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Matroid invariants from Schubert calculus

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Abstract

Grassmannians are varieties parameterizing the k-dimensional subspaces of an n-dimensional vector space. To a point in a Grassmannian we associate two objects:

- A representable matroid storing combinatorial data about linear independence among the columns of a matrix whose rowspace is the point.
- The projective toric variety obtained as the closure of the orbit of the point under the action of an algebraic torus on the Grassmannian.

We study the class of the torus orbit closure in the integral cohomology ring of the Grassmannian. Its coefficients in the Schubert basis are matroid invariants, which we call Schubert coefficients of a matroid. We show that the Schubert coefficients satisfy a linear relation and that they respect the dual matroid. If the matroid contains a loop or a coloop we describe how to obtain its Schubert coefficients from the Schubert coefficients of the matroid with the loop or coloop removed.

Next we follow [BF22] to define Schubert coefficients of non-representable matroids, that is, matroids that cannot be obtained from points in Grassmannians. We confirm the conjectured positivity of these Schubert coefficients in a special case. We use valuativity and results from the representable case to compute the Schubert coefficients of some non-representable matroids, namely the Fano, non-Pappus and Vamos matroids. In each case we confirm the positivity of the Schubert coefficients.

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Chapter 1

Introduction

This thesis aims to shed light on a problem in the intersection of three different subjects: Schubert calculus, Toric geometry and Matroid theory.

Schubert calculus [ORb].

Herman Schubert (1848-1911) was a German mathematician interested in enumerative geometry. This is the subject devoted to questions about counting geometric objects satisfying certain conditions. Schubert answered many such questions, but his calculations were based on unstable grounds. One of the famous problems Hilbert posed in 1900 was to formalize Schubert's computations. The answer is framed as the intersection theory of the Grassmannians $G(k, n)$, the projective varieties parameterizing k-dimensional subspaces of an n-dimensional vector space V over $\mathbb{C}$. Schubert varieties are certain subvarieties of $G(k, n)$ given by the k-spaces that intersect a fixed flag of V in a particular way. The data of the intersection is stored in integer partitions λ. The classes of these Schubert varieties in the cohomology ring of the Grassmannian $H^*(G(k, n))$ are called Schubert cycles, and are denoted σ_λ. Schubert cycles form a basis of $H^*(G(k, n))$. The multiplicative structure of $H^*(G(k, n))$ encodes the intersection data of Schubert varieties. This is what is known as Schubert calculus.

Toric geometry.

Toric varieties provide a rich source of examples in algebraic geometry. Due to their combinatorial nature, certain computations are easier for toric varieties than for classical algebraic varieties or schemes. A toric variety is obtained from a polyhedral object such as a cone, fan or polytope. Specifically, given a polytope Δ we obtain a projective toric variety X, together with an embedding into projective space. Many properties of X can be determined by properties of the polytope Δ.

Matroid theory [ORa].

Matroid theory was discovered independently by Japanese mathematician Takeo Nakasawa (1913-1946) and American climber and mathematician Hassler Whitney (1907–1989). The idea behind matroid theory is to unify notions of independence from many different areas of mathematics, such as: linear algebra, graph theory and algebra. From the viewpoint of linear algebra one associates a matroid to a finite set of vectors E in a vector space V. The matroid M keeps track of the linear independence among the vectors, for example by remembering which subsets of E are bases of V. Famously, there are many equivalent definitions of matroids. One which will be important for us was proven by Gelfand, Goresky, MacPherson and Serganova in [Gel+87]. Here a matroid

M is encoded in a certain matroid polytope $\Delta(M)$.

A point x in a Grassmanninan $G(k,n)$ can be described as the rowspace of a full rank k by n matrix W. From the point x we get two objects. First, the matroid M_x that stores the information of linear independence among the columns of W. Second, a toric subvariety $\overline{Tx}$ of $G(k,n)$ obtained as the Zariski closure of the orbit of x under the action of the algebraic torus $T = (\mathbb{C}^*)^n$ on $G(k,n)$. The connection between M_x and $\overline{Tx}$ is that $\overline{Tx}$ is isomorphic to the toric variety of the matroid polytope $\Delta(M_x)$. The class of $\overline{Tx}$ in the cohomology ring $H^*(G(k,n))$ can be expressed as a sum of Schubert cycles

$$[\overline{Tx}] = \sum_{\lambda} \alpha_\lambda \sigma_\lambda,$$

for positive integer coefficients α_λ.

Problem. Determine the coefficients of the class of $\overline{Tx}$ in the Schubert basis of $H^*(G(k,n))$. ♢

The coefficient α_λ is the number of points in the transverse intersection of $\overline{Tx}$ and the Schubert variety X_{λ^c}. These coefficients are matroid invariants and we wish to determine them from the matroid M_x. For a fixed partitions λ we write $\alpha_\lambda(M_x)$ for the corresponding coefficient.

In 1985 Alexander Klyachko [Kly85] gave a formula for the Schubert coefficients of the most general points, namely those where M_x is the uniform matroid $U_{k,n}$. In 2009 David E. Speyer [Spe09] showed that the Schubert coefficient corresponding to the square partition $(k-1)\times(n-k-1)$ is the well known beta invariant $\beta(M_x)$. The Schubert coefficients are only defined for representable matroids. That is, those obtained from a point in a Grassmannian. In 2022 Andrew Berget and Alex Fink [BF22] used equivariant K-theory to give similar coefficients for arbitrary matroids. They are given by $\mathrm{Sc}(M)$, a homogeneous, symmetric polynomial in $u_1,\dots,u_k$. We express $\mathrm{Sc}(M)$ in terms of Schur polynomials

$$\mathrm{Sc}(M) = \sum_{\lambda} \alpha_\lambda s_\lambda(u)$$

for integers α_λ.

Problem. Determine the coefficients of the Schur polynomial expansion of $\mathrm{Sc}(M)$. ♢

These coefficients are again matroid invariants. If $M = M_x$ for some point $x \in G(k,n)$, these coefficients are equal to those obtained from $[\overline{Tx}]$. In this way we extend the notion of Schubert coefficients to any matroid and we write $\alpha_\lambda(M)$. The positivity of the Schubert coefficients of a non-representable matroid is unknown.

Conjecture ([BF22]). *The Schubert coefficients of an arbitrary matroid are positive.* ♢

This is the simplest of several positivity conjectures in [BF22]. We investigate these two problems independently. First we consider the representable case of $[\overline{Tx}]$, then the non-representable case of $\mathrm{Sc}(M)$. Our findings in each case are as follows.

The representable case.

First we use previously known results in Schubert calculus and toric geometry to remark

Proposition (5.2.2)**.**

$$\deg([\overline{Tx}]\sigma_{(1)}^{\dim(\overline{Tx})}) = \mathrm{Vol}(\Delta(M_x)). \qquad \heartsuit$$

The product of Schubert cycles $\sigma_{(1)}^{\dim(\overline{Tx})}$ and the volume of the matroid polytope $\Delta(M_x)$ are possible to compute, so this result says that the Schubert coefficients of M_x satisfies a linear relation

$$\mathrm{Vol}(\Delta(M_x)) = \sum_{\lambda} \deg(X_\lambda)\alpha_\lambda(M_x).$$

Later we will use this result to compute the Schubert coefficients of some non-representable matroids. We then, in Proposition 5.2.6, use a well known isomorphism of Grassmannians to see that the Schubert coefficients respect the dual matroid. Next we look at the case when the matroid M_x contains a loop i. We show how to obtain the Schubert coefficients of M_x from the Schubert coefficients of the matroid $M_x \setminus i$ obtained from M_x by deleting the loop.

Theorem (5.2.9)**.** *Let $x \in G(k, n)$ be such that i is a loop of M_x. If $\lambda_k = 0$ then $\alpha_\lambda(M_x) = 0$ otherwise $\alpha_\lambda(M_x) = \alpha_{|\lambda}(M_x \setminus i)$.* $\heartsuit$

Here $|\lambda$ is the partition obtained from λ by removing the first column of its Young diagram. Similarly if i is a coloop of M_x we show how to obtain the Schubert coefficients of M_x from the Schubert coefficients of the matroid M_x/i obtained from M_x by contracting the coloop.

Theorem (5.2.10)**.** *Let $x \in G(k, n)$ be such that i is a coloop of M_x. If $\lambda_1 \neq n - k$ then $\alpha_\lambda(M_x) = 0$ otherwise $\alpha_\lambda(M_x) = \alpha_{\overline{\lambda}}(M_x/i)$.* $\heartsuit$

Here $\overline{\lambda}$ is the partition obtain from λ by removing the first row of its Young diagram. By combining these two results we describe in Corollary 5.2.11 how to obtain the Schubert coefficients of a matroid M_x, containing several loops and coloops, from the matroid with the loops and coloops removed.

The non-representable case.

Using the tools from [BF22] we observe in Proposition 6.2.5 that the matroid invariant $\mathrm{Sc}(M)$ and hence the Schubert coefficients are top-valuative. This means that given a matroid subdivision of the polytope $\Delta(M)$ with maximal dimensional pieces $\Delta(M_1), \ldots, \Delta(M_r)$ we have

$$\mathrm{Sc}(M) = \sum_i \mathrm{Sc}(M_i).$$

We use this to confirm the conjectured positivity of the Schubert coefficients in a special case.

Corollary (6.3.1)**.** *If there exists a matroid subdivision of the matroid polytope $\Delta(M)$ into pieces representable over $\mathbb{C}$, the Schubert coefficients of M are positive.* $\heartsuit$

Next we compute the Schubert coefficients of some examples of non-representable matroids, namely the Fano matroid, the non-Pappus matroid and the Vamos matroid. For example the Schubert coefficients of the Fano matroid F is

$$\mathrm{Sc}(F) = 6s_{(4,2,0)}(u) + 3s_{(4,1,1)}(u) + 3s_{(3,3,0)}(u) + 8s_{(3,2,1)}(u) + s_{(2,2,2)}(u).$$

In all the examples we confirm their positivity.

Theorem (Example 6.3.2, 6.3.3 and 6.3.4)**.** *The Schubert coefficients of the Fano, non-Pappus and Vamos matroids are positive.* ♡

To perform these calculation we use Macaulay2 [GS] to find certain matroid subdivisions. With these subdivisions and the top-valuativity of $\mathrm{Sc}(M)$ we reduce the problem to the representable case. Using the linear relation from the proposition above together with the formula for the Schubert coefficients of uniform matroids, we finish the calculations.

1.1 Notation

For a positive integer n let $[n] = \{1, \ldots, n\}$. For any set S we denote the set of subsets of S with k elements by $\binom{S}{k}$. In particular we have the set $\binom{[n]}{k}$ of all sets of k integers between 1 and n. We denote the the sets in $\binom{[n]}{k}$ by $I = \{i_1, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$. For $I \in \binom{[n]}{k}$ the the complement of I is $I^c = [n] \smallsetminus I \in \binom{[n]}{n-k}$. In examples, when n is small we may abuse notation and write $I = i_1 i_2 \ldots i_n$. For example for $n = 5$ and $k = 3$ we will write 134 instead of $\{1, 3, 4\}$, then $134^c = 25$. Also, for a singleton $\{t\}$ and a set S we write $S \cup t$ and $S \smallsetminus t$ instead of $S \cup \{t\}$ and $S \smallsetminus \{t\}$.

Throughout we let e_i and ϵ_i for $i = 1, \ldots, n$ be the standard bases of the vector spaces $\mathbb{C}^n$ and $\mathbb{R}^n$ respectively. If we want to specify the basis of a vector space over a field F by a finite set E, we write F^E. For example for a fixed $i \in [n]$ the basis of the vector space $\mathbb{C}^{[n] \smallsetminus i}$ is $\{e_1, \ldots, \hat{e}_i, \ldots, e_n\}$.

Warning: Whenever we say a point $x \in G(k, n)$ is generic we mean that the following equivalent statements holds: the matroid M_x is connected, the polytope $\Delta(M_x)$ is of dimension $n - 1$, the variety $\overline{Tx}$ is of dimension $n - 1$.

1.2 Outline

Chapter 2:

An introduction to Grassmannians and Schubert Calculus.

Chapter 3:

An introduction to toric geometry and the definition of torus orbit closures.

Chapter 4:

An introduction to matroid theory.

Chapter 5:

Results about Schubert coefficients of representable matroids.

Chapter 6:

Definition and results about Schubert coefficients of non-representable matroids and some computations.

Appendix A:

A collection of Macaulay2 programs used throughout the text and some examples of their use.

Chapter 2

Schubert calculus

2.1 Grassmannians

For two natural numbers k and n we define the Grassmannian $G(k, n)$ to be the set of k-dimensional subspaces of an n-dimensional vector space. When we want to specify the vector space we may write $G(k, V)$ where V is the n-dimensional vector space over $\mathbb{C}$. For the most part we will think of V as $\mathbb{C}^n$.

A k-dimensional vector subspace $W \subset V$ can be thought of as a $(k-1)$-dimensional linear subvariety of $\mathbb{P}V$. When we want to think of Grassmannians as the set of $(k-1)$-dimensional linear subspaces of $(n-1)$-dimensional projective space we use the notation $\mathbb{G}(k-1, \mathbb{P}V)$ or just $\mathbb{G}(k-1, n-1)$ if we do not need the vector space V and we think of $\mathbb{P}V$ as $\mathbb{P}^{n-1}$.

Example 2.1.1. The simplest example of a Grassmannian is $G(1, 2)$, the set of lines through the origin of $\mathbb{C}^2$. We recognise this as $\mathbb{P}^1$, the projective line. In general $G(1, n) = \mathbb{G}(0, n-1) \simeq \mathbb{P}^{n-1}$. $\diamondsuit$

2.2 The Plücker embedding

As seen in Example 2.1.1, the Grassmannians $G(1, n)$ have the structure of a projective variety. In fact all Grassmannians are projective varieties. We can show this by giving an embedding of $G(k, n)$ into $\mathbb{P}^{\binom{n}{k}-1}$ called the Plücker embedding. The following example will illustrate the process.

Example 2.2.1. Consider $G = G(2, V)$ the set of planes through the origin of a 4-dimensional vector space V with basis $\{e_1, e_2, e_3, e_4\}$. An element $W \in G$ spanned by two vectors $u = (a_0, a_1, a_2, a_3)$ and $v = (b_0, b_1, b_2, b_3)$ can be expressed as the rowspace of the matrix

$$M_W = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}.$$

Let $p_{ij} = a_i b_j - a_j b_i$ denote the 2×2-minor of M_W obtained from column i and j. There are 6 such minors p_{01}, p_{02}, p_{03}, p_{12}, p_{13} and p_{23}. The following map is the Plücker embedding

$$\begin{aligned} \iota\colon G &\to \mathbb{P}^5 \\ W &\mapsto (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}). \end{aligned}$$

The minors p_{ij} are called Plücker coordinates. To show that G is a projective variety we need to check that ι is a well defined embedding whose image is closed in $\mathbb{P}^5$. First, a different choice of vectors spanning W will only change all the minors by the same scalar, hence it does not change the point in $\mathbb{P}^5$. Since the rows span a 2-dimensional vector space there are 2 linearly independent columns, which shows that at least one Plücker coordinate is non-zero. For example if $p_{01} \neq 0$, row reducing M_W to the form

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$$

corresponds to scaling the coordinates of the point $\iota(W) = (1 : p'_{02} : p'_{03} : p'_{12} : p'_{13} : p'_{23})$. Now the numbers a, b, c, d are determined by $\iota(W)$, we see for example that $a = -p'_{12}$. The last thing to check is that $\iota(G)$ is closed in $\mathbb{P}^5$. The dimension of G is 4, informally this can be seen by the 4 free variables in the matrix above, so we want to show that $\iota(G)$ is a hypersurface in $\mathbb{P}^5$. Consider the matrix with repeated rows

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix},$$

whose determinant is 0. Expanding the determinant in terms of the Plücker coordinates gives us the following relation

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0. \tag{2.1}$$

Hence $\iota(G) = V(p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12})$ which shows, somewhat informally, that G is a projective variety. $\diamondsuit$

To show that all Grassmannians are projective varieties we formalize the ideas of the example above. The following is only a sketch of the construction, for a more detailed explanation see [EH16, Chapter 3.2.1].

Let $G = G(k, V)$ where V is an n-dimensional vector space with basis $\{e_1, \dots, e_n\}$. The exterior algebra $\bigwedge^k V$ is an $\binom{n}{k}$-dimensional vector space whose elements are wedges $v_1 \wedge \cdots \wedge v_k$ where $v_1, \dots v_k \in V$ with the relation that permuting two vectors changes the sign. For example $v_2 \wedge v_1 \wedge v_3 \wedge \cdots \wedge v_k = -v_1 \wedge v_2 \wedge v_3 \wedge \cdots \wedge v_k$. A basis for $\bigwedge^k V$ in terms of the basis for V is

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

An element $W \in G$ can be described as the span of k vectors $v_1, \dots v_k \in V$, and the Plücker embedding is given by

$$\begin{aligned} \iota\colon G &\to \mathbb{P}(\textstyle\bigwedge^k V) \simeq \mathbb{P}^{\binom{n}{k}-1} \\ W &\mapsto [v_1 \wedge \cdots \wedge v_k]. \end{aligned}$$

Note that expressing $v_1 \wedge \cdots \wedge v_k$ in terms of the basis of $\bigwedge^k V$ gives us

$$v_1 \wedge \cdots \wedge v_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} p_{i_1 \dots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$$

where $p_{i_1 \dots i_k}$ is the Plücker coordinate of W given by the $k \times k$-minor of W, obtained by selecting the columns $i_1, \dots, i_k$. As in the example above ι is an embedding of G into projective space giving the structure of a projective variety to G. The relations on the Plücker coordinates that describes $\iota(G)$ are called the Plücker relations. See [Man01, p. 103] for a description of the Plücker relations for a general Grassmannian.

We will denote the homogeneous coordinates on $\mathbb{P}^{\binom{n}{k}-1}$ by p_I for $I \in \binom{[n]}{k}$. For $x \in G(k,n)$ we denote the coordinates of x in $\mathbb{P}^{\binom{n}{k}-1}$ by $p_I(x)$. By a point $x \in G(k,n)$ we will mean the point in projective space i.e. $x = (p_I(x))_{I \in \binom{[n]}{k}}$, the corresponding subspace of V is denoted $L(x)$.

2.3 Schubert cells

In order to study a Grassmannian we decompose it into subsets based on how elements intersect a fixed sequence of subspaces of V.

Definition 2.3.1. A complete flag $V_\bullet$ of V is a sequence of subspaces of V

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

where the dimension of V_j is j. Given a basis $\{e_1, \dots e_n\}$ of V we define the standard flag by $V_i = \langle e_1, \dots e_i \rangle$. ♢

Given a complete flag $V_\bullet$ and a point $x \in G(k,n)$, we consider the following sequence

$$0 = \dim(L(x) \cap V_0) \leq \dim(L(x) \cap V_1) \leq \cdots \leq \dim(L(x) \cap V_n) = k.$$

At any step in this sequence the dimension can at most increase by 1. We will be especially interested in the position of the sequence where it jumps, that is where it increases by 1. The subset of points in $G(k,n)$ with the same sequence as x is what we will call the Schubert cell containing x.

Example 2.3.2. Let $\{e_1, \dots, e_4\}$ be a basis of V and $V_\bullet$ be the standard flag. Consider the point $x \in G(2,V)$ given by $L(x) = \langle e_1, e_2 + e_3 \rangle$. We obtain the following sequence

$$\begin{aligned} \dim(L(x) \cap V_0) &= 0 \\ \dim(L(x) \cap V_1) &= 1 \\ \dim(L(x) \cap V_2) &= 1 \\ \dim(L(x) \cap V_3) &= 2 \\ \dim(L(x) \cap V_4) &= 2. \end{aligned}$$

Notice that this sequence jumps in positions 1 and 3. ♢

To be precise for any $x \in G(k,n)$ we get the sequence $(\dim(L(x) \cap V_j))_{j=0}^{n}$ and a sequence $I = (j_i)_{i=1}^{k}$ such that

$$\begin{aligned}
0 &= \dim(L(x) \cap V_0) = \cdots = \dim(L(x) \cap V_{j_1 - 1}) \\
1 &= \dim(L(x) \cap V_{j_1}) = \cdots = \dim(L(x) \cap V_{j_2 - 1}) \\
&\vdots \\
k &= \dim(L(x) \cap V_{j_k}) = \cdots = \dim(L(x) \cap V_n).
\end{aligned}$$

We call I the jumping sequence associated to x with respect to the complete flag $V_\bullet$.

Definition 2.3.3. A jumping sequence is a strictly increasing sequence of k integers between 1 and n. The set of jumping sequences is

$$\mathcal{I}(k,n) = \{I = (j_1, j_2, \ldots j_k) \mid 1 \leq j_1 < j_2 < \cdots < j_k \leq n\}. \qquad \diamondsuit$$

We are now ready to give the definition of a Schubert cell associated to a jumping sequence.

Definition 2.3.4. Let $I = (j_1, \ldots, j_k) \in \mathcal{I}(k,n)$. The Schubert cell associated to I with respect to a complete flag $V_\bullet$ is

$$\begin{aligned}
\Omega_I(V_\bullet) &= \{x \in G(k,n) \mid I \text{ is the jumping sequence associated to } x\} \\
&= \{x \in G(k,n) \mid \dim(L(x) \cap V_j) = i \text{ if } j_i \leq j < j_{i+1}\}
\end{aligned}$$

We use a convention that $j_0 = 0$ and $j_{k+1} = n+1$ for the edge cases. When $V_\bullet$ is the standard flag we may write Ω_I. $\diamondsuit$

From this definition there is an immediate property of the Schubert cells.

Proposition 2.3.5. *The Schubert cells are disjoint and their union is all of $G(k,n)$,*

$$G(k,n) = \bigsqcup_{I \in \mathcal{I}(k,n)} \Omega_I(V_\bullet). \qquad \heartsuit$$

Proof. A point $x \in G(k,n)$ has a unique jumping sequence. $\square$

The following examples examine the geometric meaning of the Schubert cells.

Example 2.3.6. Consider the Grassmannian $G = G(1,4) = \mathbb{G}(0,3)$, we will think of G as the projective space $\mathbb{P}^3$. In this setting a complete flag $V_\bullet$ becomes a point, a line and a plane, $p \in L \subset H \subset \mathbb{P}^3$. And for example the conditions $\dim(L(x) \cap V_1) = 0$ and $\dim(L(x) \cap V_2) = 1$ translates to that the point in $x \in \mathbb{P}^3$ is on the line L, but not equal to p. This is the Schubert cell corresponding to the jumping sequence (2). We continue with all the 4 Schubert cells indexed by $\mathcal{I}(1,4)$.

$$\begin{aligned}
\Omega_{(1)} &= \{p\} \\
\Omega_{(2)} &= \{x \in \mathbb{P}^3 \mid p \neq x \in L\} \\
\Omega_{(3)} &= \{x \in \mathbb{P}^3 \mid x \in H \smallsetminus L\} \\
\Omega_{(4)} &= \{x \in \mathbb{P}^3 \mid x \notin H\}. \qquad \diamondsuit
\end{aligned}$$

Using the basis $\{e_1, \ldots, e_n\}$ of V and the standard flag, any point $x \in G(k,n)$ can be defined by some k by n matrix W such that $L(x)$ is the rowspace of W. Since row operations on a matrix do not change the rowspace we may use this to find a standard form for the matrices corresponding to the points belonging to a given Schubert cell.

Consider a jumping sequence $I = (j_1, \ldots, j_k) \in \mathcal{I}(k,n)$ and $x \in \Omega_I$. The subspace $L(x)$ is spanned by k linear independent vectors and the conditions $i = \dim(L(x) \cap V_{j_i}) = \cdots = \dim(L(x) \cap V_{j_{i+1}-1})$ imposed on $L(x)$ ensure that we can pick vectors of the form $w_i = e_{j_i} + \sum_{m=1}^{j_i-1} b_{i,m} e_m$ that span $L(x)$, with no restrictions on the coefficients $b_{i,m}$.

This means that we can express all $L(x)$ as the rowspace of a k by n matrix of the following form

$$\begin{pmatrix} * & \ldots & * & 1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ * & \ldots & * & * & * & \ldots & * & 1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & & & & \vdots \\ * & \ldots & * & * & * & \ldots & * & * & * & \ldots & * & 1 & 0 & \ldots & 0 \end{pmatrix} \tag{2.2}$$

The $*$'s take the place of the coefficients $b_{i,m}$, and the 1 in row i is in column j_i. Using row reduction we can replace any $*$ below a 1 by a 0 to obtain the matrix

$$\begin{pmatrix} * & \ldots & * & 1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ * & \ldots & * & 0 & * & \ldots & * & 1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & & & & & \vdots \\ * & \ldots & * & 0 & * & \ldots & * & 0 & * & \ldots & * & 1 & 0 & \ldots & 0 \end{pmatrix} \tag{2.3}$$

Each $*$ in the matrix above represents a free variable of the Schubert cell.

Example 2.3.7. Consider the Grassmannian $G(2,4)$. There are six different jumping sequences

$$\mathcal{I}(k,n) = \{(3,4), (2,4), (1,4), (2,3), (1,3)(1,2)\}.$$

The elements of each associated Schubert cell are spanned by matrices of the following form.

$$\Omega_{(3,4)} \rightsquigarrow \begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix} \rightsquigarrow \mathbb{A}^4$$

$$\Omega_{(2,4)} \rightsquigarrow \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix} \rightsquigarrow \mathbb{A}^3$$

$$\Omega_{(1,4)} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{pmatrix} \rightsquigarrow \mathbb{A}^2$$

$$\Omega_{(2,3)} \rightsquigarrow \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \mathbb{A}^2$$

$$\Omega_{(1,3)} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{pmatrix} \rightsquigarrow \mathbb{A}^1$$

$$\Omega_{(1,2)} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \rightsquigarrow \mathbb{A}^0. \qquad \diamondsuit$$

2.4 Partitions vs jumping sequences

We have now defined the Schubert cells corresponding to a jumping sequence, but there is another viewpoint which will be helpful. Jumping sequences correspond to certain integer partitions so we may index the Schubert cells by these partitions.

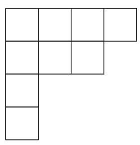
Definition 2.4.1. A partition λ is a collection of weakly decreasing positive integers. That is $\lambda = (\lambda_1, \lambda_2, \dots \lambda_l)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0$. We do not differ between partitions whose only difference is more zeros at the end (so $(3,2,1) = (3,2,1,0,0)$). We also define the following notions

- The non-zero entries of λ are called the parts of λ.
- The length $l(\lambda)$ is the number of parts.
- The size of λ is $|\lambda| = \sum_{i=1}^{l(\lambda)} \lambda_i$.
- We say λ is a partition of $|\lambda|$ and write $\lambda \vdash |\lambda|$.
- The partition $p \times q = (q, q, \dots, q)$ with length p is called the p by q rectangle. We may denote it by $(q)^p$.
- We say $\mu \subset \lambda$ if $\mu_i \leq \lambda_i$ for all i. This is a partial ordering of partitions.
- If $\lambda = (\lambda_1, \dots \lambda_l) \subset p \times q$ the complement of λ in $p \times q$ is $\lambda^c = (\lambda_1^c, \dots, \lambda_l^c)$ where $\lambda_i^c = q - \lambda_{p+1-i}$.
- For a partition λ and a positive integer b we denote

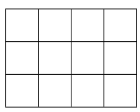
  $$\lambda \otimes b = \{\mu \mid \lambda_i \leq \mu_i \leq \lambda_{i-1} \text{ for all } i, |\mu| = |\lambda| + b\}$$

  where the condition $\mu_1 \leq \lambda_0$ is considered true. ♢

We can visualize a partition by its Young diagram (or Ferrers diagram). For a partition $\lambda = (\lambda_1, \dots \lambda_l)$ the Young diagram of λ is a set of boxes arranged in rows with λ_i boxes in row i. For example the Young diagram of $(4,3,1,1)$ is



Note that the Young diagram of the p by q rectangle $p \times q$ is exactly what we expect. For example the Young diagram of 3×4 is



The notions in Definition 2.4.1 are more natural if viewed as properties of Young diagrams.

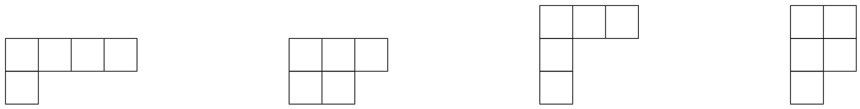
- $|\lambda|$ is the number of boxes of the Young diagram.
- $\mu \subset \lambda$ if the Young diagram of μ can fit inside the Young diagram of λ.

- If $\lambda \subset p \times q$ then λ^c is the partition corresponding to the boxes of the Young diagram of $p \times q$ that are not part of λ (rotated 180 degrees to make it a Young diagram).
- $\lambda \otimes b$ is the set of partitions μ obtained from λ by adding b boxes to the Young diagram of λ, at most one per column.

Example 2.4.2. Let $\lambda = (4, 2, 1) \subset 3 \times 4$. The Young diagrams of λ and λ^c are as follows.

