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Power Sum Decomposition of Ternary Sextic Forms

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Abstract

In this thesis we investigate power sum decomposition of ternary sextic forms by using apolarity. We find and classify all Betti tables for the resolution of an apolar ideal of a ternary sextic form. By using the Betti tables, we find a generalized notion of rank, the cactus rank, for every ternary sextic form and find the configuration of points that gives the cactus rank. Further, we use this results to give a stratification of the space of ternary sextic forms. Finally, we do explicit computations on double cubic forms and prove that every such sextic will have a cubic forms in the apolar ideal.

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Contents

1 | Introduction

Decomposition of homogeneous polynomials into power sums of linear forms has been studied for centuries, [AH95], [Syl04] and [Muk09]. A large amount of the research has been centralized around the question of finding the *rank*, that is the minimal number of linear forms needed to decompose a homogeneous polynomial $F \in S = \mathbb{C}[x_0, \dots, x_n]$ into sums of powers of linear forms. Finding an explicit minimal decomposition of F is in general very hard. Finding the rank is easier, at least with some generalization of the notion of rank. Our approach to power sum decomposition is to use the so called *cactus rank*, which we will find for any ternary sextic form.

To find the cactus rank, we will use *apolarity*. We can associate to F a homogeneous ideal $F^{\perp} \subset T = \mathbb{C}[y_0, \dots, y_n]$, referred to as the *apolar ideal* of F. The minimal length of a finite scheme Γ whose ideal is I_Γ is a subideal of $F^{\perp}$, will be the cactus rank of F. To find the cactus rank for each F, we give a classification of the possible *Betti tables* of the resolution of $T/F^{\perp}$ for a ternary sextic form, see Section 2.2, and compute the cactus rank associated to each table. Further, we use the classification to give a stratification of the space of ternary sextic forms.

Our work and methods are inspired by [Kap+21], where the same approach is used for power sum decomposition of quaternary quartic forms.

1.1 Methods and Results

A homogeneous polynomial of degree d can be considered both as an element $F \in S_d$, written as

$$F = a_0 x_0^d + a_1 x_0^{d-1} x_1 + \cdots + a_N x_n^d,$$

where $N = \binom{n+d}{n} - 1$ and $a_i \in \mathbb{C}$, and as a point $[F] \in \mathbb{P}(S_d) = \mathbb{P}^N$. Given the *Veronese embedding*

$$\begin{aligned} v_d : \mathbb{P}(S_1) &\to \mathbb{P}(S_d) \\ [L] &\mapsto [L^d], \end{aligned}$$

we have that $F = L_1^d + \ldots + L_s^d$ if and only if $[F] \in \langle v_d([L_1], \dots, [L_s]) \rangle$.

There is a duality between $\mathbb{P}(S_d)$ and $\mathbb{P}(T_d)$ given by differentiation. That is, $y_i(x_j) = \frac{\partial}{\partial x_i} x_j$ and $x_i(y_j) = \frac{\partial}{\partial y_i} y_j$. Consider the homogeneous annihilator ideal

$$F^{\perp} = \{G \in T : G(F) = 0\},$$

called the *apolar ideal* of $F^{\perp}$. Our main tool in this thesis is the following lemma:

Lemma 1.1.1 (Lemma 2.1.5 Apolarity lemma)**.** *Let $\Gamma \subset \mathbb{P}(S_1)$ be a scheme and let $F \in S_d$. Then $[F] \in \langle v_d(\Gamma) \rangle$ if and only if $F^{\perp} \supset I_\Gamma$.*

As a consequence, if $\Gamma = \{[L_1], \dots, [L_s]\}$, we have that $[F] \in \langle v_d([L_1], \dots, [L_s]) \rangle$, and hence $F = L_1^d + \ldots + L_s^d$ if and only if $I_\Gamma \subset F^{\perp}$.

Definition 1.1.2.

- The *rank* of $F \in S_d$, denoted $\mathrm{r}(F)$, is the minimal s such that $F = L_1^d + \cdots + L_s^d$, and
- The *cactus rank* of $F \in S_d$, denoted $\mathrm{cr}(F)$, is the minimal length of Γ such that $I_\Gamma \subset F^\perp$.

The *length* of a finite scheme Γ is the Hilbert polynomial of T/I_Γ. Observe that $\mathrm{cr}(F) \leq \mathrm{r}(F)$, where we have equality if we can find a subscheme of minimal length consisting of distinct points.

In some cases we can find several subideals $I_\Gamma \subset F^\perp$ of minimal length. This corresponds to finding several decompositions of F into a power sum of linear forms. The possible decompositions of a homogeneous polynomial into power sums of linear forms, are formalized in the following definition:

Definition 1.1.3. The variety of sums of powers of $F \in S_d$ is

$$\mathrm{VSP}(F, s) = \overline{\{([L_1], \ldots, [L_s]) \in \mathrm{Hilb}_s(\mathbb{P}(S_1)) : \exists\, \lambda_i \in \mathbb{C} \text{ such that } F = \lambda_1 L_1^d + \ldots \lambda_s L_s^d\}}.$$

When s is equal to the rank of F, $\mathrm{VSP}(F, \mathrm{r}(F))$ is a variety in $\mathrm{Hilb}_{\mathrm{r}(F)}(\mathbb{P}(S_1))$, where each point corresponds to a way of representing F as a power sum of $\mathrm{r}(F)$ linear forms.

Let $F \in S = \mathbb{C}[x_0, x_1, x_2]$. Then $F^\perp$ is an *Artinian Gorenstein* ideal since $T/F^\perp$ is and Artinian Gorenstein ring, by [Mac72]. A structure theorem for Gorenstein ideals I of codimension 3 proved by [BE77], gives that I is generated by the $(n-1)$*th order pfaffians*, the minors obtained by deleting the same row and column, of a skew symmetric matrix M, see Theorem 2.3.3. By the correspondence between ideals of finite schemes and matrices given by the Hilbert–Burch theorem, Theorem 5.1.1, our strategy for finding zero-dimensional ideals $I_\Gamma \subset F^\perp$ is to search for submatrices of M. We will use the following terminology:

- If $F^\perp$ is minimally generated by the $(n-1)$th order pfaffians of a matrix M, we say that M is a *Buchsbaum–Eisenbud matrix* of $F^\perp$. If F is not specified, we say that M is *a Buchsbaum–Eisenbud matrix*.
- If I_Γ is minimally generated by the maximal minors of a matrix H, we say that H is a *Hilbert–Burch matrix* of I_Γ. If I_Γ is not specified, we say that H is *a Hilbert–Burch matrix*.

We give an example of our strategy for finding a decomposition.

Example 1.1.4. Let $F \in S_6$ be the Fermat sextic, $F = x_0^6 + x_1^6 + x_2^6$. Then, by computation, $F^\perp = \langle y_0y_1, y_0y_2, y_1y_2, y_0^6 - y_1^6, y_0^6 - y_2^6\rangle$. The following matrix M is a Buchsbaum–Eisenbud matrix of $F^\perp$, where we have chosen a basis such that the M is skew symmetric:

$$M = \begin{pmatrix} 0 & -y_0^5 & y_1^5 & y_2 & -y_2 \\ y_0^5 & 0 & -y_2^5 & -y_1 & 0 \\ -y_1^5 & y_2^5 & 0 & 0 & y_0 \\ -y_2 & y_1 & 0 & 0 & 0 \\ y_2 & 0 & -y_0 & 0 & 0 \end{pmatrix}.$$

Firstly, we see that the 4th order pfaffians are the generators of $F^\perp$. Secondly, we see that we have a submatrix

$$H = \begin{pmatrix} -y_2 & y_1 & 0 \\ y_2 & 0 & -y_0 \end{pmatrix}$$

whose 2×2 minors gives three of the generators of $F^\perp$. This is the subideal I_Γ. Indeed, the points in Γ is the common zeros of y_0y_1, y_0y_2 and y_1y_2, which is $(1:0:0), (0:1:0)$ and $(0:0:1)$. By Lemma 1.1.1, we can write F as a power sum of the points in $\mathbb{P}(S_1)$ corresponding to $(1:0:0), (0:1:0)$ and $(0:0:1)$. The corresponding points in $\mathbb{P}(S_1)$ are x_0, x_1 and x_2, thus $F = x_0^6 + x_1^6 + x_2^6$.

The first result in this thesis is a classification of the Betti tables of the apolar ideals of a *non-degenerate* ternary sextic form F, for which there are no linear forms in $F^\perp$. In Theorem 3.0.4, we prove that there exists 16 different Betti tables for the apolar ideal of a ternary sextic form, summarized in Figure 3.3. From Theorem 3.0.4 we get the following corollary.

Corollary 1.1.5 (Corollary 3.0.5)**.** *The Betti table for the apolar ideal of a ternary sextic form F is determined by the number of quadratic, cubic and quartic generators of $F^\perp$.*

The number of quadratic, cubic and quartic generators of $F^\perp$ are equal to the Betti numbers b_{12}, b_{13} and b_{14}, respectively. Due to this, we write $B_{[b_{12}b_{13}b_{14}]}$ to denote a Betti table that is determined by these numbers. We also give the following definition of the *Betti strata*:

Definition 1.1.6. $\mathcal{F}_{B_{[b_{12}b_{13}b_{14}]}} = \{F \in \mathbb{P}(S_6) : T/F^\perp \text{ has Betti table } B_{[b_{12}b_{13}b_{14}]}\}$.

When the Betti table is not fixed, we write Betti strata $\mathcal{F}_B$. In Chapter 6 we prove that each $\mathcal{F}_B$ is irreducible. We also compute the rank and $\mathrm{VSP}(F, \mathrm{r}(F))$ of a general element of each $\mathcal{F}_B$. Further, we compute the configuration of points in Γ. We get the same results for every element in a fixed $\mathcal{F}_B$, except in two cases. Of this reason, we define $\mathcal{F}_{[016a]}, \mathcal{F}_{[016b]}, \mathcal{F}_{[016c]}$, and $\mathcal{F}_{[023a]}, \mathcal{F}_{[023b]}$ and $\mathcal{F}_{[023c]}$. Our results are reproduced in Figure 1.1, where we include the dimension of each $\mathcal{F}_B$.

Betti table $B_{[b_{12}b_{13}b_{14}]}$	$\mathrm{r}(F)$	$\mathrm{VSP}(F, r)$	Γ	$\dim(\mathcal{F}_B)$
[300]	3	one point	three points	8
[210]	4	one point	four points, three on a line	10
[200]	4	one point	four points	11
[202]	5	$\mathbb{P}^1$	five points, four on a line	11
[120]	5	one point	five points	14
[112]	6	$\mathbb{P}^1$	six points, four on a line	14
[111]	6	one point	six points on a conic	16
[104]	7	$\mathbb{P}^1$	seven points on a conic	17
[040]	6	one point	six points	17
[032]	7	$\mathbb{P}^1$	seven points, four on a line	17
[031]	7	one point	seven points	20
[023c]	8	$\mathbb{P}^1$	eight points, four on a line	20
[024]	8	$\mathbb{P}^1$	eight points, seven on a conic	20
[023b]	8	one point	eight points	23
[030]	9	$\mathbb{P}^2$	nine points in a CI	21
[016b]	9	$\mathbb{P}^1$	nine points, four on a line	23
[016c]	9	$\mathbb{P}^1$	nine points, seven on a conic	23
[023a]	9	one point	nine points in a CI	24
[016a]	9	two points	nine points	26
[009]	10	K3 surface	ten points	27

Figure 1.1: Rank and VSP for the Betti strata

In Chapter 7 we investigate the closure relation between the subsets $\mathcal{F}_B$ and our main results are in Proposition 7.2.4, Proposition 7.2.5 and Proposition 7.2.7. A picture of the closure relations is in Figure 1.2 on page 5, where each arrow depicts an inclusion in the closure. The Fermat sextic is of type [300], which is included in the closure of every $\mathcal{F}_B$. Since the Fermat sextic is smooth, this shows that a general element in every $\mathcal{F}_B$ is smooth.

In Chapter 8, we do explicit computations on the apolar ideal on ternary sextic forms that can be written as a double cubic form and prove the following theorem.

Theorem 1.1.7 (Theorem 8.0.8)**.** *Let Q be a irreducible ternary cubic form and let $F = Q^2$. Then $F^\perp$ contains at least one cubic form.*

An F with $\mathrm{cr}(F) = s$ lies in the sth *secant variety* to the Veronese variety. In Chapter 9 we use our results to prove Theorem 9.1.1, which describes relations between the secant varieties and *catalecticant matrices*.

1.2 Outline

The rest of the thesis is organised as follows:

In Chapter 2 we first present the apolarity construction and prove the apolarity lemma. Then we describe the structure theorem for Gorenstein ideals of codimension 3 and explain how the theorem apply for ternary sextic forms.

In Chapter 3 we classify the 16 Betti tables of the resolution of the apolar ideal of a non-degenerate ternary sextic form.

In Chapter 4 we introduce some theory about Grassmannians that we need in order to find isotropic subspaces of a skew symmetric matrix. First, we use the equation for the Grassmannian $G(2,4)$ to find the 2-dimensional isotropic subspaces of a 4×4 skew symmetric matrix with linear entries. Thereafter, we use the Chow ring and Chern classes of $G(3,6)$ and $G(4,9)$ to find the 3- and 4-dimensional isotropic subspaces of a 6×6 and a 9×9 skew symmetric matrix with linear entries, respectively. The isotropic subspaces we find correspond to Hilbert–Burch matrices of subideals of $F^{\perp}$.

In Chapter 5 we first give a description of the Hilbert–Burch matrices of the schemes that will appear as minimal subschemes of $F^{\perp}$. Then we prove that a Hilbert–Burch matrix of minimal subscheme will be a submatrix of a Buchsbaum–Eisenbud matrix M of F. At last we prove that the Hilbert–Burch matrices described actually appear as submatrices of M.

In Chapter 6 we first prove some results concerning the case when $I_{\Gamma} \subset F^{\perp}$ and Γ is contained in a line or a conic. Thereafter, we use these results to compute the rank and power sum representation of a non-degenerate ternary sextic form.

In Chapter 7 we first prove some containment relations between the schemes described in Chapter 5. Then, we use these relations to give a stratification of the space of non-degenerate ternary sextic forms. Lastly, we explain how the degenerate forms fits into the stratification.

In Chapter 8 we do explicit computations on the apolar ideal ternary sextic forms that can be written as a double cubic.

In Chapter 9 we compare our results to the secant varieties of the Veronese surface to catalecticant matrices. In addition, we raise some further questions related to power sum decomposition of homogeneous ternary forms.

We will use Macaulay2 [GS] in some of our computations. The Macaulay2 code can be found in Chapter 10.

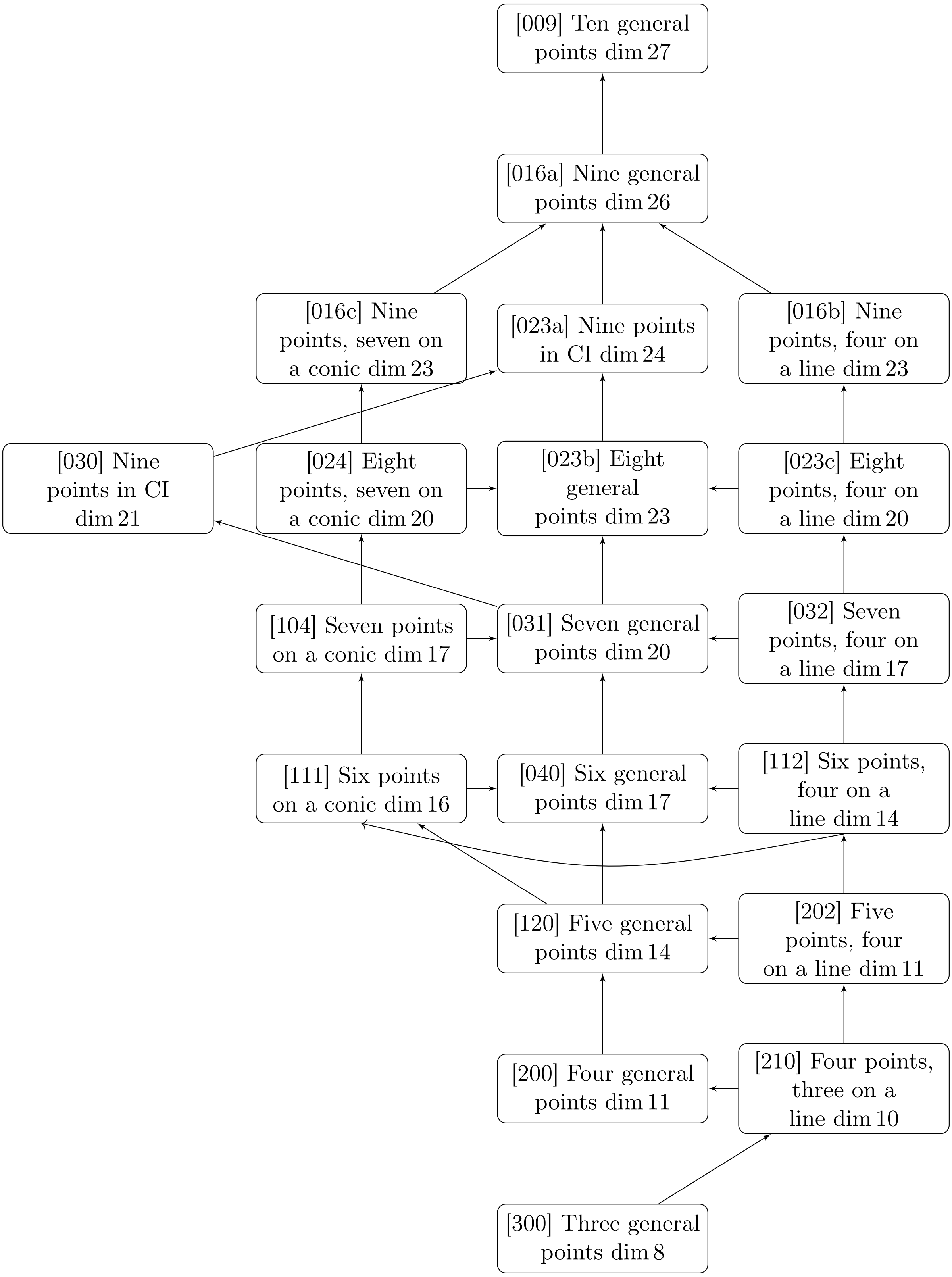


Figure 1.2: Stratification of $\mathbb{P}(S_6) = \mathbb{P}^{27}$. An arrow represents a closure relation.

2 | Preliminaries

In this chapter we introduce three concepts which form the foundation of the rest of the thesis. First, in Section 2.1, we explain what an apolar ideal is and how it can be used to find a power sum decomposition of a homogeneous polynomial. Then, in Section 2.2, we describe what a Betti table is and some of the properties it has in our case. Lastly, in Section 2.3, we introduce a structure theorem that gives a correspondence between a skew symmetric matrix and an apolar ideal.

2.1 Apolarity

Let $S = \mathbb{C}[x_0, \ldots, x_n]$ and $T = \mathbb{C}[y_0, \ldots, y_n]$. There is a duality between S_d and T_d given by differentiation, where $y_i(x_j) = \frac{\partial}{\partial x_i} x_j$ and $x_i(y_j) = \frac{\partial}{\partial y_i} y_j$. Indeed, let

$$F = \sum_{i_0+\cdots+i_n=d} a_i x_0^{i_0} \ldots x_n^{i_n} \in S_d \qquad \text{and} \qquad G = \sum_{i_0+\cdots+i_n=d} b_i y_0^{i_0} \ldots y_n^{i_n} \in T_d,$$

where $i = (i_0, \ldots, i_n)$ and $a_i, b_i \in \mathbb{C}$. Then

$$G(F) = F(G) = \sum_{i_0+\cdots+i_n=d} i_0! \ldots i_n! a_i b_i.$$

Fixing $G \in T_d$, we get the hyperplane $H_G = \{[F] : G(F) = 0\} \subset \mathbb{P}(S_d) = \mathbb{P}^N$, where $N = \binom{n+d}{n} - 1$. Hence $\mathbb{P}(S_d)^\vee = \mathbb{P}(T_d)$.

Definition 2.1.1. The ideal $F^\perp = \{G \in T : G(F) = 0\} \subset T$ is called the *apolar ideal* of $F \in S$.

We define $H_F = \{[G] : [F] \in H_G\} \subset \mathbb{P}(T_d)$ and observe that $H_F = \mathbb{P}(F_d^\perp)$. In the following, we will use the duality between S_d and T_d and the *Veronese embedding*, v_d, to investigate the relation between apolarity and power sum decomposition. Recall that

$$\begin{aligned} v_d : \mathbb{P}(S_1) &\to \mathbb{P}(S_d) \\ [L] &\mapsto [L^d]. \end{aligned}$$

We prove three lemmas before we state and prove the apolarity lemma.

Lemma 2.1.2. *Let $L = \lambda_0 x_0 + \cdots + \lambda_n x_n \in S_1$ and $G \in T_d$. Then differentiation and evaluation coincide, that is $G(L^d) = 0$ if and only if $G(\lambda_0, \ldots, \lambda_n) = 0$.*

Proof. We have

$$L^d = (\lambda_0 x_0 + \cdots + \lambda_n x_n)^d = \sum_{i_0+\cdots+i_n=d} \frac{d!}{i_0! \ldots i_n!} (\lambda_0 x_0)^{i_0} \ldots (\lambda_n x_n)^{i_n},$$

and

$$G(L^d) = \sum_{i_0+\cdots+i_n=d} b_i \frac{d!}{i_0! \ldots i_n!} i_0! \ldots i_n! (\lambda_0)^{i_0} \ldots (\lambda_n)^{i_n}$$

$$= d! \sum_{i_0+\cdots+i_n=d} b_i(\lambda_0)^{i_0}\ldots(\lambda_n)^{i_n}.$$

On the other hand, we have that

$$G(\lambda_0,\ldots,\lambda_n) = \sum_{i_0+\cdots+i_n=d} b_i(\lambda_0)^{i_0}\ldots(\lambda_n)^{i_n}.$$

Thus $G(\lambda_0,\ldots\lambda_n) = 0$ if and only if $G(L^d) = 0$. ■

Lemma 2.1.3. *Let $\Gamma \subset \mathbb{P}(S_1)$ be a scheme and I_Γ the corresponding ideal. Then $\langle v_d(\Gamma)\rangle = \mathbb{P}(I^\perp_{\Gamma,d}) \subset \mathbb{P}(S_d)$.*

Proof. Let Γ be a finite scheme and pick $G \in T_d$ such that $H_G \supset v_d(\Gamma)$. By definition of H_G, we have that $G(F) = 0$ for all $[F] \in v_d(\Gamma)$, which by Lemma 2.1.2, holds if and only if $G \in I_{\Gamma,d}$. For an arbitrary Γ, we get in the same way that $H_G \supset v_d(\Gamma)$ if and only if $G \in I_{\Gamma,d}$. In summary, we have that $I_{\Gamma_d} = \{G : H_G \supset v_d(\Gamma)\}$. Since $\langle v_d(\Gamma)\rangle$ is the intersection of all hyperplanes containing $v_d(\Gamma)$, we get that

$$\begin{aligned}\langle v_d(\Gamma)\rangle &= \{[F] : [F] \in H_G \text{ for all } G \in I_{\Gamma,d}\}\\ &= \{[F] : G(F) = 0 \text{ for all } G \in I_{\Gamma,d}\}\\ &= \{[F] : F(G) = 0 \text{ for all } G \in I_{\Gamma,d}\}\\ &= \mathbb{P}(I^\perp_{\Gamma_d})\end{aligned}$$

■

Lemma 2.1.4. *Let $\Gamma \subset \mathbb{P}(S_1)$ be scheme and let $F \in S_d$. Then $[F] \in \langle v_d(\Gamma)\rangle$ if and only if $F^\perp_d \supset I_{\Gamma,d}$.*

Proof. Let $[F] \in \mathbb{P}(S_d)$ and $G \in I_{\Gamma_d}$. By Lemma 2.1.3, $[F] \in \langle v_d(\Gamma)\rangle$ if and only if $F \in I^\perp_{\Gamma_d}$. Hence $F(G) = G(F) = 0$, thus $G \in F^\perp_d$. ■

There is a stronger version of Lemma 2.1.4 that we now can prove.

