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Advancements in Risk-Free Reference Rates and ESG-linked interest rate swaps

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Abstract

In this thesis, we look at the new SOFR rates and how they differ from the outgoing LIBOR rates. We also propose a model for pricing ESG-linked interest rate swaps.

We look deeper into SOFR futures and examine the consequences of different underlying calculation methods for 1-month and 3-month futures. This will be further studied via a particular hedge, where numerical examples are provided.

An ESG-fixed rate process is proposed yielding an expression κ_t^{ESG} for an ESG-swap rate process. We choose an OU-process $X(t)$ for modelling a firm's ESG risk score. The evolution of the ESG-swap rate process κ_t^{ESG} is illustrated using a Monte Carlo scheme.

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Chapter 1

Introduction

1.1 Overview

Following the LIBOR scandal, the Alternative Reference Rates Committee (ARRC) was established to help ensure a robust alternative (for USD-LIBOR) and came up with the Secured Overnight Financing Rate (SOFR). Other options include SONIA (Sterling Overnight Index Average) for GBP-LIBOR and €STR (Euro Short-Term Rate) for the Euro-zone.

Since LIBOR will no longer be the key benchmark, it is crucial to understand the new alternative reference rates and how they differ. For instance, LIBOR is an inter-bank rate based on a market survey, while SOFR is an overnight rate based on the U.S. Treasury repurchase market. This leads to a fundamental difference as LIBOR works as a forward-looking prediction of future rates, while the overnight rates will be backwards-looking.

This transition also requires a better understanding of available products and hedging instruments tied to Risk-Free Reference rates, and we will study SOFR futures and associated derivatives.

There is an urgent need for a green transition to address climate change. The EU has put in place a new taxonomy so that the EU can be carbon neutral by 2050. Incorporating ESG into Finance (Sustainable Finance) is becoming increasingly important, and with this in mind, we propose a framework for ESG-linked interest-rate swaps. This framework is constructed to incentivise one to achieve favourably climatic goals.

Understanding the new RFRs and ESG-linked financial products is crucial for the Insurance industry. A pension fund might have many SOFR-linked products in its portfolio. A better understanding of ESG-linked products is needed to meet stakeholder expectations, and regulatory requirements, get better risk management and provide measures for a sustainable future.

1.2 Outline

The thesis is organized as follows:

Chapter 2 The theoretical background/framework for interest rate theory is established. This includes Measure Theory, its relationship with Probability Theory, and then, finally Stochastic Analysis.

Chapter 3 Introduces important concepts from mathematical finance, like the fundamental theorems of asset pricing.

Chapter 4 Consists of interest rate theory. Here we introduce the zero coupon bond, interest rate swaps, short rate models, HJM framework, and the outgoing LIBOR rates.

Chapter 5 We look deeper into Risk-Free Reference rates, particularly SOFR. We highlight fundamental differences between SOFR and LIBOR and look further into interest rate futures. The difference between 1-month and 3-month SOFR futures, Black and Scholes Option methodology, Swaps and specific hedges are studied.

Chapter 6 An approach for incorporating ESG into Interest-rate Swaps is introduced. We take one particular case study from real life and use this as motivation for establishing a mathematical framework for ESG-linked Interest-rate swaps.

Chapter 7 We include a numerical simulation to grasp better how ESG-linked Interest-rate swaps could work. Here we benchmark different scenarios and study how the ESG framework responds.

Chapter 8 We summarize our findings and discuss shortcomings, possible model extensions and aspects for further research.

Appendix A A method for estimating parameters in the Vasicek model is presented, and how estimation can be done in an Affine Term Structure-setting.

Appendix B The Julia code used in SOFR examples includes dynamics of 1M- and 3M-SOFR futures, 3M-SOFR futures swap, and the specified SOFR hedge.

Appendix C Julia code for the Monte Carlo simulation of the ESG-linked interest rate swap.

Chapter 2

Theoretical Background

2.1 Measure Theory

The measure theory results have been gathered from [Lin17]

Definition 2.1.1 (**Sigma-algebra**). Assume that X is a non-empty set, a family $\mathcal{F}$ of subsets of X is called a sigma-algebra if the following holds:

(i) $\emptyset \in \mathcal{F}$

(ii) If $A \in \mathcal{F}$, then $A^C \in \mathcal{F}$

(iii) If $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}$

Definition 2.1.2 (**Measure**). Assume that X is a non-empty set, and that $\mathcal{F}$ is a σ-algebra on X. A measure μ on $(X, \mathcal{F})$ is a function $\mu : \mathcal{F} \to \overline{\mathbb{R}}_+ = [0, \infty) \cup \{\infty\}$ such that:

(i) $\mu(\emptyset) = 0$

(ii) if $\{A_n\}_{n\in\mathbb{N}}$ is a pairwise disjoint sequence, then:

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

We call the triplet $(X, \mathcal{F}, \mu)$ a measure space.

Proposition 2.1.3 (**Intersection of σ-algebras is a σ-algebra**). *Let $(X, \mathcal{F}, \mu)$ be a measure space, let $\mathcal{I}$ be a non-empty index set and let $\mathcal{G}_i$, $i \in \mathcal{I}$ be σ-algebras on X, then:*

$$\mathcal{G} = \bigcap_{i\in\mathcal{I}} \mathcal{G}_i = \{A \subseteq X : A \in \mathcal{G}_i,\ \forall\ i \in \mathcal{I}\}$$

is a σ-algebra on X

Proof. Since all $\mathcal{G}_i$'s are σ-algebras, we have that $\emptyset \in \mathcal{G}_i\ \forall\ i \in \mathcal{I}$, thus $\emptyset \in \mathcal{G}$.

Assume that $A \in \mathcal{G}$, meaning that $A \in \mathcal{G}_i\ \forall i \in \mathcal{I}$, now: since all $\mathcal{G}_i$'s are σ-algebras we have that $A^C \in \mathcal{G}$

Assume that $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{G}$, then we have that $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{G}_i\ \forall i \in \mathcal{I}$, and thus: $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{G}$. ■

Proposition 2.1.4 (**Continuity of measure**). *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of measurable sets in $(X, \mathcal{F}, \mu)$, then we have:*

(i) Assume that $\{A_n\}_{n\in\mathbb{N}}$ is an increasing sequence, i.e that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then:

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

(ii) Assume that $\{A_n\}_{n\in\mathbb{N}}$ is a decreasing sequence, i.e that $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$, and that $\mu(A_1) < \infty$ then:

$$\mu\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

Definition 2.1.5 (**Null set**). A set $N \subseteq X$ is called a null set, if there is a set $B \in \mathcal{F}$ such that $N \subseteq B$ and $\mu(B) = 0$.

Definition 2.1.6 (**Complete measure space**). A measure space $(X, \mathcal{F}, \mu)$ is called complete if all null sets belong to $\mathcal{F}$.

Let $\mathcal{N}$ denote the collection of all null sets.

Theorem 2.1.7 (**Complete measure space with complete measure**). *Assume that $(X, \mathcal{F}, \mu)$ is a measure space, and let:*
$\overline{\mathcal{F}} = \{A \cup N : A \in \mathcal{F}$ and $N \in \mathcal{N}\}$, define $\overline{\mu} : \overline{\mathcal{F}} \to \overline{\mathbb{R}}_+$ by:

$$\overline{\mu}(A \cup N) = \mu(A), \quad \forall A \in \mathcal{F}$$

Then $(X, \overline{\mathcal{F}}, \overline{\mu})$ is a complete measure space extending $(X, \mathcal{F}, \mu)$.

Proposition 2.1.8. *Let X be a nonempty set, and $\mathcal{A}$ a collection of subsets of X. Then there exists a smallest σ-algebra $\sigma(\mathcal{A})$ containing $\mathcal{A}$. Such that if $\mathcal{C}$ is any other σ-algebra containing $\mathcal{A}$ then $\sigma(\mathcal{A}) \subseteq \mathcal{C}$.*

Definition 2.1.9 (**Borel σ-algebra**). We define the Borel σ-algebra $\mathcal{B}$ as the smallest σ-algebra generated by all open sets on $\mathbb{R}$.

Example 2.1.10 (**Lesbegue Measure**). Let $X = \mathbb{R}$ and $\mathcal{B}$ the Borel σ-algebra, the Lesbegue measure is a measure μ such that:

$$\mu([a, b]) = b - a$$

Measurable functions

Definition 2.1.11 (**Inverse image of B under f**). Let X, Y be two non-empty sets, and let $f : X \to Y$ with $B \subseteq Y$, we then define the inverse image of B as:

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

We use the convention that $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$

Definition 2.1.12 (**Measurable function**). Let $(X, \mathcal{F}, \mu)$ be a measure space. A function $f : X \to \overline{\mathbb{R}}$ is measurable if:

$$f^{-1}([-\infty, r)) \in \mathcal{F}$$

Proposition 2.1.13. *Assume that $f, g : X \to \mathbb{R}$ are measurable functions, then:*

(i) $f + g$ *is measurable.*

(ii) $f - g$ *is measurable.*

(iii) fg *is measurable.*

Integration of non negative functions

Definition 2.1.14 (**Integration of simple function**). Assume that:

$$f(x) = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}(x)$$

is a non negative simple function on standard form i.e. $X = \bigcup_{i=1}^{n} A_i$ with $A_i = \{x \in X : f(x) = a_i\}$ disjoint and measurable. The integral of f with respect to μ is defined as:

$$\int f d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$$

We use the convention that $0 \cdot \infty = 0$

Definition 2.1.15. If $f : X \to \overline{\mathbb{R}}_+$ is measurable we define:

$$\int f d\mu = \sup \left\{ \int g d\mu : g \text{ is a non negative simple function, } g \leq f \right\}$$

Proposition 2.1.16. *If $f : X \to \overline{\mathbb{R}}_+$ is a measurable function, there exists an increasing sequence $\{h_n\}$ of simple functions converging pointwise to f. Moreover, for each n and each $x \in X$ either:*

$$f(x) - \frac{1}{2^n} < h_n(x) \leq f(x) \ \ \textit{or} \ \ h_n(x) = 2^n$$

Theorem 2.1.17 (**Monotone Convergence Theorem**). *Assume that $f : X \to \overline{\mathbb{R}}_+$ is measurable, and assume that $\{f_n\}$ is an increasing sequence of non-negative measurable functions converging pointwise to f so that $\lim\limits_{n\to\infty} f_n(x) = f$, $\forall x \in X$, then:*

$$\lim_{n\to\infty} \int f_n d\mu = \int \lim_{n\to\infty} f_n d\mu$$

Theorem 2.1.18 (**Fatou's lemma**). *Let $\{f_n\}$ be a sequence of non-negative measurable functions, then:*

$$\liminf_{n\to\infty} \int f_n d\mu \geq \int \liminf_{n\to\infty} f_n d\mu$$

Definition 2.1.19. A function $f : X \to \overline{\mathbb{R}}_+$ is said to be integrable if it is measurable and $\int f d\mu < \infty$

Integration of general functions

We would also like to integrate functions taking negative values as well, we then observe that if $f : X \to \overline{\mathbb{R}}$ then $f = f_+ - f_-$ with:

$$f_+(x) = \begin{cases} f(x) & f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad f_-(x) = \begin{cases} -f(x) & f(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.1.20. A function $f : X \to \overline{\mathbb{R}}$ is called integrable it if is measurable, and f_+ and f_- are integrable, we define the integral of f as:

$$\int f d\mu = \int f_+ d\mu - \int f_- d\mu$$

Lemma 2.1.21. *A measurable function f is integrable if and only if it's absolute value $|f|$ is integrable i.e. if and only if:*

$$\int |f| d\mu < \infty$$

Theorem 2.1.22 (**Lebesgue's Dominated Convergence Theorem**). *Assume that $g : X \to \overline{\mathbb{R}}_+$ is a non-negative, integrable function and that $\{f_n\}$ is a sequence of measurable functions converging pointwise to f. If $|f_n| \leq g$ for all n, then:*

$$\lim_{n \to \infty} \int f_n d\mu = \int f d\mu$$

Riemann and Lesbegue integration

Theorem 2.1.23. *Assume that $f : [a, b] \to [0, \infty)$ is a bounded Riemann integrable function on $[a, b]$. Then f is measurable and the Riemann and Lebesgue integral coincide:*

$$\int_a^b f(x) dx = \int_{[a,b]} f d\mu$$

L^p-spaces

Definition 2.1.24. $\mathcal{L}^p$ If $1 \leq p < \infty$ and $(X, \mathcal{F}, \mu)$ is a measure space, we define:

$$\mathcal{L}^p(X, \mathcal{F}, \mu) = \{f : X \to \overline{\mathbb{C}} : f \text{ is measurable and} \int |f|^p d\mu < \infty\}$$

furthermore, define:

$$\|f\|_p = \left(\int |f|^p d\mu\right)^{\frac{1}{p}}$$

Definition 2.1.25 (L^p). Let $p \in [1, \infty)$, and define a relation by:

$$f \sim g \iff f = g \;\; a.e.$$

Consider the equivalence class:

$$[f] := \{g \in \mathcal{L}^p : g \sim f\}$$

We then define:

$$L^p(X, \mathcal{F}, \mu) := \{[f] : f \in \mathcal{L}^p\}$$

2.2 Probability theory

Most of the results in this section are gathered from [Wal12], let $\mathcal{F}$ be a σ-algebra and let Ω denote the sample space.

Definition 2.2.1 (**Probability measure**). A probability measure P on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ such that:

(i) if $A \in \mathcal{F}$, then $P(A) \geq 0$

(ii) $P(\Omega) = 1$

(iii) if $\{A_n\}_{n \in \mathbb{N}}$ is a pairwise disjoint sequence, then:

$$P\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} P(A_n)$$

Definition 2.2.2 (**Random Variable**). Let $(\Omega, \mathcal{F}, P)$ be a probability space. A random variable X is a function $X : \Omega \to \mathbb{R}$ such that:

$$\{\omega : X(\omega) \leq x\} \in \mathcal{F}$$

Proposition 2.2.3. *Let X be a random variable, and let $A \in \mathcal{B}$ (the Borel σ-algebra), then:*

$$\{X \in A\} \in \mathcal{F}$$

Expectations and Conditional Expectations

We will now use the results from measure theory and see how it relates to the construction of expectations and conditional expectations.

Definition 2.2.4 (**Discrete random variable**). We say that a random variable X is discrete if:

$$X(\omega) = \sum_{i=1}^{\infty} x_i \mathbb{1}_{A_i}(\omega)$$

Where $A_i = \{X = x_i\}$, furthermore we assume that it is on standard-form (See Definition 2.1.14)

Definition 2.2.5 (**Expectation Discrete case**). Let X be a discrete random variable, we say that X is integrable if:

$$\sum_{i=1}^{\infty} |x_i| P(A_i) < \infty$$

If X is integrable, we define the expectation as:

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} x_i P(A_i)$$

In order to define the expectation of a general random variable X one also consider sequences of non-negative simple functions, and decomposes the expectation in two positive random variables, i.e. $X = X^+ - X^-$ whit $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$, and define the expectation as:

$$\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-]$$

We will mostly consider the expectation as a measure-theoretic integral, i.e:

$$\mathbb{E}[X] = \int_{\Omega} X(\omega) dP(\omega)$$

Definition 2.2.6 (**Conditional Expectation**). Let $(\Omega, \mathcal{F}, P)$ be a probability space, let X be an integrable random variable, and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub σ-algebra, we say that a random variable $Z = \mathbb{E}[X|\mathcal{G}]$ is the conditional expectation of X given $\mathcal{G}$ if:

(i) Z is $\mathcal{G}$-measurable, and

(ii) if $A \in \mathcal{G}$, then:

$$\int_A Z dP = \int_A X dP$$

Theorem 2.2.7. *Let X and Y be integrable random variables, let $a, b \in \mathbb{R}$ and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub σ-algebra, then:*

(i) $\mathcal{G} = \{\emptyset, \Omega\}$, *then:* $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$

(ii) $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X]$

(iii) If X is $\mathcal{G}$-measurable, $\mathbb{E}[X|\mathcal{G}] = X$ *a.e.*

(iv) $\mathbb{E}[aX + bY|\mathcal{G}] = a\mathbb{E}[X|\mathcal{G}] + b\mathbb{E}[Y|\mathcal{G}]$ *a.e.*

(v) If $X \geq 0$ a.e., $\mathbb{E}[X|\mathcal{G}] \geq 0$ *a.e.*

(vi) If $X \leq Y$ a.e., $\mathbb{E}[X|\mathcal{G}] \leq \mathbb{E}[Y|\mathcal{G}]$ *a.e.*

(vii) $|\mathbb{E}[X|\mathcal{G}]| \leq \mathbb{E}[|X||\mathcal{G}]$ *a.e.*

(viii) Suppose that Y is $\mathcal{G}$-measurable and XY are integrable, then:

$$\mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}] \text{ a.e.}$$

(ix) If X and $\mathcal{G}$ are independent, then:

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X] \text{ a.e.}$$

(x) If X_n and X are integrable, and either $X_n \uparrow X$, or $X_n \downarrow X$, then:

$$\mathbb{E}[X_n|\mathcal{G}] \to \mathbb{E}[X|\mathcal{G}] \text{ a.e.}$$

Theorem 2.2.8 (**Tower Law**). *If X is an integrable random variable, and if $\mathcal{G}_1 \subseteq \mathcal{G}_2$ are σ-algebras, then:*

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}_1]|\mathcal{G}_2] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}_2]|\mathcal{G}_1] = \mathbb{E}[X|\mathcal{G}_1]$$

Theorem 2.2.9 (**Jensen's inequality**). *Let ϕ be a convex function on an open interval (x_1, x_2) and let X be a random variable whose range is in (x_1, x_2). Suppose X and $\phi(X)$ are integrable and that $\mathcal{G} \subseteq \mathcal{F}$ are σ-algebras, then:*

$$\phi\left(\mathbb{E}[X|\mathcal{G}]\right) \leq \mathbb{E}\left[\phi(X)|\mathcal{G}\right] \text{ a.e.}$$

2.3 Stochastic Analysis

The results in this section are based on [Wal12] and [Bal17].

Stochastic processes and filtrations

Definition 2.3.1 (**Filtration**). Let $\mathcal{T}$ denote an index set either countable or a subset of $\mathbb{R}$, we say that the collection $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathcal{T}}$ of σ-algebras is a filtration if for every $s \leq t \in \mathcal{T}$:

$$\mathcal{F}_s \subseteq \mathcal{F}_t$$

Definition 2.3.2 (**Augmented Filtration**). The augmented filtration is the filtration obtained by including the collection of null sets $\mathcal{N}$ to the σ-algebra $\mathcal{F}_t = \sigma(X_u : u \leq t)$, i.e:

$$\overline{\mathcal{F}}_t = \sigma(\mathcal{F}_t \cup \mathcal{N})$$

Definition 2.3.3 (**Stochastic process**). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, P)$ denote a probability equipped with a filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$.
A stochastic process $X = (X_t)_{t\in\mathcal{T}}$ is a collection of random variables defined on $(\Omega, \mathcal{F})$ taking values in a measurable space $(E, \mathcal{E})$

Definition 2.3.4 (**Adapted process**). We say that the stochastic process $X = (X_t)_{t\in\mathcal{T}}$ is adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathcal{T}}$ if for every $t \in \mathcal{T}$ we have that X_t is $\mathcal{F}_t$-measurable.

Definition 2.3.5 (**Modification and Indistinguishable processes**). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P) = (\Omega', \mathcal{F}', (\mathcal{F}_t)_{t\geq 0}, P')$, we say that X is a modification of X' if:

$$\forall\, t \;\; P(X_t = X'_t) = 1$$

We say that X is indistinguishable from X' if:

$$P(X_t = X'_t \;\forall\, t) = 1$$

Definition 2.3.6 (**σ-finite measure [Lin17]**). We say that a measure space $(X, \mathcal{F}, \mu)$ is σ-finite if X is a countable union of sets with finite measure, i.e for $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{F}$ we have:

$$X = \bigcup_{n\in\mathbb{N}} A_n \;\text{ with }\; \mu(A_n) < \infty,\; \forall n \in \mathbb{N}$$

Theorem 2.3.7 (**[Lin17]**). *Assume that $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ are two measure spaces, and let $\mathcal{F}\otimes\mathcal{G}$ denote the σ-algebra generated by the measurable rectangles $F\times G, F \in \mathcal{F}, G \in \mathcal{G}$. Then there exists a measure $\mu \times \nu$ on $(\mathcal{F} \otimes \mathcal{G})$ such that:*

$$\mu \times \nu(F \times G) = \mu(F)\nu(G) \;\; \textit{for all } F \in \mathcal{F}, G \in \mathcal{G}$$

If μ and ν are σ-finite, this measure is unique and is called the product measure of μ and ν.

Definition 2.3.8 (**Measurable Process**). A stochastic process $X = (X_t)_{t\geq 0}$ taking values on a measurable space $(E, \mathcal{E})$ is said to be measurable if:

$$A \times \Omega \ni (t, \omega) \mapsto X_t(\omega) \in E$$

is measurable (with $A \subseteq E$), i.e:

$$\forall B \in \mathcal{B}(E) : \;\; \{(t, \omega) \in A \times \Omega : X_t(\omega) \in B\} \in \mathcal{B}(A) \otimes \mathcal{F}$$

Definition 2.3.9 (**Progressively measurable process**). A stochastic process $X = (X_t)_{t\geq 0}$ is said to be progressively measurable w.r.t $\mathcal{F}$ if:

$$\forall t: \quad [0,t] \times \Omega \ni (s,\omega) \mapsto X_s(\omega)$$

is measurable w.r.t $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$

Theorem 2.3.10 (**Kolmogorov's continuity theorem**). *Let $D \subseteq \mathbb{R}^m$, be an open set, and consider the process $X = (X_\theta)_{\theta \in D}$ and assume there exists $\alpha > 0, \beta > 0, C > 0$ such that:*

$$E[|X_{\theta_1} - X_{\theta_2}|^\beta] \leq C|\theta_1 - \theta_2|^{m+\alpha}$$

then there exists a continuous modification $\widetilde{X}$ of X. Furthermore $\widetilde{X}$ is Hölder continuous with exponent $\gamma < \frac{\alpha}{\beta}$ on all compact subsets $K \subseteq D$, i.e:

$$|\widetilde{X}_{\theta_1} - \widetilde{X}_{\theta_2}| \leq C|\theta_1 - \theta_2|^\gamma$$

Integral Spaces

Definition 2.3.11 (M^p_{loc}). Let $M^p_{loc}([a,b])$ denote the space of equivalence classes of real-valued progressively measurable processes $X = (X_t)_{t\geq 0} \in \mathbb{R}^d$ such that:

$$\int_a^b |X_s|^p ds < \infty \text{ a.s}$$

Definition 2.3.12 (M^p). Let $M^p[a,b]$ denote the subspace of $M^p_{loc}[a,b]$ such that:

$$\mathbb{E}\left[\int_a^b |X_s|^p ds\right] < \infty$$

Fubini and Stochastic Fubini

Theorem 2.3.13 (**Fubini's theorem**). *Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be two σ-finite measure spaces, and assume that $f : X \times Y \to \overline{\mathbb{R}}$ is $\mu \times \nu$-integrable, i.e.*

$$\iint |f(x,y)| d(\mu \times \nu) < \infty$$

Then:

$$x \mapsto \int f(x,y) d\nu(y) \ \ and \ \ y \mapsto \int f(x,y) d\mu(x)$$

are $\mu-$ and $\nu-$integrable, respectively. Moreover:

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \left[\int_Y f(x,y) d\nu(y) \right] d\mu(x) = \int_Y \left[\int_X f(x,y) d\mu(x) \right] d\nu(y)$$

The functions $y \mapsto \int f(x,y) d\mu(x)$ and $x \mapsto \int f(x,y) d\nu(y)$ are defined μ (a.e.) and ν (a.e.) respectively.

Theorem 2.3.14 (**Stochastic Fubini for Brownian Motion [Fil09]**). *Let $X = (X(\omega, t, s))_{[0 \leq t, s \leq T]}$ be an $\mathbb{R}^d$-valued stochastic process satisfying:*

- *X is progressively measurable w.r.t $\mathcal{F}_T \otimes \mathcal{B}([0,T])$*
- $\sup\limits_{0 \leq s,t \leq T} |X(t,s)| < \infty$

Then $\lambda(t) = \int_0^T X(t,s) ds \in M^2_{loc}[0,T]$ and there exists a $\mathcal{F}_T \otimes \mathcal{B}([0,T])$-measurable modification $\psi(s)$of $\int_0^T X(t,s) ds$ such that $\psi \in M^2_{loc}([0,T])$, moreover:

$$\int_0^T \psi(s) ds = \int_0^T \lambda(t) dW(t)$$

i.e.

$$\int_0^T \left[\int_0^T X(t,s) dW(t) \right] ds = \int_0^T \left[\int_0^T X(t,s) ds \right] dW(t)$$

Girsanov's theorem, Equivalent martingale measures and Bayes theorem

Definition 2.3.15 (**Absolutely continuous measures**). Let μ and ν be two measures defined on $(X, \mathcal{F})$, and define:

$$\mathcal{N}_\mu = \{A \in \mathcal{F} : \mu(A) = 0\}$$
$$\mathcal{N}_\nu = \{A \in \mathcal{F} : \nu(A) = 0\}$$

We say that ν is absolutely continuous w.r.t μ iff $\mathcal{N}_\mu \subseteq \mathcal{N}_\nu$ and we write $\nu \ll \mu$, i.e $\mu(A) = 0 \implies \nu(A) = 0$

Definition 2.3.16 (**Equivalent measures**). Consider the situation as described in Definition 2.3.15, we say that ν and μ are equivalent iff $\mathcal{N}_\mu \subseteq \mathcal{N}_\nu$ and $\mathcal{N}_\nu \subseteq \mathcal{N}_\mu$ i.e. $\mathcal{N}_\mu = \mathcal{N}_\nu$ and we write $\nu \sim \mu$, i.e:

$$\mu(A) = 0 \iff \nu(A) = 0$$

Theorem 2.3.17 (**Radon Nikodym derivative**). *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space. Let ν be a σ-finite measure on $(X, \mathcal{F})$ such that $\nu \sim \mu$. Then there exists a unique non-negative function f on X which is measurable w.r.t $\mathcal{F}$ for which:*

$$\nu(E) := \int_E f d\mu, \ \forall E \in \mathcal{F}$$

f is unique in the sense that if there is another non-negative measurable function g such that:

$$\nu(E) = \int_E g d\mu \implies f = g, \ \mu - a.e$$

One usually denotes:

$$f = \frac{d\nu}{d\mu}$$

Let $W = (W_t)_{t\in[0,T]} \in \mathbb{R}^m$ denote a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, furthermore let ϕ be an $\mathbb{R}^m$-valued process (also valid for $\mathbb{C}^m$) with $\phi \in M^2_{loc}([0,T])$, the process we will be interested in looks like:

$$Z_t := \mathcal{E}_t(\phi \bullet W) = \exp\left(\int_0^t \phi_s dW_s - \frac{1}{2}\int_0^t \phi_s^2 ds\right) \tag{2.1}$$

Proposition 2.3.18 (**Application of Radon-Nikodym derivative**). *Let $Q \sim P$ and define $Q(A) := \mathbb{E}[Z_T \mathbb{1}_A] = \int_A Z_T dP$, where $A \in \mathcal{F}$ and Z is defined as in Equation 2.1, furthermore require that $\mathbb{E}[Z_T] = 1$, then Q defines a new probability measure on $(\Omega, \mathcal{F})$, and*

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} = Z_T$$

Proof. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of disjoint sets in $\mathcal{F}$, we then have that Q defines a measure as:

$$Q(\emptyset) = \int_\emptyset Z_T dP = 0$$
$$1 = \mathbb{E}[Z_T] = \int_\Omega Z_T dP = Q(\Omega)$$

$$Q\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \int_{\bigcup_{n\in\mathbb{N}} A_n} Z_T dP = \int_\Omega Z_T \mathbb{1}_{\bigcup_{n\in\mathbb{N}} A_n} dP = \sum_{n\in\mathbb{N}} Q(A_n)$$

As $Z = (Z_t)_{t\geq 0}$ by construction is measurable as well as non-negative, it follows from Radon-Nikodym theorem that:

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} = Z_T$$

■

Theorem 2.3.19 (**Girsanov's theorem [Bal17]**). *Let $W = (W_t)_{t\in[0,T]}$ be an m-dimensional Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, let $Z = (Z_t)_{t\in[0,T]}$ be defined as in equation 2.1 with $\phi \in M^2_{loc}[0,T]$. Furthermore assume that Z is a martingale w.r.t P and let Q be a probability measure on $(\Omega, \mathcal{F})$ defined via the Radon-Nikodym density Z_T, then:*

$$W_t^Q = W_t - \int_0^t \phi_s ds$$

defines a $(Q, \mathcal{F})$-Brownian motion on $[0,T]$

Often what makes Girsanov's theorem hard to use is the requirement of Z being a martingale under P, therefore the next theorem is quite useful:

Theorem 2.3.20 (**[Bal17]**). *Let $\phi \in M^2_{loc}([0,T])$, define $M_t = \int_0^t \phi_s dW_s, t \in [0,T]$ with $\langle M\rangle_t = \int_0^t \phi_s^2 ds$, and let:*

$$Z_t = \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right)$$

Consider the following properties:

(i) $\mathbb{E}\left[e^{\frac{1}{2}\int_0^T |\phi_s|^2 ds}\right] < \infty$ *(Novikov's condition)*

(ii) $M = (M_t)_{t\in[0,T]}$ *is a bounded martingale in* $L^2(\Omega, \mathcal{F}, P)$ *and* $\mathbb{E}[e^{\frac{1}{2}M_T}] < \infty$

(iii) $Z = (Z_t)_{t\in[0,T]}$ *is a uniformly integrable martingale.*

Then $(i) \implies (ii) \implies (iii)$

Theorem 2.3.21 (**Bayes theorem [Øks03]**). *Let P and Q be two probability measures on $(\Omega, \mathcal{F})$ such that $\frac{dQ}{dP} = f$ with $f \in L^1(\Omega, \mathcal{F}, P)$. Let X be a random variable on $(\Omega, \mathcal{F})$ such that:*

$$\mathbb{E}_Q[|X|] = \int_\Omega |X(\omega)| f(\omega) dP(\omega) < \infty$$

Let $\mathcal{G}$ be a sigma-algebra with $\mathcal{G} \subseteq \mathcal{F}$, then:

$$\mathbb{E}_Q[X|\mathcal{G}] = \frac{\mathbb{E}[fX|\mathcal{G}]}{\mathbb{E}[f|\mathcal{G}]} \quad a.s$$

Stochastic Differential Equations

Definition 2.3.22 (**1-dimensional Ito process**). Let $F \in M^1_{loc}([a,b])$ and $G \in M^2_{loc}([a,b])$, and $W = (W_t)_{t\in[a,b]}$ be a one-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[a,b]}, P)$ then a process on the form:

$$X_t = X_a + \int_a^t F_s ds + \int_a^t G_s dW_s$$

is called an Ito process, this can also be rewritten in differential form as:

$$dX_t = F_t dt + G_t dW_t$$

Theorem 2.3.23 (**1-dimensional Ito formula [Øks03]**). *Let X_t be an Ito process, given by:*

$$dX_t = F_t dt + G_t dW_t$$

Let $g(t,x) \in C^{1,2}([0,\infty) \times \mathbb{R})$ (one time differentiable in time, and twice differentiable in space), then $Y_t = g(t, X_t)$ is again an Ito process and:

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2$$

where $(dX_t)^2 = dX_t \cdot dX_t$ is computed according to:

$$\begin{aligned} dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0 \\ dW_t \cdot dW_t = dt \end{aligned}$$

Theorem 2.3.24 (**Integral representation theorem w.r.t Brownian Motion [Bal17]**). *Let $W = (W_t)_{t\geq 0}$ be an m-dimensional Brownian motion on $(\Omega, \mathcal{F}, (\overline{\mathcal{F}}_t)_t, P)$, where $(\overline{\mathcal{F}}_t)_t$ represents the augmented natural filtration. Let $T > 0$, then we can represent every $Z \in L^2(\Omega, \overline{\mathcal{F}}_T, P)$ uniquely as:*

$$Z = \mathbb{E}[Z] + \int_0^T H_s dW_s$$

where $H \in M^2([0,T])$ is $(\overline{\mathcal{F}}_t)_t$-adapted.

Theorem 2.3.25 (**Martingale representation theorem [Bal17]**). *Let $M = (M_t)_{t\in[0,T]}$ be a square integrable martingale with w.r.t $(\overline{\mathcal{F}}_t)_t$. Then there exist a unique process $H \in M^2([0,T])$ such that:*

$$M_t = \mathbb{E}[M_T] + \int_0^T H_s dW_s = M_0 + \int_0^T H_s dW_s \quad a.s$$

Let $b(t,x) = (b_i(t,x))_{1\leq i\leq m}$ and $\sigma(t,x) = (\sigma(t,x)_{ij})_{\substack{1\leq i\leq m \\ 1\leq j\leq d}}$ be measurable functions on $[0,T] \times \mathbb{R}^m$

Definition 2.3.26 (**[Bal17]**). Let $X = (X_t)_{t\in[u,T]}$ be a stochastic process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, it is said to be a solution of the SDE (Stochastic differential equation)

$$(*) \begin{cases} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_u &= x \in \mathbb{R}^m \end{cases}$$

if:

- $W = (W_t)_{t\in[0,T]} \in \mathbb{R}^d$ is a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ and
- $\forall t \in [u, T]$ we have:

$$X_t = x + \int_u^t b(s, X_s)ds + \int_u^T \sigma(s, X_s)dW_s$$

Definition 2.3.27 (**Strong solution**). We say that equation 2.2 has strong solutions if for every standard Brownian motion $W = (W_t)_t$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$, there exists a process X that satisfies equation 2.2.

