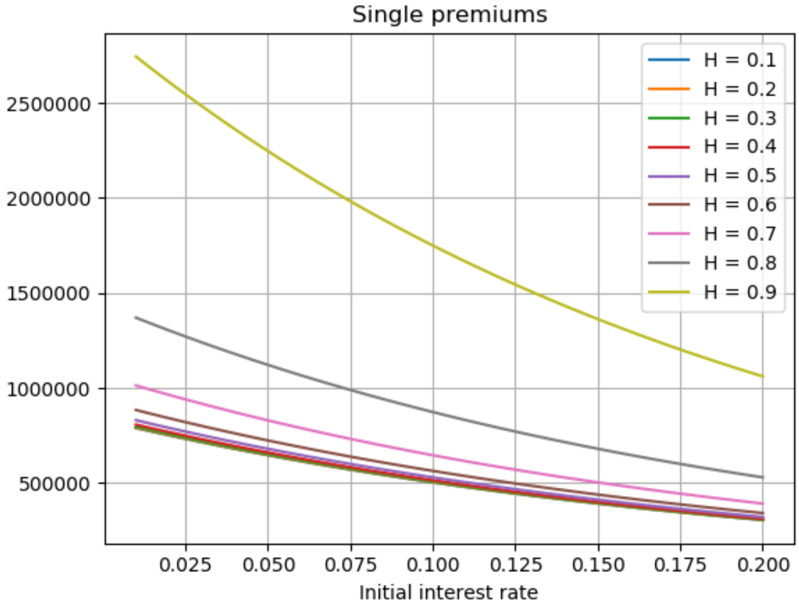


Figure 7.1: Single premiums for the policy in Example 7.1.2

7.2.1 Distributional characteristics of fractional Vasicek interest rate paths

As a preliminary investigation, we study the distribution of the fractional Vasicek interest rate model endpoints, given by Equation (6.1) and see how the Hurst-parameter influence the distribution of $r(T)$. From Theorem 6.1.4 we know that $r(T)$ is normally distributed. With $s = 0$, and $b = k/a$, the moments become:

$$\mathbb{E}[r(T)] = r(0)\exp(-aT) + b(1 - \exp(-aT))$$
$$\mathbb{V}[r(T)] = ||\sigma \exp(-a(T-x)\mathbf{1}_{[0,T]}(x)||^2_{\kappa,T}$$

In the following, we will use $H = \kappa + \frac{1}{2}$. Now we have explicit expressions for the functions connected to the distribution of $r(T)$.

We perform 5000 simulations of fractional Vasicek sample paths endpoints.

The theoretical mean and standard deviation and the mean and standard deviation obtained from the Monte-Carlo simulation is summarized in the following table:

Mean and Variance of $r(80)$				
H	Theoretical mean	Empirical mean	Theoretical S.D	Empirical S.D
0.1	0.030	0.0301	0.0069	0.0080
0.3	0.030	0.0299	0.0096	0.0108
0.5	0.030	0.0304	0.0158	0.0160
0.7	0.030	0.0303	0.0218	0.0241
0.9	0.030	0.0302	0.0356	0.0386

First, we see that the Monte-Carlo simulations are fairly accurate. More importantly is that the standard deviation increase quite significantly with the Hurst-parameter. Our suspicion, that fractional Vasicek interest rates with high Hurst-parameter values is dangerous in the sense of being more volatile, seems to find steady ground. The increased variance of higher values of H is very well illustrated in Figure 7.2.

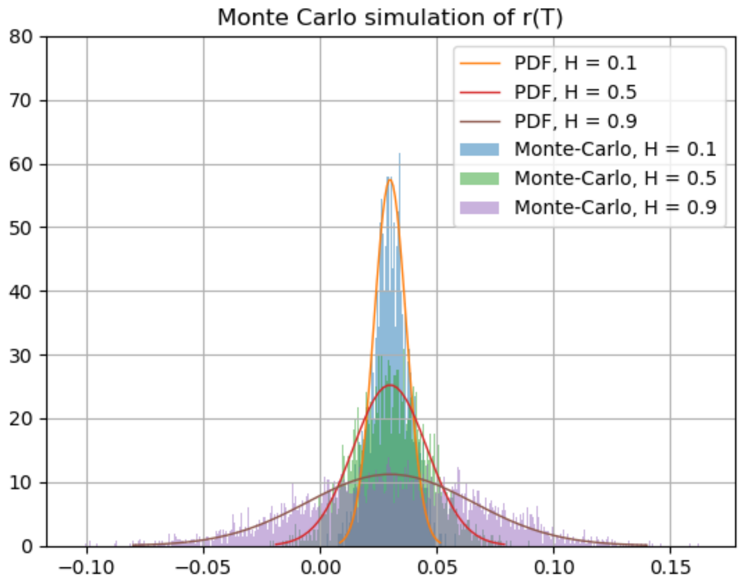


Figure 7.2

From the simulations we can also compare the theoretical and empirical probability of the interest rates ending up sub-zero. The results are summarized in the following table:

$\mathbb{P}(r(80) < 0 \mid H)$		
H	Theoretical	Empirical
0.1	0	0.000
0.3	0.0009	0.0038
0.5	0.0289	0.0266
0.7	0.0846	0.1052
0.9	0.1996	0.2126

Clearly, as H increases, so does the probability of $r(T)$ being negative. This probability increase rapidly with H. The probability of negative interest rates will of course affect bond prices and in turn single premiums. In the following we explore this dynamic further.

7.2.2 Sensitivity analysis

To see how this plays out with respect to the single premiums of a life insurance reserve, we will now look at the same pension policy as Example 7.1.2 and vary one parameter while letting the remaining parameters be fixed. We will use maturity $T = 80$ and reference values $r_0 = 2\%$, $a = 20\%$, $b = 3\%$ and $\sigma = 1\%$.

Sensitivity with respect to r_0

From Figure 7.3 we see that an increase in r_0 increase the single premiums modestly, and it seems, almost linearly. We see a breaking point for all values of r_0 when $H \approx 0.75$ where the single premiums suddenly increase exponentially.