Lemma 2.1.5 (Apolarity lemma)**.** *Let $\Gamma \subset \mathbb{P}(S_1)$ be a scheme and let $F \in S_d$. Then $[F] \in \langle v_d(\Gamma)\rangle$ if and only if $F^\perp \supset I_\Gamma$.*

Proof. Because of Lemma 2.1.4, we only need to prove that $F^\perp \supset I_\Gamma$ if and only if $F^\perp_d \supset I_{\Gamma,d}$. That $F^\perp \supset I_\Gamma$ implies $F^\perp_e \supset I_{\Gamma,e}$ for all e, so the first implication is obvious. For the second, assume $F^\perp_d \supset I_{\Gamma,d}$ and let $G \in I_{\Gamma,e}$. We want to show that $G \in F^\perp$, i.e. $G(F) = 0$. If $e = d$, then $G \in F^\perp_d \subset F^\perp$ by assumption. If $e > d$, then $G \in F^\perp$ because $G(F) = 0$ for any $G \in T_e$. Let $e < d$. Since $F \in S_d$ and $G \in T_e$, $G(F)$ has degree $d - e$. Pick any $H \in T_{d-e}$ and consider the product HG. Since $G \in I_{\Gamma,e}$, we have $HG \in I_{\Gamma,d}$. By assumption $I_{\Gamma,d} \subset F^\perp_d$, hence $HG \in F^\perp_d$. We have that

$$HG(F) = H(G(F)) = 0 \tag{2.1}$$

for all $H \in T_{d-e}$. Since both H and $G(F)$ has degree $d - e$ we have that Equation (2.1) holds if and only if $G(F) = 0$. Thus $G \in F^\perp$, which was what we wanted to prove. ■

Corollary 2.1.6. *Let $\Gamma = \{[L_1],\ldots,[L_s])\} \subset \mathbb{P}(S_1)$ and let $F \in S_d$. Then we can choose $L_i \in [L_i]$ such that $F = L_1^d + \cdots + L_s^d$ if and only if $F^\perp \supset I_\Gamma$.*

Proof. Notice that $[F] \in \langle v_d(\Gamma)\rangle$ if and only if $F = L_1^d + \ldots + L_s^d$. ■

2.2 Betti Tables

Let $I \subset T = \mathbb{C}[y_0, \dots, y_n]$ be a homogeneous ideal. A graded *free resolution* $\mathcal{F}$ of T/I is an exact sequence of the form

$$\dots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow T/I \longrightarrow 0,$$

where $F_i \simeq \bigoplus_{j\in\mathbb{Z}} T(-j)^{b_{i,j}}$ is a free T-module. If there is an l such that $F_{l+2} = F_{l+3} = \dots = 0$, but $F_{l+1} \neq 0$, we say that the resolution is *finite of length* $l+1$. For a *minimal* finite free resolution of T/I, the exponents $b_{i,j}$ is called the *Betti numbers*. They form a *Betti table*

$$\begin{array}{cccccc} 0: & b_{0,0} & b_{1,1} & b_{2,2} & \dots & b_{n+1,n+1} \\ 1: & b_{0,1} & b_{1,2} & b_{2,3} & \dots & b_{n+1,n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m: & b_{0,m} & b_{1,m+1} & b_{2,m+2} & \dots & b_{n+1,n+1+m} \end{array},$$

where m is such that $b_{i,m+p} = 1$ for one p and $b_{i,m+p} = 0$ for all other p, and $b_{i,i+j} = 0$ for $j > m$. We denote $b_{i,j} = 0$ with a $-$.

Definition-Proposition 2.2.1. [Eis95, Theorem 21.6] Let I be a homogeneous ideal of $T = \mathbb{C}[y_0, \dots, y_n]$. Then I is an *Artinian Gorenstein ideal* if and only if $I = F^\perp$ for a homogeneous $F \in S$.

Let $T = \mathbb{C}[y_0, y_1, y_2]$. Assume $I \subset T$ is an Artinian Gorenstein ideal of codimension 3 and that $\mathcal{F}$ is minimal. Then $\mathcal{F}$ has length 3 by [Eis95, Corollary 21.16], thus we have the following free resolution $\mathcal{F}$:

$$0 \longrightarrow \bigoplus_{k\in\mathbb{Z}} T(-k)^{b_{3,k}} \longrightarrow \bigoplus_{k'\in\mathbb{Z}} T(-k')^{b_{2,k'}} \longrightarrow \bigoplus_{k''\in\mathbb{Z}} T(-k'')^{b_{1,k''}} \longrightarrow T,$$

where we have chosen $b_{0,0} = 1$, and have that $b_{0,j} = 0$ for $j > 0$ since $\mathcal{F}$ is a resolution of an ideal. We now dualize with $\mathrm{Hom}(-, T)$ and get

$$\begin{aligned} \mathrm{Hom}(\bigoplus_{j\in\mathbb{Z}} T(-j)^{b_{i,j}}, T) &\simeq \\ \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}(T(-j)^{b_{i,j}}, T) &\simeq \\ \bigoplus_{j\in\mathbb{Z}} T(j)^{b_{i,j}}&. \end{aligned}$$

We get the following resolution $\mathcal{F}^\vee$:

$$T \longrightarrow \bigoplus_{k''\in\mathbb{Z}} T(k'')^{b_{1,k''}} \longrightarrow \bigoplus_{k'\in\mathbb{Z}} T(k')^{b_{2,k'}} \longrightarrow \bigoplus_{k\in\mathbb{Z}} T(k)^{b_{3,k}}.$$

By [Eis95, Corollary 21.16], $\mathcal{F} \simeq \mathcal{F}^\vee$ as complexes, which means that

$$\begin{tikzcd} \bigoplus_{k\in\mathbb{Z}} T(-k)^{b_{3,k}} \arrow[r] \arrow[d, phantom, "\wr"] & \bigoplus_{k'\in\mathbb{Z}} T(-k')^{b_{2,k'}} \arrow[r] \arrow[d, phantom, "\wr"] & \bigoplus_{k''\in\mathbb{Z}} T(-k'')^{b_{1,k''}} \arrow[r] \arrow[d, phantom, "\wr"] & T \arrow[d, phantom, "\wr"] \\ T \arrow[r] & \bigoplus_{k''\in\mathbb{Z}} T(k'')^{b_{1,k''}} \arrow[r] & \bigoplus_{k'\in\mathbb{Z}} T(k')^{b_{2,k'}} \arrow[r] & \bigoplus_{k\in\mathbb{Z}} T(k)^{b_{3,k}}. \end{tikzcd}$$

Let l be such that $b_{3,l} = 1$ and $b_{3,k} = 0$ for $k \neq l$. To get the same grading on $\mathcal{F}$ and $\mathcal{F}^\vee$, we shift $\mathcal{F}^\vee$ with $-l$. This gives that

$$\bigoplus_{k'\in\mathbb{Z}} T(-k')^{b_{2,k'}} \simeq \bigoplus_{k''\in\mathbb{Z}} T(-l+k'')^{b_{1,k''}}.$$

In other words,

$$b_{2,k'} = b_{1,k''} \qquad \text{when} \qquad k'' + k' = l \tag{2.2}$$

By the definition of the Betti table, $l = m + 3$, so we get the following symmetric Betti table:

$$\begin{array}{rcccc}
0: & 1 & - & - & - \\
1: & - & b_{1,2} & b_{1,m} & - \\
2: & - & b_{1,3} & b_{1,m-1} & - \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
m-2: & - & b_{1,m-1} & b_{1,3} & - \\
m-1: & - & b_{1,m} & b_{1,2} & - \\
m: & - & - & - & 1
\end{array}.$$

When I is Artinian Gorenstein, that is $I = F^{\perp}$ for an $F \in S_d$, then $m = \deg F = d$, [Eis95, p. 505] and [Iar99, Proposition C.22 and p. 48] . We have proven the following lemma.

Lemma 2.2.2. *Let $S = \mathbb{C}[x_0, x_1, x_2]$ and $T = \mathbb{C}[y_0, y_1, y_2]$. Let $F \in S_d$ and $F^{\perp} \subset T$. Then the Betti table of the minimal free resolution of $T/F^{\perp}$ is*

$$\begin{array}{rcccc}
0: & 1 & - & - & - \\
1: & - & b_{1,2} & b_{1,d} & - \\
2: & - & b_{1,3} & b_{1,d-1} & - \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
d-2: & - & b_{1,d-1} & b_{1,3} & - \\
d-1: & - & b_{1,d} & b_{1,2} & - \\
d: & - & - & - & 1
\end{array}.$$

2.3 Buchsbaum–Eisenbud Matrix

Let $T = \mathbb{C}[y_0, y_1, y_2]$. In this section we explain how we can relate the generators of an apolar ideal $F^{\perp} \subset T$ to a matrix. The relation follows from a general structure theorem. In order to state the theorem and to explain its consequences in our case, we give some definitions and lemmas.

Let A be an $n \times n$ matrix. Then A is *skew symmetric* if $A = -A^T$. The *pfaffian* of A, $\mathrm{Pf}(A)$, is the square root of the determinant of A, that is $(\mathrm{Pf}(A))^2 = \det A$, [Cay09]. The $(n-1)$*th order pfaffian* of A is the square root of the determinant of the matrix obtained by deleting one row and the corresponding column of A. We denote by $\mathrm{Pf}_{n-1}(A)$ the ideal generated by the $(n-1)$th order pfaffians of A. If $P = (P_0, , \ldots, P_{n-1})$ is an ordered tuple and $a = (a_0, \ldots, a_{n-1})$ is a ordered tuple such that $a_0P_0 + \cdots + a_{n-1}P_{n-1} = 0$, then a is called a *syzygy* of P. We have the following two lemmas about skew symmetric matrices and syzygies.

Lemma 2.3.1. *Let $n \geq 3$ be an odd integer and let A be an $n \times n$ skew symmetric matrix. Then* $\det A = 0$.

Proof. Since A is skew symmetric, we have that $A^T = -A$. Then

$$\det A = \det A^T = \det(-A) = (-1)^n \det A.$$

Since n is odd, we get that

$$\det A = -\det A = 0.$$

■

Lemma 2.3.2. *Let $n \geq 3$ be an odd integer and let A be an $n \times n$ skew symmetric matrix. Then the columns of A are syzygies of the $(n-1)$th order pfaffians of A ordered in the natural way.*

Proof. Let $A = (a_{ij})_{i,j=0}^{n-1}$ and let M_{ij} be the $(n-1) \times (n-1)$ minor obtained by deleting the ith row and jth column and let $P_i = M_{ii}$ be the $(n-1)$th order pfaffian obtained by deleting the ith row and ith column. By [Cay09], we have the following relation:

$$M_{ij} = (-1)^{i+j+1} P_i P_j.$$

We now compute $\det A$ by expanding along the ith row. This gives

$$\begin{aligned} \det A &= \sum_{j=0}^{n-1} (-1)^{i+j+1} a_{ij} M_{ij} \\ &= \sum_{j=0}^{n-1} (-1)^{i+j+1} (-1)^{i+j+1} a_{ij} P_i P_j \\ &= P_i \sum_{j=0}^{n-1} a_{ij} P_j \end{aligned}$$

Since $\det A = 0$ by Lemma 2.3.1, we get the following relation between the $(n-1)$th order pfaffians:

$$a_{i0} P_0 + a_{i1} P_1 + \cdots + a_{i,n-1} P_{n-1} = 0$$

Since $a_{ij} = -a_{ji}$, have showed that the ith column of A is a syzygy of the $(n-1)$th order pfaffians of A. ■

Let R be a ring and $\hat{R}$ an R-module. If $f : \hat{R}^\vee \to \hat{R}$, we say that f is an *alternating map* if there exists a basis such that the matrix A of f is skew symmetric. We denote by $\mathrm{Pf}_{n-1}(f)$ the ideal generated by the $(n-1)$*th order pfaffian* of A. We are now ready to state the structure theorem in full generality.

Theorem 2.3.3 (Buchsbaum–Eisenbud)**.** *Let R be a Noetherian local ring with maximal ideal J.*

1. *Let $n \geq 3$ be an odd integer and let $\hat{R}$ be a free R-module of rank n. Let $f : \hat{R}^\vee \to \hat{R}$ be an alternating map whose image is contained in $J\hat{R}$. Suppose $\mathrm{Pf}_{n-1}(f)$ has codimension 3. Then $\mathrm{Pf}_{n-1}(f)$ is a Gorenstein ideal, minimally generated by n elements.*
2. *Every Gorenstein ideal of codimension 3 arises as above.*

In particular this theorem holds in the polynomial ring with standard grading and a homogeneous ideal I. We will use the following graded version.

Corollary 2.3.4. *Let $n \geq 3$ be an odd integer and $T = \mathbb{C}[y_0, y_1, y_2]$ with the usual grading.*

1. *Let $A = (a_{ij})$ be a skew symmetric matrix of dimension n, where a_{ij} are homogeneous polynomials such that all $(n-1)$th order pfaffians are homogeneous. Assume $\mathrm{Pf}_{n-1}(A)$ has codimension 3. Then $\mathrm{Pf}_{n-1}(A)$ is the apolar ideal of a homogeneous $F \in S$ minimally generated by n elements.*
2. *Let $I \subset T$ be a homogeneous Gorenstein ideal of codimension 3 generated by n elements. Then I is minimally generated by $\mathrm{Pf}_{n-1}(A)$, where A is a skew symmetric matrix with homogeneous entries whose columns are a minimal basis for the syzygies of I.*

Proof. (1) Let $f : T^n \to T^n$ be the alternating map given by A. Since every entry in A is a non constant homogeneous polynomial, the image of f is in JT^n and $\mathrm{Pf}_{n-1}(A) = I$ is a homogeneous ideal. By Theorem 2.3.3, $\mathrm{Pf}_{n-1}(A)$ is a Gorenstein ideal minimally generated by n elements. Since $\dim T = 3$ and $\operatorname{codim} I = 3$, I is Artinian. By Definition-Proposition 2.2.1, $I = F^{\perp}$ for a homogeneous $F \in S$.

(2) By Theorem 2.3.3, every Gorenstein ideal of codimension 3 is generated by the $(n-1)$th order pfaffians of an $n \times n$ skew symmetric matrix A. By Lemma 2.3.2, the columns of A are syzygies of the $(n-1)$th order pfaffians. The minimality follows from Theorem 2.3.3. ∎

3 | Classification of Betti Tables

In this chapter we find and classify all Betti tables for the resolution of $T/F^{\perp}$ for a ternary sextic form F. First, we find restrictions for the Betti numbers given by the Buchsbaum–Eisenbud matrix. Then we introduce the Hilbert function of $T/F^{\perp}$ use it to find more restrictions on the Betti numbers. Lastly, we state and prove our theorem that gives the classification.

Let $S = \mathbb{C}[x_0, x_1, x_2]$ and $T = \mathbb{C}[y_0, y_1, y_2]$. Let $F \in S_6$. By Lemma 2.2.2 the Betti table of the resolution of $T/F^{\perp}$ is

$$\begin{array}{lcccc} 0: & 1 & - & - & - \\ 1: & - & b_{1,2} & b_{1,6} & - \\ 2: & - & b_{1,3} & b_{1,5} & - \\ 3: & - & b_{1,4} & b_{1,4} & - \\ 4: & - & b_{1,5} & b_{1,3} & - \\ 5: & - & b_{1,6} & b_{1,2} & - \\ 6: & - & - & - & 1. \end{array}$$

Moreover, by Equation (2.2) on page 9, $b_{1,6} = b_{2,3}$ and $b_{1,5} = b_{2,4}$. We choose to work with these numbers, therefore we write the Betti table as

$$\begin{array}{lcccc} 0: & 1 & - & - & - \\ 1: & - & b_{1,2} & b_{2,3} & - \\ 2: & - & b_{1,3} & b_{2,4} & - \\ 3: & - & b_{1,4} & b_{1,4} & - \\ 4: & - & b_{2,4} & b_{1,3} & - \\ 5: & - & b_{2,3} & b_{1,2} & - \\ 6: & - & - & - & 1. \end{array}$$

The $b_{1,j}$s are the number of minimal generators of $F^{\perp}$ of degree j, and the $b_{2,j}$ are the number of syzygies between the generators of degree less than j. The syzygy represented by $b_{2,j}$ are linear for the generators of degree $j-1$.

Now, we prove a lemma about the determinant of a matrix which we will use to prove a limitation of the Betti numbers.

Lemma 3.0.1. *Let $n \geq 3$ and let M be an $n \times n$ matrix. Assume $m < \frac{n}{2}$ and $l > m$. If M has an $l \times (n-m)$-submatrix where all the entries are zero, then* $\det M = 0$.

Proof. Write M as a block matrix with an $(n-m) \times m$ matrix A, an $(n-m) \times (n-m)$ matrix B, an $m \times m$ matrix C and an $m \times (n-m)$ block D where all the entries are zero, see Figure 3.1. Since M is a 2×2 upper triangular block matrix $\det M = \det B \cdot \det C$. Because M has an $l \times (n-m)$ zero block and $l > m$, at least one of the rows of B is zero, which gives $\det B = 0$. We then have $\det M = \det B \cdot \det C = 0$ and the lemma holds.

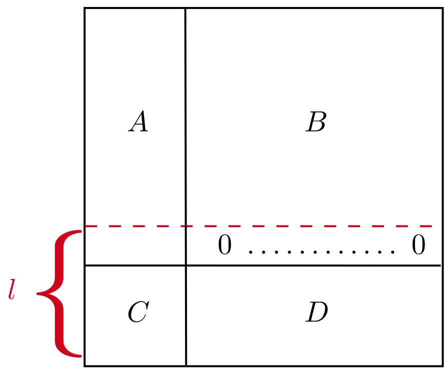


Figure 3.1: The block matrix in Lemma 3.0.1.

■

Lemma 3.0.2. *Let $F^{\perp} \subset T = \mathbb{C}[y_0, y_1, y_2]$ and let $b_{i,j}$ be the Betti numbers of the resolution of $T/F^{\perp}$. Let $k = b_{1,2} + b_{1,3}$ and $l = b_{2,3} + b_{2,4}$, then $l < k$.*

Proof. Let M be an $n \times n$ matrix and assume M is a Buchsbaums–Eisenbud matrix of $F^{\perp}$. By Theorem 2.3.3 $F^{\perp}$ is minimally generated by the $(n-1)$th order pfaffians of M. We will prove that if $l \geq k$, then at least one of the n generators are zero, contradicting the fact that n is minimal.

Assume for contradiction that $l \geq k$. Recall that k is the number of quadratic and cubic forms in $F^{\perp}$, $b_{1,4}$ the number of quartic forms and l the number of syzygies between the quadratic and the cubic forms. By symmetry, l is also the number of quintic and sextic forms in the ideal. Since the l syzygies are not syzygies between the $b_{1,4} + l$ quartic, quintic and sextic forms, M will have an $l \times (b_{1,4} + l)$ zero block. One of the pfaffians is obtained by computing the determinant of the matrix we get by deleting the first row and first column of M. We then have a $(k - 1 + b_{1,4} + l) \times (k - 1 + b_{1,4} + l)$ matrix with determinant equal to zero, by Lemma 3.0.1. Indeed, since $k \leq l$, we get $k - 1 < l \leq l + b_{1,4}$, see Figure 3.2. Both assumptions in Lemma 3.0.1 are satisfied, so the determinant is zero, and we have a contradiction. ■

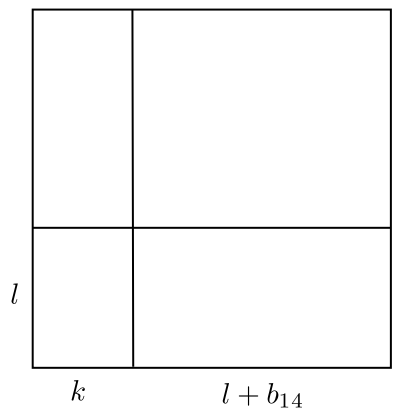


Figure 3.2: The matrix M in the proof of Lemma 3.0.2.

To compute more limitations of the Betti numbers, we introduce the Hilbert function. Let $A = \bigoplus A_i$ be a graded module. Then the *Hilbert function* $H_{A_i} = \dim_{\mathbb{C}} A_i$, that is the dimension of A_i as a vector space of $\mathbb{C}$. We will use the following relation:

$$H_{T_i} = H_{F_i^{\perp}} + H_{T/F_i^{\perp}}. \tag{3.1}$$

We find H_{T_i} be computing the number of monomials of total degree i in T, that is $H_{T_i} = \binom{i+2}{2}$. The first values are

$$H_{T_0} = 0, \quad H_{T_1} = 3, \quad H_{T_2} = 6, \qquad H_{T_3} = 10, \quad H_{T_4} = 15$$

Now, we express $H_{F^\perp,i}$ in terms of the Betti numbers. Since $F^\perp$ is a proper ideal, we obviously have $H_{F_0^\perp} = 0$, and since we have assume that there are no linear forms in $F^\perp$, also $H_{F_1^\perp} = 0$. We claim that we have the following relations:

$$\begin{aligned} H_{F_2^\perp} &= b_{1,2}, \\ H_{F_3^\perp} &= b_{1,3} + 3b_{1,2} - b_{2,3}, \\ H_{F_4^\perp} &= b_{1,4} + 6b_{1,2} + 3b_{1,3} - 3b_{2,3} - b_{24} \end{aligned}$$

Indeed, $H_{F^\perp,i}$ is the number of generators of degree i and the number of forms of degree i obtained from the generators of lower degree, minus the number of forms of degree i obtained from the syzygies between the generators of lower degree. By combining the values of H_{T_i} and $H_{F_i^\perp}$ with $H_{T/F_i^\perp}$ by using Equation (3.1), we get

$$\begin{aligned} b_{1,2} &= 6 - h_2, \\ b_{1,3} &= 10 - h_3 - 3 \cdot b_{1,2} + b_{2,3}, \\ b_{1,4} &= 15 - h_4 - 6 \cdot b_{1,2} - 3 \cdot b_{1,3} + 3 \cdot b_{2,3} + b_{2,4}, \end{aligned}$$

where $h_i = H_{T/F_i^\perp}$.

Now, we find some limitation for $H = (h_0, \ldots, h_i, \ldots)$. Because $H_{F_0^\perp} = H_{F_1^\perp} = 0$, we get that $h_0 = 1$ and $h_1 = 3$, by Equation (2.2). Since F is ternary sextic forms, $H_{F^\perp,i} = H_{T_i}$ for $i \geq 7$. This gives that $h_i = 0$ for $i \geq 7$. Since the Betti numbers are positive, Equation (2.2) also gives that $h_2 \leq 6$ and $h_3 \leq 10$. By [Iar99, Lemma 2.14], H is symmetric, that is

$$\begin{aligned} H &= (h_0, h_1, h_2, h_3, h_4, h_5, h_6) \\ &= (h_0, h_1, h_2, h_3, h_2, h_1, h_0) \\ &= (1, 3, h_2, h_3, h_2, 3, 1). \end{aligned}$$

We get a last limitation by the following lemma of Macaulay.

Lemma 3.0.3 (Macaulay). *Set $H = (h_0, \ldots, h_i, \ldots)$, where h_i are non-negative integers and write*

$$\begin{aligned} h_i &= \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \ldots \qquad \textit{and} \\ h_i^{\langle i \rangle} &= \binom{a_i + 1}{i+1} + \binom{a_{i-1}+1}{i} + \ldots, \end{aligned}$$

with $a_i > a_{i-1} > \ldots$. If H is a Hilbert function for a graded module, then $h_{i+1} \leq h_i^{\langle i \rangle}$.