Notice that they fit together to make the enitre young diagram of 3×4. $\diamondsuit$

Example 2.4.3. Let $\lambda = (2, 1)$ and $b = 2$. The partitions in $\lambda \otimes b$ have the following Young diagrams



$\diamondsuit$

We will be especially interested in the k by $n-k$ rectangle $k \times (n-k)$ and all partitions $\lambda \subset k \times (n-k)$.

Definition 2.4.4. The set of partitions contained in $k \times (n-k)$ is denoted

$$\Lambda(k, n) = \{\lambda \mid \lambda \subset k \times (n-k)\}. \qquad \diamondsuit$$

Notice that the empty partition, denoted $\emptyset$, is in $\Lambda(k, n)$. There is a natural bijection between $\Lambda(k, n)$ and $\mathcal{I}(k, n)$ given by

$$\begin{aligned} \Phi \colon \Lambda(k, n) &\to \mathcal{I}(k, n) \\ (\lambda_i)_{i=1}^k &\mapsto (n-k+i-\lambda_i)_{i=1}^k. \end{aligned} \tag{2.4}$$

Hence we can index our Schubert cells by the partitions contained in $k \times (n-k)$ and we will write $\Omega_\lambda(V_\bullet)$ for the Schubert cell associated to λ. To be explicit we get

$$\Omega_\lambda(V_\bullet) = \{x \in G(k, n) \mid \dim(L(x) \cap V_j) = i \text{ if } n-k+i-\lambda_i \leq j \leq n-k+i-\lambda_{i+1}\}$$

Here i ranges from 0 to k and we consider $\lambda_{k+1} = 0$ and $\lambda_0 = n-k$. The bijection Φ

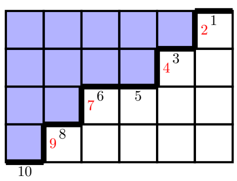


Figure 2.1: The Young diagram of λ from Example 2.4.5 in blue inside the rectangle 4×6. The jumping sequence $\Phi(\lambda)$ is the numbers in red.

might seem cumbersome, but we illustrate how to visualize it in terms of Young diagrams with an example.

Example 2.4.5. Let $\lambda \in \Lambda(k,n)$. Now picture the Young diagram of λ inside the Young diagram of the rectangle $k \times (n-k)$ and number the edges of the path from the top right to the bottom left corner of the rectangle along λ with the numbers from 1 to n. The jumping sequence $\Phi(\lambda)$ is given by the numbers on the vertical edges. Figure 2.1 pictures the procedure for $\lambda = (5,4,2,1) \subset 4 \times 6$, we see that $\Phi(\lambda) = (2,4,7,9)$. $\diamondsuit$

We will frequently switch between partitions and jumping sequences depending on the situation. Often jumping sequences will be useful in proofs, and partitions will give nice combinatorial descriptions of properties.

2.5 Schubert varieties

Definition 2.5.1. Let $V_\bullet$ be a complete flag and λ a partition contained in $k \times (n-k)$. We define the Schubert variety $X_\lambda(V_\bullet)$ associated to λ with respect to $V_\bullet$ to be the closure of the Schubert cell $\Omega_\lambda(V_\bullet)$. It can be shown that

$$X_\lambda(V_\bullet) = \{x \in G(k,n) \mid \dim(L(x) \cap V_{j_i}) \geq i\}$$

where $(j_i)_{i=1}^k = \Phi(\lambda)$. $\diamondsuit$

The following example highlights why we are interested in Schubert varieties.

Example 2.5.2. Consider the Grassmannian $G(2,4) = \mathbb{G}(1,3)$ and let $\lambda = (1,0)$. A complete flag $V_\bullet$ corresponds to a choice of a point p, a line L and a plane H in $\mathbb{P}^3$ such that $p \in L \subset H$. The Schubert variety $X_\lambda(V_\bullet) = \{q \subset \mathbb{P}^3 \mid q \text{ is a line and } q \cap L \neq \emptyset\}$. So $X_\lambda(V_\bullet)$ is the variety that parametrizes lines intersecting L. The question "How many lines in $\mathbb{P}^3$ intersect four general lines?" can be answered by finding the intersection of this Schubert variety with respect to four different complete flags. We continue in Example 2.6.1. $\diamondsuit$

The following are some important properties that will help us describe the intersection of Schubert varieties.

Proposition 2.5.3. [Man01, Proposition 3.2.3] *Let λ and λ' be partitions contained in $k \times (n-k)$, then*

1. $\Omega_\lambda(V_\bullet) \simeq \mathbb{A}^{k(n-k)-|\lambda|} = \mathbb{A}^{|\lambda^c|}$
2. $X_\lambda(V_\bullet) = \bigsqcup_{\mu \supset \lambda} \Omega_\mu(V_\bullet)$
3. $X_\lambda(V_\bullet) \subset X_{\lambda'}(V_\bullet) \iff \lambda' \subset \lambda$. $\heartsuit$

Proof. 1) For a jumping sequence $I = (j_1, \dots j_k) \in \mathcal{I}(k,n)$ the isomorphism between $\Omega_I(V_\bullet)$ and affine space is given by considering each $*$ in the matrix in (2.3) as an affine coordinate. The number of $*$'s in the ith row is $j_i - i$. Using Φ we see that the number of $*$'s in the matrix expressed in terms of the partition λ corresponding to I is

$$\sum_{i=1}^k j_i - i = \sum_{i=1}^k n - k - \lambda_i = k(n-k) - |\lambda|. \tag{2.5}$$

So we get $\Omega_\lambda(V_\bullet) \simeq \mathbb{A}^{k(n-k)-|\lambda|}$.

2) Let $(j_i)_{i=1}^k = \Phi(\lambda)$ and $(l_i)_{i=1}^k = \Phi(\mu)$ then $\mu \supset \lambda \iff \forall i \; l_i \leq j_i$. Assume $\mu \supset \lambda$ then

$$x \in \Omega_\mu(V_\bullet) \iff \forall i \dim(L(x) \cap V_{l_i}) = i \implies \forall i \dim(L(x) \cap V_{j_i}) \geq i \iff x \in X_\lambda(V_\bullet),$$

where the middle implication holds because $l_i \leq j_i$. Hence $\bigsqcup_{\mu \supset \lambda} \Omega_\mu(V_\bullet) \subset X_\lambda$. For the other inclusion assume $x \in X_\lambda(V_\bullet)$ then $x \in \Omega_\mu(V_\bullet)$ for some μ. Again let $(l_i)_{i=1}^k = \Phi(\mu)$ if there is some i such that $l_i > j_i$ then $\dim(L(x) \cap V_{j_i}) < i$ contradicting $x \in X_\lambda$, so $l_i \leq j_i$ for all i and $\mu \supset \lambda$. Hence $X_\lambda(V_\bullet) = \bigsqcup_{\mu \supset \lambda} \Omega_\mu(V_\bullet)$.

3) If $\lambda' \subset \lambda$ then $\lambda \subset \mu \implies \lambda' \subset \mu$ so $\Omega_\mu(V_\bullet) \subset X_\lambda(V_\bullet) \implies \Omega_\mu(V_\bullet) \subset X_{\lambda'}(V_\bullet)$ hence $X_\lambda(V_\bullet) \subset X_{\lambda'}(V_\bullet)$. For the other implication assume $X_\lambda(V_\bullet) \not\subset X_{\lambda'}(V_\bullet)$ then at least one partition μ is such that $\lambda \subset \mu$ and $\lambda' \not\subset \mu$. Then, for some i we must have $\lambda_i \leq \mu_i < \lambda'_i$ so $\lambda' \not\subset \lambda$. □

By "different" complete flags in Example 2.5.2, we mean that they are transverse. Loosely speaking two flags $V_\bullet$ and $W_\bullet$ are transverse if V_i and W_j intersect in the smallest dimension possible. Given a basis $\{e_1, \ldots, e_n\}$ of V the flag $V'_\bullet$ given by $V'_i = \langle e_{n+1-i}, \ldots e_n \rangle$ is called the inverse flag. The standard flag and the inverse flag is the most natural choice of a transverse pair of flags.

Definition 2.5.4. [EH16, Definition 4.4] Two flags $V_\bullet$ and $W_\bullet$ of an n-dimensional vector space V are called transverse if

$$V_i \cap W_{n-i} = 0$$

for all i. For any pair of transverse flags there is a basis of V such that the flags are the standard and inverse flags with respect to that basis. ♢

Let $V_\bullet$ be the standard flag and $V'_\bullet$ the inverse flag. In (2.3) we gave a standard form for matrices whose rowspace is $L(x)$ for some point $x \in \Omega_\lambda(V_\bullet)$. Similarly we can give the following standard form for the $k \times n$-matrices whose rowspace is $L(x)$ for some point $x \in \Omega_\lambda(V'_\bullet)$

$$\begin{pmatrix} 0 & \ldots & 0 & 1 & * & \ldots & * & 0 & * & \ldots & * & 0 & * & \ldots & * \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & 1 & * & \ldots & * & 0 & * & \ldots & * \\ \vdots & & & & & & & & & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & 1 & * & \ldots & * \end{pmatrix}. \tag{2.6}$$

Here the 1 in row i is in column $n+1-j_{k+1-i}$. This may look complicated, but it is simply rotating the matrix in (2.3) 180 degrees. In the following results for a partition λ we write $\Omega_\lambda = \Omega_\lambda(V_\bullet)$ and $\Omega'_\lambda = \Omega_\lambda(V'_\bullet)$, and similarly for the Schubert varieties $X_\lambda = X_\lambda(V_\bullet)$ and $X'_\lambda = X_\lambda(V'_\bullet)$.

Lemma 2.5.5. [Man01, Proposition 3.2.7] *For any partition $\lambda \subset k \times (n-k)$ corresponding to the jumping sequence $(j_i)_{i=1}^k$ the intersection $\Omega_\lambda \cap \Omega'_{\lambda^c}$ is the point $x \in G(k,n)$ given by $L(x) = \langle e_{j_1}, \ldots, e_{j_k} \rangle$.* ♡

Proof. Let $(j_i)_{i=1}^k = \Phi(\lambda)$ and $(l_i)_{i=1}^k = \Phi(\lambda^c)$. By using Definition 2.4.1 and Φ we have that $l_i = n+1-j_{k+1-i}$. An element of $\Omega_\lambda \cap \Omega'_{\lambda^c}$ must be expressible as a matrix of the form (2.3) and (2.6) at the same time. But in row i of each matrix the 1 of (2.3) is in column j_i and the 1 in (2.6) is in column $n+1-l_{k+1-i} = j_i$. Hence the only element in the intersection is the point x with $L(x) = \langle e_{j_1}, \ldots, e_{j_k} \rangle$. □

Lemma 2.5.6. [Man01, Proposition 3.2.7] *For partitions η, $\tau \subset k \times (n-k)$ we have $\Omega_\eta \cap \Omega'_\tau \neq \emptyset \implies \eta \subset \tau^c$.* ♡

Proof. As in the previous proof we let $(j_i)_{i=1}^k = \Phi(\eta)$ and $(l_i)_{i=1}^k = \Phi(\tau)$. As before, if $\Omega_\eta \cap \Omega'_\tau \neq \emptyset$, then an element of the intersection must be expressible as a matrix of the forms (2.3) and (2.6). For this to be possible it is necessary that $n+1-l_{k+1-i} \leq j_i$ for all i. Translating this condition from jumping sequences to partitions with Φ gives the condition $\eta_i + \tau_{k+1-i} \leq n-k$ for all i which is equivalent to $\eta \subset \tau^c$. □

Proposition 2.5.7. [Man01, Proposition 3.2.7] *Let λ, $\mu \subset k \times (n-k)$ be partitions such that $|\lambda| + |\mu| = k(n-k)$ and $\Phi(\lambda) = (j_i)_{i=1}^k$. Then the intersection $X_\lambda \cap X'_\mu$ is empty if $\mu \neq \lambda^c$ and equal to the point x given by $L(x) = \langle e_{j_1}, \ldots, e_{j_k} \rangle$ if $\mu = \lambda^c$.* ♡

Proof. We first decompose the intersection of Schubert varieties in terms of Schubert cells.

$$X_\lambda \cap X'_\mu = \bigsqcup_{\eta \supset \lambda} \Omega_\eta \cap \bigsqcup_{\tau \supset \mu} \Omega'_\tau = \bigsqcup_{\substack{\eta \supset \lambda \\ \tau \supset \mu}} \Omega_\eta \cap \Omega'_\tau$$

We will check which of the terms may contribute. Let $\eta \supset \lambda$ and $\tau \supset \mu$. Then $|\eta| \geq |\lambda|$ and $k(n-k) - |\tau^c| \geq |\mu|$. By adding these inequalities together and rearranging we get $|\eta| \geq |\tau^c|$. If at least one of the inclusions $\eta \supset \lambda$ or $\tau \supset \mu$ is strict then $|\eta| > |\tau^c|$. In this case, $\eta \not\subset \tau^c$ and by Lemma 2.5.6, $\Omega_\eta \cap \Omega'_\tau = \emptyset$. So the only term that can contribute to $X_\lambda \cap X'_\mu$ is $\Omega_\lambda \cap \Omega'_\mu$. If $\mu = \lambda^c$ we get by Lemma 2.5.5 that $X_\lambda \cap X'_\mu = \{\mathrm{x}\}$. On the other hand if $\mu \neq \lambda^c$ there is some index i such that $\lambda_i \neq \mu_i^c$. In particular $\lambda_i > \mu_i^c$. If $\lambda_i < \mu_i^c$ since $|\lambda| + |\mu| = k(n-k)$ there must be some other index j such that $\lambda_j > \mu_j^c$. Now $\lambda_i > \mu_i^c \implies \lambda_i + \mu_{k+1-i} > n-k \implies \lambda \not\subset \mu^c$ and again by Lemma 2.5.6, $\Omega_\lambda \cap \Omega'_\mu = \emptyset$. □

We can be even more precise about when the intersection $X_\lambda \cap X'_\mu$ is empty.

Proposition 2.5.8. *Let λ, μ be any partitions contained in $k \times (n-k)$. Then*

$$X_\lambda \cap X'_\mu = \emptyset \iff \lambda \not\subset \mu^c.$$ ♡

Proof. Let $(j_i)_i^k = \Phi(\lambda)$ and $(l_i)_i^k = \Phi(\mu)$. As in the proof of Lemma 2.5.5 we consider the positions of the ones in the matrices (2.3) and (2.6) and see that

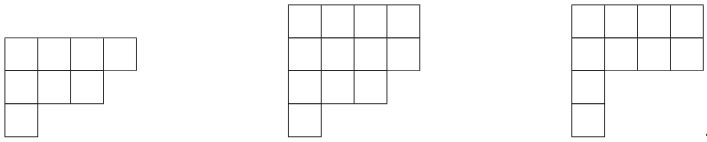
$$X_\lambda \cap X'_\mu = \emptyset \iff \exists r \;\; n+1-l_{k+1-r} > j_r.$$

by using Φ the condition on the right hand side becomes $\exists r \;\; \lambda_r + \mu_{k+1-r} > n-k$. □

Figure 2.2: The young diagrams from Example 2.5.9, λ in 4×5 in blue, the rotated μ-diagram in red and the rotated η-diagram in green.

The criterion above corresponds to rotating the Young diagram of μ, placing it in the bottom right corner of the Young diagram of the rectangle $k \times (n-k)$. Now $X_\lambda \cap X'_\mu$ is empty if and only if the rotated μ-diagram shares a box with the λ-diagram.

Example 2.5.9. In $G(4,9)$ consider the partitions $\lambda = (4,3,1)$, $\mu = (4,4,3,1)$, $\eta = (4,4,1,1)$ with young diagrams



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In Figure 2.2 we see that the λ-diagram and the rotated μ-diagram in the 4×5 rectangle share the box marked with a $*$ in position $(2,3)$, while the λ-diagram and the rotated η-diagram are disjoint, so $X_\lambda \cap X'_\mu = \emptyset$ and $X_\lambda \cap X'_\eta \neq \emptyset$. $\diamondsuit$

In the previous propositions we have taken all intersections with respect to the standard flag and inverse flag. By [EH16, Chapter 4.2] two Schubert varieties intersect generally transverse if the respective flags are transverse, so all the propositions above holds if we replace the flags with any transverse pair.

We continue with a proposition that will help on the way to learn how to intersect any Schubert varieties.

Proposition 2.5.10. [EH16, Proposition 4.9] *Let $U_\bullet$ be a complete flag transverse to $V_\bullet$ and $V'_\bullet$, λ and μ be partitions contained in $k \times (n-k)$ with $|\mu| \geq |\lambda|$ and $b = |\mu| - |\lambda|$. The intersection*

$$Y = X_\lambda(V_\bullet) \cap X_{\mu^c}(V'_\bullet) \cap X_{(b)}(U_\bullet)$$

is a point if $\mu \in \lambda \otimes b$ and empty otherwise. $\heartsuit$

The proof below closely follows the proof of [EH16, Proposition 4.9].

Proof. Let $(j_i)_{i=1}^k = \Phi(\lambda)$ and $(l_i)_{i=1}^k = \Phi(\mu)$. Then $\Phi(\mu^c) = (n+1-l_{k+1-i})_{i=1}^k$ and we have equivalences

$$\mu \in \lambda \otimes b \iff \lambda_i \leq \mu_i \leq \lambda_{i-1} \text{ for all } i \iff j_{i-1} < l_i \leq j_i \text{ for all } i. \tag{2.7}$$

Here $\mu_1 \leq \lambda_0$ and $j_0 < l_1$ are considered true. The first two Schubert varieties in the intersection above are

$$\begin{aligned} X_\lambda(V_\bullet) &= \{x \mid \dim(L(x) \cap V_{j_i}) \geq i\} \\ X_{\mu^c}(V'_\bullet) &= \{x \mid \dim(L(x) \cap V'_{n+1-l_{k+1-i}}) \geq i\}. \end{aligned}$$

Define $A_i = V_{j_i} \cap V'_{n+1-l_i}$. Note that V'_{n+1-l_i} is the subspace from the $(k+1-i)$th condition in the definition of $X_{\mu^c}(V'_\bullet)$. Now

$$x \in X_\lambda(V_\bullet) \cap X_{\mu^c}(V'_\bullet) \implies L(x) \cap A_i \neq 0. \tag{2.8}$$

This is because $L(x) \cap V_{j_i}$ and $L(x) \cap V_{n+1-l_i}$ are transverse subspaces of $L(x)$ and

$$\dim(L(x) \cap V_{j_i}) + \dim(L(x) \cap V_{n+1-l_i}) > k = \dim(L(x)),$$

so the intersection $L(x) \cap A_i$ is of dimension strictly greater than 0. Note that $A_i = \langle e_1, \ldots e_{j_i} \rangle \cap \langle e_{l_i}, \ldots, e_n \rangle$, so if $l_i > j_i$ for some i then $A_i = 0$ hence $Y = \emptyset$. We assume $l_i \leq j_i$ for all i and need to show that if $j_{i-1} \geq l_i$ then $Y = \emptyset$. Now

$A_i = \langle e_{l_i}, \ldots, e_{j_i} \rangle$ and $A_{i-1} \cap A_i = 0 \iff j_{i-1} < l_i$. Let $A = \langle A_1, \ldots, A_k \rangle$ be the span of all A_i. Since the dimension of A_i is $j_i - l_i + 1$ we have

$$\dim(A) \leq \sum_{i=1}^{k} \dim(A_i) = \sum_{i=1}^{k} j_i - l_i + 1 = k + b$$

with equality if and only if $j_{i-1} < l_i$. For $x \in X_\lambda(V_\bullet) \cap X_{\mu^c}(V'_\bullet)$ we claim that $L(x)$ is spanned by its intersections with A_i, hence $L(x) \subset A$. This follows from the fact that the flags on $L(x)$ given by $L(x) \cap V_{j_i}$ and $L(x) \cap V'_{n+1-l_{k+1-i}}$ are transverse, see [EH16, Lemma 4.5].

The last Schubert variety to consider is

$$\begin{aligned} X_{(b)}(U_\bullet) &= \{x \mid \dim(L(x) \cap U_{n-k+1-b}) \geq 1\} \\ &= \{x \mid L(x) \cap U_{n-k+1-b} \neq 0\}. \end{aligned}$$

For $x \in X_\lambda(V_\bullet) \cap X_{\mu^c}(V'_\bullet)$ it is required that $A \cap U_{n-k+1-b} \neq 0$ in order for $x \in X_{(b)}(U_\bullet)$. Since $U_{n-k+1-b}$ is general of dimension $n-k+1-b$ we must have $\dim(A) \geq k+b$. So if $j_{i-1} \geq l_i$ then $\dim(A) < k+b$ and $Y = \emptyset$. We have now shown that $Y = \emptyset$ if $\mu \notin \lambda \otimes b$. Assume $\mu \in \lambda \otimes b$, then $\dim(A) = k+b$ and A intersects $U_{n-k+1-b}$ in a line. Let v be a nonzero vector on this line. Since the A_i's are linearly independent and span A, we may write $v = v_1 + \cdots + v_k$ with $v_i \in A_i$. Let $x \in Y$ since $L(x) \subset A$ and $L(x) \cap U \neq 0$ we have that $v \in L(x)$. And since $L(x)$ is spanned by its intersections with the A_i's $L(x) = \langle v_1, \ldots v_n \rangle$, so Y is one point. □

2.6 Cohomology

This section is an informal introduction to the cohomology theory we will need to formalize the intersection of Schubert varieties. For a more rigorous introduction see [Man01, Appendix A].

Let $G = G(k,n)$. The integral cohomology ring of G is given by

$$H^*(G) = \bigoplus_{q \geq 0} H^q(G).$$

The cup product $\smile \colon H^p(G) \times H^q(G) \to H^{p+q}(G)$ gives $H^*(G)$ the structure of a graded ring. By Poincaré duality there are isomorphisms between the homology groups and cohomology groups $H_p(G) \simeq H^{2k(n-k)-p}(G)$, note that $2k(n-k)$ is the dimension of G over $\mathbb{R}$. For any subvariety $X \subset G$ of real dimension p we get a class in $H_p(X)$, and we denote the image of this class under the isomophism by $[X] \in H^{2k(n-k)-p}(G)$. In particular for a partition $\lambda \subset k \times (n-k)$ we call $[X_\lambda]$ the Schubert cycle associated to λ and denote it by σ_λ. The Schubert cycles are well defined in the sense that $\sigma_\lambda = [X_\lambda(V_\bullet)] = [X_\lambda(W_\bullet)]$ and they generate all of $H^*(G)$, that is

$$H^{2q}(G) = \bigoplus_{\substack{\lambda \subset k \times (n-k) \\ |\lambda| = q}} \mathbb{Z}\{\sigma_\lambda\}.$$

In particular the only partition λ with $|\lambda| = k(n-k)$ is $\lambda = k \times (n-k)$, so $H^{2k(n-k)}(G) = \mathbb{Z}\{\sigma_{k \times (n-k)}\} \simeq \mathbb{Z}$. We call the last isomorphism the degree map

$$\begin{aligned} \deg \colon H^{2k(n-k)}(G) &\to \mathbb{Z} \\ \sigma_{k \times (n-k)} &\mapsto 1. \end{aligned}$$

By Poincaré duality $H^{2k(n-k)}(G) \simeq H_0(G)$, so if $[X] \in H^{2k(n-k)}(G)$, then X is a finite set of points in G and $\deg([X])$ is the number of points in X. If $V_\bullet$ and $W_\bullet$ are transverse flags and λ, $\mu \subset k \times (n-k)$ then

$$[X_\lambda(V_\bullet) \cap X_\mu(W_\bullet)] = \sigma_\lambda \smile \sigma_\mu.$$

To simplify notation we will write $\sigma_\lambda \sigma_\mu = \sigma_\lambda \smile \sigma_\mu$ and $\sigma_\lambda^m = \sigma_\lambda \smile \cdots \smile \sigma_\lambda$. Using this we can translate problems of intersecting Schubert varieties into calculations in the cohomology ring $H^*(G)$.

Example 2.6.1. Continuing Example 2.5.2 we see that for $\lambda = (1,0)$ and four pairwise transverse flags $V_\bullet^1, V_\bullet^2, V_\bullet^3, V_\bullet^4$, where V_2^i correspond to a line L^i in $\mathbb{P}^3$, $X_\lambda(V_\bullet^i)$ is the set of lines in $\mathbb{P}^3$ intersecting L^i. The intersection

$$X_\lambda(V_\bullet^1) \cap X_\lambda(V_\bullet^2) \cap X_\lambda(V_\bullet^3) \cap X_\lambda(V_\bullet^4)$$

is the set of lines in $\mathbb{P}^3$ intersecting all four lines L^1, L^2, L^3 and L^4. In order to find the number of such lines we only need to compute $\deg(\sigma_\lambda^4)$. We do this in Example 2.6.4. $\diamondsuit$

The last results in the previous section give us two formulas we need in order to do computations in $H^*(G)$. The first formula is called the complementary dimension formula.

Proposition 2.6.2. *Let λ, $\mu \subset k \times (n-k)$ be partitions such that $|\lambda| + |\mu| = k(n-k)$ Then*

$$\deg(\sigma_\lambda \sigma \mu) = \begin{cases} 1 & \textit{if } \mu = \lambda^c \\ 0 & \textit{if } \mu \neq \lambda^c. \end{cases} \qquad \heartsuit$$

Proof. This follows from Proposition 2.5.7. $\square$

The second is called Pieri's formula.

Proposition 2.6.3. *Let $\lambda \subset k \times (n-k)$ be a partition and, b be a positive integer. Then*

$$\sigma_\lambda \sigma_{(b)} = \sum_{\substack{\mu \subset k \times (n-k) \\ \mu \in \lambda \otimes b}} \sigma_\mu. \qquad \heartsuit$$

Proof. We know that $\sigma_\lambda \sigma_{(b)} \in H^{2(|\lambda|+b)}(G)$, so

$$\sigma_\lambda \sigma_{(b)} = \sum_{\substack{\mu \subset k \times (n-k) \\ |\mu| = |\lambda| + b}} \alpha_\mu \sigma_\mu.$$

To find the the coefficients α_μ we fix $\mu \subset k \times (n-k)$ with $|\mu| = |\lambda| + b$ and notice that by the complementary dimension fomula $\deg(\sigma_\lambda \sigma_{(b)} \sigma_{\mu^c}) = \alpha_\mu$. The fact that α_μ is 1 if $\mu \in \lambda \otimes b$ and 0 if $\mu \notin \lambda \otimes b$ follows from Proposition 2.5.10. $\square$

We use this to finish the example counting the number of lines through four general lines in $\mathbb{P}^3$.

Example 2.6.4. In Example 2.6.1 we saw that in order to find the number of lines in $\mathbb{P}^3$ through four general lines we compute $\deg(\sigma_{(1,0)}^4)$. By Pieri's formula $\sigma_{(1,0)}^2 = \sigma_{(1,1)} + \sigma_{(2,0)}$, so

$$\sigma_{(1,0)}^4 = (\sigma_{(1,1)} + \sigma_{(2,0)})^2 = \sigma_{(1,1)}^2 + 2\sigma_{(1,1)}\sigma_{(2,0)} + \sigma_{(2,0)}^2.$$

And by the complementary dimension formula $\deg(\sigma_{(1,1)}^2) = \deg(\sigma_{(2,0)}^2) = 1$ and $\deg(\sigma_{(1,1)}\sigma_{(2,0)}) = 0$, so $\deg(\sigma_{(1,0)}^4) = 2$. $\diamondsuit$

Example 2.6.5. The Grassmannian $G(1,N)$ is isomorphic to projective space $\mathbb{P}^{N-1}$. The only Schubert classes are $\sigma_{(l)}$ for $0 \leq l \leq N-1$, so the description of the cohomology ring is especially simple. The ring $H^*(\mathbb{P}^{N-1})$ is generated by $\sigma_{(1)}$, and we have $\sigma_{(1)}^l = \sigma_{(l)}$ and $\sigma_1^N = 0$. Hence the cohomology ring can be described as the polynomial ring $H^*(\mathbb{P}^{N-1}) = \mathbb{Z}[\sigma_{(1)}]/(\sigma_{(1)}^N)$. $\diamondsuit$

The complementary dimension formula together with Pieri's formula is enough to compute the product $\sigma_\lambda\sigma_\mu$ for any pair of partitions. This gives us full control over computations in $H^*(G)$, hence any intersection of Schubert varieties. Some such computations can be made simpler by Giambelli's formula see [EH16, chapter 4.7], but we will not need it.

The Schubert class $\sigma_{(1)}$ can be used to find the degree of a subvariety of $G(k,n)$ in $\mathbb{P}^{\binom{n}{k}-1}$. Degree is a notion associated not to an abstract variety, but with a particular embedding of a projective variety into projective space. For our purposes, we will use the classical definition.

Definition 2.6.6. Let X be a projective variety of dimension s, with an embedding into projective space $\mathbb{P}^N$. The degree of X in $\mathbb{P}^N$ is the number of points in the intersection $X \cap H$, where H is a generic plane of codimension s in $\mathbb{P}^N$. $\diamondsuit$

Lemma 2.6.7. *Consider $X_{(1)}(V_\bullet)$ and $G(k,n)$ as subvarieties of $\mathbb{P}^{\binom{n}{k}-1}$ under the Plücker embedding. Then $X_{(1)}(V_\bullet) = G(k,n) \cap H$ for a hyperplane H of $\mathbb{P}^{\binom{n}{k}-1}$ that depends on $V_\bullet$.* $\heartsuit$

This is well known, but we could not find a proof in the literature, so we provide one here.