Definition 2.3.28 (**Uniqueness in distribution**). We say that for the SDE in 2.2, there is uniqueness in distribution if given two solutions X^i on $(\Omega^i, \mathcal{F}^i, (\mathcal{F}^i_t)_t, P^i), i = 1, 2$ have the same distribution, i.e.

$$X^1 \overset{d}{=} X^2$$

Theorem 2.3.29 (**[Bal17]**). *Let $X = (X_t)_{t\in[u,T]}$ be a stochastic process defined on $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, P)$, furthermore let $\eta \in L^2(\Omega, \mathcal{F}, P)$ be $\mathcal{F}_u$-measurable and consider the SDE:*

$$\begin{cases} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_u &= \eta \end{cases} \tag{2.2}$$

where b, σ satisfies:

- *b, σ are measurable functions such that: $\exists L > 0, M > 0$ such that $\forall x, y \in \mathbb{R}^m, \forall t \in [u, T]$*

$$\begin{aligned} |b(t, x)| &\leq M(1 + |x|) \\ |\sigma(t, x)| &\leq M(1 + |x|) \\ |b(t, x) - b(t, y)| &\leq L|x - y| \\ |\sigma(t, x) + \sigma(t, y)| &\leq L|x - y| \end{aligned}$$

Then $\exists X \in M^2([u, T])$ satisfying 2.2 and the solution is strong and strongly unique.

Assumption 2.3.30. Throughout this thesis, unless otherwise specified, we will assume that our probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ are such that:

- $\mathbb{F}$ is augmented with it's respective measure, i.e $\mathcal{F}_t = \overline{\mathcal{F}}_t = \sigma(\mathcal{F}_t \cup \mathcal{N})$ and
- $\mathbb{F}$ is right-continuous, i.e.

$$\mathcal{F}_t = \mathcal{F}_{t^+} := \bigcap_{u>t} \mathcal{F}_u$$

2.4 Levy processes

Definition 2.4.1 (**Levy process [Ken]**). A stochastic process $X = (X_t)_{t\geq 0}$ is a Levy process if:

1. $X_0 = 0$
2. X has independent increments, i.e $\forall\ 0 \leq t < u$:

$$X_u - X_t \text{ is independent of } X_s - X_r \ \forall\ 0 \leq r < s \leq t$$

3. X has stationary increments, i.e.

$$\forall\ 0 \leq s < t : \ X_t - X_s \overset{d}{=} X_{t-s}$$

4. X is stochastically continuous:

$$\forall\ \epsilon > 0 : \ \lim_{h\to 0} P(|X_{t+h} - X_t| \geq \epsilon) = 0$$

5. X has càdlàg sample paths

Definition 2.4.2 (**Infinitely divisible [App+04]**). A random variable X is said to be infinitely divisible if $\forall\ n \in \mathbb{N}$ there exist $X_{n,1}, \dots, X_{n,n}$ such that:

$$X \overset{d}{=} \sum_{k=1}^{n} X_{n,k}$$

Proposition 2.4.3. *If $X = (X_t)_{t\geq 0}$ is a Levy-process, then $\forall\ t \geq 0$ X_t is infinitely divisible.*

Definition 2.4.4 (**Levy-measure**). A Levy-measure is a Borel-measure ν defined on $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$ such that:

$$\int_{\mathbb{R}_0^d} 1 \wedge |x|^2 \nu(dx) < \infty$$

Theorem 2.4.5 (**Levy-Khintchine theorem [App+04]**). *Let μ be a probability measure on $\mathbb{R}^d$, then there exist:*

- $\gamma \in \mathbb{R}^d$
- $A \in \mathbb{R}^{d\times d}$: *Positive semi-definite symmetric matrix* ($u^{Tr} Au \geq 0, \forall\ u \in \mathbb{R}^d$)
- ν *a Levy-measure on* $\mathbb{R}_0^d$ *such that* $\forall\ u \in \mathbb{R}^d$:

$$\varphi_\mu(u) = \exp\left(i\langle \gamma, u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}_0^d} \left[e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle \mathbb{1}(|x| < 1) \right] \nu(dx) \right)$$

(γ, A, ν) is called the characteristic triplet of X.

Definition 2.4.6 (**Characteristic exponent**). The function $\Psi : \mathbb{R}^d \to \mathbb{C}$:

$$\Psi(u) = i\langle \gamma, u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}_0^d} \left[e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle \mathbb{1}(|x| < 1)\right] \nu(dx)$$

is called the characteristic exponent of the Levy-process X

Theorem 2.4.7 ([**App+04**]). *If $X = (X_t)_{t\geq 0}$ is a Levy process, then the characteristic triplet of each random variable X_t takes the following form:*

$$(\gamma_{(t)}, A_{(t)}, \nu_{(t)}) = (t\gamma, tA, t\nu)$$

where γ, A and ν are as described in Theorem 2.4.5, the characteristic function takes the following form:

$$\begin{aligned}\mathbb{E}[e^{iuX(t)}] &= \exp\Big(\Psi_{(t)}(u)\Big)\\ &= \exp\Big(t\Psi_{(1)}(u)\Big)\end{aligned}$$

One uses the following convetion:

$$\Psi(u) := \Psi_{(1)}(u) = i\langle \gamma, u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}_0^d} \left[e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle \mathbb{1}(|x| < 1)\right] \nu(dx)$$

2.4.1 Compound Poisson Process (CPP)

Definition 2.4.8 (**Compound Poisson process**). A compound Poisson process (CPP) with intensity $\lambda > 0$ and jump size distribution $F_J(dx)$ is a stochastic process:

$$Y(t) = \sum_{i=1}^{N(t)} J_k$$

where J_k are iid with distribution $F_J(dx)$ and $N(t)$ is a Poisson process with intensity λ, independent of $(J_k)_{k\geq 1}$

Proposition 2.4.9 (**Charactersitc function of CPP [Tan03]**). *The characteristic function of a CPP $I(t)$ is given by:*

$$\mathbb{E}[e^{iuI(t)}] = \exp\left(\lambda t \int_{\mathbb{R}} (e^{iux} - 1) F_J(dx)\right)$$

Proof.

$$\begin{aligned}
\mathbb{E}[e^{iuI(t)}] &= \mathbb{E}\left[e^{iu\sum_{k=1}^{N_t} J_k}\right] \\
&= \mathbb{E}\left[\mathbb{E}[e^{iu\sum_{k=1}^{N_t} J_k} | N_t = n]\right] \\
&= \sum_{n\in\mathbb{N}_0} \mathbb{E}\left[e^{iu\sum_{k=1}^{n} J_k}\right] P(N_t = n) \\
&= \sum_{n\in\mathbb{N}} \prod_{k=1}^{n} \mathbb{E}[e^{iuJ_k}] \cdot e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\
&= \sum_{n\in\mathbb{N}} \left(\mathbb{E}\left[e^{iuJ_1}\right]\right)^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\
&= e^{-\lambda t} \sum_{n\in\mathbb{N}} \frac{(\lambda t \mathbb{E}[e^{iuJ}])^n}{n!} \\
&= e^{-\lambda t} e^{\lambda t \mathbb{E}[e^{iuJ}]} \\
&= \exp\left(\lambda t (\mathbb{E}[e^{iuJ}] - 1)\right) \\
&= \exp\left(\lambda t \left(\int_{\mathbb{R}} e^{iux} F_J(dx) - 1\right)\right) \\
&= \exp\left(\lambda t \int_{\mathbb{R}} \left[e^{iux} - 1\right] F_J(dx)\right)
\end{aligned}$$

∎

Proposition 2.4.10 (**Characteristic triplet of CPP**). *Let $I(t)$ be a CPP as described in Definition 2.4.8, we then have that the characteristic triplet of I is given by:*

$$(\gamma, A, \nu) = \left(\lambda \int_{|x|<1} x F_J(dx), 0, \lambda F_J\right)$$

Proof. From Proposition 2.4.9, we have:

$$\begin{aligned}\mathbb{E}[e^{iuI(t)}] &= \exp\left(\lambda t \int_{\mathbb{R}} (e^{iux} - 1) F_J(dx)\right) \\ &= \exp\left(t\Psi(u)\right)\end{aligned}$$

Now:

$$\begin{aligned}\int_{\mathbb{R}} \left[e^{iux} - 1 - iux\mathbb{1}(|x| < 1)\right] \lambda F_J(dx) &= \lambda \int_{\mathbb{R}} [e^{iux} - 1] F_J(dx) - iu\lambda \int_{\mathbb{R}} x\mathbb{1}(|x| < 1) F_J(dx) \\ &= \lambda \int_{\mathbb{R}} [e^{iux} - 1] F_J(dx) - i\left\langle \lambda \int_{\mathbb{R}} x\mathbb{1}(|x| < 1) F_J(dx), u\right\rangle\end{aligned}$$

From this, we infer that:

$$\begin{gathered} i\langle \gamma, u\rangle - i\left\langle \lambda \int_{\mathbb{R}} x\mathbb{1}(|x| < 1) F_J(dx), u\right\rangle = 0 \\ \Updownarrow \\ \gamma = \lambda \int_{|x|<1} x F_J(dx)\end{gathered}$$

■

Proposition 2.4.11 ([**BBK08**]). *Assume that I is a CPP, g a continuous function and that $s \mapsto \Psi(ug(s)) \in L^1([0,t], \mathcal{F}, P)$, then:*

$$\mathbb{E}\left[\exp\left(i\theta \int_s^t g(u) dI(u)\right)\right] = \exp\left(\int_s^t \Psi(\theta g(u)) du\right)$$

Where $\Psi(x)$ is the cumulant function of $I(1)$ i.e:

$$\Psi(x) = \lambda \int_{\mathbb{R}} (e^{iyx} - 1) F_J(dy)$$

Proof. Since g is a continuous function on $[s,t]$ we know that there exist $M > 0$ such that $|g(u)| \leq M$, $\forall u \in [s,t]$, furthermore from Proposition 2.1.16, we know that g may be approximated by simple functions:

$$h(u) = \sum_{k=1}^{n} a_k \mathbb{1}_{(u_{k-1}, u_k]}(u), \text{ where: } s = u_0 < u_1 < \cdots < u_n = t$$

$$\mathbb{E}\left[\exp\left(i\theta \int_s^t h(u) dI(u)\right)\right] = \mathbb{E}\left[\exp\left(i\theta \sum_{k=1}^{n} a_k [I(u_k) - I(u_{k-1})]\right)\right]$$

Now as I is a CPP (and therefore a Levy-process), we know that it has independent increments and has a stationary distribution, meaning that: $I(u_k) - I(u_{k-1}) \stackrel{d}{=} I(u_k - u_{k-1}) = I(\Delta_k)$, leaving us with:

$$\begin{aligned}
\mathbb{E}\left[\exp\left(i\theta\sum_{k=1}^{n}a_k[I(u_k)-I(u_{k-1})]\right)\right] &= \prod_{k=1}^{n}\mathbb{E}\left[\exp\left(i\theta I(\Delta_k)\right)\right] \\
&= \prod_{k=1}^{n}\exp\left(\Psi(\theta a_k)\Delta_k\right) \\
&= \exp\left(\sum_{k=1}^{n}\Psi(\theta a_k)\Delta_k\right)
\end{aligned}$$

Thus:

$$\begin{aligned}
\mathbb{E}\left[\exp\left(i\theta\int_s^t g(u)dI(u)\right)\right] &= \lim_{\Delta_k\to 0}\mathbb{E}\left[\exp\left(i\theta\int_s^t h(u)dI(u)\right)\right] \\
&= \lim_{\Delta_k\to 0}\exp\left(\sum_{k=1}^{n}\Psi(\theta a_k)\Delta_k\right) \\
&\overset{\text{DCT}}{=}\exp\left(\lim_{\Delta_k\to 0}\sum_{k=1}^{n}\Psi(\theta a_k)\Delta_k\right) \\
&= \exp\left(\int_s^t\Psi(\theta g(u))du\right)
\end{aligned}$$

■

2.4.2 Esscher Transform for CPP

Let $Y \sim F_Y(dy)$, we then have:

$$\mathbb{E}[e^{\theta Y}] = \int_{\mathbb{R}} e^{\theta y} F_Y(dy)$$

Furthermore, we denote:

$$Z^\theta(T) := \frac{dQ^\theta}{dP} = \frac{e^{\theta Y}}{\mathbb{E}[e^{\theta Y}]}$$

Now let $I(t)$ denote a CPP with intensity λ and jump-distribution $J \sim F_J(dx)$, in this case we get:

$$Z^\theta(T) = \frac{e^{\theta I(T)}}{\mathbb{E}[e^{\theta I(T)}]}$$

In order for $Z^\theta(T)$ to be well defined, we need $\mathbb{E}[e^{\theta I(T)}] < \infty$, now from Theorem 2.4.5 in combination with Proposition 2.4.9, we have:

$$\mathbb{E}[e^{\theta I(T)}] = \exp(T\Psi(-i\theta)) = \exp\Big(\lambda T(\mathbb{E}[e^{\theta J}] - 1)\Big)$$

Meaning that we need $\mathbb{E}[e^{\theta J}] < \infty$ for $Z^\theta(T)$ to be well defined.

Notation 2.4.12. For simplicity and ease of notation, we define:

$$\xi(\theta) := \Psi(-i\theta) = \lambda(\mathbb{E}[e^{\theta J}] - 1)$$

We can now rewrite $Z^\theta(T)$ as:

$$Z^\theta(T) = e^{\theta I(T) - \xi(\theta) T}$$

Proposition 2.4.13. *$Z^\theta = (Z^\theta(t))_{t\in[0,T]}$ is a $(P, \mathbb{F})$-martingale.*

Proof. Let $0 \le s \le t \le T$:

$$\begin{aligned}
\mathbb{E}[Z^\theta(t)|\mathcal{F}_s] &= \mathbb{E}\left[e^{\theta I(t) - \xi(\theta)t}\middle|\mathcal{F}_s\right] \\
&= e^{-\xi(\theta)t}\mathbb{E}\left[e^{\theta[I(s)+(I(t)-I(s))]}\middle|\mathcal{F}_s\right] \\
&= e^{-\xi(\theta)t}e^{\theta I(s)}\mathbb{E}\left[e^{\theta I(t-s)}\right] \\
&= e^{-\xi(\theta)t}e^{\theta I(s)}e^{\xi(\theta)(t-s)} \\
&= e^{\theta I(s) - \xi(\theta)s} \\
&= Z^\theta(s)
\end{aligned}$$

■

Proposition 2.4.14 ([**BBK08**]). *$I(t)$ is a CPP under Q^θ with intensity $\lambda_{Q^\theta} = \lambda\mathbb{E}[e^{\theta J}]$*

Proof. We start off by calculating the characteristic function under Q^θ:

$$\begin{aligned}
\mathbb{E}_{Q^\theta}\left[e^{iuI(t)}\right] &= \mathbb{E}\left[e^{iuI(t)}Z^\theta(t)\right] \\
&= \mathbb{E}\left[e^{iuI(t)}e^{\theta I(t)-\xi(\theta)t}\right] \\
&= \exp(-\xi(\theta)t)\exp\left(\lambda t(\mathbb{E}[e^{(iu+\theta)J}] - 1)\right) \\
&= \exp\left(\lambda t(\mathbb{E}[e^{(iu+\theta)J}]) - \lambda t(\mathbb{E}[e^{\theta J}] - 1)\right) \\
&= \exp\left(\lambda t\mathbb{E}[e^{\theta J}(e^{iuJ} - 1)] \cdot \frac{\mathbb{E}[e^{\theta J}]}{\mathbb{E}[e^{\theta J}]}\right) \\
&= \exp\left(\lambda t\mathbb{E}[e^{\theta J}]\mathbb{E}[Z^\theta(1)(e^{iuJ} - 1)]\right) \\
&= \exp\left(t\,\underbrace{\lambda\mathbb{E}[e^{\theta J}]}_{=\lambda_{Q^\theta}}(\mathbb{E}_{Q^\theta}[e^{iuJ}] - 1)\right)
\end{aligned}$$

Thus:

$$\mathbb{E}_{Q^\theta}\left[e^{iuI(t)}\right] = \exp\left(t\Psi_{Q^\theta}(u)\right), \text{ where: } \Psi_{Q^\theta}(u) = \lambda_{Q^\theta}\left(\mathbb{E}_{Q^\theta}[e^{iuJ}] - 1\right)$$

■

Lemma 2.4.15 ([**Exercise MAT4770, Spring 2021**]). *Let $I(t)$ be a CPP under P with intensity λ, and jump distribution $J \sim Exp(\mu)$, then for $\theta < \mu$, we have:*

$$\lambda_Q = \frac{\lambda\mu}{\mu - \theta} \text{ and } J_1 \sim Exp(\mu - \theta)$$

Proof. For the Esscher transform to be well-defined, we must have that $\mathbb{E}[e^{\theta J}] < \infty$, now:

$$\begin{aligned}
\mathbb{E}[e^{\theta J}] &= \int_0^\infty e^{\theta x}\mu e^{-\mu x}dx \\
&= \frac{\mu}{\theta - \mu}e^{(\theta-\mu)x}\Big|_0^\infty \\
&= \begin{cases} \infty & \theta \geq \mu \\ \frac{\mu}{\theta-\mu} & \theta < \mu \end{cases}
\end{aligned}$$

To find the distribution of J under Q, we can derive it's characteristic function:

$$\mathbb{E}_Q[e^{iuJ}] = \mathbb{E}_Q[e^{iuI(1)}] = \exp\left(\xi(iu+\theta) - \xi(\theta)\right)$$

$$\begin{aligned}
\xi(iu+\theta) - \xi(\theta) &= \lambda\left(\mathbb{E}[e^{(iu+\theta)J}] - 1\right) - \lambda\left(\mathbb{E}[e^{\theta J}] - 1\right) \\
&= \lambda\mathbb{E}[e^{\theta J}(e^{iuJ} - 1)] \\
&= \lambda\int_0^\infty e^{\theta x}[e^{iux} - 1]F_J(dx) \\
&= \lambda\int_0^\infty e^{\theta x}[e^{iux} - 1]\mu e^{-\mu x}dx \cdot \frac{\theta - \mu}{\theta - \mu}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda\mu}{\theta-\mu}\int_0^{\infty}[e^{iux}-1](\mu-\theta)e^{-(\mu-\theta)x}dx \\
&= \lambda_Q\int_0^{\infty}[e^{iux}-1]F_J^Q(dx)
\end{aligned}
$$

Meaning that $J \overset{Q}{\sim} Exp(\mu-\theta)$ ■

Chapter 3

Mathematical Finance

3.1 Market Model

For the time being consider $r = (r_t)_{t\in[0,T]}$ to be a deterministic interest rate process. Furthermore assume that we have the following probability space $(\Omega, \mathcal{F}, (\bar{\mathcal{F}}_t)_{t\in[0,T]}, P)$

Definition 3.1.1 (Money market account). We define the money market account $B(t)$ as a solution to the ODE:

$$dB(t) = r(t)B(t)dt$$

with initial condition $B(0) = 1$, this gives the solution:

$$B(t) = e^{\int_0^t r(s)ds}$$

Consider the following processes:

- $B = (B_t)_{t\in[0,T]}$ the money market account
- $dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t$ the risky asset.

Let μ and σ be defined so that the conditions in Theorem 2.3.29 are met.

Let $\phi^i = \{\phi^i_t, t \in [0, T]\}$ be two stochastic processes defined on the above probability space. Denote $\phi = (\phi^0, \phi^1)$, where:

- ϕ^0_t represents the number of units invested in the money market account at time t.
- ϕ^1_t represents the number of units invested in the risky asset S at time t.

Definition 3.1.2 (**Trading strategy**). We say that $\phi = (\phi^0, \phi^1)$ is a trading strategy is it is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted and:

$$\phi^0 rB \in M^1([0, T]), \;\; \phi\mu S \in M^1([0, T]) \;\; \phi^1\sigma S \in M^2([0, T])$$

Definition 3.1.3 (**Value of portfolio**). The value of a portfolio with trading strategy ϕ is given by:

$$V^\phi(t, S_t) = \phi^0_t B_t + \phi^1_t S_t, \;\; t \in [0, T]$$

Definition 3.1.4 (**Self-financing strategy**). We say that the trading strategy ϕ is self-financing if:

$$dV^\phi(t, S_t) = \phi^0_t dB_t + \eta^1_t dS_t, \;\; t \in [0, T]$$

Definition 3.1.5 (**Arbitrage opportunity**). An arbitrage opportunity is a self-financing strategy ϕ with:

$$V^\phi(0, S_0) = 0, \;\; V^\phi(T, S_T) \geq 0, \;\; P(V^\phi(T, S_T) \geq 0) > 0$$

3.2 Fundamental theorems of asset pricing

Theorem 3.2.1 (**First Fundamental theorem of asset pricing**). *The following are equivalent:*

i There are no arbitrage opportunities

ii there exists an equivalent martingale measure $Q \sim P$ such that the process $(\tilde{S})_{t\in[0,T]} = \left(\frac{S_t}{B_t}\right)_{t\in[0,T]}$ is a $(Q,\mathbb{F})$-martingale.

Definition 3.2.2 (**Attainable claim**). We say that a claim H is attainable if there exists a trading strategy $\phi = (\phi^0, \phi^1)$ such that:

$$V^{\phi}(T, S_T) = H \ a.s$$

We assume that H is $\mathcal{F}_T$-measurable as well as $H \in M^2([0,T])$

Definition 3.2.3 (**Complete market**). We say the market is complete if all contingent claims in Definition 3.2.2 are attainable.

Theorem 3.2.4 (**Second Fundamental Theorem of Asset Pricing**). *An arbitrage-free market is complete if and only if there exists a unique equivalent martingale measure $Q \sim P$.*

Chapter 4

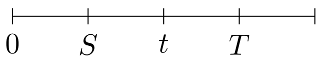
Interest rate theory

4.1 Zero Coupon Bonds and interest rates

Definition 4.1.1 (**Zero Coupon Bond [Fil09]**). A zero coupon bond (ZCB) with maturity T guarantees the holder one dollar to be paid out at maturity T. We denote the time t price of the zero coupon bond as $P(t,T)$

We will assume the following:

- There is a frictionless market for T-bonds for all $T > 0$
- $P(T,T) = 1$ for all T
- $P(t,T)$ is differentiable in T.



Definition 4.1.2 (**Simple forward rate [Fil09]**). The simple forward rate for $[S,T]$ prevailing at time $t \leq T$ is defined as:

$$F(t,S,T) = \frac{1}{T-S}\frac{P(t,S) - P(t,T)}{P(t,T)}$$

Definition 4.1.3 (**Continuously compounding forward rate [Fil09]**). The continuously compounded forward rate for $[S,T]$ prevailing at $t \leq T$ is given by:

$$R(t;S,T) = -\frac{\ln P(t,T) - \ln P(t,S)}{T-S}$$

Definition 4.1.4 (**Instantaneous forward rate [Fil09]**). The instantaneous forward rate with maturity T, prevailing at time t is defined as:

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T}$$

Definition 4.1.5 (**Short rate [Fil09]**). The instantaneous short rate at time t is defined as:

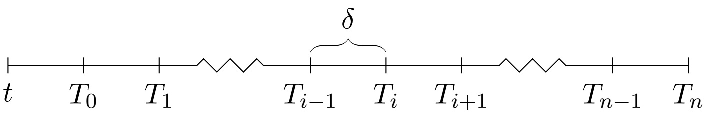
$$r(t) = f(t,t) = \left(-\frac{\partial \log P(t,T)}{\partial T}\right)\Big|_{T=t}$$

4.2 Swaps

Definition 4.2.1 (**Fixed Interest rate swap**). An interest rate swap is a forward contract in which one stream of future interest payments is exchanged for a fixed interest rate.

Some clarification:

- N represents the nominal value, think of it as the amount you loan/lend.
- $0 < T_0 < T_1 < \cdots < T_n$ a sequence of future dates.
- $\delta = T_i - T_{i-1}$ a fixed leg between payments
- κ a fixed rate.



We use the following notation for the simple forward rate:

$$F(t,T) := F(t,t,T) = \frac{1}{T-t}\left(\frac{1}{P(t,T)} - 1\right)$$

This means that we can write:

$$F(T_{i-1},T_i) = \frac{1}{T_i - T_{i-1}}\left(\frac{1}{P(T_{i-1},T_i)} - 1\right) = \frac{1}{\delta}\left(\frac{1}{P(T_{i-1},T_i)} - 1\right)$$

Exchanging a floating rate with a fixed-rate payer-swap contract has the following specification:

- Pay $\kappa\delta N$ (-)
- Receive $F(T_{i-1},T_i)\delta N$ (+)

Cash flow at time T_i:

$$F(T_{i-1},T_i)\delta N - \kappa\delta N = [F(T_{i-1},T_i) - \kappa]\delta N$$

Time t-value for $t \leq T_0$ at time T_i:

$$\begin{aligned} P(t,T_i)[F(T_{i-1},T_i) - \kappa]\delta N &= P(t,T_i)\left(\frac{1}{\delta}\left(\frac{1}{P(T_{i-1},T_i)} - 1\right) - K\right)\delta N \\ &= \frac{P(t,T_i)}{P(T_{i-1},T_i)}N - P(t,T_i)N - P(t,T_i)\kappa\delta N \end{aligned}$$

Proposition 4.2.2. *We have the following relationship:*

$$\frac{P(t,T_i)}{P(T_{i-1},T_i)} = P(t,T_{i-1})$$

Proof. We use a classical arbitrage argument:

First, we note that:

$$\frac{P(t,T_i)}{P(T_{i-1},T_i)} = P(t,T_i)\frac{1}{P(T_{i-1},T_i)}$$

This is the time t-value of receiving $\frac{1}{P(T_{i-1},T_i)}$ at time T_i. Our strategy will be:

- at time t: buy T_{i-1}-bond $(-P(t,T_{i-1}))$
- at time T_{i-1}: receive (+\$1), and immediately reinvest in T_i-bonds, we buy $\frac{1}{P(T_{i-1},T_i)}$ number of T_i-bonds.
- at time T_i: we have $\frac{1}{P(T_{i-1},T_i)}$

$$\begin{array}{ccc} -P(t,T_i) & +\$1, \left(-\frac{1}{P(T_{i-1},T_i)}\right) & +\frac{1}{P(T_{i-1},T_i)} \\ \hline t & T_{i-1} & T_i \end{array}$$

This means that we have a risk-free profit of $\frac{1}{P(T_{i-1},T_i)}$, meaning that in order to avoid arbitrage we must have that:

$$\frac{P(t,T_i)}{P(T_{i-1},T_i)} = P(t,T_{i-1})$$

■

Thus from the above proposition, we get the following time t-value for $t \leq T_0$:

$$N[P(t,T_{i-1}) - P(t,T_i)] - \kappa\delta N P(t,T_i)$$

Total payer cash flow:

$$\begin{aligned}\mathcal{C}_p(t) &= \sum_{i=1}^{n}[N[P(t,T_{i-1}) - P(t,T_i)] - \kappa\delta N P(t,T_i)] \\ &= N(P(t,T_0) - P(t,T_n)) - \kappa\delta N\sum_{i=1}^{n}P(t,T_i)\end{aligned}$$

A receiver interest rate swap corresponds to changing the sign of the cash flows, this yields:

$$\mathcal{C}_p(t) = -\mathcal{C}_r(t)$$

Result 4.2.3. The "fair" fixed rate $\kappa = R_{swap}(t)$ should be chosen such that $\mathcal{C}_p(t) = -\mathcal{C}_r(t) = 0$, this gives:

$$R_{Swap}(t) = \frac{P(t,T_0) - P(t,T_n)}{\delta\sum_{i=1}^{n}P(t,T_i)}$$

4.3 Short rate models

Consider the following probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, the market model consists of:

- Money market account $B = (B(t))_t$ with $B(t) = e^{\int_0^t r(u)du}$
- Short rate process $r = (r(t))_t$

We assume the following short-rate dynamics:

$$dr(t) = b(t)dt + \sigma(t)dW(t)$$

Where $r = (r(t))_{t\geq 0}$ is a process satisfying the necessary conditions given in Theorem 2.3.29.

Furthermore, the market is assumed to be arbitrage-free, meaning that there $\exists\, Q \sim P$ such that:

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \mathcal{E}_t\left(\gamma \bullet W\right)$$

Result 4.3.1 (**Relationship between zero coupon bonds and the short rate**). We can express the Zero Coupon Bond price as follows:

$$P(t,T) = \mathbb{E}_Q\left[e^{-\int_t^T r(u)du}\middle|\mathcal{F}_t\right],\ \forall\, t \in [0,T]$$

Proof. By the First Fundamental theorem of asset pricing (Theorem 3.2.1), we have that to avoid arbitrage, all tradable assets should be $(Q, \mathbb{F})$-martingales after discounting, meaning that:

$$\mathbb{E}_Q\left[\frac{P(T,T)}{B(T)}\middle|\mathcal{F}_t\right] = \frac{P(t,T)}{B(t)}$$

Now $P(T,T) = 1$, and $B(t)$ is $\mathcal{F}_t$-measurable, this gives us:

$$P(t,T) = \mathbb{E}_Q\left[\frac{B(t)}{B(T)}\middle|\mathcal{F}_t\right] = \mathbb{E}_Q\left[e^{-\int_t^T r(u)du}\middle|\mathcal{F}_t\right]$$

■

Proposition 4.3.2 ([**Fil09**]). *Considering the above setting, then we have that the process* $r = (r(t))_{t\in[0,T]}$ *have the following* Q*-dynamics:*

$$dr(t) = \left(b(t) + \sigma(t)\gamma(t)^{Tr}\right) dt + \sigma(t)dW^Q(t) \ (Q)$$

Proof. Let $W = (W_t)_{t\in[0,T]} \in \mathbb{R}^m$, and let $\mathcal{F}_t = \sigma(W_s : s \leq t)$, also let $\gamma \in \mathbb{R}^m$, as well as $\gamma \in M^2_{loc}([0,T])$. By assumption there are no arbitrage, thus $\mathcal{E}_t(\gamma \bullet W) \in M^2([0,T])$, this is also a $(P,\mathbb{F})$-martingale. Girsanov's Theorem 2.3.19, then tells us that the Q-dynamics takes the following form:

$$dW^Q(t) = dW(t) - \gamma(t)^{Tr}dt$$

This yields:

$$\begin{aligned} dr(t) &= b(t)dt + \sigma(t)dW(t) \\ &= b(t)dt + \sigma(t)[dW^Q(t) + \gamma(t)^{Tr}dt] \\ &= \left(b(t) + \sigma(t)\gamma^{Tr}(t)\right) dt + \sigma(t)dW^Q(t) \end{aligned}$$

■

4.4 Affine Term Structures

Consider the general SDE:

$$dr(t) = b(t, r(t))dt + \sigma(t, r(t))dW^Q(t), \ (Q) \tag{4.1}$$

Assume that b and σ are such that they satisfy the necessary conditions given in Theorem 2.3.29, meaning that a solution exists and is strongly unique.

Definition 4.4.1 (**Affine Term Structure**). A short rate model $r = (r(t))_{t\geq 0}$ is said to provide an ATS (Affine Term Structure) if the Zero Coupon Bond $P(t, T)$, can be expressed as:

$$P(t, T) = \exp\left(-A(t, T) - B(t, T)r(t)\right)$$

Where A, B are smooth C^1-functions, meaning they are continuous and have continuous first derivatives.

Proposition 4.4.2 ([**Fil09**]). *The short rate model $r = (r(t))_{t\geq 0}$ provides an ATS if and only if the diffusion and drift terms take the form:*

$$\begin{aligned} \sigma^2(t, r(t)) &= a(t) + \alpha(t)r(t) \\ b(t, r(t)) &= b(t) + \beta(t)r(t) \end{aligned}$$

a, α, b, β are continuous functions, furthermore A and B solve the Ricatti equations:

$$\begin{aligned} \partial_t A(t, T) &= \frac{1}{2}a(t)B^2(t, T) - b(t)B(t, T), \quad A(T, T) = 0 \\ \partial_t B(t, T) &= \frac{1}{2}\alpha(t)B^2(t, T) - \beta(t)B(t, T), \quad B(T, T) = 0 \end{aligned}$$

4.4.1 Vasicek model

Consider the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, Q)$ and let the dimension $d = 1$.

Proposition 4.4.3 (**Vasicek Model**). *The Vasicek model is an Ornstein–Uhlenbeck process with the following dynamics:*

$$dr(t) = \alpha[m - r(t)]dt + \sigma dW^Q(t)$$

Here α, m, σ are real-valued constants, with $\sigma > 0$.