As a consequence of Lemma 3.0.3, we get the following limitations.

$$\begin{aligned} H &= (1, 3, 5, h_3, 5, 3, 1), & h_3 &\leq 10, \\ H &= (1, 3, 5, h_3, 5, 3, 1), & h_3 &\leq 7, \\ H &= (1, 3, 4, h_3, 4, 3, 1), & h_3 &\leq 5, \\ H &= (1, 3, 3, h_3, 3, 3, 1), & h_3 &\leq 4, \end{aligned}$$

$$H = (1,3,2,h_3,2,3,1), \qquad h_3 \le 2,$$
$$H = (1,3,1,h_3,1,3,1), \qquad h_3 \le 1,$$
$$H = (1,3,0,h_3,0,3,1), \qquad h_3 \le 0.$$

We are now ready to state and prove our theorem.

Theorem 3.0.4. *Let F be a ternary sextic form. Assume that there is no linear form apolar to F. Then $T/F^{\perp}$ has one of the 16 Betti tables in Figure 3.3. We also give the Hilbert polynomial of $T/F^{\perp}$.*

Before we prove the theorem, we state a corollary of the theorem.

Corollary 3.0.5. *The Betti table for the apolar ideal of a ternary sextic form F is determined by the number of quadratic, cubic and quartic generators of $F^{\perp}$.*

Proof of Theorem 3.0.4. We recall the relations between the Betti numbers and the Hilbert function of $T/F^{\perp}$.

$$\begin{aligned} b_{1,2} &= 6 - h_2, \\ b_{1,3} &= 10 - h_3 - 3 \cdot b_{1,2} + b_{2,3}, \\ b_{1,4} &= 15 - h_4 - 6 \cdot b_{1,2} - 3 \cdot b_{1,3} + 3 \cdot b_{2,3} + b_{2,4}, \end{aligned}$$

In the proof, we will use these relations together with the limitation for the Betti numbers and the Hilbert function of $T/F^{\perp}$ from Lemma 3.0.2 and Lemma 3.0.3. Since the Betti numbers and the h_is are positive, we get that $0 \le b_{1,2} \le 6$. We go through each of these cases.

Case $b_{1,2} = 0$:

Since $b_{1,2} = 0$, $b_{2,3} = 0$, indeed $b_{2,3}$ represents the number of syzygies between the quadratic forms in $F^{\perp}$, and there are no quadratic forms in the ideal. The number of quartic generators in the ideal is $b_{1,4} = 15 - 6 - 3b_{1,3} + b_{2,4} = 9 - 3b_{1,3} + b_{2,4}$. The shape of the Betti table in this case is:

0:	1	-	-	-
1:	-	0	0	-
2:	-	$b_{1,3}$	$b_{2,4}$	-
3:	-	$9 - 3b_{1,3} + b_{2,4}$	$9 - 3b_{1,3} + b_{2,4}$	-
4:	-	$b_{2,4}$	$b_{1,3}$	-
5:	-	0	0	-
6:	-	-	-	1

When $b_{1,3} \in \{0, 1\}$, $b_{2,4} = 0$ because of Lemma 3.0.2. We have the two Betti tables in Figure 3.3 and they correspond to the Hilbert functions $(1, 3, 6, 10, 6, 3, 1)$ and $(1, 3, 6, 9, 6, 3, 1)$. When $b_{1,3} = 2$, Lemma 3.0.2 gives $b_{2,4} \in \{0, 1\}$. We have both of the Betti tables in Figure 3.3 and they correspond to the Hilbert function $(1, 3, 6, 8, 6, 3, 1)$. For $b_{1,3} = 3$ the possible values for $b_{2,4} \in \{0, 1, 2\}$ by Lemma 3.0.2. We have the three Betti tables in Figure 3.3, which correspond to the Hilbert function $(1, 3, 6, 7, 6, 3, 1)$. When $b_{1,3} = 4$ Lemma 3.0.2 gives that $b_{2,4} \in \{0, 1, 2, 3\}$. Only case $b_{2,4} = 3$ is possible, indeed $b_{1,4} = 9 - 3 \cdot 4 + b_{2,4} = b_{2,4} - 3$, so $b_{2,4} \ge 3$. We have the Betti table in Figure 3.3 and the corresponding Hilbert function is $(1, 3, 6, 6, 6, 3, 1)$.

The cases $b_{1,3} \ge 5$ are not realizable, indeed Lemma 3.0.2 gives that $b_{2,4} < b_{1,3}$, but $b_{1,4} = 9 - 3 \cdot b_{1,3} + b_{2,4} < 9 - 2b_{1,3}$ which gives $b_{1,4} < 0$ when $b_{1,3} \ge 5$.

$\begin{array}{cccc} 1 & - & - & - \\ - & 3 & 2 & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & 2 & 3 & - \\ - & - & - & 1 \end{array}$ $(1,3,3,3,3,3,1)$	$\begin{array}{cccc} 1 & - & - & - \\ - & 2 & - & - \\ - & - & 1 & - \\ - & - & - & - \\ - & 1 & - & - \\ - & - & 2 & - \\ - & - & - & 1 \end{array}$ $(1,3,4,4,4,3,1)$	$\begin{array}{cccc} 1 & - & - & - \\ - & 2 & 1 & - \\ - & 1 & 1 & - \\ - & - & - & - \\ - & 1 & 1 & - \\ - & 1 & 2 & - \\ - & - & - & 1 \end{array}$ $(1,3,4,4,4,3,1)$	$\begin{array}{cccc} 1 & - & - & - \\ - & 2 & 1 & - \\ - & - & - & - \\ - & 2 & 2 & - \\ - & - & - & - \\ - & 1 & 2 & - \\ - & - & - & 1 \end{array}$ $(1,3,4,5,4,3,1)$
$\begin{array}{cccc} 1 & - & - & - \\ - & 1 & - & - \\ - & 2 & 2 & - \\ - & - & - & - \\ - & 2 & 2 & - \\ - & - & 1 & - \\ - & - & - & 1 \end{array}$ $(1,3,5,5,5,3,1)$	$\begin{array}{cccc} 1 & - & - & - \\ - & 1 & - & - \\ - & 1 & - & - \\ - & 1 & 1 & - \\ - & - & 1 & - \\ - & - & 1 & - \\ - & - & - & 1 \end{array}$ $(1,3,5,6,5,3,1)$	$\begin{array}{cccc} 1 & - & - & - \\ - & 1 & - & - \\ - & 1 & - & - \\ - & 2 & 2 & - \\ - & - & 1 & - \\ - & - & 1 & - \\ - & - & - & 1 \end{array}$ $(1,3,5,6,5,3,1)$	$\begin{array}{cccc} 1 & - & - & - \\ - & 1 & - & - \\ - & - & - & - \\ - & 4 & 4 & - \\ - & - & - & - \\ - & - & 1 & - \\ - & - & - & 1 \end{array}$ $(1,3,5,7,5,3,1)$
$\begin{array}{cccc} 1 & - & - & - \\ - & - & - & - \\ - & 4 & 3 & - \\ - & - & - & - \\ - & 3 & 4 & - \\ - & - & - & - \\ - & - & - & 1 \end{array}$ $(1,3,6,6,6,3,1)$	$\begin{array}{cccc} 1 & - & - & - \\ - & - & - & - \\ - & 3 & - & - \\ - & - & - & - \\ - & - & 3 & - \\ - & - & - & - \\ - & - & - & 1 \end{array}$ $(1,3,6,7,6,3,1)$	$\begin{array}{cccc} 1 & - & - & - \\ - & - & - & - \\ - & 3 & 1 & - \\ - & 1 & 1 & - \\ - & 1 & 3 & - \\ - & - & - & - \\ - & - & - & 1 \end{array}$ $(1,3,6,7,6,3,1)$	$\begin{array}{cccc} 1 & - & - & - \\ - & - & - & - \\ - & 3 & 2 & - \\ - & 2 & 2 & - \\ - & 2 & 3 & - \\ - & - & - & - \\ - & - & - & 1 \end{array}$ $(1,3,6,7,6,3,1)$
$\begin{array}{cccc} 1 & - & - & - \\ - & - & - & - \\ - & 2 & - & - \\ - & 3 & 3 & - \\ - & - & 2 & - \\ - & - & - & - \\ - & - & - & 1 \end{array}$ $(1,3,6,8,6,3,1)$	$\begin{array}{cccc} 1 & - & - & - \\ - & - & - & - \\ - & 2 & 1 & - \\ - & 4 & 4 & - \\ - & 1 & 2 & - \\ - & - & - & - \\ - & - & - & 1 \end{array}$ $(1,3,6,8,6,3,1)$	$\begin{array}{cccc} 1 & - & - & - \\ - & - & - & - \\ - & 1 & - & - \\ - & 6 & 6 & - \\ - & - & 1 & - \\ - & - & - & - \\ - & - & - & 1 \end{array}$ $(1,3,6,9,6,3,1)$	$\begin{array}{cccc} 1 & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & 9 & 9 & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & 1 \end{array}$ $(1,3,6,10,6,3,1)$

Figure 3.3: The 16 Betti tables of the resolution of $T/F^{\perp}$

Case $b_{1,2} = 1$:

Since $b_{1,2} = 1$, $b_{2,3} = 0$, indeed there are no syzygies between the quadratic forms when there are just one quadratic generator in $F^\perp$. The number of quartic generators of the ideal is $b_{1,4} = 15 - 5 - 6 \cdot 1 - 3b_{1,3} + b_{2,4} = 4 - 3b_{1,3} + b_{2,4}$. The shape of the Betti table in this case is:

0:	1	-	-	-
1:	-	1	0	-
2:	-	$b_{1,3}$	$b_{2,4}$	-
3:	-	$4 - 3b_{1,3} + b_{2,4}$	$4 - 3b_{1,3} + b_{2,4}$	-
4:	-	$b_{2,4}$	$b_{1,3}$	-
5:	-	0	1	-
6:	-	-	-	1

When $b_{1,3} = 0$, $b_{2,4} = 0$ by Lemma 3.0.2. We have the Betti table in Figure 3.3 and it corresponds to the Hilbert function $(1, 3, 5, 7, 5, 3, 1)$. For $b_{1,3} = 1$ we get that $b_{2,4} \in \{0, 1\}$ and we have the Betti tables in Figure 3.3. They correspond to the Hilbert function $(1, 3, 5, 6, 5, 3, 1)$. When $b_{1,3} = 2$, Lemma 3.0.2 gives that $b_{2,4} \leq 2$. Only $b_{2,4} = 2$ is realizable, indeed $b_{1,4} = 4-3\cdot 2+b_{2,4} = b_{2,4}-2$. We have this Betti table in Figure 3.3 and it corresponds to the Hilbert function $(1, 3, 5, 5, 5, 3, 1)$.

The cases $b_{1,3} \geq 3$ are not realizable. Lemma 3.0.2 gives that $b_{2,4} \leq b_{1,3}$ and $b_{1,4} = 4 - 3b_{1,3} + b_{2,4} \leq 4 - 2b_{1,3}$ is negative when $b_{1,3} \geq 3$.

Case $b_{1,2} = 2$: The only possible values for $b_{2,3}$ is 0 and 1. Indeed, assume for contradiction that $b_{2,3} = 2$. Then we have that

$$p_0x_0 = p_1x_1, \tag{3.2}$$
$$p_0x_0' = p_1x_1', \tag{3.3}$$

where the p_is are quadratic generators of the ideal and x_i, x_i' are linear forms where no two are equal. Since T is a unique factorization domain and $x_0 \neq x_1$, $p_0 = x_1x_0''$ and $p_1 = x_0x_1''$. In the same way we get that $p_0 = x_1'x_0'''$ and $p_1 = x_0'x_1'''$, where all x_i^p are linear forms. Substituting this into Equation (3.3), we get that

$$x_1x_0''x_0' = x_0'x_1'''x_1',$$

and $x_0'' = x_1'''$. Denote x_0'' and x_1''' by x. Then we get that

$$x_1xx_0 = x_0'xx_1,$$

where we have substituted $p_0 = x_1x$ and $p_1 = x_0'x$ into Equation (3.2). This gives that $x_0 = x_0'$, a contradiction. In summary, we have $b_{2,3} = \{0, 1\}$ and work through each case.

Subcase $b_{2,3} = 0$:

The number of cubic generators in $F^\perp$ is $b_{1,3} = 10 - 2 \cdot 3 - h_3 = 4 - h_3$ and the number of quartic generators is $b_{1,4} = 15 - 4 - 6 \cdot 2 - 3b_{1,3} + b_{2,4} = b_{2,4} - 3b_{1,3} - 1$. We have the following shape of the Betti table of this case:

0:	1	-	-	-
1:	-	2	0	-
2:	-	$4 - h_3$	$b_{2,4}$	-
3:	-	$b_{2,4} - 3b_{1,3} - 1$	$b_{2,4} - 3b_{1,3} - 1$	-
4:	-	$b_{2,4}$	$4 - h_3$	-
5:	-	0	2	-
6:	-	-	-	1

We see that $h_3 \leq 4$. If we let $h_3 = 4$, we get $b_{1,3} = 0$ and, by Lemma 3.0.2, $b_{2,4} \in \{0, 1\}$. We also get that $b_{1,4} = b_{2,4} - 1 - 3 \cdot 0 = b_{2,4} - 1$, so $b_{1,2} = 1$. We have the Betti table in Figure 3.3 and the corresponding Hilbert function is $(1, 3, 4, 4, 4, 3, 1)$.

The cases $h_3 \leq 3$ are not realizable. Indeed, we get $b_{1,3} = 4 - h_3$ and $b_{2,4} < 6 - h_3$. Then $b_{1,4} = b_{2,4} - 1 - 3b_{1,3} < 2h_3 - 7$, which is negative for $h_3 \leq 3$.

Subcase $b_{2,3} = 1$:

The number of cubic generators in the ideal is $b_{1,3} = 10 - 2 \cdot 3 - h_3 + 1 = 5 - h_3$ and the number of quartic generators is $b_{1,4} = 15 - 4 - 6 \cdot 2 - 3b_{1,3} + b_{2,4} + 3 \cdot 1 = b_{2,4} - 3b_{1,3} + 2$. In this case we have the following shape of the Betti table:

0:	1	-	-	-
1:	-	2	1	-
2:	-	$5 - h_3$	$b_{2,4}$	-
3:	-	$b_{2,4} - 3b_{1,3} + 2$	$b_{2,4} - 3b_{1,3} + 2$	-
4:	-	$b_{2,4}$	$5 - h_3$	-
5:	-	1	2	-
6:	-	-	-	1

We see that $h_3 \leq 5$. If we let $h_3 = 5$ we get that $b_{1,3} = 0$ and $b_{2,4} = 0$ by Lemma 3.0.2. We have the Betti table in Figure 3.3 and the corresponding Hilbert function is $(1, 3, 4, 5, 4, 3, 1)$. When $h_3 = 4$, $b_{1,3} = 1$. From Lemma 3.0.2, we get that $b_{2,4} \leq 1$. We also have that $b_{1,4} = b_{2,4} + 2 - 3 \cdot 1 = b_{2,4} - 1$, so $b_{2,4} = 1$. We have this Betti table in Figure 3.3 which corresponds to the Hilbert function $(1, 3, 4, 4, 4, 3, 1)$.

The cases $h_3 \leq 3$ are not realizable. Indeed, we get that $b_{1,3} = 5 - h_3$ and $b_{2,4} < 6 - h_3$. Then $b_{1,4} = b_{2,4} + 2 - 3b_{1,3} < 2h_3 - 10$ which is negative for $h_3 \leq 3$.

Case $b_{1,2} = 3$:

We get that $b_{2,3} \leq 3$ because there cannot be more that three linearly independent linear syzygies when we work with three variables. We will show that $b_{2,3} \neq 3$. Indeed, if $b_{2,3} = 3$ we have three linear syzygies between three quadratic forms and get the following equations:

$$
\begin{aligned}
a_1 q_1 + a_2 q_2 + a_3 q_3 &= 0, \\
b_1 q_1 + b_2 q_2 + b_3 q_3 &= 0, \\
c_1 q_1 + c_2 q_2 + c_3 q_3 &= 0,
\end{aligned}
$$

where a_i, b_i, c_i are linear forms and q_i are quadratic forms. This system can be written in matrix form,

$$
\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \cdot \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
$$

For this matrix equation to have a non-trivial solution, the (3×3) matrix must have determinant equal to zero, i.e. there must be a linear relation between the rows. This linear relation is linear secondary syzygy between the quadratic forms, and can be found in the Betti table as $b_{3,4} = 1$, but $b_{3,4} = 0$ in our case, so we have contradiction. The only possible values for $b_{2,3}$ is $0, 1$ and 2. We go through each case.

Subcase $b_{2,3} = 0$:

The number of cubic generators in the ideal is $b_{1,3} = 10 - 3 \cdot 3 - h_3 = 1 - h_3$ and the number of quartic generators is $b_{1,4} = 15 - 3 - 6 \cdot 3 - 3b_{1,3} + b_{2,4} = b_{2,4} - 3b_{1,3} - 6$. In this case the Betti table has the following shape:

$$\begin{array}{lcccc}
0: & 1 & - & - & \\
1: & - & 3 & 0 & - \\
2: & - & 1 - h_3 & b_{2,4} & - \\
3: & - & b_{2,4} - 3b_{1,3} - 6 & b_{2,4} - 3b_{1,3} - 6 & - \\
4: & - & b_{2,4} & 1 - h_3 & - \\
5: & - & 0 & 3 & - \\
6: & - & - & - & 1
\end{array}$$

We get that $h_3 \leq 1$ and, by Lemma 3.0.2, $b_{2,4} < 4 - h_3$. Then $b_{1,4} = b_{2,4} - 3b_{1,3} - 6 < 2h_3 - 5$ which is negative for $h_3 \leq 1$. There are no Betti table for $b_{2,3} = 0$.

Subcase $b_{2,3} = 1$:

The number of cubic generators in the ideal is $b_{1,3} = 10 - 3 \cdot 3 - h_3 + 1 = 2 - h_3$ and the number of quartic generators is $b_{1,4} = 15 - 3 - 6 \cdot 3 - 3b_{1,3} + b_{2,4} + 3 \cdot 1 = b_{2,4} - 3b_{1,3} - 3$. In this case the Betti table has the following shape:

$$\begin{array}{lcccc}
0: & 1 & - & - & - \\
1: & - & 3 & 1 & - \\
2: & - & 2 - h_3 & b_{2,4} & - \\
3: & - & b_{2,4} - 3b_{1,3} - 3 & b_{2,4} - 3b_{1,3} - 3 & - \\
4: & - & b_{2,4} & 2 - h_3 & - \\
5: & - & 1 & 3 & - \\
6: & - & - & - & 1
\end{array}$$

We get that $h_3 \leq 2$ and, by Lemma 3.0.2, $b_{2,4} < 4 - h_3$. Then $b_{1,4} = b_{2,4} - 3b_{1,3} - 3 < 2h_3 - 5$ which is negative for $h_3 \leq 2$. There are no Betti table for $b_{2,3} = 1$.

Subcase $b_{2,3} = 2$:

The number of cubic generators in the ideal is $b_{1,3} = 10 - 3 \cdot 3 - h_3 + 2 = 3 - h_3$ and the number of quartic generators is $b_{1,4} = 15 - 3 - 6 \cdot 3 - 3b_{1,3} + b_{2,4} + 3 \cdot 2 = b_{2,4} - 3b_{1,3}$. In this case the Betti table has the following shape:

$$\begin{array}{lcccc}
0: & 1 & - & - & - \\
1: & - & 3 & 2 & - \\
2: & - & 3 - h_3 & b_{2,4} & - \\
3: & - & b_{2,4} - 3b_{1,3} & b_{2,4} - 3b_{1,3} & - \\
4: & - & b_{2,4} & 3 - h_3 & - \\
5: & - & 2 & 3 & - \\
6: & - & - & - & 1
\end{array}$$

We see that $h_3 \leq 3$. Let $h_3 = 3$. Then $b_{1,3} = 0$ and $b_{2,4} = 0$ by Lemma 3.0.2. We have the Betti table in Figure 3.3 and the corresponding Hilbert function is $(1, 3, 3, 3, 3, 3, 1)$.

If we let $h_3 \leq 2$ we get, by Lemma 3.0.2, that $b_{2,4} \leq 4 - h_3$. Then $b_{1,4} = b_{2,4} - 3b_{1,3} < 2h_3 - 5$ which is negative for $h_3 \leq 2$.

Case $b_{1,2} = 4$:

We still have $b_{2,3} \leq 3$. The number of cubic forms in the ideal is $b_{1,3} = 10 - h_3 - 4 \cdot 3 + b_{2,3} = b_{2,3} - 2 - h_3$. We have $h_2 = 6 - b_{1,2} = 2$. From Lemma 3.0.3 we know that $h_3 \leq 2$ for the sequence $(1,3,2,h_3,2,3,1)$. If $h_3 = 2$ we get that $b_{1,3} = b_{2,3} - 4$, so we must have $b_{2,3} \geq 4$ for $b_{1,3}$ to be positive. Since $b_{2,3} \leq 3$, this doesn't work. If $h_3 = 1$ we get $b_{1,3} = b_{2,3} - 3$. We check what happens when $b_{2,3} = 3$. The number of quartic generators is $15 - 2 - 6 \cdot 4 + 3 \cdot 3 + b_{2,4} = b_{2,4} - 2$, so $b_{2,4} \geq 2$. From Lemma 3.0.2 we get that $b_{2,4} = 0$. This doesn't work. The same happens for $h_3 = 0$. There are no Betti tables with $b_{1,2} = 4$.

Case $b_{1,2} = 5$**:**

We still have $b_{2,3} \leq 3$. The number of cubic generators in the ideal is $b_{1,3} = 10 - h_3 - 5 \cdot 3 + b_{2,3} = b_{2,3} - 5 - h_3$. We have $h_2 = 6 - b_{1,2} = 1$. From Lemma 3.0.3 we know that $h_3 \leq 1$ for the sequence $(1,3,1,h_3,1,3,1)$. We get that $b_{1,3}$ is negative for $b_{2,3} \leq 3$ and $h_3 \leq 1$. There are no Betti tables with $b_{1,2} = 5$.

Case $b_{1,2} = 6$**:**

We still have $b_{2,3} \leq 3$. The number of cubic generators in the ideal is $b_{1,3} = 10 - h_3 - 6 \cdot 3 + b_{2,3} = b_{2,3} - 8 - h_3$. We have $h_2 = 6 - b_{1,2} = 0$. From Lemma 3.0.3 we know that $h_3 = 0$ for the sequence $(1,3,0,h_3,0,3,1)$. We get that $b_{1,3}$ is negative for $b_{2,3} \leq 3$ and $h_3 = 0$. There are no Betti tables with $b_{1,2} = 6$. ■

4 | Grassmannians and Skew Symmetric Matrices

In this chapter we will use *Grassmannians* to find isotropic subspaces to some skew symmetric matrices. We will see in Chapter 5, that the isotropic subspaces correspond to finite schemes Γ such that $I_\Gamma \subset F^\perp$.

This chapter is organized as follows. First, in Section 4.1, we introduce the concept of Grassmannians. Then, in Section 4.2, we use the Grassmannian $G(2,4)$ to find 2-dimensional isotropic subspaces to a skew symmetric matrix of dimension 4. Then, in Section 4.3 we introduce the *Chow ring* of a Grassmannian and the *Chern classes* of vector bundles of a Grassmannian and explain how these concepts are related to isotropic subspaces of skew symmetric matrices. Finally, in Section 4.4, we use the Chern classes related to the Grassmannians $G(3,6)$ and $G(4,9)$ to find 3- and 4-dimensional isotropic subspaces to skew symmetric matrix of dimension 6 and 9, respectively.