Proof. We have that

$$X_{(1)}(V_\bullet) = \{x \in G(k,n) \mid L(x) \cap V_{n-k} \neq 0\}.$$

That is $X_{(1)}(V_\bullet)$ depends only on the $n-k$ dimensional part of the flag, and consists of those subspaces intersecting V_{n-k} in at least a line. Pick vectors $v_1, \ldots, v_{n-k}$ spanning V_{n-k}, and consider $v_1 \wedge \cdots \wedge v_{n-k}$ in $\bigwedge^{n-k}\mathbb{C}^n$. The vector space $\bigwedge^{n-k}\mathbb{C}^n$ has a basis of vectors of the form $e_J = \bigwedge_{i\in J} e_i$ for $J \in \binom{[n]}{n-k}$. So we have

$$v_1 \wedge \cdots \wedge v_{n-k} = \sum_{J \in \binom{[n]}{n-k}} a_J e_J.$$

Similarly for a point $x \in G(k,n)$ we get $\sum_{I \in \binom{[n]}{k}} p_I(x)e_I$ in $\bigwedge^k \mathbb{C}^n$. Now consider the map

$$\bigwedge^{n-k} \mathbb{C}^n \times \bigwedge^{k} \mathbb{C}^n \to \bigwedge^{n} \mathbb{C}^n = \mathbb{C}\{e_{[n]}\}$$
$$(u,v) \mapsto u \wedge v = c \cdot e_{[n]}.$$

In this language we have that $L(x) \cap V_{n-k} \neq 0$ if and only if

$$\sum_{J \in \binom{[n]}{n-k}} a_J e_J \wedge \sum_{I \in \binom{[n]}{k}} p_I(x) e_I = 0 \cdot e_{[n]}. \tag{2.9}$$

Notice that $e_J \wedge e_I = 0 \cdot e_{[n]} \iff J \cap I \neq \emptyset$, so the only contributions on the left hand side of (2.9) are those where $J = I^c = [n] \setminus I$. Hence we get

$$\begin{aligned} &\sum_{I \in \binom{[n]}{k}} (a_{I^c} p_I(x)) e_{I^c} \wedge e_I = 0 \cdot e_{[n]} \\ \iff &\sum_{I \in \binom{[n]}{k}} (sgn(I) a_{I^c} p_I(x)) e_{[n]} = 0 \cdot e_{[n]} \\ \iff &\sum_{I \in \binom{[n]}{k}} sgn(I) a_{I^c} p_I(x) = 0. \end{aligned} \tag{2.10}$$

Here $sgn(I)$ is the sign of the permutation σ given by $I^c I$ in one line notation. To be precise if $I = \{i_1, \dots, i_k\}$ with $i_1 < \dots < i_k$ and $I^c = \{j_1, \dots, j_{n-k}\}$ with $j_1 < \dots < j_{n-k}$ consider the n-tuple $t = (j_1, \dots j_{n-k}, i_1, \dots, i_k)$. Then σ is the permutation of $[n]$ that sends r to the rth entry of t.

We have now shown that $L(x) \cap V_{n-k} \neq 0$ if and only if the Plücker coordinates $(p_I(x))_{I \in \binom{[n]}{k}}$ satisfy the linear equation (2.10). In other words $x \in X_{(1)}(V_\bullet) \iff x \in H$ where $H = V(\sum_{I \in \binom{[n]}{k}} sgn(I) a_{I^c} p_I)$.

One point to notice is that we made a choice of vectors v_i spanning V_{n-k} which does impact the coefficients a_{I^c}. Luckily these coefficients are precisely the Plücker coordinates of V_{n-k} as a point in $G(n-k,n)$ which we know only changes by a common multiple with a different choice of vectors spanning V_{n-k}. So the hyperplane H is well defined. □

We look at an example of the construction of H from V_{n-k}.

Example 2.6.8. In $G(2,4)$ let V_2 be the subspace spanned by $e_1 + e_2$ and e_3. Then $(e_1 + e_2) \wedge e_3 = e_1 \wedge e_3 + e_2 \wedge e_3$, so

$$a_J = \begin{cases} 1 \text{ if } J = 13, 23 \\ 0 \text{ else .} \end{cases}$$

We only need to check the sign of I for 24 and 14, and we see that $sgn(24) = sgn((2,3)) = -1$ and $sgn(14) = sgn((1,3)(1,2)) = 1$. Hence $H = V(p_{14} - p_{24})$. ♢

Proposition 2.6.9. *Let X be a subvariety of $G(k,n)$ of dimension s embedded into $\mathbb{P}^{\binom{n}{k}-1}$ via the Plücker embedding. The degree of X in $\mathbb{P}^{\binom{n}{k}-1}$ is* $\deg([X]\sigma_{(1)}^s)$. ♡

Proof. By applying Lemma 2.6.7 s times we see that $[X]\sigma_{(1)}^s$ is the class of the intersection of X with a generic plane of codimension s in $\mathbb{P}^{\binom{n}{k}-1}$. So $\deg([X]\sigma_{(1)}^s)$ is the number of points in the intersection, i.e. the degree of X in $\mathbb{P}^{\binom{n}{k}-1}$. $\square$

We can use this to calculate the degree of any subvariety of $G(k,n)$ by the coefficients of the class of X in $H^*(G(k,n))$ in the Schubert basis. We have already seen one of these computations for the Schubert variety $X_{(1)}(V_\bullet)$ in $G(2,4)$ in Example 2.6.4.

Example 2.6.10. Consider $G(3,7)$. We aim to calculate the degree of all Schubert varieties of dimension 6 in $\mathbb{P}^{34}$. By using Pieri's formula 6 times we see that

$$\sigma_{(1)}^6 = 9\sigma_{(4,2)} + 10\sigma_{(4,1,1)} + 5\sigma_{(3,3)} + 16\sigma_{(3,2,1)} + 5\sigma_{(2,2,2)}.$$

By the complementary dimension formula and Proposition 2.6.9 the coefficients above are precisely the degrees of the Schubert varieties associated to the complement of the partitions. For example the complement of $(3,3)$ in 3×4 is $(4,1,1)$ so the degree of $X_{(3,3)}(V_\bullet)$ in $\mathbb{P}^{34}$ is $\deg(\sigma_{(3,3)}\sigma_{(1)}^6) = 10$. $\diamondsuit$

Remark 2.6.11. The Chow ring of Grassmannians are isomorphic to the cohomology ring, $A^\bullet(G(k,n)) \simeq H^*(G(k,n))$. So we could have introduced Schubert calculus as computations in the Chow ring instead of the cohomology ring. See [EH16] for a reference with this approach. $\diamondsuit$

Chapter 3

Toric varieties

This section is a brief introduction to toric varieties, and in particular projective toric varieties associated to convex polytopes. For a more in-depth introduction see [CLS12] or [Ful93].

3.1 Affine toric varieties

In this setting, the torus will not mean the torus from topology, but an algebraic torus. That is an affine variety T which is isomorphic to $(\mathbb{C}^*)^n$ viewed as a variety with a group structure inherited from multiplication on $\mathbb{C}^*$. A toric variety is an irreducible variety X containing a torus T as an open, dense subset such that the group action of T on itself extends to X. The theory of toric varieties is parallel to convex geometry in a vector space over $\mathbb{R}$. We start by surveying affine toric varieties following chapter 1 of [CLS12].

Let $T \simeq (\mathbb{C}^*)^n$ be a torus. A character of T is a morphism from T to $\mathbb{C}^*$, which is also a group homomorphism. Given a point $m = (a_1, \dots a_n) \in \mathbb{Z}^n$ the map

$$\begin{aligned} \chi^m \colon T &\to \mathbb{C}^* \\ (t_1, t_2, \dots t_n) &\mapsto t_1^{a_1} t_2^{a_2} \dots t_n^{a_n} \end{aligned}$$

is a character of T. In fact all characters arise in this manner, so the group of characters, denoted M is isomorphic to $\mathbb{Z}^n$. Expressions of the form $t_1^{a_1} t_2^{a_2} \dots t_n^{a_n}$ are called Laurent monomials. A one-parameter subgroup of T is a morphism from $\mathbb{C}^*$ to T which is also a group homomorphism. Similarly a point $u = (b_1, \dots b_n) \in \mathbb{Z}^n$ gives the one-parameter subgroup

$$\begin{aligned} \lambda^u \colon \mathbb{C}^* &\to T \\ t &\mapsto (t^{b_1}, t^{b_2}, \dots, t^{a_n}) \end{aligned}$$

and again all one-parameter subgroups arise like this. So the group of one-parameter subgroups, denoted N, is isomorphic to $\mathbb{Z}^n$.

Toric varieties will be described by convex geometry in the $\mathbb{R}$-vector spaces given by the lattices M and N. That is the vector spaces $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$. We also define a bilinear pairing between N and M by $\langle \chi^m, \lambda^u \rangle = a_1 b_1 + \cdots + a_n b_n \in \mathbb{Z}$.

This is motivated by considering the composition

$$\begin{aligned}\chi^m \circ \lambda^u \colon \mathbb{C}^* &\to \mathbb{C}^* \\ t &\mapsto t^{a_1 b_1 + \cdots + a_n b_n}\end{aligned}$$

as a character of $\mathbb{C}^*$. In most examples we will take $M = N = \mathbb{Z}^n$ and the bilinear pairing is then the normal inner product on $\mathbb{R}^n$.

We now describe the affine toric variety associated to a finite set of points in the lattice M. Let $\mathcal{A} = \{m_1, \dots m_r\} \subset M$ and consider the map

$$\begin{aligned}\Phi_{\mathcal{A}} \colon T &\to \mathbb{A}^r \\ t &\mapsto (\chi^{m_1}(t), \dots, \chi^{m_r}(t)).\end{aligned}$$

That is, $\Phi_{\mathcal{A}}$ is given by certain Laurent monomials specified by points in the lattice M.

Definition 3.1.1. [CLS12, Definition 1.1.7] The affine toric variety $Y_{\mathcal{A}}$ is the Zariski closure of the image of $\Phi_{\mathcal{A}}$. $\diamondsuit$

The following examples give explicit equations for some affine toric varieties.

Example 3.1.2. [CLS12, Example 1.1.4] Let $T = \mathbb{C}^*$ and $\mathcal{A} = \{2, 3\} \in \mathbb{Z}$ then

$$\Phi_{\mathcal{A}}(t) = (t^2, t^3) \in \mathbb{A}^2$$

and the affine variety $Y_{\mathcal{A}}$ is the curve $V(x^3 - y^2)$. $\diamondsuit$

Example 3.1.3. Let $T = (\mathbb{C}^*)^2$ and $\mathcal{A} = \{(2, 0), (1, 1), (0, 2)\} \subset \mathbb{Z}^2$ then

$$\Phi_{\mathcal{A}}(s, t) = (s^2, st, t^2) \in \mathbb{A}^3.$$

The surface $Y_{\mathcal{A}}$ is called the rational normal cone of degree 2 and is given by $V(y^2 - xz)$. $\diamondsuit$

Affine toric varieties can be described in terms of maps as above, by certain affine semigroups or by toric ideals. For our purposes the description above will be sufficient. See [CLS12, Chapter 1] for more details.

3.2 Polytopes

Toric varieties are obtained from objects in convex geometry. Specifically projective toric varieties are obtained from convex polytopes. In this section we review some basic properties of convex polytopes. See [CLS12, Chapter 2.2] for details. Throughout, fix real vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ containing the lattices M and N as in the previous section such that $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ are dual by the bilinear pairing $\langle \cdot, \cdot \rangle$.

Definition 3.2.1. [CLS12, Definition 1.2.2] A polytope in $M_{\mathbb{R}}$ is a set of the form

$$\Delta = \operatorname{Conv}(S) = \{\sum_{u \in S} \lambda_u u \mid u \geq 0, \sum_{u \in S} \lambda_u = 1\},$$

where S is a finite set of points in $M_{\mathbb{R}}$. We say Δ is the convex hull of S. If $S \subset M$ then Δ is called a lattice polytope. $\diamondsuit$

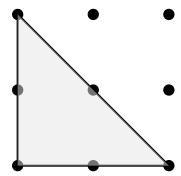


Figure 3.1: A 2-simplex

Strictly speaking this is the definition of a convex polytope, but as we are only interested in convex polytopes, we only write polytope.

The dimension of a polytope Δ in $M_\mathbb{R}$ is the dimension of the smallest affine linear subspace of $M_\mathbb{R}$ containing Δ. Given a vector $u \in N_\mathbb{R}$ and $b \in \mathbb{R}$ we get an affine hyperplane $H_{u,b}$ and a half-space $H^+_{u,b}$ given by

$$\begin{aligned} H_{u,b} &= \{m \in M_\mathbb{R} \mid \langle m, u\rangle = b\} \\ H^+_{u,b} &= \{m \in M_\mathbb{R} \mid \langle m, u\rangle \geq b\}. \end{aligned}$$

If $\Delta \subset H^+_{u,b}$, the intersection $\Lambda = H_{u,b} \cap \Delta$ is called a face of Δ, written $\Lambda \preceq \Delta$. If $\Lambda \neq \Delta$ it is called a proper face and we write $\Lambda \prec \Delta$. If $\Delta = \mathrm{Conv}(S)$ then $\Lambda = \mathrm{Conv}(S \cap \Lambda)$ which shows that each face of Δ is a polytope. We call a face of Δ a facet, edge or vertex if it is of codimension 1 in Δ, dimension 2 or dimension 1 respectively. Each vertex of Δ lies in S and Δ is the convex hull of the set of vertices. Furthermore Δ can be written as a finite intersection of half-spaces

$$\Delta = \bigcap_i H^+_{u_i,b_i}.$$

Every proper face $\Lambda \prec \Delta$ is the intersection of the facets of Δ containing Λ.

These properties can be seen in [CLS12, Proposition 2.2.1]. We now look at some examples to get a feel for these notions.

Example 3.2.2. Suppose $M = N = \mathbb{Z}^2$ and $S = \{(0,0),(2,0),(0,2)\}$. The polytope $\Delta = \mathrm{Conv}(S)$ is the triangle pictured in Figure 3.1. Let $u_1 = (0,1)$, $b_1 = 0$, $u_2 = (1,0)$, $b_2 = 0$ and $u_3 = (-1,-1)$, $b_3 = -2$. Then we have

$$\begin{aligned} H^+_{u_1,b_1} &= \{m = (m_1, m_2) \mid m_1 \geq 0\} \\ H^+_{u_2,b_2} &= \{m = (m_1, m_2) \mid m_2 \geq 0\} \\ H^+_{u_3,b_3} &= \{m = (m_1, m_2) \mid m_1 + m_2 \leq 2\}. \end{aligned}$$

That is, $H^+_{u_1,b_1}$ is the upper half-plane, $H^+_{u_2,b_2}$ is the right half-plane, and $H^+_{u_3,b_3}$ is the set of points below the line between $(2,0)$ and $(0,2)$. Considering this we see that $\Delta = \cap_{i=1}^3 H^+_{u_i,b_i}$. The intersections $\Lambda_i = H_{u_i,b_i} \cap \Delta$ are the three facets/edges of Δ given by the line segments between $(0,0)$ and $(2,0)$, $(0,0)$ and $(0,2)$ and $(2,0)$ and $(0,2)$. To obtain the vertices as faces of Δ we need to pick other values for u and b. For example, the origin $(0,0)$ is a vertex of Δ obtained as $(0,0) = H_{u,b} \cap \Delta$ if $u = (1,1)$ and $b = 0$. We also see that the facets containing the origin is Λ_1 and Λ_2 and that $(0,0) = \Lambda_1 \cap \Lambda_2$. $\diamondsuit$

Example 3.2.3. A d-simplex is a polytope of dimension d with $d+1$ vertices. The triangle in Figure 3.1 is 2 dimensional with 3 vertices, hence a 2-simplex. In general a d-simplex can be attained in $\mathbb{R}^{d+1}$ as the convex hull of the $d+1$ coordinate unit vectors. This is called the standard d-simplex, often denoted Δ_d and can be written explicitly as

$$\Delta_d = \{r = (r_1, \dots, r_{d+1}) \mid \sum_i r_i = 1\}. \qquad \diamondsuit$$

We will be interested in the volume of lattice polytopes with respect to the lattice M. Let Δ be a full dimensional lattice polytope in $M_{\mathbb{R}} \simeq \mathbb{R}^n$. We denote the n-dimensional volume of Δ by $\mathrm{Vol}'_n(\Delta)$.

Definition 3.2.4. The normalized n-dimensional volume of Δ is

$$\mathrm{Vol}_n(\Delta) = n!\,\mathrm{Vol}'_n(\Delta). \qquad \diamondsuit$$

When the dimension is understood we will write $\mathrm{Vol}(\Delta)$.

3.3 Projective toric varieties

Let $\mathcal{A}$ be a finite subset of M as before. Notice that the origin of $\mathbb{A}^r$ is not contained in the image of $\Phi_{\mathcal{A}}$. Hence we can compose with the projection $\pi\colon \mathbb{A}^r \smallsetminus 0 \to \mathbb{P}^{r-1}$.

Definition 3.3.1. [CLS12, Definition 2.1.1] The projective toric variety $X_{\mathcal{A}}$ is the Zariski closure of the image of $\phi_{\mathcal{A}} = \pi \circ \Phi_{\mathcal{A}}$ in $\mathbb{P}^{r-1}$. $\diamondsuit$

We will be especially interested in the varieties $X_{\mathcal{A}}$ when $\mathcal{A}$ is the set of lattice points of a lattice polytope in $M_{\mathbb{R}}$.

Definition 3.3.2. Let Δ be a lattice polytope in $M_{\mathbb{R}}$. The projective toric variety X_Δ associated to Δ is $X_{\Delta\cap M}$. $\diamondsuit$

As mentioned in [CLS12, Chapter 2.2] the definition above requires that Δ has "enough" lattice points. For our purposes the definition above will suffice. Since all polytopes we deal with will be normal as in [CLS12, Definition 2.2.9].

Example 3.3.3. Consider the 2-simplex $\Delta = \mathrm{Conv}((0,0),(0,1),(1,0)) \subset \mathbb{R}^3$. Let $\mathcal{A} = \Delta \cap \mathbb{Z}^2 = \{(0,0),(1,0),(0,1)\}$. We see that $\phi(t,s) = (1 : t : s)$. So the image of ϕ is $D_+(x) \smallsetminus V(yz) = \mathbb{P}^2 \smallsetminus V(xyz)$ and the closure is all of $\mathbb{P}^2$. $\diamondsuit$

Example 3.3.4. Let 2Δ be the polytope in Example 3.2.2. By Figure 3.1 we see that $\mathcal{A} = 2\Delta \cap \mathbb{Z}^2 = \{(0,0),(2,0),(0,2),(1,0),(0,1),(1,1)\}$ so

$$\phi_{\mathcal{A}}(t,s) = (1 : t^2 : s^2 : t : s : ts) \in \mathbb{P}^5.$$

The toric variety X_Δ is the Veronese surface. That is a degree 4 embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$. $\diamondsuit$

In the two examples above we saw that scaling the polytope Δ resulted in isomorphic varieties, with different embeddings, this holds in general. We can extract a lot of information about the variety X_Δ from the polytope Δ.

Proposition 3.3.5. [CLS12, Proposition 2.1.2] *The dimension of X_Δ is the dimension of Δ.* $\heartsuit$

Proposition 3.3.6. [Ful93, Section 5.3] *The normalized volume of Δ is the degree of the X_Δ embedded in $\mathbb{P}^{r-1}$.* $\heartsuit$

In Example 3.3.4 we saw that the degree of $X_{2\Delta}$ in $\mathbb{P}^5$ is 4, and by Figure 3.1 we see that this matches the normalized volume of 2Δ.

3.4 The torus action on Grassmannians

In this section we describe the toric subvarieties of $G(k,n)$ we are interested in, though we postpone the description of the corresponding polytope to chapter 5.

We will think of $\mathbb{C}^n$ as the ambient space in the Grassmannian $G = G(k,n) = G(k,\mathbb{C}^n)$. The algebraic torus $(\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$ by scaling the coordinates. That is if $t = (t_1, ..., t_n) \in (\mathbb{C}^*)^n$ the action is defined by $te_i = t_i e_i$.

The linearity of this action ensures that it extends to an action on G given by $L(tx) = \{tv \mid v \in L(x)\}$. Note that $L(tx)$ is also a k-dimensional linear subspace of $\mathbb{C}^n$ since if $a, b \in \mathbb{C}$ and $tv, tw \in L(tx)$ then $atv + btw = t(av + bw) \in L(tx)$, and if $v_1, \ldots v_k$ span $L(x)$ then $tv_1, \ldots tv_k$ span $L(tx)$. Explicitly if $L(x)$ is the rowspace of a $k \times n$ matrix with column vectors $v_1, \ldots, v_n$ then $L(tx)$ is the rowspace of the matrix $(t_1 v_1 \ldots t_n v_n)$. Note that the diagonal subgroup $D = \{(t_0, \ldots, t_0) \in (\mathbb{C}^*)^n \mid t_0 \in \mathbb{C}^*\}$ acts trivially on G, as $(t_0 v_1 \ldots t_0 v_n)$ has the same rowspace as $(v_1 \ldots v_n)$. So we might as well consider the action of the $(n-1)$-dimensional torus $T = (\mathbb{C}^*)^n / D \simeq (\mathbb{C}^*)^{n-1}$ on G.

We will consider the orbit Tx of a point $x \in G$ and its Zariski closure $\overline{Tx}$ as a subvariety of G.

Example 3.4.1. Let $t = (t_1, t_2, t_3, t_4) \in T$ and let $x \in G(2,4)$. If we assume that x is in the biggest Schubert cell $\Omega_\emptyset$ we can express $L(x)$ as the rowspace of the matrix

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix}.$$

Now $L(tx)$ is the rowspace of the matrix

$$\begin{pmatrix} t_1 a & t_2 b & t_3 & 0 \\ t_1 c & t_2 d & 0 & t_4 \end{pmatrix} \sim \begin{pmatrix} \frac{t_1}{t_3} a & \frac{t_2}{t_3} b & 1 & 0 \\ \frac{t_1}{t_4} c & \frac{t_2}{t_4} d & 0 & 1 \end{pmatrix}.$$

As points in $\mathbb{P}^5$

$$x = (ad - bc : -c : a : -d : b : 1).$$

and

$$tx = (t_1 t_2 (ad - bc) : -t_1 t_3 c : t_1 t_4 a : -t_2 t_3 d : t_2 t_4 b : t_3 t_4). \qquad \diamondsuit$$

In general we have that

$$tx = (t_I p_I(x))_{I \in \binom{[n]}{k}} \in \mathbb{P}^{\binom{n}{k}-1} \tag{3.1}$$

where $t_I = \prod_{i \in I} t_i$.

To understand the orbit closures we need to understand the action. One question we may ask is what the fixed points are. That is, points $x \in G$ such that $tx = x$ for all $t \in T$.

Proposition 3.4.2. *For $J \in \binom{[n]}{k}$ the point $x_J \in G(k,n)$ given by $L(x_J) = \langle e_i \mid i \in J \rangle$ is a fixed point of T. Moreover these are the only fixed points.* $\heartsuit$

Proof. Notice that the Plücker coordinates of x_J are all zeros except for a 1 in the Jth position, i.e. $p_I(x_J) = \delta_{I,J}$. By (3.1) we see that $\iota(tx_J) = \iota(x_J)$ so x_J is a fixed point. Any other point $x \in G$ will have at least two non-zero Plücker coordinates, $p_I(x), p_{I'}(x) \neq 0$. By (3.1) we see that x can not be a fixed point. $\square$

The description of the torus action in terms of Plücker coordinates also gives us the following result.

Proposition 3.4.3. *If $x \in G$ where $L(x)$ is contained in some plane H in $\mathbb{C}^n$ spanned by coordinate vectors, then any point $y \in \overline{Tx}$ is such that $L(y)$ is contained in H.* ♡

Proof. Since any coordinate plane H can be described as the intersection of coordinate hyperplanes $H_i = \langle e_1, \dots \hat{e_i}, \dots e_n \rangle$ it suffices to check these. Note that $L(x) \subset H_i \iff p_I(x) = 0$ for all $I \ni i$. So the set of points in G contained in H_i is the closed set $A = V(p_I \mid i \in I) \cap G$. Assume $L(x) \subset H_i$ and $y \in Tx$, by (3.1) $p_I(y) = t_I p_I(x)$ so $p_I(y) = 0$ for all $I \ni i$ and $Tx \subset V(p_I \mid i \in I) \implies \overline{Tx} \subset V(p_I \mid i \in I)$. □

The orbit closures $\overline{Tx}$ are projective toric varieties, so as seen in the previous section, we can get a lot of information about $\overline{Tx}$ by studying the corresponding polytope. We postpone describing these polytopes until we have defined the matroid corresponding to a point in the Grassmannian in the next chapter.

Let s be the codimension of $\overline{Tx}$ in $G(k, n)$. We may express the class of $\overline{Tx}$ in $H^*(G(k, n))$ as

$$[\overline{Tx}] = \sum_{\substack{\lambda \subset k \times (n-k) \\ |\lambda| = s}} \alpha_\lambda \sigma_\lambda$$

for integer coefficients α_λ. These coefficients are the main object of study in this thesis.

Chapter 4

Matroids

4.1 Definitions

A matroid is a mathematical structure that generalizes the notion of independence, such as linear independence of a set of vectors. In this section we follow [Kat16, Chapter 3 and 5]. Many of the definitions below were first introduced by Whitney in [Whi35]. There are many equivalent definitions, we start with the following.

Definition 4.1.1 (Independent sets description). A matroid M is a pair $(E, \mathcal{I})$ where E is a finite set called the ground set and $\mathcal{I}$ is a family of subsets of E satisfying

1. $\emptyset \in \mathcal{I}$
2. If $A, B \in \mathcal{I}$ then $A \subset B \implies A \in \mathcal{I}$
3. If A,$B \in \mathcal{I}$ and $|A| < |B|$ then there exists $x \in B \smallsetminus A$ such that $A \cup \{x\} \in \mathcal{I}$. $\diamondsuit$

A subset of E is called independent if it belongs to $\mathcal{I}$, and dependent if not. A maximal independent set is called a basis of M. It is easy to see that the size of all bases are the same, this number is called the rank of M. The subset $\mathcal{B}$ of $\mathcal{I}$ consisting of all bases is enough to determine the whole matroid. Any subset of $A \subset E$ is either contained in some basis, or not, and $A \in \mathcal{I} \iff A \subset B$ for some $B \in \mathcal{B}$. So we could also define a matroid to be the pair $(E, \mathcal{B})$. This gives the following definition, which will be most useful for our purposes.

Definition 4.1.2 (Bases description). A matroid M is a pair $(E, \mathcal{B})$ where E is a finite set and $\mathcal{B}$ is a family of subsets of E satisfying

1. $\mathcal{B} \neq \emptyset$
2. If A,$B \in \mathcal{B}$ and $x \in B \smallsetminus A$, then there exists $y \in A \smallsetminus B$ s.t. $(B \smallsetminus x) \cup y \in \mathcal{B}$. $\diamondsuit$

The second axiom is called the basis exchange property.

Example 4.1.3. The uniform matroid of rank k on n elements is the matroid $U_{k,n} = (E, \mathcal{B})$ with $|E| = n$ and $\mathcal{B} = \binom{E}{k} = \{B \subset E \mid |B| = k\}$. $\diamondsuit$

Definition 4.1.4. An isomorphism of two matroids $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ is a bijection $f\colon E_1 \to E_2$ such that $f(B) \in \mathcal{B}_2 \iff B \in \mathcal{B}_1$. If there exists an isomorhism between M_1 and M_2 we say they are isomorphic. $\diamondsuit$

Almost all matroids we consider will be on the ground set $[n]$. This simply corresponds to labeling the elements of the matroid by the numbers 1 to n. If M_1 and M_2 are matroids of rank k on $[n]$, the sets of bases $\mathcal{B}_1$ and $\mathcal{B}_2$ are subsets of $\binom{[n]}{k}$. We will write $M_1 = M_2$ when $\mathcal{B}_1 = \mathcal{B}_2$ and $M_1 \simeq M_2$ when the matroids are isomorphic, but may not have the exact same bases. An isomorphism in this case is simply a permutation $\pi \in S_n$ of $[n]$ such that $\pi(I) \in \mathcal{B}_2 \iff I \in \mathcal{B}_1$.

We will be especially interested in a particular class of matroids that are called representable matroids. These are matroids obtained from a set of vectors in a vector space. Let F be a field, V a vector space over F and E a finite set of vectors in V. We obtain a matroid on the ground set E by specifying the bases of M as the subsets of E that are bases of $\operatorname{span}(E)$.

Definition 4.1.5. A matroid M is called representable over F if it is isomorphic to a matroid obtained from a set of vectors in a vector space over F as above. $\diamondsuit$

Most of the matroids we consider will be representable over $\mathbb{C}$.