Let $0 \leq t \leq T$ we then have an explicit solution given by:

$$r(T) = e^{-\alpha(T-t)}r(t) + m[1 - e^{-\alpha(T-t)}] + \sigma \int_t^T e^{-\alpha(T-u)}dW^Q(u)$$

Furthermore, the Vasicek model belongs to the class of Affine term structures where:

$$\begin{aligned} B(t, T) &= \frac{1}{\alpha}\left[1 - e^{-\alpha(T-t)}\right] \\ A(t, T) &= -mB(t, T) - m(T - t) - \frac{\sigma^2}{2}\int_t^T B^2(u, T)du \end{aligned}$$

Proof. This follows from applying Ito's Formula on $g(t,x) = e^{\alpha t}x$:

$$\begin{aligned} d[e^{\alpha t}r(t)] &= \alpha e^{\alpha t}r(t)dt + e^{\alpha t}dr(t) \\ &= \alpha e^{\alpha t}r(t)dt + e^{\alpha t}\left(\alpha[m - r(t)]dt + \sigma dW^Q(t)\right) \\ &= \alpha m e^{\alpha t}dt + \sigma e^{\alpha t}dW^Q(t) \end{aligned}$$

Thus:

$$\begin{aligned} r(T) &= e^{-\alpha(T-t)}r(t) + \alpha m\int_t^T e^{-\alpha(T-u)}du + \sigma\int_t^T e^{-\alpha(T-u)}dW^Q(u) \\ &= e^{-\alpha(T-t)}r(t) + m[1 - e^{-\alpha(T-t)}] + \sigma\int_t^T e^{-\alpha(T-u)}dW^Q(u) \qquad (4.2) \end{aligned}$$

We want to find an expression for $-\int_t^T r(u)du$:

$$r(T) - r(t) = \alpha m(T-t) - \alpha\int_t^T r(u)du + \sigma\int_t^T dW^Q(u)$$

$$\Downarrow$$

$$-\alpha\int_t^T r(u)du = r(T) - r(t) - \alpha m(T-t) - \sigma\int_t^T dW^Q(u) \qquad (4.3)$$

By plugging in the expression for $r(T)$ as found in Equation 4.2, into Equation 4.3 and dividing by α yields:

$$-\int_t^T r(u)du = \underbrace{\frac{1}{\alpha}[e^{-\alpha(T-t)} - 1]}_{=-B(t,T)}r(t) + \underbrace{\frac{m}{\alpha}[1 - e^{-\alpha(T-t)}] + m(T-t)}_{=-b(T-t)} + \int_t^T \underbrace{\frac{\sigma}{\alpha}[e^{-\alpha(T-u)} - 1]}_{=-c(T-u)}dW^Q(u) \qquad (4.4)$$

$$= -B(t,T) - b(T-t) - \int_t^T c(T-u)dW^Q(u)$$

Now:

$$\begin{aligned} P(t,T) &= \mathbb{E}_Q\left[e^{-\int_t^T r(u)du}\middle|\mathcal{F}_t\right] \\ &= e^{-B(t,T)r(t)-b(T-t)}\mathbb{E}_Q\left[e^{-\int_t^T c(T-u)dW^Q(u)}\right] \end{aligned}$$

$c(T-u)$ is a deterministic function, we thus have:

$$-\int_t^T c(T-u)dW^Q(u) \sim \mathcal{N}\left(0, \int_t^T c^2(T-u)du\right)$$

Furthermore if $X \sim \mathcal{N}(\mu, \sigma^2)$, we have:

$$\varphi_X(t) = \mathbb{E}[e^{itX}] = e^{iut - \frac{1}{2}\sigma^2 t^2}$$

Thus:

$$\mathbb{E}_Q\left[e^{-\int_t^T c(T-u)dW^Q(u)}\right] = e^{-\frac{1}{2}\int_t^T c^2(T-u)}$$

Leaving us with:

$$P(t,T) = \exp\left(\underbrace{-b(T-t) - \frac{1}{2}\int_t^T c^2(T-u)du}_{=-A(t,T)} - B(t,T)r(t)\right)$$

■

4.5 HJM-modelling

We have seen how the short rate and the zero coupon bond are related, however, we also have the relation:

$$P(t,T) = e^{-\int_t^T f(t,u)du}$$

where f represents the forward rate, the Heath-Jarrow-Morton (HJM) approach consists of modelling the forward rate directly:

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t) \\ f(t,T) &= f(0,T) + \int_0^T \alpha(s,T)ds + \int_0^T \sigma(s,T)dW(s) \end{aligned}$$

Consider $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ as the objective probability space, and let $\alpha = (\alpha(t,T))_{t\in[0,T]}$ denote an $\mathbb{R}$-valued process and let $\sigma = (\sigma(t,T))_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued process, i.e $\sigma(t,T) = (\sigma_1(t,T), \ldots, \sigma_d(t,T))$. We impose the following conditions:

- **(HJM.1)** α and σ are progressively measurable w.r.t $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$
- **(HJM.2)**
$$\int_0^T \int_0^T |\alpha(s,u)|dsdu < \infty, \ \forall\, T$$
- **(HJM.3)**
$$\sup_{s,u\leq T} \|\sigma(s,u)\| < \infty, \ a.e. \ \forall\, T$$

4.5.1 P-dynamics

Proposition 4.5.1 (**Dynamics of** $\ln P(t,T)$). *We have that the dynamics of* $\ln P(t,T)$ *is given by:*

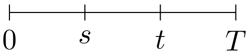
$$d\ln P(t,T) = r(t)dt - \int_t^T \alpha(t,u)dudt - \int_t^T \sigma(t,u)dudW(t)$$

Proof. Quite involved, so we start on the next page.

We have the following relationship: $\ln P(t,T) = -\int_t^T f(t,u)du$, this can be rewritten as:

$$\begin{aligned}
\ln P(t,T) &= -\int_t^T f(0,u)du - \int_t^T \int_0^t \alpha(s,u)duds - \int_t^T \int_0^t \sigma(s,u)dW(s)du \\
&= -\int_t^T f(0,u)du - \int_0^t \int_t^T \alpha(s,u)dsdu - \int_0^t \int_t^T \sigma(s,u)dudW(s)
\end{aligned}$$

In order to get cleaner expressions, we split up the integral in the following way:



Hence $0 \leq s \leq t \leq T$, and:

$$\int_t^T = \int_s^T - \int_s^t$$

We will now replace the integral parts containing $\int_t^T$ with the above:

$$\begin{aligned}
\ln P(t,T) = & \underbrace{-\int_0^T f(0,u)du}_{=\ln P(0,T),(1)} + \underbrace{\int_0^t f(0,u)du}_{=(2)} \underbrace{- \int_0^t \int_s^T \alpha(s,u)duds}_{=(3)} + \underbrace{\int_0^t \int_s^t \alpha(s,u)duds}_{=(4)} \\
& \underbrace{- \int_0^t \int_s^T \sigma(s,u)dudW(s)}_{=(5)} + \underbrace{\int_0^t \int_s^t \sigma(s,u)dudW(s)}_{=(6)}
\end{aligned}$$

Now let's rewrite this again, here $(x)'$ means that one used Fubini on the following part:

$$\ln P(t,T) = (1) + (2) + (4)' + (6)' + (3) + (5)$$

This means that Fubini is applied to (4) and (6):

$$(4) = \int_0^t \int_s^t \alpha(s,u)duds = \int_0^t \int_0^u \alpha(s,u)dsdu = (4)'$$

and:

$$(6) = \int_0^t \int_s^t \sigma(s,u)dudW(s) = \int_0^t \int_0^u \sigma(s,u)dW(s)du = (6)'$$

Here we used Fubini (Theorem 2.3) and Stochastic Fubini (Theorem 2.3.14), with:

$$\begin{cases} s \leq u \leq t \\ 0 \leq s \leq t \end{cases} \iff \begin{cases} 0 \leq s \leq u \\ 0 \leq u \leq t \end{cases}$$

We would also like to recall, that:

$$r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \sigma(s,t)dW(s)$$
$$\Downarrow$$
$$r(u) = f(0,u) + \int_0^u \alpha(s,u)ds + \int_0^u \sigma(s,u)dW(s)$$
$$\Downarrow$$
$$\int_0^t r(u)du = \underbrace{\int_0^t f(0,u)du}_{=(2)} + \underbrace{\int_0^t \int_0^u \alpha(s,u)dsdu}_{=(4)'} + \underbrace{\int_0^t \int_0^u \sigma(s,u)dW(s)du}_{=(6)'}$$

This yields:

$$\begin{aligned}\ln P(t,T) &= (1) + \int_0^t r(u)du + (3) + (5)\\ &= \ln P(0,T) + \int_0^t r(u)du - \int_0^t \int_s^T \alpha(s,u)duds - \int_0^t \int_s^T \sigma(s,u)dudW(s)\end{aligned}$$

Which then finally yields:

$$d\ln P(t,T) = r(t)dt - \int_t^T \alpha(t,u)dudt - \int_t^T \sigma(t,u)dudW(t)$$

■

Notation 4.5.2 (**Volatility process**). We will use the following notation for the volatility process:

$$v(t,T) := -\int_t^T \sigma(t,u)du$$

Lemma 4.5.3. *We have that for every maturity T, that the dynamics of $P(t,T)$ can be expressed as:*

$$\frac{dP(t,T)}{P(t,T)} = \left[r(t) - \int_t^T \alpha(t,u)du + \frac{1}{2}\|v(t,T)\|^2\right] dt + v(t,T)dW(t)$$

Proof. For simplicity, consider $d = 1$, we will use Ito's formula on e^x with $x = \ln P(t,T)$:

$$\begin{aligned}dP(t,T) &= P(t,T)d\ln P(t,T) + \frac{1}{2}P(t,T)[d\ln P(t,T)]^2\\ &= P(t,T)\left[r(t)dt - \int_t^T \alpha(t,u)dudt + v(t,T)dW(t)\right]\\ &+ \frac{1}{2}P(t,T)\left(v(t,T)\right)^2 dt\end{aligned}$$

Collecting dt-terms and dividing by $P(t,T)$ gives:

$$\frac{dP(t,T)}{P(t,T)} = \left[r(t) - \int_t^T \alpha(t,u)du + \frac{1}{2}\left(v(t,T)\right)^2\right] dt + v(t,T)dW(t)$$

■

Corollary 4.5.4. *The discounted zero coupon bond process has the following dynamics:*

$$d\left[\frac{P(t,T)}{B(t)}\right] = \frac{P(t,T)}{B(t)}\left(\frac{1}{2}\|v(t,T)\|^2 - \int_t^T \alpha(t,u)du\right)dt + \frac{P(t,T)}{B(t)}v(t,T)dW(t) \quad (4.5)$$

Proof. This is just an application of Ito's product rule:

$$\begin{aligned}
d\left[\frac{P(t,T)}{B(t)}\right] &= dP(t,T)\frac{1}{B(t)} + P(t,T)d\left[\frac{1}{B(t)}\right] + dP(t,T)d\left[\frac{1}{B(t)}\right] \\
&= r(t)\frac{P(t,T)}{B(t)}dt - \frac{P(t,T)}{B(t)}\int_t^T \alpha(t,u)dudt + \frac{1}{2}\frac{P(t,T)}{B(t)}\|v(t,T)\|^2 dt \\
&+ \frac{P(t,T)}{B(t)}v(t,T)dW(t) - r(t)\frac{P(t,T)}{B(t)}dt \\
&= \frac{P(t,T)}{B(t)}\left(\frac{1}{2}\|v(t,T)\|^2 - \int_t^T \alpha(t,u)du\right)dt + \frac{P(t,T)}{B(t)}v(t,T)dW(t)
\end{aligned}$$

■

4.5.2 Q-dynamics and absence of arbitrage

Let $\gamma(t) = (\gamma_1(t), \dots, \gamma_d(t)) \in M^2([0,T])$ be an $\mathcal{F}_t$-adapted process, furthermore assume that $\mathcal{E}_t(\gamma \bullet W)$ is a $(P, \mathbb{F})$-martingale, then from Girsanov's Theorem (2.3.19) we know that $\exists\, Q \sim P$ such that

$$dW^Q(t) = dW(t) - \gamma(t)^{Tr} dt$$

is a Q-Brownian motion, yielding the following Q-dynamics for f:

$$df(t,T) = [\alpha(t,T) + \sigma(t,T)\gamma(t)^{Tr}]dt + \sigma(t,T)dW^Q(t) \tag{4.6}$$

Plugging the Q-Brownian motion into Equation 4.5 yields:

$$\begin{aligned} d\left[\frac{P(t,T)}{B(t)}\right] &= \frac{P(t,T)}{B(t)}\left(\frac{1}{2}\|v(t,T)\|^2 - \int_t^T \alpha(t,u)du + v(t,T)\gamma(t)^{Tr}\right)dt \\ &\quad + \frac{P(t,T)}{B(t)} v(t,T) dW^Q(t) \end{aligned}$$

Theorem 4.5.5 ([**Fil09**]). *We have that $\frac{P(t,T)}{B(t)}$ is a Q-martingale if and only if:*

$$-v(t,T)\gamma(t)^{Tr} = \frac{1}{2}\|v(t,T)\|^2 - \int_t^T \alpha(t,u)du \tag{4.7}$$

and the Q-dynamics of f are given by:

$$df(t,T) = \int_t^T \sigma(t,u)du \cdot \sigma(t,T)^{Tr} dt + \sigma(t,T)dW^Q(t)$$

And the discounted T-bond price satisfy:

$$\frac{P(t,T)}{P(0,T)B(t)} = \mathcal{E}_t(v(\cdot,T) \bullet W^Q)$$

Proof. As a consequence of the Martingale Representation Theorem (2.3.25), we get that in order for Q-martingality there cannot be any drift, i.e.

$$\begin{aligned} &\frac{1}{2}\|v(t,T)\|^2 - \int_t^T \alpha(t,u)du + v(t,T)\gamma(t)^{Tr} = 0 \\ &\Updownarrow \\ &-v(t,T)\gamma(t)^{Tr} = \frac{1}{2}\|v(t,T)\|^2 - \int_t^T \alpha(t,u)du \end{aligned}$$

In order to get the desired dynamics, we take the partial derivative w.r.t T on LHS and RHS of Equation 4.7 where:

$$\frac{\partial}{\partial T}\left[-v(t,T)\gamma(t)^{Tr}\right] = \sigma(t,T)\gamma(t)^{Tr}$$

and:

$$\begin{aligned}
\frac{\partial}{\partial T}\left[\frac{1}{2}\|v(t,T)\|^2\right] &= \frac{\partial}{\partial T}\left[\frac{1}{2}\sum_{i=1}^{d}\left(\int_t^T \sigma_i(t,u)du\right)^2\right] \\
&= \sum_{i=1}^{d}\int_t^T \sigma_i(t,u)du * \sigma_i(t,T) \\
&= \int_t^T \sigma(t,u)du \cdot \sigma(t,T)^{Tr}
\end{aligned}$$

This leaves us with:

$$\sigma(t,T)\gamma(t)^{Tr} = \int_t^T \sigma(t,u)du \cdot \sigma(t,T)^{Tr} - \alpha(t,T)$$

Now plugging this into Equation 4.6, yields:

$$\begin{aligned}
df(t,T) &= \left[\alpha(t,T) + \int_t^T \sigma(t,u)du \cdot \sigma(t,T)^{Tr} - \alpha(t,T)\right]dt + \sigma(t,T)dW^Q(t) \\
&= \int_t^T \sigma(t,u)du \cdot \sigma(t,T)^{Tr}dt + \sigma(t,T)dW^Q(t)
\end{aligned}$$

Suppose that the arbitrage condition in Equation 4.7 holds, then:

$$\begin{aligned}
d\left[\frac{P(t,T)}{B(t)}\right] &= \frac{P(t,T)}{B(t)}v(t,T)dW^Q(t) \\
&\Downarrow \\
d\left[\frac{P(t,T)}{P(0,T)B(t)}\right] &= \frac{P(t,T)}{P(0,T)B(t)}v(t,T)dW^Q(t) \\
&\Updownarrow \\
\frac{P(t,T)}{P(0,T)B(t)} &= \mathcal{E}_t(v(\cdot,T)\bullet W^Q)
\end{aligned}$$

■

4.6 Estimating the Term Structure

In the market we can only observe: $P(0, T_1), \ldots, P(0, T_n)$ for maturities $T_1, \ldots, T_n$, however it could be that we need $P(0, T_r)$ where the ZCB with maturity T_r is not observable.

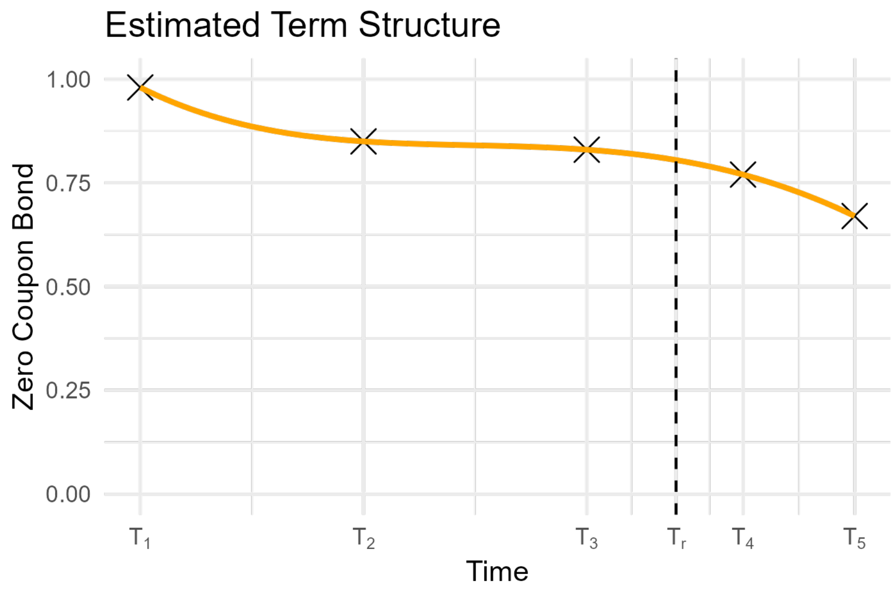


Figure 4.1: Example of estimated term structure

Typical methods to estimate these term structures are regressions and interpolation methods. We will look at parametric estimation methods, in particular, exponential-polynomial families as these methods are often used by central banks. For instance, the Norwegian Central Bank uses the Svensson method [21c].

4.6.1 Exponential-Polynomial Families

Let $P_1, \ldots, P_n$ be the observed ZCB's with maturities $T_1, \ldots, T_n$, the goal will be the following:

$$\min_{\theta} |P_\theta(T_i) - P_i|^2$$

One proposal is the Nelson-Siegel Curve

$$f_{NS}(T, \underbrace{z_1, z_2, z_3, z_4}_{\theta}) = z_1 + (z_2 + z_3 T)e^{-z_4 T}$$

where one has the following link between P_θ and f_{NS}:

$$P_\theta(T) = \exp\left(-\int_0^T f_{NS}(u; \theta)du\right)$$

The Svensson curve is given by:

$$f_S(T, \theta) = z_1 + (z_2 + z_3 T)e^{-z_5 T} + z_4 T e^{-z_6 T}$$

4.7 Forward Measures

Consider the following probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, Q))$, furthermore let $X \in L^1(\Omega, \mathcal{F}, Q)$ as well as $\mathcal{F}_T$-measurable. The goal of this section is to study:

$$\pi(t) = \mathbb{E}_Q\left[\frac{B(t)}{B(T)}X\middle|\mathcal{F}_t\right]$$

Notation 4.7.1. We use the following notation:

$$Z^T(t) := \frac{P(t,T)}{P(0,T)B(t)}$$

Proposition 4.7.2. *Assume that* $\mathbb{E}_Q\left[e^{\frac{1}{2}\int_0^T \|v(s,T)\|^2 ds}\right] < \infty\ \forall T$*, then we have that:*

$$Z^T(t) = \mathcal{E}_t(v(\cdot,T) \bullet W^Q),\ t \leq T$$

is a Q*-martingale, furthermore there* $\exists\, Q^T \sim Q$ *such that:*

$$\left.\frac{dQ^T}{dQ}\right|_{\mathcal{F}_t} = Z^T(t)$$

and:

$$dW^T(t) = dW^Q(t) - v(t,T)dt$$

defines a Q^T*-Brownian motion.*

Proof. Since we assume that $\mathbb{E}_Q\left[e^{\frac{1}{2}\int_0^T \|v(s,T)\|^2 ds}\right] < \infty\ \forall T$, it follows from Novikov's condition (2.3.20) and Theorem 4.7, that $Z^T(t)$ is a Q-martingale. Girsanov's Theorem (2.3.19) justifies that $\exists\, Q^T \sim Q$ such that

$$\left.\frac{dQ^T}{dQ}\right|_{\mathcal{F}_t} = Z^T(t)$$

and that $dW^T(t) = dW^Q(t) - v(t,T)dt$ defines a Q^T-Brownain motion. ■

Proposition 4.7.3 (**[Fil09]**). *Let* $X \in L^1(\Omega, \mathcal{F}, Q)$ *as well as* $\mathcal{F}_T$*-measurable, we then have that:*

$$\mathbb{E}_{Q^T}[|X|] < \infty$$

and:

$$\pi(t) = P(t,T)\mathbb{E}_{Q^T}[X|\mathcal{F}_t]$$

Proof. We get from Bayes theorem (2.3.21) the following:

$$\begin{aligned}
\mathbb{E}_{Q^T}[|X|] &= \frac{\mathbb{E}_Q\left[|X|\frac{dQ^T}{dQ}\right]}{\mathbb{E}_Q\left[\frac{dQ^T}{dQ}\right]} \\
&= \frac{\mathbb{E}_Q\left[|X|Z^T(T)\right]}{Z^T(0)} \\
&= \mathbb{E}_Q\left[\frac{|X|}{P(0,T)B(T)}\right] \\
&\leq \mathbb{E}_Q[|X|] < \infty
\end{aligned}$$

The second part also relies on Bayes theorem:

$$\mathbb{E}_{Q^T}[X|\mathcal{F}_t] = \frac{\mathbb{E}_Q\left[X\frac{dQ^T}{dQ}\middle|\mathcal{F}_t\right]}{\mathbb{E}_Q\left[\frac{dQ^T}{dQ}\middle|\mathcal{F}_t\right]}$$
$$= \frac{\mathbb{E}_Q\left[XZ^T(T)\middle|\mathcal{F}_t\right]}{Z^T(t)} \tag{4.8}$$

We recall that $Z^T(t) = \frac{P(t,T)}{P(0,T)B(t)}$, now from Equation 4.8 we get:

$$Z^T(t)\mathbb{E}_{Q^T}[X|\mathcal{F}_t] = \mathbb{E}_Q[XZ^T(T)|\mathcal{F}_t] \iff P(t,T)\mathbb{E}_{Q^T}[X|\mathcal{F}_t] = \mathbb{E}_Q\left[\frac{B(t)}{B(T)}X\middle|\mathcal{F}_t\right] := \pi(t)$$

■

Lemma 4.7.4 ([**Fil09**]). *Let $S > 0$ and $S \leq T$. Then the T-bond discounted S-bond price process:*

$$\frac{P(t,S)}{P(t,T)} = \frac{P(0,S)}{P(0,T)}\mathcal{E}_t(\sigma_{S,T} \bullet W^T), \quad t \leq S \leq T$$

is a Q^T-martingale. Where we define:

$$\sigma_{S,T}(t) = -\sigma_{T,S}(t) = v(t,S) - v(t,T) = \int_S^T \sigma(t,u)du$$

Moreover, the T- and S-forward measures are related by:

$$\left.\frac{dQ^S}{dQ^T}\right|_{\mathcal{F}_t} = \frac{P(t,S)}{P(t,T)}\frac{P(0,T)}{P(0,S)} = \mathcal{E}_t(\sigma_{S,T} \bullet W^T)$$

Proof. Quite involved, so we start on the next page:

Q^T**-martingality**:

Let $u \le t \le S \le T$, we then get:

$$\mathbb{E}_{Q^T}\left[\frac{P(t,S)}{P(t,T)}\middle|\mathcal{F}_u\right] \overset{\text{Bayes:2.3.21}}{=} \underbrace{\frac{\mathbb{E}_Q\left[\frac{P(t,S)}{P(t,T)}Z^T(T)\middle|\mathcal{F}_u\right]}{\mathbb{E}_Q\left[Z^T(T)|\mathcal{F}_u\right]}}_{(*)}$$

Now from Proposition 4.7.2 and the Tower law of conditional expectation (Theorem 2.2.8), we get that:

$$\begin{aligned}
(*) &= \frac{\mathbb{E}_Q\left[\mathbb{E}_Q\left[\frac{P(t,S)}{P(t,T)}Z^T(T)\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_u\right]}{Z^T(u)} \\
&= \frac{\mathbb{E}_Q\left[\frac{P(t,S)}{P(t,T)}\frac{P(t,T)}{P(0,T)B(t)}\middle|\mathcal{F}_u\right]}{\frac{P(u,T)}{P(0,T)B(u)}} \\
&= \frac{\mathbb{E}_Q\left[\frac{P(t,S)}{B(t)}\middle|\mathcal{F}_u\right]}{\frac{P(u,T)}{B(u)}} = \frac{P(u,S)}{P(u,T)}
\end{aligned}$$

Where the last equality comes from the fact that the discounted zero coupon bond process is a Q-martingale.

Explicit expression:

From Proposition 4.7.2, we have that: $P(t,T) = B(t)P(0,T)\mathcal{E}_t(v(\cdot,T) \bullet W^Q)$, this yields:

$$\begin{aligned}
\frac{P(t,S)}{P(t,T)} &= \frac{P(0,S)}{P(0,T)}\frac{\exp\left(\int_0^t v(u,S)dW^Q(u) - \frac{1}{2}\int_0^t \|v(u,S)\|^2 du\right)}{\exp\left(\int_0^t v(u,T)dW^Q(u) - \frac{1}{2}\int_0^t \|v(u,T)\|^2 du\right)} \\
&= \frac{P(0,S)}{P(0,T)}\exp\left(\underbrace{\int_0^t [v(u,S)-v(u,T)]dW^Q(u) - \frac{1}{2}\int_0^t \left(\|v(u,S)\|^2 - \|v(u,T)\|^2\right)du}_{=(*)}\right)
\end{aligned}$$

Now: $dW^Q(u) = dW^T(u) + v(u,T)du$, this leaves us with:

$$(*) = \int_0^t [v(u,S)-v(u,T)]dW^T(u) + \int_0^t [v(u,S)-v(u,T)]v(u,T)du - \frac{1}{2}\int_0^t \left(\|v(u,S)\|^2 - \|v(u,T)\|^2\right)du$$

Now let's collect the du-terms into one integral, and work with the inner expression

$$\begin{aligned}
&[v(u,S)-v(u,T)]v(u,T)-\frac{1}{2}\left(\|v(u,S)\|^2-\|v(u,T)\|^2\right)\\
=&v(u,S)v(u,T)-\|v(u,T)\|^2-\frac{1}{2}\|v(u,S)\|^2+\frac{1}{2}\|v(u,T)\|^2\\
=&v(u,S)v(u,T)-\frac{1}{2}\|v(u,T)\|^2-\frac{1}{2}\|v(u,S)\|^2\\
=&-\frac{1}{2}\|v(u,S)-v(u,T)\|^2\\
=&-\frac{1}{2}\|\sigma_{S,T}(u)\|^2
\end{aligned}$$

Thus:

$$\begin{aligned}
\frac{P(t,S)}{P(t,T)}&=\frac{P(0,S)}{P(0,T)}\exp\left(\int_0^t\sigma_{S,T}(u)dW^T(u)-\frac{1}{2}\int_0^t\|\sigma_{S,T}(u)\|^2du\right)\\
&=\frac{P(0,S)}{P(0,T)}\mathcal{E}_t\left(\sigma_{S,T}\bullet W^T\right)
\end{aligned}$$

Radon-Nikodym derivative:

$$\begin{aligned}
\left.\frac{dQ^S}{dQ^T}\right|_{\mathcal{F}_t}&=\left.\frac{dQ^S}{dQ}\right|_{\mathcal{F}_t}\bullet\left(\left.\frac{dQ^T}{dQ}\right|_{\mathcal{F}_t}\right)^{-1}\\
&=Z^S(t)[Z^T(t)]^{-1}\\
&=\frac{P(t,S)}{P(0,S)B(t)}\frac{P(0,T)B(t)}{P(t,T)}\\
&=\frac{P(t,S)P(0,T)}{P(t,T)P(0,S)}\\
&=\mathcal{E}_t\left(\sigma_{S,T}\bullet W^T\right)
\end{aligned}$$

■

4.8 The LIBOR market model

4.8.1 Introduction

Definition 4.8.1 (**LIBOR-rate**). The LIBOR-rate $L(t,T)$ is defined as:

$$L(t,T) = F(t,T,T+\delta) = \frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)} - 1\right)$$

Typically LIBOR has the following Tenors:

- O/N (Overnight)
- 1 Week
- 1 Month
- 2 Months
- 3 Months
- 6 Months
- 12 Months

The rates are calculated by a trimmed average provided by Panel Banks, meaning that it works as a market survey. Let's say that there were 16 Panel Banks: $B_1, \dots, B_{16}$. Each Bank B_i would submit a borrowing rate r_i to Intercontinental Exchange Benchmark Administration (ICE) . It would then be sorted, and then one would cut the highest 25%, and the lowest 25%. For further information, one can consult [Exc23].

Example 4.8.2. Assume that the Panel Banks provide the data below for a 3-month tenor. First, one collects the data, then sort it in ascending order, followed by trimming the data:

$$\begin{bmatrix} 0.043 & 0.056 & 0.049 & 0.050 \\ 0.046 & 0.058 & 0.052 & 0.041 \\ 0.039 & 0.037 & 0.045 & 0.044 \\ 0.046 & 0.042 & 0.034 & 0.057 \end{bmatrix} \to \begin{bmatrix} 0.034 & 0.037 & 0.039 & 0.041 \\ 0.042 & 0.043 & 0.044 & 0.045 \\ 0.046 & 0.046 & 0.049 & 0.050 \\ 0.052 & 0.056 & 0.057 & 0.058 \end{bmatrix} \to \begin{bmatrix} 0.042 & 0.043 & 0.044 & 0.045 \\ 0.046 & 0.046 & 0.049 & 0.050 \end{bmatrix}$$

After the data is trimmed, one takes the mean:

$$\frac{1}{8}(0.042 + 0.043 + 0.044 + 0.045 + 0.046 + 0.049 + 0.050) = 0.04550$$

One also has conventions for the number of decimals, which are currency specific.

In a **Market model**, one is interested in modelling only the relevant T's, meaning that one finds a model for each T_i. In the market, there are essentially three types of interest rate derivatives: **caps**, **floors** and **swaptions**. By swaption, we mean a call/put on a swap.

Definition 4.8.3 (**LIBOR-Caplet**). A caplet with reset date $\mathbf{T}$ and settlement $\mathbf{T}+\delta$ pays the holder: LIBOR minus strike κ if it is positive:

$$\delta(F(T;T,T+\delta)-\kappa)^+ = \delta(L(T,T)-\kappa)^+$$

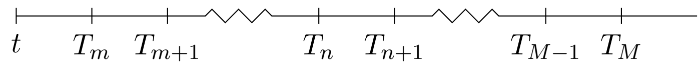
Definition 4.8.4 (**LIBOR-Floorlet**). The opposite of a caplet, it has the following payoff:

$$\delta(\kappa - L(T,T))^+$$

We assume equidistant times $T_m = m\delta, m = 0,1,\dots,M$, furthermore we will work on: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T_M]}, Q^{T_M})$, $W^{T_M}(t)$ is a Q^{T_M}-BM. In addition $L(0,T_i)\geq 0$ are given for $m = 1,\dots,M-1$:

$$L(0,T_m) = \frac{1}{\delta}\left(\frac{P(0,T_m)}{P(0,T_{m+1})}-1\right), \quad m = 0,\dots,M-1$$

We get the following timeline:



The dynamics of $L(t,T_{M-1})$ are given by:

$$dL(t,T_{M-1}) = L(t,T_{M-1})\lambda(t,T_{M-1})dW^{T_M}(t), \quad t\leq T_{M-1}$$
$$\Downarrow$$
$$L(t,T_{M-1}) = L(0,T_{M-1})\exp\left(\int_0^t \lambda(s,T_{M-1})dW^{T_M}(s) - \frac{1}{2}\int_0^t \|\lambda(s,T_{M-1})\|^2 ds\right)$$

Here: $t\mapsto \lambda(t,T_{M-1})$ is assumed to be an $\mathbb{R}^d$-valued bounded deterministic measurable function.

Since

$$\mathbb{E}_{Q^{T_M}}\left[e^{\frac{1}{2}\int_0^{T_{M-1}}\|\lambda(s,T_{M-1})\|^2 ds}\right] < \infty \;\text{ and }\; \frac{L(t,T_{M-1})}{L(0,T_{M-1})} = \mathcal{E}_t\left(\lambda(\cdot,T_{M-1})\bullet W^{T_M}\right)$$

We have that $L(t,T_{M-1})$ is a Q^{T_M}-martingale. The idea will be to iterate backwards, so that $L(t,T_{m-1})$ will be martingales under Q^{T_m} for $m\geq 2$, we thus need valid measure changes from Q^{T_m} to $Q^{T_{m-1}}$ on $\mathcal{F}_{T_{m-1}}$

One defines:

$$\left.\frac{dQ^{T_{M-1}}}{dQ^{T_M}}\right|_{\mathcal{F}_{T_{M-1}}} = \mathcal{E}_{T_{M-1}}\left(\sigma_{T_{M-1},T_M} \bullet W^{T_M}\right)$$

Where:

$$\sigma_{T_{M-1},T_M}(t) := \frac{\delta L(t, T_{M-1})}{\delta L(t, T_{M-1}) + 1}\lambda(t, T_{M-1}),\ t \leq T_{M-1}$$

Let $K \in \mathbb{R}$, now as:

$$\begin{aligned} \left\|\sigma_{T_{M-1},T_M}(t)\right\|^2 &\leq \|\lambda(t, T_{M-1})\|^2 \leq K \\ &\Downarrow \\ \mathbb{E}_{Q^{T_M}}\left[e^{\frac{1}{2}\int_0^{T_{M-1}}\left\|\sigma_{T_{M-1},T_M}(s)\right\|^2 ds}\right] &\leq e^{\frac{1}{2}T_{M-1}K} < \infty \end{aligned}$$

Furthermore:

$$\mathbb{E}_{Q^{T_M}}\left[\left.\frac{dQ^{T_{M-1}}}{dQ^{T_M}}\right|_{\mathcal{F}_{T_{M-1}}}\right] = \mathbb{E}_{Q^{T_M}}\left[\left.\frac{dQ^{T_{M-1}}}{dQ^{T_M}}\right|_{\mathcal{F}_{T_{M-1}}}\middle|\mathcal{F}_0\right] = \mathcal{E}_0\left(\sigma_{T_{M-1},T_M} \bullet W^{T_M}\right) = 1$$

We thus have that $Q^{T_{M-1}} \sim Q^{T_M}$ and from Girsanov's Theorem we have that:

$$dW^{T_{M-1}}(t) = dW^{T_M}(t) - \sigma_{T_{M-1},T_M}(t)dt$$

defines a $Q^{T_{M-1}}$ Brownian Motion on $\mathcal{F}_{T_{M-1}}$.