4.1 Grassmannians

In projective space $\mathbb{P}^n$ we have coordinates $(x_0 : \cdots : x_n)$, where $(x_0 : \cdots : x_n) = \lambda(x_0 : \cdots : x_n)$ for $\lambda \in \mathbb{C}^*$. A linear subspace of $\mathbb{P}^n$ is defined as the set of points satisfying a set of linear equations. When these points satisfy $n-d$ linear independent equations, we say that the linear space is d-dimensional. The set of all d-dimensional linear spaces in $\mathbb{P}^n$ is called the *Grassmannian* of d-planes in n-space and denoted $\mathbb{G}(d,n)$. The set of $(d+1)$-dimensional linear subspaces of a $(n+1)$-dimensional vector space is equivalent to $\mathbb{G}(d,n)$ and is denoted $G(d+1,n+1)$. We will also use the notation $G(d+1,V)$, where V is a given vector space.

We will now show that $\mathbb{G}(d,n)$ can be represented by a certain smooth subvariety of $\mathbb{P}^N$, where $N = \binom{n+1}{d+1} - 1$. Let L be a d-plane in $\mathbb{P}^n$ and pick $d+1$ points $x_i = (x_{01} : \cdots : x_{ni})$ that span L and form the $(d+1)(n+1)$ matrix

$$\begin{pmatrix} x_{00} & \cdots & x_{0n} \\ \vdots & \ddots & \vdots \\ x_{d0} & \cdots & x_{dn} \end{pmatrix}.$$

Pick $(d+1)$ integers $j_0, \ldots, j_d$, where $0 \le j_0 < \cdots < j_d \le n$, and let $p_{j_0 \cdots j_d}$ be the $(d+1) \times (d+1)$-minor of the submatrix consisting of the j_ith columns for $i = 0, \ldots, d$. There are $\binom{n+1}{d+1}$ choices of picking the j_is. Since the points x_i are assumed to span L, at least one of $p_{j_0 \cdots j_d}$ has to be different from zero. In this way, the $p_{j_0 \cdots j_d}$ defines a point $(\cdots : p_{j_0 \cdots j_d} : \ldots)$ in $\mathbb{P}^N$, where we order the coordinates by lexicographic ordering. The $p_{j_0 \cdots j_d}$s are called the Plücker coordinates of L.

Example 4.1.1. Let $d = 1$ and $n = 3$. Then $X = G(2,4)$ is the Grassmannian of lines in $\mathbb{P}^3$. Given a line L and two points $x = (x_0 : x_1 : x_2 : x_3)$ and $x' = (x'_0 : x'_1 : x'_2 : x'_3)$ on L, the Plücker coordinates of L is the 2×2-minors of

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_0' & x_1' & x_2' & x_3' \end{pmatrix}.$$

We get

$$\begin{aligned} p_{01} &= x_0x_1' - x_0'x_1 & p_{12} &= x_1x_2' - x_1'x_2 \\ p_{02} &= x_0x_2' - x_0'x_2 & p_{13} &= x_1x_3' - x_1'x_3 \\ p_{03} &= x_0x_3' - x_0'x_3 & p_{23} &= x_2x_3' - x_2'x_3 \end{aligned}$$

By computation, we have the that Plücker coordinates of L satisfy the following relation:

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0. \tag{4.1}$$

We want to show that the points in $G(2,4)$ are exactly the points in $\mathbb{P}^5$ that fulfill the relation in 4.1. Indeed, let $p = (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) \in \mathbb{P}^5$ be a point such that $p_{01}p_{23} - p_{02}p_{13} + p_{12}p_{03} = 0$. Since the p_{ij}s are projective coordinates, one of them has to be non-zero. Assume $p_{01} = 1$. We want to show that the p_{ij}s are Plücker coordinates of a line $L \in \mathbb{P}^3$, and we claim that L is the line spanned by $(1 : 0 : -p_{12} : -p_{13})$ and $(0 : 1 : p_{02} : p_{03})$. Indeed, the minors of

$$A = \begin{pmatrix} 1 & 0 & -p_{12} & -p_{13} \\ 0 & 1 & p_{02} & p_{03} \end{pmatrix}$$

gives that the Plücker coordinates of L is $(1 : p_{02} : p_{03} : p_{12} : p_{13} : -p_{12}p_{03} + p_{02}p_{13})$. Since $-p_{12}p_{03} + p_{02}p_{13} = p_{01}p_{23}$ and $p_{01} = 1$ by assumption, the Plücker coordinates can be written $(p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23})$, thus the p_{ij}s are Plücker coordinates of L, which was what we wanted to show.

We will show that a similar relation as 4.1 between the Plücker coordinates of a d-plane in $\mathbb{P}^n$ holds in general.

Proposition 4.1.2. *Let $0 \le j_0 < \cdots < j_{d-1} \le n$ and $0 \le k_0 < \cdots < k_{d+1} \le n$ be two sequences of integers. Then*

$$\sum_{\lambda=0}^{d+1} (-1)^{\lambda} p_{j_0 \ldots j_{d-1} k_\lambda} p_{k_0 \ldots k_\lambda^* \ldots k_{d+1}} = 0,$$

where k_λ^* means that k_λ is not in the sequence $k_0, \ldots, k_{d+1}$.

Proof. First we write the relation we are going to prove in terms of determinants.

$$\sum_{\lambda=0}^{d+1} (-1)^{\lambda} \begin{vmatrix} x_{0,j_0} & \cdots & x_{0,j_{d-1}} & x_{0,k_\lambda} \\ \vdots & \vdots & \vdots & \vdots \\ x_{i,j_0} & \cdots & x_{i,j_{d-1}} & x_{i,k_\lambda} \\ \vdots & \vdots & \vdots & \vdots \\ x_{d,j_0} & \cdots & x_{d,j_{d-1}} & x_{d,k_\lambda} \end{vmatrix} \begin{vmatrix} x_{0,k_0} & \cdots & x_{0,k_\lambda}^* & \cdots & x_{0,k_{d+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{i,k_0} & \cdots & x_{i,k_\lambda}^* & \cdots & x_{i,k_{d+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{d,k_0} & \cdots & x_{d,k_\lambda}^* & \cdots & x_{d,k_{d+1}} \end{vmatrix} = 0$$

We now expand the first determinants along the last column and get

$$\sum_{\lambda=0}^{d+1} (-1)^{\lambda} \left[\sum_{i=0}^{d} (-1)^{d+i} \begin{vmatrix} x_{0,j_0} & \cdots & x_{0,j_d} \\ \vdots & \vdots & \vdots \\ x_{i,j_0}^* & \cdots & x_{i,j_d}^* \\ \vdots & \vdots & \vdots \\ x_{d,j_0} & \cdots & x_{d,j_d} \end{vmatrix} x_{i,k_\lambda} \right] \begin{vmatrix} x_{0,k_0} & \cdots & x_{0,k_\lambda}^* & \cdots & x_{0,k_{d+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{i,k_0} & \cdots & x_{i,k_\lambda}^* & \cdots & x_{i,k_{d+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{d,k_0} & \cdots & x_{d,k_\lambda}^* & \cdots & x_{d,k_{d+1}} \end{vmatrix} = 0$$

We rearrange the terms and get the relation

$$\sum_{i=0}^{d}(-1)^{d+i}\begin{vmatrix} x_{0,j_0} & \dots & x_{0,j_d} \\ \vdots & \vdots & \vdots \\ x^*_{i,j_0} & \dots & x^*_{i,j_d} \\ \vdots & \vdots & \vdots \\ x_{d,j_0} & \dots & x_{d,j_d} \end{vmatrix}\left[\sum_{\lambda=0}^{d+1}(-1)^{\lambda}x_{i,k_\lambda}\begin{vmatrix} x_{0,k_0} & \dots & x^*_{0,k_\lambda} & \dots & x_{0,k_{d+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{i,k_0} & \dots & x^*_{i,k_\lambda} & \dots & x_{i,k_{d+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{d,k_0} & \dots & x^*_{d,k_\lambda} & \dots & x_{d,k_{d+1}} \end{vmatrix}\right] = 0$$

This relation is obtained from expanding the second determinant in the following relation along the first row.

$$\sum_{i=0}^{d}(-1)^{d+i}\begin{vmatrix} x_{0,j_0} & \dots & x_{0,j_d} \\ \vdots & \vdots & \vdots \\ x^*_{i,j_0} & \dots & x^*_{i,j_d} \\ \vdots & \vdots & \vdots \\ x_{d,j_0} & \dots & x_{d,j_d} \end{vmatrix}\begin{vmatrix} x_{i,k_0} & \dots & x_{i,k_\lambda} & \dots & x_{i,k_{d+1}} \\ x_{0,k_0} & \dots & x_{0,k_\lambda} & \dots & x_{0,k_{d+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{i,k_0} & \dots & x_{i,k_\lambda} & \dots & x_{i,k_{d+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{d,k_0} & \dots & x_{d,k_\lambda} & \dots & x_{d,k_{d+1}} \end{vmatrix} = 0.$$

The second determinant is zero since two rows are equal, so we are done. ■

Conversely, every point in $\mathbb{P}^N$ that fulfill the relation in Proposition 4.1.2 corresponds to a d-plane in $\mathbb{P}^n$, [KL72, Theorem 1]. We now show that $\mathbb{G}(d,n)$ is smooth. In the same way as we did in Example 4.1.1 we can assume that for at least one choice of $j_0,\dots,j_d$ we have $p_{j_0\dots j_d} = 1$. To simplify computation, we assume that $p_{01\dots d} = 1$. This means that the submatrix given by the columns $0,1,\dots,d$ is the $(d+1)(d+1)$-identity matrix. Every d-plane in $\mathbb{P}^n$ that has $p_{01\dots d} = 1$ can be represented by a matrix of the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 & x_{0,d+1} & \dots & x_{0n} \\ 0 & 1 & 0 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & x_{d,d+1} & \dots & x_{dn} \end{pmatrix}.$$

The set of all such matrices corresponds to an affine space of dimension $(d+1)(n-d)$, [KL72, Proposition 2]. Since this holds for any choice of $j_0,\dots,j_d$ we have that $\mathbb{G}(d,n)$ represented as a subvariety of $\mathbb{P}^N$ is covered by $(N+1)$ copies of the affine space of dimension $(d+1)(n-d)$. This shows that $\mathbb{G}(d,n)$ is a smooth variety of dimension $(d+1)(n-d)$.

4.2 Skew Symmetric Matrices of Dimension 4

In this section we will prove a theorem about 2-dimensional isotropic subspaces of 4×4 skew symmetric matrices. First, we introduce some properties of a skew symmetric matrix and how such a matrix can be related to $G(2,4)$. Then we state and prove our the theorem in this section. Lastly, we discuss some geometric interpretation of our results.

Let A be an $n\times n$ skew symmetric matrix. Recall that the *pfaffian* of A, $\mathrm{Pf}(A)$ is the square root of the determinant of A, that is $(\mathrm{Pf}(A))^2 = \det A$. If n is odd, then $\det A = 0$, by Lemma 2.3.1, thus $\mathrm{Pf}(A) = 0$. When n is even, we get that $\mathrm{Pf}(A)$ is a polynomial in the entries of A. We observe that if $n = 2r$, then the pfaffian has degree r. A minor obtained from a submatrix where the indices of the rows and columns are the same, are called a *principal minor*. The even submatrices that form the principal minors of a skew symmetric matrix will again be skew symmetric and the principal minors will therefore also be the square of a pfaffian. By the *order of a minor* we mean the dimension of the corresponding submatrix.

Now, we give two lemmas about skew symmetric matrices that we will use to prove a correspondence between skew symmetric matrices of rank 2 ad points in $G(2,n)$.

Lemma 4.2.1. *[Hey69, Equation (3.42)-(3.45)] Let A be an $n \times n$ skew symmetric matrix. Then*

- *a minor of even order $2r$ is a quadratic form in pfaffians of order r, and*
- *a minor of odd order $2r-1$ is a quadratic form in pfaffians of order r and $r-1$.*

Lemma 4.2.2. *Let A be an $n \times n$ skew symmetric matrix. If all the pfaffians of order r vanishes, then the matrix has rank at most $2r-2$.*

Proof. Let A be $n \times n$ skew symmetric matrix and assume that all the pfaffians of order r vanishes. Then, by Lemma 4.2.1, the $2r \times 2r$ minors and the $(2r-1) \times (2r-1)$ minors vanishes. Since all the $(2r-1) \times (2r-1)$ minors vanishes, A has rank at most $2r-2$. ∎

The space of $n \times n$ skew symmetric matrices are in correspondence with the space of the Plücker coordinates of $G(2,n)$. That is, a matrix can be identified with a point in $\mathbb{P}^N$, where $N = \binom{n}{2} = \frac{n(n-1)}{2}$. Indeed, counting the entries in the lower triangle of an $n \times n$ skew symmetric matrix gives $\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}$ entries. Further, we have that the subspace of the $n \times n$ skew symmetric matrices consisting of rank 2 matrices corresponds to points on $G(2,n)$. We show this first for 4×4 skew symmetric matrices and then for a general n.

Lemma 4.2.3. *The 4×4 skew symmetric matrices of rank 2 are in $1-1$ correspondence with points in $G(2,4)$.*

Proof. For the first implication, let L be a line in $\mathbb{P}^3$ and $P = (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23})$ the corresponding point in $\mathbb{P}^5$. The Plücker coordinates fulfill the relation $p_{01}p_{23} - p_{02}p_{13} + p_{12}p_{03} = 0$. If we set

$$A = \begin{pmatrix} 0 & p_{01} & p_{02} & p_{03} \\ -p_{01} & 0 & p_{12} & p_{13} \\ -p_{02} & -p_{12} & 0 & p_{23} \\ -p_{03} & -p_{13} & -p_{23} & 0 \end{pmatrix},$$

we get that

$$\det A = (p_{01}p_{23} - p_{02}p_{13} + p_{12}p_{03})^2.$$

Since

$$p_{01}p_{23} - p_{02}p_{13} + p_{12}p_{03} = 0,$$

we have $\det A = 0$. Since $\det A = 0$, $\operatorname{rank} A \leq 3$, but since all the principle 3×3 minors are skew symmetric, they are zero by Lemma 2.3.1. Therefore, $\operatorname{rank} A \leq 2$. Since none of the 2×2 principle minors of A are zero, $\operatorname{rank} A = 2$.

For the other implication, let $p = (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) \in \mathbb{P}^5$, and let A be the corresponding rank 2 skew symmetric matrix. This means that $\det A = 0$, thus the entries fulfill the relation $p_{01}p_{23} - p_{02}p_{13} + p_{12}p_{03} = 0$. Since the p_{ij}s are projective coordinates, one of them has to be non zero. Assume $p_{01} = 1$. We want to show that the p_{ij}s are Plücker coordinates of a line $L \in \mathbb{P}^3$, and we claim that L is the line spanned by $(1 : 0 : -p_{12} : -p_{13})$ and $(0 : 1 : p_{02} : p_{03})$. Indeed, the minors of

$$A = \begin{pmatrix} 1 & 0 & -p_{12} & -p_{13} \\ 0 & 1 & p_{02} & p_{03} \end{pmatrix}$$

gives that the Plücker coordinates of L is $(1 : p_{02} : p_{03} : p_{12} : p_{13} : -p_{12}p_{03} + p_{02}p_{13})$. Since $-p_{12}p_{03} + p_{02}p_{13} = p_{01}p_{23}$ and $p_{01} = 1$, the Plücker coordinates can be written $(p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23})$, thus the p_{ij}s are Plücker coordinates of L, which was what we wanted to show. ∎

Now, we show the correspondence for a general n. We have from Lemma 4.2.2 that if the all the pfaffians of order 2 of an $n \times n$ skew symmetric matrix vanishes, then the matrix has rank at most 2. On the other hand, if A has rank at most 2, then all the pfaffians of order 2 vanishes. Since the square of a pfaffian of order 2 is the determinant of a 4×4 skew symmetric submatrix of A, we will get a relation between the Plücker coordinates for each pfaffian of order 2. In summary, we will get the relations in Proposition 4.1.2. Thus an $n \times n$ skew symmetric matrix of rank 2 will indeed correspond to a point in $G(2, n)$. Now, we introduce the notion of an isotropic subspace and prove a lemma that we will use in the proof of our theorem.

Definition 4.2.4. Let A be an $n \times n$ skew symmetric matrix. An *isotropic* subspace to A is a subspace of $\mathbb{C}^n$, such that for every $u, v \in U$ we have $uAv^T = 0$. If V is a space of $n \times n$ skew symmetric matrices and U is isotropic to every matrix in V, we say that U is an isotropic subspace to V.

Lemma 4.2.5. *Let A be a 4×4 skew symmetric matrix and H_A be the set of all 2-dimensional isotropic subspaces to A. Then H_A defines a hyperplane in $\mathbb{P}^5$ that intersects $G(2, 4)$.*

Proof. First we prove that given an $U \in H_A$, then U corresponds to a point in $\mathbb{P}^5$ that lies on a hyperplane that intersects $G(2, 4)$. Let $u = (u_0, u_1, u_2, u_3)$ and $v = (v_0, v_1, v_2, v_3)$, where $u, v \in U$. Let

$$A = \begin{pmatrix} 0 & a_{01} & a_{02} & a_{03} \\ -a_{01} & 0 & a_{12} & a_{13} \\ -a_{02} & -a_{12} & 0 & a_{23} \\ -a_{03} & -a_{13} & -a_{23} & 0 \end{pmatrix}.$$

We get that

$$uAv^T = a_{01}p_{01} + a_{02}p_{02} + a_{03}p_{03} + a_{12}p_{12} + a_{13}p_{13} + a_{23}p_{23}, \tag{4.2}$$

where p_{ij} are the Plücker coordinates of the line through u and v when considered as points in $\mathbb{P}^3$. When $uAv^T = 0$, we get an equation in the Plücker coordinates, i.e. a hyperplane in $\mathbb{P}^5$ that intersects $G(2, 4)$. We need to show that given $s, t \in U$, we get the same equation. Indeed, since U is 2-dimensional, $s = au + bv$ and $t = cu + dv$. We get that

$$\begin{aligned} sAt^T &= (au + bv)A(cu + dv)^T, \\ &= aduAv^T + bcvAu^T, \\ &= (ad - bc)uAv^T, \end{aligned}$$

where we have used the linearity and that $uAv^T = -vAu^T$. Thus, we get the same equation with different choices of representatives for U.

Now, we prove that given a point in $\mathbb{P}^5$ that satisfies Equation (4.2), we can find a corresponding $U \in H_A$. Let $p = (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) \in \mathbb{P}^5$ and assume that p satisfies Equation (4.2). Let A be the skew symmetric matrix with entries given by the a_{ij}s in Equation (4.2). Since the p_{ij}s are projective coordinates, one of them has to be non-zero. Assume $p_{01} = 1$. We want to show that $u = (1, 0, -p_{12}, -p_{13})$ and $v = (0, 1, p_{02}, p_{03})$ generates an U. From the proof of Lemma 4.2.9 we have that the Plücker coordinates of the line through u and v is given by the coefficients of p. We need to show that $uAv^T = 0$. Indeed, we get that

$$\begin{aligned} uAv^T &= a_{01} + a_{02}p_{02} + a_{03}p_{03} + a_{12}p_{12} + a_{13}p_{13} + a_{23}(p_{02}p_{13} - p_{03}p_{12}) \\ &= a_{01} + a_{02}p_{02} + a_{03}p_{03} + a_{12}p_{12} + a_{13}p_{13} + a_{23}p_{01}p_{23} \\ &= 0, \end{aligned}$$

where we have used that $p_{01} = 1$, that the Plücker coordinates satisfies the relation $p_{01}p_{23} = p_{02}p_{13} - p_{03}p_{12}$, and the we assumed that Equation (4.2) was satisfied. ∎

We are now ready to state and prove the theorem of this section.

Theorem 4.2.6. *Let W be a 3-dimensional vector space of 4×4 skew symmetric matrices. Then there exists a conic of 2-dimensional isotropic subspaces to W.*

Proof. We want to prove that there is a common 2-dimensional isotropic subspace for a basis of matrices in W. Let A_{H_1},A_{H_2} and A_{H_3} be such a basis and let H_1, H_2 and H_3 be the corresponding hypersurfaces in $\mathbb{P}^5$ we get from Lemma 4.2.5. The basis matrices A_{H_1},A_{H_2} and A_{H_3} has a common isotropic subspace if the intersection

$$G(2,4) \cap H_1 \cap H_2 \cap H_3$$

is non-empty, which is the case. Indeed, $G(2,4)$ is a 4-dimensional quadric, and the intersection with each H_i reduces the dimension by 1. We get that we have a conic of common 2-dimensional isotropic subspaces for V. ∎

When we have an isotropic subspace of a 4×4 skew symmetric matrix, we can choose a row basis such that $a_{23} = 0$. We prove this in the following lemma.

Lemma 4.2.7. *Let W be a vector space of 4×4 skew symmetric matrices and let U be an isotropic subspace for W. Then we can choose a row basis for W such that $a_{23} = 0$ for all matrices in W.*

Proof. Let the row basis be such that for $u, v \in U$ we have $u = (0,0,1,0)$ and $v = (0,0,0,1)$. Then the Plücker coordinates of L through u and v are zero for $i \neq 2, j \neq 3$, and $p_{23} = 1$. If $uAv^T = 0$, then $a_{23} = 0$, by Equation (4.2). ∎

4.2.1 Geometric Interpretation

We will now use the correspondence between 4×4 skew symmetric matrices of rank 2 and lines in $\mathbb{P}^3$ to get a geometric interpretation of 2-dimensional isotropic subspaces to a vector space of 4×4 skew symmetric matrices.

Lemma 4.2.8. *Let W be a general 3-dimensional vector space of 4×4 skew symmetric matrices. Then there is a basis for W given by rank 2 matrices.*

Proof. Since W is a 3-dimensional vector space, W is spanned by three points in $\mathbb{P}^5$, by Lemma 4.2.3, which forms a $\mathbb{P}^2 \subset \mathbb{P}^5$. Let W' be the space of 4×4 skew symmetric matrices of rank 2. If $W \subset W'$ the lemma is obviously true. Assume $W \not\subset W'$. As shown in Lemma 4.2.9, the entries of a matrix $A \in W'$ satisfies the Plücker relation

$$p_{01}p_{23} - p_{02}p_{13} + p_{12}p_{03} = 0,$$

so the points in $\mathbb{P}^5$ that corresponds to a matrix of rank 2 lies on the quadric $G(2,4) \subset \mathbb{P}^5$. The skew symmetric matrices of rank 2 in W lies in the intersection of $G(2,4)$ and $\mathbb{P}^2$, which is a conic or two lines when W is general. Since three points on a conic or two lines span a $\mathbb{P}^2$, we can choose a basis for W given by three rank 2 matrices. ∎

When W is not general, the intersection between $G(2,4)$ and $\mathbb{P}^2$ in the proof of Lemma 4.2.8 might be a double line. Then we will not get a basis of rank 2 matrices.

Now, we can show that for a general W, the problem of finding an isotropic subspace to W is equivalent to finding a line that intersect the three lines i $\mathbb{P}^3$ that corresponds to three basis matrices for W of rank 2. We state a lemma that we will use to describe this equivalence.

Lemma 4.2.9. *Two lines in $\mathbb{P}^3$ intersect if and only if*

$$p_{01}q_{23} - p_{02}q_{13} + p_{03}q_{12} + p_{12}q_{03} - p_{13}q_{02} + p_{23}q_{01} = 0, \tag{4.3}$$

where p_{ij}, q_{ij} are the Plücker coordinates of the two lines.