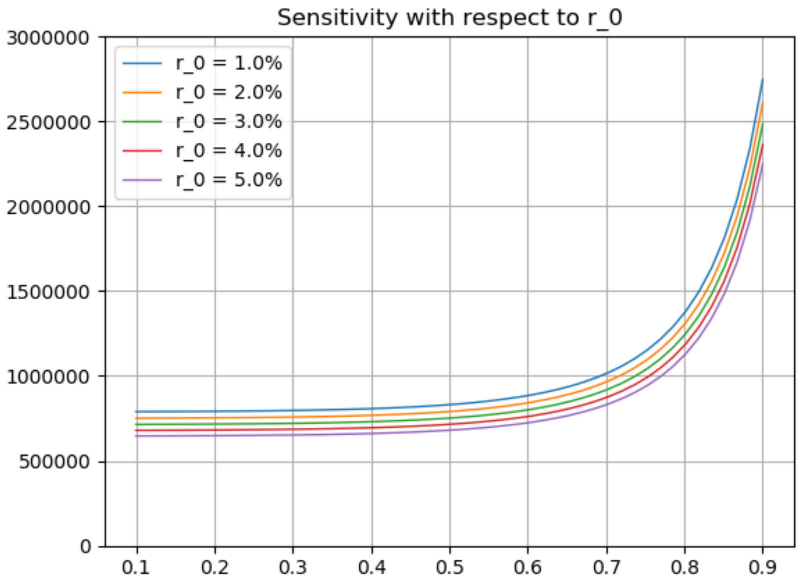


Figure 7.3

Sensitivity with respect to a

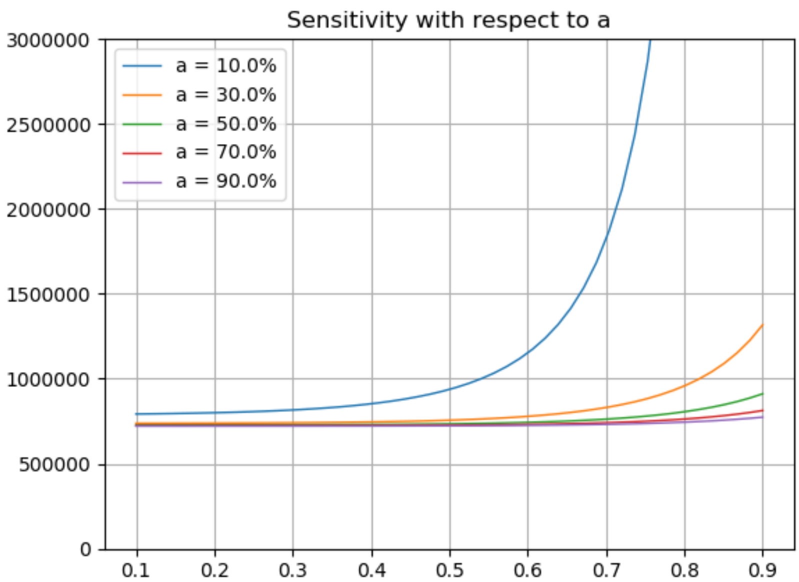


Figure 7.4

Recall that a determines the mean-reversion speed of the fractional Vasicek model. In Figure 7.4 we see that values of a will direct the interest rate toward the long term mean (and thus away from ending up sub zero) with a higher force. Therefore we see that the single premium is higher with lower values of a and vice versa as the risk of going sub-zero decreases with higher values of a. The increase in single premiums increase rapidly with small values of a and the breaking point for $a = 0.1$ comes quickly at $H \approx 0.6$. On the other hand, the breaking point is less pronounced or non-existent for the other, higher values of a.

Sensitivity with respect to b

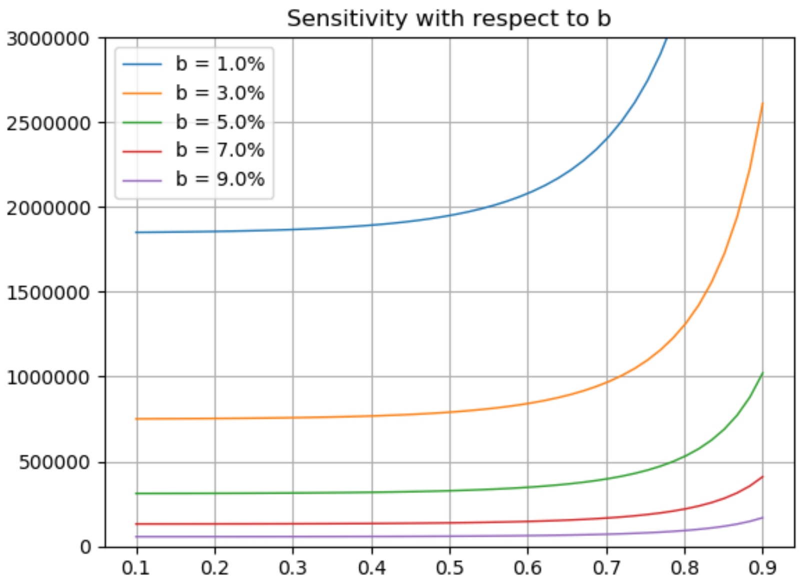


Figure 7.5

It is worth to note that the maturities of the bonds involved in the computations are very long (up to 80 years) which again increase the possibility of negative interest rate, especially if the long-term mean is low. However, we see from Figure 7.5 that it does not affect the single premium in as a dramatic way as a; a decrease in b shifts the value upwards while lowering the breaking point about 0.1 units of H. However, the shift upwards seem to be almost exponential, which does correlate well with the fact that a lower long-term mean obviously increase the probability of negative interest rates.

Sensitivity with respect to σ

The influence of σ on the single premium is quite dramatic. We see from Figure 7.6 that higher values of σ increases the single premium significantly. We see that σ clearly influence the single premiums the most and we see a rapid increase already at $H \approx 0.4$ for all values bar $\sigma = 0.5\%$. Also, the explosions comes earlier, even as early as $H \approx 0.55$ for the highest value. The explanation is straight forward; for higher the value of σ, the more volatile the process is and the more likely it is to turn negative or be lower than the interest rate at $t = T$.

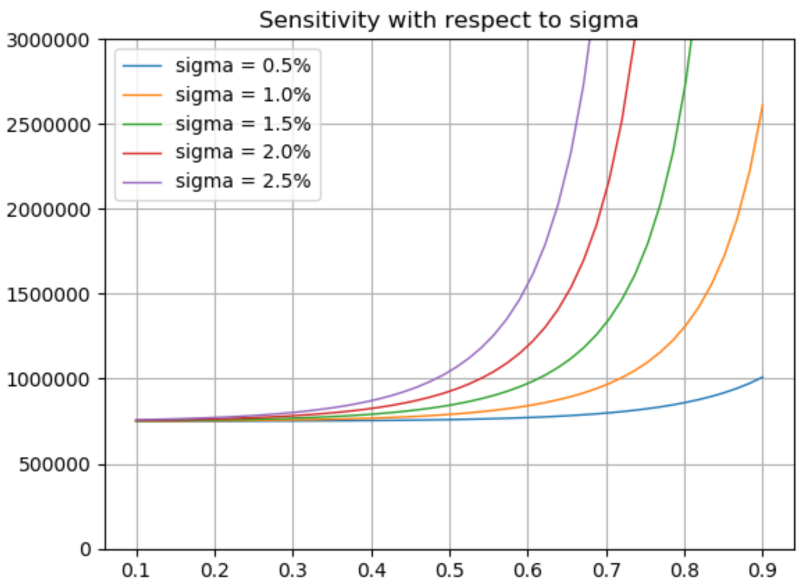

Sensitivity with respect to sigma
sigma = 0.5%
sigma = 1.0%
sigma = 1.5%
sigma = 2.0%
sigma = 2.5%
3000000
2500000
2000000
1500000
1000000
500000
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

Figure 7.6

Chapter 8

Conclusion and further work

After an introduction to mathematical finance and life insurance mathematics we discussed fractional Brownian motion and reviewed arbitrage theory in markets with transaction costs, including a fractional version of the HJM framework (due to [Oha09]). We then followed [FKZ13] and [Fin11] in deriving short rates under transaction costs and priced a zero-coupon bond in this model. We then implemented these results in life insurance policies and studied an example pension policy under the fractional Vasicek model.