Lemma 4.8.5 ([**Fil09**]). *Let X be a T_m-contingent claim, we then have that for $t \leq T_m \leq T_n$*

$$\frac{\pi(t)}{P(t, T_m)} = \frac{P(t, T_n)}{P(t, T_m)}\mathbb{E}_{Q^{T_n}}\left[\frac{X}{P(T_m, T_n)}\middle|\mathcal{F}_t\right]$$

Proof. We already have that $\pi(t) = P(t, T_m)\mathbb{E}_{Q^{T_m}}[X|\mathcal{F}_t]$, and from Lemma 4.7.4, by using $S = T_n$ and $T = T_m$, we have:

$$\left.\frac{dQ^{T_m}}{dQ^{T_n}}\right|_{\mathcal{F}_t} = \mathcal{E}_t\left(\sigma_{T_m,T_n} \bullet W^{T_n}\right)$$

This is a Q^{T_n}-martingale, now in combination with Bayes Theorem, we get:

$$\begin{aligned} \mathbb{E}_{Q^{T_m}}\left[X|\mathcal{F}_t\right] &= \frac{\mathbb{E}_{Q^{T_n}}\left[\frac{P(T_m,T_m)}{P(T_m,T_n)}\frac{P(0,T_n)}{P(0,T_m)}\middle|\mathcal{F}_t\right]}{\frac{P(t,T_m)}{P(t,T_n)}\frac{P(0,T_n)}{P(0,T_m)}} \\ &= \frac{P(t, T_n)}{P(t, T_m)}\mathbb{E}_{Q^{T_n}}\left[\frac{X}{P(T_m, T_n)}\middle|\mathcal{F}_t\right] \end{aligned}$$

■

4.8.2 LIBOR-caplets

We will consider a caplet with reset time T_{n-1} and settlement T_n and derive the T_m-price. Meaning that we will be interested in:

$$
\begin{aligned}
Cpl(T_m, T_{n-1}, T_n) :&= \pi(T_m) \\
&= P(T_m, T_n)\mathbb{E}_{Q^{T_n}}\left[\delta\left(L(T_{n-1}, T_{n-1}) - \kappa\right)^+ \middle| \mathcal{F}_{T_m}\right]
\end{aligned}
$$

Proposition 4.8.6 (**Price of LIBOR-caplet [Fil09]**).

$$
Cpl(T_m, T_{n-1}, T_n) = \delta P(T_m, T_n)\left[L(T_m, T_{n-1})\Phi(d_+(T_m, T_{n-1})) - \kappa\Phi(d_-(T_m, T_{n-1})\right]
$$

where:

$$
d_\pm(T_m, T_{n-1}) = \frac{\ln\left(\frac{L(T_m, T_{n-1})}{\kappa}\right) \pm \frac{1}{2}\int_{T_m}^{T_{n-1}} \|\lambda(s, T_{n-1})\|^2 ds}{\left(\int_{T_m}^{T_{n-1}} \|\lambda(s, T_{n-1})\|^2 ds\right)^{1/2}}
$$

Proof. We have the following dynamics for $L(t, T_{n-1})$:

$$
dL(t, T_{n-1}) = L(t, T_{n-1})\lambda(t, T_{n-1})dW^{T_n}(t)
$$

where W^{T_n} is a Q^{T_n}-Brownian motion and $t \leq T_{n-1}$

We also recall that:

$$
\begin{aligned}
L(t, T_{n-1}) &= L(0, T_{n-1})\mathcal{E}_t\left(\lambda(\cdot, T_{n-1}) \bullet W^{T_n}\right) \\
L(T_{n-1}, T_{n-1}) &= L(0, T_{n-1})\mathcal{E}_{T_{n-1}}\left(\lambda(\cdot, T_{n-1}) \bullet W^{T_n}\right)
\end{aligned}
$$

Leaving us with:

$$
\begin{aligned}
L(T_{n-1}, T_{n-1}) &= L(t, T_{n-1})\mathcal{E}_t^{T_{n-1}}\left(\lambda(\cdot, T_{n-1}) \bullet W^{T_n}\right) \\
&= L(t, T_{n-1})\exp\left(\int_t^{T_{n-1}} \lambda(s, T_{n-1})dW^{T_n}(s) - \frac{1}{2}\int_t^{T_{n-1}} \|\lambda(s, T_{n-1}(s)\|^2 ds\right)
\end{aligned}
$$

We are interested in the T_m-price, giving us:

$$
L(T_{n-1}, T_{n-1}) = \underbrace{L(T_m, T_{n-1})}_{\mathcal{F}_{T_m}\text{- measurable}} \underbrace{\mathcal{E}_{T_M}^{T_{n-1}}\left(\lambda(\cdot, T_{n-1}) \bullet W^{T_n}\right)}_{\mathcal{F}_{T_m}\text{ independent}}
$$

Furthermore:

$$
\int_{T_m}^{T_{n-1}} \lambda(s, T_{n-1})dW^{T_n}(s) \overset{Q^{T_n}}{\sim} \mathcal{N}\left(0, \int_{T_m}^{T_{n-1}} \|\lambda(s, T_{n-1})\|^2 ds\right)
$$

Now let $b^2 = \int_{T_m}^{T_{n-1}} \|\lambda(s, T_{n-1})\|^2 ds$, and $Z \sim \mathcal{N}(0,1)$, then:

$$
\int_{T_m}^{T_{n-1}} \lambda(s, T_{n-1})dW^{T_n}(s) - \frac{1}{2}\int_{T_m}^{T_{n-1}} \|\lambda(s, T_{n-1}(s)\|^2 ds \overset{\mathrm{d}}{=} bZ - \frac{1}{2}b^2
$$

This leaves us with:

$$
\mathbb{E}_{Q^{T_n}}\left[\left(L(T_{n-1}, T_{n-1}) - \kappa\right)^+ \middle| \mathcal{F}_{T_m}\right] = \mathbb{E}_{Q^{T_n}}\left[\left(x\exp\left(bZ - \frac{1}{2}b^2\right) - \kappa\right)^+\right]_{x = L(T_{n-1}, T_{n-1})}
$$

$$\left(x\exp\left(bZ-\frac{1}{2}b^2\right)-\kappa\right)\geq 0 \iff Z\geq \frac{\ln\left(\frac{\kappa}{x}\right)+\frac{1}{2}b^2}{b} := d_1$$

Now this gives us:

$$\left(x\exp\left(bZ-\frac{1}{2}b^2\right)-\kappa\right)^+ = \left(x\exp\left(bZ-\frac{1}{2}b^2\right)-\kappa\right)\mathbb{1}_{\{Z\geq d_1\}}$$

Taking the expectation yields:

$$\begin{aligned}\mathbb{E}_{Q^{T_n}}\left[\left(x\exp\left(bZ-\frac{1}{2}b^2\right)-\kappa\right)^+\right] &= \int_{d_1}^{\infty}\left(x\exp\left(bz-\frac{1}{2}b^2\right)-\kappa\right)f_Z(z)dz \\ &= \underbrace{x\int_{d_1}^{\infty}\exp\left(bz-\frac{1}{2}b^2\right)f_Z(z)dz}_{=(1)} - \kappa\underbrace{\int_{d_1}^{\infty}f_Z(z)dz}_{=(2)}\end{aligned}$$

Let's rewrite (1):

$$\begin{aligned}(1) = \int_{d_1}^{\infty} e^{bz-\frac{1}{2}b^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz &= \int_{d_1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(z^2-2bz+b^2\right)}dz \\ &= \int_{d_1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-b)^2}dz\end{aligned}$$

This is just an ordinary u-substitution, whit $u=(z-b)$, now $z=d_1$ gives $u=d_1-b=d_1'$, so we have:

$$\int_{d_1'}^{\infty} f_U(u)du = P(U\geq d_1') = P(U\leq -d_1') = \Phi(-d_1')$$

$$d_+(T_m,T_{n-1}) := -d_1' = \frac{\ln\left(\frac{x}{\kappa}\right)+\frac{1}{2}\int_{T_m}^{T_{n-1}}\|\lambda(s,T_{n-1})\|^2ds}{\left(\int_{T_m}^{T_{n-1}}\|\lambda(s,T_{n-1})\|^2ds\right)^{1/2}}$$

And then we calculate (2):

$$(2) = \int_{d_1}^{\infty} f_Z(z)dz = P(Z\geq d_1) = P(Z\leq -d_1) = \Phi(-d_1)$$

With $-d_1 = d_-(T_m,T_{n-1})$.

Plugging all together, yields:

$$\begin{aligned}Cpl(T_m,T_{n-1},T_n) &= P(T_m,T_n)\delta\mathbb{E}_{Q^{T_n}}\left[\left(x\exp\left(bZ-\frac{1}{2}b^2\right)-\kappa\right)^+\right]_{x=L(T_{n-1},T_{n-1})} \\ &= \delta P(T_m,T_n)\left[L(T_m,T_{n-1})\Phi(d_+(T_m,T_{n-1}))-\kappa\Phi(d_-(T_m,T_{n-1})\right]\end{aligned}$$

■

Chapter 5

SOFR- Secured Overnight Financing Rate

5.1 Introduction

Following the LIBOR scandal, the Federal Reserve and regulators in the U.K. have come up with a replacement called the Secured Overnight Financing Rate (SOFR) . There are also other RFR alternatives (Risk-Free Reference Rates) who work similarly: SONIA (Sterling Overnight Index Average) managed by The Bank of England. One could also mention €STR (Euro Short-Term Rate) [Gro23].

On November 30, 2020, the Federal Reserve announced that the LIBOR would be phased out and eventually replaced by June 2023. Banks were also instructed to stop writing contracts using the LIBOR by the end of 2021, and that all contracts using the LIBOR should wrap up by June 30, 2023 [Hay22].

SOFR is fundamentally different from LIBOR. The Federal Reserve Bank of New York collects transaction data from the overnight Treasury Repo market. It then calculates a volume-weighted median interest rate. Which then gets published at 08.00 AM (Eastern Time) the following business day [ARR21].

This means that SOFR is backwards looking as it is based upon overnight transactions. Furthermore, it cannot look beyond 24 hours.

Key differences between LIBOR and SOFR/RFR's

1. Calculation Method: LIBOR is calculated based on submissions from Panel Banks. SOFR is based on the overnight repo market.
2. Tenors: LIBOR has multiple tenors, while SOFR has one: overnight. Meaning that LIBOR is forward-looking while SOFR is backwards-looking.
3. Validity: Following the LIBOR scandal, one has seen that LIBOR has been more prone to manipulation, as one can give higher or lower rate submissions altering the trimmed mean. SOFR is transaction based, meaning that it is harder to manipulate.
4. Secured vs unsecured. Any collateral does not back up the loans that involve LIBOR. In the SOFR situation, the repo transaction is collateralized by a high-quality bond such as a US-Treasury note [HSB22].

Definition 5.1.1 (**Discrete overnight SOFR** [SS20]). The discrete overnight SOFR is defined as:

$$R_{d_i}(T_i) = \frac{1}{d_i}\left(\frac{1}{P(T_i, T_i + d_i)} - 1\right)$$

Where:

- d_i: denotes the day count fraction multiplied by the number of days the overnight rate applies. I.e. $d_i = 1/360$ on business days, and $d_i = 3/360$ on fridays.

Definition 5.1.2 (**Backward-looking compounding SOFR-average** [SS20]). The backwards-looking compounded average over the period $[S, T]$ is defined as:

$$R^B(S, T) = \frac{1}{T - S}\left(\prod_{i=1}^{N}[1 + d_i R_{d_i}(T_i)] - 1\right)$$

- N: total number of days in the applicable period.
- $S \leq T_1 \leq \cdots \leq T_N \leq T$

Figure 5.1: O/N, 1M and 3M SOFR-rates

This figure is collected from [Fed23] and displays the following SOFR rates: green: O/N, blue: 1M-average and red: 3M-average. Here the 1M-average and 3M-average are calculated according to Definition 5.1.2.

If we look at the green O/N rates, we see a spike in September 2019. This was related to quarterly corporate tax payments due September 16. This led to a demand-supply mismatch [AAS20].

As discussed in [SS20], one has that $R^B(S,T)$ is $\mathcal{F}_T$-measurable. This is unsuitable as it does not incorporate market expectations about future rates. This leads to the following definition.

Definition 5.1.3 (**Forward-looking term-SOFR rate [SS20]**). The forward-looking term SOFR rate over the period $[S,T]$ is defined as:

$$R^F(S,T) = \frac{1}{T-S}\left[\frac{1}{P(S,T)} - 1\right]$$

We see that the term SOFR rate is just a simple forward rate evaluated at time $t = S$ (See Definition 4.1.2 p.27, i.e $R^F(S,T) = F(S,S,T)$).

Proposition 5.1.4 ([**SS20**]). *We have the following relationship between the forward-looking term SOFR rate $R^F(S,T)$ and the backwards-looking SOFR rate $R^B(S,T)$:*

$$R^F(S,T) = \mathbb{E}_{Q^T}[R^B(S,T)|\mathcal{F}_S]$$

Proof. We start by calculating the expectation:

$$\begin{aligned}\mathbb{E}_{Q^T}\left[R^B(S,T)|\mathcal{F}_S\right] &= \mathbb{E}_{Q^T}\left[\frac{1}{T-S}\left(\prod_{i=1}^{N}[1+d_iR_{d_i}(T_i)] - 1\right)\middle|\mathcal{F}_S\right] \\ &= \frac{1}{T-S}\left(\mathbb{E}_{Q^T}\left[\prod_{i=1}^{N}[1+d_iR_{d_i}(T_i)]\middle|\mathcal{F}_S\right] - 1\right) \qquad (5.1)\end{aligned}$$

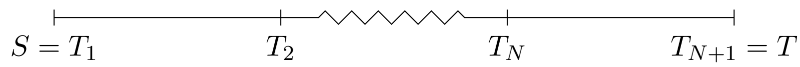
Now: $R_{d_i}(T_i) = R^F(T_i, T_i + d_i)$, this means that:

$$1 + d_iR_{d_i}(T_i) = \frac{1}{P(T_i, T_i + d_i)}$$

This yields:

$$\mathbb{E}_{Q^T}\left[\prod_{i=1}^{N}[1+d_iR_{d_i}(T_i)]\middle|\mathcal{F}_S\right] = \mathbb{E}_{Q^T}\left[\prod_{i=1}^{N}\frac{1}{P(T_i,T_i+d_i)}\middle|\mathcal{F}_S\right]$$

For simplicity we let $T_{i+1} = T_i + d_i$, this gives us the following timeline:



We observe that:

$$P(S,T) = P(T_1, T_{N+1}) = \prod_{i=1}^{N} P(T_i, T_{i+1})$$

This gives us:

$$\mathbb{E}_{Q^T}\left[\prod_{i=1}^{N}\frac{1}{P(T_i,T_i+d_i)}\middle|\mathcal{F}_S\right] = \mathbb{E}_{Q^T}\left[\frac{1}{P(S,T)}\middle|\mathcal{F}_S\right] = \frac{1}{P(S,T)}$$

Plugging this into Equation 5.1, yields:

$$\mathbb{E}_{Q^T}\left[R^B(S,T)|\mathcal{F}_S\right] = \frac{1}{T-S}\left(\frac{1}{P(S,T)} - 1\right) := R^F(S,T)$$

■

It should also be mentioned that SOFR is closely related to EFFR (Effective Federal Funds Rate). This is an overnight rate reflecting the rate banks and depository institutions lend/borrow to maintain the reserve requirements given by the Federal Reserve.

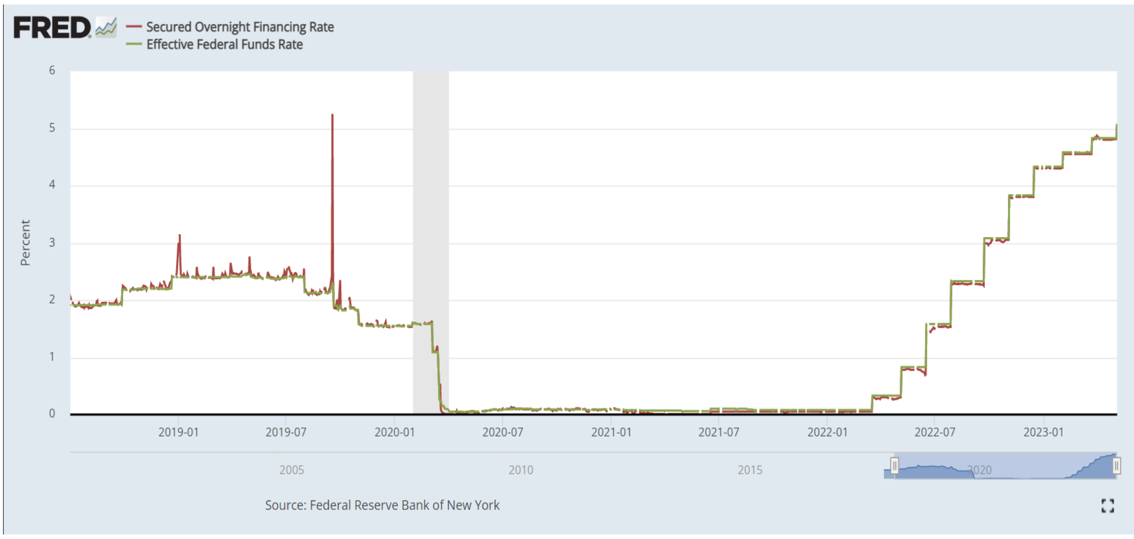


Figure 5.2: SOFR and EFFR

As mentioned earlier, we have that SOFR is backwards-looking and cannot give any indications beyond 24 hours. However, the CME Group publishes daily a set of forward-looking interest rate estimates called CME Term SOFR Reference Rates Benchmarks. They have the following tenors: 1M, 3M, 6M and 12M.

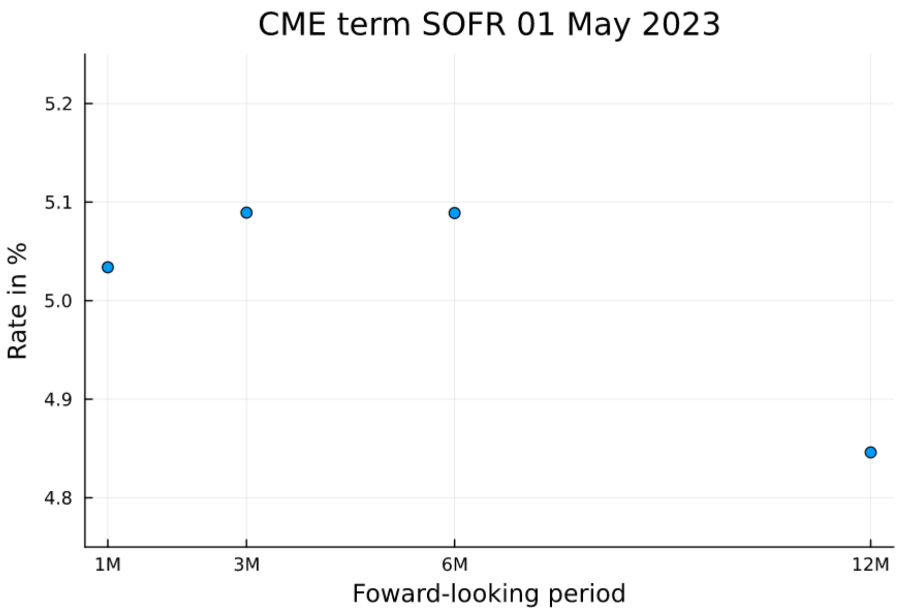


Figure 5.3: CME term SOFR

They use SOFR futures for calculating the term rates. The calculation method can be found at [Lim23]. Some of the reasons why futures were chosen were because of their liquidity. This represents the market's view on the rate; furthermore, this does not require expert judgment or a survey of market participants.

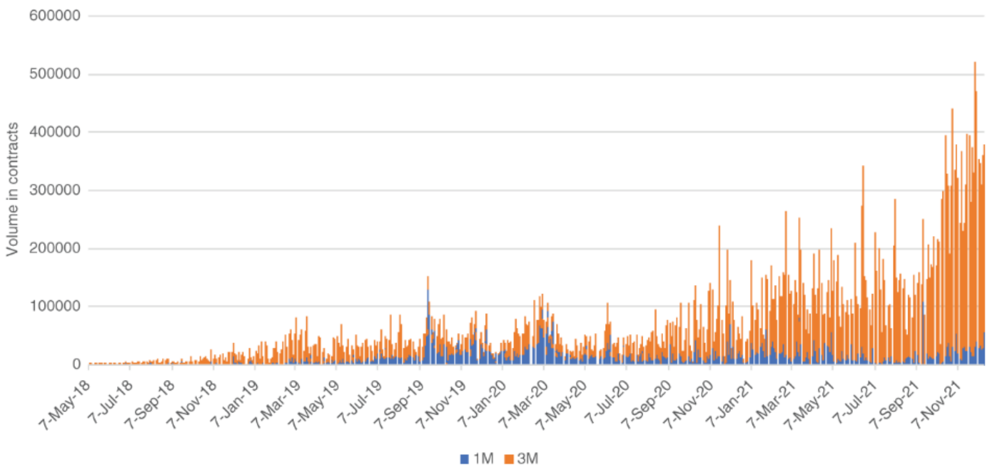


Figure 5.4: Daily volume of SOFR futures, source: [HSB22]

We see that there are some significant differences between the volumes. This is further discussed in [HSB22], where some of the explanation could be of the calculation methods for SOFR futures, i.e. arithmetic and geometric. However, it is also mentioned that there may be a boost in demand for 1M-SOFR futures when LIBOR is fully phased out.

5.2 SOFR futures

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q)$ denote our probability space, and let Q be the risk-neutral probability measure defined via the money market account $B(t)$ as numeraire.

Definition 5.2.1 (**Futures contract [Bjö19]**). A futures contract on X with delivery T, is a financial asset with following properties:

1. For all t with $0 \leq t \leq T$, there exists in the market a quoted object $f(t, T; X)$ known as the futures price of X at time t, with delivery T.
2. At time T of delivery, the holder of the contract pays: $f(T, T; X)$ and receives the claim X.
3. During an arbitrary interval $(s, t]$, the holder of the contract receives:

$$f(t, T; X) - f(s, T; X)$$

In typical futures contracts, the underlying asset X could be oil barrels, corn, cattle etc. In our situation, the underlying asset is an interest rate. The typical settlement is a cash settlement.

Some reasons to enter a futures contract:

- Speculation: By trading futures, one can make money by differences in quotes.
- Hedging: One could hedge against higher/lower interest rates. For instance, someone paying a floating rate might buy futures to lock in a future interest rate.

We will be interested in SOFR futures, and such futures can be found at CME (Chicago Mercantile Exchange) . CME uses the following convention for quoting interest rate futures:

$$100 - R$$

Here R will represent the implied SOFR rate. Let's say we observe a 3M-SOFR futures today (March 23) with settlement Jun 23, quoted at 96.6. This would mean that the implied $3M$-SOFR rate (annualized) over the period March 23 - June 23 would be:

$$(100 - 96.6)\% = 3.4\%$$

Further specifications on how CME quotes 3M-SOFR futures can be found in [Gro19].

When dealing with SOFR futures, one distinguishes between 1-month- and 3-month futures, as they are not based upon the same calculation methodology:

Definition 5.2.2 (**SOFR 1-month arithmetic average [SS20]**). The 1-month SOFR arithmetic average of the daily reference rate observed over the period $[S, T]$ is defined as:

$$R^{1M}(S,T) = \frac{1}{N}\sum_{i=1}^{N} R_{d_i}(T_i)$$

Where:

- N: total number of days in the month
- $S \leq T_1 \leq \cdots \leq T_N \leq T$

Definition 5.2.3 (**SOFR 3-month geometric average**). The 3-month SOFR geometric average of the daily reference rate observed over the period $[S, T]$ is defined as:

$$R^{3M}(S,T) = \frac{1}{T-S}\left(\prod_{i=1}^{N}(1 + d_i R_{d_i}(T_i)) - 1\right)$$

As futures contracts are free to enter, we get that:

$$\mathbb{E}_Q\left[R^{\ell M}(S,T) - f^{\ell M}(t,S,T)\middle|\mathcal{F}_t\right] = 0, \quad \ell = 1, 3$$

Furthermore, one uses the following convention:

$$R^{1M}(S,T) \approx \frac{1}{T-S}\int_S^T r(s)ds \;\text{ and }\; R^{3M}(S,T) \approx \frac{1}{T-S}\left(e^{\int_S^T r(s)ds} - 1\right)$$

This gives rise to the following definitions:

Definition 5.2.4 (**1-month SOFR futures [SS20]**). We denote the time t rate of the 1-month futures starting to accrue at time S and with settlement on time T as:

$$f^{1M}(t,S,T) = \frac{1}{T-S}\mathbb{E}_Q\left[\int_S^T r(s)ds\middle|\mathcal{F}_t\right]$$

Definition 5.2.5 (**3-month SOFR futures [SS20]**). We denote the time t rate of the 3-month futures starting to accrue at time S and with settlement on time T as:

$$f^{3M}(t,S,T) = \frac{1}{T-S}\left(\mathbb{E}_Q\left[e^{\int_S^T r(s)ds}\middle|\mathcal{F}_t\right] - 1\right)$$

Proposition 5.2.6 (**Vasicek dynamics of 1M/3M-SOFR futures, [Exercise STK4530, Autumn 2021]**). *Assume that the short rate has the following dynamics:*

$$dr(t) = \alpha[m - r(t)]dt + \sigma dW^Q(t)$$

Then the dynamics of $f^{1M}(t, S, T)$ is given by:

$$df^{1M}(t, S, T) = \frac{1}{T - S} B(t, S, T)\sigma dW^Q(t)$$

and the dynamics of $f^{3M}(t, S, T)$ is given by:

$$df^{3M}(t, S, T) = \left(f^{3M}(t, S, T) + \frac{1}{T - S}\right) B(t, S, T)\sigma dW^Q(t)$$

Where:

$$B(t, S, T) = \frac{1}{\alpha}\left[e^{-\alpha(S-t)} - e^{-\alpha(T-t)}\right]$$

Proof. From Proposition 4.4.3, we have that:

$$r(s) = e^{-\alpha(s-t)}r(t) + m[1 - e^{-\alpha(s-t)}] + \sigma\int_t^s e^{-\alpha(s-u)}dW^Q(u)$$

Now: $f^{1M}(t, S, T) = \frac{1}{T-S}\mathbb{E}_Q\left[\int_S^T r(s)ds|\mathcal{F}_t\right]$:

$$\mathbb{E}_Q\left[r(s)|\mathcal{F}_t\right] = r(t)e^{-\alpha(s-t)} + m\left(1 - e^{-\alpha(s-t)}\right)$$
$$\Downarrow$$
$$f^{1M}(t, S, T) = \frac{e^{-\alpha S} - e^{-\alpha T}}{\alpha(T - S)}[r(t) - m]e^{\alpha t} + m$$

Giving rise to the following dynamics:

$$df^{1M}(t, S, T) = \frac{e^{-\alpha S} - e^{-\alpha T}}{\alpha(T - S)} d\left[(r(t) - m)e^{\alpha t}\right]$$

Let's work with the differential part first:

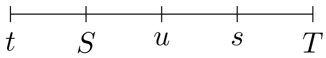
$$\begin{aligned} d[(r(t) - m)e^{\alpha t}] &= d[r(t)e^{\alpha t}] - md(e^{\alpha t}) \\ &= \alpha m e^{\alpha t}dt + \sigma e^{\alpha t}dW^Q(t) - \alpha m e^{\alpha t}dt \\ &= \sigma e^{\alpha t}dW^Q(t) \end{aligned}$$

Thus:

$$\begin{aligned} df^{1M}(t, S, T) &= \frac{e^{-\alpha S} - e^{-\alpha T}}{\alpha(T - S)}\sigma e^{\alpha t}dW^Q(t) \\ &= \frac{1}{T - S}B(t, S, T)\sigma dW^Q(t) \end{aligned}$$

Now for $f^{3M}(t,S,T)$ we must study $\int_S^T r(s)ds$:

We have the following timeline:



namely $t \le S \le u \le s \le T$, this gives us:

$$\begin{aligned}\int_S^T r(s)ds = \frac{r(t)}{\alpha}\left[e^{-\alpha(S-t)} - e^{-\alpha(T-t)}\right] + m(T-S) - \frac{m}{\alpha}\left[e^{-\alpha(S-t)} - e^{-\alpha(T-t)}\right] \\ + \sigma \underbrace{\int_S^T \int_t^s e^{-\alpha(s-u)} dW^Q(u) ds}_{=(*)}\end{aligned}$$

Now by additivity of the integral, we see that:

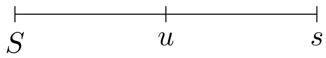
$$\int_t^s = \int_t^S + \int_S^s$$

This leaves us with:

$$\begin{aligned}(*) &= \int_S^T \left(\int_t^S e^{-\alpha(s-u)} dW^Q(u) + \int_s^S e^{-\alpha(s-u)} dW^Q(u) \right) ds \\ &= \underbrace{\int_S^T \int_t^S e^{-\alpha(s-u)} dW^Q(u) ds}_{=(1)} + \underbrace{\int_S^T \int_S^s e^{-\alpha(s-u)} dW^Q(u) ds}_{=(2)}\end{aligned}$$

By Stochastic Fubini, we get:

$$(1) = \int_S^T \int_t^S e^{-\alpha(s-u)} dW^Q(u) ds = \int_t^S \int_S^T e^{-\alpha(s-u)} ds dW^Q(u) = (1)'$$



$$\begin{cases} S \le s \le T \\ S \le u \le s \end{cases} \iff \begin{cases} u \le s \le T \\ S \le u \le T \end{cases}$$

Now this leaves us with the following:

$$(2) = \int_S^T \int_S^s e^{-\alpha(s-u)} dW^Q(u) ds = \int_S^T \int_u^T e^{-\alpha(s-u)} ds dW^Q(u) = (2)'$$

We now calculate the inner integral in $(1)'$ and $(2)'$ respectively:

$$\begin{aligned}\int_S^T e^{-\alpha(s-u)} ds &= \frac{1}{\alpha}\left[e^{\alpha(S-u)} - e^{-\alpha(T-u)}\right] \\ \int_u^T e^{-\alpha(s-u)} ds &= \frac{1}{\alpha}\left[1 - e^{-\alpha(T-u)}\right]\end{aligned}$$

We can then define:

$$\Sigma(u,t,S,T) = \begin{cases} e^{\alpha(S-u)} - e^{-\alpha(T-u)}, & u \in [t,S) \\ 1 - e^{-\alpha(T-u)}, & u \in [S,T] \end{cases}$$

We are thus left with:

$$
\begin{aligned}
\int_S^T r(s)ds &= \frac{r(t)}{\alpha}\left[e^{-\alpha(S-t)} - e^{-\alpha(T-t)}\right] + m(T-S) - \frac{m}{\alpha}\left[e^{-\alpha(S-t)} - e^{-\alpha(T-t)}\right] \\
&\quad + \frac{\sigma}{\alpha}\int_t^T \Sigma(u,t,S,T)dW^Q(u) \\
&= \left(\frac{r(t)-m}{\alpha}\right)\left[e^{-\alpha(S-t)} - e^{-\alpha(T-t)}\right] + m(T-S) + \underbrace{\frac{\sigma}{\alpha}\int_t^T \Sigma(u,t,S,T)dW^Q(u)}_{\mathcal{F}_t-\text{independent}}
\end{aligned}
\tag{5.2}
$$

As the last part is $\mathcal{F}_t$-independent,we get:

$$
\begin{aligned}
\mathbb{E}_Q\left[\exp\left(\int_S^T r(s)ds\right)\Big|\mathcal{F}_t\right] &= \exp\left[\left(\frac{r(t)-m}{\alpha}\right)\left[e^{-\alpha(S-t)-e^{-\alpha(T-t)}}\right] + m(T-S)\right] \\
&\quad \times \mathbb{E}_Q\left[\exp\left(\frac{\sigma}{\alpha}\int_t^T \Sigma(u,t,S,T)dW^Q(u)\right)\right]
\end{aligned}
$$

Since Σ is deterministic, we have that:

$$
\mathbb{E}_Q\left[\exp\left(\frac{\sigma}{\alpha}\int_t^T \Sigma(u,t,S,T)dW^Q(u)\right)\right] = \exp\left(\frac{1}{2}\frac{\sigma^2}{\alpha^2}\int_t^T \Sigma^2(u,t,S,T)du\right)
$$

This leaves us with the following expression:

$$
\mathbb{E}_Q\left[\exp\left(\int_S^T r(s)ds\right)\Big|\mathcal{F}_t\right] = \exp\left(A(t,S,T) + B(t,S,T)r(t)\right) := g(t,r(t)) \tag{5.3}
$$

where:

$$
\begin{aligned}
A(t,S,T) &= m(T-S) - \frac{m}{\alpha}\left[e^{-\alpha(S-t)} - e^{-\alpha(T-t)}\right] + \frac{1}{2}\frac{\sigma^2}{\alpha^2}\int_t^T \Sigma^2(u,t,S,T)du \\
B(t,S,T) &= \frac{1}{\alpha}\left[e^{-\alpha(S-t)} - e^{-\alpha(T-t)}\right]
\end{aligned}
$$

This means that we have:

$$
f^{3M}(t,S,T) = \frac{1}{T-S}\left[g(t,r(t)) - 1\right] \tag{5.4}
$$

We note that $\mathbb{E}_Q\left[\exp\left(\int_S^T r(s)ds\right)\Big|\mathcal{F}_t\right]$ is a Q-martingale, thus from the Martingale Representation Theorem 2.3.25, we can neglect the dt-terms of $g(t,r(t))$:

We apply Ito's Formula as $g(t,x) \in C^{1,2}([0,\infty] \times \mathbb{R})$, giving us:

$$
\begin{aligned}
\partial_t g(t,x) = 0, \quad \partial_x g(t,x) &= g(t,x)B(t,S,T), \quad \partial_{xx} g(t,x) = g(t,x)B^2(t,S,T) \\
dr(t)^2 &= \sigma^2 dt \\
&\Downarrow \\
dg(t,r(t)) &= B(t,S,T)g(t,r(t))\sigma dW^Q(t)
\end{aligned}
$$

This gives the following dynamics for $f^{3M}(t,S,T)$:

$$\begin{aligned} df^{3M}(t,S,T) &= \frac{1}{T-S} dg(t,r(t)) \\ &= \frac{1}{T-S} B(t,S,T) \exp\left(A(t,S,T) + B(t,S,T)r(t)\right) \sigma dW^Q(t) \\ &= \frac{1}{T-S} B(t,S,T) \left[(T-S) f^{3M}(t,S,T) + 1\right] \sigma dW^Q(t) \\ &= \left(f^{3M}(t,S,T) + \frac{1}{T-S}\right) B(t,S,T) \sigma dW^Q(t) \end{aligned}$$

■

To get a bit better grasp of the SOFR futures rates f^{1M} and f^{3M}, we include a graph of possible Q-dynamics:

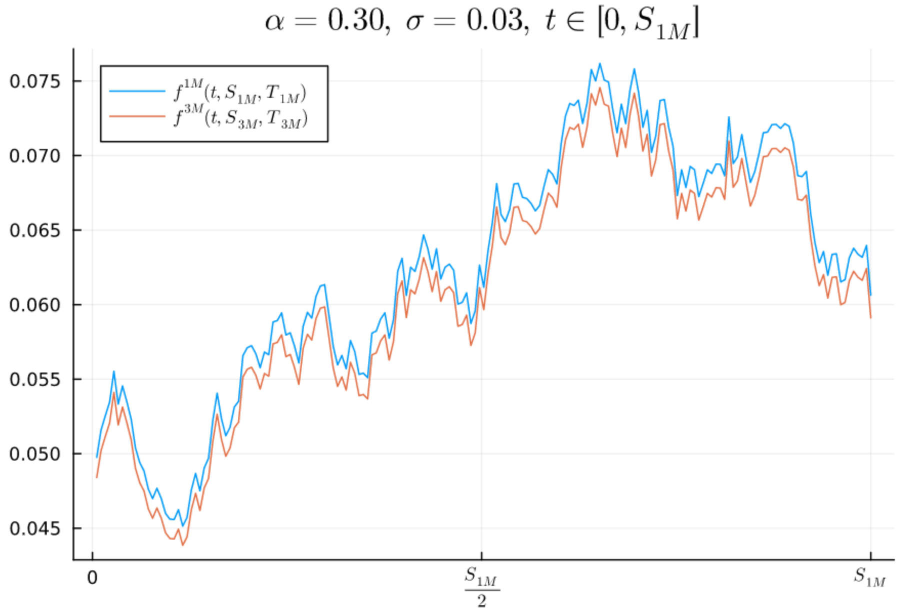


Figure 5.5: Realization of $f^{1M}(t, S_{1M}, T_{1M})$ and $f^{3M}(t, S_{3M}, T_{3M})$ for $t \in [0, S_{1M}]$

Here we have assumed that the quotes for 1M and 3M futures differ. We took:

$$f^{1M}(0, S_{1M}, T_{1M}) = (100 - 95.025) \cdot \frac{1}{100} \approx 0.0498$$
$$f^{3M}(0, S_{3M}, T_{3M}) = (100 - 95.160) \cdot \frac{1}{100} \approx 0.0484$$

Furthermore, we have $S_{1M} = S_{3M} = 6$ months, and f^{1M}, f^{3M} are generated by the same Brownian motion.