Proof. Assume $L_1, L_2 \in \mathbb{P}^3$ and $x_1, x_1' \in L_1$ and $x_2, x_2' \in L_2$. Considering the points as vectors in $\mathbb{C}^4$, they span a plane if and only if the lines intersect. Then the two lines intersect if and only if the determinant of the following matrix is zero:

$$\begin{pmatrix} x_{10} & x_{11} & x_{12} & x_{13} \\ x'_{10} & x'_{11} & x'_{12} & x'_{13} \\ x_{20} & x_{21} & x_{22} & x_{23} \\ x'_{20} & x'_{21} & x'_{22} & x'_{23} \end{pmatrix},$$

where the rows are the coordinates of x_1, x_1', x_2, x_2'. The determinant in terms of the Plücker coordinates p_{ij} of L_1 and q_{ij} of L_2 is $p_{01}q_{23} - p_{02}q_{13} + p_{03}q_{12} + p_{12}q_{03} - p_{13}q_{02} + p_{23}q_{01}$. ∎

From the lemma we get that when A is of rank 2, the points in hyperplane of 2-dimensional isotropic subspaces from Lemma 4.2.5 corresponds to lines intersecting the line corresponding to A. Indeed, let

$$A = \begin{pmatrix} 0 & q_{01} & q_{02} & q_{03} \\ -q_{01} & 0 & q_{12} & q_{13} \\ -q_{02} & -q_{12} & 0 & q_{23} \\ -q_{03} & -q_{13} & -q_{23} & 0 \end{pmatrix}.$$

We have from Lemma 4.2.5 that H_A consists of the points $p \in \mathbb{P}^5$ such that

$$q_{01}p_{01} + q_{02}p_{02} + q_{03}p_{03} + q_{12}p_{12} + q_{13}p_{13} + q_{23}p_{23} = 0 \tag{4.4}$$

Let $p = (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23})$ be a point that satisfies 4.4. Let $r = (r_{01} : r_{02} : r_{03} : r_{12} : r_{13} : r_{23})$, where

$$\begin{aligned} r_{01} &= p_{23}, & r_{12} &= p_{03}, \\ r_{02} &= -p_{13}, & r_{13} &= -p_{02}, \\ r_{03} &= p_{12}, & r_{23} &= p_{01}. \end{aligned}$$

First, we have that r satisfies the Plücker relation. Indeed

$$\begin{aligned} r_{01}r_{23} - r_{02}r_{13} + r_{03}r_{12} &= p_{23}p_{01} - (-p_{13})(-p_{02}) + p_{12}p_{03} \\ &= p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} \\ &= 0. \end{aligned}$$

Then, we have that r corresponds to a line that intersect the line that corresponds to A. Indeed,

$$\begin{gathered} r_{01}q_{23} - r_{02}q_{13} + r_{03}q_{12} + r_{12}q_{03} - r_{13}q_{02} + r_{23}q_{01} = \\ p_{23}q_{23} - (-p_{13})q_{13} + p_{12}q_{12} + p_{03}q_{03} - (-p_{02})q_{02} + p_{01}q_{01} = \\ q_{01}p_{01} + q_{02}p_{02} + q_{03}p_{03} + q_{12}p_{12} + q_{13}p_{13} + q_{23}p_{23} = 0. \end{gathered}$$

This shows that a rank 2 matrix A has an isotropic subspace if and only there exits a line that intersect the corresponding line. To have a common isotropic subspace for three basis matrices then implies that there must exists a line that intersects all three lines.

Now, we show that there indeed exists a pencil of lines that intersects L_1, L_2 and L_3. Indeed, given L_1, L_2 and L_3, take any point p on L_1 and consider the plane spanned by p and L_2. This plane will intersect L_3. Let L be the line through p in this plane such that L intersect L_3. Since we can choose any point on L_1, there is a pencil of lines L that intersect the three lines.

The conic of 4×4 skew symmetric matrices of rank 2 and the conic of 2-dimensional isotropic subspaces are related. The points on each conic corresponds to the two families of lines on the quadric in $\mathbb{P}^3$. We show this below, but first we need a lemma.

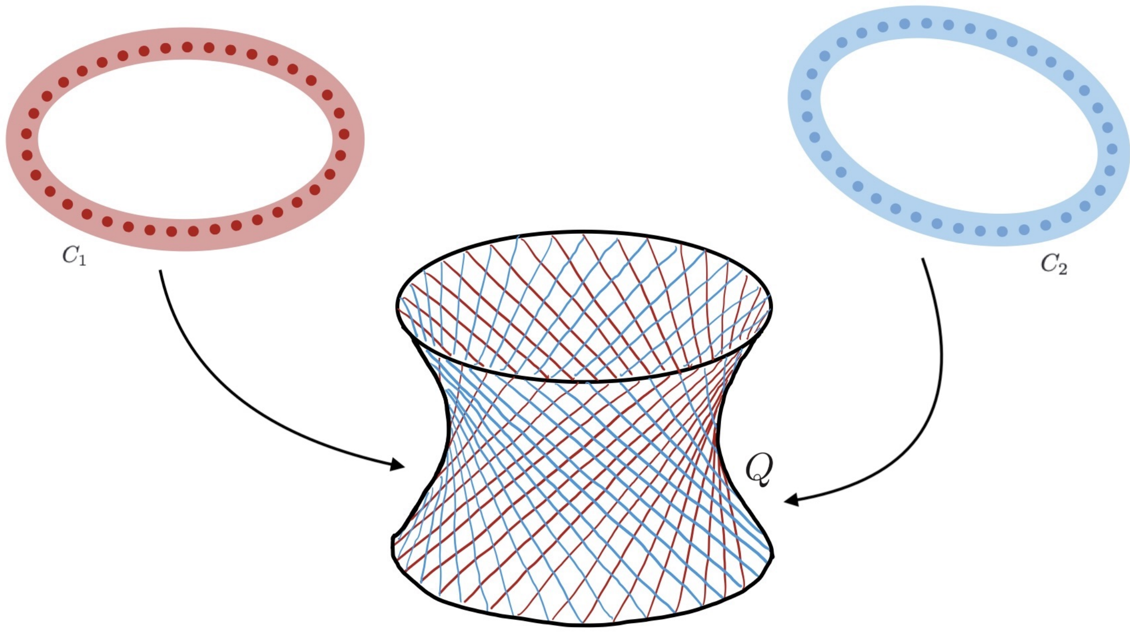


Figure 4.1: The points of each conic correspond to lines on the quadric in $\mathbb{P}^3$.

Lemma 4.2.10. *Let C_1 be a conic in $G(2,4) \subset \mathbb{P}^5$, $p_1, p_2 \in C_1$, and L_1 and L_2 the lines in $\mathbb{P}^3$ corresponding to p_1 and p_2. Then $L_1 \cap L_2 = \emptyset$.*

Proof. Assume for contradiction that $L_1 \cap L_2 \neq \emptyset$ and let $P \in \mathbb{P}^3$ be the intersection point. The lines through P that lie in the plane spanned by L_1 and L_2 corresponds to points on the line in $G(2,4)$ through p_1 and p_2. Since C_1 does not contain the line through p_1 and p_2, we have a contradiction. ■

Let $C_1 \in \mathbb{P}^5$ be the conic of skew symmetric matrices of rank 2 from the proof of Lemma 4.2.8. Let Q be the union of the lines in $\mathbb{P}^3$ that correspond to points on C_1. Then Q is a surface in $\mathbb{P}^3$. We want to show that the degree of Q is two. Indeed, let $L \in \mathbb{P}^3$ be a general line that intersects Q. We want to find the number of intersection points, i. e. the number of lines on Q that intersect L. From Lemma 4.2.9, we have that to lines intersect if and only if their Plücker coordinates satisfy Equation (4.3), i.e. the lines that intersect L correspond to points in a hyperplane in $\mathbb{P}^5$. The number of lines on Q that intersect L is equal to the number of intersection points between the hyperplane and the conic in $\mathbb{P}^5$, which is two.

Now, let C_2 be the conic that parameterize the lines intersecting every line parameterized by C_1. By the same arguments as above, the union of the lines parameterized by C_2 is a degree two surface in $\mathbb{P}^3$. We want to show that this is Q. Indeed, since no two lines parameterized by C_1 intersect, but are intersecting all the lines parameterized by C_2, the two family of lines have to lie on the same surface. We get that Q has degree two and is the quadric in $\mathbb{P}^3$ where one family of lines corresponds to lines parameterized by C_1 and the other family corresponds to lines parameterized by C_2, see Figure 4.1.

4.3 More on Grassmannians and Skew Symmetric Matrices

We will prove that there exists similar isotropic subspaces for skew symmetric matrices of dimension 6 and 9. In these dimensions, there are difficult to compute directly as we did for skew symmetric matrices of dimension 4. We therefore need more theory about Grassmannians and skew symmetric matrices. First, in Section 4.3.1, we introduce the *Chow ring* of a Grassmannian and the *Schubert cycles*. Then, in Section 4.3.2, we introduce the *Chern classes* of some *vector bundles* on the Grassmannian. In Section 4.3.3, generalize the relation between isotropic subspaces and a Grassmannian. Lastly, in Section 4.3.4 we do computations on the Chern classes that we will use to prove our theorems in the last section of this chapter.

4.3.1 Chow Ring of a Grassmannian

In this section, we will describe the *Chow ring* of a Grassmannian, following [Ful98] and [Eis16]. We start with the general definition of the Chow ring. Let X be an algebraic variety of dimension n. A *k-cycle* is a finite formal sum

$$\sum_{Y_i} n_i Y_i,$$

where $n_i \in \mathbb{Z}$ and Y_i is a k-dimensional subvariety of X. The group of k-cycles of a variety X is denoted $Z_k(X)$ and the *group of cycles* on X is $Z(X) = \bigoplus Z_k(X)$. Two cycles Y_1 and Y_2 are *rationally equivalent* if there exists a cycle on $\mathbb{P}^1 \times X$ whose restrictions to two fibers $t_0 \times X$ and $t_1 \times X$ is Y_0 and Y_1. The cycles that are rationally equivalent form a subgroup of $Z_k(X)$ denoted $\operatorname{Rat}_k(X)$, and we can form the quotient group $A_k(X) = Z_k(X)/\operatorname{Rat}_k(X)$ of k-cycles modulo rational equivalence. We call $A_*(X) = \bigoplus A_k(X)$ the *Chow group* of X. We denote the equivalence class of a subvariety $Y \subset X$ as $[Y] \in A_*(X)$.

For a smooth variety X there also exists a product on $A_*(X)$ which in special cases corresponds to taking the intersection of two subvarieties of X. Let $Y_1, Y_2 \subset X$ be subvarieties of a smooth variety X such that every irreducible component Z of the intersection $Y_1 \cap Y_2$ satisfies $\operatorname{codim} Z = \operatorname{codim} Y_1 + \operatorname{codim} Y_2$. If Y_1 and Y_2 intersect transversely, then $[Y_1 \cap Y_2] = \sum [Z]$. Generally, for each such component Z there is a positive integer $m_Z(Y_1, Y_2)$ called the *intersection multiplicity* of Y_1 and Y_2 along Z, such that

$$[Y_1 \cap Y_2] = \sum m_Z(Y_1, Y_2)][Z].$$

Theorem 4.3.1. *[Ful98, Proposition 8.1.1] Let X be a smooth variety of dimension n and let $Y_1, Y_2 \subset X$ be subvarieties of X. There is a unique product structure on $A_*(X)$, i.e. for $[Y_1] \in A_{n-k_1}(X)$ and $[Y_2] \in A_{n-k_2}(X)$, then $[Y_1][Y_2] \in A_{n-k_1-k_2}(X)$. If every irreducible component Z of the intersection $Y_1 \cap Y_2$ has codimension* $\operatorname{codim} Z = \operatorname{codim} Y_1 + \operatorname{codim} Y_2$*, then*

$$[Y_1][Y_2] = [Y_1 \cap Y_2].$$

The product makes $A_*(X)$ into an associative, commutative ring, called the Chow ring *of X.*

Now, we describe the Chow ring of a Grassmannian. First, we note that the Grassmannian is indeed a smooth variety as seen in Section 4.1. Let V be an $(n+1)$-dimensional vector space. We will describe the Chow ring of the Grassmannian $X = G(d+1, V)$ by describing a special kind of subvarieties whose equivalence classes generate $A_*(X)$ under addition. Let $W \subset V$ be a $(d+1)$-dimensional vector space, and let $[W]$ be the corresponding point in X. Fix a *flag* $U_1 \subset \cdots \subset U_{n+1} = V$ of vector spaces, where $\dim U_i = i$. Choose $d+1$ integers i_j such that $0 \le i_0 \le \cdots \le i_d \le n-d$. We define

$$\omega(i_d, \ldots, i_0) = \{[W] \in X : \dim(W \cap U_{n+1-k-i_k}) \ge d+1-k \text{ for all } k \text{ such that } 0 \le k \le d\}. \tag{4.5}$$

The subset $\omega(i_d, \ldots, i_0)$ is a subvariety of $G(d+1, V)$ by [KL72, Corollary 5], and we call it a Schubert cycle. The equivalence class $[\omega(i_d, \ldots, i_0)]$ depends solely on the choice of the i_js and not on the choice of the flag U_i, [KL72, p. 1070]. We therefore denote the equivalence class $[\omega(i_d,\dots,i_0)] = \Omega(i_d,\dots,i_0) \in A_*(X)$. With addition as binary operation, $A_*(X)$ is a free abelian group and $\Omega(i_d,\dots,i_0)$ form a basis, [KL72, p. 1071]. The element $\Omega(i_d,\dots,i_0)$ is called a *Schubert class*.

For $X = G(2,4)$, we fix a flag $U_1 \subset U_2 \subset U_3 \subset U_4 = V$ and let W be a 2-dimensional subspace of V. Then, the Schubert cycles are

$$\omega(i_1,i_0) = \{[W] \in X : \dim(W \cap U_{4-i_0}) \geq 2, \dim(W \cap U_{3-i_1}) \geq 1\},$$

where $0 \leq i_0 \leq i_1 \leq 2$. We describe the Schubert cycles of $G(2,4)$ in more detail.

Example 4.3.2. Let $X = G(2,4)$, W be a 2-dimensional vector space and fix a flag $U_1 \subset U_2 \subset U_3 \subset U_4$. The Schubert cycles in $G(2,4)$ correspond to the following sets:

$$\begin{aligned}
\omega(0,0) &= \{[W] \in X : \dim(W \cap U_4) \geq 2, \dim(W \cap U_3) \geq 1\} \\
\omega(1,0) &= \{[W] \in X : \dim(W \cap U_4) \geq 2, \dim(W \cap U_2) \geq 1\} \\
\omega(1,1) &= \{[W] \in X : \dim(W \cap U_3) \geq 2, \dim(W \cap U_2) \geq 1\} \\
\omega(2,0) &= \{[W] \in X : \dim(W \cap U_4) \geq 2, \dim(W \cap U_1) \geq 1\} \\
\omega(2,1) &= \{[W] \in X : \dim(W \cap U_3) \geq 2, \dim(W \cap U_1) \geq 1\} \\
\omega(2,2) &= \{[W] \in X : \dim(W \cap U_2) \geq 2, \dim(W \cap U_1) \geq 1\}
\end{aligned}$$

First, we describe the lines in $\mathbb{P}^3$ that fulfill the conditions in each Schubert cycle and then we describe the corresponding Schubert class. We note that $\mathbb{P}(U_i)$ is isomorphic to $\mathbb{P}^{i-1}$, so projectively we have the fixed flag $p \subset \mathbb{P}^1 \subset \mathbb{P}^2 \subset \mathbb{P}^3$.

$$\begin{aligned}
\omega(0,0) &= \{[W] \in X : l_W \text{ intersects } \mathbb{P}^2 \text{ and is contained in } \mathbb{P}^3\} \\
&= \{[W] \in X\},
\end{aligned}$$

since both conditions are satisfied for all lines in $\mathbb{P}^3$. This shows that $\Omega(0,0)$ is the equivalence class of X.

$$\begin{aligned}
\omega(1,0) &= \{[W] \in X : l_W \text{ intersects } \mathbb{P}^1 \text{ and is contained in } \mathbb{P}^3\} \\
&= \{[W] \in X : l_W \text{ intersects } \mathbb{P}^1\},
\end{aligned}$$

since the second condition is satisfied for all lines in $\mathbb{P}^3$. The set of lines intersecting a line l in $\mathbb{P}^3$ corresponds to a hyperplane in $X \in \mathbb{P}^5$. Indeed, let p_{ij} be the Plücker coordinates of l. By Lemma 4.2.9, we have that a line l' intersects l if and only if

$$p_{01}q_{23} - p_{02}q_{13} + p_{03}q_{12} + p_{12}q_{03} - p_{13}q_{02} + p_{23}q_{01} = 0, \tag{4.6}$$

where q_{ij} are the Plücker coordinates of l'. The set of points in X satisfying 4.6 is a hyperplane. This shows that $\Omega(1,0)$ is the equivalence class of a hyperplane.

$$\begin{aligned}
\omega(1,1) &= \{[W] \in X : l_W \text{ intersects } \mathbb{P}^1 \text{ and is contained in } \mathbb{P}^2\} \\
&= \{[W] \in X : l_W \text{ is contained in } \mathbb{P}^2\},
\end{aligned}$$

since two lines in $\mathbb{P}^2$ always intersect. The set of lines contained in a $\mathbb{P}^2$ corresponds to a plane in $X \in \mathbb{P}^5$. Indeed, a line in $\mathbb{P}^2$ is determined by two points in $\mathbb{P}^2$ up to scalar multiplication, thus the dimension of the parameter space of lines in a $\mathbb{P}^2$ is $2+2-1-1=2$. This plane is called an α-plane. This shows that $\Omega(2,0)$ is the equivalence class of an α-plane.

$$\omega(2,0) = \{[W] \in X : l_W \text{ intersects } p \text{ and is contained in } \mathbb{P}^3\}$$

$$= \{[W] \in X : l_W \text{ intersects } p\},$$

since the second condition is satisfied for all lines in $\mathbb{P}^3$. The set of lines through a point in $\mathbb{P}^3$ is a plane in $X \in \mathbb{P}^5$. Indeed, a line through a point p is determined by p and another point in $\mathbb{P}^3$ up to scalar multiplication, thus the dimension of the parameter space of lines through $p \in \mathbb{P}^3$ is $3-1=2$. We call this plane a β-plane. This shows that $\Omega(2,0)$ is the equivalence class of a β-plane. Later, we will show that an α-plane and a β-plane does not intersect.

$$\omega(2,1) = \{[W] \in X : l_W \text{ intersects } p \text{ and is contained in } \mathbb{P}^2\}.$$

The set of lines that intersects a point p in a $\mathbb{P}^2$ is a line in $X \in \mathbb{P}^5$. Indeed, a line through a point in a $\mathbb{P}^2$ is determined by p and another point in $\mathbb{P}^2$ up to scalar multiplication, thus the dimension of the parameter space is $2-1=1$. This shows that $\Omega(2,1)$ is the equivalence class of a line.

$$\begin{aligned}\omega(2,2) &= \{[W] \in X : l_W \text{ intersects } p \text{ and is contained in } \mathbb{P}^1\} \\ &= \{[W] \in X : l_W = \mathbb{P}^1\},\end{aligned}$$

because $\mathbb{P}^1$ is the only line containing $\mathbb{P}^1$. This shows that $\Omega(2,2)$ is the equivalence class of a point.

The example shows that $\operatorname{codim} \omega(i_1, i_0) = i_1 + i_0$. This equality holds in general for a Grassmannian $G(n+1, d+1)$. That is $\operatorname{codim} \omega(i_d, \dots, i_0) = i_d + \dots + i_0$, [KL72, p. 1071]. We prove the results for $G(2,4)$ in another way. The proof can be generalized to a general Grassmannian.

Lemma 4.3.3. *Let* $X = G(2,4)$. *Then* $\operatorname{codim} \omega(i_1, i_0) = i_1 + i_0$.

Proof. We will prove the lemma in two parts. First, we construct subvariaties S_{i_1,i_0} of X where $\operatorname{codim} S(i_1,i_0) = i_1 + i_0$. Then, we prove that S_{i_1,i_0} is an open subset of $\omega(i_1,i_0)$. Thereafter, we show that $\overline{S_{i_1,i_0}} = \omega(i_1,i_0)$, which gives that $\operatorname{codim} \omega(i_1,i_0) = \operatorname{codim} S_{i_1,i_0} = i_1 + i_0$.

Let S_{i_1,i_0} be the subvariety of X given by the minors of the 2×4 matrix A_{i_1,i_0} constructed in the following way. Let $a_d = i_d + d$ for $d = 0, 1$. The a_dth column of A_{i_1,i_0} is a pivot column with the pivot element in the $(1-d)$th row. The two remaining columns, denoted the c_1th and the c_2th columns, have entries in $\mathbb{C}$, except when $c_j < a_d$ for some d. Then the dth entry of c_j is zero. We denote variable elements by $*$ and get the following matrices.

$$A_{0,0} = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}, \qquad A_{2,0} = \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_{1,0} = \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}, \qquad A_{2,1} = \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_{1,1} = \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}, \qquad A_{2,2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The minors of A_{i_1,i_0} is indeed a subvariety of X by construction. Furthermore, we have $S_{i_1,i_0} \simeq \mathbb{A}^{i_1+i_0}$. We show that S_{i_1,i_0} is an open subset of a Schubert cycle $\omega(i_1,i_0)$. Fix the variables in A_{i_1,i_0} and let U_{i_1,i_0} be the corresponding point in X. Let V be a vector space with the basis given by

$$e_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, e_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let $U_i = \text{Span}(e_1, \ldots, e_i)$ and consider the flag $U_1 \subset U_2 \subset U_3 \subset U_4 = V$. We will show that $\dim(U_{i_1,i_0} \cap U_{4-i_0}) \geq 2$ and $\dim(U_{i_1,i_0} \cap U_{3-i_1}) \geq 1$. Indeed, by construction, U_{i_1,i_0} is contained in U_{4-i_0}, since the first row of A_{i_1,i_0} has a pivot element in the i_0th column. In the same way, U_{i_1,i_0} intersects U_{3-i_1} in a 1-dimensional subspace, since the second row of A_{i_1,i_0} has a pivot element in the (i_1+1)th column. This shows that S_{i_1,i_0} satisfies the conditions for the Schubert cycle $\omega(i_1, i_0)$, which gives that $S_{i_1,i_0} \subset \omega(i_1, i_0)$. For each S_{i_1,i_0}, the Plücker coordinate $p_{a_0a_1} = 1$. Indeed, the 2×2-minor obtained from the a_0th and a_1th column is 1. This gives that S_{i_1,i_0} is an affine open subset of $\omega(i_1, i_0)$, thus $\overline{S_{i_1,i_0}} = \omega(i_1, i_0)$. ■

In the following, we show some of the intersection products between Schubert cycles. In order to compute the intersection we need two flags. Let $\mathcal{F}$ and $\mathcal{G}$ be two flags. We say that $\mathcal{F}$ and $\mathcal{G}$ are *generally transversely* if each component of the flags intersect in *intersect transversely*. This means that either $\text{codim}\, Z_{\mathcal{F}} \cap \text{codim}\, Z_{\mathcal{G}} = \text{codim}\, Z_{\mathcal{F}} + \text{codim}\, Z_{\mathcal{G}} - n$ or $Z_{\mathcal{F}} \cap Z_{\mathcal{G}} = \emptyset$, where $Z_{\mathcal{F}}$ and $Z_{\mathcal{G}}$ are subsets of the flags $\mathcal{F}$ and $\mathcal{G}$, respectively.