Investigating the example pension policy and the distribution of $r(T)$, we saw that an increase in the Hurst-parameter H is dangerous with respect to single premiums. The sensitivity analysis also showed how an increase in H made the reserves a lot more sensitive to changes in variables, especially the volatility, which easily exploded for high values of H. It is, however, doubtful that interest rates show that much persistence. In any case, it suggests that more persistence in the interest rates equal a higher single premium for life insurance policies. As persistence is what has been reported ([McC+04]), this could be of interest to practitioners in the industry.

During the work of this thesis, several related topics that could prove fruitful has come to mind. First, further research on the persistency or roughness of financial assets is warranted in order to assess the usefulness of fractional interest rate models in life insurance and finance. The existing literature on the persistence of interest rates referenced in this thesis is from 2004 and earlier and could be outdated and new analyses should be conducted.

From the authors of [FKZ13] and [Fin11], whose work we have utilized extensively in this thesis, there is more implementations of fractional short rates that can be used. For instance one could implement defaultable bonds driven by fractional Hull-White models in life insurance policies to account for credit risk driven by fractional Brownian motion (see [BFK13]).

Another topic to explore could be regime-switching interest rate models based on fractional Brownian motion, where for instance the long-term mean $b(t)$ in the Hull-White model would change when $r(t)$ exceeded a threshold. This could be used to better model shifts such as the interest rates before and after the financial crisis of 2008. In the same vein, studying the implementation of fractional Brownian motion with time-dependent Hurst-parameters as driving noise in interest rate models could also prove fruitful as one could imagine that the persistency of the interest rates decreases or increases during the 'lifetime' of a policy.

Finally, as seems to be mandatory when writing a thesis in a field bordering on statistics, I must mention George Box's principle that all models are wrong, but some are useful. Life insurance reserves under fractional Hull-White interest rates is, I believe, a useful model. Arguably, one downside of the model is the lack of Markiovanity. This leads to grizzly expressions and often less than elegant mathematics. However, as is often the case, while we lose some elegance, we gain some realisticness. Transaction costs (occurs in real life) is added to the model as well as persistence or roughness in interest rates (might occur in real life).

In sum, modelling life insurance reserves with fractional interest rates seem to highlight a substantial risk not shown in the classic Markovian models. One could say, with apologies to Dermon and Willmott, "while all models sweep dirt under the rug, the non-Markovian model here presented also makes some of the dirt in the Markovian case visible". And, I believe, again with apologies, that it does so without excessively sacrifice reality for elegance. In any case I hope it could be a useful addition to the actuary's toolbox.

Appendix A

Theory

A.1 Probability theory

Definition A.1.1 (Characteristic function). *The characteristic function of a random variable X is given by*

$$\phi(t) = \mathbb{E}(e^{itX})$$

Definition A.1.2 ($\mathbb{P}$-a.s). *Let $(\mathbb{P}, \Omega, \mathcal{F})$ be a probability space. An event $E \in \mathcal{F}$ happens $\mathbb{P}$-almost surely if $\mathbb{P}(E) = 1$. It is abbreviated $\mathbb{P}$-a.s*

$\mathbb{P}$-almost surely is used interchangeably with its measure theoretical counterpart $\mathbb{P}$-almost everywhere. We write *a.s* when it is obvious which measure it is referred to.

Definition A.1.3 (Convergence in probability). *The sequence of random variables $\{X_n\}$ converges to X in probability if for all $\varepsilon \geq 0$*

$$\mathbb{P}\{|X_n - X| > \varepsilon\} \longrightarrow 0 \text{ as } n \to \infty$$

A.2 Spaces, measures and norms

Definition A.2.1 (Banach space). *A Banach space is a complete normed space $(X, \|\cdot\|)$*

Definition A.2.2 (Hilbert-space). *A complete inner product space is called a Hilbert space*

Definition A.2.3 (L^p-space). *L^p is a Hilbert space of functions f for which the following holds*

$$\int |f|^p \, dx < \infty$$

with norm given by

$$||f||_p = \left(\int |f| \, d\mu \right)^{\frac{1}{p}} \tag{A.1}$$

Note that the space L^2 has an inner product given by

$$\langle f, g \rangle_{L^2} = \int f g \, d\mu \tag{A.2}$$

L^2 is the only L^p space that is an inner product space.

Definition A.2.4 (Sobolev-space). *The Sobolev space $W^{s,p}$ is defined by*

$$W^{s,p}(\Omega) = \{f \in L^p(\Omega) : \forall|\alpha| \leq s, \, \partial^\alpha f \in L^p(\Omega)\}$$

where Ω is an open subset of R^d, $p \in [1, \infty]$ and $s \in \mathbb{N}$. Also $\alpha = (\alpha_1, ..., \alpha_d)$, $|\alpha| = \alpha_1 + ... + \alpha_d$ and the derivatives $\partial_x^\alpha f = \partial_{x_1}^{\alpha_1} ... \partial_{x_d}^{\alpha_d} f$ are taken in a weak sense.

Note that the Sobolev space is a Banach space when endowed with the norm

$$||f||_{s,p,\Omega} = \sum_{|\alpha| \leq s} ||\partial_x^\alpha f||_{L^p(\Omega)}$$

and a Hilbert space when $p = 2$.

Definition A.2.5 (Hilbert-Schmidt operator). *Let $\mathcal{H}$ be a Hilbert space and $\{e_i, i \in I\}$ an orthonormal basis for $\mathcal{H}$. An operator T for which $\sum_{i \in I} ||Te_i||^2 < \infty$ is a self-adjoint ideal of $\mathcal{B}(\mathcal{H})$ is called an Hilbert-Schmidt operator on $\mathcal{H}$.*

Definition A.2.6 (Locally integrable function and L_{loc}-space). *A locally integrable function is a function which is integrable on every compact subset of its domain of definition. The space of these functions is denoted L_{loc}.*

Definition A.2.7 (Finite signed measure). *Let X be a set and Σ a σ-algebra such that (X, Σ) is a measurable space. A finite signed measure is a set function*

$$\mu : \Sigma \to \mathbb{R} \setminus (-\infty, \infty)$$

such that the usual requirements on a measure is fulfilled:

1. $\mu(\emptyset) = 0$
2. *For disjoint sets $A_1, A_2, ... A_n$ of Σ we have*

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

Definition A.2.8 (Total variation norm). *Let μ be a signed measure on a measurable space (X, Σ) and define for all $E \in \Sigma$ the upper- and lower variation*

$$W^+(\mu, E) = \sup\{\mu(A) | A \in \Sigma \text{ and } A \subset E\}$$
$$W^-(\mu, E) = \inf\{\mu(A) | A \in \Sigma \text{ and } A \subset E\}$$

then the total variation norm is given by

$$||\mu||_{TV} = W^+(\mu, E) - W^-(\mu, E)$$

A.3 Miscellaneous definitions

Definition A.3.1 (Semigroup). *Let $L(X)$ be the Banach space of bounded linear operators on X. A family $S(t) \in L(X), t \geq 0$ of bounded operators on a Banach space X is called a strongly continuous semigroup if*

1. $S(0) = I$*, where I is the identity operator on X*

2. $S(t+s) = S(t)S(s)$ *for every* $t, s \geq 0$
3. $\lim_{t\to 0^+} S(t)x = x$ *for every* $x \in X$

[GM11] def.1.1

Definition A.3.2 (Infinitesmal generator). *Let $S(t)$ be a strongly continuous semigroup on a Banach space X. The linear operator A with domain*

$$D(A) = \left\{ x \in X : \lim_{t\to 0} \frac{S(t)x - x}{t} \text{ exists} \right\}$$

defined by

$$Ax = \lim_{t\to 0} \frac{S(t)x - x}{t}$$

is called the infinitesmal generator of $S(t)$ [GM11] def.1.2

Definition A.3.3 ($\mathbb{P}$-full support). *Let $\mathcal{X}$ be a Polish space. A random element $\xi : \Omega \to \mathcal{X}$ has $\mathbb{P}$-full support when $\mathbb{P}_\xi := \mathbb{P} \circ \xi^{-1}(\mathcal{U}) > 0$ for every nonempty open set $\mathcal{U}$ in $\mathcal{X}$ ([Oha09], pp.10).*

Definition A.3.4 (γ-Hölder continuous.). *A function f is (γ-)Hölder continuous when there are constants $C, \gamma > 0$ s.t*

$$|f(x) - f(y)| \leq C||x - y||^\gamma$$

for all x and y in the domain of f.

Appendix B

Computation of the conditional variance of a integrated fractional Hull-White-process

The conditional variance term in Equation (6.7) is given by

$$[||c(x)\mathbf{1}_{[t,T]}(x)||^2_{\kappa,T^\star} - ||\Psi^\kappa_c(t,T,x)\mathbf{1}_{[0,t]}(x)||_{\kappa,T^\star}]$$

and as noted in [Fin11] it is hard to compute due to the instability of the fractional integration. We follow [Gao+23], where an algorithm has been developed and tested to compute the conditional variance for a fractional Hull-White process, as in Proposition 6.1.5, but not for an *integrated* fractional Hull-White process. We can, however, easily expand the algorithm to account for this.

In [Gao+23], the term $c(r)$ in Algorithm 1 is given by

$$c(r) = \sigma e^{-a(T-r)}$$

which is expanded into

$$c(r) = \sum_{n=0}^{\infty} c_n r^n \tag{B.1}$$

where

$$c_n = \sigma e^{-aT} \frac{a^n}{n!} \tag{B.2}$$

This needs some modification as we use the following expression for $c(r)$

$$c(r) = \sigma D(r,T) = \sigma \int_r^T \exp\left\{-\int_r^v a\, dw\right\} dv = \sigma \frac{1-e^{-a(T-r)}}{a} \tag{B.3}$$

we therefore need to write a similar expansion as done in the paper. We get

$$\begin{aligned} c(r) &= \sigma D(t,T) \\ &= \sigma \frac{1-e^{-a(T-r)}}{a} \\ &= \frac{\sigma}{a} - \frac{\sigma}{a} e^{-a(T-r)} \\ &= \frac{\sigma}{a} - \frac{1}{a} \sum_{n=0}^{\infty} \sigma e^{-aT} \frac{a^n}{n!} r^n \end{aligned}$$

$$= \frac{\sigma}{a} - \frac{1}{a}\sum_{n=0}^{\infty} c_n r^n$$

Inserting into equation (3.13) in [Gao+23];

$$\begin{aligned}
\kappa \int_z^t r^\kappa c(r)(r-z)^{\kappa-1}\,dr = &\int_z^t r^\kappa \Big(\frac{\sigma}{a} - \frac{1}{a}\sum_{n=0}^{\infty} \sigma e^{-aT}\frac{a^n}{n!} r^n\Big)(r-z)^{\kappa-1}\,dr \\
=&\kappa \int_z^t r^\kappa \frac{\sigma}{a}(r-z)^{\kappa-1}\,dr \\
&-\kappa \int_z^t r^\kappa \Big(\frac{1}{a}\sum_{n=0}^{\infty} \sigma e^{-aT}\frac{a^n}{n!} r^n\Big)(r-z)^{\kappa-1}\,dr \\
&\frac{\sigma}{a}\kappa \int_z^t r^\kappa (r-z)^{\kappa-1}\,dr - \frac{1}{a}\sum_{n=0}^{\infty} \sigma e^{-aT}\frac{a^n}{n!}\kappa \int_z^t r^{\kappa+n}(r-z)^{\kappa-1}\,dr \\
&= \frac{\sigma}{a} R_0(\kappa, z) - \frac{1}{a} c_n R_n(\kappa, z)
\end{aligned}$$

Denoting

$$h_\star := \frac{\sigma}{a} R_0 - \frac{1}{a} h(k, z)$$

jumping into section 3.3 of [Gao+23] we adjust Algorithm 1 so that we insert under line 21:

$h_\star \leftarrow \frac{\sigma}{a} R_0 - \frac{1}{a} h$

and change line 22 to

$I \leftarrow I + m e^{(-m \cdot i)^2} h_\star^2$

The variance when $\kappa = 0$ is also different. Remembering that when $\kappa = 0$ the norm

$$||c(x)\mathbf{1}_{[t,T]}(x)||^2_{\kappa,T^\star} - ||\Psi_c^\kappa(t,T,x)\mathbf{1}_[0,t](x)||_{\kappa,T^\star}$$

is equal to the L^2-norm given by

$$||f||^2_{0,T} = \int_0^T f^2(r)\,dr$$

We get

$$\begin{aligned}
||\left(\frac{\sigma}{a} - \frac{\sigma}{a}e^{-a(T-t)}\right)\mathbf{1}_{[t,T]}||^2_{0,T} = &\int_t^T \left(\frac{\sigma}{a} - \frac{\sigma}{a}e^{-a(T-t)}\right)^2 dr \\
=&\frac{\sigma^2}{a^2}\left(\frac{2Ta - 3 - e^{-2Ta}\left(-4e^{as+Ta} + e^{2as} + 2ase^{2Ta}\right)}{2a}\right)
\end{aligned}$$

and so line 2 will result in an error message while line 3 will return

Var $= \frac{\sigma^2}{a^2}\left(\frac{2Ta-3-e^{-2Ta}\left(-4e^{as+Ta}+e^{2as}+2ase^{2Ta}\right)}{2a}\right)$

Figure B.1 shows that the norm plotted via this modified scheme yields the same result as in [Fin11] pp.64.