5.3 Interest rate swap with SOFR-futures as floating

Proposition 5.3.1. *The fixed swap rate $\kappa_t^{\ell M-SOFR}$ in a swap with 1M/3M-SOFR as floating is given by:*

$$\kappa_t^{\ell M-SOFR} = \frac{\sum_{i=1}^n P(t,T_i) f^{\ell M}(t,T_{i-1},T_i)}{\sum_{i=1}^n P(t,T_i)}, \quad \ell = 1,3$$

Proof. In this swap, we have the following specification, at time T_i:

- Pay $\kappa_t^{\ell M-SOFR}\delta N$ (-)
- Receive $f^{\ell M}(t,T_{i-1},T_i)\delta N, \quad \ell = 1,3$ (+)

Cash flow at time T_i:

$$f^{\ell M}(t,T_{i-1},T_i)\delta N - \kappa_t^{\ell M-SOFR}\delta N = [f^{\ell M}(t,T_{i-1},T_i) - \kappa_t^{\ell M-SOFR}]\delta N, \quad \ell = 1,3$$

Time t-value for $t \leq T_0$ at time T_i:

$$P(t,T_i)[f^{\ell M}(t,T_{i-1},T_i) - \kappa_t^{SOFR}]\delta N, \quad \ell = 1,3$$

Total payer cash flow:

$$\mathcal{C}_P^{\ell M-SOFR}(t) = \delta N \sum_{i=1}^n P(t,T_i)[f^{\ell M}(t,T_{i-1},T_i) - \kappa_t^{\ell M-SOFR}], \quad \ell = 1,3$$

$\kappa_t^{\ell M-SOFR}$ should be chosen such that:

$$\mathbb{E}_Q[\mathcal{C}_P^{\ell M-SOFR}(t)|\mathcal{F}_t] = 0$$

Thus:

$$\sum_{i=1}^n P(t,T_i) f^{\ell M}(t,T_{i-1},T_i) = \sum_{i=1}^n P(t,T_i)\kappa_t^{\ell M-SOFR}$$

$$\Downarrow$$

$$\kappa_t^{\ell M-SOFR} = \frac{\sum_{i=1}^n P(t,T_i) f^{\ell M}(t,T_{i-1},T_i)}{\sum_{i=1}^n P(t,T_i)}$$

∎

Let us consider the case where we look at $\ell = 3$, $n = 3$, and $\delta = \frac{3}{12}$. For simplicity, we choose the Vasicek model as this gives an explicit formula for $f^{3M}(t, S, T)$ as described in Equation 5.4.

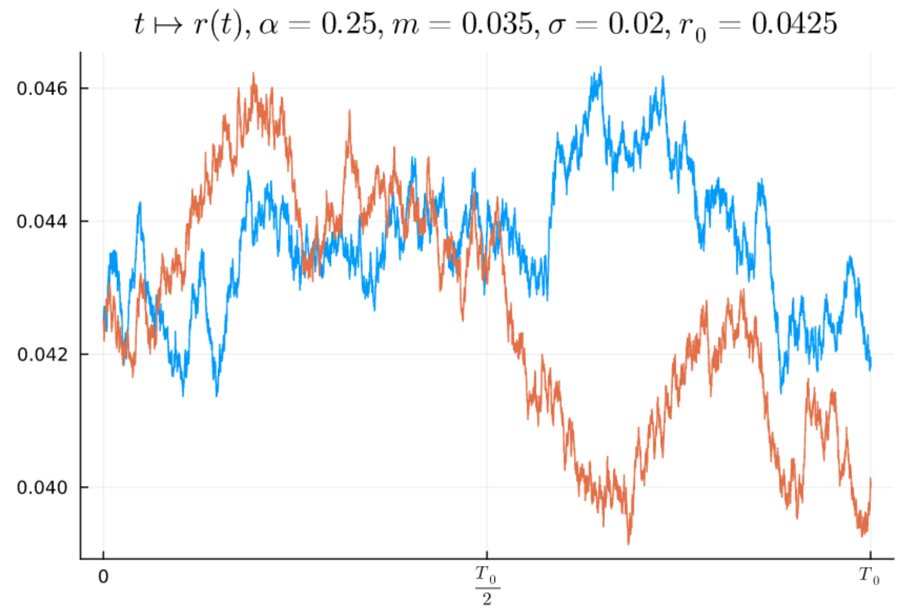


Figure 5.6: Realizations of $t \mapsto r(t), t \in [0, T_0]$

Here we see two realizations of $[0, T_0] \ni t \mapsto r(t)$. In the graph below, we see the effect of the time horizon and the starting point for the fixed rate $\kappa_t^{3M-SOFR}$:

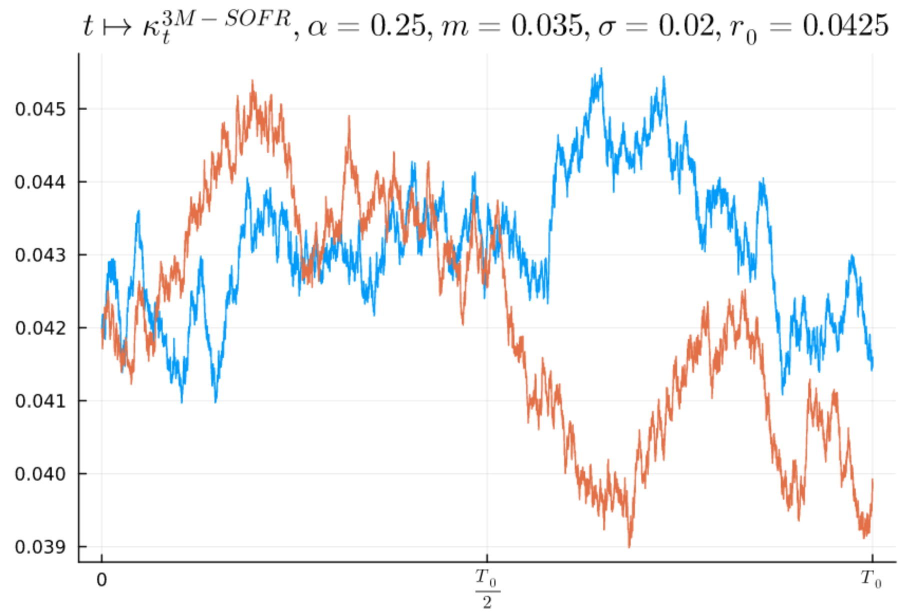


Figure 5.7: Realizations of $t \mapsto \kappa_t^{3M-SOFR}, t \in [0, T_0]$

Here we took two realizations of $[0, T_0] \ni t \mapsto \kappa_t^{3M-SOFR}$. For $t = 0$, we got:

$$\kappa_0^{3M-SOFR} = \frac{\sum_{i=1}^{3} P(0, T_i) f^{3M}(0, T_{i-1}, T_i)}{\sum_{i=1}^{3} P(0, T_i)} = 0.042$$

5.4 Options on SOFR futures

Consider a call option on SOFR futures, with exercise time $\tau \leq S \leq T$ and strike κ, the price at time $t \leq \tau$ for $\ell = 1,3$ is given by:

$$C^{\ell M}(t,\tau) := \mathbb{E}_Q\left[\frac{B(t)}{B(\tau)}\left(f^{\ell M}(\tau,S,T)-\kappa\right)^+ \middle| \mathcal{F}_t\right] \overset{\text{Prop 4.7.3}}{=} P(t,\tau)\mathbb{E}_{Q^\tau}\left[\left(f^{\ell M}(\tau,S,T)-\kappa\right)^+ \middle| \mathcal{F}_t\right]$$

Here we use the convention that $(x)^+ = \max(x,0)$. From Theorem 4.7 p 38, we have:

$$\left.\frac{dQ^\tau}{dQ}\right|_{\mathcal{F}_t} = \mathcal{E}_t(v(\cdot,\tau) \bullet W^Q)$$

Assuming Novikov's condition holds, i.e: $\mathbb{E}_Q\left[e^{\frac{1}{2}\int_0^T \|v(s,\tau)\|^2 ds}\right] < \infty$ we get from Girsanov's theorem, that

$$dW^\tau(t) = dW^Q(t) - v(t,\tau)dt$$

defines a Q^τ-Brownian motion.

Proposition 5.4.1 (**1M-SOFR futures Caplet [Lecture STK4530]**). *Consider a call option on the 1M-SOFR futures with exercise time $\tau \leq T$ and strike κ. Let*

$$df^{1M}(t,S,T) = \Sigma^{1M}(t,S,T)dW^Q(t)$$

Where $\Sigma^{1M}(t,S,T)$ is assumed to be a deterministic and bounded function. The price at time $t \leq \tau$ is given by:

$$C^{1M}(t,\tau) = P(t,\tau)\sqrt{\int_t^\tau \Sigma^{1M}(u,S,T)^2 du}\,[d\Phi(d) + \varphi(d)]$$

Where:

$$d = \frac{f^{1M}(t,S,T) + \int_t^\tau \Sigma^{1M}(u,S,T)v(u,\tau)du - \kappa}{\sqrt{\int_t^\tau \Sigma^{1M}(u,S,T)^2 du}}$$

And Φ, φ represents the cumulative and density function of a standard-normal distribution respectively.

Proof. The Q^τ-dynamics are given by:

$$df^{1M}(t,S,T) = \Sigma^{1M}(t,S,T)v(t,\tau)dt + \Sigma^{1M}(t,S,T)dW^\tau(t)$$
$$\Downarrow$$
$$f^{1M}(\tau,S,T) = \underbrace{f^{1M}(t,S,T)}_{x} + \underbrace{\int_t^\tau \Sigma^{1M}(u,S,T)v(u,\tau)du}_{m} + \int_t^\tau \Sigma^{1M}(u,S,T)dW^\tau(u)$$

Let

$$b^2 = \int_t^\tau \Sigma^{1M}(u,S,T)^2 du$$

We are then left with the following:

$$\mathbb{E}_{Q^\tau}\left[(f^{1M}(\tau,S,T)-\kappa)^+\middle|\mathcal{F}_t\right]=\mathbb{E}\left[(x+m+bZ-\kappa)^+\right]\Big|_{x=f^{1M}(t,S,T)}$$

where $Z\sim\mathcal{N}(0,1)$, this yields:

$$\mathbb{E}\left[(x+m+bZ-\kappa)^+\right]\Big|_{x=f^{1M}(t,S,T)}=\int_{\mathbb{R}}(x+m+bz-\kappa)^+\varphi(z)dz$$

Furthermore:

$$x+m+bz-\kappa\geq 0\iff z\geq\frac{\kappa-x-m}{b}:=d'$$

This yields:

$$\int_{\mathbb{R}}(x+m+bz-\kappa)^+\varphi(z)dz=\underbrace{(x+m-\kappa)\int_{d'}^{\infty}\varphi(z)dz+b\int_{d'}^{\infty}z\varphi(z)dz}_{(A)}$$

By symmetry of the normal distribution we have: $P(Z>d')=P(Z\leq -d')$, where we define:

$$d:=-d'=\frac{x+m-\kappa}{b}$$

furthermore $z\varphi(z)=-\varphi'(z)$, thus:

$$\int_{d'}^{\infty}z\varphi(z)dz=-\int_{d'}^{\infty}\varphi'(z)dz=-(\varphi(\infty)-\varphi(d'))=\varphi(d')=\varphi(d)$$

Leaving us with:

$$\begin{aligned}(A)&=(x+m-\kappa)\Phi(d)+b\varphi(d)\\&=b[d\Phi(d)+\varphi(d)]\end{aligned}$$

Now from Proposition 4.7.3 p.41, we get:

$$\begin{aligned}C^{1M}(t,\tau)&=P(t,\tau)\mathbb{E}_{Q^\tau}\left[(f^{1M}(\tau,S,T)-\kappa)^+\middle|\mathcal{F}_t\right]\\&=P(t,\tau)\sqrt{\int_t^\tau\Sigma^{1M}(u,S,T)^2du}\,[d\Phi(d)+\varphi(d)]\end{aligned}$$

■

5.5 Hedging with SOFR-futures

For SOFR futures, we have two pricing approaches: arithmetic (1M) and geometric (3M). In this section, we will look at some relationships between them.

5.5.1 Hedging 3-month arithmetic with 3-month geometric

Consider the case where we want to hedge:

$$X^{3M_A}(S,T) = \frac{1}{T-S}\int_S^T r_u du \tag{5.5}$$

We want to hedge an arithmetic interest rate over the 3M period $[S,T]$. The only available product for hedging a 3M period in the market is the 3M futures contract $f^{3M}(t,S,T)$.

Proposition 5.5.1.

$$\operatorname*{arg\,min}_{a_t\in\mathbb{R}} \mathbb{E}_Q\left[\left(X^{3M_A}(S,T) - a_t f^{3M}(t,S,T)\right)^2 \middle| \mathcal{F}_t\right] = \frac{\int_S^T \mathbb{E}_Q[r(u)|\mathcal{F}_t]du}{(T-S)f^{3M}(t,S,T)}$$

Meaning that the optimal weighting $\hat{a}_t^{3M}$ in 3M-SOFR futures will be:

$$\hat{a}_t^{3M} = \frac{\int_S^T \mathbb{E}_Q[r(u)|\mathcal{F}_t]du}{(T-S)f^{3M}(t,S,T)}$$

Proof. We have that the 3M-SOFR futures $f^{3M}(t,S,T)$ is based upon a geometric average, giving us the following hedge:

$$\operatorname*{arg\,min}_{a_t\in\mathbb{R}} \mathbb{E}_Q\left[\left(X^{3M_A}(S,T) - a_t f^{3M}(t,S,T)\right)^2 \middle| \mathcal{F}_t\right]$$

Now, fix t and denote:

$$G(a_t) := \mathbb{E}_Q\left[\left(X^{3M_A}(S,T) - a_t f^{3M}(t,S,T)\right)^2 \middle| \mathcal{F}_t\right]$$

Expanding the square yields:

$$G(a_t) = \mathbb{E}_Q\left[(X^{3M_A}(S,T))^2|\mathcal{F}_t\right] - 2a_t f^{3M}(t,S,T)\mathbb{E}_Q\left[X^{3M_A}(S,T)\middle|\mathcal{F}_t\right] + a_t^2[f^{3M}(t,S,T)]^2$$

Taking the derivative w.r.t. a_t yields:

$$\frac{d}{da_t}G(a_t) = -2f^{3M}(t,S,T)\mathbb{E}_Q\left[X^{3M_A}(S,T)\middle|\mathcal{F}_t\right] + 2a_t[f^{3M}(t,S,T)]^2$$

Since:

$$\frac{d^2}{da_t^2}G(a_t) = 2[f^{3M}(t,S,T)]^2 > 0$$

We have that the minimum is obtained by setting the derivative equal to zero, i.e.

$$\frac{d}{da_t}G(a_t) = 0$$

Now:

$$\frac{d}{da_t}G(a_t) = 0$$
$$\Updownarrow$$
$$a_t = \frac{\mathbb{E}_Q[X^{3M_A}(S,T)|\mathcal{F}_t]}{f^{3M}(t,S,T)}$$

Furthermore:

$$\mathbb{E}_Q[X^{3M_A}(S,T)|\mathcal{F}_t] = \frac{1}{T-S}\int_S^T \mathbb{E}_Q[r(u)|\mathcal{F}_t]$$

Yielding:

$$a_t = \frac{\int_S^T \mathbb{E}_Q[r(u)|\mathcal{F}_t]du}{(T-S)f^{3M}(t,S,T)}$$

■

5.5.2 Affine Term Structure-setting

Proposition 5.5.2. *Consider the above setting, and let* $r = (r(t))_{t\geq 0}$ *be a model that provides ATS, meaning that:*

$$dr(t) = [b(t) + \beta(t)r(t)]dt + \sqrt{a(t) + \alpha(t)r(t)}dW^Q(t)$$

a, α, b, β *are continuous and deterministic functions. This gives us the following:*

$$\underset{a_t \in \mathbb{R}}{\arg\min} \mathbb{E}_Q \left[\left(X^{3M_A}(S,T) - a_t f^{3M}(t,S,T) \right)^2 \middle| \mathcal{F}_t \right]$$

$$= \frac{r(t)(T-S) + \int_S^T \int_t^u b(s)dsdu + \int_S^T \int_t^u \beta(s)g(s)dsdu}{(T-S)f^{3M}(t,S,T)}$$

Where:

$$g(s) = \exp\left(\int_t^s \beta(v)dv \right) \left(\int_t^s e^{-\int_t^w \beta(v)dv} b(w)dw + \mathbb{E}_Q[r(t)] \right)$$

Proof. Since $r = (r(t))_{t\geq 0}$ is a model that provides ATS (Affine Term Structure), as described in Proposition 4.4.2, as well as above, we have the following dynamics:

$$dr(t) = [b(t) + \beta(t)r(t)]dt + \sqrt{a(t) + \alpha(t)r(t)}dW^Q(t)$$

Here b, β, a, α are deterministic continuous functions. Now from the dynamics, we get that for $u \geq t$:

$$r(u) = r(t) + \int_t^u b(s)ds + \int_t^u [\beta(s)r(s)]ds + \int_t^u \sqrt{a(s) + \alpha(s)r(s)}dW^Q(s) \tag{5.6}$$

Each term is assumed to be Ito-integrable, i.e in $M^2([0,T])$, and by $\mathcal{F}_t$-independence, we get:

$$\mathbb{E}_Q \left[\int_t^u \sqrt{a(s) + \alpha(s)r(s)}dW^Q(s) \right] = 0$$

And again by $\mathcal{F}_t$-independence in combination with Fubini, we get:

$$\mathbb{E}_Q \left[\int_t^u [\beta(s)r(s)]ds \right] = \int_t^u \beta(s)\mathbb{E}_Q[r(s)]ds$$

This leaves us with the following:

$$\int_S^T \mathbb{E}_Q[r(u)|\mathcal{F}_t]du = r(t)(T-S) + \int_S^T \int_t^u b(s)dsdu + \int_S^T \int_t^u \beta(s)\mathbb{E}_Q[r(s)]dsdu \tag{5.7}$$

Overview of time-interval:

Thus in our setting we have: $t \leq S \leq s \leq u \leq T$, proceeding as in Equation 5.6, we have:

$$r(s) = r(t) + \int_t^s b(v)dv + \int_t^s \beta(v)r(v)dv + \int_t^s \sqrt{a(v) + \alpha(v)r(v)}dW^Q(v) \tag{5.8}$$

Now let $g(s) := \mathbb{E}_Q[r(s)]$, applying the expectation to 5.8 yields:

$$g(s) = r(t) + \int_t^s b(v)dv + \int_t^s \beta(v)g(v)dv$$

Taking the derivative w.r.t. s and using the fundamental theorem of calculus gives us the following:

$$g'(s) = b(s) + \beta(s)g(s)$$

We have initial condition $g(t) = \mathbb{E}_Q[r(t)]$, this is an ordinary differential equation with solution:

$$g(s) = \exp\left(\int_t^s \beta(v)dv\right)\left(\int_t^s e^{-\int_t^w \beta(v)dv} b(w)dw + g(t)\right)$$

We have that $g(s) = \mathbb{E}_Q[r(u)]$ appears in Equation 5.7, this gives us the desired result.

■

5.5.3 Bounding the hedge with available instruments in the market

We now denote:

$$X^{3M_A}(S,T) = \frac{1}{T-S}\int_S^T r_u du = \frac{1}{T-S}Z(S,T)$$
$$f^{3M_A}(t,S,T) = \frac{1}{T-S}\mathbb{E}_Q\left[\int_S^T r_u du \middle| \mathcal{F}_t\right] = \frac{1}{T-S}\mathbb{E}_Q[Z(S,T)|\mathcal{F}_t]$$

Now from Jensen's Inequality 2.2.9, we have that for $Z, \varphi(Z) \in L^1(\Omega, \mathcal{F}, Q)$, with $\varphi(x) = e^x$

$$\exp\left(\mathbb{E}_Q[Z(S,T)|\mathcal{F}_t]\right) \leq \mathbb{E}_Q\left[\exp(Z(S,T))|\mathcal{F}_t\right]$$
$$\Updownarrow$$
$$\exp\left((T-S)f^{3M_A}(t,S,T)\right) \leq \mathbb{E}_Q\left[\exp(Z(S,T))|\mathcal{F}_t\right] \tag{5.9}$$

Now from definition 5.2.5, we have:

$$f^{3M}(t,S,T) = \frac{1}{T-S}\left(\mathbb{E}_Q\left[\underbrace{e^{\int_S^T r_u du}}_{=e^{Z(S,T)}}\middle|\mathcal{F}_t\right] - 1\right)$$
$$\Downarrow$$
$$\mathbb{E}_Q[\exp(Z(S,T))|\mathcal{F}_t] = (T-S)f^{3M}(t,S,T) + 1 \tag{5.10}$$

Now by inserting 5.10 into 5.9 yields:

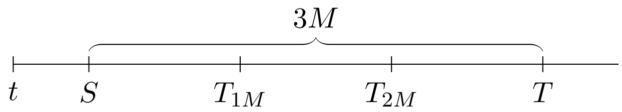
$$\exp\left((T-S)f^{3M_A}(t,S,T)\right) \leq (T-S)f^{3M}(t,S,T) + 1$$
$$\Updownarrow$$
$$f^{3M_A}(t,S,T) \leq \frac{\ln[(T-S)f^{3M}(t,S,T) + 1]}{(T-S)}$$

5.5.4 Hedging three-month arithmetic with 1M-SOFR futures

We still want to hedge:

$$X^{3M_A}(S,T) = \frac{1}{T-S}\int_S^T r_u du$$

However, now we will not use the available 3M-SOFR future contract, rather we will hedge by buying $(\hat{a}_t, \hat{b}_t, \hat{c}_t)$ 1M-SOFR future contracts at time t. Here $[S,T]$ will still denote a 3M period, we get the following timeline:



Our hedge will in this case look like:

$$\underset{(a_t,b_t,c_t)\in\mathbb{R}^3}{\arg\min}\ \mathbb{E}_Q\left[\left(X^{3M_A}(S,T) - \left[a_t f^{1M}(t,S,T_{1M}) + b_t f^{1M}(t,T_{1M},T_{2M}) + c_t f^{1M}(t,T_{2M},T)\right]\right)^2 \middle| \mathcal{F}_t\right]$$

Denote:

$$G(a_t,b_t,c_t) := \mathbb{E}_Q\left[\left(X^{3M_A}(S,T) - \left[a_t f^{1M}(t,S,T_{1M}) + b_t f^{1M}(t,T_{1M},T_{2M}) + c_t f^{1M}(t,T_{2M},T)\right]\right)^2 \middle| \mathcal{F}_t\right]$$

Expanding the square yields:

$$\begin{aligned}
G(a_t,b_t,c_t) &= \mathbb{E}_Q\left[(X^{3M_A}(S,T))^2|\mathcal{F}_t\right] \\
&- 2\mathbb{E}_Q[X^{3M_A}(S,T)|\mathcal{F}_t]\left[a_t f^{1M}(t,S,T_{1M}) + b_t f^{1M}(t,T_{1M},T_{2M}) + c_t f^{1M}(t,T_{2M},T)\right] \\
&+ a_t^2[f^{1M}(t,S,T_{1M})]^2 \\
&+ 2a_t f^{1M}(t,S,T_{1M})\left[b_t f^{1M}(t,T_{1M},T_{2M}) + c_t f^{1M}(t,T_{2M},T)\right] \\
&+ b_t^2[f^{1M}(t,T_{1M},T_{2M})]^2 \\
&+ 2b_t c_t\left[f^{1M}(t,T_{1M},T_{2M}) f^{1M}(t,T_{2M},T)\right] \\
&+ c_t^2[f^{1M}(t,T_{2M},T)]^2
\end{aligned}$$

Fix t, to ease the notation we denote $(a_t,b_t,c_t) = (x_1(t),x_2(t),x_3(t)) = \mathbf{x}_t$ furthermore let:

$$\left(f^{1M}(t,S,T_{1M}), f^{1M}(t,T_{1M},T_{2M}), f^{1M}(t,T_{2M},T)\right) = (\alpha_t,\beta_t,\gamma_t)$$

We also let:

$$\begin{aligned}
\mathbb{E}_Q\left[(X^{3M_A}(S,T))^2|\mathcal{F}_t\right] &= p_t \\
\mathbb{E}_Q[X^{3M_A}(S,T)|\mathcal{F}_t] &= q_t
\end{aligned}$$

This leaves us with:

$$\begin{aligned}
G(\mathbf{x}_t) &= p_t \\
&- 2q_t\left[\alpha_t x_1(t) + \beta_t x_2(t) + \gamma_t x_3(t)\right] \\
&+ \alpha_t^2 x_1(t)^2 \\
&+ 2\alpha x_1(t)\left[\beta_t x_2(t) + \gamma_t x_3(t)\right] \\
&+ \beta_t^2 x_2(t)^2 \\
&+ 2\beta_t\gamma_t x_2(t)x_3(t) \\
&+ \gamma_t^2 x_3(t)^2
\end{aligned}$$

To get a bit better grasp of $G(\mathbf{x}_t)$ we include a level plot:

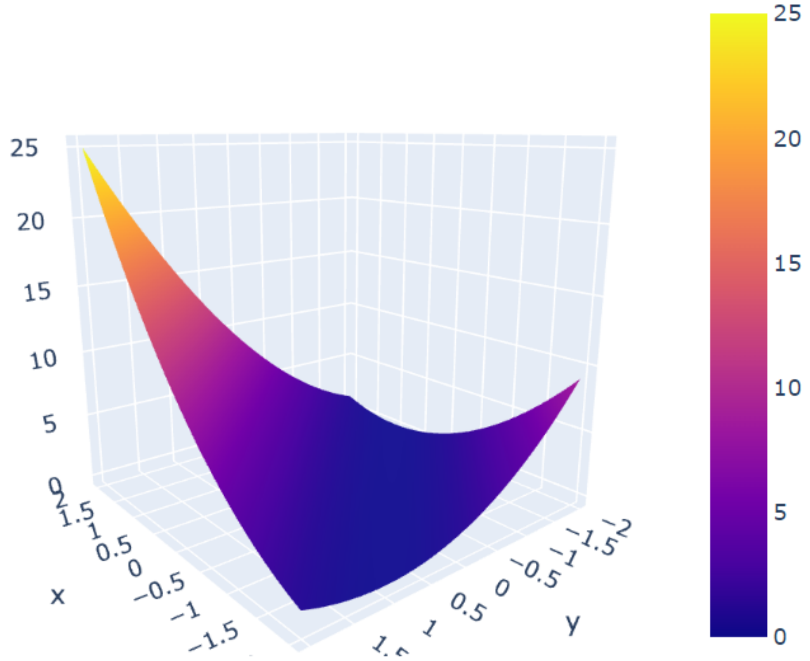


Figure 5.8: level Curve $G(\mathbf{x}_t) = k$, where all constants are set to one

NB! Figure 5.8, is purely for illustration purposes, this will not be a realistic representation of $G(\mathbf{x}_t)$.

We will be interested in obtaining a minimum of $G(\mathbf{x}_t)$, meaning that we will be interested in the following:

$$\nabla G(\mathbf{x}_t) = (\partial_{x_1} G(\mathbf{x}_t), \partial_{x_2} G(\mathbf{x}_t), \partial_{x_3} G(\mathbf{x}_t))$$

Where:

$$\partial_{x_1} G(\mathbf{x}_t) = -2q_t\alpha_t + 2\alpha_t^2 x_1(t) + 2\alpha_t \left[\beta_t x_2(t) + \gamma_t x_3(t)\right]$$

$$\partial_{x_2} G(\mathbf{x}_t) = -2q_t\beta_t + 2\beta_t^2 x_2(t) + 2\beta_t \left[\alpha_t x_1(t) + \gamma_t x_3(t)\right]$$

$$\partial_{x_3} G(\mathbf{x}_t) = -2q_t\gamma_t + 2\gamma_t^2 x_3(t) + 2\gamma_t \left[\alpha_t x_1(t) + \beta_t x_2(t)\right]$$

To verify that G obtains a minimum we need the Hessian matrix $H(G)$ of G. This will be a 3×3 matrix with entries:

$$[H(G)]_{i,j} = \frac{\partial^2 G}{\partial x_i \partial x_j}, \; i = 1, 2, 3, \; j = 1, 2, 3$$

Meaning that our Hessian matrix looks the following:

$$H(G) = \begin{bmatrix} 2\alpha_t^2 & 2\alpha_t\beta_t & 2\alpha_t\gamma_t \\ 2\alpha_t\beta_t & 2\beta_t^2 & 2\beta_t\gamma_t \\ 2\alpha_t\gamma_t & 2\beta_t\gamma_t & 2\gamma_t^2 \end{bmatrix}$$

Now as $\frac{\partial^2 G(\mathbf{x})}{\partial x_i^2} > 0$ for $i = 1, 2, 3$ we know that the minimum should be obtained by setting $\partial_{x_i} G(\mathbf{x}) = 0$, i.e we must solve:

$$\nabla G(\mathbf{x}_t) = \mathbf{0} \tag{5.11}$$

Now Equation 5.11, gives arise to the following matrix equation:

$$\underbrace{\begin{bmatrix} \alpha_t^2 & \alpha_t\beta_t & \alpha_t\gamma_t \\ \beta_t^2 & \alpha_t\beta_t & \beta_t\gamma_t \\ \gamma_t^2 & \alpha_t\gamma_t & \beta_t\gamma_t \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}}_{\mathbf{x}_t} = q \underbrace{\begin{bmatrix} \alpha_t \\ \beta_t \\ \gamma_t \end{bmatrix}}_{\mathbf{b}} \iff M\mathbf{x}_t = \mathbf{b}$$

Now:

$$\begin{aligned} \det(M) &= \alpha_t^2 \left[\alpha_t\beta_t^2\gamma_t - \alpha_t\beta_t\gamma_t^2\right] - \alpha_t\beta_t \left[\beta_t^3\gamma_t - \beta_t\gamma_t^3\right] + \alpha_t\gamma_t \left[\beta_t^2\alpha_t\gamma_t - \alpha_t\beta_t\gamma_t^2\right] \\ &= \alpha_t\beta_t\gamma_t(\beta_t - \gamma_t)\left[\alpha_t(\alpha_t + \gamma_t) - \beta_t(\beta_t + \gamma_t)\right] \end{aligned}$$

In order for $\det(M) \neq 0$, we must have:

$$\boxed{\begin{array}{l} \alpha_t \neq 0,\ \beta_t \neq 0,\ \gamma_t \neq 0 \\ \beta_t \neq \gamma_t,\ \gamma_t \neq -(\alpha_t + \beta_t) \end{array}} \tag{5.12}$$

Thus the optimal weight $\hat{\mathbf{x}}_t$ will then be:

$$\hat{\mathbf{x}}_t = M^{-1}\mathbf{b}$$

Now if the Condition 5.12 does not hold, one approach could be:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{3} x_i(t) \\ \text{subject to} & M\mathbf{x}_t = \mathbf{b} \end{array} \tag{5.13}$$

Assume that $\tilde{\mathbf{x}}_t$ is the optimal solution to 5.13, this method is very much related to Basis Pursuit [CDS98].