Example 4.1.6. Let $V = \mathbb{C}^3$. The matroid M on the ground set $[4]$ with bases $\mathcal{B} = \{123, 134, 234\}$ is represented by the vectors $\{e_1, e_2, e_3, e_1 + e_2\}$ via the matroid isomorphism

$$
\begin{aligned}
f\colon [4] &\to \{e_1, e_2, e_3, e_1 + e_2\} \\
1 &\mapsto e_1 \\
2 &\mapsto e_2 \\
3 &\mapsto e_3 \\
4 &\mapsto e_1 + e_2.
\end{aligned}
$$

We will often represent matroids by the set of column vectors of a matrix. In this example M is represented by the matrix

$$
\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \qquad \diamondsuit
$$

Given a matroid M of rank k, another way of saying that M is represented by a set of vectors E is that the bases of M are the sets $S \in \binom{E}{k}$ such that $\dim(\operatorname{span}(S)) = k$, in general we can consider the function

$$
\begin{aligned}
r\colon \mathcal{P}(E) &\to \mathbb{Z} \\
S &\mapsto \dim(\operatorname{span}(S)).
\end{aligned}
$$

This is the idea behind the next equivalent definition of a matroid.

Definition 4.1.7 (Rank function description). A matroid of rank k on a finite set E is a rank function $r\colon \mathcal{P}(E) \to \mathbb{Z}$ satisfying

1. $0 \le r(S) \le |S|$,
2. $S \subset T \implies r(S) \le r(T)$,
3. $r(E) = k$,

4. $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$. ♢

We may recover the other definitions by specifying the independent sets and bases as

$$\mathcal{I} = \{S \subset E \mid r(S) = |S|\}$$
$$\mathcal{B} = \{S \subset E \mid r(S) = k\}.$$

With the rank function definition, a matroid isomorphism as in Definition 4.1.4 is simply a bijection $f\colon E_1 \to E_2$ such that $r_1(S) = r_2(f(S))$.

Another motivating example comes from graph theory. Let G be a graph. We can define a matroid $M(G)$ on the set of edges of G such that the bases of $M(G)$ are the spanning forests of G.

Example 4.1.8. Consider the graph G in Figure 4.1. The bases of $M(G)$ are

$$1345, 1346, 1356, 2345, 2346, 2356$$

Notice that 7 is in no bases, 3 is in all bases and 1 and 2 are never contained in the same basis. ♢

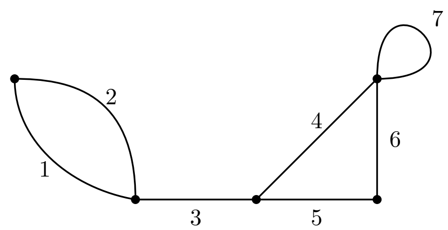


Figure 4.1: The graph from Example 4.1.8 whose matroid contain a loop, a coloop and parallel points.

The example above motivates the following definition.

Definition 4.1.9. Let M be a matroid. An element e of M is called a loop if it is not contained in any bases, and e is called a coloop if it is contained in all bases. A pair a, b of elements of M are called parallel points if they are not loops and no bases contain both a and b. A matroid is called simple if it contains no loops or parallel points. ♢

For representable matroids a loop corresponds to the zero vector, parallel points to parallel vectors and coloops to vectors that are not contained in the subspace spanned by all the other vectors.

If a simple matroid M of rank k is represented by a set of vectors E in a vector space V of dimension k we may think of each element of E as a point in the projective space $\mathbb{P}(V)$. A set $I \in \binom{E}{k}$ is then a basis of M if and only if the corresponding points in $\mathbb{P}(V)$ are not contained in a hyperplane of $\mathbb{P}(V)$.

Example 4.1.10. The matroid M of rank 3 on 7 elements represented over $\mathbb{C}$ by the columns of the matrix

$$W = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

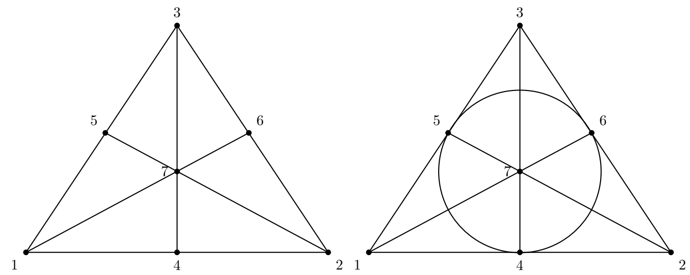


Figure 4.2: The Fano and non-Fano matroids

is illustrated as points in $\mathbb{P}^2$ on the left in Figure 4.2. The lines are the hyperplanes of $\mathbb{P}^2$ through 3 points. So we see that M is given by the set of bases

$$\binom{[7]}{3} \setminus \{124, 135, 167, 236, 257, 347\}.$$

This matroid is called the non-Fano matroid. $\diamondsuit$

We may extend this idea to obtain matroids from other such arrangements of hyperplanes in $\mathbb{P}^{k-1}$. In particular we get rank 3 matroids from line arrangements in $\mathbb{P}^2$ satisfying that any two distinct lines meet in at most one point, see [Oxl06, Condition 1.5.9].

Example 4.1.11. The matroid obtained from the hyperplane arrangement on the right in Figure 4.2 is known as the Fano matroid and is given by the bases

$$\binom{[7]}{3} \setminus \{124, 135, 167, 236, 257, 347, 456\}.$$

The Fano matroid is not representable over $\mathbb{C}$, but is representable over the finite field of characteristic two, $\mathbb{F}_2$. Indeed column 4,5 and 6 in the matrix in Example 4.1.10 are linearly dependent over $\mathbb{F}_2$, which corresponds to the circle in Figure 4.2. $\diamondsuit$

Definition 4.1.12. Let M_i be a matroid on the set E_i with bases $\mathcal{B}_i$ for $i = 1, 2$. We define the direct sum matroid $M_1 \oplus M_2$ on the ground set $E_1 \sqcup E_2$ by the bases

$$\mathcal{B} = \{B_1 \sqcup B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}. \qquad \diamondsuit$$

If M_i is defined by the rank function r_i, then $M_1 \oplus M_2$ is defined by the rank function

$$\begin{aligned} r \colon \mathcal{P}(E_1 \sqcup E_2) &\to \mathbb{Z} \\ S_1 \sqcup S_2 &\mapsto r_1(S_1) + r_2(S_2). \end{aligned}$$

If M_i is represented by a set of vectors E_i in a vector space V_i, then $M_1 \oplus M_2$ is represented by $E_1 \sqcup E_2$ in the vector space $V_1 \oplus V_2$. We say M is connected if it cannot be written as the direct sum of two non-empty matroids. Any matroid M can be written uniquely as a sum of connected matroids $M = \bigoplus_i M_i$. The matroids M_i are called the connected components of M.

Definition 4.1.13. The number of connected components is denoted $\kappa(M)$. $\diamondsuit$

We may define an equivalence relation on $[n]$ generated by $i \sim j$ if there exists bases B_1, B_2 such that $B_2 = (B_1 \smallsetminus i) \cup j$. That is, B_1 and B_2 are separated by precisely one basis exchange. Now i and j are in the same connected component if and only if $i \sim j$, so the connected components of M are obtained from the equivalence classes of $\sim$, see [Kat16, Chapter 5].

Example 4.1.14. Let M be the matroid of rank 2 on $[4]$ given by the bases $\mathcal{B} = \{13, 14, 23, 24\}$. We see that $1 \sim 2$ since $(14 \smallsetminus 1) \cup 2 = 24$ or $(13 \smallsetminus 1) \cup 2 = 23$ and $3 \sim 4$ since $(13 \smallsetminus 3) \cup 4 = 14$ or $(23 \smallsetminus 3) \cup 4 = 24$. Indeed the connected components of M are the uniform matroids of rank 1 on $\{1, 2\}$ and $\{3, 4\}$. $\diamondsuit$

Notice that a graph G being connected is not equivalent to the matroid $M(G)$ being connected. To see this, let G be a connected graph with a cut vertex c. Then any spanning tree T of G can be partitioned $T = T_1 \sqcup T_2$ with each T_i being a spanning tree of the parts of G cut by c.

Given a graph and an edge we can delete or contract that edge to get a new graph. This motivates the following definition of two operations on matroids called deletion and contraction.

Definition 4.1.15. Let $M = (E, \mathcal{B})$ be a matroid and $e \in E$. If e is not a coloop we define the deletion matroid $M \smallsetminus e = (E \smallsetminus e, \mathcal{B} \smallsetminus e)$ by

$$\mathcal{B} \smallsetminus e = \{B \in \mathcal{B} \mid e \notin B\}.$$

If e is not a loop we define the contraction matroid $M/e = (E \smallsetminus e, \mathcal{B}/e)$ by

$$\mathcal{B}/e = \{B \smallsetminus e \mid e \in B \in \mathcal{B}\}. \qquad \diamondsuit$$

Notice that if e is a coloop or loop then $\mathcal{B} \smallsetminus e$ and $\mathcal{B}/e$ are respectively empty.

Example 4.1.16. The uniform matroid $U_{0,1}$ consists of a single loop, and $U_{1,1}$ consists of a single coloop. Any matroid M can be decomposed as

$$M = M' \oplus \bigoplus_{\#\text{loops}} U_{0,1} \oplus \bigoplus_{\#\text{coloops}} U_{1,1}.$$

Where M' is a matroid without loops and coloops. To show this let i be a coloop of M, then any basis B can be written $B = i \sqcup B'$ where B' is a basis of M/i. Similarly if j is a loop of M, then B can be written $B = \emptyset \sqcup B$ where B is a basis of $M \smallsetminus i$ and since $U_{0,1}$ is of rank 0, $\emptyset$ is a basis of $U_{0,1}$. $\diamondsuit$

We can extend the deletion and contraction operations to any subset X of E, even if X contains a loop or coloop.

Definition 4.1.17. Let M be a matroid on the ground set E with rank function r, and let X be a subset of E. The deletion matroid $M \smallsetminus X$ is a matroid on the ground set $E \smallsetminus X$ with rank function

$$r_{M \smallsetminus X}(S) = r(S).$$

The contraction matroid M/X is a matroid on the ground set $E \smallsetminus X$ with rank function

$$r_{M/X}(S) = r(S \cup X) - r(X). \qquad \diamondsuit$$

The next example is an exercise in squeezing as many definitions as possible into one example.

Example 4.1.18. Let G_1 and G_2 be two connected graphs with matroids M_1 and M_2. Graphical matroids are also representable over any field, see [Kat16, Example 4.7], so assume M_i is represented by a finite set of vectors E_i in the vector space V_i. Consider the graph G given by connecting a vertex of G_1 to a vertex of G_2 with an edge e. Notice that e is a coloop of $M = M(G)$ since any spanning tree of G must contain e. The matroid M is represented by the finite set $E_1 \sqcup E_2 \sqcup \{e\}$ of vectors in the vector space $V_1 \oplus V_2 \oplus \mathbb{C}\{e\}$. For a subset S of edges of G let G_S be the subgraph of G spanned by S. Then the rank function of M is given by $r(S) = \#(\text{ vertices in } G_S) - \#(\text{connected components of } G_S)$. Now consider the matroid $M \smallsetminus e$. Any set S of edges of G not containing e can be split into $S = S_1 \sqcup S_2$ with S_i a subset of the edges of G_i. Now the graph $G_S = G_{S_1} \sqcup G_{S_2}$ so we have

$$\begin{aligned} r_{M \smallsetminus e}(S) &= \#(\text{ vertices in } G_S) - \#(\text{connected components of } G_S) \\ &= \#(\text{ vertices in } G_{S_1}) - \#(\text{connected components of } G_{S_2}) \\ &\quad + \#(\text{ vertices in } G_{S_2}) - \#(\text{connected components of } G_{S_2}) \\ &= r_1(S_1) + r_2(S_2) \end{aligned}$$

if r_i is the rank function of M_i. This shows that $M \smallsetminus e = M_1 \oplus M_2$, hence $M \smallsetminus e$ is disconnected. This can also be seen since $M \smallsetminus e$ is the matroid represented by $E_1 \sqcup E_2$ in the vector space $\pi(V_1 \oplus V_2 \oplus \mathbb{C}\{e\}) = V_1 \oplus V_2$ where π is the projection along e. $\diamondsuit$

We have one more operation on matroids to discuss, the notion of the dual matroid.

Definition 4.1.19. The dual matroid M^* of a matroid M on E with bases $\mathcal{B}$ is a matroid on E given by the bases

$$\mathcal{B}^* = \{E \smallsetminus B \mid B \in \mathcal{B}\}. \qquad \diamondsuit$$

The dual matroid satisfies many nice properties. An element $i \in E$ is a loop of M if and only if i is a coloop in M^*. And since $(M^*)^* = M$, i is a coloop of M if and only if i is a loop of M^*. And the dual interchanges deletion and contraction in the sense that for $X \subset E$ we have $(M \smallsetminus X)^* = M^*/X$ and $(M/X)^* = M^* \smallsetminus X$. The dual matroid of a graphical matroid $M(G)$ is the matroid of the dual graph of G.

4.2 Matroid polytopes

In this section we establish the convex polytopes in $\mathbb{R}^n$ associated to a matroid on $[n]$.

Definition 4.2.1. [Gel+87, Chapter 4] Let $M = ([n], \mathcal{B})$ be a matroid. For each basis $B \in \mathcal{B}$ we define the point in $\mathbb{R}^n$, $\epsilon_B = \sum_{i \in B} \epsilon_i$. The convex polytope corresponding to M is

$$\Delta(M) = \operatorname{Conv}(\epsilon_B \mid B \in \mathcal{B}). \qquad \diamondsuit$$

The following examples will be especially useful.

Example 4.2.2. The polytope corresponding to the uniform matroid of rank 1 on n elements is

$$\Delta(U_{1,n}) = \operatorname{Conv}(\epsilon_1, \dots, \epsilon_n) \subset \mathbb{R}^n.$$

We recognize this as the standard $(n-1)$-simplex, Δ_{n-1} embedded into $\mathbb{R}^n$. $\diamondsuit$

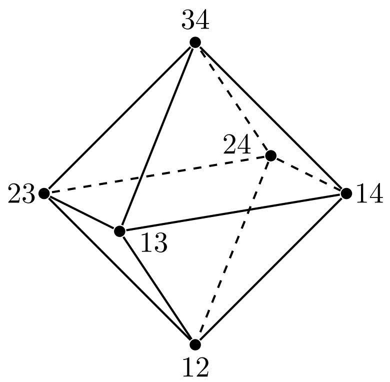


Figure 4.3: The hypersimplex $\Delta_{2,4}$

Example 4.2.3. Consider the uniform matroid $U_{k,n}$. The polytope $\Delta(U_{k,n})$ is the convex hull of all points $\epsilon_B = \sum_{j=1}^{k} \epsilon_{i_j}$ for $B = \{i_1, \ldots, i_k\} \in \binom{[n]}{k}$. This is the hypersimplex denoted $\Delta_{k,n}$. To be precise

$$\Delta_{k,n} = \{r = (r_1, \ldots, r_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} r_i = k, r_i \in [0,1]\}.$$

It is a $(n-1)$-dimensional polytope. Notice it is contained in the hyperplane of $\mathbb{R}^n$ given by the equation $\sum_{i=1}^{n} r_i = k$. ♢

Example 4.2.4. The hypersimplex $\Delta_{2,4} = \Delta(U_{2,4})$ is an octahedron in $\mathbb{R}^4$ with vertices labeled by the bases of $U_{2,4}$, $\mathcal{B} = \{12, 13, 14, 23, 24, 34\}$. It is pictured in Figure 4.3. ♢

Definition 4.2.5 (Polytope description). A matroid polytope Δ is a convex polytope contained in $\Delta_{k,n}$ for some k and n such that the vertices of Δ are vertices of $\Delta_{k,n}$ and the edges of Δ are translates of the vectors $\epsilon_i - \epsilon_j$ for $i \neq j$. ♢

Theorem 4.2.6. [Gel+87] *A polytope Δ is a matroid polytope if and only if $\Delta = \Delta(M)$ for some matroid M.* ♡

This means that matroid polytopes can be taken as an alternative definition of a matroid. For proofs of this see [Gel+87, chapter 4] or [Kat16, Chapter 8].

Given a matroid polytope Δ we can recover the bases of the matroid M by the vertices of Δ. If an edge of Δ from ϵ_{B_1} to ϵ_{B_2} is parallel to $\epsilon_j - \epsilon_i$ then $B_2 = B_1 \setminus i \cup j$. That is the edges of Δ indicate which bases of M are separated by one basis exchange.

We can use this to find all matroids on rank 2 on 4 elements.

Example 4.2.7. As in Example 4.2.4 consider the octahedron $\Delta_{2,4}$ with vertices labeled by $\mathcal{B} = \{12, 13, 14, 23, 24, 34\}$. If we consider a subset $\mathcal{A} \subset \mathcal{B}$ then $\mathcal{A}$ is the set of bases of a matroid of rank 2 on [4] if and only if the edges of $\text{Conv}(\mathcal{A})$ are edges of $\Delta_{2,4}$. For example by referring to Figure 4.3 we see that $\mathcal{A} = \{12, 13, 14, 24\}$ does not define a matroid since the edge between 13 and 24 is not an edge of $\Delta_{2,4}$. Indeed $\mathcal{A}$ does not satisfy the basis exchange property since $1 \in 13$ and $2, 4 \in 24$, but neither $(13 \setminus 1) \cup 2 = 23$ nor $(13 \setminus 1) \cup 4 = 34$ lie in $\mathcal{A}$. By looking at Figure 4.3 we can easily count that there are 36 matroids of rank 2 on [4]. ♢

If the matroid M is disconnected, say $M = M_1 \oplus M_2$ where M_i is a matroid on E_i. Then for each basis B the vector ϵ_B can be decomposed $\epsilon_B = \epsilon_{B_1} + \epsilon_{B_2}$ where $\epsilon_{B_i} \in \mathbb{R}^{E_i}$. So the matroid polytope $\Delta(M)$ is the sum $\Delta(M_1) + \Delta(M_2)$. The dimension of $\Delta(M_i)$ is at most $|E_i| - 1$, so the dimension of $\Delta(M)$ is at most $|E_1 \sqcup E_2| - 2$. In fact each connected component of M decreases the dimension of $\Delta(M)$ by one.

Proposition 4.2.8. [Spe09, Chapter 4] *The dimension of $\Delta(M)$ is $n - \kappa(M)$.* ♡

Proof. The dimension of $\Delta(M)$ is the dimension of the vector space V spanned by all the edge directions of $\Delta(M)$. Let $\sim$ be the equivalence relation on $[n]$ generated by $i \sim j$ if there exists an edge of $\Delta(M)$ parallel to $\epsilon_i - \epsilon_j$. Now each equivalence class of $\sim$ corresponds to a vector $\epsilon_i - \epsilon_j$ that is not in the span of V. Notice that this equivalence relation is the same as the one that gives the connected components of M, so $\dim(\Delta(M)) = n - \kappa(M)$. □

In particular if M is connected $\Delta(M)$ is of dimension $n-1$, i.e. full dimensional, in the sense that it is full dimensional in the hyperplane all matroid polytopes are contained in.

4.3 Connection to the Grassmannian

Matroid theory relates to Grassmannians since any point x in a Grassmannian gives rise to a representable matroid M_x. We will see in Theorem 5.1.1 that $\Delta(M_x)$ is the polytope of the projective toric variety $\overline{Tx}$.

Definition 4.3.1. Let $x \in G(k,n)$ and $W = (v_1, \dots v_n)$ be a matrix with column vectors $v_i \in \mathbb{C}^k$ such that the rowspace of W is $L(x)$. The matroid M_x on the ground set $[n]$ is given by the bases $\mathcal{B}_x = \{J = \{i_1, \dots i_n\} \in \binom{[n]}{k} \mid \langle v_{i_1}, \dots v_{i_k}\rangle = \mathbb{C}^k\}$. ♢

In other words, M_x is the matroid represented over $\mathbb{C}$ by the columns of W. Since W spans a k-dimensional subspace of $\mathbb{C}^n$ there is at least one choice of k linearly independent column vectors. So the rank of M_x is k. It is not clear that M_x is well-defined, that is, if M_x depends on the choice of W or not. One way to see that it is well-defined is by considering the Plücker coordinates of x. Let $I \in \binom{[n]}{k}$ then I is a basis for M_x if and only if the Plücker coordinate $p_I(x)$ is non-zero. So M_x depends only on which Plücker coordinates of x are non-zero, which we know is independent of the choice of W. Clearly any matroid representable over $\mathbb{C}$ can be realized as the matroid of a point in some Grassmannian.

Another way to view M_x that is more inherent to the subspace $L(x)$ rather than the matrix W is given in [Ard21, Chapter 2]. For $I \subset [n]$ let V_I be the coordinate plane in $\mathbb{C}^n$

$$\begin{aligned} V_I &= \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i = 0 \text{ for } i \in I\} \\ &= \langle e_i \mid i \notin I\rangle. \end{aligned}$$

Now I is independent in M_x if $L(x)$ intersect V_I transversally, i.e. $\dim(L(x) \cap V_I) = \dim(L(x)) - |I| = k - |I|$. In particular, we may define the matroid M_x by

$$I \in \binom{[n]}{k} \text{ is a basis of } M_x \iff L(x) \cap V_I = 0. \tag{4.1}$$

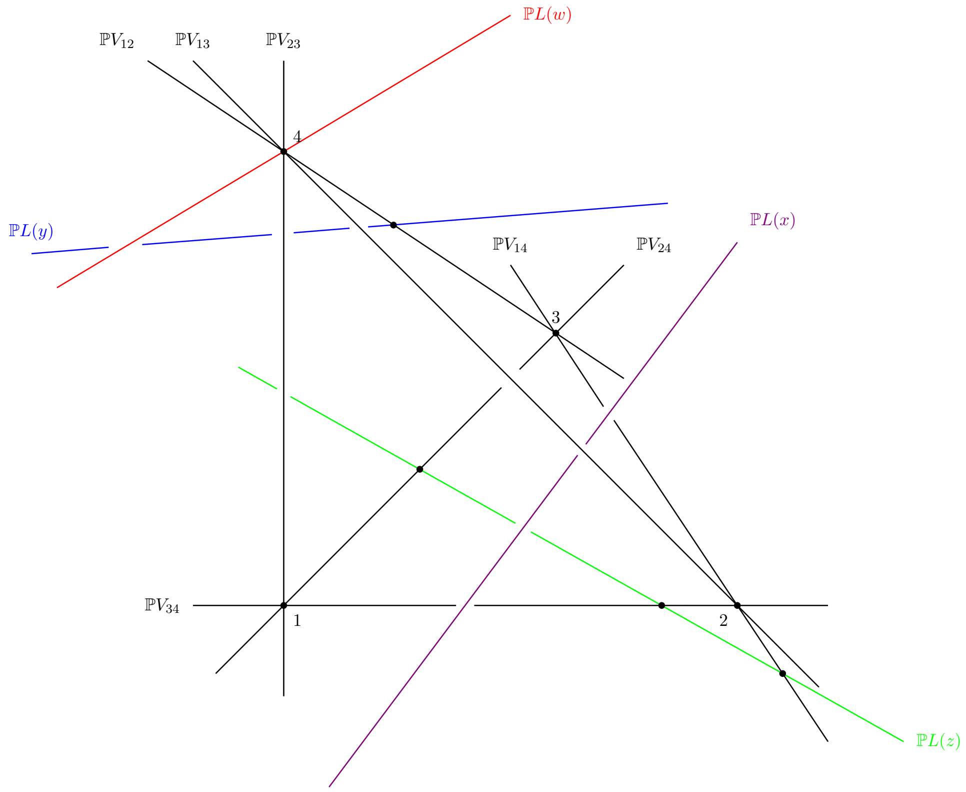


Figure 4.4: Intersection of $L(x)$ for $x \in G(2,4)$ with coordinate planes.

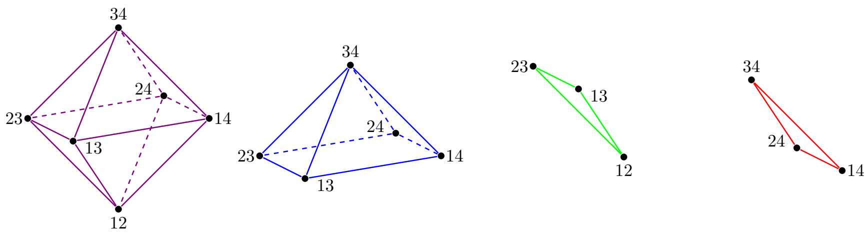


Figure 4.5: The matroid polytopes $\Delta(M_x)$, $\Delta(M_y)$, $\Delta(M_z)$, $\Delta(M_w)$

We have already seen this fact in disguise in the proof of Lemma 2.6.7. Recall that Lemma 2.6.7 states that $x \in H \iff L(x) \cap V_{n-k} \neq 0$ for some hyperplane H of $\mathbb{P}^{\binom{n}{k}-1}$ depending on V_{n-k}. By examining the proof we see that if H is the coordinate hyperplane $H = V(p_I)$ for some $I \in \binom{[n]}{k}$ then $V_{n-k} = V_I$. So we have

$$\begin{aligned} I \text{ is a basis of } M_x &\iff p_I(x) \neq 0 \\ &\iff x \notin V(p_I) \\ &\iff L(x) \cap V_I = 0. \end{aligned}$$

Example 4.3.2. Consider $G(2,4)$. The matroids M_x for $x \in G(2,4)$ are rank 2 matroids on $[4]$. For $I \in \binom{[4]}{2}$ we can draw the projectivization $\mathbb{P}V_I$ of V_I as a line in $\mathbb{P}^3$. The collection of all such lines are pictured in Figure 4.4. Similarly we can draw $\mathbb{P}L(x)$ as a line in $\mathbb{P}^3$ and by (4.1) the matroid M_x is determined by the intersections $\mathbb{P}L(x) \cap \mathbb{P}V_I$. Figure 4.4 shows four lines $\mathbb{P}L(x)$ in violet, $\mathbb{P}L(y)$ in blue, $\mathbb{P}L(z)$ in green and $\mathbb{P}L(w)$ in red. Notice that $\mathbb{P}L(x)$ does not intersect any of the coordinate lines $\mathbb{P}V_I$ hence all I are bases of M_x so

$$M_x = U_{2,4}.$$

The line $\mathbb{P}L(y)$ only intersect $\mathbb{P}V_{12}$ so the bases of M_y are $\{13, 14, 23, 24, 34\}$. We see that 1 and 2 are parallel points of M_y. The line $\mathbb{P}L(z)$ is contained in the coordinate plane containing the lines $\mathbb{P}V_{14}$, $\mathbb{P}V_{24}$ and $\mathbb{P}V_{34}$ so the bases of M_z are $\{12, 13, 23\}$. That is, 4 is a loop of M_z and

$$M_z \simeq U_{2,3} \oplus U_{0,1}.$$

Lastly $\mathbb{P}L(w)$ intersects the lines $\mathbb{P}V_{12}$, $\mathbb{P}V_{13}$ and $\mathbb{P}V_{23}$ so the bases of M_w are $\{14, 24, 34\}$. So 4 is a coloop of M_w and

$$M_w \simeq U_{1,3} \oplus U_{1,1}.$$

The corresponding matroid polytopes are pictured in Figure 4.5. $\diamondsuit$

For $x \in G(k, n)$, as in Definition 4.3.1, let $W = (v_1, \dots, v_n)$ be a matrix whose rowspace is $L(x)$. An element $i \in M_x$ is a loop if and only if $p_I(x) = 0$ for all $I \ni i$. This happens if and only if the column vector $v_i = 0$. Also i is a coloop if and only if $p_I(x) = 0$ for all $I \not\ni i$. In terms of the matrix W up to row reduction, i being a coloop corresponds to v_i having all entries 0 except for one 1, and that the row with the 1 is otherwise all zeros. That is W can be taken to have the form

$$\begin{pmatrix} & & & 0 & & & \\ & * & & \vdots & & * & \\ & & & 0 & & & \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ & & & 0 & & & \\ & * & & \vdots & & * & \\ & & & 0 & & & \end{pmatrix}. \tag{4.2}$$

For $i \in [n]$ fix the line $l_i = \langle e_i \rangle$ and the hyperplane $H_i = \langle e_1, \dots, \hat{e_i}, \dots, e_n \rangle$. In these terms we have the following equivalences

$$L(x) \subset H_i \iff i \text{ is a loop in } M_x \tag{4.3}$$
$$l_i \subset L(x) \iff i \text{ is a coloop in } M_x. \tag{4.4}$$

We can do even better than this. Let $S \subset [n]$. Since $\langle e_i \mid i \notin S \rangle = \bigcap_{i \in S} H_i$ and $\langle e_i \mid i \in I \rangle = \bigoplus_{i \in S} l_i$ we have that

$$L(x) \subset \langle e_i \mid i \notin S \rangle \iff \text{all } i \in S \text{ are loops of } M_x$$
$$\langle e_i \mid i \in S \rangle \subset L(x) \iff \text{all } i \in S \text{ are coloops of } M_x.$$

Example 4.3.3. Let $\lambda = (n-k)$ and $\mu = (1)^k$ be partitions in $k \times (n-k)$ and let $V_\bullet$ be a flag with $V_1 = l_i$ and $W_\bullet$ be a flag with $W_{n-1} = H_i$. Then $X_\mu(W_\bullet) = \{x \mid L(x) \subset H_i\}$ and $X_\lambda(V_\bullet) = \{x \mid l_i \subset L(x)\}$. The fact that $X_\mu(W_\bullet) \cap X_\lambda(V_\bullet) = \emptyset$, which can be seen by Pieri's formula or Proposition 2.5.8, is equivalent to the statement that i cannot be both a loop and a coloop in M_x. $\diamondsuit$

We now define functions on Grassmannians mimicking deletion and contraction on matroids, as in Definition 4.1.15. Let $p\colon \mathbb{C}^{[n]} \to H_i = \mathbb{C}^{[n] \setminus i}$ be the projection onto

H_i we define the following deletion function

$$\begin{aligned} G(k,n) \setminus \{y \mid l_i \subset L(y)\} &\longrightarrow G(k,n-1) \\ x &\longmapsto x \setminus i \end{aligned} \tag{4.5}$$

given by $L(x \setminus i) = p(L(x))$. Notice that if $l_i \subset L(x)$ then $p(L(x))$ would be of dimension $k-1$, so the deletion function would not be well-defined. Similarly we define the contraction function

$$\begin{aligned} G(k,n) \setminus \{y \mid L(y) \subset H_i\} &\longrightarrow G(k-1,n-1) \\ x &\longmapsto x/i \end{aligned} \tag{4.6}$$

where $L(x/i) = L(x) \cap H_i$. Here we think of the ambient space in the Grassmannians $G(k,n-1)$ and $G(k-1,n-1)$ as H_i. Notice that if $L(x) \subset H_i$ then $L(x) \cap H_i$ is of dimension k so the contraction function would not be well-defined.