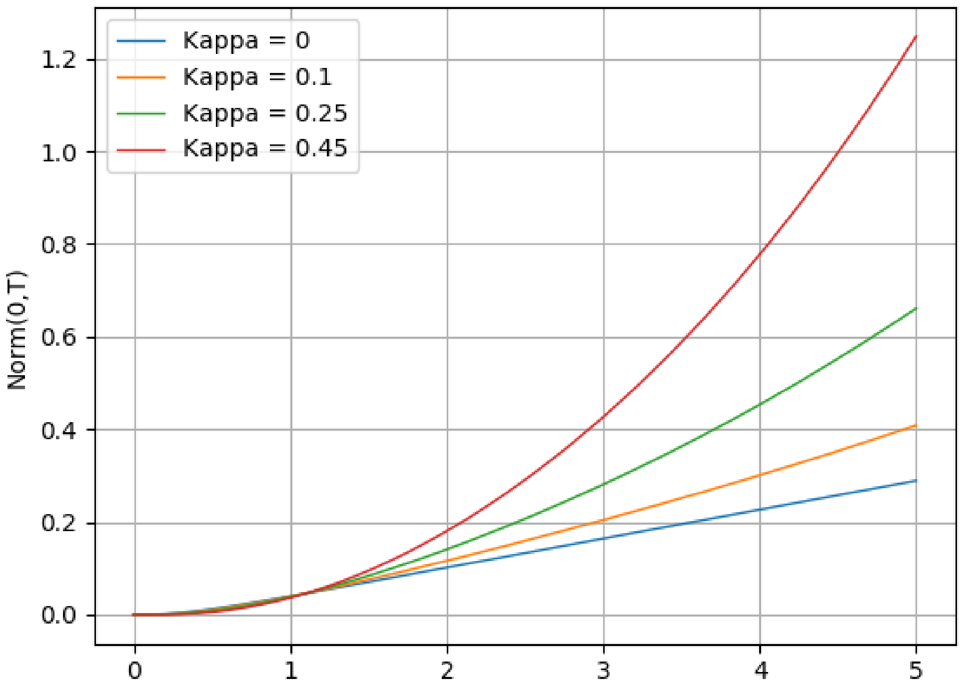


Figure B.1: The norm in Equation (6.7) calculated via a modification of Algorithm 1 in [Gao+23]

Appendix C

Python code

Simulations of fBm, fractional Gaussian noise, Brownian motion and Gaussian noise are done using the "stochastic" Python package by Christopher Flynn, under the MIT license. Documentation can be found at: https://stochastic.readthedocs.io/en/stable/