5.5.5 Simulation of Error distribution

To say something about how good the hedge in Proposition 5.5.1, we study:

$$\begin{aligned} ER_1(t) :&= X^{3M_A}(S,T) - \hat{a}_t^{3M} f^{3M}(t,S,T) \\ &= \frac{1}{T-S}\int_S^T r(u)du - \frac{1}{T-S}\int_S^T \mathbb{E}_Q[r(u)|\mathcal{F}_t]du \\ &= \frac{1}{T-S}\left(\int_S^T r(u)du - \int_S^T \mathbb{E}_Q[r(u)|\mathcal{F}_t]du\right) \end{aligned}$$

We will also study the Error distribution from Section 5.5.4, now this will correspond to:

$$ER_2(t) = X^{3M_A}(S,T) - \left(\hat{a}_t f^{1M}(t,S,T_{1M}) + \hat{b}_t f^{1M}(t,T_{1M},T_{2M}) + \hat{c}_t f^{1M}(t,T_{2M},T)\right)$$

We choose the Vasicek model for simulation:

$$dr(t) = \alpha[m - r(t)]dt + \sigma dW^Q(t)$$

For calculating $X^{3M_A}(S,T)$, we use the closed expression for $\int_S^T r(u)du$ as described in Equation 5.2 p.60:

$$\int_S^T r(u)du = \left(\frac{r(t)-m}{\alpha}\right)\left[e^{-\alpha(S-t)} - e^{-\alpha(T-t)}\right] + m(T-S) + \frac{\sigma}{\alpha}\int_t^T \Sigma(u,t,S,T)dW^Q(u)$$

Where:

$$\Sigma(u,t,S,T) = \left[e^{-\alpha(S-u)} - e^{-\alpha(T-u)}\right]\mathbb{1}_{[t,S)}(u) + \left[1 - e^{-\alpha(T-u)}\right]\mathbb{1}_{[S,T]}(u)$$

For simplicity we fix $t = 0$, meaning that we will study: $ER_1(0)$ and $ER_2(0)$, using the Vasicek model, where the parameters are as shown below:

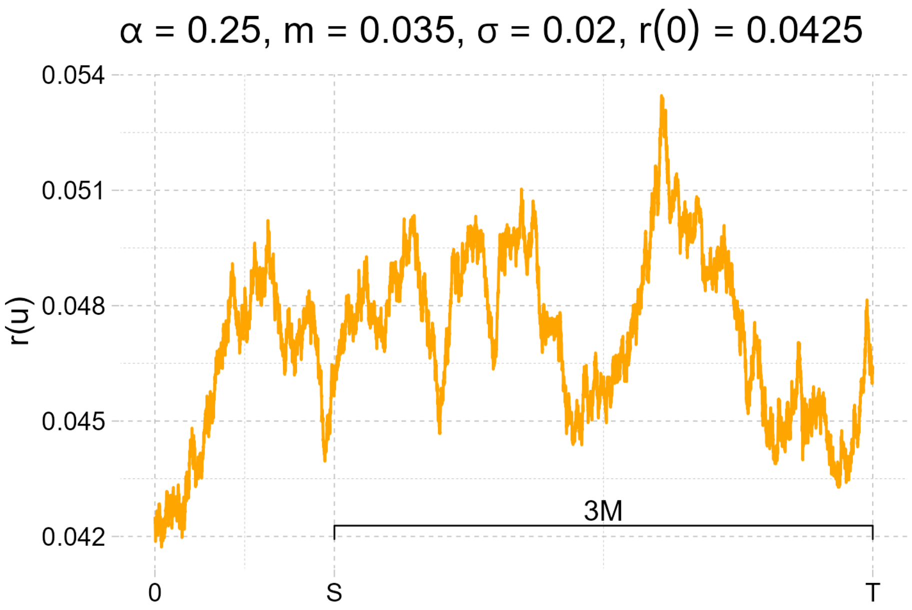


Figure 5.9: Path of Vasicek model for $u \in [0, T]$

3M-arithmetic vs $\hat{a}_0^{3M}$ 3M-SOFR futures calculations:

We have that:

$$\hat{a}_t^{3M} = \frac{\int_S^T \mathbb{E}_Q[r(u)|\mathcal{F}_t]du}{(T-S)f^{3M}(t,S,T)}$$

For calculating $f^{3M}(t,S,T)$ we have from Equation 5.4 p.60, that:

$$f^{3M}(t,S,T) = \frac{1}{T-S}\left[\exp\left(A(t,S,T) + B(t,S,T)r(t)\right) - 1\right]$$

Where:

$$A(t,S,T) = m(T-S) - \frac{m}{\alpha}\left[e^{-\alpha(S-t)} - e^{-\alpha(T-t)}\right] + \frac{1}{2}\frac{\sigma^2}{\alpha^2}\int_t^T \Sigma^2(u,t,S,T)du$$
$$B(t,S,T) = \frac{1}{\alpha}\left[e^{-\alpha(S-t)} - e^{-\alpha(T-t)}\right]$$

We are interested in the case where $t = 0$, giving us:

$$\hat{a}_0^{3M} = \frac{\int_S^T \mathbb{E}_Q[r(u)]du}{(T-S)f^{3M}(0,S,T)} = 0.995$$

3M-arithmetic vs $(\hat{a}_0, \hat{b}_0, \hat{c}_0)$ 1M-SOFR futures calculations

In this simulation we had that Condition 5.12 were met, as we got:

$$\alpha_0 = 0.0423$$
$$\beta_0 = 0.0421$$
$$\gamma_0 = 0.0420$$

Furthermore, the optimal weight $\hat{\mathbf{x}}_0$ turned out to be:

$$\hat{\mathbf{x}}_0 = \begin{bmatrix} \alpha_0^2 & \alpha_0\beta_0 & \alpha_0\gamma_0 \\ \beta_0^2 & \alpha_0\beta_0 & \beta_0\gamma_0 \\ \gamma_0^2 & \alpha_0\gamma_0 & \beta_0\gamma_0 \end{bmatrix}^{-1} \begin{bmatrix} q_0\alpha_0 \\ q_0\beta_0 \\ q_0\gamma_0 \end{bmatrix} = \begin{bmatrix} 0.501 \\ 0.501 \\ -0.004 \end{bmatrix}$$

Here:

$$q_0 = \mathbb{E}_Q[X^{3M_A}(S,T)] = 0.0421$$

3M-arithmetic vs 3M-geometric SOFR futures:

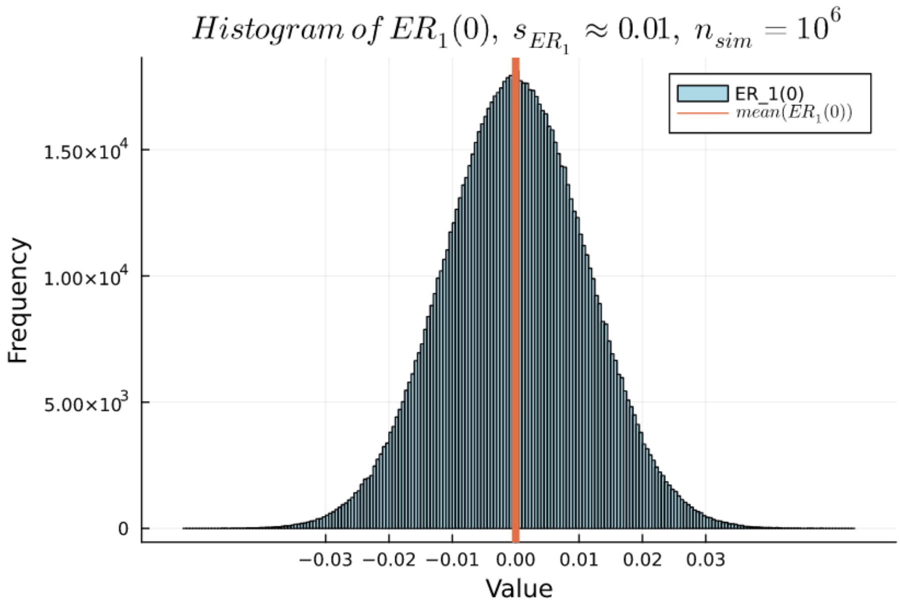


Figure 5.10: Simulation of $ER_1(0)$

Here we see the distribution $ER_1(0)$, we see that the mean of $ER_1(0)$, $\overline{ER_1(0)} = 0$, this indicates that one average hedging $X^{3M_A}(S,T)$ where one takes $\hat{a}_0 = 0.995$-positions in $3M$-SOFR futures would be a good hedge. We also have that $\mathbb{E}_Q[ER_1(0)] = 0$.

3M-arithmetic vs $(\hat{a}_0, \hat{b}_0, \hat{c}_0)$ **1M-SOFR futures**

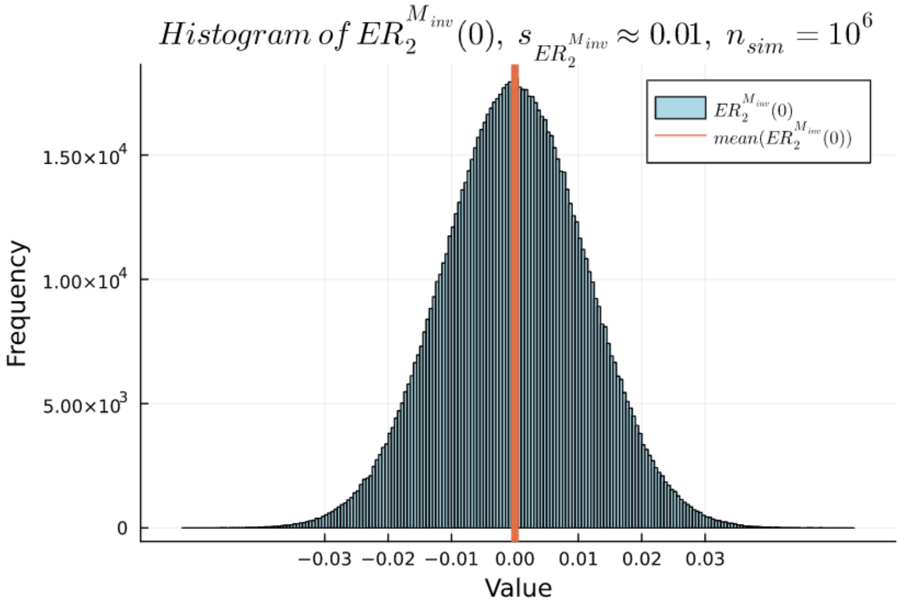


Figure 5.11: Simulation of $ER_2^{inv}(0)$

Here we see the $ER_2(0)$ distribution under the assumption that M is invertible. Here we have taken the following position in $1M$-SOFR futures:

$$(\hat{a}_0, \hat{b}_0, \hat{c}_0) = (0.501, 0.501, -0.004)$$

We see that the mean of $ER_2^{M_{inv}}(0)$, $\overline{ER_2^{M_{inv}}(0)} = 0$, which then again indicated that on average taking the above position at time $t = 0$ would be a good hedge.

To illustrate this a bit further, we can take a look at the error distribution where one naively chooses a strategy in 1M-SOFR futures, for instance, the following weighting:

$$(\hat{a}_0, \hat{b}_0, \hat{c}_0) = (0.33, -0.33, 0.33)$$

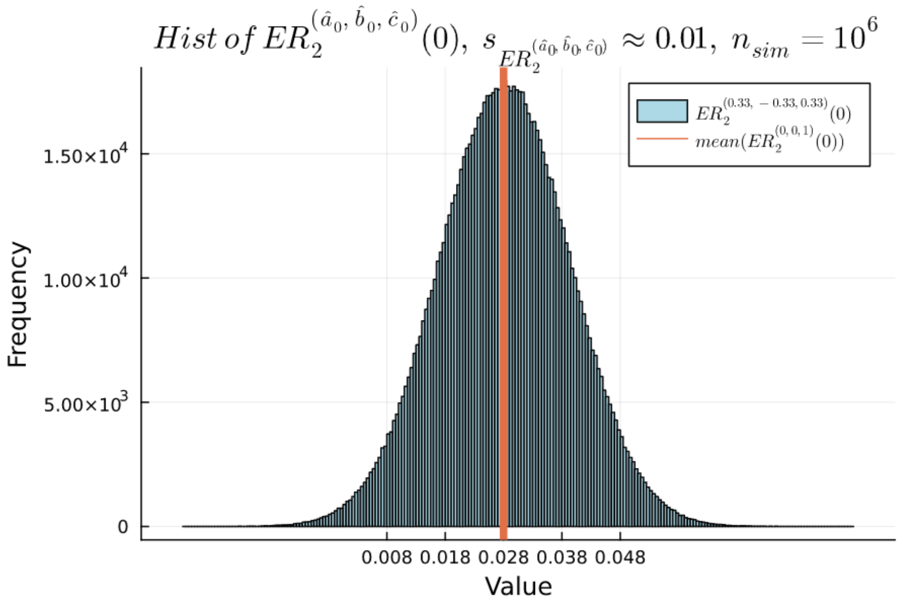


Figure 5.12: Simulation of $ER_2^{(\hat{a}_0,\hat{b}_0,\hat{c}_0)}(0)$

Now, the weights that we have calculated are weighing in SOFR futures rates, not the actual position one would take at CME, from earlier we remember that SOFR futures at CME are quoted the following way (modified to our example with decimals):

$$Q^{\ell M}(t, S, T) = 1 - f^{\ell M}(t, S, T),\ \ell = 1, 3$$

To get more intuition about the weighing in SOFR futures, we construct an example:

Example 5.5.3. Assume that we have a loan of 30 million over a 3M period. From the contract, it is agreed upon that the rate we will pay is the floating 3M-arithmetic rate $X^{3M_A}(S, T)$ plus an additional risk-premium of 200 bp (basis points).

We want to hedge against rising interest rates, so if $f^{\ell M}$ increases, then $Q^{\ell M}$ decreases.

Case 1: Hedge by taking position in 3M-SOFR futures. From previous calculations, we got $\hat{a}_0^{3M} = 0.995$. This will correspond to the position in the 3M SOFR futures rate. Now a 3M SOFR futures are based upon a notional of 1 million dollars. Meaning that we would need 30 contracts to cover our loan amount. We can not take a fractional position, meaning that we, in this case, would take one position in the futures rate. Since we want to hedge against rising interest rates, we would take 30 short positions here.

Case 2: Hedge by taking position in 1M-SOFR futures. From our simulation we got $(\hat{a}_0, \hat{b}_0, \hat{c}_0) = (0.501, 0.501, -0.004) \approx (0.5, 0.5, 0.0)$. Now 1M-SOFR futures has a notional of 5 million dollars. So to cover our loan, we would need six contracts. These would be covered by taking three positions in the first futures contract, three in the middle futures contract, and zero in the last. The type of position would be a short position.

We see that for all our simulations, the error distributions seem to be normally distributed. This is not surprising as in each simulation, and we take a normal random variable: $\int_S^T r(u)du$ as described in Equation 5.2 p.60, and subtract a constant/linear combination of constants. This is a new, normally distributed random variable with an altered mean. We can also address the normality via a QQplot:

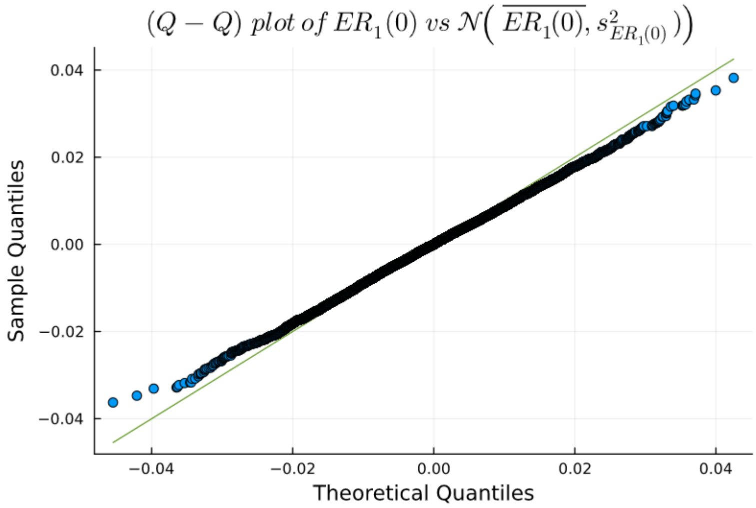


Figure 5.13: (Q-Q) plot of $ER_1(0)$

Chapter 6

ESG swaps

6.1 ESG

ESG stands for Environmental, Social, and Governance. ESG metrics are becoming more critical for the financial sector in the European Union. The European Union have established a new taxonomy for sustainable activities [21a].

This taxonomy is implemented so the EU can be carbon neutral by 2050. The goal is to help investors and companies contribute to the Paris Climate Agreement. It will enable companies and issuers to access financing consistent with these goals.

One such contribution could be to access cheaper financing, for instance, by lowering the fixed swap rate if certain ESG criteria are met. In our case, this will be when a firm manages to get below a certain ESG risk score. We take a look at what the different letters could represent:

- Environmental: Carbon emissions, usage of renewable energy, waste generation etc.
- Social: workforce rights, human-rights policies, customer satisfaction etc.
- Governance: management structure and diversity, CEO pay ratio, crisis management, transparency, codes of conduct etc.

We will not go into detail regarding the ESG risk score, however, we propose a model and highlight some implications.

6.2 Case study

Our modelling approach will be based on the ISDA document [21b], particularly the interest rate swap between SBM Offshore and ING, where Sustainalytics set the ESG risk score. The deal has the following specifications:

- It gets added a positive or negative spread to the fixed rate set at the inception of the swap, based upon SBM's ESG score set by Sustainalytics.
- At the beginning of every year during the contract's life, ING sets a target ESG score.
- If the score has been met, a discount of 5-10 basis points is applied to the fixed rate.
- If the score has not been met, a 5-10 basis points penalty is applied to the fixed rate.

We consider the ESG criteria to be $\mathcal{F}_0$-measurable. We will also take a constant discount d. To exemplify this deal pretend that the original fixed rate was set at $\kappa = 0.07$, with a discount/penalty $d = 0.005$ and the contract length is $n = 4$ years.

Assume that SBM Offshore met the criteria the first three times but did not meet the last time. That would give rise to the following ESG fixed rate sequence:

$$K_1^{ESG} = 0.065, \quad K_2^{ESG} = 0.060, \quad K_3^{ESG} = 0.055, \quad K_4^{ESG} = 0.060$$

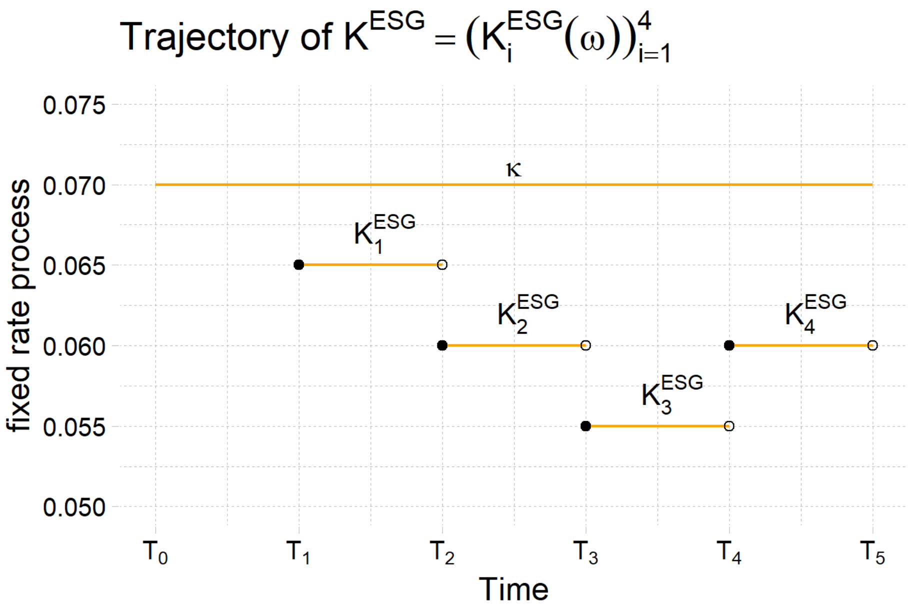


Figure 6.1: ESG-fixed rate trajectory

6.3 General setup

We will take this deal as motivation but generalize a bit further:

Assumption 6.3.1. We assume the following:

- N represents the nominal value. Think of it as the amount you loan/lend.
- $0 < T_0 < T_1 < \cdots < T_n$ a sequence of future dates.
- $\delta = T_i - T_{i-1}$ a fixed leg between payments
- κ the fixed rate from the original swap, i.e. without ESG-link.
- d basis points added or subtracted to the fixed rate κ
- $\{A_i\}_{i=1}^n$ sequence of events, where: $A_i = \{X_{T_i} \leq C_{T_i}^{ESG}\}$ i.e. the sequence of events measuring if the ESG-risk score at time T_i: X_{T_i}, is below the ESG-criteria C_{T_i} or not.

Definition 6.3.2 (**ESG fixed rate process**). Let $K^{ESG} = (K_i^{ESG}(\omega))_{i=1}^n$ denote the ESG fixed rate process, we define it recursively as:

$$K_i^{ESG}(\omega) = (K_{i-1}^{ESG}(\omega) - d)\mathbb{1}_{A_i}(\omega) + (K_{i-1}^{ESG}(\omega) + d)\mathbb{1}_{A_i^C}(\omega), \quad i \geq 2$$

Where:

$$K_1^{ESG}(\omega) = (\kappa - d)\mathbb{1}_{A_1}(\omega) + (\kappa + d)\mathbb{1}_{A_1^C}(\omega)$$

Notation 6.3.3. Let $\mathcal{I} = \{k_1, \ldots, k_n\}$ represent an index set. Furthermore, let $k_1 < \cdots < k_l < \cdots < k_m < \cdots < k_n$ We then define:

$$\left(\bigcap_{i\in\mathcal{I}} A_i\right)^{\{(k_1,k_l,k_m)\}} = A_{k_1}^C \cap A_{k_2} \cap \cdots \cap A_{k_l}^C \cap A_{k_{l+1}} \cap \cdots \cap A_{k_m}^C \cap A_{k_{m+1}} \cap \cdots \cap A_{k_n}$$

Result 6.3.4. Let $n \in \mathbb{N}_2 := \{k : k \geq 2, k \in \mathbb{N}\}$, consider the above situation. Let $\mathcal{I}_n = \{1, \ldots, n\}$ and $\mathcal{I}_{2n}^{Even} = \{2, \ldots, 2n\}$, $(A_i)_{i\in\mathcal{I}}$ denotes the event that the ESG criteria are met. Let $j_1 < j_2 < \cdots < j_{|\mathcal{I}_\alpha^{Even}|} \in \mathbb{N}$ furthermore $|\mathcal{I}_{2n}^{Even}|$ and $|\mathcal{I}_n|$ denotes the cardinality of the respective sets.

We can then express $K_n^{ESG}(\omega)$ as:

$$K_n^{ESG}(\omega) = [\kappa - dn]\mathbb{1}\left[\bigcap_{i\in\mathcal{I}_n} A_i\right](\omega) + \sum_{\alpha\in\mathcal{I}_{2n}^{Even}} \left([\kappa - d(n-\alpha)]\mathbb{1}\left[\bigcup_{j_1\neq\ldots\neq j_{|\mathcal{I}_\alpha^{Even}|}\in\mathcal{I}_n} \left(\bigcap_{i\in\mathcal{I}_n} A_i\right)^{\{(j_1,\ldots,j_{|\mathcal{I}_\alpha^{Even}|})\}}\right]\right)(\omega)$$

To get a better grasp of Result 6.3.4, we include the following graph:

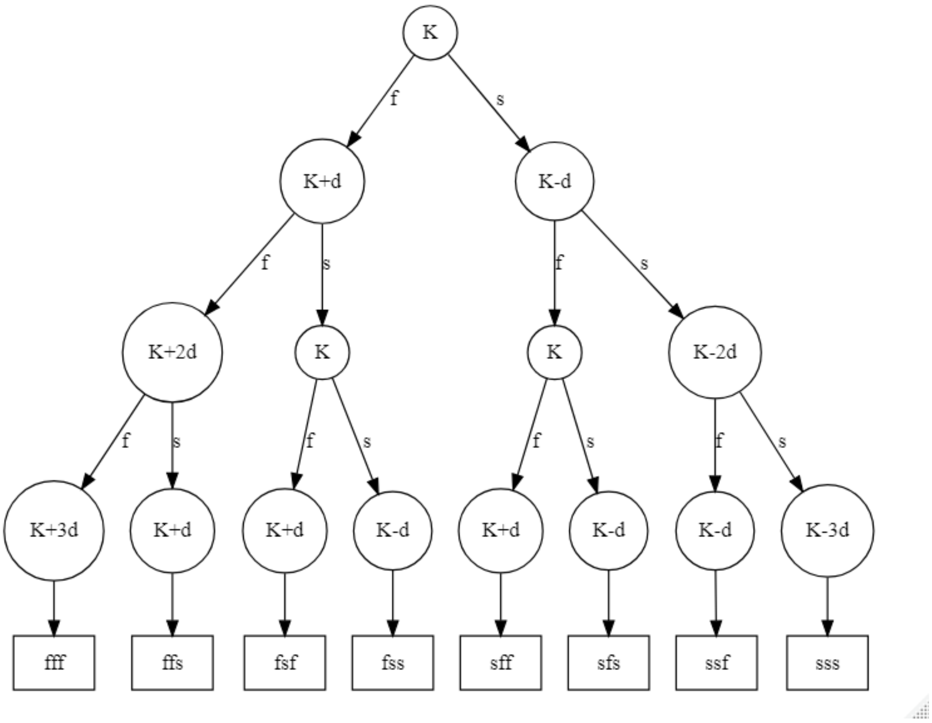


Figure 6.2: Possible outcomes of $K_n^{ESG}(\omega)$

For a general n, we have 2^n possible paths, and the unique values will be

$$\kappa + id, i \in \{n - 2k : k = 0, \ldots, n\}$$

We will have $|\{n - 2k : k = 0, \ldots, n\}| = n + 1$ unique values. With $\binom{n}{|i|}$ number of possible paths leading to the value $\kappa + id$.

Thus for $n = 3$, we have $2^3 = 8$ possible paths, and $3 + 1 = 4$ unique values:

$$\kappa + 3d,\ \kappa + d,\ \kappa - d\ \kappa - 3d$$

We see that there are $\binom{3}{1} = 3$ different paths yielding a result of $\kappa + d$, which in this case are: ffs, fsf and sff.

$$
\begin{aligned}
K_3^{ESG}(\omega) = {} & [\kappa - 3d]\mathbb{1}\left[\bigcap_{i\in\mathcal{I}_3} A_i\right](\omega) \\
& + \sum_{\alpha\in\{2,4,6\}}\left([\kappa - d(3-\alpha)]\mathbb{1}\left[\bigcup_{\{j_1\neq\ldots\neq j_{|\mathcal{I}_\alpha^{Even}|}\}\in\{1,2,3\}}\left(\bigcap_{i=1}^{3} A_i\right)^{(j_1,\ldots,j_{|\mathcal{I}_\alpha^{Even}|})}\right]\right)(\omega)
\end{aligned}
$$

The number of unique possible outcomes will be $|\{2,4,6\}|+1 = 4$. If we fix α, say, $\alpha = 4$, we get the value $\kappa + d$, and there are 3 paths leading to this result. Now for $j_1 < j_2$, we see that $j_1, j_2 \in \{1,2,3\}$, leads to the following set-combinations:

$$
\{1,2\},\ \{1,3\} \text{ and } \{2,3\}
$$

Namely 3 sets, where we got the index of where the criteria were not met. And this corresponds to the following expression:

$$
\begin{aligned}
& [\kappa - d(3-4)]\mathbb{1}\left[\bigcup_{j_1\neq j_2\in\{1,2,3\}}\left[\bigcap_{i=1}^{3} A_i\right]^{\{(j_1,j_2)\}}\right](\omega) \\
= & [\kappa + d]\left[\underbrace{\mathbb{1}(A_1^C\cap A_2^C\cap A_3)(\omega)}_{=ffs} + \underbrace{\mathbb{1}(A_1^C\cap A_2\cap A_3^C)(\omega)}_{=fsf} + \underbrace{\mathbb{1}(A_1\cap A_2^C\cap A_3^C)(\omega)}_{=sff}\right]
\end{aligned}
$$

This shows that the expression provided in Result 6.3.4 is reasonable and behaves as one would like.

Proposition 6.3.5. *Let us denote* $\kappa_t^{ESG}(i) := \mathbb{E}_Q[K_i^{ESG}(\omega)|\mathcal{F}_t]$. *We then have that for* $t \le T_0$ *that:*

$$\kappa_t^{ESG}(i) = \kappa - d \cdot D(i)$$

Where:

$$D(i) = i \cdot \mathbb{E}_Q\left[\prod_{l=1}^{i} \mathbb{1}(A_l)\middle|\mathcal{F}_t\right] + \sum_{\alpha \in \mathcal{I}_{2i}^{Even}} [i-\alpha] \sum_{j_1 \neq \ldots \neq j_{|\mathcal{I}_\alpha^{Even}|}} \mathbb{E}_Q\left[\left(\prod_{l=1}^{i} \mathbb{1}(A_l)\right)^{\{(j_1,\ldots,j_{|\mathcal{I}_\alpha^{Even}|})\}}\middle|\mathcal{F}_t\right]$$

Proof. $K_i^{ESG}(\omega)$ is as described in Result 6.3.4:

$$\begin{aligned}\mathbb{E}_Q\left[K_i^{ESG}(\omega)\middle|\mathcal{F}_t\right] &= [\kappa - d\cdot i]\mathbb{E}_Q\left[\mathbb{1}\left(\bigcap_{l=1}^{i} A_l\right)(\omega)\middle|\mathcal{F}_t\right] \\ &+ \sum_{\alpha \in \mathcal{I}_{2i}^{Even}}\left([\kappa - d(i-\alpha)]\mathbb{E}_Q\left[\mathbb{1}\left[\bigcup_{\{j_1\neq\ldots\neq j_{|\mathcal{I}_\alpha^{Even}|}\}}\left(\bigcap_{l=1}^{i} A_l\right)^{\{(j_1,\ldots,j_{|\mathcal{I}_\alpha^{Even}|})\}}\right](\omega)\middle|\mathcal{F}_t\right]\right)\end{aligned}$$

Now:

$$\begin{aligned}&\mathbb{E}_Q\left[\mathbb{1}\left[\bigcup_{\{j_1\neq\ldots\neq j_{|\mathcal{I}_\alpha^{Even}|}\}}\left(\bigcap_{l=1}^{i} A_l\right)^{\{(j_1,\ldots,j_{|\mathcal{I}_\alpha^{Even}|})\}}\right](\omega)\middle|\mathcal{F}_t\right] \\ &= \sum_{j_1\neq\ldots\neq j_{|\mathcal{I}_\alpha^{Even}|}} \mathbb{E}_Q\left[\mathbb{1}\left(\bigcap_{l=1}^{i} A_l\right)^{\{(j_1,\ldots,j_{|\mathcal{I}_\alpha^{Even}|})\}}(\omega)\middle|\mathcal{F}_t\right]\end{aligned}$$

Furthermore:

$$\mathbb{E}_Q\left[\mathbb{1}\left(\bigcap_{l=1}^{i} A_l\right)(\omega)\middle|\mathcal{F}_t\right] = \underbrace{\mathbb{E}_Q\left[\prod_{l=1}^{i}\mathbb{1}(A_l)\middle|\mathcal{F}_t\right]}_{=s(l=1,i)}$$

and:

$$\mathbb{E}_Q\left[\mathbb{1}\left(\bigcap_{l=1}^{i} A_l\right)^{\{(j_1,\ldots,j_{|\mathcal{I}_\alpha^{Even}|})\}}(\omega)\middle|\mathcal{F}_t\right] = \underbrace{\mathbb{E}_Q\left[\left(\prod_{l=1}^{i}\mathbb{1}(A_l)\right)^{\{(j_1,\ldots,j_{|\mathcal{I}_\alpha^{Even}|})\}}(\omega)\middle|\mathcal{F}_t\right]}_{=f(l=1,i,j_{|\mathcal{I}_\alpha|})}$$

$$\begin{aligned}\mathbb{E}_Q[K_i^{ESG}(\omega)|\mathcal{F}_t] &= [\kappa - d\cdot i]s(l=1,i) \\ &+ \kappa \sum_{\alpha\in\mathcal{I}_{2i}^{Even}}\left(\sum_{j_1\neq\ldots\neq j_{|\mathcal{I}_\alpha^{Even}|}} f(l=1,i,j_{|\mathcal{I}_\alpha^{Even}|})\right) \\ &- d\sum_{\alpha\in\mathcal{I}_{2i}^{Even}}[i-\alpha]\left(\sum_{j_1\neq\ldots\neq j_{|\mathcal{I}_\alpha^{Even}|}} f(l=1,i,j_{|\mathcal{I}_\alpha^{Even}|})\right)\end{aligned}$$

We collect κ and d-terms:

$$\mathbb{E}_Q[K_i^{ESG}(\omega)|\mathcal{F}_t] = \kappa \underbrace{\left[s(l=1,i) + \sum_{\alpha\in\mathcal{I}_{2i}^{Even}} \sum_{j_1\neq\ldots\neq j_{|\mathcal{I}_\alpha^{Even}|}} f(l=1,i,j_{|\mathcal{I}_\alpha^{Even}|}) \right]}_{=M(i)}$$
$$- d \underbrace{\left[i\cdot s(l=1,i) + \sum_{\alpha\in\mathcal{I}_{2i}^{Even}} \left([i-\alpha] \sum_{j_1\neq\ldots\neq j_{|\mathcal{I}_\alpha^{Even}|}} f(l=1,i,j_{|\mathcal{I}_\alpha^{Even}|}) \right) \right]}_{=D(i)}$$
$$= \kappa M(i) - d\cdot D(i)$$

Where:

$$D(i) = i\cdot \mathbb{E}_Q\left[\prod_{l=1}^{i} \mathbb{1}(A_l) \middle| \mathcal{F}_t \right] + \sum_{\alpha\in\mathcal{I}_{2i}^{Even}} [i-\alpha] \sum_{j_1\neq\ldots\neq j_{|\mathcal{I}_\alpha^{Even}|}} \mathbb{E}_Q\left[\left(\prod_{l=1}^{i} \mathbb{1}(A_l) \right)^{\{(j_1,\ldots,j_{|\mathcal{I}_\alpha^{Even}|})\}} \middle| \mathcal{F}_t \right]$$

and:

$$M(i) = \mathbb{E}_Q\left[\prod_{l=1}^{i} \mathbb{1}(A_l) \middle| \mathcal{F}_t \right] + \sum_{\alpha\in\mathcal{I}_{2i}^{Even}} \sum_{j_1\neq\ldots\neq j_{|\mathcal{I}_\alpha^{Even}|}} \mathbb{E}_Q\left[\left(\prod_{l=1}^{i} \mathbb{1}(A_l) \right)^{\{(j_1,\ldots,j_{|\mathcal{I}_\alpha^{Even}|})\}} \middle| \mathcal{F}_t \right]$$

Let's rewrite $M(i)$ on set notation again:

$$M(i) = \mathbb{E}_Q\left[\mathbb{1}\left(\bigcap_{l\in\mathcal{I}_i} A_l \right) \middle| \mathcal{F}_t \right] + \sum_{\alpha\in\mathcal{I}_{2i}^{Even}} \mathbb{E}_Q\left[\mathbb{1}\left[\bigcup_{j_1\neq\ldots\neq j_{|\mathcal{I}_\alpha^{Even}|}} \left(\bigcap_{l\in\mathcal{I}_i} A_l \right)^{\{(j_1,\ldots,j_{|\mathcal{I}_\alpha^{Even}|})\}} \right] \middle| \mathcal{F}_t \right]$$

For $i = 1,\ldots,n$, we have:

$$\Omega_i = \underbrace{\left[\sum_{\alpha\in\mathcal{I}_{2i}^{Even}} \bigcup_{j_1\neq\ldots\neq j_{|\mathcal{I}_\alpha^{Even}|}} \left(\bigcap_{l\in\mathcal{I}_i} A_l \right)^{\{(j_1,\ldots,j_{|\mathcal{I}_\alpha^{Even}|})\}} \right]}_{=\mathcal{H}_i} \bigcup \underbrace{\left[\bigcap_{l\in\mathcal{I}_i} A_l \right]}_{=\mathcal{L}_i} \quad \text{with} \quad Q(\Omega_i) = 1$$

Now as $\mathcal{H}_i \cap \mathcal{L}_i = \emptyset$, and $\mathcal{H}_i \cup \mathcal{L}_i = \Omega_i$, in addition to exploiting the linearity of the expectation operator we get:

$$M(i) = \mathbb{E}_Q\left[\mathbb{1}\left[\left[\sum_{\alpha\in\mathcal{I}_{2i}^{Even}} \bigcup_{j_1\neq\ldots\neq j_{|\mathcal{I}_\alpha^{Even}|}} \left(\bigcap_{l\in\mathcal{I}_i} A_l \right)^{\{(j_1,\ldots,j_{|\mathcal{I}_\alpha^{Even}|})\}} \right] \bigcup \left[\bigcap_{l\in\mathcal{I}_i} A_l \right] \right] \middle| \mathcal{F}_t \right]$$
$$= \mathbb{E}_Q\left[\mathbb{1}(\Omega_i) | \mathcal{F}_t \right]$$
$$= 1$$

Leaving us with:

$$\mathbb{E}_Q[K_i^{ESG}(\omega)|\mathcal{F}_t] = \kappa - d\cdot D(i) := \kappa_t^{ESG}(i)$$

■

6.4 Zero Coupon Bond ESG-swap

Proposition 6.4.1. *Consider a zero coupon bond swap, i.e where the situation is as described in Section Section 4.2, we then get that the ESG-swap rate process* $\kappa_t^{ESG} = (\kappa_t^{ESG}(i))_{i=1,\dots,n}$ *is given by:*

$$\kappa_t^{ESG}(i) = \kappa_t^{ZCB} - d \cdot D(i)$$

Where:

$$\kappa_t^{ZCB} = \frac{P(t,T_0) - P(t,T_n)}{\delta \sum_{i=1}^{n} P(t,T_i)}$$

And $D(i)$ *is as described in Proposition 6.3.5*

Proof. Now from Section Section 4.2 we have that:

$$\kappa_t^{ZCB} = \frac{P(t,T_0) - P(t,T_n)}{\delta \sum_{i=1}^{n} P(t,T_i)}$$

Thus:

$$\kappa_t^{ESG}(i) = \kappa_t^{ZCB} - d \cdot D(i)$$

■

Chapter 7

Numerical Simulation

7.1 Introduction

To grasp the ESG-swap rate process κ_t^{ESG}, we need a model for the ESG-risk score process of the company. It could be logical with a downward trending score since the company will be incentivised to enter such an agreement.