If $L(x)$ is the rowspace of $W = (v_1 \dots v_n)$ then $L(x \setminus i)$ is the rowspace of the k by $n-1$ matrix $(v_1 \dots \hat{v}_i \dots v_n)$. And $L(x/i)$ is the rowspace of the $k-1$ by n matrix obtained from W in the following way:

Since $L(x) \not\subset H_i$, $v_i \neq 0$. Pick a pivot, that is a non-zero entry of v_i, say the pivot is contained in the rth row of W. Now row reduce W so that the ith column has one 1 in the pivot position and otherwise zeros. Now $L(x/i)$ is the rowspace of the $k-1$ by n matrix W' obtained by removing the rth row of the row reduces matrix. Notice that the ith column of W' is all zeros, so we may also remove that column, to obtain a $k-1$ by $n-1$ matrix, without changing the rowspace. One can show that x/i is independent of choice of pivot element.

Example 4.3.4. Let x be the point in $G(2,3)$ given by $L(x) = \langle e_1 + e_3, e_2 + e_3 \rangle$. Then $L(x)$ is the rowspace of

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Now $L(x \setminus 3)$ is H_3 and $L(x/3)$ is the line in H_3 spanned by $(-1,1)$. ♢

The following proposition justifies the notation above.

Proposition 4.3.5. *$M_{x \setminus i} = M_x \setminus i$ and $M_{x/i} = M_x/i$.* ♡

Proof. For the first statement assume $l_i \not\subset L(x)$. By (4.3) i is not a coloop in M_x, so $M_x \setminus i$ is well-defined. The subspace $L(x \setminus i)$ is the rowspace of the matrix $(v_1 \dots \hat{v}_i \dots v_n)$ so the ground set of $M_{x \setminus i}$ is $[n] \setminus \{i\}$ and $I \in \binom{[n] \setminus \{i\}}{k}$ is a basis for $M_{x \setminus i}$ if and only if I is a basis for M_x. So the bases of $M_{x \setminus i}$ are the bases of M_x not containing i, just as in Definition 4.1.15, so $M_{x \setminus i} = M_x \setminus i$.

For the second statement assume that $L(x) \not\subset H_i$. Again by (4.4) i is not a loop in M_x so M_x/i is well-defined. Since i is not a loop, $v_i \neq 0$. Up to row reduction we may assume the entries of v_i are all zeros except for one 1 in the rth position. Recall that x/i is the rowspace of the matrix $(v'_1 \dots v'_{i-1} v'_{i+1} \dots v'_n)$ where v'_j is obtained from v_j by removing the rth entry. For any $I = \{j_1, \dots, j_n\} \in \binom{[n] \setminus \{i\}}{k-1}$, the corresponding set $\{v'_{j_1}, \dots, v'_{j_{k-1}}\}$ of vectors is linearly independent if and only if the set $\{v_{j_1}, \dots, v_{j_{k-1}}, v_i\}$ is linearly independent, since $\det(v'_{j_1} \dots v'_{j_{k-1}}) = \det(v_{j_1} \dots v_i \dots v_{j_{k-1}})$. Comparing this to Definition 4.1.15 we see that $M_{x/i} = M_x/i$. □

As mentioned in [GH94, Chapter 1.5] the inclusion of vector spaces $\mathbb{C}^{[n]\setminus i} \hookrightarrow \mathbb{C}^{[n]}$ induces two inclusions of Grassmannians in the opposite directions of (4.5) and (4.6),

$$\iota_1 \colon G(k, \mathbb{C}^{[n]\setminus i}) \hookrightarrow G(k, \mathbb{C}^{[n]}) \tag{4.7}$$
$$\iota_2 \colon G(k-1, \mathbb{C}^{[n]\setminus i}) \hookrightarrow G(k, \mathbb{C}^{[n]}). \tag{4.8}$$

The first is given by $L(\iota_1(x)) = L(x)$ considered as a subspace of $\mathbb{C}^{[n]}$ and the second by $L(\iota_2(x)) = L(x) \oplus l_i$. By (4.3) and (4.4) we see that $M_{\iota_1(x)} = M_x \oplus U_{0,1}$ and $M_{\iota_2(x)} = M_x \oplus U_{1,1}$. Recall the isomorphism

$$\begin{aligned} G(k,n) &\longrightarrow G(n-k,n) \\ x &\longmapsto x^{\perp} \end{aligned}$$

where $L(x^{\perp})$ is the orthogonal complement $L(x)^{\perp}$.

Proposition 4.3.6. *Let $x \in G(k,n)$ then $M_{x^{\perp}} = M_x^*$.* ♡

Proof. Since $V_I^{\perp} = V_{I^c}$ and $(V \cap W)^{\perp} = V^{\perp} \oplus W^{\perp}$, by applying (4.1) we get for $I \in \binom{[n]}{k}$ that

$$\begin{aligned} I \text{ is a basis of } M_x &\iff L(x) \cap V_I = 0 \\ &\iff (L(x) \cap V_I)^{\perp} = 0^{\perp} \\ &\iff L(x)^{\perp} \oplus V_{I^c} = \mathbb{C}^n \\ &\iff L(x^{\perp}) \cap V_{I^c} = 0 \\ &\iff I^c \text{ is a basis of } M_{x^{\perp}}. \end{aligned}$$

Here the fourth equivalence holds since $L(x)^{\perp}$ and V_{I^c} are of complementary dimension. Comparing this with Definition 4.1.19 we see that $M_{x^{\perp}} = M_x^*$. □

4.4 Matroid invariants

In this section we follow the discussion on valuative matroid invariants in [FS12] and [Kat16, Chapter 7].

Let $\mathcal{M}_{k,n}$ denote the set of rank k matroids on $[n]$, $\mathcal{M}_n = \bigsqcup_{k=0}^{n} \mathcal{M}_{k,n}$ the set of matroids on $[n]$ and $\mathcal{M} = \bigsqcup_n \mathcal{M}_n$ the set of all matroids.

Definition 4.4.1. A matroid invariant valued in a set A is a function $f \colon \mathcal{M} \to A$ that takes the same value on isomorphic matroids. ♢

If f is a function from $\mathcal{M}_{k,n}$ or $\mathcal{M}_n$ and A is a group we may extend f to $\mathcal{M}$ by specifying $f(M) = 0$ if M is not in $\mathcal{M}_{k,n}$ or $\mathcal{M}_n$. One important matroid invariant is the Crapo beta invariant.

Example 4.4.2. [Cra67] Let M be a matroid on $[n]$ with rank function r. The beta invariant $\beta \colon \mathcal{M} \to \mathbb{Z}$ is defined as

$$\beta(M) = (-1)^{r(M)} \sum_{S \subset [n]} (-1)^{|S|} r(S).$$

The beta invariant can be defined by satisfying the recursion $\beta(M) = 0$ if M contains a loop or coloop and $\beta(U_{1,2}) = 1$ and for an element $i \in [n]$ that is not a loop or colooop $\beta(M) = \beta(M/i) + \beta(M \setminus i)$. It is also well known that $\beta(M)$ is positive and $\beta(M) = 0 \iff M$ is not connected (excluding the base case $U_{0,1}$), see [Cra67, Theorem 2]. ♢

Definition 4.4.3. [Kat16, Definition 7.2] A matroid invariant $f\colon \mathcal{M} \to R$ with values in a commutative ring R is called a Tutte-Grothendieck invariant if it satisfies

1. $f(M) = f(M \setminus i) + f(M/i)$ for i not a loop or coloop in M
2. $f(M_1 \oplus M_2) = f(M_1)f(M_2)$. ♢

An important examples of a Tutte-Grothendieck invariant is the Tutte polynomial.

Example 4.4.4. [Kat16, Proposition 7.4] The Tutte polynomial is a matroid invariant valued in $\mathbb{Z}[x,y]$ defined by

$$T_M(x,y) = \sum_{S \subset [n]} (x-1)^{r(M)-r(S)}(y-1)^{|S|-r(S)}. \qquad ♢$$

It is the unique Tutte-Grothendieck invariant valued in $\mathbb{Z}[x,y]$ such that $T_{U_{1,1}}(x,y) = x$ and $T_{U_{0,1}}(x,y) = y$.

One may recover the beta invariant from the Tutte polynomial as the coefficient of x. Next we define valuative functions of matroids, that is functions that behave well with respect to matroid subdivisions.

Definition 4.4.5. A polyhedral complex is a set $\mathcal{D}$ of polytopes in $\mathbb{R}^n$ such that

1. for any $\Delta \in \mathcal{D}$ all faces of Δ are in $\mathcal{D}$
2. for $\Delta_1, \Delta_2 \in \mathcal{D}$, the intersection $\Delta_1 \cap \Delta_2$ is a face of both Δ_1 and Δ_2. ♢

Definition 4.4.6. [FS12, Chapter 4] A subdivision of a polytope Δ is a polyhedral complex such that $\Delta = \cup_{\Lambda \in \mathcal{D}} \Lambda$. In the case when Δ is a matroid polytope, $\mathcal{D}$ is called a matroid subdivision if all polytopes in $\mathcal{D}$ are matroid polytopes. ♢

For any such subdivision $\mathcal{D}$ where $\Delta_1, \dots, \Delta_r$ are the elements of $\mathcal{D}$ of maximal dimension, we denote $\Delta_J = \cap_{j \in J} \Delta_j$ for any $J \subset [r]$. We also fix $\Delta_\emptyset = \Delta$.

Definition 4.4.7. [FS12, Chapter 4] We identify the set $\mathcal{M}$ with the set of matroid polytopes. A function $f\colon \mathcal{M} \to A$, valued in an additive abelian group A is called valuative if for any matroid M and any matroid subdivision $\mathcal{D}$ of $\Delta(M)$ as above

$$\sum_{J \subset [r]} (-1)^{|J|} f(\Delta_J) = 0. \tag{4.9}$$

♢

Note that (4.9) is equivalent to

$$f(M) = \sum_{\emptyset \neq J \subset [r]} (-1)^{|J|+1} f(\Delta_J). \tag{4.10}$$

That is, f being valuative means that it satisfies an inclusion-exclusion property with respect to all matroid subdivisions. If (4.10) only depend on the maximal dimensional elements $\Delta_1, \ldots, \Delta_r$, that is, if

$$\sum_{\substack{\emptyset \neq J \subset [r] \\ |J| \neq 1}} (-1)^{|J|+1} f(\Delta_J) = 0,$$

we call f top-valuative.

Example 4.4.8. For a fixed n, the normalized volume $\mathrm{Vol}_{n-1} \colon \mathcal{M}_n \to \mathbb{Z}$ is a top-valuative matroid invariant since it is 0 on any polytope of dimension different from $n-1$ and $\mathrm{Vol}_{n-1}(\Delta) = \sum_{i=1}^{r} \mathrm{Vol}_{n-1}(\Delta_i)$. $\diamondsuit$

The Tutte polynomial and the beta invariant are also examples of valuative matroid invariants. Moreover the beta invariant is top-valuative. To see this notice that any polytope in a subdivision of dimension smaller than the dimension of $\Delta(M)$ is the matroid polytope of a disconnected matroid, hence with $\beta = 0$. So we have that

$$\beta(\Delta(M)) = \sum_{i=1}^{r} \beta(\Delta_i). \tag{4.11}$$

Chapter 5

Schubert coefficients of representable matroids

In this chapter we continue the discussion on the torus orbits closure of a point $x \in G(k, n)$ from Section 3.4 and its connection to the matroid M_x of the point x.

5.1 The torus orbit closure

Theorem 5.1.1. *[Gel+87] Let $x \in G(k, n)$. The variety $\overline{Tx}$ is a toric translation of $X_{\Delta(M_x)}$.* ♡

By toric translation we mean that there is an automorphism of $\mathbb{P}^{\binom{n}{k}-1}$, given by $p_I \mapsto p_I(x)p_I$ when I is a basis of M_x and $p_I \mapsto p_I$ otherwise, which takes $X_{\Delta(M_x)}$ to $\overline{Tx}$.

Proof. Recall from (3.1) that $\overline{Tx}$ is defined as the closure of the image of the map

$$\begin{aligned} T &\to \mathbb{P}^{\binom{n}{k}-1} \\ t &\mapsto (p_I(x)t_I)_{I \in \binom{[n]}{k}} \end{aligned}$$

where $t_I = \prod_{i \in I} t_i$. On the other hand $X_{\Delta(M_x)}$ is defined as the closure of the image of the map $\phi_{\mathcal{A}}$ as in Definition 3.3.1 where $\mathcal{A} = \Delta(M_x) \cap \mathbb{Z}^n$. The matroid polytope $\Delta(M_x)$ is contained in the n-cube $[0, 1]^n$ so the only possible lattice points in $\mathcal{A}$ are the vertices ϵ_I which correspond to the bases I of M_x. Let $\mathcal{B}$ denote the set of bases of M_x. In this language $X_{\Delta(M_x)}$ is the closure of the image of the map

$$\begin{aligned} \phi_{\mathcal{A}} \colon (\mathbb{C}^*)^n &\to \mathbb{P}^{|\mathcal{B}|-1} \\ t &\mapsto (t_I)_{I \in \mathcal{B}}. \end{aligned}$$

Since $\mathcal{B} \subset \binom{[n]}{k}$ we may compose $\phi_{\mathcal{A}}$ with the inclusion $\mathbb{P}^{|\mathcal{B}|-1} \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$ given by mapping into the coordinate plane $V(p_I \mid I \notin \mathcal{B})$ of $\mathbb{P}^{\binom{n}{k}-1}$. Also the diagonal torus acts trivially through this map, and the coefficients $p_I(x)$ are zero if and only if I is not a basis of M_x, so the only difference between the maps above is the translate by the non-zero coefficients $p_I(x)$. □

A result of this theorem is that a lot of information about the variety $\overline{Tx}$ is encoded in the matroid M_x. We emphasis this fact with some corollaries.

Corollary 5.1.2. *The dimension of $\overline{Tx}$ is the dimension of $\Delta(M_x)$ which is $n-\kappa(M_x)$. In particular for a generic point x the matroid M_x is connected, so the polytope $\Delta(M_x)$ and the variety $\overline{Tx}$ are $n-1$ dimensional.* ♡

Proof. This follows from Theorem 5.1.1 and Proposition 4.2.8. □

Corollary 5.1.3. *The degree of $\overline{Tx}$ embedded in $\mathbb{P}^{\binom{n}{k}-1}$ is the normalized volume of $\Delta(M_x)$.* ♡

Proof. This follows from Theorem 5.1.1 and Proposition 3.3.6 since the automorphism in Theorem 5.1.1 does not change the degree. □

Example 5.1.4. Recall from Example 2.6.5 that the cohomology ring of $\mathbb{P}^{N-1}$ is isomorphic to the polynomial ring $\mathbb{Z}[H]/(H^N)$ where $H = \sigma_{(1)}$ is the class of a hyperplane. The class of a subvariety Y of codimension s and degree d is $[Y] = dH^s$. By Corollary 5.1.2 and Corollary 5.1.3 it follows that the class of $\overline{Tx}$ thought of as a subvariety of $\mathbb{P}^{\binom{n}{k}-1}$ through the Plücker embedding is

$$[\overline{Tx}]_{\mathbb{P}^{\binom{n}{k}-1}} = \mathrm{Vol}(\Delta(M_x))H^{\binom{n}{k}-1-(n-\kappa(M_x))}.$$

Hence the class of $\overline{Tx}$ in $H^*(\mathbb{P}^{\binom{n}{k}-1})$ is a matroid invariant only depending on the number of connected components of M_x and the normalized volume of $\Delta(M_x)$. ♢

A finer matroid invariant, and the property of $\overline{Tx}$ we are most interested in, is the class of $\overline{Tx}$ in $H^*(G(k,n))$ and specifically the coefficients of $[\overline{Tx}]$ in the Schubert basis of $H^*(G(k,n))$.

Definition 5.1.5. If $\overline{Tx}$ is of codimension s in $G(k,n)$ the class $[\overline{Tx}]$ is in

$$H^{2s}(G(k,n)) = \mathbb{Z}\{\sigma_\lambda \mid \lambda \subset k \times (n-k), |\lambda| = s\}$$

so we have

$$[\overline{Tx}] = \sum_{\substack{\lambda \subset k\times(n-k) \\ |\lambda|=s}} \alpha_\lambda \sigma_\lambda$$

for integer coefficients α_λ. We call these coefficients the Schubert coefficients of M_x. For a fixed partition λ we may write $\alpha_\lambda(M_x)$ for the Schubert coefficient associated to λ. If $\lambda \not\subset k \times (n-k)$ or $|\lambda| \neq k(n-k) - (n-\kappa(M_x))$ we define $\alpha_\lambda(M_x) = 0$. ♢

We postpone the proof of the fact that the Schubert coefficients are matroid invariants to Corollary 6.2.3, but we may notice the following.

Remark 5.1.6. Let $\pi \in S_n$ be a permutation of $[n]$, and consider the map $\mathbb{C}^{[n]} \to \mathbb{C}^{[n]}$ that sends e_i to $e_{\pi(i)}$. This extends to an automorphism of $G(k,n)$ whose induced map on cohomology is the identity. Also $\pi(\overline{Tx}) = \overline{T\pi(x)}$ so the classes $[\overline{Tx}]$ and $[\overline{T\pi(x)}]$ are equal, and π is a matroid isomorphism $M_x \simeq M_{\pi(x)}$. ♢

We want to see what we can say about these Schubert coefficients. We know they are positive, but are they valuative? Do they in some sense respect known operations on matroids like duality, deletion, contraction or direct sum? And is it feasible to compute them from M_x?

By the complementary dimension formula, Proposition 2.5.7, we can describe $\alpha_\lambda(M_x)$ as

$$\alpha_\lambda(M_x) = \deg([\overline{Tx}] \cdot \sigma_{\lambda^c}) = \#(\overline{Tx} \cap X_{\lambda^c}(V_\bullet)).$$

Hence the Schubert coefficients $\alpha_\lambda(M_x)$ are positive integers, as they count the number of points in a transverse intersection of varieties. Later in Definition 6.2.4 we define Schubert coefficients of arbitrary matroids, not just those representable over $\mathbb{C}$. In this case the it is unknown if the Schubert coefficients are positive.

For small n and k we can use the Macaulay2, [GS], Program A.1.1 to compute all these coefficients from the matrix whose rowspace represents $x \in G(k,n)$. The program uses the packages [Che] for computations with matroids, [Sta] for Schubert calculus, [SYP] for computing toric ideals, [BK] for polytopes and some code from [Man22] to compute the beta invariant.

Example 5.1.7. By running Program A.1.1 we get the Schubert coefficients of the uniform matroid $U_{3,6}$.

```
Point x in G(3,6):
| 1 0 0 2 1 1 |
| 0 1 0 1 2 1 |
| 0 0 1 1 1 2 |

M_x connected:    true
M_x = U_{3,6}:    true
Beta(M_x):        6
Vol(Delta(M_x)):  66
[Tx]:             3s       + 6s       + 3s
                    3,1,0      2,2,0      2,1,1
```

We also get some additional information, namely the beta invariant and the normalized volume of $\Delta(U_{3,6})$, which by Corollary 5.1.3 is the degree of $\overline{Tx}$ in $\mathbb{P}^{19}$. ♢

5.2 Results about Schubert coefficients

For a generic point x, by Corollary 5.1.2, $\overline{Tx}$ is of dimension $n-1$ so of codimension

$$s := k(n-k) - (n-1) = (k-1)(n-k-1)$$

in $G(k,n)$. In this case there is one coefficient we can compute in general, namely the one corresponding to the partition $\lambda = (k-1) \times (n-k-1)$. In this case $\lambda^c = (n-k, 1, \dots, 1)$ is the so called hook-partition and the Schubert variety $X_{\lambda^c}(V_\bullet)$ only depends on the line $l = V_1$ and hyperplane $H = V_{n-1}$. Explicitly

$$X_{\lambda^c}(l, H) = \{x \in G(k,n) \mid l \subset L(x) \subset H\}.$$

In [Spe09, chapter 5], Speyer proves the following theorem.

Theorem 5.2.1. [Spe09, Theorem 5.1] *For a generic $x \in G(k,n)$ the Schubert coefficient $\alpha_{(k-1)\times(n-k-1)}$ is the beta invariant $\beta(M_x)$.* ♡

Recall from Example 4.4.2 that the beta via a recursion relation. The proof of Theorem 5.2.1 relies on showing that $\#(\overline{Tx} \cap X_{\lambda^c}(l, H))$ satisfies the same recursion. The fact that the intersection is empty if M_x contains a loop or a coloop is true due to Corollary 5.1.2, since in that case M_x is not connected, so the dimension of $\overline{Tx}$ is too small. One can also check that for a generic $x \in G(1, 2)$, $M_x = U_{1,2}$ and $\overline{Tx} \cap X_{\lambda^c}(l, H)$ is a single point. To show the additive property Speyer partitions $\overline{Tx}$ into two closed parts and one open

$$
\begin{aligned}
X_1 &= \{y \in \overline{Tx} \mid L(y) \subset H_i\} \\
X_2 &= \{y \in \overline{Tx} \mid l_i \subset L(y)\} \\
U &= \overline{Tx} \setminus (X_1 \sqcup X_2)
\end{aligned}
$$

and shows that

$$
\begin{aligned}
X_1 &= \overline{T'(x \setminus i)} \\
X_2 &= \overline{T'(x/i)} \\
U \cap X_{\lambda^c}(l, H) &= \emptyset
\end{aligned}
$$

where T' is the $(n-2)$-dimensional torus acting on $G(k, \mathbb{C}^{[n] \setminus i})$ and $G(k-1, \mathbb{C}^{[n] \setminus i})$. See [Spe09, chapter 5] for the details of the proof.

The next observation will aid in computing Schubert coefficients.

Proposition 5.2.2. *Let $x \in G(k, n)$ and $d = n - \kappa(M_x)$, then*

$$\deg([\overline{Tx}]\sigma_{(1)}^d) = \mathrm{Vol}_d(\Delta(M_x)). \qquad \heartsuit$$

Proof. By Proposition 3.3.6 and Proposition 2.6.9 we have two ways of computing the degree of a toric subvariety of $G(k, n)$ which along with Theorem 5.1.1 gives us the result. □

This gives us a linear equation among the Schubert coefficients α_λ. In particular, if there are only two coefficients we can determine both by Theorem 5.2.1 together with Proposition 5.2.2 as shown in the next example for $G(2, 5)$ and $G(2, 6)$.

Example 5.2.3. Let x be a generic point in $G(2, 5)$, such that $dim(\overline{Tx}) = 4$, and of codimension 2 in $G(2, 5)$. The class of $\overline{Tx}$ is

$$[\overline{Tx}] = \alpha\sigma_{(1,1)} + \beta\sigma_{(2)}.$$

By Theorem 5.2.1 $\beta = \beta(M_x)$. By applying Pieri's formula 4 times we see that

$$\sigma_{(1)}^4 = 2\sigma_{(2,2)} + 3\sigma_{(3,1)}.$$

Now by the complementary dimension formula we have that

$$\deg([\overline{Tx}]\sigma_{(1)}^4) = 2\alpha + 3\beta.$$

Which by Proposition 5.2.2 gives us

$$
\begin{aligned}
2\alpha + 3\beta &= \mathrm{Vol}(\Delta(M_x)) \\
\Longrightarrow \alpha &= \frac{1}{2}(\mathrm{Vol}(\Delta(M_x)) - 3\beta(M_x)).
\end{aligned}
$$

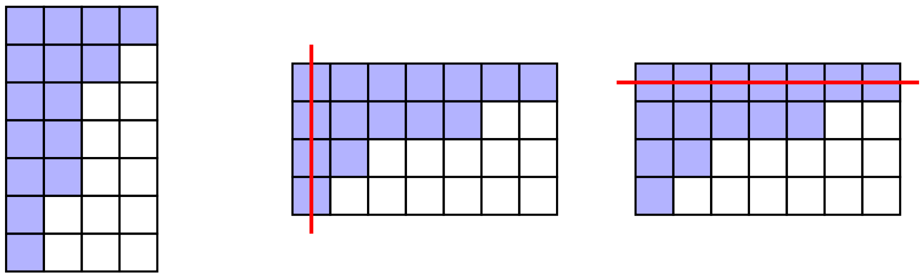


Figure 5.1: The Young diagrams of $\lambda^\top$ on the left, $|\lambda$ in the middle and $\overline{\lambda}$ on the right from Example 5.2.5.

In particular since we know α has to be positive $\mathrm{Vol}(\Delta(M_x)) \leq 3\beta(M_x)$.

The same example works for a generic point $x \in G(2,6)$. Then we get

$$\begin{aligned} [\overline{Tx}] &= \alpha\sigma_{(2,1)} + \beta\sigma_{(3)} \\ \beta &= \beta(M_x) \\ \sigma_{(1)}^5 &= 5\sigma_{(3,2)} + 4\sigma_{(4,1)} \\ \deg([\overline{Tx}]\sigma_{(1)}^5) &= 5\alpha + 4\beta \\ \alpha &= \frac{1}{5}(\mathrm{Vol}(\Delta(M_x)) - 4\beta(M_x)). \end{aligned}$$ ♢

We will look at the behaviour of Schubert coefficients under duality and when M_x contains a loop or coloop, but first we need to introduce some notation.

Definition 5.2.4. Let $\lambda = (\lambda_i)_{i=1}^k \subset k \times (n-k)$. The partitions $\lambda^\top \subset (n-k) \times k$, $|\lambda \subset k \times (n-k-1)$ and $\overline{\lambda} \subset (k-1) \times (n-k)$ are defined as follows.

- $\lambda^\top = (\lambda_j^\top)_{j=1}^{n-k}$ where $\lambda_j^\top = |\{i \mid \lambda_i \geq j\}|$
- If $\lambda_k \neq 0$ then $|\lambda = (\lambda_i - 1)_{i=1}^k$
- If $\lambda_1 = n-k$ then $\overline{\lambda} = (\lambda_2, \ldots, \lambda_k)$. ♢

In terms of their Young diagrams $\lambda^\top$ is obtained by transposing the Young diagram of λ. If the the first column of the Young diagram of λ is full in the rectangle $k \times (n-k)$, $|\lambda$ is obtained by removing that column. Similarly if the first row is full, $\overline{\lambda}$ is obtained by removing that row.

Example 5.2.5. Let $\lambda = (7,5,2,1) \subset 4 \times 7$. The Young diagrams of $\lambda^\top$, $|\lambda$ and $\overline{\lambda}$ are pictured in Figure 5.1. ♢

Proposition 5.2.6. *For $x \in G(k,n)$ consider $x^\perp \in G(n-k,n)$ and let*

$$[\overline{Tx}] = \sum_{\substack{\lambda \subset k \times (n-k) \\ |\lambda| = s}} \alpha_\lambda \sigma_\lambda \quad \text{and} \quad [\overline{Tx^\perp}] = \sum_{\substack{\mu \subset (n-k) \times k \\ |\mu| = s}} \beta_\mu \sigma_\mu.$$

Then $\alpha_\lambda = \beta_{\lambda^\top}$. ♡

In this sense, the Schubert coefficients respect taking the dual of the matroid M_x.

Proof. For $t \in T$ and $y \in G(k,n)$, $(ty)^\perp = t(y^\perp)$ so the isomorphism $x \mapsto x^\perp$ restricts to an isomorphism between $\overline{Tx}$ and $\overline{Tx^\perp}$. We will prove that the induced map on cohomology of the isomorphism $G(k,n) \simeq G(n-k,n)$ takes σ_λ to $\sigma_{\lambda^\top}$. This is mentioned in [GH94, Chapter 1.5], but we give a more detailed proof. It follows that

$$\alpha_\lambda = \#(\overline{Tx} \cap X_\lambda(V_\bullet)) = \#(\overline{Tx^\perp} \cap X_{\lambda^\top}(W_\bullet)) = \beta_{\lambda^\top}.$$

First notice that any Schubert variety $X_\lambda(V_\bullet)$ only depends on the corners of the Young diagram of λ. This holds, since if $(j_i)_{i=1}^k = \Phi(\lambda)$ and $j_{i-1} = j_i - 1$ for some i then $\dim(L(x) \cap V_{j_{i-1}}) \geq \dim(L(x) \cap V_{j_i}) - 1$ so $\dim(L(x) \cap V_{j_i}) \geq i \implies \dim(L(x) \cap V_{j_{i-1}}) \geq i-1$.