C.1 Functions

```
import numpy as np
import math
import scipy.integrate
import scipy.stats as stats
from stochastic.processes.noise import GaussianNoise, FractionalGaussianNoise
from stochastic.processes.continuous import FractionalBrownianMotion

#-----------------------------------------------------------------------------------------
# Gompertz-Makeham survival probability
#-----------------------------------------------------------------------------------------
def surv_prob(t, s, a=-11.693,b=0.1092,c=0.000063):
    mu = b / (2*c)
    sigma = np.sqrt(1/(2*c))
    return np.exp(-sigma*np.exp(a+(b**2)/(4*c))*np.sqrt(2*np.pi)\
                  *(stats.norm.cdf((s-mu)/sigma) - stats.norm.cdf((t-mu)/sigma)))



#-----------------------------------------------------------------------------------------
# Conditional variance of a fractional HW-process and integrated fractional HW-process
#-----------------------------------------------------------------------------------------
def f_2F1(alfa, beta, gamma, zeta, N=20):
    s = [(math.gamma(alfa + n) / math.gamma(alfa) * math.gamma(beta + n) /
    math.gamma(beta))/(math.factorial(n) * math.gamma(gamma + n)
    / math.gamma(gamma)) * (zeta **n) for n in range(N)]
    return sum(s)

# Variance of the fractional Hull-White model
def var_fHW(a, sigma, kappa, s,T, stepsize=0.5, sumrange=5, expansterms=20):
    tol = 10e-8
    if abs(kappa) < tol:
        if a < tol:
            var = sigma**2 * (T-s)
        else:
```

```
            var = ((sigma**2 * (1- np.exp(-2*a*(T-s)))) / (2*a) )
    else:
        if abs(T-s) < tol:
            var = 0
        else:
            I = 0
            M = math.floor(sumrange / stepsize)
            M_list = np.arange(-M, M+1, 1)
            for i in M_list:
                w = stepsize * i
                y = 1/2 * math.erfc(-w)
                x = (T**(1-2*kappa) - s**(1-2*kappa))*y + s**(1-2*kappa)
                z = x ** (1 / (1-2*kappa))
                h = 0
                R = np.zeros(expansterms+1)
                for n in range(expansterms+1):
                    c_n = sigma * np.exp(-a*T) * ((a**n) / math.factorial(n))
                    if n < tol:
                        if s < tol and z/T < 1/2:
                            R[0] = (math.gamma(kappa + 1)**2 * z**(2*kappa))\
                                /(2*math.cos(math.pi*kappa) * math.gamma(1 + 2*kappa))\
                                 + (kappa*(T-z)**kappa * T**kappa) / (2*kappa) *
                                 f_2F1(-kappa, 1, 1-2*kappa, z/T)
                        else:
                            R[0] = z**(2*kappa) * (1 - z/T)**kappa *
                            f_2F1(2*kappa+1, kappa, kappa+1, 1-z/T)
                    else:
                        R[n] = (kappa*T**(kappa + n) * (T-z)**kappa + z*(kappa + n) *
                        R[n-1]) / (2*kappa + n)
                    h += c_n * R[n]
                I += stepsize * np.exp(- (stepsize*i)**2) * h**2
            var = (math.gamma(1-kappa)/(math.gamma(2-2*kappa) * math.gamma(kappa + 1)))\
                * (((1 + 2*kappa)*(T**(1-2*kappa) - s**(1-2*kappa)))/ math.sqrt(math.pi)) * I
    return var

# Variance of the integrated fractional Hull-White model
def var_integrated_fHW(a, sigma, kappa, s,T, stepsize=0.5, sumrange=5, expansterms=20):
    tol = 10e-8
    if abs(kappa) < tol:
        if a < tol:
            print("UNDEFINED FOR a=0")
        else:
            var = (sigma**2/a**2) * (1/(2*a)) * (2*T*a-3 - math.exp(-2*T*a)\
                 * (-4*math.exp(a*(T-s)) + math.exp(2*a*s) + 2*a*math.exp(2*T*a) * s))
    else:
        if abs(T-s) < tol:
            var = 0
        else:
            I = 0
            M = math.floor(sumrange / stepsize)
            M_list = np.arange(-M, M+1, 1)
            for i in M_list:
                w = stepsize * i
                y = 1/2 * math.erfc(-w)
                x = (T**(1-2*kappa) - s**(1-2*kappa))*y + s**(1-2*kappa)
                z = x ** (1 / (1-2*kappa))
```

```
                h = 0
                R = np.zeros(expansterms+1)
                for n in range(expansterms+1):
                    c_n = sigma * np.exp(-a*T) * ((a**n) / math.factorial(n))
                    if n < tol:
                        if s < tol and z/T < 1/2:
                            R[0] = (math.gamma(kappa + 1)**2 * z**(2*kappa))\
                                /(2*math.cos(math.pi*kappa) * math.gamma(1 + 2*kappa))\
                                 + (kappa*(T-z)**kappa * T**kappa) / (2*kappa) *
                                 f_2F1(-kappa, 1, 1-2*kappa, z/T)
                        else:
                            R[0] = z**(2*kappa) * (1 - z/T)**kappa *
                            f_2F1(2*kappa+1, kappa, kappa+1, 1-z/T)
                    else:
                        R[n] = (kappa*T**(kappa + n) * (T-z)**kappa + z*(kappa + n) *
                        R[n-1]) / (2*kappa + n)
                    h += c_n * R[n]
                h_star = (sigma/a) * R[0] - 1/a * h
                I += stepsize * np.exp(- (stepsize*i)**2) * h_star**2
            var = (math.gamma(1-kappa)/(math.gamma(2-2*kappa) * math.gamma(kappa + 1)))\
                * (((1 + 2*kappa)*(T**(1-2*kappa) - s**(1-2*kappa)))/ math.sqrt(math.pi)) * I
    return var

#-------------------------------------------------------------------------------------------
# Vasicek interest rate model path
#-------------------------------------------------------------------------------------------
def vasicek_path(T, r0, a, b, sigma, n):
    r = np.zeros(T*n)
    r[0] = r0
    dt = 1/n
    gn = GaussianNoise(T)
    dWt = gn.sample(T*n)
    for i in range(1, T*n):
        r[i] = r[i-1] + a * (b - r[i-1]) * dt + sigma * dWt[i]
    return r


#-------------------------------------------------------------------------------------------
# Price of a Zero-Coupon Bond under Vasicek interest rate model
#-------------------------------------------------------------------------------------------
def B(t,T, a):
    return 1/a * (1 - np.exp(-a*(T-t)))

def A(t,T, a, b, sigma):
    return np.exp((b - (sigma**2)/(2*a**2)) * (B(t,T,a) - T + t)
                - (sigma**2)/(4*a)*B(t,T,a)**2)

def vasicek_ZCB(t,T,rt, a, b, sigma):
    return A(t,T,a,b,sigma) * np.exp(- B(t,T,a) * rt)


#-------------------------------------------------------------------------------------------
# Fractional Vasicek interest rate model path
#-------------------------------------------------------------------------------------------
def frac_vasicek_path(T, r0, b, a, sigma, H, n=1000):
    r = np.zeros(T*n)
```

```
    r[0] = r0
    dt = 1/n
    fgn = FractionalGaussianNoise(H, T)
    dWHt = fgn.sample(T*n)
    for i in range(1, T*n):
        r[i] = r[i-1] + a*(b - r[i-1]) * dt + sigma * dWHt[i]
    return r


#------------------------------------------------------------------------------------------
# Price of a Zero-Coupon Bond under the fractional Vasicek interest rate model
#------------------------------------------------------------------------------------------
def frac_vasicek_D(t,T,a):
    return 1/a - 1/a * math.exp(-a*(T-t))

def frac_vasicek_ZCB(t,T, r0, k, a, sigma, kappa):
    part1 = frac_vasicek_D(0,T,a)*r0
    part2 = k * scipy.integrate.quad(frac_vasicek_D, 0, T, args=(T,a))[0]
    norm = var_integrated_fHW(a, sigma, kappa, 0, T)
    return math.exp(- part1 - part2 + 1/2 * norm)


#------------------------------------------------------------------------------------------
# Present value of a pension policy under deterministic interest rate
#------------------------------------------------------------------------------------------
# Policy function
def adot_benefit(benefit):
    return benefit

# Discounting
def v(t,s,r):
    f = lambda s: r
    return np.exp(-scipy.integrate.quad(f, 0, s)[0] + scipy.integrate.quad(f, 0, t)[0])

# Present value of benefits
def PV_integrand_det(s, x, t, r, benefit):
    return surv_prob(x+t,x+s) * adot_benefit(benefit) * v(t,s,r)

def PV_det(x, t, lower, upper, r, benefit):
    return scipy.integrate.quad(PV_integrand_det, lower, upper,
                                args=(x, t, r, benefit))[0]


#------------------------------------------------------------------------------------------
# Present value of a pension policy under the Vasicek interest rate model
#------------------------------------------------------------------------------------------
# Discounting under the Vasicek model
def P_V(t,s,r, a, b, sigma):
    return vasicek_ZCB(t, s, r, a, b, sigma)

# Reserves
def PV_vasicek_integrand(s, x, t, r, benefit, a, b, sigma):
    return adot_benefit(benefit) * surv_prob(x+t,x+s) *  P_V(t,s,r, a, b, sigma)

def PV_vasicek(x, t, lower, upper, r, benefit, a, b, sigma):
    return scipy.integrate.quad(PV_vasicek_integrand, lower, upper,
```

```
                                args=(x, t, r, benefit, a, b, sigma))[0]


#-----------------------------------------------------------------------------------------
# Present value of a pension policy under the fractional Vasicek interest rate model
#-----------------------------------------------------------------------------------------
# Discounting under the fractional Vasicek model
def P_fV(t,s,r, b, a, sigma, H):
    k = b*a
    return frac_vasicek_ZCB(t,s, r, k, a, sigma, H-0.5)

# Present value of reserves at time t
def PV_frac_vasicek_integrand(s, x, t, r, benefit, b, a, sigma, H):
    return adot_benefit(benefit) * surv_prob(x+t,x+s) *  P_fV(t,s,r, b, a, sigma, H)

def PV_frac_vasicek(x, t, lower, upper, r, benefit,b, a, sigma, H):
    return scipy.integrate.quad(PV_frac_vasicek_integrand, lower, upper,
                                args=(x, t, r, benefit, b, a, sigma, H))[0]


#-----------------------------------------------------------------------------------------
# Expectation of fractional Vasicek interest rate model
#-----------------------------------------------------------------------------------------
def psi_c_kappa(s,t,v,kappa,a, sigma):
    tol = 10e-8
    if v > tol and (s-v) > tol:
        p1 = np.sin(np.pi * kappa) / np.pi
        p2 = v**(-kappa) * (s-v)**(-kappa)
        integrand = lambda r: (r**kappa * (r-s)**kappa)
                    / (r-v) * (sigma * np.exp(-a*(t-r)))
        I = scipy.integrate.quad(integrand, s, t)[0]
        return p1 * p2 * I
    else:
        return 0

def expectation_fHW(r0, a, b, T):
    return r0 * np.exp(-a*T) + b*(1-np.exp(-a*T))


#-----------------------------------------------------------------------------------------
# CDF of the fractional Vasicek model
#-----------------------------------------------------------------------------------------
def cdf_frac_Vasicek(x, r0, b, a, sigma, T, H):
    k = a*b
    kappa = H-0.5
    mean = expectation_fHW(r0, a, b, T)
    sd = np.sqrt(var_fHW(a, sigma, kappa, 0, T))
    return stats.norm.cdf(x, loc=mean, scale=sd)


#-----------------------------------------------------------------------------------------
# PPF of the fractional Vasicek model
#-----------------------------------------------------------------------------------------
def ppf_frac_Vasicek(q, r0, b, a, sigma, T, H):
    k = a*b
    kappa = H-0.5
```

```
    mean = expectation_fHW(r0, a, b, T)
    sd = np.sqrt(var_fHW(a, sigma, kappa, 0, T))
    return stats.norm.ppf(q, loc=mean, scale=sd)


#-------------------------------------------------------------------------------------------
# PPF of the fractional  Vasicek model
#-------------------------------------------------------------------------------------------
def pdf_frac_Vasicek(x_arr, r0, b, a, sigma, T, H):
    kappa = H-0.5
    mean = expectation_fHW(r0, a, b, T)
    sd = np.sqrt(var_fHW(a, sigma, kappa, 0, T))
    return stats.norm.pdf(x_arr, loc=mean, scale=sd)
```