Proposition 4.3.4.

$$\Omega(1,1) \cap \Omega(1,1) = \Omega(2,2) \tag{4.7}$$
$$\Omega(2,0) \cap \Omega(2,0) = \Omega(2,2) \tag{4.8}$$
$$\Omega(2,1) \cap \Omega(1,0) = \Omega(2,2) \tag{4.9}$$

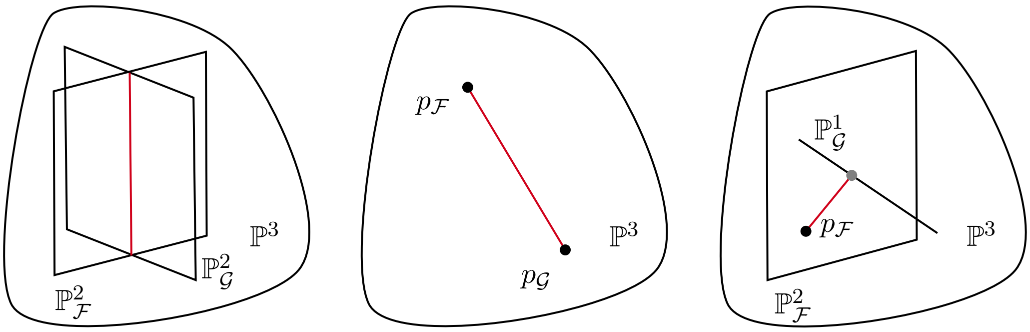
Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be two general flags, that is $p_{\mathcal{F}} \subset \mathbb{P}^1_{\mathcal{F}} \subset \mathbb{P}^2_{\mathcal{F}} \subset \mathbb{P}^3$ and $p_{\mathcal{G}} \subset \mathbb{P}^1_{\mathcal{G}} \subset \mathbb{P}^2_{\mathcal{G}} \subset \mathbb{P}^3$.

4.7 The points in $\omega_F(1,1) \cap \omega_{\mathcal{G}}(1,1)$ are the points corresponding to the lines in $\mathbb{P}^3$ that are contained in $\mathbb{P}^2_{\mathcal{F}}$ and $\mathbb{P}^2_{\mathcal{G}}$. There is only one line satisfying this, that is the line in the intersection.

4.8 The points in $\omega_{\mathcal{F}}(2,0) \cap \omega_{\mathcal{G}}(2,0)$ are the points corresponding to the lines in $\mathbb{P}^3$ that intersects $p_{\mathcal{F}}$ and $p_{\mathcal{G}}$. There is only one such line, that is the unique line through the points.

4.9 The points in $\omega_{\mathcal{F}}(2,1) \cap \omega_{\mathcal{G}}(1,0)$ are the points corresponding to the lines i $\mathbb{P}^3$ that intersect $p_{\mathcal{F}} \subset \mathbb{P}^2_{\mathcal{F}}$ and $\mathbb{P}^1_{\mathcal{G}}$. There is only one such line. Indeed, $\mathbb{P}^2_{\mathcal{F}}$ and $\mathbb{P}^1_{\mathcal{G}}$ intersect in a point p, and the line through p and $p_{\mathcal{F}}$ satisfies the conditions.

The three cases are summarized in the following figure.



■

Proposition 4.3.5.

$$\Omega(2,0) \cap \Omega(1,0) = \Omega(2,1) \tag{4.10}$$
$$\Omega(1,1) \cap \Omega(1,0) = \Omega(2,1) \tag{4.11}$$
$$\Omega(1,1) \cap \Omega(2,0) = \emptyset \tag{4.12}$$

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be two general flags, that is $p_{\mathcal{F}} \subset \mathbb{P}^1_{\mathcal{F}} \subset \mathbb{P}^2_{\mathcal{F}} \subset \mathbb{P}^3$ and $p_{\mathcal{G}} \subset \mathbb{P}^1_{\mathcal{G}} \subset \mathbb{P}^2_{\mathcal{G}} \subset \mathbb{P}^3$.

4.10 The points in $\omega_{\mathcal{F}}(2,0) \cap \omega_{\mathcal{G}}(1,0)$ are the points corresponding to the lines that intersect $p_{\mathcal{F}}$ and $\mathbb{P}^2_{\mathcal{G}}$. The lines intersecting $p_{\mathcal{F}}$ correspond to a β-plane $\mathcal{P}_{\mathbb{P}^1_{\mathcal{F}}} \subset X$ and the lines that intersect $\mathbb{P}^2_{\mathcal{G}}$ correspond to a hyperplane $\mathcal{H}_{\mathbb{P}^2_{\mathcal{G}}} \subset X$. Since $X \subset \mathbb{P}^5$, $\mathcal{P}_{\mathbb{P}^1_{\mathcal{F}}}$ and $\mathcal{H}_{\mathbb{P}^2_{\mathcal{G}}}$ intersect in a line $\mathcal{L}$. Since $\Omega(2,1)$ is the equivalence class of a line, we are done.

4.11 The points in $\omega_{\mathcal{F}}(1,1) \cap \omega_{\mathcal{G}}(1,0)$ are the points corresponding to the lines that are contained in $\mathbb{P}^2_{\mathcal{F}}$ and that intersect $\mathbb{P}^2_{\mathcal{G}}$. The lines contained in $\mathbb{P}^2_{\mathcal{F}}$ correspond to an α-plane $\mathcal{P}_{\mathbb{P}^2_{\mathcal{F}}} \subset X$, and the lines that intersect $\mathbb{P}^2_{\mathcal{G}}$ correspond to a hyperplane $\mathcal{H}_{\mathbb{P}^2_{\mathcal{G}}} \subset X$. Since $X \subset \mathbb{P}^5$, $\mathcal{P}_{\mathbb{P}^2_{\mathcal{F}}}$ and $\mathcal{H}_{\mathbb{P}^2_{\mathcal{G}}}$ intersect in a line $\mathcal{L}$. Since $\Omega(2,1)$ is the equivalence class of a line, we are done.

4.12 The points in $\omega_{\mathcal{F}}(1,1) \cap \omega_{\mathcal{G}}(2,0)$ are the points corresponding to lines that are contained in $\mathbb{P}^2_{\mathcal{F}}$ and that intersect $p_{\mathcal{G}}$. A general point in $\mathbb{P}^3$ is not contained in a given $\mathbb{P}^2$, so there is no line satisfying the condition. ∎

We will now use the previous propositions to intersect two, three and four hyperplanes, respectively. First, we have that

$$\Omega(1,0) \cap \Omega(1,0) = a\Omega(1,1) + b\Omega(2,0), \tag{4.13}$$

for $a, b \in \mathbb{Z}$. Indeed, $\operatorname{codim}(\omega(1,0)) = 1$, thus $\operatorname{codim}(\omega(1,0) \cap \omega(1,0)) = 2$, by Theorem 4.3.1. This gives that $\Omega(1,0) \cap \Omega(1,0) \in A_2(X)$ and is therefore a sum of the two generators of $A_2(X)$. To find a and b, we intersect Equation (4.13) with $\Omega(2,0)$ and $\Omega(1,1)$ in turn. On one hand we get

$$\begin{aligned}
\Omega(2,0) \cap [\Omega(1,0) \cap \Omega(1,0)] &= \ \Omega(2,0) \cap [a\Omega(1,1) + b\Omega(2,0)] \\
&= a\Omega(2,0) \cap \Omega(1,1) + b\Omega(2,0) \cap \Omega(2,0) \\
&= b, \\
\Omega(1,1) \cap [\Omega(1,0) \cap \Omega(1,0)[&= \ \Omega(1,1) \cap [a\Omega(1,1) + b\Omega(1,1)] \\
&= a\Omega(1,1) \cap \Omega(1,1) + b\Omega(1,1) \cap \Omega(2,0) \\
&= a,
\end{aligned}$$

where we have used the previous results. On the other hand we get

$$\begin{aligned}
\Omega(2,0) \cap [\Omega(1,0) \cap \Omega(1,0)] &= [\Omega(2,0) \cap \Omega(1,0)] \cap \Omega(1,0) \\
&= \Omega(2,1) \cap \Omega(1,0) \\
&= \Omega(2,2) \\
&= \text{one point} \\
\Omega(1,1) \cap [\Omega(1,0) \cap \Omega(1,0)] &= [\Omega(1,1) \cap \Omega(1,0)] \cap \Omega(1,0) \\
&= \Omega(2,1) \cap \Omega(1,0) \\
&= \Omega(2,2) \\
&= \text{one point}
\end{aligned}$$

To summarize, we get that $a = b = 1$, which was what we were going to show.

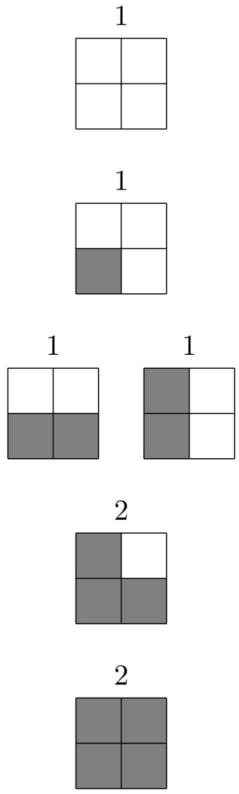
We observe that the intersection of two hyperplanes are the sum of the two generators for $A_2(X)$. Intersecting three hyperplanes, we get

$$\begin{aligned}
\Omega(1,0) \cap \Omega(1,0) \cap \Omega(1,0) &= \Omega(1,0) \cap [\Omega(1,1) + \Omega(2,0)] \\
&= \Omega(1,0) \cap \Omega(1,1) + \Omega(1,0) \cap \Omega(2,0) \\
&= \Omega(2,1) + \Omega(2,1) \\
&= 2\Omega(2,1),
\end{aligned}$$

a multiple of the generator for $A_1(X)$. Intersecting four hyperplanes gives

$$\begin{aligned}\Omega(1,0)\cap\Omega(1,0)\cap\Omega(1,0)\cap\Omega(1,0) &= \Omega(1,0)\cap 2\Omega(2,1)\\ &= 2\Omega(2,2).\end{aligned}$$

To conclude, the intersection of four hyperplane is two points. Since X is of dimension 4 we get that the degree of X is 2, which we need from the alternative proof of Theorem 4.2.6. The intersection of hyperplanes can be visualized by coloring of boxes of dimension 2×2. Each box corresponds to a Schubert class $\Omega(i_1, i_0)$, where the number of colored boxes in the first and second column is i_1 and i_0, respectively. We organize the boxes such that the ith row of boxes (counted from zero) corresponds to the Schubert classes in A_{4-i}. We also put a number next to each box such that the sum of the Schubert classes in the ith row is the intersection of i hyperplanes. We get the following boxes.

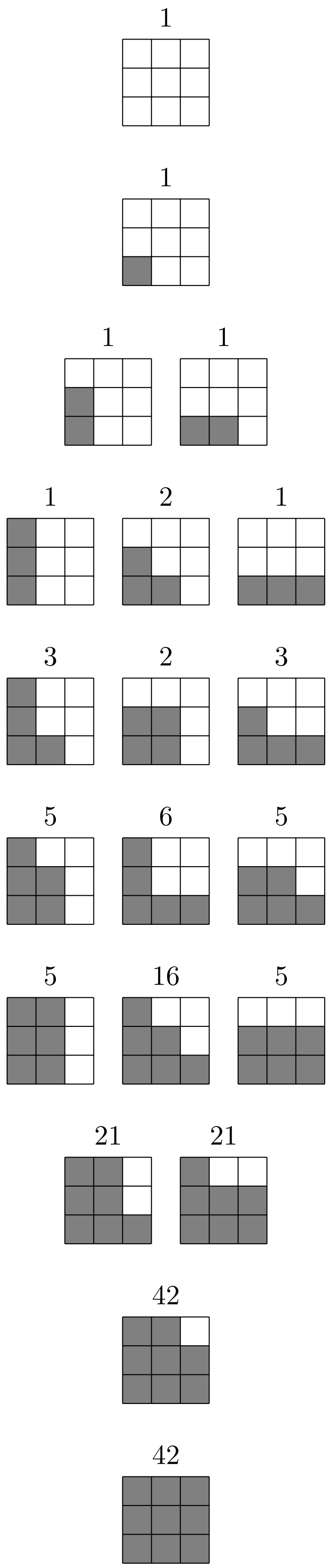


For a general $G(d+1, n+1)$, the boxes are of dimension $d(n-d)$. In the proofs of Theorem 4.4.1 and Theorem 4.4.7 we will use a subvariety of $G(3,6)$ and $G(4,9)$. Therefore, we calculate the similar scheme of boxes for $G(3,6)$ and $G(4,9)$. In this calculation, we will need a general result about intersection of hyperplanes. Let $\Omega_a = \Omega(a_d, \ldots, a_0)$, where $a_d + \cdots + a_0 = a$. We have the following

Proposition 4.3.6. *[KL72, p. 1073](Pieri's formula) Let $X = G(d+1, n+1)$. Then for any Schubert class $\Omega_a \in A_*(X)$ we have*

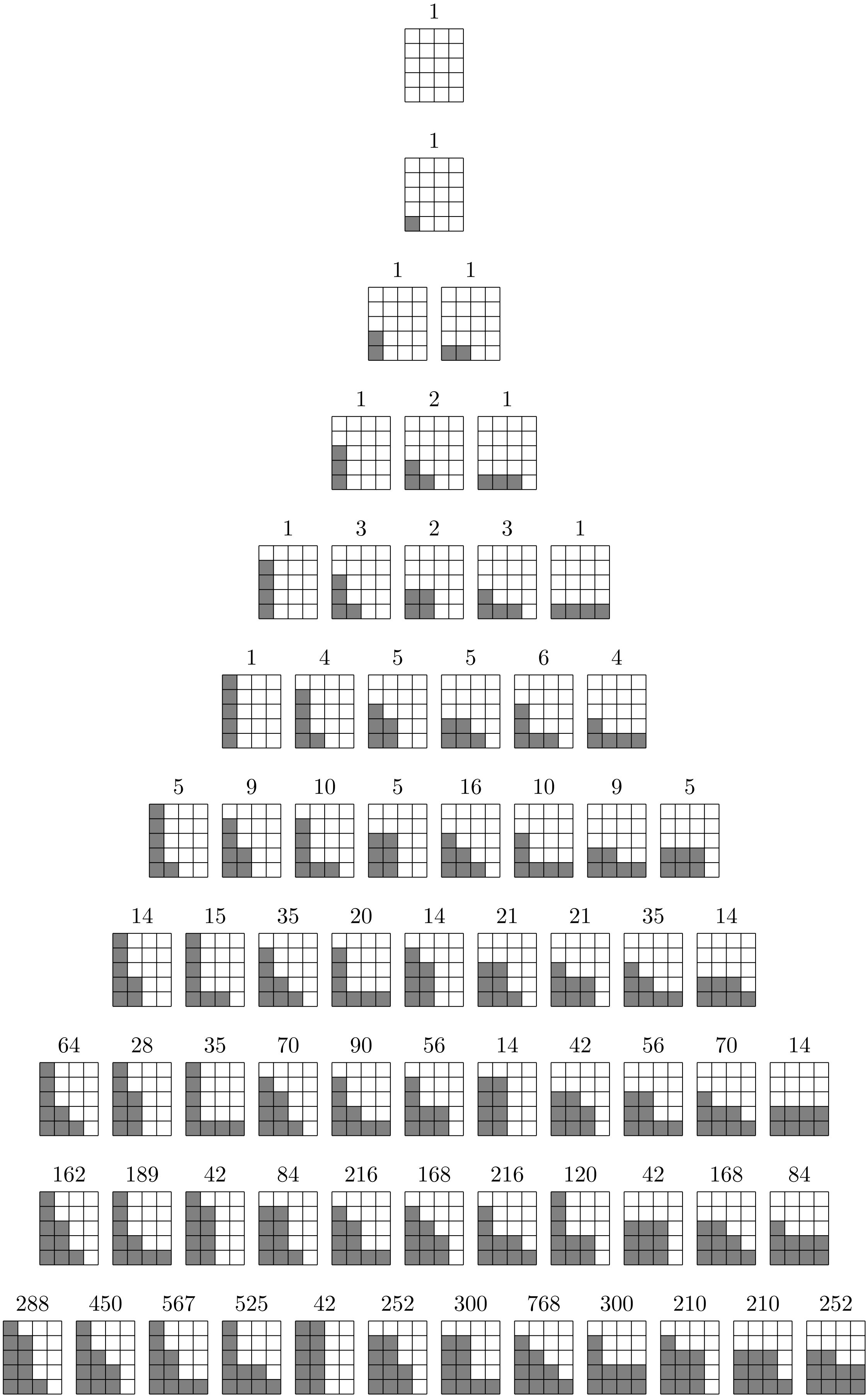
$$\Omega_a \cap \Omega_1 = \sum_{\substack{|c|=|a|+1\\ a_j\le c_j\le a_{j-1}\forall j}} \Omega_c$$

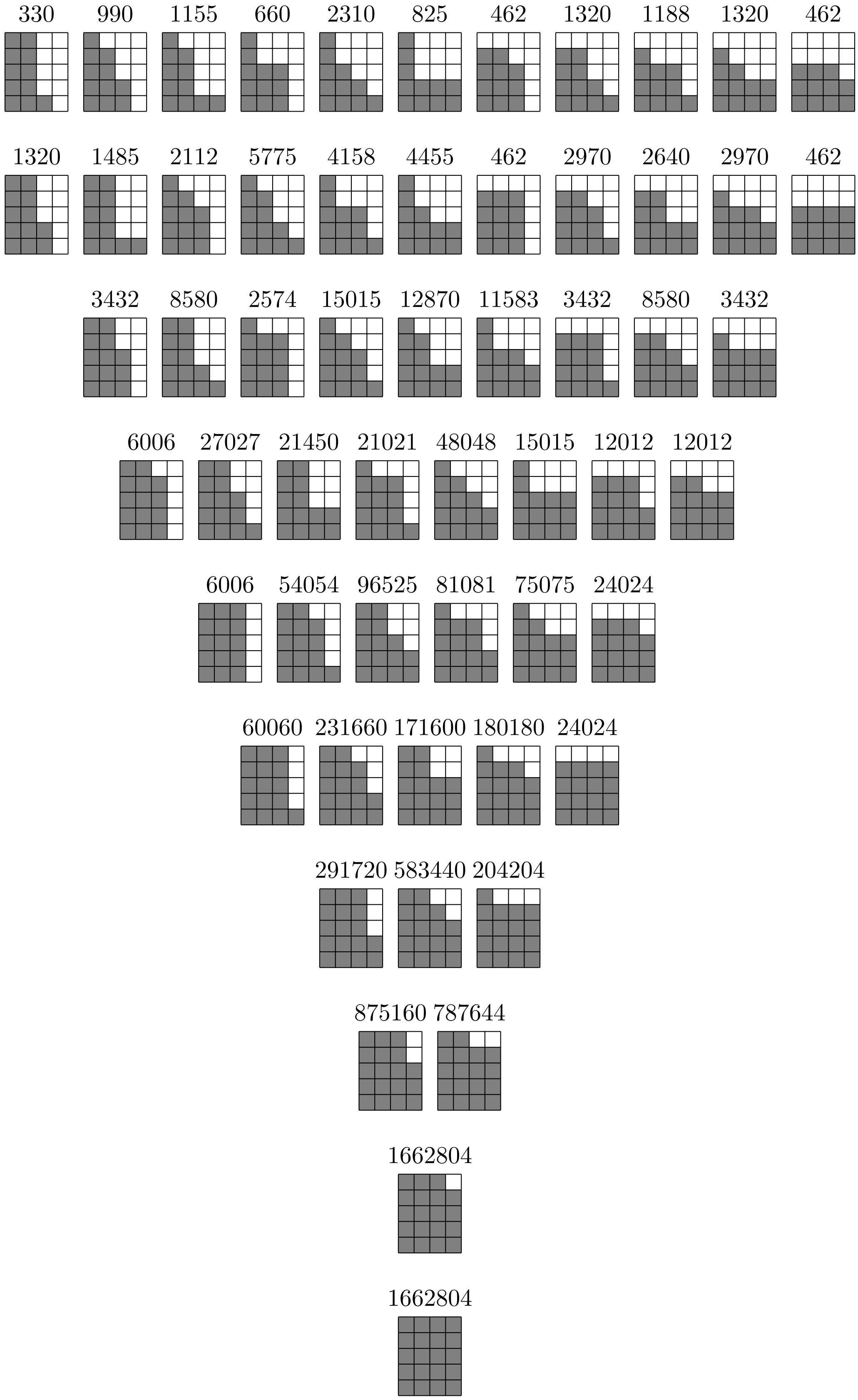
The proposition says that the intersection between a general Schubert cycle Ω_a and a hyperplane Ω_1 is the sum of all Schubert cycles that can appear by coloring the box of Ω_a in all ways such that the number of colored boxes in the left box is greater than or equal to the number of colored boxes to the right. We get the following boxes for $\mathbb{G}(3,6)$.



We observe that the degree of $G(3,6)$ is 42.

Lastly, we compute the boxes for $G = G(4, 9)$.





We observe that the degree of $G(4,9)$ is 1662804.

4.3.2 Chern Classes of Vector Bundles on Grassmannians

Some of the $\Omega(i_d, \ldots, i_0)$ are special in the sense that they generate $A_*(X)$ under multiplication. We will see that this will be the ones that correspond to the so called Chern classes of some vector bundles on X. Before we investigate this relationship, we need the definition of a general vector bundle on a variety X.

Definition 4.3.7. Let X be a variety and V an n-dimensional vector space over $\mathbb{C}$. A *vector bundle* V_X on X of rank n is a variety together with a morphism $f : V_X \to X$ such that for an open affine covering $X = \bigcup_{i \in I} U_i$ then $f^{-1}(U_i) \simeq U_i \times V$. In addition, for

$$(U_i \cap U_j) \times V \xleftarrow{\phi_j} f^{-1}(U_i \cap U_j) \xrightarrow{\phi_i} (U_i \cap U_j) \times V$$

$$\begin{aligned} \phi_j \circ \phi_i^{-1} : (U_i \cap U_j) \times V &\to (U_i \cap U_j) \times V \\ (x, v) &\mapsto (x, \psi_{ij}(v)), \end{aligned}$$

ψ_{ij} has to be a linear map.

A *global section* on X is a morphism $s : X \to V_X$ such that $f \circ s = \mathrm{id}_X$. Explicitly, for all $x \in X$, $s(x) = (x, v)$ for a fixed $v \in V$. We say that $s_1(x) = (x, v_1)$ and $s_2(x) = (x, v_2)$ are linearly dependent if $v_1, v_2 \in V$ are linearly dependent. We are now ready to define the Chern classes of a vector bundle on X.

Definition 4.3.8. Let X be a variety of dimension $N = (n-d)(d+1)$, V_X a vector bundle on X of rank n and $s_i : X \to V_X$ for $i \in \{1, \ldots, n\}$ general global sections on X. We define the *Chern classes* of V_X, denoted $c_i(V_X)$ for $i \in \{1, \ldots, n\}$, as

$$\begin{aligned} c_1(V_X) &= [\{x \in X : s_1(x), \ldots, s_n(x) \text{ are linearly dependent}\}], \\ c_2(V_X) &= [\{x \in X : s_1(x), \ldots, s_{n-1}(x) \text{ are linearly dependent}\}], \\ &\vdots \\ c_i(V_X) &= [\{x \in X : s_1(x), \ldots, s_{n-i+1}(x) \text{ are linearly dependent}\}], \\ &\vdots \\ c_n(V_X) &= [\{x \in X : s_1(x) = 0\}]. \end{aligned}$$

For $i > n$, we set $c_i(V_X) = 0$. We also set $c_0(V_X) = 1$.