Let $(j_l)_{l=1}^k = \Phi(\lambda)$ and $(j'_{l'})_{l'=1}^{n-k} = \Phi(\lambda^\top)$. The partitions λ and $\lambda^\top$ clearly have the same number of corners so let s be the common number of corners. And let $(j_{l_g})_{g=1}^s$ and $(j'_{l'_g})_{g=1}^s$ be the subsequences of the jumping sequences marking the corners. That is, l_g are the indices such that $j_{l_g+1} \neq j_{l_g} + 1$, and similarly for l'_g.

Recall how to obtain $(j_l)_{l=1}^k$ as in Example 2.4.5 and notice that $(j'_{l'})_{l'=1}^{n-k}$ is obtained in a similar way, by numbering the edges of the path along the Young diagram of λ from the bottom left corner to the top right corner of the Young diagram of $k \times (n-k)$. Then $(j'_{l'})_{l'=1}^{n-k}$ are the numbers on the horizontal edges.

Pick a corner of the Young diagram of λ and let g be the index such that j_{l_g} and $j'_{l'_{s+1-g}}$ marks that corner. Then j_{l_g} is the number of edges on the path from the top right to the corner, and $j'_{l'_{s+1-g}}$ is the number of edges on the path from the bottom left to the corner, so their sum is the number of edges in the path from the top right to the bottom left i.e.

$$j_{l_g} + j'_{l'_{s+1-g}} = n. \tag{5.1}$$

We also see that l'_{s+1-g} is the number of horizontal edges in the path from the bottom left to the corner so

$$\begin{aligned} l'_{s+1-g} &= \lambda_{l_g} = n - k + l_g - j_{l_g} \\ \iff l_g &= l'_{s+1-g} - n + k + j_{l_g}. \end{aligned} \tag{5.2}$$

Let $V_\bullet$ be a complete flag of $\mathbb{C}^n$ and let $W_\bullet$ be the orthogonal complete flag given by $W_j = V_{n-j}^\perp$. In the language above we have

$$\begin{aligned} X_\lambda(V_\bullet) &= \{x \in G(k,n) \mid \dim(L(x) \cap V_{j_{l_g}}) \geq l_g \text{ for } g = 1, \ldots, s\} \\ X_{\lambda^\top}(W_\bullet) &= \{y \in G(n-k,n) \mid \dim(L(y) \cap W_{j'_{l'_g}}) \geq l'_g \text{ for } g = 1, \ldots, s\}. \end{aligned}$$

We will show that $x \in X_\lambda(V_\bullet) \implies x^\perp \in X_{\lambda^\top}(W_\bullet)$. Since $(x^\perp)^\perp = x$ and $(\lambda^\top)^\top = \lambda$ this is enough to show that the induced map on cohomology takes σ_λ to $\sigma_{\lambda^\top}$.

In particular we will show that $\dim(L(x) \cap V_{j_{l_g}}) \geq l_g \implies \dim(L(x^\perp) \cap W_{j'_{l'_{s+1-g}}}) \geq l'_{s+1-g}$. We have

$$\begin{aligned} (L(x) \cap V_{j_{l_g}})^\perp &= L(x)^\perp \oplus V_{j_{l_g}}^\perp \\ &= L(x^\perp) \oplus W_{n-j_{l_g}} \\ &= L(x^\perp) \oplus W_{j'_{l'_{s+1-g}}}. \end{aligned} \tag{5.3}$$

Here the last equality holds by (5.1). Assume $\dim(L(x) \cap V_{j_{l_g}}) \geq l_g$ then by (5.3) and (5.2) we have

$$\begin{aligned}
\dim(L(x^\perp) \cap W_{j'_{l'_{s+1-g}}}) &= \dim(L(x^\perp)) + \dim(W_{j'_{l'_{s+1-g}}}) - \dim(L(x^\perp) \oplus W_{j'_{l'_{s+1-g}}}) \\
&= n - k + j'_{l'_{s+1-g}} - \dim((L(x) \cap V_{j_{l_g}})^\perp) \\
&= n - k + j'_{l'_{s+1-g}} - (n - \dim(L(x) \cap V_{j_{l_g}})) \\
&\geq -k + j'_{l'_{s+1-g}} + l_g \\
&= -k + j'_{l'_{s+1-g}} + l'_{s+1-g} - n + k + j_{l_g} \\
&= j_{l_g} + j'_{l'_{s+1-g}} - n + l'_{s+1-g} = l'_{s+1-g}.
\end{aligned}$$

So $\dim(L(x^\perp) \cap W_{j'_{l'_{s-g}}}) \geq l'_{s+1-g}$. □

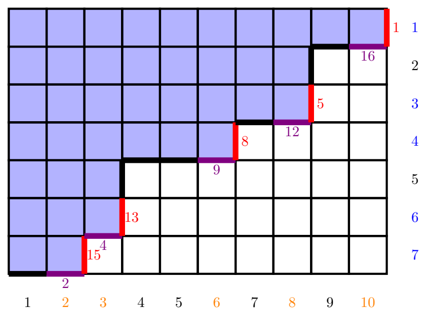


Figure 5.2: Young diagram of $\lambda = (10, 8, 8, 6, 3, 3, 2) \subset 7 \times 10$

The following example may aid in reading the index-heavy proof above.

Example 5.2.7. Let $\lambda = (10, 8, 8, 6, 3, 3, 2) \subset 7 \times 10$. Figure 5.2 shows the Young diagram of λ with 5 corners along with the sequences $(j_{l_g})_{g=1}^5$, $(j'_{l'_g})_{g=1}^5$, $(l_g)_{g=1}^5$ and $(l'_g)_{g=1}^5$ in red, violet, blue and orange respectively. That is,

$$\begin{aligned}
(j_{l_g})_{g=1}^5 &= (1, 5, 8, 13, 15) \\
(j'_{l'_g})_{g=1}^5 &= (2, 4, 9, 12, 16) \\
(l_g)_{g=1}^5 &= (1, 3, 4, 6, 7) \\
(l'_g)_{g=1}^5 &= (2, 3, 6, 8, 10).
\end{aligned}$$
♢

To prove our next results we will need the following lemma.

Lemma 5.2.8. *Let Y be a closed subvariety of X, B_1,B_2 be subvarieties of Y and A_1, A_2 be subvarieties of X such that $B_1 = A_1$ and $B_2 = Y \cap A_2$. If the intersections $B_1 \cap B_2$ and $Y \cap A_2$ are transverse, so is $A_1 \cap A_2$.* ♡

Proof. Let $x \in A_1 \cap A_2$. We need to show that $T_xA_1 + T_xA_2 = T_xX$. Since $Y \cap A_2$ is transverse, $T_xY + T_xA_2 = T_xX$ and since $B_1 \cap B_2$ is transverse, $T_xB_1 + T_xB_2 = T_xY$ so $T_xB_1 + T_xB_2 + T_xA_2 = T_xX$ but since $B_2 \subset A_2$, $T_xB_2 \subset T_xA_2$, and since $B_1 = A_1$, $T_xB_1 = T_xA_1$ so $T_xA_1 + T_xA_2 = T_xX$. □

The next result determines the Schubert coefficients of $[\overline{Tx}]$ when x has a loop i.e. x is in the image of the map ι_1 defined in (4.7).

Theorem 5.2.9. *Let x be a generic point in $G(k, \mathbb{C}^{[n]\smallsetminus i})$ so $\iota_1(x) \in G(k, \mathbb{C}^{[n]})$. Let T be the torus acting on $G(k, \mathbb{C}^{[n]})$ and T' be the subtorus acting on $G(k, \mathbb{C}^{[n]\smallsetminus i})$. The codimension of $\overline{T\iota_1(x)}$ in $G(k, \mathbb{C}^{[n]})$ is $s := (k-1)(n-k-1)+1$ and the codimension of $\overline{T'x}$ in $G(k, \mathbb{C}^{[n]\smallsetminus i})$ is $s-k = (k-1)(n-k-2)$. Let the respective classes be*

$$[\overline{T'x}] = \sum_{\substack{\mu\subset k\times(n-k-1)\\ |\mu|=s-k}} \beta_\mu \sigma_\mu \quad \text{and} \quad [\overline{T\iota_1(x)}] = \sum_{\substack{\lambda\subset k\times(n-k)\\ |\lambda|=s}} \alpha_\lambda \sigma_\lambda.$$

Then we have

1. *ι_1 restricts to an isomorphism $\overline{T'x} \simeq \overline{T\iota_1(x)}$*
2. *if $\lambda_k = 0$ then $\alpha_\lambda = 0$*
3. *if $\lambda_k \neq 0$ then $\alpha_\lambda = \beta_{|\lambda}$, where $|\lambda$ is as in Definition 5.2.4.* ♡

Proof. 1) Let $x = (p_J(x))_{J\in\binom{[n]\smallsetminus i}{k}}$ and $\iota_1(x) = (p_I(\iota_1(x)))_{I\in\binom{[n]}{k}}$. Since i is a loop of $M_{\iota_1(x)}$ we have $p_I(\iota_1(x)) = 0$ for all $I \ni i$ and if $I \not\ni i$ we have up to a common multiple that $p_I(\iota_1(x)) = p_I(x)$. Let $t = (t_1, \dots, t_n) \in T$ and $t' = (t_1, \dots, \hat{t_i}, \dots, t_n) \in T'$. Now we have

$$\begin{aligned} t'x &= (t_J p_J(x))_{J\in\binom{[n]\smallsetminus i}{k}}, \\ t\iota_1(x) &= (t_I p_I(\iota_1(x)))_{I\in\binom{[n]}{k}} \\ &= (t_I p_I(x) \colon \text{ zeros })_{i\notin I\in\binom{[n]}{k}} \\ &= (t_J p_J(x) \colon \text{ zeros })_{J\in\binom{[n]\smallsetminus i}{k}}. \end{aligned}$$

So $T\iota_2(x)$ is contained in the plane $V(p_I \mid i \in I)$ while on all other coordinates it is equal to $T'x$ hence $\overline{T'x} \simeq \overline{T\iota_2(x)}$.

2) Assume $\lambda_k = 0$ then $\lambda_1^c = n-k$, so any point $z \in X_{\lambda^c}(V_\bullet)$ satisfies

$$\dim(L(z) \cap V_1) \geq 1 \iff V_1 \subset L(z).$$

By 1) any point $z \in \overline{T\iota_1(x)}$ is of the form $z = \iota_1(y)$ for some $y \in \overline{T'x}$ so $L(z) = L(y) \subset \mathbb{C}^{[n]\smallsetminus i}$. A generic line V_1 is not contained in $\mathbb{C}^{[n]\smallsetminus i}$ so $\overline{T\iota_1(x)} \cap X_{\lambda^c}(V_\bullet) = \emptyset$ and $\alpha_\lambda = \#(\overline{T\iota_1(x)} \cap X_{\lambda^c}(V_\bullet)) = 0$.

3) Assume $\lambda_k \neq 0$. We denote the complement of $|\lambda$ in $k \times (n-k-1)$ by $(|\lambda)^c$. Let $W_\bullet$ be a complete flag of $\mathbb{C}^{[n]\smallsetminus i}$ such that the intersection

$$B := \overline{T'x} \cap X_{(|\lambda)^c}(W_\bullet)$$

is transverse, then $\beta_{|\lambda} = |B|$. Let $V_\bullet$ be the complete flag of $\mathbb{C}^{[n]}$ given by $V_0 = 0$ and $V_j = W_{j-1} \oplus l_i$ for $j = 1, \dots, n$. We claim that the intersection

$$A := \overline{T\iota_1(x)} \cap X_{\lambda^c}(V_\bullet)$$

is transverse and that $|A| = |B|$.

Let $(j_l)_{l=1}^k$ be the jumping sequence corresponding to λ^c in $k \times (n-k)$. The jumping sequence corresponding to $(|\lambda)^c$ in $k \times (n-k-1)$ is $(j_l - 1)_{l=1}^k$, so we have

$$X_{\lambda^c}(V_\bullet) = \{z \in G(k, \mathbb{C}^{[n]}) \mid \dim(L(z) \cap V_{j_l}) \geq l \quad \text{for } l = 1, \dots, k\} \tag{5.4}$$

$$X_{(|\lambda)^c}(W_\bullet) = \{y \in G(k, \mathbb{C}^{[n] \smallsetminus i}) \mid \dim(L(y) \cap W_{j_l - 1}) \geq l \text{ for } l = 1, \dots, k\}. \tag{5.5}$$

Let $y \in B$, by 1) $\iota_1(y) \in \overline{T\iota_1(x)}$. For $l = 1, \dots, k$ we have

$$L(\iota_1(y)) \cap V_{j_l} = L(y) \cap (W_{j_l - 1} \oplus l_i) = L(y) \cap W_{j_l - 1}. \tag{5.6}$$

Comparing this with (5.4) and (5.5) we see that $\iota_1(y) \in A$. Now let $z \in A$, by 1) $z = \iota_1(y)$ for some $y \in \overline{T'x}$ and by (5.6) we see that $y \in B$.

It remains to see that the intersection $A = \overline{T\iota_1(x)} \cap X_{\lambda^c}(V_\bullet)$ is transverse. If we can show $X_{\lambda^c}(V_\bullet)$ intersect $G(k, \mathbb{C}^{[n] \smallsetminus i})$ transversally this follows from Lemma 5.2.8. Let $\mu = (1, \dots, 1) = (1)^k$ and $U_\bullet$ be a flag with $U_{n-1} = \mathbb{C}^{[n] \smallsetminus i}$ then

$$X_\mu(U_\bullet) = \{y \in G(k, \mathbb{C}^{[n]}) \mid L(y) \subset U_{n-1}\} = G(k, \mathbb{C}^{[n] \smallsetminus i}).$$

We know Schubert varieties intersect transversally if the flags are transverse according to Definition 2.5.4 which holds in this case since $U_{n-1} \cap V_1 = \mathbb{C}^{[n] \smallsetminus i} \cap l_i = 0$. Hence we have that $\alpha_\lambda = |A| = |B| = \beta_{|\lambda}$. □

We get a similar result for when M_x has a coloop i.e if x is in the image of ι_2 defined in (4.8).

Theorem 5.2.10. *Let x be a generic point of $G(k-1, \mathbb{C}^{[n] \smallsetminus i})$ so $\iota_2(x) \in G(k, \mathbb{C}^{[n]})$. Let T be the torus acting on $G(k, \mathbb{C}^{[n]})$ and T' the subtorus acting on $G(k-1, \mathbb{C}^{[n] \smallsetminus i})$. The codimension of $\overline{T\iota_2(x)}$ in $G(k, \mathbb{C}^{[n]})$ is $s := (k-1)(n-k-1)+1$ and the codimension of $\overline{T'x}$ in $G(k-1, \mathbb{C}^{[n] \smallsetminus i})$ is $s - (n-k) = (k-2)(n-k-1)$. Let the respective classes be*

$$[\overline{T'x}] = \sum_{\substack{\mu \subset (k-1) \times (n-k) \\ |\mu| = s - (n-k)}} \beta_\mu \sigma_\mu \quad \text{and} \quad [\overline{T\iota_2(x)}] = \sum_{\substack{\lambda \subset k \times (n-k) \\ |\lambda| = s}} \alpha_\lambda \sigma_\lambda.$$

Then we have

1. *ι_2 restricts to an isomorphism $\overline{T'x} \simeq \overline{T\iota_2(x)}$,*
2. *if $\lambda_1 \neq n-k$ then $\alpha_\lambda = 0$,*
3. *if $\lambda_1 = n-k$ then $\alpha_\lambda = \beta_{\overline{\lambda}}$, where $\overline{\lambda}$ is as in Definition 5.2.4.* ♡

Proof. 1) Let $x = (p_J(x))_{J \in \binom{[n] \smallsetminus i}{k-1}}$ and $\iota_2(x) = (p_I(\iota_2(x)))_{I \in \binom{[n]}{k}}$. Since i is a coloop of $M_{\iota_2(x)}$ we have $p_I(\iota_2(x)) = 0$ for all $I \not\ni i$ and if $I \ni i$ we have up to a common multiple that $p_I(\iota_2(x)) = p_{I \smallsetminus i}(x)$. Let $t = (t_1, \dots, t_n) \in T$ and $t' = (t_1, \dots, \hat{t_i}, \dots, t_n) \in T'$. Now we have

$$\begin{aligned} t'x &= (t_J p_J(x))_{J \in \binom{[n] \smallsetminus i}{k-1}} \\ t\iota_2(x) &= (t_I p_I(\iota_2(x)))_{I \in \binom{[n]}{k}} \\ &= (t_I p_{I \smallsetminus i}(x) \colon \text{ zeros })_{i \in I \in \binom{[n]}{k}} \\ &= (t_i t_{I \smallsetminus i} p_I(\iota_2(x)) \colon \text{ zeros })_{i \in I \in \binom{[n]}{k}} \\ &= (t_J p_J(x) \colon \text{ zeros })_{J \in \binom{[n] \smallsetminus i}{k-1}}. \end{aligned}$$

So $T\iota_2(x)$ is contained in the plane $V(p_I \mid i \notin I)$ while on all other coordinates it is equal to $T'x$, hence $\overline{T'x} \simeq \overline{T\iota_2(x)}$.

We may prove the rest of the proposition by a similar argument as for Theorem 5.2.9, but it also follows from the fact that i is a coloop of $M_{\iota_2(x)}$ if and only if it is a loop of the dual matroid $M^*_{\iota_2(x)}$.

Recall from Proposition 4.3.6 that $M^*_{\iota_2(x)} = M_{\iota_2(x)^\perp}$. Notice that

$$L(\iota_2(x)^\perp) = (L(x) \oplus l_i)^\perp = L(x)^\perp \cap l_i^\perp = L(x)^\perp \cap \mathbb{C}^{[n]\smallsetminus i} = L(x^\perp).$$

The last $\perp$ above refers to the orthogonal complement in $\mathbb{C}[n] \smallsetminus i$, while the others refer to the orthogonal complement in $\mathbb{C}^{[n]}$. We now see that $\iota_2(x)^\perp = \iota_1(y)$. That is, the diagram below commutes.

$$\begin{tikzcd}
G(k-1, \mathbb{C}^{[n]\smallsetminus i}) \arrow[r, hook, "\iota_2"] \arrow[d, "\perp"] & G(k, \mathbb{C}^{[n]}) \arrow[d, "\perp"] \\
G(n-k, \mathbb{C}^{[n]\smallsetminus i}) \arrow[r, hook, "\iota_1"] & G(n-k, \mathbb{C}^{[n]})
\end{tikzcd}$$

Let the classes of $\overline{T'x^\perp}$ in $G(n-k, \mathbb{C}^{[n]\smallsetminus i})$ and $\overline{T\iota_1(x^\perp)}$ in $G(n-k, \mathbb{C}^{[n]})$ be

$$[\overline{T'x^\perp}] = \sum_{\substack{\mu \subset (n-k)\times(k-1) \\ |\mu| = s-(n-k)}} \delta_\mu \sigma_\mu \quad \text{and} \quad [\overline{T\iota_1(x^\perp)}] = \sum_{\substack{\lambda \subset (n-k)\times k \\ |\lambda| = s}} \gamma_\lambda \sigma_\lambda.$$

Now by Proposition 5.2.6 for any λ and μ we have $\alpha_\lambda = \gamma_{\lambda^\top}$ and $\beta_\mu = \delta_{\mu^\top}$.

2) Suppose $\lambda_1 \neq n-k$, then $\lambda^\top_k = 0$ and by Theorem 5.2.9 $\alpha_\lambda = \gamma_{\lambda^\top} = 0$.

3) Suppose $\lambda_1 = n-k$, then $\lambda^\top_k \neq 0$ and by Theorem 5.2.9 we have

$$\alpha_\lambda = \gamma_{\lambda^\top} = \delta_{|(\lambda^\top)} = \beta_{(|(\lambda^\top))^\top} = \beta_{\overline{\lambda}}. \qquad \square$$

The proofs of Theorem 5.2.9 and Theorem 5.2.10 easily extend to the case when x is not chosen generically. So we can extend these results to when M_x contain several loops and coloops. Let $x \in G(k,n)$ be such that M_x contains l loops and c coloops, that is

$$M_x \simeq M_y \oplus \bigoplus_{i\in[l]} U_{0,1} \oplus \bigoplus_{i\in[c]} U_{1,1}.$$

Then x is in the image of the map $G(k',n') \to G(k,n)$ which is the composition of l copies of ι_1 and c copies of ι_2. Let $y \in G(k',n')$ be the point mapping to x, then the Schubert coefficients of M_x can be recovered from the Schubert coefficients of M_y.

Corollary 5.2.11. *In the situation above let T' be the subtorus of T acting on $G(k',n')$ and let $s = k'(n'-k') - (n' - \kappa(M_y))$ be the codimension of $\overline{T'y}$ in $G(k',n')$. Let the respective classes be*

$$[\overline{T'y}] = \sum_{\substack{\mu \subset k'\times(n'-k') \\ |\mu| = s}} \beta_\mu \sigma_\mu \quad \textit{and} \quad [\overline{Tx}] = \sum_{\substack{\lambda \subset k\times(n-k) \\ |\lambda| = lc+k'l+c(n'-k')}} \alpha_\lambda \sigma_\lambda.$$

then we have

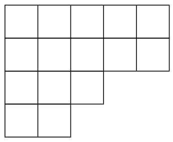
1. *If $\lambda_k < l$ or $\lambda_c \neq n-k$ then $\alpha_\lambda = 0$,*

2. *otherwise* $\alpha_\lambda = \beta_\eta$, *where* $\eta = (\lambda_{c+i} - l)_{i=1}^{k'}$.

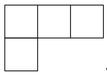
Here η *is the partition obtained from* λ *by removing the* l *first columns and* c *first rows of its Young diagram.* ♡

Proof. This follows from applying Theorem 5.2.9 l times and Theorem 5.2.10 c times. □

Example 5.2.12. Let $x \in G(4,9)$ and suppose $M_x = M_y \oplus U_{0,1} \oplus U_{0,1} \oplus U_{1,1}$ for a generic $y \in G(3,6)$. That is M_x contains two loops and one coloop. The codimension of $\overline{Tx}$ in $G(4,9)$ is $k(n-k) - (n-4) = 15$. Let $\lambda \subset 4 \times 5$ with $|\lambda| = 15$. If $\lambda_1 \neq 5$ or $\lambda_4 < 2$ then $\alpha_\lambda(M_x) = 0$. Otherwise $\alpha_\lambda(M_x) = \alpha_\mu(M_y)$ where $\mu \subset 3 \times 3$ is the partition obtained from lambda by removing the first row and the first two columns of its Young diagram. For example if $\lambda = (5,5,3,2)$ with Young diagram



then $\mu = (3,1)$ with Young diagram



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Chapter 6

Schubert coefficients of non-representable matroids

We would like to define Schubert coefficients for any matroid, not just those representable over $\mathbb{C}$. To do this we go via K-theory.

6.1 K-theory of Grassmannians

We follow the K-theoretic approach first from [FS12] and then from [BF22]. In [FS12] Fink and Speyer study the class of the structure sheaf $\mathcal{O}_{\overline{Tx}}$ in the equivariant K-theory of the Grassmannian, and extends this to a function defined for any matroid. Next in [BF22] Berget and Fink study the lift of $\overline{Tx}$ to the affine variety $\mathbb{A}^{k\times n}$ of k by n matrices and the class of this lift in the equivariant K-theory of $\mathbb{A}^{k\times n}$. This will give us a way to define Schubert coefficients for arbitraty matroids. But first: a short recap of definitions regarding equivariant K-theory and T-equivariat localization, which is also given in [FS12, Chapter 2].

For a variety X we denote $K^0(X)$ the free abelian group generated by isomorphism classes of locally free sheaves (vector bundles) on X modulo the relation $[B] = [A] + [C]$ whenever A, B and C fit into a short exact sequence of locally free sheaves

$$0 \to A \to B \to C \to 0.$$

This is often called the Grothendieck group of locally free sheaves on X. We give $K^0(X)$ a ring structure by $[A][B] = [A \otimes B]$. We give a similar definition of $K_0(X)$ by replacing locally free sheaf by coherent sheaf. Since all locally free sheaves are coherent we get an inclusion $K^0(X) \hookrightarrow K_0(X)$. If X is smooth this inclusion is an isomorphism. For any proper morphism of varieties $f\colon Y \to X$ there is a pushforward map $f_*\colon K_0(Y) \to K_0(X)$. If X is smooth and Y is a closed subvariety of X the structure sheaf of Y gives a class in $K^0(Y)$ and hence in $K_0(Y)$. We abuse notation and write $[\mathcal{O}_Y]$ for the pushforward of the class of the structure sheaf of Y through the closed immersion $Y \hookrightarrow X$.

If X comes with an action of an algebraic group G we denote the G-equivariant K-theory by $K^0_G(X)$ ($K^G_0(X)$). The definition is the same as before, replacing locally free (coherent) sheaf by T-equivariant locally free (coherent) sheaf. By using the localization described in [FS12, Section 2.3] we get an explicit description of $K^0_T(G(k,n))$.

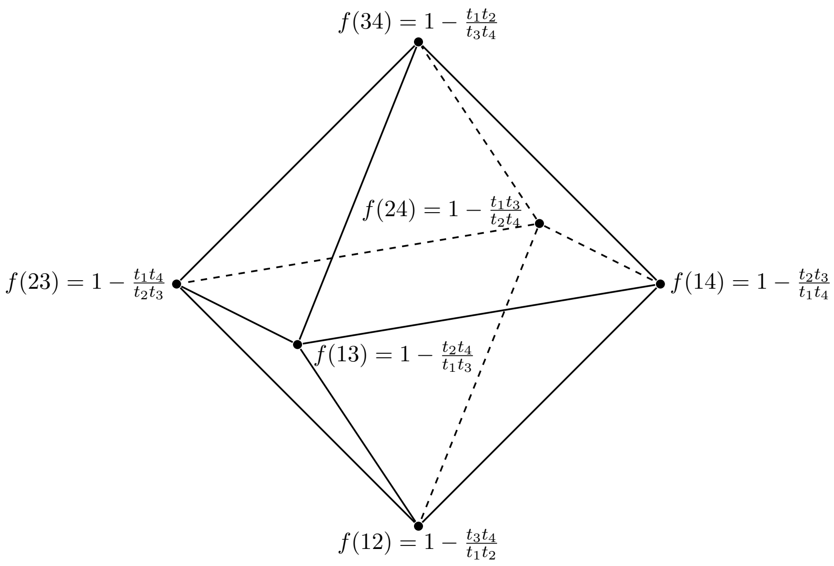


Figure 6.1: An element of $K_T^0(G(2,4))$

Proposition 6.1.1. [FS12, Example 2.10] *The ring $K_T^0(G(k,n))$ is isomorphic to the ring of functions $f\colon \binom{[n]}{k} \to \mathbb{Z}[t_1^{\pm}, \dots t_n^{\pm}]$ satisfying $f(S \sqcup i) \equiv f(S \sqcup j) \mod 1 - t_i/t_j$ for all $S \in \binom{[n]}{k-1}$ and $i, j \in [n] \smallsetminus S$.* ♡

This follows from the discussion in [FS12, Section 2.3] and the fact that $G(k,n)$ is a smooth projective variety with finitely many T-fixed points and finitely many 1-dimensional T-orbits whose closures are isomorphic to $\mathbb{P}^1$. See [FS12, Condition 2.3, 2.4 and 2.8]. We give an example of the description above for $G(2,4)$.

Example 6.1.2. To give an example of an element of $K_T^0(G(2,4))$ we give a Laurent polynomial $f(I) \in \mathbb{Z}[t_1^{\pm}, t_2^{\pm}, t_3^{\pm}, t_4^{\pm}]$ for each $I \in \binom{[4]}{2}$ such that for any $i \in I$ and $j \notin I$. A convenient way to obtain a valid $f(I \smallsetminus i \cup j)$ from $f(I)$ is by applying the transposition (i,j) to $f(I)$. We may visualize this by placing a Laurent polynomial on each vertex of the octahedron $\Delta_{2,4}$ and the direction of the edge between two vertices $\epsilon_j - \epsilon_i$ specifies the transposition (i,j). If we want an element of $K_T^0(G(2,4))$ with $f(12) = 1 - \frac{t_3t_4}{t_1t_2}$ then the other $f(I)$ obtained like this are pictured in Figure 6.1. To verify that this gives a well-defined element of $K_T^0(G(2,4))$, as in Proposition 6.1.1, we look at the example when $S = 1$, $i = 2$ and $j = 3$. We need to find a Laurent polynomial q such that

$$f(12) - f(13) = \frac{t_2t_4}{t_1t_3} - \frac{t_3t_4}{t_1t_2} = q(1 - \frac{t_2}{t_3}).$$

One may check that the Laurent polynomial $q = \frac{-t_4(t_1+t_3)}{t_1t_2}$ works. ♢

For a matroid M of rank k on $[n]$ we now describe the matroid invariant $y(M) \in K_T^0(G(k,n))$ as in [FS12, Chapter 3]. This invariant satisfies that $y(M) = [\mathcal{O}_{\overline{Tx}}]$ whenever M is representable over $\mathbb{C}$ and $M_x = M$. Let I be a basis of M, hence ϵ_I is a vertex of $\Delta(M)$. Let $\mathrm{Cone}_I(\Delta(M))$ be the positive real span of the direction of the edges of $\Delta(M)$ from ϵ_B. For any cone C in $\mathbb{R}^n$ we define the Hilbert series of C to be

$$\mathrm{hilb}(C) = \sum_{a \in C \cap \mathbb{Z}^n} t^a \in \mathbb{Z}[t_1^{\pm}, \dots, t_n^{\pm}].$$

As mentioned in [FS12, Chapter 3] the monoid $\mathrm{Cone}_I(\Delta(M)) \cap \mathbb{Z}^n$ is generated by the edge directions, i.e. the vectors of the form $\epsilon_j - \epsilon_i$ for $j \notin I$ and $i \in I$ such that $(I \setminus i) \cup j$ is a basis of M.