C.2 Chapter 2

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl; mpl.rcParams["axes.grid"] = True; mpl.rcParams['lines.linewidth'] = 1
from stochastic.processes.continuous import BrownianMotion

"""Brownian motion path realization"""
N  = 1000
bm = BrownianMotion()
x = np.linspace(0, 1, N)
plt.plot(x, bm.sample(N-1))
plt.title("Standard Brownian Motion")
plt.show()
```

C.3 Chapter 3

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl; mpl.rcParams["axes.grid"] = True; mpl.rcParams['lines.linewidth'] = 1

""" Present value of a pension policy with deterministic interest rate """
# Contract specifications
x0 = 30                 # age at start of contract
T0 = 40                 # pension age from start of contract
T = 80                  # pension end from start of contract
benefit = 200000        # yearly pension
r0 = 0.02               # interest rate

# Single premium
pi0 = PV_det(x0, 0, T0, T, r0, benefit)
print(f"Single premium of pension policy with deterministic interest rate = {pi0}")

# Present value plot
t_arr = np.linspace(0, T, T*10)
PV_det_benefit = [PV_det(x0, t, max(T0, t), T, r0, benefit) for t in t_arr]
```

```
plt.plot(t_arr, PV_det_benefit)
plt.title("Present value of benefits")
plt.xlabel("Age of contract")
plt.show()


""" Present value of premiums in a pension policy with deterministic interest rate """
# Premium
pi = PV_det(x0, 0, T0, T, r0, benefit)/PV_det(x0, 0, 0, T0, r0, 1)
print(f"Premium of pension policy with deterministic interest rate = {pi}")


""" Reserves of a pension policy with deterministic interest rate """
# Reserve plot
PV_det_premium = [-PV_det(x0, t, min(t,T0), T0, r0, pi) for t in t_arr]
plt.plot(t_arr, PV_det_benefit, label = "PV Benefit")
plt.plot(t_arr, PV_det_premium, label = "PV Premium")
PV_reserve = [(PV_det_benefit[i] + PV_det_premium[i]) for i in range(len(t_arr))]
plt.plot(t_arr, PV_reserve, label="PV Reserve")
plt.title("Present value of reserve")
plt.xlabel("Age of contract")
plt.legend()
plt.show()

"""Vasicek interest rate paths"""
# Vasicek parameters
a = 0.2         # mean-reversion speed
b = 0.03        # long-term mean
sigma = 0.01    # volatility
r0 = 0.02       # initial interest rate
t = 0

# Plot
T_maturity = 10
x = np.linspace(0, T_maturity, T_maturity*1000)
for i in range(5):
    plt.plot(x, vasicek_path(1,r0, a, b,sigma,len(x)))
plt.title("Vasicek interest rate path realizations")
plt.show()


"""Vasicek interest rate bond prices"""
# Plot of ZCB-prices wrt different initial interest rates
T_maturity = 80 #Maturity
T_arr = np.linspace(t,T_maturity,T_maturity, 1000)
y = np.zeros(T)
for i in range(len(T_arr)):
    y[i] = vasicek_ZCB(t, T_arr[i], r0, a, b, sigma)
plt.plot(T_arr,y)
plt.title("Vasicek ZCB price")
plt.xlabel("Maturity")
plt.show()


"""Life insurance reserves under the Vasicek model"""
# Contract specifications, policy function and vasicek parameters as above.
```

```
# Plots of single premium wrt different initial interest rates
r_arr = np.linspace(0.01,0.2, 100)
V = []
for r in r_arr:
    V.append(PV_vasicek(x0, 0, T0, T, r, benefit, a, b, sigma))
plt.vlines(r0,0,max(V), linestyles="--")
plt.plot(r_arr,V)
plt.title("Single premiums")
plt.xlabel("Initial interest rate")
plt.show()

# Single premium r0 = 0.02
pi0 = PV_vasicek(x0, 0, T0, T, r0, benefit, a, b, sigma)
print(f"Single premium of a pension policy under the Vasicek model = {pi0}")

# Plot of present value
for r in [0.01, 0.03, 0.05, 0.07]:
    t_arr = np.linspace(0, T, T*10)
    PV_vasicek_dev = [PV_vasicek(x0, t, max(T0, t), T, r, benefit, a, b, sigma)
                for t in t_arr]
    plt.plot(t_arr, PV_vasicek_dev, label=f"Vasicek, r0 = {r}")
plt.plot(t_arr, PV_det_benefit, label=f"Deterministic, r0 = {r0}", color="black",
            linestyle="--")
plt.title("Present value")
plt.xlabel("Age of contract")
plt.legend()
plt.show()
```

C.4 Chapter 4

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl; mpl.rcParams["axes.grid"] = True; mpl.rcParams['lines.linewidth'] = 1
from stochastic.processes.continuous import FractionalBrownianMotion

"""fBm path realizations"""
N  = 1000
H = [0.1, 0.5, 0.9]
x = np.linspace(0, 1, N)
for i in range(len(H)):
    fbm_i = FractionalBrownianMotion(H[i])
    plt.subplot(3,1,i+1)
    plt.plot(x, fbm_i.sample(N-1))
    plt.title(f'H={H[i]}')
plt.tight_layout()
plt.show()
```

C.5 Chapter 6

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl; mpl.rcParams["axes.grid"] = True; mpl.rcParams['lines.linewidth'] = 1
```

```
# Vasicek parameters
a = 0.2             # mean-reversion speed
b = 0.03            # long-term mean
sigma = 0.01        # volatility
r0 = 0.02           # initial interest rate
T = 80              # Maturity

"""fractional Vasicek path realizations"""
N  = 1000
H = [0.1, 0.5, 0.9]
t = np.linspace(0, T, num=T*1000)
for i in range(len(H)):
    plt.subplot(3,1,i+1)
    plt.plot(t, frac_vasicek_path(T, r0, b, a, sigma, H[i]))
    plt.title(f'H={H[i]}')
plt.tight_layout()
plt.show()

"""Fractional Vasicek interest rate bond prices"""
N = 10
x = np.linspace(0,T,T*N)
P_fV = np.zeros(T*N)
for H in [0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9]:
    for i in range(T*N):
        P_fV[i] = frac_vasicek_ZCB(0, x[i], r0, a*b, a, sigma, H-0.5)
    plt.plot(x, P_fV, label=f"H = {H}")
plt.legend()
plt.title("Fractional Vasicek ZCB price")
plt.xlim(0, T); plt.ylim(0, 1)
plt.xlabel("Maturity")
plt.show()
```