Let V be a vector space and let $G = G(d+1, V)$. We will construct three vector bundles on G. Firstly, we define the trivial vector bundle $V_G = V \times G = \{(v, [U]) : v \in V, U \subset V\}$. It is called the trivial vector bundle since the fiber over every point $[U] \in G$ is V. Next, we define the sub-bundle $\mathcal{U} = \{(v, [U]) : v \in U \subset V\} \subset V_G$. Since the fiber over a point $[U] \in G$ is U, the rank of $\mathcal{U}$ is $d+1$. The sub-bundle is indeed a vector bundle, [Eis16, Proposition 3.3]. Third, we have the quotient bundle $\mathcal{Q} = V_G/\mathcal{U}$ of rank $n-d$. Since both V_G and $\mathcal{U}$ are vector bundles, $\mathcal{Q}$ is obviously also a vector bundle. By construction, these vector bundles fit in a short exact sequence:

$$0 \longrightarrow \mathcal{U} \longrightarrow V_G \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Dualizing, we also get the vector bundles $\mathcal{Q}^*, V_G^*$ and $\mathcal{U}^*$. Since the dualizing is exact on vector bundles, we have the following short exact sequence:

$$0 \longrightarrow \mathcal{Q}^* \longrightarrow V_G^* \longrightarrow \mathcal{U}^* \longrightarrow 0.$$

We are going to prove that there is a correspondence between the Chern classes of $\mathcal{Q}$ and $\mathcal{U}^*$ and the Schubert cycles of G. In order to do this, we look at the global sections on G.

Let $s : G \to V_G$ be a global section and let $\phi : V_G \to \mathcal{Q}$ be the quotient map. We set $\phi(v) = [v]$. Since $s([U]) = (v, [U])$ for a fixed $v \in V$, we get that $\phi \circ s : G \to \mathcal{Q}$ is a global section. Indeed, $\phi \circ s([U]) = ([v], [U])$. The argument is summarized in the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{U} \arrow[r] & V_G \arrow[r, "\phi"] & \mathcal{Q} \arrow[r] & 0. \\ & & G \arrow[u, "s"] \arrow[ur, "\phi\circ s"'] & & \end{tikzcd}$$

Now, let $s : G \to V_G^*$ be a global section and let $\psi : V_G^* \to \mathcal{U}^*$ be the quotient map. By abuse of notation we say that $s([U]) = ([U], s)$, where $s : V \to \mathbb{C}$ is a linear map. The composition $\psi \circ s$ is a global section. Indeed, $\psi \circ s([U]) = ([U], s|_U)$, where $s|_U : U \to \mathbb{C}$ is the restriction of s to U. The argument is summarized in the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{Q}^* \arrow[r] & V_G^* \arrow[r, "\psi"] & \mathcal{U}^* \arrow[r] & 0. \\ & & G \arrow[u, "s"] \arrow[ur, "\psi\circ s"'] & & \end{tikzcd}$$

We are now ready to prove the relationship between the Chern classes of $\mathcal{U}^*$ and $\mathcal{Q}$ and the Schubert classes of G. We begin with the Chern classes of $\mathcal{U}^*$.

Proposition 4.3.9. *Let V be an $(n+1)$-dimensional vector space and let $G = G(d+1, V)$. Then*

$$\begin{aligned} &(1) \qquad c_{d+1}(\mathcal{U}^*) = \Omega(1, \dots, 1) \quad \in A_*(G), \\ &(2) \qquad c_d(\mathcal{U}^*) = \Omega(1, \dots 1, 0) \ \in A_*(G), \\ &(3) \qquad c_1(\mathcal{U}^*) = \Omega(1, 0, \dots, 0) \in A_*(G). \end{aligned}$$

Proof. Let $s_i : G \to V_G^*$ and $s_i|_U = \psi \circ s_i : G \to \mathcal{U}^*$ for $i \in \{1, \dots, d+1\}$ be general global section as described above. By Definition 4.3.8, we have that

$$c_e(\mathcal{U}^*) = \left[[U] \in G : s_1\big|_U([U]), \dots, s_{d-e+2}\big|_U([U]) \text{ are linearly dependent} \right].$$

The global sections are linearly dependent if the linear maps $s_i|_U : U \to \mathbb{C}$ are linearly dependent. Therefore, we get that

$$c_e(\mathcal{U}^*) = \left[[U] \in G : s_1\big|_U, \dots, s_{d-e+2}\big|_U \text{ are linearly dependent} \right], \tag{4.14}$$

where $s_i|_U$ are linear maps. Given $s_1 : V \to \mathbb{C}$, the kernel of s_1 is a n-dimensional subspace of V. Generally, $n+1$ linear maps $s_i : V \to \mathbb{C}$ induce a flag $U_1 \subset \cdots \subset U_{n-1} \subset U_n \subset V$, where U_i is the i-dimensional subspace of V that disappears on $s_i, s_{i-1}, \dots, s_1$. Assume we have $n+1$ such linear maps and the corresponding flag.

(1) Let $e = d+1$. From Equation (4.14) we get that

$$c_{d+1}(\mathcal{U}^*) = \left[\{[U] \in G : s_1\big|_U = 0\} \right].$$

In other words, $c_{d+1}(\mathcal{U}^*)$ is the equivalence class of the points $[U] \in G$ such that for all $u \in U$ we have $s_1(u) = 0$, i.e. the set of points $[U] \in G$ such that $U \subset \ker s_1 = U_n$. We have established that

$$c_{d+1}(\mathcal{U}^*) = [\{[U] \in G : U \subset U_n\}].$$

Now, we want want to show that the set of $[U] \in G$ such that $U \in U_n$ is a Schubert cycle $\omega(1,\ldots,1)$. Indeed, we get from Equation (4.5) on page 29 that

$$\omega(1,\ldots,1) = \{[U] \ \in G : \dim(U \cap U_{n-k}) \geq d+1-k \ \text{ for all } \ k \ \text{such that } 0 \leq k \leq d\}.$$

For $k = 0$ we get the requirement $\dim(U \cap U_n) \geq d+1$, which is fulfilled when $U \subset U_n$. For $k \geq 1$ the requirement is is trivially fulfilled. Indeed, since $U \subset U_n$ we get that the intersection of a $d+1$-dimensional subspace U and a subspace of U_n of dimension $n-k$ has dimension at least $(d+1)+(n-k)-n = d+1-k$. To summarize, $\omega(1,\ldots,1)$ is the points $[U] \in G$ such that $U \subset U_n$, which was what we were going to show. Consequently, $c_{d+1} = \Omega(1,\ldots,1)$.

(2) Let $e = d$. From Equation (4.14) we get that

$$c_d(\mathcal{U}^*) = \left[\{[U] \in G : s_1\big|_U = \lambda s_2\big|_U,\ \lambda \in \mathbb{C}\}\right].$$

Let $(a_{i1},\ldots,a_{i,d+1})$, $a_{ij} \in \mathbb{C}$, be the matrix representation of $s_i|_U$ and let

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1,d+1} \\ a_{21} & \ldots & a_{2,d+1} \end{pmatrix}.$$

Assume $U \subset V$ such that $s_1|_U = \lambda s_2|_U$. We then have that $\dim(\ker A) \geq d$. Since $\ker A \subset U$ and $\ker A \subset U_{n-1}$, this means that $\dim(U \cap U_{n-1}) \geq d$. We have now proved that

$$c_d(\mathcal{U}^*) = [\{[U] \in G : \dim U \cap U_{n-1}) \geq d\}].$$

Now, we want to show that the set of points $[U] \in G$ such that $U \in U_{n-1}$ is a Schubert cycle $\omega(1,\ldots,1,0)$. Indeed, we get from Equation (4.5) on page 29 that

$$\begin{aligned}\omega(1,\ldots,1,0) = \{[U] \in G :\, & \dim(U \cap U_{n+1}) \geq d+1 \text{ and} \\ & \dim(U \cap U_{n-k}) \geq d+1-k \ \text{ for all } \ k \ \text{such that } 1 \leq k \leq d\}.\end{aligned}$$

The condition that $\dim(U \cap U_{n+1}) \geq d+1$ is obviously fulfilled since U is assumed to be a subspace of $V = U_{n+1}$. For $k = 1$, we get the condition that $\dim(U \cap U_{n-1}) \geq d$. For $k \geq 2$, we get the condition that $\dim(U \cap U_{n-k}) \geq d-1-k$, which is trivially fulfilled when $\dim(U \cap U_{n-1}) \geq d$. Indeed, the dimension of the intersection between U and $U_{n-k} \subset U_{n-1}$ of dimension $n-k$ is at least $d+(n-k)-(n-1) = d+1-k$. In summary, $\omega(1,\ldots,1,0)$ is the set of points $[U] \in G$ such that $\dim(U \cap U_{n-1}) \geq d$. Consequently, we get that $c_d(\mathcal{U}^*) = \Omega(1,\ldots,1,0)$.

(3) Let $e = 1$. From Equation (4.14) we get that

$$c_1(\mathcal{U}^*) = \left[\{[U] \in G : s_1\big|_U,\ldots,s_{d+1}\big|_U \text{ are linearly dependent}\}\right].$$

As above, we let $(a_{i1},\ldots,a_{i,d+1})$, $a_{ij} \in \mathbb{C}$ be the matrix representation of $s_i|_U$ and let

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1,d+1} \\ \vdots & \ddots & \vdots \\ a_{d+1,1} & \ldots & a_{d+1,d+1} \end{pmatrix}.$$

Assume $U \subset V$ such that $s_1|_U,\ldots,s_{d+1}|_U$ are linearly dependent. Then $\dim(\ker A) \geq 1$. Since $\ker A \subset U$ and $\ker A \subset U_{n-d}$, this means that $\dim(U \cap U_{n-d} \geq 1)$. We have proved that

$$c_1(\mathcal{U}^*) = [\{[U] \in G : \dim(U \cap U_{n-d}) \geq 1\}].$$

Now, we want to show that the set of points $[U] \in G$ such that $\dim U \cap U_{n-d}) \geq 1\}$ is a Schubert cycle $\omega(1,0,\ldots,0)$. Indeed, from Equation (4.5) on page 29, we get that

$$\omega(1,0,\ldots,0) = \{[U] \in G : \dim(U \cap U_{n-d}) \geq 1 \text{ and}$$

$$\dim(U \cap U_{n+1-k}) \geq d+1-k \text{ for all } k \text{ such that } 0 \leq k < d\}.$$

For $k > d$, the requirement is fulfilled. Indeed, the dimension of the intersection of a $d+1$-dimensional subspace U and a subspace of dimension $n+1-k$ is at least $d+1+(n+1-k)-(n+1) = d+1-k$. The requirement for $k = d$ is that $\dim(U \cap U_{n-d}) \geq 1$. Therefore, we get that $\omega(1, 0, \ldots, 0)$ is the set of points $[U] \in G$ such that $\dim(U \cap U_{n-d}) \geq 1$, which was what we were going to show. As a consequence, $c_1(\mathcal{U}^*) = \Omega(1, 0, \ldots, 0)$. ■

Now, we prove the relationship between the Chern classes of $\mathcal{Q}$ and the Schubert cycles of G.

Proposition 4.3.10. *Let V be an $(n+1)$-dimensional vector space and let $G = G(d+1, V)$. Then*

$$\begin{array}{lrll} (1) & c_{n-d}(\mathcal{Q}) & = \Omega(n-d, 0, \ldots, 0) & \in A_*(G), \\ (2) & c_{n-d-1}(\mathcal{Q}) & = \Omega(n-d-1, 0, \ldots, 0) & \in A_*(G), \\ (3) & c_1(\mathcal{Q}) & = \Omega(1, 0, \ldots, 0) & \in A_*(G). \end{array}$$

Proof. Let $s_i : G \to V_G$ and $\psi \circ s_i : G \to \mathcal{Q}$ for $i \in \{1, \ldots, n-d\}$ be general global sections as described above. By Definition 4.3.8, we have that

$$c_e(\mathcal{Q}) = [\{[U] \in G : \psi \circ s_1([U]), \ldots, \psi \circ s_{n-d-e+1}([U]) \text{ are linearly dependent}\}].$$

Recall that $\psi \circ s_i([U]) = ([v_i], [U])$ for $v_i \in V$. We have that $\psi \circ s_i([U])$ are linearly dependent if $[v_i]$ are linearly dependent. Therefore, we get that

$$c_e(\mathcal{Q}) = [\{[U] \in G : [v_1], \ldots, [v_{n-d-e+1}] \text{ are linearly dependent}\}]. \tag{4.15}$$

Given $v_1 \in V$. Then v_1 span a 1-dimensional subspace U_1. Generally, $n+1$ vectors $v_i \in V$ induce a flag $U_1 \subset \ldots U_{n-1} \subset U_n = V$, where U_i the span of $v_1, \ldots v_i$.

(1) Let $e = n-d$. From Equation (4.15) we get that

$$c_{n-d}(\mathcal{Q}) = [\{[U] \in G : [v_1] = 0\}].$$

In other words, $c_{n-d}(\mathcal{Q})$ is the equivalence class of the set of points $[U] \in G$ such that $U_1 \in U$. We have established that

$$c_{n-d}(\mathcal{Q}) = [\{[U] \in G : U_1 \subset U\}].$$

We want to show that the set of $[U] \in G$ such that $U_1 \subset U$ is a Schubert cycle $\omega(n-d, 0, \ldots, 0)$. Indeed, from Equation (4.5) on page 29, we get that

$$\begin{aligned} \omega(n-d, 0, \ldots, 0) = \{[U] \ \in G : & \dim(U \cap U_1) \geq 1 \text{ and} \\ & \dim(U \cap U_{n+1-k}) \geq d+1-k \text{ for all } k \text{ s.t. } 0 \leq k < d\}. \end{aligned}$$

The requirement for $k = d$ is that $\dim(U \cap U_1) \geq 1$, i.e. U contains U_1. The requirement for $k < d$ is trivially fulfilled. Indeed, since $U \subset U_{n+1}$ by assumption, we get that the intersection of a $d+1$-dimensional subspace U and a subspace U_{n+1-k} of dimension $(n+1-k)$ is at least $(d+1)+(n+1-k)-(n+1) = d+1-k$. To summarize, $\omega(n-d, 0, \ldots, 0)$ is the set of points $[U] \in G$ such that $U_1 \subset U$, which was what we were going to show. Consequently, $\Omega(n-d, 0, \ldots, 0) = c_{n-d}(\mathcal{Q})$.

(2) Let $e = n-d-1$. From Equation (4.15), we get that

$$c_{n-d-1}(\mathcal{Q}) = [\{[U] \in G : [v_1], [v_2] \text{ are linearly dependent}\}].$$

In other words, $c_{n-d-1}(\mathcal{Q})$ is the equivalence class of the set of points $[U] \in G$ such that there exists an $\lambda \in \mathbb{C}$ such that $[v_1] + \lambda[v_2] = 0$. This is equivalent to require that $v_1 + \lambda v_2 \in U$. This show that $c_{n-d-1}(\mathcal{Q})$ is the equivalence class of the set of U such that $\dim(U \cap U_2) \geq 1$. We have established that

$$c_{n-d-1}(\mathcal{Q}) = [\{[U] \in G : \dim(U \cap U_2) \geq 1\}] .$$

We want to show that the set of points $[U]$ such that $\dim(U \cap U_2) \geq 1$ is a Schubert cycle $\omega(n-d-1, 0, \dots, 0)$. Indeed, from Equation (4.5) on page 29, we get that

$$\begin{aligned}\omega(n-d-1,0,\dots,0) = \{[U] \in G : &\dim(U \cap U_2) \geq 1 \text{ and} \\ &\dim(U \cap U_{n+1-k}) \geq d+1-k \text{ for all } k \text{ s.t. } 0 \leq k < d\}.\end{aligned}$$

The requirement for $k > d$ is as above trivially fulfilled. The requirement for $k = d$ is that $\dim(U \cap U_2) \geq 1$. Consequently, $\Omega(n-d-1, 0, \dots, 0) = c_{n-d-1}(\mathcal{Q})$.

(3) Let $e = 1$. From Equation (4.15), we get that We have by definition that

$$c_1(\mathcal{Q}) = [\{[U] \in G : [v_1], \dots, [v_{n-d}] \text{ are linearly dependent}\}]$$

In other words, $c_1(\mathcal{Q})$ is the equivalence class of the set of points $[U] \in G$ such that there exists $\lambda_i \in \mathbb{C}$ such that $\lambda_1[v_1] + \lambda_2[v_2] + \cdots + \lambda_{n-d}[v_{n-d}] = 0$. This is equivalent to require that $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{n-d} v_{n-d} \in U$. This shows that $c_1(\mathcal{Q})$ is the equivalence class of the set of U such that $\dim(U \cap U_{n-d}) \geq 1$. We have established that

$$c_1(\mathcal{Q}) = [\{[U] \in G : \dim(U \cap U_{n-d}) \geq 1\}] .$$

We want to show that the set of points such that $\dim(U \cap U_{n-d}) \geq 1$ is a Schubert cycle $\omega(1, 0, \dots, 0)$. Indeed, we showed in the proof of Proposition 4.3.9 that $\omega(1, 0, \dots, 0)$ is the set of points $[U] \in G$ such that $\dim(U \cap U_{n-d} \geq 1$. Consequently, $\Omega(1, 0, \dots, 0) = c_1(\mathcal{Q})$. ■

Corollary 4.3.11. $c_{n-d}(\mathcal{Q})^{d+1} = \Omega(n-d, \dots, n-d) = c_{d+1}(\mathcal{U}^*)^{n-d}$.

Proof. For the first equality, we use that

$$c_{n-d}(\mathcal{Q}) = \Omega(n-d, 0 \dots, 0),$$

which is the equivalence class of the points $[U] \in G$ such that U contains a given 1-dimensional subspace of V. Then $c_{n-d}(\mathcal{Q})^{d+1}$ is the equivalence class of the points $[U] \in G$ such that U contains $d+1$ general 1-dimensional subspaces. There is only one $d+1$-dimensional subspace U that fulfills this condition, that is the vector space spanned by the 1-dimensional subspaces.

For the second equality, we use that

$$c_{d+1}(\mathcal{U}) = \Omega(1, \dots, 1),$$

which is the equivalence class of the points $[U] \in G$ such that U is contained in a n-dimensional subspace. Then $c_{d+1}(\mathcal{U})^{n-d}$ is the equivalence class of the points $[U] \in G$ such that U is contained in $n-d$ n-dimensional subspaces. There is only one such $d+1$-dimensional U, that is the vector space in the intersection of the n-dimensional vector spaces. ■

4.3.3 The set of Isotropic Subspaces as a Subvariety of a Grassmannian

In the proof of Theorem 4.2.6 we used that the set of 2-dimensional isotropic subspaces of a 4×4 skew symmetric matrix corresponds to a hyperplane in $G(2,4)$. Now, we will show the general correspondence between the set of k-dimensional isotropic subspaces of an $(n+1) \times (n+1)$ skew symmetric matrix and $G(k, n+1)$.

To describe the set of isotropic subspaces as a subvariety of a Grassmannian we need the concept of the second wedge product. Let V be a vector space of dimension $n+1$. Then the *second wedge product* of V is

$$\wedge^2 V = V \otimes V / v \otimes w + w \otimes v,$$

for $v, w \in V$. We have further that $\dim(\wedge^2 V) = \frac{(n+1)n}{2}$. Indeed, we have that $V \otimes V = \operatorname{Sym} V \oplus \wedge^2 V$, where $\operatorname{Sym} V$ is the symmetric algebra over V, i.e. $\operatorname{Sym} V = V \oplus V / v \oplus w - w \oplus v$. We have that $\dim(V \ \oplus V) = (n+1)^2$ and $\dim(\operatorname{Sym} V) = \frac{(n+1)^2 + (n+1)}{2}$. Then

$$\begin{aligned}
\dim(\wedge^2 V) &= \dim(V \otimes V) - \dim(\operatorname{Sym} V) \\
&= (n+1)^2 - \frac{(n+1)^2 + (n+1)}{2} \\
&= \frac{(n+1)n}{2}.
\end{aligned}$$

We have the following correspondence between skew symmetric matrices and $\wedge^2(V)$.

Lemma 4.3.12. *Let V be a vector space of dimension $n+1$. Then there is a $1-1$ correspondence between $(n+1) \times (n+1)$ skew symmetric matrices and elements in $\wedge^2(V^*)$.*

Proof. Let A be an $(n+1) \times (n+1)$ skew symmetric matrix and consider

$$\begin{aligned}
\phi : \wedge^2 V &\to \mathbb{C} \\
(v, w) &\mapsto vAw^T
\end{aligned}$$

This shows that $\phi \in \wedge^2(V^*)$. Consider now $\psi \in \wedge^2(V^*)$. Then ψ is a bilinear map, which indeed is represented by a unique matrix. Further we have that $\psi(v,w) = -\psi(w,v)$. This gives that $vAw^T = -wAv^T$. On the other hand we also have that

$$\begin{aligned}
vAw^T &= (vAw^T)^T \\
&= wA^T v^T.
\end{aligned}$$

In summary, we get that $wA^T v^T = -wAv^T$, which is fulfilled if and only if $A = -A^T$. This shows that A is skew symmetric. ∎

We will prove that the set of isotropic subspaces corresponds to one of the Chern classes of $\wedge^2(\mathcal{U}^*)$. In order to do this, we first look at the global sections $s : G \to \wedge^2(\mathcal{U}^*)$. In the same way as a global section $s_U : G \to \mathcal{U}^*$ is induced by a global section $s : G \to V_G^*$, a global section $s_U : G \to \wedge^2 \mathcal{U}^*$ is induced by a global section $s : G \to \wedge^2 V_G^*$. Indeed, we have that the following sequence is exact.

$$0 \longrightarrow (\mathcal{U}^* \otimes \mathcal{Q}^*) \oplus \wedge^2 \mathcal{Q}^* \longrightarrow \wedge^2 V_G^* \longrightarrow \wedge^2 \mathcal{U}^* \longrightarrow 0 \tag{4.16}$$

Let $s : G \to \wedge^2(V_G^*)$ be a the global section such that $s([U]) = ([U], s)$, where $s : \wedge^2 V_G \to \mathbb{C}$. Let $\phi : \wedge^2 V_G^* \to \wedge^2 \mathcal{U}^*$ be the quotient map we get from Equation (4.16). Then $\phi \circ s$ is a global section. Indeed, $\phi \circ s([U]) = ([U], s|_U)$, where $s|_U : \wedge^2 U \to \mathbb{C}$.

We are now ready to prove the following

Proposition 4.3.13. *Let V be an $(n+1)$-dimensional vector space, A be an $(n+1)\times(n+1)$ skew symmetric matrix, H_A^{d+1} the set of $(d+1)$-dimensional isotropic subspaces to A and let $G = G(d+1, V)$. Then $[\mathcal{H}_A^{d+1}] = c_m(\wedge^2\mathcal{U}^*)$, where $m = \frac{n(n-1)}{2}$ and $\mathcal{H}_A^{d+1}$ is the subset of G corresponding to H_A^{d+1} .*

Proof. Let $s : G \to \wedge^2 V_G^*$ be a section as described above, where $s : \wedge^2 V_G^*$ is represented by A and let $s|_U = \phi \circ s$. Then

$$\begin{aligned} c_m(\wedge^2\mathcal{U}^*) &= \left[\{[U] \in G : s|_U = 0\}\right] \\ &= \left[\{[U] \in G : A|_U = 0\}\right]. \end{aligned}$$

Since $\{[U] \in G : A|_U = 0\}$ is the set of $(d+1)$-dimensional isotropic subspaces to A, we are done. ■

4.3.4 Computation with Chern Classes

The Chern classes of $\wedge^2(\mathcal{U}^*)$ corresponds to the Chern classes of $\mathcal{U}^*$. In the this section, we show how we can find the explicit correspondence. We will also describe how to compute with Chern classes.

Let V_X be a vector bundle of rank n. We then define the *Chern polynomial* as the polynomial

$$c_t(V_X) = \sum_{i=0}^{n} c_i(V_X)t^i \in A_*(X)[t].$$

When $n = 1$ we call V_X a line bundle, denoted $\mathcal{L}$, and we have $c_t(\mathcal{L}) = 1 + c_1(\mathcal{L})t$. When $\mathcal{E}$ is a direct sum of line bundles $\mathcal{L}_i$ we have that

$$c_t(\mathcal{E}) = \prod c_t(\mathcal{L}_i), \tag{4.17}$$

[Ful98, p. 51]. This relation will give a way of computing with Chern classes. The vector bundles we are considering are however not direct sums of line bundles, but because of *the splitting principle* introduced below we can use the same techniques for computing as we have for a direct sum of line bundles.