To define $y(M)$ we specify the values on each $I \in \binom{[n]}{k}$ as follows

$$y(M)(I) = \begin{cases} 0, & \text{if } I \text{ is not a basis of } M \\ \mathrm{hilb}(\mathrm{Cone}_I(\Delta(M))) \prod_{i \in I} \prod_{j \notin I} (1 - \frac{t_j}{t_i}), & \text{if } I \text{ is a basis of } M. \end{cases}$$

A priori it is not clear the $y(M)(I)$ is a Laurent polynomial, but as shown in [FS12, Proposition 3.3] $y(M)$ is a well defined class in $K_T^0(G(k,n))$ as in Proposition 6.1.1. If $M = M_x$ for some $x \in G(k,n)$ then $y(M)$ is the class of $\mathcal{O}_{\overline{Tx}}$. Moreover proposition 12.5 says that y is a matroid invariant, and proposition 4.3 says that y is valuative. In the rest of [FS12] Fink and Speyer use $y(M)$ to give a K-theoretic interpretation of the Tutte polynomial $T_M(x,y)$.

We look at an example for $U_{2,4}$.

Example 6.1.3. [FS12, example 3.5] We start by computing $y(U_{2,4})(12)$. The cone $\mathrm{Cone}_{12}(\Delta_{2,4})$ is generated by the edge directions $\epsilon_3 - \epsilon_2, \epsilon_4 - \epsilon_2, \epsilon_3 - \epsilon_1, \epsilon_4 - \epsilon_1$. We want to calculate $\mathrm{hilb}(\mathrm{Cone}_{12}(\Delta_{2,4}))$ so we need to find a nice way to describe the monoid $\mathrm{Cone}_{12}(\Delta_{2,4}) \cap \mathbb{Z}^4$. A problem is that the cone is generated by more vectors than the dimension of its real span (it is not simplicial), so thinking of the monoid as the $\mathbb{Z}_{\geq 0}$ span of the edge directions we will count some lattice points more than once. To rectify this we triangulate $\mathrm{Cone}_{12}(\Delta_{2,4})$ into simplicial cones

$$\begin{aligned} C &= \mathrm{Cone}(\epsilon_3 - \epsilon_2, \epsilon_4 - \epsilon_2, \epsilon_3 - \epsilon_1) \\ C' &= \mathrm{Cone}(\epsilon_4 - \epsilon_2, \epsilon_3 - \epsilon_1, \epsilon_4 - \epsilon_1). \end{aligned}$$

Now we have

$$\mathrm{hilb}(\mathrm{Cone}_{12}(\Delta_{2,4})) = \mathrm{hilb}(C) + \mathrm{hilb}(C') - \mathrm{hilb}(C \cap C').$$

To compute the right hand side notice that

$$\begin{aligned} \mathrm{hilb}(C) &= \sum_{a \in C \cap \mathbb{Z}^4} t^a \\ &= \sum_{a,b,c \in \mathbb{Z}_{\geq 0}} t^{a(\epsilon_3-\epsilon_2)+b(\epsilon_4-\epsilon_2)+c(\epsilon_3-\epsilon_1)} \\ &= \sum_{a,b,c \in \mathbb{Z}_{\geq 0}} \left(\frac{t_3}{t_2}\right)^a \left(\frac{t_4}{t_2}\right)^b \left(\frac{t_3}{t_1}\right)^c \\ &= \frac{1}{(1-\frac{t_3}{t_2})(1-\frac{t_4}{t_2})(1-\frac{t_3}{t_1})}. \end{aligned}$$

By a similar computation for C' and $C \cap C'$ we get

$$\begin{aligned} \mathrm{hilb}(\mathrm{Cone}_{12}(\Delta_{2,4})) &= \frac{1}{(1-\frac{t_3}{t_2})(1-\frac{t_4}{t_2})(1-\frac{t_3}{t_1})} + \frac{1}{(1-\frac{t_4}{t_2})(1-\frac{t_3}{t_1})(1-\frac{t_4}{t_1})} - \frac{1}{(1-\frac{t_4}{t_2})(1-\frac{t_3}{t_1})} \\ &= \frac{(1-\frac{t_4}{t_1}) + (1-\frac{t_3}{t_2}) - (1-\frac{t_4}{t_1})(1-\frac{t_3}{t_2})}{(1-\frac{t_3}{t_2})(1-\frac{t_4}{t_2})(1-\frac{t_3}{t_1})(1-\frac{t_4}{t_1})} \\ &= \frac{1-\frac{t_3t_4}{t_1t_2}}{(1-\frac{t_3}{t_2})(1-\frac{t_4}{t_2})(1-\frac{t_3}{t_1})(1-\frac{t_4}{t_1})} \end{aligned}$$

The denominator above is precisely $\prod_{i\in 12}\prod_{j\notin 12}(1-\frac{t_j}{t_i})$, so we have

$$y(U_{2,4})(12) = 1 - \frac{t_3t_4}{t_1t_2}.$$

From the symmetry of the octahedron $\Delta_{2,4}$ we see that a similar computation works for the other bases of $U_{2,4}$. Indeed the class $y(U_{2,4})$ in $K_T^0(G(2,4))$ is the f in Figure 6.1. $\diamondsuit$

One may obtain an element of the cohomology ring of $G(k,n)$ from $y(M)$ by taking its image in non-equivarient K-theory $K^0(G(k,n))$ and then applying the Chern character. If $M = M_x$ for $x \in G(k,n)$ the degree $k(n-k)-(n-\kappa(M_x))$ part of this element is the class of $\overline{Tx}$. Instead of this approach we will follow the work of Berget and Fink in [BF18], [BF17] and [BF22] which builds on [FS12] by defining a certain lift of $y(M)$.

6.2 General Schubert coefficients

Instead of studying the Grassmannian $G(k,n)$, they study the affine variety $\mathbb{A}^{k\times n}$ of k by n matrices over $\mathbb{C}$ and the action of the algebraic group $G = GL_k(\mathbb{C}) \times (\mathbb{C}^*)^n$ on $\mathbb{A}^{k\times n}$. Given a matrix $v \in \mathbb{A}^{k\times n}$, if it is of rank k, then the rowspace of v gives a point $x \in G(k,n)$. In [BF22] they study the closure of the G-orbit of v denoted X_v, and its class in $K_0^G(\mathbb{A}^{k\times n})$. We summarize the results of [BF22].

The G-equivariant K-theory of $\mathbb{A}^{k\times n}$ is given by

$$K_0^G(\mathbb{A}^{k\times n}) = K_G^0(\mathbb{A}^{k\times n}) = \mathbb{Z}[u_1^{\pm},\dots,u_k^{\pm},t_1^{\pm},\dots,t_n^{\pm}]^{S_k}$$

where the symmetric group S_k acts by permuting the u_i's. So that an element of $K_0^G(\mathbb{A}^{k\times n})$ is a Laurent polynomial symmetric in the u_i's. To any matroid M of rank k on $[n]$ as in [BF22, Chapter 8] we may give an element of $K_0^G(\mathbb{A}^{k\times n})$ by

$$\mathcal{K}(M) = \sum_{w\in S_n}\prod_{j\notin B(w)}\prod_{i\in[k]}(1-u_it_j)\prod_{i=1}^{n-1}\frac{1}{1-t_{w_{i+1}}/t_{w_i}}. \tag{6.1}$$

Here the permutation $w \in S_n$ is described in one line notation $(w_1,\dots w_i, w_{i+1},\dots w_n)$ and the set $B(w)$ is the lexicographically first basis of M in w. The class of X_v in $K_0^G(\mathbb{A}^{k\times n})$ only depends on the matroid M represented by v [BF22, theorem 6.2], and it is given by $\mathcal{K}(M)$ [BF22, theorem 8.1]. Moreover $\mathcal{K}(M)$ is a polynomial in $u_1,\dots,u_k,t_1,\dots,t_n$ [BF22, theorem 8.3] and $\mathcal{K}$ is a valuative matroid invariant. A simpler formula for $\mathcal{K}(U_{2,n})$ is given in [BF18, Proposition 5.2]. Program A.1.2 computes the polynomial $\mathcal{K}(U_{k,n})$ by (6.1) for small n and k, see Example A.2.5 for some computations.

The G-equivariant Chow ring of $\mathbb{A}^{k\times n}$ is given by

$$A_G^\bullet(\mathbb{A}^{k\times n}) = \mathbb{Z}[u_1,\dots,u_k,t_1,\dots,t_n]^{S_k}.$$

As described in [BF22, Proposition 9.9], or [BF17, Proposition 2.4], we obtain a class in $A_G^\bullet(\mathbb{A}^{k\times n})$ from the polynomial $\mathcal{K}(M)$ by substituting u_i by $1-u_i$, t_i by $1-t_i$ and gathering the lowest degree terms after expanding. In particular the expression $1-u_it_j$ becomes $1-(1-u_i)(1-t_j) = u_i+t_j-u_it_j$ and gathering the lowest degree terms gives u_i+t_j. Similarly the term $\frac{1}{1-t_{w_{i+1}}/t_{w_i}}$ becomes

$$\frac{1}{1-(1-t_{w_{i+1}})/(1-t_{w_i})} = \frac{(1-t_{w_i})}{(1-t_{w_i})-(1-t_{w_{i+1}})} = \frac{1}{t_{w_{i+1}}-t_{w_i}} - \frac{t_{w_i}}{t_{w_{i+1}}-t_{w_i}}$$

and gathering the lowest degree terms gives $\frac{1}{t_{w_{i+1}}-t_{w_i}}$. In total $\mathcal{K}(M)$ gives the class in $A^\bullet_G(\mathbb{A}^{k\times n})$

$$\mathcal{C}(M) = \sum_{w\in S_n} \prod_{j\notin B(w)} \prod_{i\in[k]} (u_i + t_j) \prod_{i=1}^{n-1} \frac{1}{t_{w_{i+1}} - t_{w_i}}.$$

As shown in [BF22, Proposition 9.10], if $v \in \mathbb{A}^{k\times n}$ is a matrix of rank k with matroid M then $\mathcal{C}(M)$ is in fact the class of X_v in $A^\bullet_G(\mathbb{A}^{k\times n})$. Since $\mathcal{K}(M)$ is a polynomial so is $\mathcal{C}(M)$ and in particular it is homogeneous. Evaluating $\mathcal{C}(M)$ at $t_1 = \cdots = t_n = 0$ gives a symmetric polynomial in $u_1, \ldots u_k$.

We pause to review the ring of symmetric functions, following [Man01, Chapter1]. The subring of $\mathbb{Z}[u_1, \ldots, u_k]$ consisting of symmetric polynomials is denoted Λ_k. An important basis of Λ_k is the set of Schur polynomials. Given a partition λ a λ-tableau is a filling of the boxes of the Young diagram of λ with integers $1, \ldots, k$. Such a tableau is called semistandard if it is weakly increasing in the rows and strictly increasing in columns. Given a λ-tableau T we obtain a monomial $u^T = u_1^{T(1)} u_2^{T(2)} \ldots u_k^{T(k)}$ where $T(i)$ is the number of i's in the tableau T. The Schur polynomial corresponding to λ is defined as

$$s_\lambda(u) = \sum_T u^T$$

where the sum is over all semistandard λ-tableaux.

Example 6.2.1. Let $\lambda = (3, 1)$ and $k = 2$. There are three semistandard λ-tableaux

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array},$$

so the Schur polynomial is $s_\lambda(u) = u_1^3 u_2 + u_1^2 u_2^2 + u_1 u_2^3$. ♢

These Schur polynomials are symmetric, homogeneous polynomials of degree $|\lambda|$ and they form a basis of Λ_k. Moreover as is [Man01, Corollary 3.2.9] the map

$$\phi_k \colon \Lambda_k \to H^*(G(k, n))$$

defined by $\phi_k(s_\lambda(u)) = \sigma_\lambda$ if $\lambda \subset k \times (n-k)$ and $\phi_k(s_\lambda(u)) = 0$ otherwise, is a surjective ring homomorphism.

So given a matroid M, as described in [BF22, Conjecture 9.13], there are unique integer coefficients α_λ such that

$$\mathcal{C}(M)|_{t_i=0} = \sum_{\substack{\lambda\subset k\times(n-k)\\ |\lambda|=k(n-k)-(n-\kappa(M))}} \alpha_\lambda s_\lambda(u). \tag{6.2}$$

To ease notation we write $\mathrm{Sc}(M)$ for $\mathcal{C}(M)|_{t_i=0}$.

Proposition 6.2.2. [BF22, Corollary 6.3 and Chapter 9] *When M is represented by a rank k matrix $v \in \mathbb{A}^{k\times n}$ and $x \in G(k, n)$ is the point such that $L(x)$ is the rowspace of v then the coefficients α_λ above are the Schubert coefficients obtained from $[\overline{Tx}] \in H^*(G(k, n))$.* ♡

In other words $\phi_k(Sc(M_x)) = [\overline{Tx}]$.

Corollary 6.2.3. *The class of $\overline{Tx}$ in $H^*(G(k, n))$ is a matroid invariant.* ♡

Proof. Let $x, y \in G(k,n)$ with $M_x \simeq M_y$. There must be some permutation π of $[n]$ such that $M_{\pi(x)} = M_y$. Recall from Remark 5.1.6 that $[\overline{Tx}] = [\overline{T\pi(x)}]$, so since $\mathrm{Sc}(M_{\pi(x)}) = \mathrm{Sc}(M_y)$ we have

$$[\overline{Tx}] = [\overline{T\pi(x)}] = [\overline{Ty}]. \qquad \square$$

Moreover if π is a matroid isomorphism between M and M' then $\mathcal{K}(M') = \pi\mathcal{K}(M)$ where S_n acts on the polynomial ring $\mathbb{Z}[u_1, \dots, u_k, t_1, \dots t_n]$ by permuting the t_i's. When evaluating $\mathcal{C}(M)$ in $t_i = 0$ we get $\mathrm{Sc}(M) = \mathrm{Sc}(M')$, so Sc is also a matroid invariant. In this way we extend the notion of Schubert coefficients of a matroid to matroids not representable over $\mathbb{C}$.

Definition 6.2.4. Let M be a matroid of rank k on $[n]$ and $\lambda \subset k \times (n-k)$ a partition with $|\lambda| = k(n-k) - (n - \kappa(M))$. We denote the corresponding coefficient in (6.2) by $\alpha_\lambda(M)$. If $\lambda \not\subset k \times (n-k)$ or $|\lambda| \neq k(n-k) - (n-\kappa(M))$ we define $\alpha_\lambda(M) = 0$. We call these integers the Schubert coefficients of M. ♢

The positivty of the Schubert coefficients for matroids not representable over $\mathbb{C}$ is conjectured in [BF22, Conjeucture 9.13]. In the next section we do some computations to verify their positivity in some small examples.

Proposition 6.2.5. $\mathrm{Sc}(M)$ *and hence the Schubert coefficients of M are top-valuative matroid invariants.* ♡

Proof. Since $\mathcal{K}(-)$ is valuative, so is $\mathrm{Sc}(-)$. Notice from (6.2) that $\mathrm{Sc}(M)$ is a homogeneous polynomial of degree $k(n-k)-(n-\kappa(M))$. Let $\mathcal{D}$ be a matroid subdivision of $\Delta(M)$ with maximal dimensional pieces $\Delta(M_1), \dots, \Delta(M_r)$. Then as in (4.10)

$$\mathrm{Sc}(M) = \sum_{\emptyset \neq J \subset [r]} (-1)^{|J|+1} \mathrm{Sc}(M_J) \tag{6.3}$$

where M_J is the matroid of the polytope $\cap_{i\in J}\Delta(M_i)$. For any J with $|J| > 1$, $\kappa(M_J) > \kappa(M)$ so $\mathrm{Sc}(M_J)$ would contribute with higher degree terms to $\mathrm{Sc}(M)$ in (6.3). But $\mathrm{Sc}(M)$ is homogeneous of degree $k(n-k-(n-\kappa(M))$ so

$$\mathrm{Sc}(M) = \sum_{j=1}^{r} \mathrm{Sc}(M_j). \tag{6.4}$$

$\square$

As mentioned in [BF17, Section 5.5], using the notation above, a theorem of Klyachko gives us the Schubert coefficients of arbitrary uniform matroids.

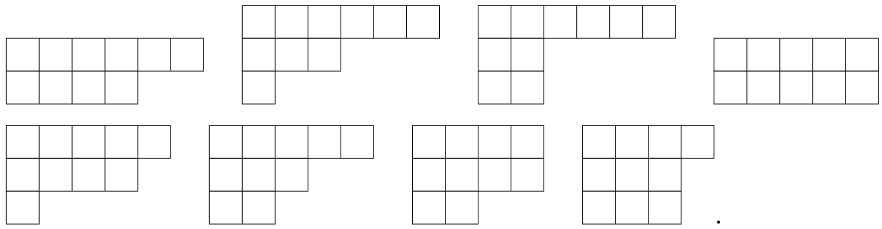
Theorem 6.2.6. [Kly85, theorem 6] *Let $\lambda \subset k \times (n-k)$ be a partition with $|\lambda| = (k-1)(n-k-1)$. The associated Schubert coefficient of $U_{k,n}$ is*

$$\alpha_\lambda(U_{k,n}) = \sum_{i=0}^{k} (-1)^i \binom{n}{i} s_{\lambda^c}(1^{k-i}). \qquad ♡$$

Here $s_{\lambda^c}(u)$ is the Schur polynomial of the complement of λ in the variables $u_1, \dots, u_k$ and by $s_{\lambda^c}(1^{k-i})$ we mean the evaluation of $s_{\lambda^c}(u)$ in the point $(1, \dots, 1, 0, \dots, 0)$ with $k-i$ ones and i zeros.

We use this to compute the Schubert coefficients of $U_{3,9}$.

Example 6.2.7. There are 8 partitions $\lambda \subset 3 \times 6$ with $|\lambda| = 10$. Their young diagrams are pictured below.



.

To determine the Schubert coefficient of, for example, the second partition $\lambda = (6,3,1)$ we see that $\lambda^c = (5,3)$ and the relevant evaluations of the Schur polynomial $s_{\lambda^c}(u_1, u_2, u_3)$ are

$$\begin{aligned} s_{\lambda^c}(1,1,1) &= 42 \\ s_{\lambda^c}(1,1,0) &= 3 \\ s_{\lambda^c}(1,0,0) &= s_{\lambda^c}(0,0,0) = 0. \end{aligned}$$

Hence we get

$$\alpha_\lambda(U_{3,9}) = \binom{9}{0} \cdot 42 - \binom{9}{1} \cdot 3 = 15.$$

We do a similar computation for the other partitions above in the Program A.1.7 using the Macaulay2 package [Niñ]. This gives the class

$$\begin{aligned} \mathrm{Sc}(U_{3,9}) = {} & 15 s_{(6,4,0)}(u) + 15 s_{(6,3,1)}(u) + 6 s_{(6,2,2)}(u) + 21 s_{(5,5,0)}(u) \\ & + 24 s_{(5,4,1)}(u) + 15 s_{(5,3,2)}(u) + 6 s_{(4,4,2)}(u) + 3 s_{(4,3,3)}(u). \end{aligned}$$ ♢

6.3 Computations for non-representable matroids

In this section we use the valuativity of $\mathrm{Sc}(-)$ to compute the Schubert coefficients of some small examples of matroids not representable over $\mathbb{C}$. Suppose M is such a matroid. If we find a matroid subdivision of M where the pieces of maximal dimension correspond to matroids M_i that are representable over $\mathbb{C}$, we can use Proposition 6.2.5 to compute the Schubert coefficients of M in terms of the Schubert coefficients the M_i's. Since we know the Schubert coefficients of the M_i's are positive, we can confirm the conjectured positivity of the Schubert coefficients of M in this case.

Corollary 6.3.1. *If a matroid M has a matroid subdivision into matroids representable over $\mathbb{C}$ the Schubert coefficients of M are positive.* ♡

Another approach we will use is to find a subdivision of some matroid containing M. We may still use (6.4) to compute the Schubert coefficients of M by the Schubert coefficients of the other pieces of the subdivision, but in this case we cannot ensure their positivity. In the following examples we use the regular subdivision, as implemented in the Macaulay2 package [BK] to find subdivisions. This method does not ensure that we get a matroid subdivision, so in each case we confirm this.

Example 6.3.2 (The Fano matroid). Recall that the Fano matroid F as defined in Example 4.1.11 is not representable over $\mathbb{C}$, but it is representable over $\mathbb{F}_2$. It is the smallest example of a matroid not representable over $\mathbb{C}$. By [OPS19, Corollary 31] any matroid representable over $\mathbb{F}_2$ has no non-trivial matroid subdivision. By lifting the vertices of $\Delta(U_{3,7})$ that do not correspond to a basis of F to height 1 the regular subdivision gives us a matroid subdivision of $\Delta(U_{3,7})$ containing $\Delta(F)$ as a maximal dimensional piece. Hence by (6.4) we get

$$\begin{aligned} \mathrm{Sc}(U_{3,7}) &= \mathrm{Sc}(F) + \sum_i \mathrm{Sc}(M_i) \\ \implies \mathrm{Sc}(F) &= \mathrm{Sc}(U_{3,7}) - \sum_i \mathrm{Sc}(M_i) \end{aligned} \tag{6.5}$$

where $\Delta(M_i)$ are the other maximal dimensional pieces of the subdivision. Program A.1.3 checks this and verifies that the 7 maximal dimensional pieces of the subdivision that are not $\Delta(F)$ are all the matroid polytope of matroids isomorphic to the matroid M_x represented over $\mathbb{C}$ by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

That is, M_x is a matroid with 4 parallel points such that the simplification (removing loops and parallel points) is $U_{3,4}$. Since both M_x and $U_{3,7}$ are representable over $\mathbb{C}$, say $M_y = U_{3,7}$, by using Program A.1.1 we get the following outputs.

```
Point x in G(3,7):
| 1 0 0 2 1 1 3 |
| 0 1 0 1 2 1 5 |
| 0 0 1 1 1 2 7 |

M_x connected:    true
M_x = U_{3,7}:    true
Beta(M_x):        10
Vol(Delta(M_x)):  302
[Tx]:             6s      + 3s      + 10s      + 8s      + s
                    4,2,0     4,1,1      3,3,0      3,2,1     2,2,2
```

```
Point x in G(3,7):
| 1 0 0 1 1 1 1 |
| 0 1 0 1 1 1 1 |
| 0 0 1 1 1 1 1 |

M_x connected:    true
M_x = U_{3,7}:    false
Beta(M_x):        1
Vol(Delta(M_x)):  10
[Tx]:             s
                   3,3,0
```

So the Schubert coefficients of $U_{3,7} = M_y$ and M_x are

$$\begin{aligned} [\overline{Ty}] &= 6\sigma_{(4,2,0)} + 3\sigma_{(4,1,1)} + 10\sigma_{(3,3,0)} + 8\sigma_{(3,2,1)} + \sigma_{(2,2,2)} \\ [\overline{Tx}] &= \sigma_{(3,3,0)}. \end{aligned}$$

Note that we could also have used Theorem 6.2.6 to compute the Schubert coefficients of $U_{3,7}$. Now by Proposition 6.2.2 we get

$$\begin{aligned} \mathrm{Sc}(U_{3,7}) &= 6s_{(4,2,0)}(u) + 3s_{(4,1,1)}(u) + 10s_{(3,3,0)}(u) + 8s_{(3,2,1)}(u) + s_{(2,2,2)}(u) \\ \mathrm{Sc}(M) &= s_{(3,3,0)}(u). \end{aligned}$$

Together with (6.5) this gives the Schubert coefficients of the Fano matroid

$$\begin{aligned}\mathrm{Sc}(F) &= 6s_{(4,2,0)}(u) + 3s_{(4,1,1)}(u) + 10s_{(3,3,0)}(u) + 8s_{(3,2,1)}(u) + s_{(2,2,2)}(u) - 7s_{(3,3,0)}(u) \\ &= 6s_{(4,2,0)}(u) + 3s_{(4,1,1)}(u) + 3s_{(3,3,0)}(u) + 8s_{(3,2,1)}(u) + s_{(2,2,2)}(u). \qquad \diamondsuit\end{aligned}$$

The next natural example to check is the so called non-Pappus matroid. The Pappus matroid P is the matroid obtained from the line arrangement in $\mathbb{P}^2$ pictured on the left in Figure 6.2. That is P is the matroid of rank 3 on $[9]$ defined by the bases

$$\binom{[9]}{3} \setminus \{123, 456, 789, 157, 168, 247, 269, 348, 359\}.$$

The pappus matroid is representable over $\mathbb{C}$. Similarly the non-Pappus matroid NP is the matroid from the line arrangement on the right in Figure 6.2. Notice that the points 7, 8 and 9 are not colinear, so the bases of NP are the bases of P plus 789. The non-Pappus matroid is not representable over any field, [Oxl06, Proposition 6.1.10].

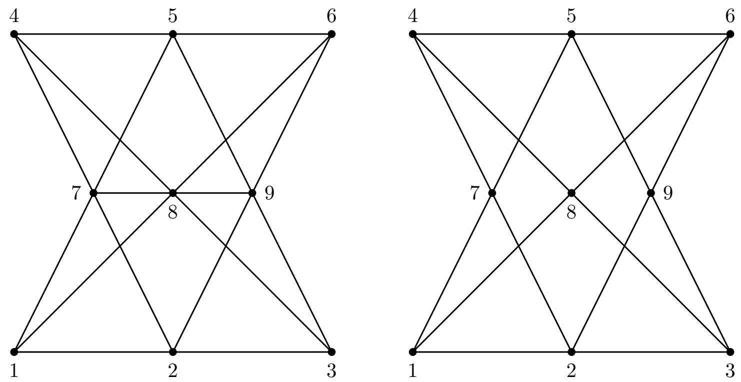


Figure 6.2: The Pappus and non-Pappus matroids

Example 6.3.3 (The Pappus and non-Pappus matroids). As demonstrated in Program A.1.4, by lifting the one vertex of $\Delta(NP)$ that is not a vertex of $\Delta(P)$ to height 1, the regular subdivision gives a matroid subdivision of $\Delta(NP)$ into two pieces given by the Pappus matroid P and the matroid M_x represented over $\mathbb{C}$ by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{6.6}$$

That is, M_x is a matroid with simplification $U_{3,4}$ and 6 parallel points. By Corollary 6.3.1 we conclude that the Schubert coefficients of NP are positive. And we have

$$\mathrm{Sc}(NP) = \mathrm{Sc}(P) + \mathrm{Sc}(M_x) \tag{6.7}$$

These matroids are too large to compute the Schubert coefficients directly with Program A.1.1. As in Example 6.3.2 by lifting the vertices of $U_{3,9}$ not corresponding to bases of NP to height 1 we get a matroid subdivision of $\Delta(U_{3,9})$ where the maximal dimension pieces are $\Delta(NP)$ and 8 pieces isomorphic to $\Delta(M_x)$. Program A.1.4 verifies this. Now we have

$$\mathrm{Sc}(U_{3,9}) = \mathrm{Sc}(NP) + 8\,\mathrm{Sc}(M_x).$$

We computed the Schubert coefficients of $U_{3,9}$ in Example 6.2.7, so it only remains to find the Schubert coefficients of M_x. To do this we first see from Program A.1.4 that the normalized volume of $\Delta(M_x)$ is 21. Also, by applying Pieri's formula 8 times in $G(3,9)$ we get

$$\begin{aligned}\sigma_{(1)}^8 =& 20\sigma_{(6,2,0)} + 21\sigma_{(6,1,1)} + 28\sigma_{(5,3,0)} + 50\sigma_{(5,2,1)} \\ &+ 14\sigma_{(4,4,0)} + 70\sigma_{(4,3,1)} + 56\sigma_{(4,2,2)} + 42\sigma_{(3,3,2)}.\end{aligned}$$

By Proposition 5.2.2 we have $\deg([\overline{Tx}]\sigma_{(1)}^8) = \mathrm{Vol}(\Delta(M_x))$, expanding this gives

$$\begin{aligned}21 =\ & 20\alpha_{(6,4,0)} + 21\alpha_{(5,5,0)} + 28\alpha_{(6,3,1)} + 50\alpha_{(5,4,1)} \\ &+ 14\alpha_{(6,2,2)} + 70\alpha_{(5,3,2)} + 56\alpha_{(4,4,2)} + 42\alpha_{(4,3,3)}.\end{aligned} \tag{6.8}$$

Since M_x is connected, by Theorem 5.2.1 we know $\alpha_{(5,5,0)} = \beta(M_x) = 1$. Now since all other Schubert coefficients in (6.8) are positive they must all be equal to 0. Hence we have

$$\begin{aligned}\mathrm{Sc}(NP) =& \mathrm{Sc}(U_{3,9}) - 8\,\mathrm{Sc}(M) \\ =& 15s_{(6,4,0)}(u) + 15s_{(6,3,1)}(u) + 6s_{(6,2,2)}(u) + 13s_{(5,5,0)}(u) \\ &+ 24s_{(5,4,1)}(u) + 15s_{(5,3,2)}(u) + 6s_{(4,4,2)}(u) + 3s_{(4,3,3)}(u).\end{aligned}$$

Combining this with (6.7) we see that the Pappus matroid P have the same Schubert coefficients as NP and $U_{3,9}$ except for the beta invariant

$$\alpha_{(5,5,0)}(P) = \beta(P) = \beta(NP) - 1 = 12. \qquad \diamondsuit$$

We can perform a very similar computation for the Vamos matroid. The Vamos matroid V is a rank 4 matroid on [8]. It is not representable over any field. See [Oxl06, Example 2.1.25] for the definition of V.