There are many possible alternatives to model such a model, so we impose the following model for the **ESG-risk score**:

$$X(t) = 100\exp(-Z(t))$$

Here $Z(t)$ is an OU-process given by:

$$dZ(t) = -\beta Z(t)dt + \sigma dW^Q(t) + dI^Q(t)$$

Where:

$$I^Q(t) = \sum_{k=1}^{N(t)} J_k, \quad J_k \sim Exp(\mu), \quad N(t) \sim Pois(\lambda t)$$

Furthermore I^Q and W^Q are assumed to be independent, now for $J \sim Exp(\mu)$, we have:

$$f_J(x) = \mu e^{-\mu x}\mathbb{1}_{[0,\infty)}(x), \quad \mathbb{E}[J] = \frac{1}{\mu}, \text{ and } Var[J] = \frac{1}{\mu^2}$$

An explicit solution is given by:

$$\begin{aligned}
d[e^{\beta t}Z(t)] &= d[e^{\beta t}]Z(t) + e^{\beta t}dZ(t) \\
&= \beta e^{\beta t}Z(t)dt + e^{\beta t}[-\beta Z(t)dt + \sigma dW^Q(t) + dI^Q(t)] \\
&= \sigma e^{\beta t}dW^Q(t) + e^{\beta t}dI^Q(t) \\
&\Downarrow \\
Z(t) &= Z(0)e^{-\beta t} + \int_0^t e^{-\beta(t-s)}dW^Q(s) + \int_0^t e^{-\beta(t-s)}dI^Q(s) \qquad (7.1)
\end{aligned}$$

Proposition 7.1.1 (**Characteristic function of** $Z(t)$). *The characteristic function of* $Z(t)$ *is given by:*

$$\mathbb{E}_Q\left[\exp(iuZ(t))\right] = \exp\left(iuZ(0)e^{-\beta t}\right)\left(-\frac{u^2}{4\beta}[1-e^{-2\beta t}]\right)\left(\frac{1-iue^{-\beta t}1/\mu}{1-iu1/\mu}\right)^{\frac{\lambda}{\beta}}$$

Proof. Since I^Q and W^Q are independent we have that:

$$\mathbb{E}_Q[e^{iuZ(t)}] = \exp\left(iuZ(0)e^{-\beta t}\right)\mathbb{E}_Q\left[\exp\left(iu\int_0^t e^{-\beta(t-s)}dW^Q(s)\right)\right]\mathbb{E}_Q\left[\exp\left(iu\int_0^t e^{-\beta(t-s)}dI^Q(s)\right)\right]$$

The normality of deterministic Ito-integrals gives us the following:

$$iu\int_0^t e^{-\beta(t-s)}dW^Q(s) \sim \mathcal{N}\left(0, -u^2\int_0^t e^{-2\beta(t-s)}ds\right)$$
$$\Downarrow$$
$$\mathbb{E}_Q\left[\exp\left(iu\int_0^t e^{-\beta(t-s)}dW^Q(s)\right)\right] = \exp\left(-\frac{u^2}{4\beta}[1-e^{-2\beta t}]\right)$$

From Proposition 2.4.11 p.19, we have:

$$\mathbb{E}\left[\exp\left(iu\int_0^t e^{-\beta(t-s)}dI(s)\right)\right] = \exp\left(\int_0^t \Psi(ue^{-\beta s})ds\right)$$

To ease some notation, we write:

$$\Psi(x) = \lambda(\varphi_F(x)-1), \text{ where: } \varphi_F(x) = \int_{\mathbb{R}} e^{iyx}F_J(dy)$$

We start with calculating $\varphi_F(x)$ with $F_J(dy) = \mu e^{-\mu y}\mathbb{1}_{[0,\infty)}(y)dy$:

$$\varphi_F(x) = \int_0^\infty e^{ixy}\mu e^{-\mu y}dy = \frac{1}{1-ix\frac{1}{\mu}}$$

Giving us:

$$\varphi_F(x) - 1 = \frac{1}{1-ix\frac{1}{\mu}} - \frac{1-ix\frac{1}{\mu}}{1-ix\frac{1}{\mu}} = \frac{ix\frac{1}{\mu}}{1-ix\frac{1}{\mu}}$$

Now:

$$\begin{aligned}\Psi(ue^{-\beta s}) &= \lambda(\varphi_F(ue^{-\beta s}))-1) \\ &= \lambda\left(\frac{iue^{-\beta s}1/\mu}{1-iue^{\beta s}1/\mu}\right)\cdot\frac{\beta}{\beta} \\ &= \frac{\lambda}{\beta}\left(\frac{\beta iue^{-\beta s}1/\mu}{1-iue^{\beta s}1/\mu}\right)\end{aligned}$$

We observe:

$$h(s) := \ln[1 - iue^{-\beta s}1/\mu]$$
$$\Downarrow$$
$$h'(s) = \frac{\beta iue^{-\beta s}1/\mu}{1 - iue^{\beta s}1/\mu}$$

Leaving us with:

$$\int_0^t \Psi(ue^{-\beta s})ds = \frac{\lambda}{\beta}\int_0^t h'(s)ds = \frac{\lambda}{\beta}[h(t) - h(0)] = \frac{\lambda}{\beta}\ln\left[\frac{1 - iue^{-\beta t}1/\mu}{1 - iu1/\mu}\right]$$

■

By using Proposition 7.1.1 we can find the expectation of $X(t)$:

$$\begin{aligned}\mathbb{E}_Q[X(t)] &= \mathbb{E}_Q\left[e^{i(-i)Z(t)}\right] \\ &= \exp\left(Z(0)e^{-\beta t}\right)\exp\left(\frac{1}{4\beta}[1 - e^{-2\beta t}]\right)\left(\frac{1 + e^{-\beta t}1/\mu}{1 + 1/\mu}\right)\end{aligned}$$

Now if $\beta > 0$, one can find:

$$\lim_{t\to\infty}\mathbb{E}_Q[X(t)] = \exp\left(\frac{1}{4\beta} + \frac{1}{1 + 1/\mu}\right)$$

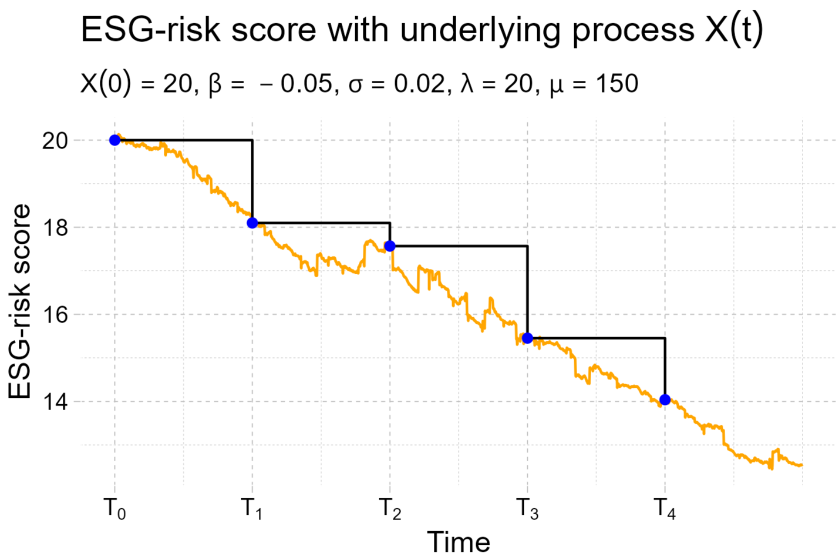


Figure 7.1: ESG-risk score with underlying process $X(t)$

The blue dots represent the observed ESG-risk score at the relevant observation times. The solid dark line represents the ESG-risk score between observation times, and the underlying orange process describes our continuous time ESG-risk score process $X(t)$.

7.2 Simulation of Zero Coupon Bond ESG-swap

In this section, we will look at a numerical simulation of the ESG-swap rate process $\kappa_t^{ESG} = (\kappa_t^{ESG}(i))_{i=1,\dots,4}$. We will simulate different scenarios. These will be when the ESG-criteria C^{ESG} is reasonable and when the ESG criteria are unreasonable.

By unreasonable, we will look at the extremes, i.e. where the criteria are met all the time and where the criteria are never satisfied. This will show the effect of the discount and penalty, respectively.

After analysis, one has concluded that the model described in Figure 7.1 is a good fit for the counterparty company in this ESG swap.

Parameters

- $X(0) = 20 \implies Z(0) = -\ln\left(\frac{20}{100}\right)$
- $\beta = -0.05$
- $\sigma = 0.02$
- $\lambda = 20$
- $\mu = 150$

We will use a stepsize of $dt = \frac{1}{360}$, and since the calculation is based upon Monte Carlo simulations, we will establish the analysis on 1 Million simulations.

Agreement/specifications

- For simplicity, we will assume that our ESG criteria process $C^{ESG} = (C_{T_i}^{ESG})_{\{i=1,\dots,4\}}$ to be $\mathcal{F}_0$-measurable.
- $\delta = 1$ meaning that the time between observations times T_i and T_{i-1} is one year.
- We assume that the penalty/discount $d = 0.005$.
- We will also, for simplicity, set:

$$\kappa_t^{ZCB} = \frac{P(t,T_0) - P(t,T_4)}{\delta \sum_{i=1}^{4} P(t,T_i)} = 0.07$$

Reasonable criteria

- $C^{ESG} = (17.8, 16.8, 15.8, 14.8)$

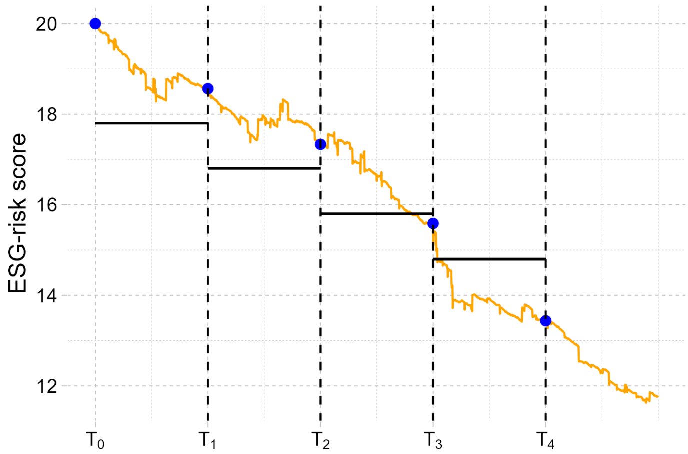


Figure 7.2: ESG-risk score where ESG-criteria is reasonable

In this figure, we see the underlying ESG risk score process. Here the dark-solid lines represent the criteria $C_{T_i}^{ESG}$. The blue dots represent the ESG-risk score at time T_i. In this particular realization, we see that the criteria are not met at T_1 and T_2, and then met at T_3 and T_4.

After 1 Million simulations, we got:

	$C_{T_i}^{ESG}$	κ_t^{ZCB}	κ_t^{ESG}
T_1	17.8	0.070	0.071
T_2	16.8	0.070	0.070
T_3	15.8	0.070	0.068
T_4	14.8	0.070	0.065

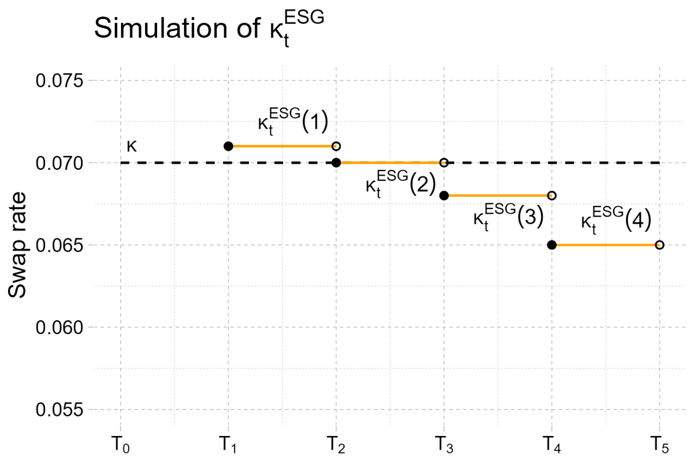


Figure 7.3: ESG-swap rate when ESG-criteria is reasonable

unreasonable criteria, where criteria are always met

- $C^{ESG} = (24, 23, 22, 21)$

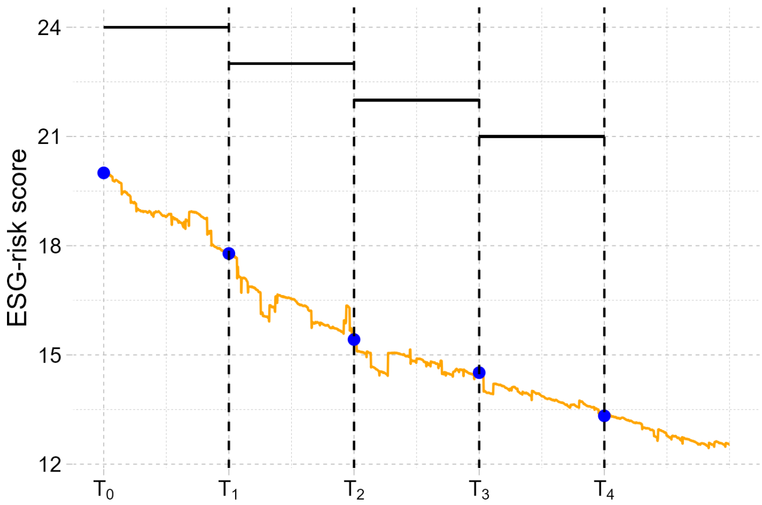


Figure 7.4: ESG-risk score where ESG-criteria is unfavourable for lender

The blue dots are always under the dark solid lines, meaning the criteria are always met.

After 1 Million simulations, we got:

	$C^{ESG}_{T_i}$	κ^{ZCB}_t	κ^{ESG}_t
T_1	24	0.070	0.065
T_2	23	0.070	0.060
T_3	22	0.070	0.055
T_4	21	0.070	0.050

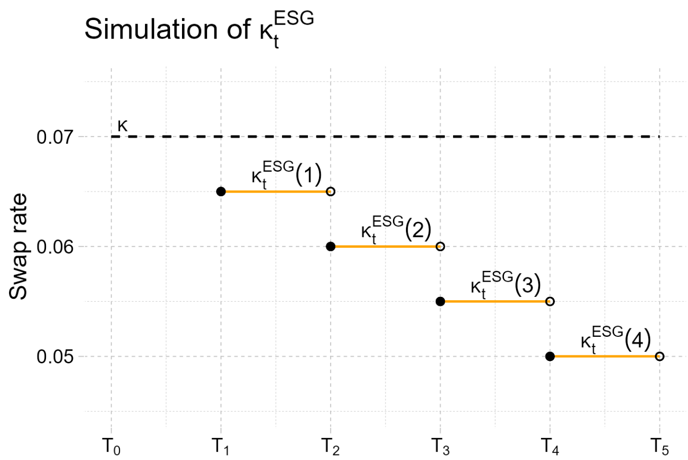


Figure 7.5: ESG-swap rate, when ESG-criteria is not reasonable for the lender

Here we see the effect of how the discount d works. For each T_i, we see that κ^{ESG}_t goes down by the discount d.

unreasonable criteria, where criteria are never met

- $C^{ESG} = (15, 14, 13, 12)$

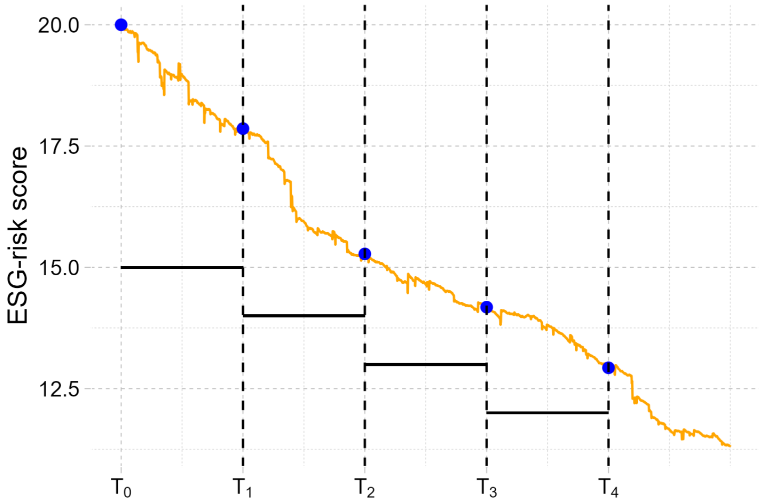


Figure 7.6: ESG-risk score where ESG-criteria is unfavourable for borrower

The blue dots are consistently above the criteria, meaning that the criteria will not be met.

After 1 Million simulations, we got:

	$C_{T_i}^{ESG}$	κ_t^{ZCB}	κ_t^{ESG}
T_1	24	0.070	0.075
T_2	23	0.070	0.080
T_3	22	0.070	0.085
T_4	21	0.070	0.090

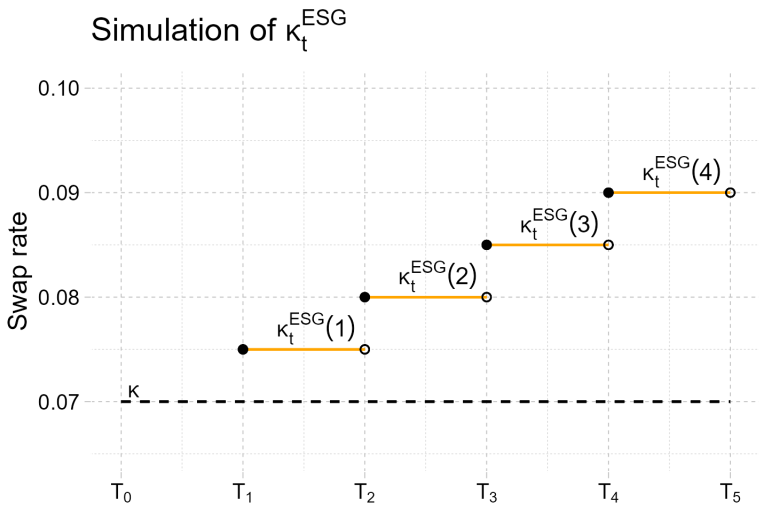


Figure 7.7: ESG-swap rate, when ESG-criteria is not reasonable for the borrower

In this situation, d works like a penalty, and correspondingly κ_t^{ESG} increases.

Chapter 8

Conclusion and further work

8.1 SOFR

In the SOFR studies, we examined the implications of different underlying calculation methods for the SOFR futures. We studied a 3M-arithmetic interest rate:

$$X^{3M_A}(S,T) = \frac{1}{T-S}\int_S^T r_u du$$

We bench marked taking a $\hat{a}_t - f^{3M}(t,S,T)$ 3M-SOFR futures position against taking a $(\hat{a}_t, \hat{b}_t, \hat{c}_t) - f^{1M}(t,S,T)$ 1M-SOFR futures position.

For simulation purposes, we chose the following model for the integral of $r = (r(t))_{t\geq 0}$:

$$\int_S^T r(u)du = \left(\frac{r(t)-m}{\alpha}\right)\left[e^{-\alpha(S-t)} - e^{-\alpha(T-t)}\right] + m(T-S) + \frac{\sigma}{\alpha}\int_t^T \Sigma(u,t,S,T)dW^Q(u)$$

Where:

$$\Sigma(u,t,S,T) = \left[e^{-\alpha(S-u)} - e^{-\alpha(T-u)}\right]\mathbb{1}_{[t,S)}(u) + \left[1 - e^{-\alpha(T-u)}\right]\mathbb{1}_{[S,T]}(u)$$

It should be mentioned that this may not be a realistic representation of r, and should be addressed accordingly. As discussed in [BM13], a solution could be multi-factor models, as one source of uncertainty could be too restrictive for realistic modelling approaches.

In [SS20], they consider 1-, 2- and 3-factor versions of Gaussian arbitrage-free short-rate models. One should also take into account the market price of risk λ_t, coming from Girsanov's Theorem: $dW^Q(t) = dW(t) - \lambda_t dt$, to get suitable P-dynamics of the futures rates: $f^{\ell M}, \ell = 1, 3$

It would also be interesting to study term-SOFR dynamics further. For instance, in [GS21] the authors discuss how SOFR is related to EFFR, which then again is heavily affected by US monetary policy rates. It would also be interesting to further study how the CME term SOFR is inferred from the futures market.

From simulations we got $\hat{a}_t^{3M} = 0.95$ for the position in 3M-SOFR futures. An ideal hedge here would be if this number equalled 1. Pretend that instead of hedging $X^{3M_A}(S,T)$, we wanted to hedge:

$$X^{3M_G}(S,T) := \frac{1}{T-S}\left[e^{\int_S^T r(u)du} - 1\right]$$

Namely a geometric average over the period $[S,T]$, then:

$$G(a_t) := \arg\min_{a_t \in \mathbb{R}} \mathbb{E}_Q\left[\left(X^{3M_G}(S,T) - a_t f^{3M}(t,S,T)\right)^2 \middle| \mathcal{F}_t\right]$$

Now following the same arguments as on p.66, one has:

$$\hat{a}_t^{3M} = \frac{\mathbb{E}_Q[X^{3M_G}(S,T)|\mathcal{F}_t]}{f^{3M}(t,S,T)} = \frac{f^{3M}(t,S,T)}{f^{3M}(t,S,T)} = 1$$

8.2 ESG

We imposed a model for ESG-linked swaps, which led to an ESG swap rate process $\kappa_t^{ESG} = (\kappa_t^{ESG}(i))_{i=1}^n$ giving a penalty/discount depending on whether the criteria were met or not.

We remember that the criteria A_i looked like the following:

$$A_i = \{X_{T_i} \leq C_{T_i}^{ESG}\}$$

This means that our ESG-fixed rate process heavily depends upon the OU-process $X(t)$ and the criteria $C_{T_i}^{ESG}$. It could be hard to establish "reasonable" criteria. In our simulation, we took $C^{ESG} = (C_{T_i}^{ESG})_{i\geq 1}$ to be $\mathcal{F}_0$-measurable, this could lead to some uncertainty as one would have to "know" even more about the company's development. Maybe a more reasonable approach would be to take C^{ESG} to be $\mathcal{F}_{T_{i-1}}$-measurable. However, this would again add to the complexity.

In our modelling approach, we modelled directly under Q. However, the market one operates in is under P, meaning that it would have been suitable with an Esscher-transform of $X(t)$, which then by Proposition 2.4.14 implies that we would still have a CPP, but with altered intensity λ_Q and jump-size distribution $F_J^Q(dx)$.

In our case we have $I(t) = \sum_{k=1}^{N(t)} J_k$ with $N(t) \sim Pois(\lambda t)$ and $J \sim Exp(\mu)$, and from Lemma 2.4.15, with $\theta \in (-\infty, \mu)$, we have:

$$\lambda_Q = \frac{\lambda\mu}{\theta - \mu} \text{ and } J \overset{Q}{\sim} Exp(\mu - \theta)$$

Furthermore, $X(t)$ is company dependent, meaning that to get reasonable estimates of the necessary parameters included, data accessibility is crucial to impose a suitable model.

As highlighted in [BKR22], there are several problems associated with ESG scoring, including:

- Measurement: measure the same indicator using different ESG metrics.
- Scope: ratings based upon different sets of ESG indicators.
- Weight: different views on the relative importance of ESG indicators.

Consider the case where the ESG score $\mathcal{S}$ is defined as a linear combination of weighted ESG metrics X_j i.e.

$$\mathcal{S} = \sum_{j=1}^{m} w_j X_j$$

How many metrics $X_1, \ldots, X_m$ should one choose? And how should one choose the weights $w_1, \ldots w_m$? In [Bil+21], it is even highlighted that rating agencies can have opposite opinions on the same evaluated companies.

If one looks at the expression $K_n^{ESG}(\omega)$:

$$\begin{aligned} K_n^{ESG}(\omega) = [\kappa - dn]\mathbb{1}\left[\bigcap_{i\in\mathcal{I}_n} A_i\right](\omega) \\ + \sum_{\alpha\in\mathcal{I}_{2n}^{Even}} \left([\kappa - d(n-\alpha)]\mathbb{1}\left[\bigcup_{j_1\neq\ldots\neq j_{|\mathcal{I}_\alpha^{Even}|}\in\mathcal{I}_n} \left(\bigcap_{i\in\mathcal{I}_n} A_i\right)^{\{(j_1,\ldots,j_{|\mathcal{I}_\alpha^{Even}|})\}}\right]\right)(\omega) \end{aligned}$$

We see that it tracks every path, and for each n there are 2^n-possible paths, meaning that as n increases, the complexity increases. Furthermore, this expression is rather general, meaning that one must rely upon Monte Carlo simulations to get an estimate of $\kappa_t^{ESG}(i)$. At the same time, its generality also gives greater flexibility for other types of stochastic models for the ESG-risk score.

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Nomenclature

κ_t^{ESG}	ESG-swap rate process
$\overline{\mathbb{R}}$	Extended real-line i.e $[-\infty, \infty]$
$\overline{\mathbb{R}}_+$	Positive real line including infinity, i.e $[0, \infty]$
$\mathbb{R}_0^d$	$\mathbb{R}^d \setminus \{0\}$
ξ	Transformed characteristic exponent of Levy process, $\xi(\theta) := \Psi(-i\theta)$
$f^{1M}(t, S, T)$	1-month SOFR futures
$f^{3M}(t, S, T)$	3-month SOFR futures
K^{ESG}	ESG fixed rate process
$L(t, T)$	LIBOR-rate
$v(t, T)$	Volatility process, where: $v(t, T) = -\int_t^T \sigma(t, u)du$
$Z^T(t)$	Radon Nikodym derivative where: $Z^T(t) = \frac{P(t,T)}{P(0,T)B(t)}$
1M	1-month
3M	3-months
ARRC	Alternative Reference Rates Committee
ATS	Affine Term Structure
Borel-measure	Measure defined on the σ-algebra of Borel sets
CME	Chicago Mercantile Exchange
CPP	Compound Poisson process
càdlàg	right continuous with existing left limits
DCT	Dominated Convergence Theorem
EFFR	Effective Federal Funds Rate
ESG	Environmental, Social, and Governance
HJM	Heath-Jarrow-Morton
ICE	Intercontinental Exchange Benchmark Administration

LIBOR	London Interbank Offered Rate
O/N	Overnight
OU	Ornstein Uhlenbeck
RFR	Risk-Free Reference Rates
SDE	Stochastic Differential Equation
SOFR	Secured Overnight Financing Rate
SONIA	Sterling Overnight Index Average
ZCB	Zero Coupon Bond
€STR	Euro Short-Term Rate

Appendix A

Estimating parameters for interest rate models

Definition A.0.1 (**ARMA(p,q)**). A stochastic process $Y = (Y_i)_{i \geq 1}$ is called an $ARMA(p,q)$ process, if it has the following representation:

$$Y_i = \mu + \sum_{j=1}^{p} \phi_j (Y_{i-j} - \mu) + \epsilon_i - \sum_{k=1}^{q} \theta_k \epsilon_{i-k}$$

Where:

- ϵ_i are iid with $\mathbb{E}[\epsilon_i] = 0$ and $Var[\epsilon_i] = \sigma_\epsilon^2$ as well as independent of $\mathcal{Y}_{i-1} = (Y_1, \dots, Y_{i-1})$

A special case of an $ARMA(p,q)$ is $AR(p)$, we have the the following relationship:

$$AR(p) = ARMA(p,0)$$

We will be interested in $AR(1)$, meaning that:

$$Y_i = \mu + \phi[Y_{i-1} - \mu] + \epsilon_i$$

Maximum likelihood AR(1)
Consider the case when $\epsilon_i \sim \mathcal{N}(0, \sigma_w^2)$, in order to find the MLE estimates: $\boldsymbol{\theta} = (\mu, \phi, \sigma_w^2)$, one can use the conditional likelihood function $L(\boldsymbol{\theta}|Y_1)$. We note that $Y_i|Y_{i-1}, i \geq 2$ is Markovian, meaning that the conditional likelihood takes the following form:

$$L(\boldsymbol{\theta}|Y_1) = \prod_{i=2}^{n} f_{Y_i|Y_{i-1}}(y_i|y_{i-1})$$

Furthermore $Y_i|Y_{i-1} \sim \mathcal{N}(\mu + \phi[Y_{i-1} - \mu], \sigma_w^2)$, meaning that:

$$L(\boldsymbol{\theta}|Y_1) = \left(\frac{1}{\sqrt{2\pi\sigma_w^2}}\right)^{n-1} \prod_{i=2}^{n} \exp\left(-\frac{1}{2\sigma_w^2}\left[y_i - (\mu + \phi[y_{i-1} - \mu])\right]^2\right)$$

$$\Downarrow$$

$$l(\boldsymbol{\theta}|Y_1) = (n-1)\ln\left(\frac{1}{\sqrt{2\pi\sigma_w^2}}\right) - \frac{1}{2\sigma_w^2}\sum_{i=2}^{n}\left[y_i - \mu - \phi[y_{i-1} - \mu]\right]^2$$

Now from [Rem13], it follows that the MLE estimates are approximate:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \overline{Y}$$
$$\hat{\phi} = \frac{\sum_{i=2}^{n}(Y_i - \overline{Y})(Y_{i-1} - \overline{Y})}{\sum_{i=1}^{n}(Y_i - \overline{Y})^2}$$
$$\hat{\sigma}_w^2 = \frac{1}{n-1}\sum_{i=2}^{n}[Y_{i-1} - \overline{Y} - \hat{\phi}(Y_{i-1} - \overline{Y})]^2$$

Now recall that the Vasicek model looks like:

$$dr(t) = \alpha[m - r(t)]dt + \sigma dW^Q(t)$$

With explicit solution:

$$\begin{aligned} r(T) &= e^{-\alpha(T)}r(0) + m[1 - e^{-\alpha(T)}] + \sigma\int_0^T e^{-\alpha(T-u)}dW^Q(u) \\ &= m + e^{-\alpha T}\left[r(0) - m\right] + \sigma\int_0^T e^{-\alpha(T-u)}dW^Q(u) \end{aligned} \tag{A.1}$$

Proposition A.0.2 ([**Rem13**]). *One can express the Vasicek model as an $AR(1)$-process:*

$$r_k = m + \phi(r_{k-1} - m) + \epsilon_k,\ k = 1, \ldots, n \tag{A.2}$$

Where:

- $r_k := r(kh)$*, here* r *is as described in Equation A.1.*
- h *is an equidistant time-interval between* r_i *and* r_{i-1}*.*
- $\phi = e^{-\alpha h}$
- $\epsilon_k \sim \mathcal{N}\left(0, \frac{\sigma^2}{2\alpha}\left[1 - \phi^2\right]\right)$

Thus in order to estimate m, ϕ and $\sigma_w^2 = \frac{\sigma^2}{2\alpha}\left[1 - \phi^2\right]$ from Equation A.2 one can plug it into the approximate MLE $AR(1)$-estimates.