C.6 Chapter 7

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl; mpl.rcParams["axes.grid"] = True

""" Pension policy with fractional Vasicek interest rates """
# Contract specifications
x0 = 30                 # age at start of contract
T0 = 40                 # pension age from start of contract
T = 80                  # pension end from start of contract
benefit = 200000        # yearly pension

# Vasicek parameters
a = 0.2             # mean-reversion speed
b = 0.03            # long-term mean
sigma = 0.01        # volatility

H = [0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9]
r = np.linspace(0.01, 0.2, 50)
for Hi in H:
    V = np.zeros(len(r))
    for i in range(len(r)):
```

```
        V[i] = PV_frac_vasicek(x0, 0, T0, T, r[i], benefit, b, a, sigma, Hi)
    plt.plot(r, V, label=f"H = {Hi}")
plt.title("Single premiums")
plt.xlabel("Initial interest rate")
plt.legend()
plt.show()




""" Numerical analysis of the distribution of r(T)"""
# Vasicek parameters
r0 = 0.02
a = 0.2
b = 0.03
sigma = 0.01

H = [0.1, 0.3, 0.5, 0.7, 0.9]
T = 80

# Monte-Carlo simulation
numsim = 5000
frac_vasicek_endpoints = np.zeros((len(H), numsim))

for i in range(len(H)):
    for j in range(numsim):
        frac_vasicek_endpoints[i][j] = frac_vasicek_path(T, r0, b, a, sigma, H[i])[-1]

# Theoretical and empirical moments from simulation
for i in range(len(H)):
    mean_empirical = np.mean(frac_vasicek_endpoints[i])
    sd_empirical = np.std(frac_vasicek_endpoints[i])
    mean_theoretical = expectation_fHW(r0, a, b, T)
    sd_theoretical = np.sqrt(var_fHW(a, sigma, H[i]-0.5, 0, T))
    print(f"H = {H[i]} : theoretical mean = {mean_theoretical:.4f},
            empirical mean = {mean_empirical:.4f}, theoretical sd = {sd_theoretical:.4f},
            empirical sd = {sd_empirical:.4f}")

# P(r(80)<0|H)
for i in range(len(H)):
    sum_over_zero = (frac_vasicek_endpoints[i] > 0).sum()
    prob_empirical = (numsim - sum_over_zero)/numsim
    prob_theoretical = cdf_frac_Vasicek(0, r0, b, a, sigma, T, H[i])
    print(f"H = {H[i]} : Prob sub zero theoretical = {prob_theoretical:.4f},
                empirical = {prob_empirical:.4f}")


# Plot
for i in [0, 2, 4]:
    plt.hist(frac_vasicek_endpoints[i], bins=int(numsim/10), alpha=0.5,
                density=True, label=f"Monte-Carlo, H = {H[i]}")
    x_lower = ppf_frac_Vasicek(0.001, r0, b, a, sigma, T, H[i])
    x_upper = ppf_frac_Vasicek(0.999, r0, b, a, sigma, T, H[i])
    x = np.linspace(x_lower, x_upper)
    plt.plot(x, pdf_frac_Vasicek(x, r0, b, a, sigma, T, H[i]), label=f"PDF, H = {H[i]}")
plt.title("Monte Carlo simulation of r(T)")
```

```
plt.legend()
plt.show()



""" Pension policy with fractional Vasicek interest rates """
# Contract specifications
x0 = 30             # age at start of contract
T0 = 40             # pension age from start of contract
T = 80              # pension end from start of contract
benefit = 200000    # yearly pension


""" Sensitivity analysis for a pension policy with fractional Vasicek interest rates """
# Contract spesifications as above
# Baseline Vasicek parameters:
a = 0.2             # mean-reversion speed
b = 0.03            # long-term mean
sigma = 0.01        # volatility
r0 = 0.02           # initial interest rate

# Parameters to check
H = np.linspace(0.1, 0.9)
r0_arr = np.arange(0.01, 0.06, step=0.01)
a_arr = np.arange(0.1, 1, step=0.2)
b_arr = np.arange(0.01, 0.1, step=0.02)
sigma_arr = np.arange(0.005, 0.03, step=0.005)

# Sensitivity analysis computation
Va, Vb, Vsigma, Vr0 = [], [], [], []
param_arr = [Vr0,Va,Vb,Vsigma]
param_name = ["r_0","a", "b", "sigma"]
param_value = [r0_arr, a_arr, b_arr, sigma_arr]
for Hi in H:
    Vr0.append([[PV_frac_vasicek(x0, 0, T0, T, ri, benefit, b, a, sigma, Hi)
                for Hi in H] for ri in r0_arr])
    Va.append([[PV_frac_vasicek(x0, 0, T0, T, r0, benefit, b, ai, sigma, Hi)
                for Hi in H] for ai in a_arr])
    Vb.append([[PV_frac_vasicek(x0, 0, T0, T, r0, benefit, bi, a, sigma, Hi)
                for Hi in H] for bi in b_arr])
    Vsigma.append([[PV_frac_vasicek(x0, 0, T0, T, r0, benefit, b, a, sigmai, Hi)
                for Hi in H] for sigmai in sigma_arr])

# Plot of sensitivities
for k in range(len(param_arr)):
    V_ = param_arr[k]
    p = param_value[k]
    p_name = param_name[k]
    for i in range(len(p)):
        p_val = p[i]*100
        plt.plot(H, V_[k][i], label = f"{p_name} = {p_val:.1f}%")
        plt.ylim(0,3000000)
    plt.title(f"Sensitivity with respect to {p_name}")
    plt.legend()
    plt.show()
```

C.7 Appendix B

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl; mpl.rcParams["axes.grid"] = True; mpl.rcParams['lines.linewidth'] = 1

a = 4
sigma = 1
t = 0
T = 5

T_ = np.linspace(t,T)
for kpa in [0, 0.1, 0.25, 0.45]:
    norm_ = np.zeros(len(T_))
    for i in range(len(norm_)):
        norm_[i] = var_integrated_fHW(a, sigma, kpa, t, T_[i])/(sigma**2)
    plt.plot(T_, norm_, label=f"Kappa = {kpa}")
    plt.ylabel("Norm(0,T)")
    plt.legend()
plt.show()
```

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