Example 6.3.4 (The Vamos matroid). By lifting the vertices of $\Delta(U_{4,8})$ that are not vertices of $\Delta(V)$ to height 1 the regular subdivision gives a subdivision of $\Delta(U_{4,8})$ into $\Delta(V)$ and 5 other pieces all isomorphic to the matroid M_x with simplification $U_{4,5}$ and 4 parallel points. This is verified in Program A.1.6. We have

$$\mathrm{Sc}(V) = \mathrm{Sc}(U_{4,8}) - 5\,\mathrm{Sc}(M_x)$$

We use Theorem 6.2.6 to compute the Schubert coefficients of $U_{4,8}$ in Program A.1.7 and get

$$\begin{aligned}\mathrm{Sc}(U_{4,8}) =& 4s_{(4,4,1,0)}(u) + 20s_{(4,3,2,0)}(u) + 12s_{(4,3,1,1)}(u) \\ &+ 12s_{(4,2,2,1)}(u) + 20s_{(3,3,3,0)}(u) + 20s_{(3,3,2,1)}(u) + 4s_{(3,2,2,2)}(u).\end{aligned} \tag{6.9}$$

As for the non-Pappus matroid we can also compute the Schubert coefficients of M_x. The coefficient of $\sigma_{(4,1,1,1)}$ in the expansion of $\sigma_{(1)}^7$ is 20 and from Program A.1.6 we see that $\mathrm{Vol}(\Delta(M_x)) = 20$. Since $\alpha_{(3,3,3,0)}(M_x) = \beta(M_x) = 1$, by Proposition 5.2.2 all other Schubert coefficients of M_x are zero. So the Schubert coefficients of V are the same as those of $U_{4,8}$ except for the beta invariant

$$\alpha_{(3,3,3,0)}(V) = \alpha_{(3,3,3,0)}(U_{4,8}) - 5\alpha_{(3,3,3,0)}(M) = 15. \qquad \diamondsuit$$

The reason we were able to compute the Schubert coefficients of NP and V was that the only non-zero Schubert coefficient of M_x is the beta invariant in both cases. This is not always the case, even for matroids with uniform simplification.

Example 6.3.5. Consider the matroid M with simplification $U_{2,4}$ and two parallel points. It is representable over $\mathbb{C}$, say by $x \in G(2,5)$, and the Schubert coefficients are

$$[\overline{Tx}] = \alpha\sigma_{(1,1)} + \beta\sigma_{(2)}.$$

Recall from Example 5.2.3 that $\alpha = \frac{1}{2}(\mathrm{Vol}(\Delta(M)) - 3\beta(M))$. Here $\beta(M) = 2$ and $\mathrm{Vol}(\Delta(M)) = 8$ so $\alpha = 1$. ♢

Also as we saw, the Schubert coefficients of F, NP and V are all equal to the Schubert coefficients of the respective uniform matroids, except for the beta invariant. This is not true in general.

Example 6.3.6. Recall from Example 5.1.7 that the Schubert coefficients of $U_{3,6}$ are

$$3\sigma_{(3,1,0)} + 6\sigma_{(2,2,0)} + 3\sigma_{(2,1,1)}.$$

In the following output we see that the all Schubert coefficients are different from those of $U_{3,6}$.

```
Point x in G(3,6):
| 1 0 0 1 1 1 |
| 0 1 0 1 2 0 |
| 0 0 1 0 0 1 |

M_x connected:    true
M_x = U_{3,6}:    false
Beta(M_x):        2
Vol(Delta(M_x)):  27
[Tx]:             2s       + 2s       + s
                    3,1,0      2,2,0     2,1,1
```

♢

6.4 Further research

An overarching question is whether the properties we know hold for Schubert coefficients of representable matroids in fact extend to the non-representable case.

The most obvious continuation is to try to prove the conjectured positivity of the Schubert coefficients. It might even be possible to find a formula, as in the case of the beta invariant and the uniform matroids. Note that from the formula for the Schubert coefficients of uniform matroids in Theorem 6.2.6, it is not obvious that these are in fact positive. So such a formula for a general Schubert coefficient may not prove [BF22, Conjecture 9.13]. On the other hand, an enumerative description of the Schubert coefficients, for example $\alpha_\lambda(M)$ is the number of λ-tableaux satisfying a certain condition, would prove the conjecture.

A question which remains unanswered is how the Schubert coefficients behave with respect to direct sums. Given two Grassmannians $G(k_1, n_1)$ and $G(k_2, n_2)$ we can embed their product as

$$G(k_1, n_1) \times G(k_2, n_2) \hookrightarrow G(k_1 + k_2, n_1 + n_2) \tag{6.10}$$

by sending a pair (x, y) to the point $x \oplus y$ given by $L(x \oplus y) = L(x) \oplus L(y)$. Let T be the torus acting on $G(k_1 + k_2, n_1 + n_2)$ and T_i the subtorus acting on $G(k_i, n_i)$. Then (6.10) restricts to an isomorphism of $\overline{T_1 x} \times \overline{T_2 y} \simeq \overline{Tx \oplus y}$ and $M_{x \oplus y} = M_x \oplus M_y$. The image of (6.10) can be described as the intersection of Schubert varieties

$$X_{k_1 \times (n_2 - k_2)}(V_\bullet) \cap X_{k_2 \times (n_1 - k_1)}(W_\bullet) \tag{6.11}$$

for flags with $V_{n_1} = \langle e_1, \dots e_{n_1} \rangle$ and $W_{n_2} = \langle e_{n_1+1}, \dots e_{n_1+n_2} \rangle$. The product of Schubert cycles of such square partitions $\sigma_{k_1 \times (n_2 - k_2)} \sigma_{k_2 \times (n_1 - k_1)}$ is particularly nice, as described in [Oka98]. The codimension of $\overline{Tx \oplus y}$ in $G(k_1 + k_2, n_1 + n_2)$ is the sum of the codimensions of the intersection (6.11), $\overline{T_1 x}$ and $\overline{T_2 y}$. We suspect that the Schubert coefficients of $M_x \oplus M_y$ can be computed from the product $\sigma_{k_1 \times (n_2 - k_2)} \sigma_{k_2 \times (n_1 - k_1)}$ and the Schubert coefficients of M_x and M_y. If such a formula is found Theorem 5.2.9 and Theorem 5.2.10 will be special cases.

As we saw in Example 6.3.5, the Schubert coefficients of a matroid is not equal to the corresponding Schubert coefficients of its simplification. Nevertheless it seems that there is some connection between the Schubert coefficients of a matroid, and those of its simplification. Since we already know from Theorem 5.2.9 that adding a loop does not change the Schubert coefficients, it only remains to see what happens when adding parallel points.

Appendix A

Computations with Macaulay2

A.1 Programs

All the programs below are made in Macaulay2 [GS].

A.1.1 Program 1

Given a point $x \in G(k, n)$, here represented by a full rank k by n matrix, the following program represents the torus orbit closure $\overline{Tx}$ as a subvariety of $G(k, n)$. We then print some information about the matroid M_x and the Schubert coefficients of the class of $\overline{Tx}$ in the cohomology ring of $G(k, n)$. This program uses the packages [BK], [Che], [SYP] and [Sta].

```
restart
loadPackage "Polyhedra"
loadPackage "Matroids"
loadPackage ("FourTiTwo", Reload => true)
loadPackage "MultiprojectiveVarieties"

--Eline Mannino's code for beta invariant.
betaInvariant = Matroid -> (
    d = rank(Matroid);
    T := characteristicPolynomial Matroid;
    R = ring T;
    Q = frac R;
    lift (T, R);
    g = T/(R_0 - 1);
    beta = (-1)^(d-1)*sub(g, Q_0=>1);
    return beta
)

--Return list of all minors of matrix x sorted reverse lexicographic
--and list of indexes with non-zero minor.
pluckers = x -> (
  k = numgens target x;
  n = numgens source x;
  L = subsets(toList(0..n-1),k);
  output1 = new MutableList from L;
  output2 = {};
  for i to #L-1 do(
    curMinor = det(x_(L#i));
```

```
    output1#i = curMinor;
    if curMinor != 0 then output2 = append(output2, i);
  );
  return (toList(output1), toList(output2))
)

--Point in Grassmanian.
x = matrix{{1,0,0,1,1,0},{0,1,0,1,0,1},{0,0,1,0,1,1}}
k = numgens target x
n = numgens source x

--Construct Grassmannian with "MultiprojectiveVarieties".
kk = ZZ/3331
Gr = GG Grass(k-1,n-1,kk,Variable=>"p")
Rng = ring(ambient(Gr)) --reverse lexicographic ordering.

--Construct matroid assosiated to x with "Matroids".
M = matroid(x)
U = uniformMatroid(k,n)
indMat = basisIndicatorMatrix(M) --reverse lexicographic ordering.
Delta = convexHull indMat -- Make matroid polytope with "Polyhedra"

--Make subring of Rng with relevant coordinates, and ringmap.
(minorList, nonZeros) = pluckers(x)
(Rng',noInUse) = selectVariables(nonZeros,Rng)
g = map(Rng',Rng)

--Compute toric ideal of Delta(M_x) with "FourTiTwo".
I' = toricMarkov(indMat, Rng')

--Translate by Plucker coordinates of x.
pluck = matrix{delete(0,minorList)} --non-zero Plucker coordinates.
tranMat = vars Rng' * diagonalMatrix(pluck)
f = map(Rng',Rng',tranMat)

--Composing f and g and taking preimage of I'.
I = preimage(f*g,I')

--Torus orbit closure as subvariety of Gr.
Tx = projectiveVariety(I)%Gr

--Print
<< "Point x in G(" << k << "," << n <<"): "<< endl
<< x << endl << endl
<< "M_x connected:    " << isConnected(M) << endl
<< "M_x = U_{" << k << "," << n << "}:    " << M == U << endl
<< "Beta(M_x):        " << betaInvariant(M) << endl
<< "Vol(Delta(M_x)):  " << latticeVolume(Delta) << endl
<< "[Tx]:             " << cycleClass(Tx) << endl
```

A.1.2 Program 2

The following program computes the polynomial $\mathcal{K}(U_{k,n})$ and uses this to compute $\mathcal{C}(U_{k,n})$ and $\mathrm{Sc}(U_{k,n}) = \mathcal{C}(U_{k,n})|_{t_i=0}$ for small n and k.

```
restart
--Compute K(M) for M = U_{k,n}
k = 2
n = 4
```

```
R = ZZ[u_1..u_k,t_1..t_n]
--Computes the first product in K(M).
Prod1 = Bwc -> (
  p = 1;
  for j to #Bwc-1 do(
    for i from 1 to k do(
      p = p*(1 - u_i*t_(Bwc#j));
    );
  );
  return p
)
--Computes the second product in K(M)
Prod2 = w -> product(for i to #w-2 list ((t_(w#i))/(t_(w#i)-t_(w#(i+1)))))
--Computes the sum in K(M)
K = L -> sum(for i to #L-1 list Prod1(L#i_{k..n-1})*Prod2(L#i))

perms = permutations(toList(1..n))
KM = K(perms)
KM = substitute(KM,R) --K(M)

usub = for i from 1 to k list u_i => (1-u_i)
tsub = for i from 1 to n list t_i => (1-t_i)
KMsub = sub(KM, join(usub, tsub))
CM = part((k-1)*(n-k-1), KMsub) -- C(M)

zerosub = for i from 1 to n list t_i => 0
CM0 = sub(CM, zerosub) -- C(M)|_{t_i=0} = Sc(M)
```

A.1.3 Program 3

Subdividing $\Delta(U_{3,7})$ to a subdivision containing the Fano matroid and 7 matroid polytopes isomorphic to a matroid with simplification $U_{3,4}$ and 4 parallel points. This program uses the packages [Che] and [BK].

```
restart
loadPackage "Polyhedra"
loadPackage "Matroids"
--Make matroid subdivision of U_{3,7} containing the Fano matroid.

E = toList(0..6) --Ground set
F  = specificMatroid "fano"
U = uniformMatroid(3,7)
x = matrix{{1,0,0,1,1,1,1},{0,1,0,1,1,1,1},{0,0,1,1,1,1,1}}
Mx = matroid(x) --Matroid with U_{3,4} as simplification and 4 parallel points.

Ubases = bases(U)
Fnonbases = nonbases(F)

--Regular subdivision of Delta U by lifting nonbases of F to height 1.
L = for i to #Ubases-1 list (if member(Ubases#i, Fnonbases) then 1 else 0)
FnonbasisIndex = matrix{L}
subdiv = regularSubdivision(basisIndicatorMatrix(U), FnonbasisIndex)

--Make list of matroids in the subdivision.
makeMatroid = S -> matroid(E, Ubases_S)
subdivMatroids = apply(subdiv, makeMatroid)

--Pick out first matroid in subdivision and the other in a list.
S1 = subdivMatroids#0
```

```
other = subdivMatroids_(toList(1..7))
isoCheck = M -> areIsomorphic(M,Mx)

<< "Matroid subdivision: " << all(subdivMatroids, isWellDefined) << endl
<< "F == S1:             " << areIsomorphic(F,S1) << endl
<< "other == Mx:         " << all(other, isoCheck)
```

A.1.4 Program 4

Subdividing $\Delta(U_{3,9})$ to a subdivision containing the non-Pappus matroid and 8 matroids isomorphic to a matroid with simplification $U_{3,4}$ and 6 parallel points. This program uses the packages [Che] and [BK].

```
restart
loadPackage "Polyhedra"
loadPackage "Matroids"
--Make matroid subdivision of U_{3,9} containing the nonpappus matroid.

E = toList(0..8) --Ground set
NP  = specificMatroid "nonpappus"
U = uniformMatroid(3,9)
x = matrix{{1,0,0,1,1,1,1,1,1},{0,1,0,1,1,1,1,1,1},{0,0,1,1,1,1,1,1,1}}
Mx = matroid(x) --Matroid with U_{3,4} as simplification and 6 parallel points.

Ubases = bases(U)
NPnonbases = nonbases(NP)
DeltaMx = convexHull(basisIndicatorMatrix(Mx))


--Regular subdivision of Delta U by lifting nonbases of F to height 1.
L = for i to #Ubases-1 list (if member(Ubases#i, NPnonbases) then 1 else 0)
NPnonbasisIndex = matrix{L}
subdiv = regularSubdivision(basisIndicatorMatrix(U), NPnonbasisIndex)

--Make list of matroids in the subdivision.
makeMatroid = S -> matroid(E, Ubases_S)
subdivMatroids = apply(subdiv, makeMatroid)

--Pick out second matroid in subdivision and the other in a list.
M2 = subdivMatroids#1
other = subdivMatroids_({0,2,3,4,5,6,7,8})
isoCheck = M -> areIsomorphic(M,Mx)

<< "Vol(Delta(Mx)) =      " << latticeVolume(DeltaMx) << endl
<< "Matroid subdivision: " << all(subdivMatroids, isWellDefined) << endl
<< "NP == M2:             " << areIsomorphic(NP,M2) << endl
<< "other == Mx:         " << all(other, isoCheck)
```

A.1.5 Program 5

Subdividing $\Delta(NP)$ into two pieces, the Pappus matroid, and a matroids isomorphic to a matroid with simplification $U_{3,4}$ and 6 parallel points. This program uses the packages [Che] and [BK].

```
restart
loadPackage "Polyhedra"
```

```
loadPackage "Matroids"
--Make matroid subdivision of nonpappus matroid into pappus matroid and one more.

E = toList(0..8) --Ground set
NP  = specificMatroid "nonpappus"
P = specificMatroid "pappus"
x = matrix{{1,0,0,1,1,1,1,1,1},{0,1,0,1,1,1,1,1,1},{0,0,1,1,1,1,1,1,1}}
Mx = matroid(x) --Matroid with U_{3,4} as simplification and 6 parallel points.

NPbases = bases NP
Pnonbases = nonbases P

--Regular subdivision of Delta NP by lifting nonbase of P to height 1.
L = for i to #NPbases-1 list (if member(NPbases#i, Pnonbases) then 1 else 0)
PnonbasisIndex = matrix{L}
subdiv = regularSubdivision(basisIndicatorMatrix(NP), PnonbasisIndex)

makeMatroid = S -> matroid(E, NPbases_S)
(M1, M2) = toSequence(apply(subdiv, makeMatroid))

<<"Matroid subdivision: " << isWellDefined(M1) and isWellDefined(M2) << endl
<<"M1 == P: " << areIsomorphic(M1,P) << endl
<<"M2 == Mx: " << areIsomorphic(M2,Mx)
```

A.1.6 Program 6

Subdividing $\Delta(U_{4,8})$ into $\Delta(V)$ and 4 copies of $\Delta(M_x)$ where M_x is the matroid with simplification $U_{4,5}$ and 4 parallel points.

```
restart
loadPackage "Polyhedra"
loadPackage "Matroids"
--Make matroid subdivision of U_{4,8} containing the vamos matroid.

E = toList(0..7) --Ground set
V  = specificMatroid "vamos"
U = uniformMatroid(4,8)
x = matrix{{1,0,0,0,1,1,1,1},{0,1,0,0,1,1,1,1},{0,0,1,0,1,1,1,1},{0,0,0,1,1,1,1,1}}
Mx = matroid(x) --Matroid with U_{3,4} as simplification and 4 parallel points.

Ubases = bases(U)
Vnonbases = nonbases(V)
DeltaMx = convexHull(basisIndicatorMatrix(Mx))

--Regular subdivision of Delta U by lifting nonbases of V to height 1.
L = for i to #Ubases-1 list (if member(Ubases#i, Vnonbases) then 1 else 0)
VnonbasisIndex = matrix{L}
subdiv = regularSubdivision(basisIndicatorMatrix(U), VnonbasisIndex)

--Make list of matroids in the subdivision.
makeMatroid = S -> matroid(E, Ubases_S)
subdivMatroids = apply(subdiv, makeMatroid)

--Pick out second matroid in subdivision and the other 5 in a list.
M2 = subdivMatroids#1
other = subdivMatroids_({0,2,3,4,5})
isoCheck = M -> areIsomorphic(M,Mx)

<< "Vol(Delta(Mx)) =       " << latticeVolume(DeltaMx) << endl
```

```
<< "Matroid subdivision: " << all(subdivMatroids, isWellDefined) << endl
<< "V == S1:             " << areIsomorphic(V,M2) << endl
<< "other == Mx:         " << all(other, isoCheck)
```

A.1.7 Program 7

Calculating the Schubert Coefficients of $U_{k,n}$ using Theorem 6.2.6. This program uses the package [Niñ].

```
restart
loadPackage "SpechtModule"
--Calculate the Schubert coefficients of U_{k,n}
k = 3
n = 6

--Make partitions.
lambdas' = partitions((k-1)*(n-k-1), (n-k))
notInBox = p -> #p>k
inBoxLambdas = for i to #lambdas'-1 list (if notInBox(lambdas'#i) then  continue; lambd
addZeros = p -> if #p == k then p else new Partition from join(toList(p),(k-#p):0)
lambdas = apply(inBoxLambdas, addZeros)

--Complement of a partition in k x (n-k).
comp = p -> new Partition from (for i to #p-1 list (try n-k-p#(k-1-i) else n-k))
compLambdas = apply(lambdas, comp)


R = ZZ[u_1..u_k]
makeSchurPoly = p -> schurPolynomial(toList(0..(k-1)),p,R)
iZeros = i -> toList(join((k-i):1,i:0))
evaluate = (s, L) -> sub(s,for i from 1 to k list u_i => L#(i-1))
--Compute the Schubert coefficient of p^c as in Klyachko thm.
SchubertCoeff = p -> (
  s = makeSchurPoly(p);
  L = for i to k list (-1)^i*binomial(n,i)*evaluate(s, iZeros(i));
  return sum(L)
)

Sc = apply(compLambdas, SchubertCoeff)
for i to #Sc-1 do << Sc#i << "s_" << toList(lambdas#i) << " + "
```

A.2 Examples

Example A.2.1. We examine the Schubert coefficients in the case when M_x is the non-Fano matroid as in Example 4.1.10. By running Program A.1.1 we get the following output.

```
Point x in G(3,7):
| 1 0 0 1 1 0 1 |
| 0 1 0 1 0 1 1 |
| 0 0 1 0 1 1 1 |

M_x connected:   true
M_x = U_{3,7}:   false
Beta(M_x):       4
Vol(Delta(M_x)): 242
```

```
[Tx]:               6s        + 3s        + 4s        + 8s        + s
                      4,2,0     4,1,1       3,3,0       3,2,1      2,2,2
```

As seen in Example 2.6.10 by using Pieri's formula, Proposition 2.6.3, 6 times in $H^*(G(3,7))$ we get

$$\sigma_{(1)}^6 = 9\sigma_{(4,2,0)} + 5\sigma_{(3,3,0)} + 10\sigma_{(4,1,1)} + 16\sigma_{(3,2,1)} + 5\sigma_{(2,2,2)}.$$

To verify Proposition 5.2.2 in this example we use the complimentary dimension formula, Proposition 2.6.2, to compute the degree of $[\overline{Tx}]\sigma_{(1)}^6$

$$\deg([\overline{Tx}]\sigma_{(1)}^6) = 6\cdot 9 + 3\cdot 5 + 4\cdot 10 + 8\cdot 16 + 1\cdot 5 = 242 = \mathrm{Vol}(\Delta(M_x)). \qquad \diamondsuit$$

Example A.2.2. For a generic $x \in G(2,5)$ and $x^\perp \in G(3,5)$ we have

$$[\overline{Tx}] = A\sigma_{(2,0)} + B\sigma_{(1,1)}$$
$$[\overline{Tx^\top}] = a\sigma_{(1,1,0)} + b\sigma_{(2,0,0)}.$$

Since $(2,0)^\top = (1,1,0)$ and $(1,1)^\top = (2,0,0)$ by Proposition 5.2.6 we have $A = a$ and $B = b$. In the output from Program A.1.1 below we see that $A = a = 3$ and $B = b = 1$.

```
Point x in G(2,5):
| 1 0 2 1 1 |
| 0 1 1 2 1 |

M_x connected:    true
M_x = U_{2,5}:    true
Beta(M_x):        3
Vol(Delta(M_x)):  11
[Tx]:             3s     + s
                    2,0     1,1
```

```
Point x in G(3,5):
| -2 -1 1 0 0 |
| -1 -2 0 1 0 |
| -1 -1 0 0 1 |

M_x connected:    true
M_x = U_{3,5}:    true
Beta(M_x):        3
Vol(Delta(M_x)):  11
[Tx]:             s       + 3s
                   2,0,0      1,1,0
```

$\diamondsuit$

Example A.2.3. Let $x \in G(2,7)$ be contained in the hyperplane $H_7 = \mathbb{C}^{[6]}$, then $x \setminus 7 \in G(2,6)$. Suppose $dim(\overline{T'x \setminus 7}) = dim(\overline{Tx}) = 5$ so $\overline{Tx}$ is of codimension 5 in $G(2,7)$ and $\overline{T'x \setminus 7}$ is of codimension 3 in $G(2,6)$. The classes in cohomology are

$$[\overline{Tx}] = A\sigma_{(4,1)} + B\sigma_{(3,2)} + C\sigma_{(5,0)}$$
$$[\overline{T'x \setminus 7}] = a\sigma_{(3,0)} + b\sigma_{(2,1)}.$$

by Theorem 5.2.9 we have $C = 0$, $A = a$ and $B = b$. In the output below we have $A = a = 4$, $B = b = 2$ and $C = 0$.

```
Point x in G(2,7):
| 1 0 2 1 1 5 0 |
| 0 1 1 2 1 3 0 |

M_x connected:   false
M_x = U_{2,7}:   false
Beta(M_x):       0
Vol(Delta(M_x)): 26
[Tx]:            4s     + 2s
                   4,1      3,2
```

```
Point x in G(2,6):
| 1 0 2 1 1 5 |
| 0 1 1 2 1 3 |

M_x connected:   true
M_x = U_{2,6}:   true
Beta(M_x):       4
Vol(Delta(M_x)): 26
[Tx]:            4s     + 2s
                   3,0      2,1
```

◇

Example A.2.4. Let $x \in G(3,6)$ contain the line l_6, so $x/6 \in G(2,5)$. Suppose $dim(\overline{T'x/6}) = dim(\overline{Tx}) = 4$ so $\overline{Tx}$ is of codimension 5 in $G(3,6)$ and $\overline{T'x/6}$ is of codimension 2 in $G(2,5)$. If

$$[\overline{Tx}] = A\sigma_{(3,2,0)} + B\sigma_{(3,1,1)} + C\sigma_{(2,2,1)}$$
$$[\overline{T'x/6}] = a\sigma_{(2,0)} + b\sigma_{(1,1)}$$

then by Theorem 5.2.10 we have $C = 0$, $A = a$ and $B = b$. In the following output we see that $A = a = 3$, $B = b = 1$ and $C = 0$.

```
Point x in G(3,6):
| 0 0 0 0 0 1 |
| 1 0 2 1 1 0 |
| 0 1 1 2 1 0 |

M_x connected:   false
M_x = U_{3,6}:   false
Beta(M_x):       0
Vol(Delta(M_x)): 11
[Tx]:            3s       + s
                   3,2,0    3,1,1
```

```
 Point x in G(2,5):
| 1 0 2 1 1 |
| 0 1 1 2 1 |

M_x connected:   true
M_x = U_{2,5}:   true
Beta(M_x):       3
Vol(Delta(M_x)): 11
[Tx]:            3s     + s
                   2,0    1,1
```

◇

Example A.2.5. We get the following computations of $\mathcal{K}(M)$ and $\mathcal{C}(M)|_{t_i=0}$ for various uniform matroids from Program A.1.2.

$$
\begin{aligned}
\mathcal{K}(U_{2,3}) &= 1,\\
\mathcal{K}(U_{2,4}) &= 1 - u_1^2u_2^2t_1t_2t_3t_4,\\
\mathcal{C}(U_{2,4})|_{t_i=0} &= 2u_1 + 2u_2,\\
\mathcal{K}(U_{2,5}) &= 1 - u_1^2u_2^2(t_1t_2t_3t_4 + t_1t_2t_3t_5 + t_1t_2t_4t_5 + t_1t_3t_4t_5 + t_2t_3t_4t_5)\\
&\quad + 2(u_1^3u_2^2 + u_1^2u_2^3)t_1t_2t_3t_4t_5,\\
\mathcal{C}(U_{2,5})|_{t_i=0} &= 3u_1^2 + 4u_1u_2 + 3u_2^2,\\
\mathcal{K}(U_{2,6}) &= 1 - u_1^2u_2^2(\sum_{S\in\binom{[6]}{4}} t_S) + 2(u_1^3u_2^2 + u_1^2u_2^3)(\sum_{S\in\binom{[6]}{5}} t_S) - (3u_1^4u_2^2 + 4u_1^3u_2^3 + 3u_1^2u_2^4)t_1t_2t_3t_4t_5t_6,\\
\mathcal{C}(U_{2,6})|_{t_i=0} &= 4u_1^3 + 6u_1^2u_2 + 6u_1u_2^2 + 4u_2^3,\\
\mathcal{K}(U_{3,5}) &= 1 - 2(u_1^2u_1^2u_3 + u_1^2u_2u_3^2 + u_1u_2^2u_3^2)t_1t_2t_3t_4t_5\\
&\quad + u_1^2u_2^2u_3^2(t_1^2t_2t_3t_4t_5 + t_1t_2^2t_3t_4t_5 + t_1t_2t_3^2t_4t_5 + t_1t_2t_3t_4^2t_5 + t_1t_2t_3t_4t_5^2),\\
\mathcal{C}(U_{3,5})|_{t_i=0} &= u_1^2 + u_2^2 + u_3^2 + 4u_1u_2 + 4u_1u_3 + 4u_2u_3. \qquad \diamondsuit
\end{aligned}
$$

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