There are also other expressions for the MLE estimates like the following from [FP15]:

$$\hat{m} = \frac{S_1 S_{00} - S_0 S_{01}}{S_0 S_1 - S_0^2 - S_{01} + S_{00}}$$

$$\hat{\alpha} = \frac{1}{h}\ln\left(\frac{S_0 - \hat{m}}{S_1 - \hat{m}}\right)$$

$$\hat{\sigma}^2 = \frac{1}{n\beta(\hat{\alpha})[1 - \frac{1}{2}\hat{\alpha}\beta(\hat{\alpha})]}\sum_{k=1}^{n}[r_k - \ell_{k-1}(k)]^2$$

Where:

$$S_0 = \frac{1}{n}\sum_{k=1}^{n} r_k, \quad S_1 = \frac{1}{n}\sum_{k=1}^{n} r_{k-1}$$
$$S_{00} = \frac{1}{n}\sum_{k=1}^{n} r_{k-1}r_{k-1}, \quad S_{01} = \frac{1}{n}\sum_{k=1}^{n} r_{k-1}r_k$$

Furthermore:

- $\beta(\alpha) = \frac{1}{\alpha}\left[1 - e^{-\alpha h}\right]$
- $\ell_s(t) = m \cdot \alpha \cdot B(s,t) + r_s[1 - \alpha B(s,t)]$
- $B(s,t) = \frac{1}{\alpha}\left[1 - e^{-\alpha(t-s)}\right]$

Typically we will not observe the short-rate r. However, zero coupon bond yields are observable, often called the yield term structure. The idea is to establish the connection between the zero coupon yields and Affine term structures:

$$P(t,T) = e^{-R(t,T)(T-t)}, \text{ and } P(t,T) = \exp\left(-A(t,T) - B(t,T)r(t)\right)$$

This gives us the following relationship between ATS and zero-coupon yields:

$$R(t,T) = \frac{A(t,T) + B(t,T)r(t)}{T-t}$$

Let $\tau = T - t$; we can then express the short rate. For the Vasicek model, we have the following:

$$B_\tau(\boldsymbol{\theta}) = \frac{1}{\alpha}[1 - e^{-\alpha\tau}]$$
$$A_\tau(\boldsymbol{\theta}) = \left(\frac{\sigma^2}{2\alpha} - m\right)[\tau - B_\tau(\boldsymbol{\theta})] + \frac{\sigma^2}{4\alpha}B_\tau^2(\boldsymbol{\theta})$$

This again gives us the following expression for the short-rate:

$$r(t) := r_{\boldsymbol{\theta}}(t) = \frac{\tau R(t, t+\tau) - A_\tau(\boldsymbol{\theta})}{B_\tau(\boldsymbol{\theta})}$$

Let $R_1, \ldots, R_n$ represent the observed annualized zero coupon yields with maturities $\tau_1, \ldots, \tau_n$, meaning that:

$$R_k := R(kh, kh + \tau_k)$$

This yields:

$$r_k := r_k^{\boldsymbol{\theta}} = \frac{\tau_k R_k - A_{\tau_k}(\boldsymbol{\theta})}{B_{\tau_k}(\boldsymbol{\theta})}$$

We have that $\boldsymbol{R} = (R_1, \ldots, R_n)$ are observable, while $\boldsymbol{r} = (r_1, \ldots, r_n)$ are not. We note that:

$$R_k = \frac{A_{\tau_k}(\boldsymbol{\theta}) + B_{\tau_k}(\boldsymbol{\theta})r_k}{\tau_k} := g(r_k)$$

Here g is a linear function of r_k, meaning it is invertible and strictly increasing. This means that:

$$f_{\boldsymbol{\theta}}(\boldsymbol{R}|R_1) = \frac{f_{\boldsymbol{\theta}}(\boldsymbol{r}|r_1)}{|J_g(\boldsymbol{r})|}$$

Here:

$$J_g(\boldsymbol{r}) = \det\left(\frac{\partial g_i(\boldsymbol{r})}{\partial r_k}\right)_{i,k=1}^{n}$$

In our case we have $g_i = g$, with:

$$\frac{\partial g(\boldsymbol{r})}{\partial r_l} = \begin{cases} \frac{B_{\tau_k(\boldsymbol{\theta})}}{\tau_k} = \frac{1}{a\tau_k}[1 - e^{-a\tau_k}] \neq 0 & , \ l = k \\ 0 & , \ l \neq k \end{cases}$$

This means that J_g is a diagonal matrix, giving us the following determinant:

$$|J_g(\boldsymbol{r})| = \prod_{k=1}^{n} \left|\frac{\partial g(\boldsymbol{r})}{\partial r_k}\right| = \prod_{k=1}^{n} \frac{B_{\tau_k(\boldsymbol{\theta})}}{\tau_k}$$

And again from [Rem13] we get that:

$$\begin{aligned} l(\boldsymbol{\theta}|R_1) &= \ln\left[\frac{f_{\boldsymbol{\theta}}(\boldsymbol{r}|r_1)}{|J_g(\boldsymbol{r})|}\right] \\ &= l(\boldsymbol{\theta}|g^{-1}(R_1)) - \ln(|J_g(\boldsymbol{r})|) \\ &= (n-1)\ln\left(\frac{1}{\sqrt{2\pi\sigma_\epsilon^2}}\right) - \frac{1}{2\sigma_\epsilon^2}\sum_{k=2}^{n}\left[r_k^{\boldsymbol{\theta}} - m - \phi(r_{k-1}^{\boldsymbol{\theta}} - m)\right]^2 - \sum_{k=1}^{n}\frac{B_{\tau_k(\boldsymbol{\theta})}}{\tau_k} \end{aligned}$$

Appendix B

Scripts Chapter 5

B.1 SOFR: Dynamics of $f^{1M}(t, S_{1M}, T_{1M})$ and $f^{3M}(t, S_{3M}, T_{3M})$

```
using Revise

#stats etc.
using Random
using Distributions

#plotting histogram and LaTeX labels:
using Plots
using LaTeXStrings

#Vasicek dynamics of SOFR futures rates:
alpha = 0.30
sigma = 0.03

function B(t,S,T)
    return (1/alpha)*(exp(-alpha*(S-t))-exp(-alpha*(T-t)))
end


function f1M(t,S,T, dt, initial)
    "
    Simulates 1M-futures rates for t<= S
    Args:
        t {Float64}: initial start point
        S {Float64}: start observation period
        T {Float65}: end observation period
        dt {Float64}: stepsize
        inital {Float64}: inital futures rate
    Returns:
        df1M(t,S,T) = 1/(T-S)*B(t,S,T)*sigma*dW^{Q}(t)
    "
    time = range(t, S, step=dt)
    n = length(time)
```

```
    Random.seed!(1)
    Z = rand(Normal(0,1), n)
    f1_r = zeros(n)
    f1_r[1] = initial
    for i in 2:n
        f1_r[i] = f1_r[i-1] + (1/(T-S))*B(time[i-1],
        ↪  S,T)*sigma*sqrt(dt)*Z[i]
    end

    return f1_r
end


function f3M(t,S,T, dt, initial)
    "
    Simulates 3M-futures rates for t<= S
    Args:
        t {Float64}: initial start point
        S {Float64}: start observation period
        T {Float65}: end observation period
        dt {Float64}: stepsize
        inital {Float64}: inital futures rate
    Returns:
        df3M(t,S,T) = (f3M(t,S,T)+1/(T-S))*B(t,S,T)*sigma*dW^{Q}(t)
    "
    time = range(t, S, step=dt)
    n = length(time)

    Random.seed!(2)
    #W(t) d= sqrt(t)Z, Z ~ N(0,1)
    Z = rand(Normal(0,1), n)
    f3_r = zeros(n)
    f3_r[1] = initial
    for i in 2:n
        f3_r[i] = f3_r[i-1] +
        (f3_r[i-1] + 1/(T-S))*B(time[i-1], S,T)*sigma*sqrt(dt)*Z[i]
    end

    return f3_r
end

Random.seed!(3)
#time params:
t = 0
dt = 1/360
#1M-futures:
S1M = 6/12
T1M = S1M + 1/12
#3M-futures:
```

```
S3M = S1M
T3M = S3M + 3/12


#simulation
time = range(t, S1M, step=dt)
n = length(time)
Z = rand(Normal(0,1), n)
f1 = zeros(n)
f3 = zeros(n)


f1[1] = (100-95.025)*1/100
f3[1] = (100-95.16)*1/100

for i in 2:n
    f1[i] = f1[i-1] + (1/(T1M-S1M))*B(time[i-1],
    ↪  S1M,T1M)*sigma*sqrt(dt)*Z[i]
    f3[i] = f3[i-1] + (f3[i-1] + 1/(T3M-S3M))*B(time[i-1],
    ↪  S3M,T3M)*sigma*sqrt(dt)*Z[i]
end

plot(f1, label = L"f^{1M}(t, S_{1M},T_{1M})", title =
L"\alpha = 0.30,\; \sigma = 0.03,\; t\in [0,S_{1M}]", legend= :topleft)
plot!(f3, label = L"f^{3M}(t, S_{3M}, T_{3M})")
xticks!([0, n/2 ,n], ["0", L"\frac{S_{1M}}{2}", L"S_{1M}"])
```

B.2 SOFR: Simulation of $\kappa_t^{3M-SOFR}$

```julia
using Revise

#statistics and distributions
using Random
using Distributions
using Statistics

#data-wrangeling:
using DataFrames

#for numerical integration:
using QuadGK

#plotting histogram and LaTeX labels:
using Plots
using LaTeXStrings

#----------------------------------------------------------------------
#time parameters
T0 = 1/12
T1 = 4/12
T2 = 7/12
T3 = 10/12
timepoints = [T0, T1, T2, T3]
n_steps = 10000

#We use that Vasicek is ATS, i.e P(t,T) = exp(-A(t,T)-B(t,T)r(t))

function r_Vasicek(alpha, m,sigma, r_t, time_interval, n_steps)
    """
    Args:
        #Vasicek parameters:
        alpha{Float64}: speed of reversion
        m{Float64}: long term mean level
        sigma{Float64}: volatility
        r_t{Float64}: initial value of r = (r(u))

        #time:
        time_interval (vector): the time interval we model measured in
 ↪  years.
        n_steps (int): number of timesteps we partition over

    Returns:
        it simulates the process: r = (r(u)), for u in [t_start, t_end]
    """

    #time partition:
```

```
    t_start = time_interval[1]
    t_end = time_interval[2]
    dt = (t_end-t_start)/n_steps
    n_steps = length(collect(t_start:dt:t_end))

    #initializing r:
    r = zeros(n_steps)
    r[1] = r_t

    #Standard normal rv's
    Z = rand(Normal(0,1), n_steps)

    for i in 2:n_steps
        r[i] = r[i-1] - alpha*(m-r[i-1])*dt + sigma*sqrt(dt)*Z[i]
    end

    return r
end

function B_ZCB(t,T)
    ans = -(1/alpha)*(exp(-alpha*(T-t))-1)
    return ans
end

function A_ZCB(t,T)
    integral, _ = quadgk(u -> B_ZCB(u,T)^(2), t,T)
    ans = m*B_ZCB(t,T) - m*(T-t) - (1/2)*sigma^(2)*integral
    return ans
end

function P(t,T, r_t)
    ans = exp(-A_ZCB(t,T) -B_ZCB(t,T)*r_t)
    return ans
end

#---------------------------------------------------------------
#Calculating f^3M(t,S,T), again using ATS structure:
#f^3M(t,S,T) = 1/(T-S)*(exp(A(t,S,T) + B(t,S,T)r(t))-1)
function Sigma1(u,t,S,T)
    ans = exp(-alpha*(S-u)) - exp(-alpha*(T-u))
    return ans
end

function Sigma2(u,t,S,T)
    ans = 1-exp(-alpha*(T-u))
    return ans
end

function B(t,S,T)
```

```
    ans = (1/alpha)*(exp(-alpha*(S-t))-exp(-alpha*(T-t)))
    return ans
end

function A(t,S,T)
    first_part = m*(T-S) - m*B(t,S,T)
    c1_2, _  = quadgk(u -> Sigma1(u, t, t, S)^(2), t,S)
    c2_2, _  = quadgk(u -> Sigma2(u, t, S, T)^(2), S,T)

    ans = first_part + (1/2)*(sigma^2/alpha^2)*(c1_2 + c2_2)

    return ans
end


function f_3M(t,S,T, r_t)
    ans = (1/(T-S))*(exp(A(t,S,T) + B(t,S,T)*r_t) - 1)
    return ans
end

#time t-value of kappa in 3M SOFR-futures rate swap
function kappa_t(t, r_t)
    "
    Args:
        t{Float64}: vector of time points i.e [0,T1]
        r_t{Float64}: vector of realization of interest rate model
    Returns:
        above = sum(P(t,T_{i}*f^{3M}(t,T_{i-1}, T_{i})), i = 1:n)
        below = sum(P(t,T_{i}), i = 1:3)
        kappa_t_3M_SOFR = above/below
    "
    ZCB_prices = map(T -> P(t,T, r_t), timepoints[2:end])
    f_3M_rates = map((x, y) -> f_3M(t, x, y, r_t), timepoints[1:end-1],
    ↪  timepoints[2:end])
    above = sum(ZCB_prices.*f_3M_rates)
    below = sum(ZCB_prices)
    ans = above/below
    return ans
end


#Vasicek parameters:
alpha = 0.25
m = 0.035
r_0 = 0.0425
sigma = 0.02

#time:
t_start = 0
```

```
t_end = T0
dt = (t_end-t_start)/n_steps
t = collect(t_start:dt:t_end)

#initialization of simulation
n_sim = 2

R = zeros(length(t), n_sim) #Vasicek rates
K = zeros(length(t), n_sim) #fixed rate kappa for each pair (t,r_t)

Random.seed!(1234)
for i in 1:n_sim
    #Vasicek realization:
    r = r_Vasicek(alpha, m, sigma, r_0, [t_start,t_end], n_steps)
    #collect the time t rate and time t kappa:
    R[:, i] = r
    K[:, i] = map((x,y)-> kappa_t(x,y), t, r)
end

R
K[1]

#plot of rates
plot(R, layout = (1,1),
        legend = false,
        title = L"t \mapsto r(t),\alpha = 0.25, m = 0.035, \sigma =
        ↪  0.02, r_{0} = 0.0425 "
        )
xticks!([0, 10_000/2 ,10_000], ["0", L"\frac{T_{0}}{2}", L"T_{0}"])


#plot of kappa_t
plot(K, layout=(1,1),
        legend = false,
        title = L"t \mapsto \kappa_{t}^{3M-SOFR},\alpha = 0.25, m =
        ↪  0.035, \sigma = 0.02, r_{0} = 0.0425"
        )
xticks!([0, 10_000/2 ,10_000], ["0", L"\frac{T_{0}}{2}", L"T_{0}"])
```

B.3 SOFR: Hedging 3M-arithmetic SOFR

```
using Random
using Distributions
using Statistics
using StatsPlots #qqplot

#for matrix operations and linear programming
using LinearAlgebra
using JuMP  #lp-problem setup
using HiGHS #lp-solver

#data-wrangeling:
using DataFrames

#for numerical integration:
using QuadGK

#rerun calculations easier:
using Revise

#plotting histogram and LaTeX labels:
using Plots
using LaTeXStrings

#--------------------------------------------------------------------------#
#Vasicek parameters:
alpha = 0.25
m = 0.035
r_t = 0.0425
sigma = 0.02

#time parameters
t = 0
S = 1/12
T1M = 2/12
T2M = 3/12
T = 4/12

function Sigma1(u,t,S,T)
    ans = exp(-alpha*(S-u)) - exp(-alpha*(T-u))
    return ans
end

function Sigma2(u,t,S,T)
    ans = 1-exp(-alpha*(T-u))
    return ans
end
```

```
function int_r_start_stop(low,up, t)
    "
    The integral: integral(r(u)du, low, up) as described in Eq (5.5)
↪   p.66
    Args:
        low{Float64}: lower integration limit
        up{Float64}:  upper integtation limit
    Returns:
        the integral: int_low_up r(u)du
    "
    if t > low
        return "Please chose t <=low"
    end

    #Sigma1 is N(0, int_t_low c1_2 du), Sigma2 is N(0, int_low_up c2_2
    ↪  du)
    c1_2, _  = quadgk(u -> Sigma1(u, t, low,up)^(2), t,low)
    c2_2, _  = quadgk(u -> Sigma2(u, t, low,up)^(2), low,up)

    c1 = sqrt(c1_2)
    c2 = sqrt(c2_2)
    Z = rand(Normal(0,1))
    ans = ((r_t-m)/alpha)*(exp(-alpha*(low-t))-exp(-alpha*(up-t))) +
                        m*(up-low) + sigma/alpha*(c1*Z + c2*Z)
    return ans
end

function integrand_E_Q_r(u, r_t, t)
    ans = exp(-alpha*(u-t))*r_t + m*(1-exp(-alpha*(u-t)))
    return ans
end

#int_S_T E_Q[r(u)|F_t]du:
integral_E_Q_r, _ = quadgk(u -> integrand_E_Q_r(u, r_t, t), S,T)

integral_E_Q_r
#--------------------------------------------------------------------------#
# Calculating a_hat_3M:
function B(t,S,T)
    ans = (1/alpha)*(exp(-alpha*(S-t))-exp(-alpha*(T-t)))
    return ans
end

function A(t,S,T)
    first_part = m*(T-S) - m*B(t,S,T)
    c1_2, _  = quadgk(u -> Sigma1(u, t, t, S)^(2), t,S)
    c2_2, _  = quadgk(u -> Sigma2(u, t, S, T)^(2), S,T)

    ans = first_part + (1/2)*(sigma^2/alpha^2)*(c1_2 + c2_2)
```

```
    return ans
end


function f_3M(t,S,T)
    "
    Vasicek representation of f^{3M}(t,S,T) as described in Eq. (5.4)
↪   p.60
    "
    ans = (1/(T-S))*(exp(A(t,S,T) + B(t,S,T)*r_t) - 1)
    return ans
end

f_3M(0,S,T)

a_hat = integral_E_Q_r/((T-S)*f_3M(0,S,T))

#-----------------------------------------------------------------------------------------------
# 3M-arithmetic vs (a,b,c) 1M-SOFR futures:
integral1 , _ = quadgk(u -> integrand_E_Q_r(u, r_t, t), S,T1M)
integral2 , _ = quadgk(u -> integrand_E_Q_r(u, r_t, t), T1M,T2M)
integral3 , _ = quadgk(u -> integrand_E_Q_r(u, r_t, t), T2M,T)

f_1M_S_T1M = (1/(T1M-S))*integral1
f_1M_T1M_T2M = (1/(T2M-T1M))*integral2
f_1M_T2M_T = (1/(T-T2M))*integral3

#variable naming to be more consistent with MSc Thesis:
a = f_1M_S_T1M        #alpha, I use alpha in Vasicek, hence a:
beta = f_1M_T1M_T2M   #beta
gamma = f_1M_T2M_T    #gamma

futures = [a,beta,gamma]

#E_Q[X^(3M_A)(S,T)|F_t] = q:
q = (1/(T-S))*integral_E_Q_r

#matrix of coeff:
M = [a^(2)      a*beta   a*gamma;
     beta^(2)   a*beta   beta*gamma;
     gamma^(2)  a*beta   beta*gamma]

#vector of values:
b = q.*[a;
        beta;
        gamma]

#optimal weight of futures:
```

```
x_hat = inv(M)*b
#----------------------------------------------------------------------------------------------------
# incase M is not invertible:
# Define optimization problem
model = Model(HiGHS.Optimizer)
@variable(model, x[1:3])
@objective(model, Min, sum(x))
@constraint(model, M * x .== b)

# Solve optimization problem
optimize!(model)
# optimal value
x_tilde = value.(x)

#----------------------------------------------------------------------------------------------------
# Simulations:
n_sim = 10^(6)
#constants:
#int_S_T E_Q[r(u)|F_t]du:
integral_E_Q_r, _ = quadgk(u -> integrand_E_Q_r(u, r_t, t), S,T)

futures_weighted_M_inv = x_hat'futures
futures_weighted_BP = x_tilde'futures


Random.seed!(1234)
X_3MA = zeros(n_sim)
for i in 1:n_sim
    #aritmetic interest rate relaization:
    X_3MA[i] = (1/(T-S))*(int_r_start_stop(S,T,t))
end

mean(X_3MA)
#elementwise substraction:
ER_1 = X_3MA .-(1/(T-S))*integral_E_Q_r
ER_2_M_inv = X_3MA .-futures_weighted_M_inv
ER_2_random = X_3MA .-[0.33, -0.33, 0.33]'futures

#------------------------------------------------------------
# plotting of histograms:
#hedge with a_{t}^{3M}- f^{3M}
mean_ER_1 = round(mean(ER_1), digits = 3)
sigma_ER_1 = round(std(ER_1), digits = 2)

#hedge with optimal (a_{t}^{1M}, b_{t}^{1M}, c_{t}^{1M})-f^{1M}
mean_ER_2_M_inv = round(mean(ER_2_M_inv), digits = 3)
sigma_ER_2 = round(std(ER_2_M_inv), digits = 2)

#not optimal 1M hedges, naive strategy:
```

```
mean_ER_2_random = round(mean(ER_2_random), digits = 3)
sigma_ER_2_random = round(std(ER_2_random), digits = 2)

#ER_1
ticks_ER_1 = round.([mean_ER_1 + i*sigma_ER_1 for i in -3:1:3], digits =
↪  3)

histogram(ER_1,
          color =:lightblue,
          xlabel="Value",
          ylabel="Frequency",
          title =
          L"Histogram\; of\; ER_{1}(0), \; s_{ER_{1}}\approx
          ↪  0.01,\;n_{sim} = 10^{6}",
          labels = "ER_1(0)",
          xticks = ticks_ER_1
          )
vline!([mean_ER_1], lw = 5, labels = L"mean(ER_{1}(0))" )


#ER_2_M_inv:
ticks_ER_2_M_inv = round.([mean_ER_2_M_inv + i*sigma_ER_2 for i in
↪  -3:1:3], digits = 3)

histogram(ER_2_M_inv,
          color =:lightblue,
          xlabel="Value",
          ylabel="Frequency",
          title =
          L"Histogram\; of\; ER_{2}^{M_{inv}}(0), \;
          ↪  s_{ER_{2}^{M_{inv}}} \approx 0.01, \;n_{sim} = 10^{6}",
          labels = L"ER_{2}^{M_{inv}}(0)",
          xticks = ticks_ER_2_M_inv
          )
vline!([mean_ER_2_M_inv], lw = 5, labels = L"mean(ER_{2}^{M_{inv}}(0))")


#Naive strategy ER_2_random (0.33, -0.33, 0.33)
ticks_random = round.([mean_ER_2_random + i*sigma_ER_2_random for i in
↪  -2:1:2], digits = 3)

histogram(ER_2_random,
          color =:lightblue,
          xlabel="Value",
          ylabel="Frequency",
          title =
          L"Hist\; of\; ER_{2}^{(\hat{a}_{0}, \hat{b}_{0},
          ↪  \hat{c}_{0})}(0), \; s_{ER_{2}^{(\hat{a}_{0}, \hat{b}_{0},
          ↪  \hat{c}_{0})}} \approx 0.01, \;n_{sim} = 10^{6}",
```

```
          labels = L"ER_{2}^{(0.33,-0.33,0.33)}(0)",
          xticks = ticks_random
          )
vline!([mean_ER_2_random], lw = 5, labels =
↪  L"mean(ER_{2}^{(0,0,1)}(0))")



#adressing normality:
#-----------------------------------------
x = ER_1[1:10^(6)]
y = rand(Normal(mean_ER_1, sigma_ER_1), 10^(6))

qqplot(x,y, title =
            L"(Q-Q)\; plot\; of\; ER_{1}(0) \;vs\;
            ↪  \mathcal{N}\left(\overline{ER_1(0)},
            ↪  s_{ER_1(0)}^{2})\right)",
            xlabel = "Theoretical Quantiles",
            ylabel = "Sample Quantiles")
```

Appendix C

Scripts Chapter 7

C.1 Numerical Simulation For ESG swap rate

```
using Revise

using Random
using Distributions
using Statistics

using LinearAlgebra
using DataFrames

#using Combinatorics
using Plots

function create_array(dims::Array{Tuple{Int,Int},1})
    "
    Args:
        dims{Array{Int64}}:
        vector of tuples, where each element corresponds to
↪   matrix-dimension

    Returns:
        Array of matrices A = (M_{1}, ..., M_{n})
        Each matrix can take on different dimensions, i.e.:
        dimensions = [(m,n), (k,l), (r,q), ...]
        The array will return an initialization of zero matrices
    "
    arr = Array{Array{Float64,2}}(undef, length(dims))

    for (i, dim) in enumerate(dims)
        arr[i] = zeros(dim...)
    end

    return(arr)
end
```

```
function all_perm(xs, n)
    "
    Args:
        xs{Vector}: input vector of what should be permutated
        n{Int64}: desired length of vector

    Returns:
        generates permutation of elements in vector xs of length n
        all_perm([0.0, 1.0], 2) = [0.0, 0.0],[1.0, 0.0], [0.0, 1.0],
↪   [1.0, 1.0]
        all_perm([0.0, 1.0], 3) = [0.0, 0.0, 0.0], [0.0, 1.0, 0.0],
↪   [1.0, 0.0, 0.0], ...
    "
    return(vec(map(collect, Iterators.product(ntuple(_ -> xs, n)...))))
end


function OU_CPP(z0::Float64, beta::Float64, sigma::Float64,
↪   lambda::Float64, mu::Float64, dt::Float64,T_end::Float64)
    "
    Description:
        dZ(t) = -beta*Z(t)dt + sigma*dW(t) + dI(t)
        I(t): I(t) = sum_{i=1}^{N(t)}J_k, J_k ~ Exp(mu),
        NB! Julia parametrize with 1/mu

    Args:
        z0{Float64}: inital value of the process Z(t)
        beta{Float64}: mean-retreving parameter
        sigma{Float64}: volatility parameter of Brownian Motion
        lambda{Float64}: jump intensity of process, N(t)~ Pois(lambda*t)
        dt{Float64}: stepsize
        T_end{Float64}: for how long the simulation should go

    Returns:
        X(t) = 100exp(-Z(t))
    "

    time = collect(0:dt:T_end)
    n = length(time)

    #number of jumps from [0,T_end] on each dt: N(dt) ~ Pois(lambda*dt)
    N = rand(Poisson(lambda*dt),n)

    #Brownian motion W~N(0,dt) on [0,dt]
    W = rand(Normal(0,1) ,n)

    #intialising Z(t)
    z = zeros(n)
```

```
    z[1] = z0

    #dZ(t) = -beta*Z(t)dt + sigma*dW(t) + dI(t)
    for i in 2:n
        dI = sum(rand(Exponential(mu), N[i])) -
        ↪  sum(rand(Exponential(mu), N[i-1]))
        z[i] = z[i-1] - beta*z[i-1]*dt + sigma*W[i]*sqrt(dt) + dI
    end

    #X(t) = 100exp(-Z(t))
    x = 100*exp.(-z)
    df = DataFrame(time = time, score = x)
    return df
end

function simulation(n_sim, params, C_ESG, T_end, relevant_times)
    "
    Args:
        n_sim{Int64}: number of simulations
        params{Vector{Float64}}: the parameters in OU_CPP
        C_ESG{Vector{Float64}}: ESG-criteria at time T_{i}
        relevant_times{Vector{Float64}}: vector of relevant times, [T1,
↪   T2, T3, ...],
        expressed as a percentage of the year.
    "

    "
    returns:
        m{Matrix{Float64}}: matrix checking if criteria at time T_{i} is
↪   met or not.
        Each row in the matrix corresponds to a simulation, i.e.
        m = [0,0,0; did not meet any criteria
             0,1,1; met criteria at T2 and T3
             ...  ]
    "
    z0 = params[1]
    beta = params[2]
    sigma = params[3]
    lambda = params[4]
    mu = params[5]

    #store matrix of zeros, row = simulation number, col = agreed
    ↪  observation times
    m = zeros(n_sim, length(relevant_times))
    for i in 1:n_sim
        #df_tmp: general simulation
        df_tmp = OU_CPP(z0, beta, sigma, lambda, mu ,1/360, T_end)
        #get the relevant time points as df:
        df_relevant = filter(row -> row.time in relevant_times, df_tmp)
```

```
        #get the score:
        relevant_score = df_relevant.score

        #check if X_{T_{i}} <= C^_{T_{i}}^{ESG} for T_{1}, ..., T_{n}
        ESG_criteria = relevant_score .<= C_ESG
        ESG_criteria = Float64.(ESG_criteria)

        #store ESG_criteria:
        m[i, :] = ESG_criteria
    end

    return(m)
end

function D(i::Int, m::Matrix)
    "
    Args:
        i{Int64}: index in sequence
        m{Matrix}: matrix with measurements of whether criteria were met
↪   or not.

    Returns:
        D(i)-term in in E_{Q}[K_{i}^{ESG}(omega)|F_{t}] = kappa_t -
↪   d*D(i)
    "
    if i > size(m)[2]
        return println("You cannot evaluate D outside of agreed
         ↪  contract")
    end

    #adjusting for column dimensions in Boolean check:
    m_adj = m[:, 1:i]

    v = all_perm([0.0, 1.0], i)
    possible_patterns = mapreduce(permutedims, vcat, v)

    #use the row sum to determine how many errors/fails there are:
    success_sum = collect(0.0:Float64(i))

    #p is the indicator of successes for the trial:
    p = zeros(size(possible_patterns)[1], i+1)
    for k in 1:(i+1)
        p[:, k] = Bool[success_sum[k] == sum(possible_patterns[j, :])
         ↪  for j=1:size(possible_patterns,1)]'
    end
    #turn p into Boolean object so that we can use findall:
    p = Bool.(p)

    #store row dimensions, so that we can initialize array later
```

```
row_dims = zeros(Int, i+1)
for i in 1:(i+1)
    row_dims[i] = size(possible_patterns[findall(p[:, i]), :])[1]
end

#initializing the needed dimensions
dimensions = [(row_dims[k], i) for k in 1:(i+1)]
"
A: array of matricies, A=(M_1, ..., M_i)
Let i = 3:
M_1: matrix of patterns giving zero successes  [0,0,0] (1x3)
M_2: matrix of patterns giving one succeses    [0,0,1;
                                                0,1,0;
                                                1,0,0] (3x3)
M_3: matrix of patterns giving two succeses    [1,1,0;
                                                1,0,1;
                                                0,1,1] (3x3)
etc.
"
A = create_array(dimensions)

for l in 1:(i+1)
    A[l] = possible_patterns[findall(p[:, l]), :]
end

"
E_fails: (vector) Expecation of all linear combinations where:
E_fails[1]: expectation of all linear combinations giving all fails
↪ (1 path)
E_fails[2]: expectation of all linear combinaition giving fails, but
↪ 1 success (multiple paths)
Let i=3: E_fails[1] = E[fff|F_t]
         E_fails[2] = E[ssf|F_t] + E[sfs|F_t] + E[fss|F_t]
etc.
"
E_fails = zeros(i+1)

for l in 1:(i+1)
    s = 0
    for j in 1:size(A[l],1)
        s += mean(Bool[A[l][j, :] == m_adj[r, :] for
         ↪  r=1:size(m_adj,1)])
    end
    E_fails[l] = s
end

#represents I_{2i}^{Even} = {2,...,2i}
I_2_Even = collect(2.0:2.0:Float64(2*i))
```

```
    #represents vector of sum_{alpha in I_{2i}^{Even}}[i-alpha],
    weight = i .- I_2_Even

    #all success, all success but one, all success but two, ...
    E_success = reverse(E_fails)

    #=
    i*E_{Q}[\cap 1(A_{l})|F_t] +
    sum_[alpha \in I_2_Even]sum_[j_1 != ... != j_I_alpha_Even]x
    E_{Q}[(\cap 1(A_{l}))^[[j_{1} != ... != j_I_alpha_Even]]|F_t]
    =#

    ans = i*E_success[1] + sum(weight.*E_success[2:length(E_success)])

    return(ans)
end

#nice parameters:
z0 = -log(20/100)
beta = -0.05
sigma = 0.02
lambda = 20.0
mu = 1/150
dt = 1/360

OU_params = [z0, beta, sigma, lambda, mu]

#ESG criteria:
C_ESG_reas = [17.8, 16.8, 15.8, 14.8]
C_ESG_wins = [24.0, 23.0, 22.0, 21.0]
C_ESG_loss = [5.0, 5.0, 5.0, 5.0]


T_end = 5.0
relevant_times = [1.25, 2.25, 3.25, 4.25]
n_sim = 10^(6)
n_sim_unreas = 10^(6)


m_reas = simulation(n_sim, OU_params, C_ESG_reas, 5.0, relevant_times)
m_wins = simulation(n_sim_unreas, OU_params, C_ESG_wins, 5.0,
↪ relevant_times)
m_loss = simulation(n_sim_unreas, OU_params, C_ESG_loss, 5.0,
↪ relevant_times)

#---------------------------------------------------------------------------
#kappa_{t}^{ESG} and kappa_{t}^{ZCB}:
d = 0.005
kappa_t_ZCB = 0.070
```

```
function kappa_t_ESG(i, m)
    "
    The ESG swap rate process
    Args:
        i (Int64): corresponds to T_{i}
        m (Matrix): matrix showing wheter or not criteria were met at
 ↪  relevant T_{i}'s
    Returns:
        The ESG swap rate process in ZCB case
    "
    ans = kappa_t_ZCB-d*D(i,m)
    return round(ans, digits = 3)
end

#reasonable criteria:
println("(t, kappa_t_ZCB, kappa_t_ESG, C_ESG_reas, relevant_times)")
for i in 1:length(relevant_times)
    println((i, kappa_t_ZCB ,kappa_t_ESG(i, m_reas), C_ESG_reas,
     ↪  relevant_times))
end

#wins all the time:
println("(t, kappa_t_ZCB, kappa_t_ESG, C_ESG_wins, relevant_times)")
for i in 1:length(relevant_times)
    println((i, kappa_t_ZCB ,kappa_t_ESG(i, m_wins), C_ESG_wins,
     ↪  relevant_times))
end

#loss all the time
println("(t, kappa_t_ZCB, kappa_t_ESG, C_ESG_loss, relevant_times)")
for i in 1:length(relevant_times)
    println((i, kappa_t_ZCB ,kappa_t_ESG(i, m_loss), C_ESG_loss,
     ↪  relevant_times))
end
```