Theorem 4.3.14. *[Ful98, p. 51](Splitting principle) Any identity among Chern classes of bundles that is true for bundles that are direct sums of line bundles is true in general.*

Before we use the splitting principle to compute the relations between a vector bundle $\mathcal{E}$ and $\wedge^2\mathcal{E}$ we need the following.

Lemma 4.3.15. *Let $\mathcal{E} = \bigoplus \mathcal{L}_i$. Then $\wedge^2\mathcal{E} = \bigoplus_{i<j,} \mathcal{L}_i \oplus \mathcal{L}_j$.*

Proof. Given $\mathcal{E} = \bigoplus_{i=0}^n \mathcal{L}_i$ we have that $\mathcal{E} \bigotimes \mathcal{E} = \bigoplus_{0\le i,j\le n} \mathcal{L}_i \oplus \mathcal{L}_j$. Let $u = (l_1, \dots, l_n)$ and $u' = (l'_1, \dots, l'_n)$. Then

$$\begin{aligned} u \otimes u' &= (l_1 l'_1, \dots, l_i l'_j, \dots, l_n l'_n) \\ u' \otimes u &= (l'_1 l_1, \dots, l'_i l_j, \dots, l'_n l_n) \end{aligned}$$

Then $u \otimes u' + u' \otimes u = 0$ gives that $l_i l'_j + l'_i l_j = 0$ for $0 \le i, j \le n$. Assume $i = j$. Then $l_i l'_i + l'_i l_i = 0$ for every $0 \le i \le n$, which is fulfilled if and only if $\mathcal{L}_i \oplus \mathcal{L}_i = 0$. Assume $i \ne j$. Then we have that $l_i l'_j + l'_i l_j = 0$. This is fulfilled if and only if $\mathcal{L}_i \oplus \mathcal{L}_j = -\mathcal{L}_j \oplus \mathcal{L}_i$. We therefore have that

$$\wedge^2\mathcal{E} = \mathcal{E} \otimes \mathcal{E}/v \oplus w + w \oplus v = \bigoplus_{i<j,} \mathcal{L}_i \oplus \mathcal{L}_j.$$

■

Let now $\mathcal{E}$ be a vector bundle of rank n. By the splitting we can assume the $\mathcal{E}$ splits into a direct sum of n line bundles. We write the line bundles as $\mathcal{O}_G(\alpha_i)$, where α_i is called the Chern root. The Chern roots are used to compute relation between Chern classes. When $\mathcal{E}$ is a sum of line bundles, then $\alpha_i = c_1(\mathcal{L}_i)$, but when $\mathcal{E}$ is not a sum of line bundles, then the Chern roots has no geometric origin.

$$\mathcal{E} = \mathcal{O}_G(\alpha_1) \oplus \mathcal{O}_G(\alpha_2) \oplus \cdots \oplus \mathcal{O}_G(\alpha_n).$$

By 4.17 and the splitting principle we have that

$$c_t(\mathcal{E}) = (1+\alpha_1 t)(1+\alpha_2 t)\dots(1+\alpha_n t),$$

which gives

$$\begin{aligned} c_1 &= \alpha_1 + \alpha_2 + \cdots + \alpha_n, \\ c_2 &= \alpha_1\alpha_2 + \cdots + \alpha_i\alpha_j + \cdots + \alpha_{n-1}\alpha_n, \\ &\vdots \\ c_n &= \alpha_1\alpha_2\dots\alpha_n \end{aligned}$$

By Lemma 4.3.15 we have that

$$\wedge^2(\mathcal{E}) = \mathcal{O}_G(\alpha_1+\alpha_2) \oplus \cdots \oplus \mathcal{O}_G(\alpha_i+\alpha_j) \oplus \cdots \oplus \mathcal{O}_G(\alpha_{n-1}+\alpha_n),$$

and

$$d_t(\wedge^2(\mathcal{E})) = (1+(\alpha_1+\alpha_2)t)\dots\ (1+(\alpha_i+\alpha_j)t)\dots(1+(\alpha_{n-1}+\alpha_n)t).$$

This gives

$$\begin{aligned} d_1 &= \sum_{i<j}(\alpha_i+\alpha_j) \\ d_2 &= \sum_{i<j,k<l}(\alpha_i+\alpha_j)(\alpha_k+\alpha_l) \\ &\vdots \\ d_m &= \prod_{i<j}(\alpha_i+\alpha_j), \end{aligned}$$

where $m = \frac{n(n-1)}{2}$. We have now expressed both c_i and d_i in terms of the Chern roots of $\mathcal{E}$. From these equations we can express d_i in terms of c_i. We give an example in the following

Lemma 4.3.16. *Let $\mathcal{E}$ be a rank 2 vector bundle. Let c_i be the Chern classes for $\mathcal{E}$ and d_i the Chern classes for $\wedge^2\mathcal{E}$. Then $c_1 = d_1$.*

Proof. By the splitting principle we can assume that $\mathcal{E}$ splits into a direct sum of two line bundles, that is,

$$\mathcal{E} = \mathcal{O}_G(\alpha) \oplus \mathcal{O}_G(\beta).$$

We then have that

$$c_t(\mathcal{E}) = (1+\alpha t)(1+\beta t),$$

which gives

$$\begin{aligned} c_1 &= \alpha + \beta \\ c_2 &= \alpha\beta \end{aligned}$$

We have further that

$$\wedge^2(\mathcal{E}) = \mathcal{O}_G(\alpha + \beta)$$

and

$$d_t(\wedge^2(\mathcal{E})) = (1 + (\alpha + \beta)t).$$

This shows that $d_1 = \alpha + \beta = c_1$. ■

We observe that c_1 is the same hyperplane that we found in the proof of Lemma 4.2.5 on page 25.

We now explain how we can find relations among the Chern classes of $\mathcal{U}^*$. First we need a proposition.

Proposition 4.3.17. *[Ful98, Theorem 3.2 e)] If*

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_3 \longrightarrow 0$$

is a short exact sequence of vector bundles, then $c_t(\mathcal{E}_2) = c_t(\mathcal{E}_1)c_t(\mathcal{E}_3)$.

Corollary 4.3.18. $c_t(\mathcal{U}^*)c_t(\mathcal{Q}^*) = 1$.

Proof. Since

$$0 \longrightarrow \mathcal{Q}^* \longrightarrow V_G^* \longrightarrow \mathcal{U}^* \longrightarrow 0$$

is exact, we have $c_t(\mathcal{U}^*)c_t(\mathcal{Q}^*) = c_t(V_G^*)$. We have to prove that $c_e(V_G^*) = 0$ for all $e \neq 0$. Indeed, let $s_i : G \to V_G^*$ be general global sections, where $s_i([U]) = ([U], s_i)$ for $s_i : V \to \mathbb{C}$. Then

$$\begin{aligned} c_e(V_G^*) &= [[U] \in G : s_1([U]), \ldots s_{n-e+2}([U]) \text{ are linearly dependent}] \\ &= [[U] \in G : s_1, \ldots s_{n-e+2} \text{ are linearly dependent}] \end{aligned}$$

There are no subsets $[U]$ where $n - e + 2$ general linear maps on an $n + 1$-dimensional vector space is linearly dependent. This gives that $c_e(V_G^*) = 0$ for $e \neq 0$. ■

We show how we can use this correspondence to compute relations between the Chern classes of $\mathcal{U}^*$.

Example 4.3.19. We use the technique on $G(2, 4)$. We want to find the relations between the Chern classes of $\mathcal{U}^*$.

$$\begin{aligned} c_t(\mathcal{Q}^*)c_t(\mathcal{U}^*) &= (1 + d_1t + d_2t^2)(1 + c_1t + c_2t^2) \\ &= 1 + (d_1 + c_1)t + (d_2 + c_1d_1 + c_2)t^2 + (c_1d_2 + c_2d_1)t^3 + (c_2d_2)t^4 \\ &= 1. \end{aligned}$$

This equality is fulfilled if and only if

$$\begin{aligned} d_1 + c_1 &= 0 \\ d_2 + c_1d_1 + c_2 &= 0 \end{aligned}$$

$$\begin{aligned} c_1d_2 + c_2d_1 &= 0 \\ c_2d_2 &= 0 \end{aligned}$$

The first equation gives $d_1 = -c_1$. Using this relation in the second equation gives $d_2 = c_1^2 - c_2$. The two last equation then gives

$$\begin{aligned} c_1^3 - 2c_1c_2 &= 0 \\ c_1^2c_2 - c_2^2 &= 0. \end{aligned}$$

Multiplying the first of the relation with c_1 gives that

$$c_1^4 = 2c_1^2c_2 = 2c_2^2,$$

and using that $c_2^2 = 1$ by Corollary 4.3.11 we get back the relations

$$\begin{aligned} c_1^4 &= \Omega(1,0)^4 = 2, \\ c_1^2c_2 &= \Omega(1,0)^2 \cap \Omega(1,1) = 1. \end{aligned}$$

4.4 Application to Skew Symmetric Matrices of Dimension 4, 6 and 9

We are now ready to prove two theorems about skew symmetric matrices of dimension 6 and 9 and give another proof of Theorem 4.2.6. We will prove the theorems separately, but in all cases we will use the techniques developed in the previous section.

4.4.1 Skew Symmetric Matrices of Dimension 4 - revisited

We use the same strategy that we used in the alternative proof of Theorem 4.2.6 in Section 4.2 on page 26, that is we want to show that there exists a common 2-dimensional isotropic subspace for a basis for a 3-dimensional vector space W of 4×4 skew symmetric matrices.

Alternative proof of Theorem 4.2.6. Let W be a 4-dimensional vector space and let W be a 3-dimensional vector space of 4×4 skew symmetric matrices. Let A_1, A_2 and A_3 be a basis for W and let $X = G(2, V)$. We have that $A_i|_U \in \wedge^2\mathcal{U}^*$. Since $\dim \wedge^2\mathcal{U}^* = 1$, the isotropic subspaces $U \subset V$ to $A_i|_U$ is $c_1(\wedge^2\mathcal{U}^*)$. We have from Lemma 4.3.16 that $c_1(\wedge^2\mathcal{U}^*) = c_1(\mathcal{U}^*)$. We have a common isotropic subspace we for A_1, A_2 and A_3 is $c_1(\mathcal{U}^*)^3$ is non empty. Indeed, we have that

$$\begin{aligned} c_1(\mathcal{U}^*)^3 &= c_1^3 \\ &= 2\Omega(2,1). \end{aligned}$$

Since $c_1^3c_1 = 2$, we have that there exists a conic of common isotropic subspaces for A_1, A_2 and A_3. ■

4.4.2 Skew Symmetric Matrices of Dimension 6

The result in dimension 6 is the following

Theorem 4.4.1. *Let W be a 3-dimensional vector space of 6×6 skew symmetric matrices. Then there exists two 3-dimensional isotropic subspaces to W.*

We will now prove this theorem, but first we need four lemmas.

Lemma 4.4.2. *Let W be a 3-dimensional vector space of 6×6 skew symmetric matrix and let U be a 3-dimensional isotropic subspace for W. Then we can choose a row basis for W such that $a_{34} = a_{35} = a_{45} = 0$ for every matrix in W.*

Proof. Let the row basis be such that for $u, v, w \in U$ we have that

$$\begin{aligned} u &= (0,0,0,1,0,0), \\ v &= (0,0,0,0,1,0), \\ w &= (0,0,0,0,0,1). \end{aligned}$$

Pick $A \in W$. Then

$$\begin{aligned} uAv^T &= a_{34}, \\ uAw^T &= a_{35}, \\ vAw^T &= a_{45}. \end{aligned}$$

As a consequence, $a_{34} = a_{35} = a_{45} = 0$ if $uAv^T = 0$ for all $u, v \in U$. ■

By Proposition 4.3.13, the equivalence class of the set of all 3-dimensional isotropic subspaces to a 6×6 skew symmetric matrix is $c_3(\wedge^2 \mathcal{U}^*)$. We now express $c_3(\wedge^2 \mathcal{U}^*)$ in terms of the Chern classes of $\mathcal{U}^*$ by using the following

Lemma 4.4.3. *Let $\mathcal{E}$ be a rank 3 vector bundle. Let c_i be the Chern classes for $\mathcal{E}$ and d_i the Chern classes for $\wedge^2 \mathcal{E}$. Then $d_3 = c_1 c_2 - c_3$.*

Proof. We use the strategy we used in Lemma 4.3.16. Since $\mathcal{E}$ is a rank 3 vector bundle, we get that

$$\begin{aligned} c_1 &= \alpha + \beta + \gamma, \\ c_2 &= \alpha\beta + \alpha\gamma + \beta\gamma, \\ c_3 &= \alpha\beta\gamma, \end{aligned}$$

where α, β, γ are the Chern roots of $\mathcal{E}$. We then have that

$$\begin{aligned} d_3 &= (\alpha+\beta)(\alpha+\gamma)(\beta+\gamma) \\ &= (\alpha+\beta)(\alpha\beta+\alpha\gamma+\beta\gamma+\gamma\gamma) \\ &= \alpha(\alpha\beta+\alpha\gamma+\beta\gamma)+\beta(\alpha\beta+\alpha\gamma+\beta\gamma)+\gamma(\alpha\beta+\alpha\gamma+\beta\gamma)-\alpha\beta\gamma \\ &= (\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)-\alpha\beta\gamma \\ &= c_1 c_2 - c_3 \end{aligned}$$

■

Lemma 4.4.4. *Let $G = G(3,6)$ and let c_1, c_2, c_3 be the Chern classes for $\mathcal{U}^*$. We then have the following relations.*

$$\begin{aligned} 3c_1^2 c_2 - 2c_1 c_3 - c_1^4 - c_2^2 &= 0 \\ 2c_1 c_2^2 - c_1^3 c_2 - 2c_2 c_3 + c_1^2 c_3 &= 0 \\ 2c_1 c_2 c_3 - c_1^3 c_3 - c_3^2 &= 0 \end{aligned}$$

Proof. We use the strategy described above and get:

$$\begin{aligned} c_t(\mathcal{Q})c_t(\mathcal{U}^*) = 1 &+ (c_1+d_1)t \\ &+ (d_2+c_1d_1+c_2)t^2 \\ &+ (d_3+c_1d_2+c_2d_1+c_3)t^3 \\ &+ (c_1d_3+c_2d_2+c_3d_1)t^4 \end{aligned}$$

$$
\begin{aligned}
&+(c_2d_3 + c_3d_2)t^5 \\
&\qquad\qquad +c_3d_3t^6 = 1,
\end{aligned}
$$

where d_i are the Chern classes of $\mathcal{Q}$. The equality holds if and only if the following equalities hold:

$$
\begin{aligned}
c_1 + d_1 &= 0, \\
d_2 + c_1d_1 + c_2 &= 0, \\
c_1d_2 + c_2d_1 + c_3 &= 0, \\
c_1d_3 + c_2d_2 + c_3d_1 &= 0, \\
c_2d_3 + c_3d_2 &= 0, \\
c_3d_3 &= 0.
\end{aligned}
$$

Solving these equation recursively gives the relations in the lemma. ■

Proposition 4.4.5. *Let $G = G(3,6)$ and let c_1, c_2, c_3 be the Chern classes for $\mathcal{U}^*$. We then have the following number of intersection points.*

$$
\begin{aligned}
c_1^9 &= 42 \\
c_1^7c_2 &= 21 \\
c_1^5c_2^2 &= 11 \\
c_1^6c_3 &= 5 \\
c_1^3c_2^3 &= 6 \\
c_1^4c_2c_3 &= 3
\end{aligned}
\qquad
\begin{aligned}
c_1c_2^4 &= 3 \\
c_1^2c_2^2c_3 &= 2 \\
c_1^3c_3^2 &= 1 \\
c_2^3c_3 &= 1 \\
c_1c_2c_3^2 &= 1
\end{aligned}
$$

Proof. Consider the relations given in Lemma 4.4.4. The first relations we multiply with the five monomials of degree three, i.e. $c_1^5, c_1^3c_2, c_1c_2^2, c_1^2c_3, c_2c_3$. The second and third relation we multiply with the monomials of degree four, $c_1^4, c_1^2c_2, c_2^2, c_1c_3$, and three, c_1^3, c_1c_2 and c_3, respectively. We get the following equations.

$$
\begin{aligned}
3c_1^7c_2 - 2c_1^6c_3 - c_1^9 - c_1^5c_2^2 &= 0 \\
3c_1^5c_2^2 - 2c_1^4c_2c_3 - c_1^7c_2 - c_1^3c_2^3 &= 0 \\
3c_1^3c_2^3 - 2c_1^2c_2^2c_3 - c_1^5c_2^2 - c_1c_2^4 &= 0 \\
3c_1^4c_2c_3 - 2c_1^3c_3^2 - c_1^6c_3 - c_1^2c_2^2c_3 &= 0 \\
3c_1^2c_2^2c_3 - 2c_1c_2c_3^2 - c_1^4c_2c_3 - c_2^3c_3 &= 0 \\
2c_1^5c_2^2 - c_1^7c_2 - 2c_1^4c_2c_3 + c_1^6c_3 &= 0 \\
2c_1^3c_2^3 - c_1^5c_2^2 - 2c_1^2c_2^2c_3 + c_1^4c_2c_3 &= 0 \\
2c_1c_2^4 - c_1^3c_2^3 - 2c_2^3c_3 + c_1^2c_2^2c_3 &= 0 \\
2c_1^2c_2^2c_3 - c_1^4c_2c_3 - 2c_1c_2c_3^2 + c_1^3c_3^2 &= 0 \\
2c_1^4c_2c_3 - c_1^6c_3 - c_1^3c_3^2 &= 0 \\
2c_1^2c_2^2c_3 - c_1^4c_2c_3 - c_1c_2c_3^2 &= 0 \\
2c_1c_2c_3^2 - c_1^3c_3^2 - c_3^3 &= 0
\end{aligned}
$$

$$\begin{pmatrix} -1 & 3 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 3 & 0 & -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 3 & 0 & -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 3 & 0 & -1 & -2 & 0 \\ 0 & -1 & 2 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} c_1^9 \\ c_1^7c_2 \\ c_1^5c_2^2 \\ c_1^6c_3 \\ c_1^3c_2^3 \\ c_1^4c_2c_3 \\ c_1c_2^4 \\ c_1^2c_2^2c_3 \\ c_1^3c_3^2 \\ c_2^3c_3 \\ c_1c_2c_3^2 \\ c_3^3 \end{pmatrix} = 0.$$

Row reduction gives

$$\begin{aligned} c_1^9 &= 42c_3^3 \\ c_1^7c_2 &= 21c_3^3 \\ c_1^5c_2^2 &= 11c_3^3 \\ c_1^6c_3 &= 5c_3^3 \\ c_1^3c_2^3 &= 6c_3^3 \\ c_1^4c_2c_3 &= 3c_3^3 \end{aligned} \qquad \begin{aligned} c_1c_2^4 &= 3c_3^3 \\ c_1^2c_2^2c_3 &= 2c_3^3 \\ c_1^3c_3^2 &= c_3^3 \\ c_2^3c_3 &= c_3^3 \\ c_1c_2c_3^2 &= c_3^3 \end{aligned}$$

Since $c_3^3 = 1$ by Corollary 4.3.11 we have the relations in the lemma. ■

We are now ready to give a proof of Theorem 4.4.1 on page 47.

Proof of Theorem 4.4.1. We want to prove that there is a common 3-dimensional isotropic subspace for a basis of matrices in W. Let A_1, A_2 and A_3 be a basis for W. The basis matrices A_1, A_2 and A_3 has a common isotropic subspace if the intersection

$$c_3(\wedge^2(\mathcal{U}^*))^3$$

is non empty, which is the case. Indeed,

$$\begin{aligned} (c_1c_2 - c_3)^3 &= c_1^3c_2^3 - 3c_1^2c_2^2c_3 + 3c_1c_2c_3^2 - c_3^3 \\ &= 6c_3^3 - 3 \cdot 2c_3^3 + 3c_3^3 - c_3^3 \\ &= 2c_3^3 \\ &= 2. \end{aligned}$$

This means that A_1, A_2 and A_3 have two common 3-dimensional isotropic subspaces. ■

Remark 4.4.6. Let I be the ideal generated by the relations in Lemma 4.4.4. Then I is the is the apolar ideal of a homogeneous polynomial f in c_1, c_2 and c_3 of degree 6. We want to show this by finding the polynomial f. Our strategy is to use the relations in Lemma 4.4.4 as a differentiation operator on a polynomial with unknown coefficients. We then get a linear system of relations between the coefficients of the polynomial which we solve using row reduction. Let

$$f = a_0c_1^9 + a_1c_1^7c_2 + a_2c_1^6c_3 + a_3c_1^5c_2^2 + a_4c_1^4c_2c_3 + a_5c_1^3c_2^3 + a_6c_1^3c_3^2 + a_7c_1^2c_2^2c_3 + a_8c_1c_2^4 + a_9c_1c_2c_3^2 + a_{10}c_2^3c_3 + a_{11}c_3^3$$

be a polynomial, where $a_i \in \mathbb{Z}$. We will now differentiate f with respect to each of the relations in Lemma 4.4.4. We consider the relations as polynomials in $\mathbb{Z}[d_1, d_2, d_3]$. We start with the first relation.

Let $g_1 = 3d_1^2d_2 - 2d_1d_3 - d_1^4 - d_2^2$. Then

$$\begin{aligned} f(g_1) =& 126a_1c_1^5 + 120a_3c_1^3c_2 + 36a_4c_1^2c_3 + 54a_5c_1c_2^2 + 12a_7c_2c_3 \\ &- 12a_2c_1^5 - 8a_4c_1^3c_2 - 12a_6c_1^2c_3 - 4a_7c_1c_2^2 - 4a_9c_2c_3 \\ &- 3024a_0c_1^5 - 840a_1c_1^3c_2 - 360a_2c_1^2c_3 - 120a_3c_1c_2^2 - 24a_4c_2c_3 \\ &- 2a_3c_1^5 - 6a_5c_1^3c_2 - 2a_7c_1^2c_3 - 12a_8c_1c_2^2 - 6a_{10}c_2c_3 \end{aligned}$$

Gathering the coefficients of each of the monomials $c_1^5, c_1^3c_2, c_1^2c_3, c_1c_2^2$ and c_2c_3 gives the following relations between the coefficients for g_1 to be apolar to f.

$$-3024a_0 + 126a_1 - 12a_2 - 2a_3 = 0, \tag{4.18}$$
$$-840a_1 + 120a_3 - 8a_4 - 6a_5 = 0, \tag{4.19}$$
$$-360a_2 + 36a_4 - 12a_6 - 2a_7 = 0, \tag{4.20}$$
$$-120a_3 + 54a_5 - 4a_7 - 12a_8 = 0, \tag{4.21}$$
$$-24a_4 + 12a_7 - 4a_9 - 6a_{10} = 0. \tag{4.22}$$

Let now $g_2 = d_1d_2^2 - d_1^3d_2 - 2d_2d_3 + d_1^2d_3$. Then

$$\begin{aligned} f(g_2) =& 20a_3c_1^4 + 36a_5c_1^2c_2 + 8a_7c_1c_3 + 24a_8c_2^2 \\ &- 210a_1c_1^4 - 120a_3c_1^2c_2 - 24a_4c_1c_3 - 18a_5c_2^2 \\ &- 2a_4c_1^4 - 4a_7c_1^2c_2 - 4a_9c_1c_3 - 6a_{10}c_2^2 \\ &+ 30a_2c_1^4 + 12a_4c_1^2c_2 + 12a_6c_1c_3 + 2a_7c_2^2 \end{aligned}$$

Gathering the coefficients of each of the monomials $c_1^4, c_1^2c_2, c_1c_3$ and c_2^2 gives the following relations between the coefficients for g_2 to be apolar to f.

$$-210a_1 + +30a_2 + 20a_3 - 2a_4 = 0, \tag{4.23}$$
$$-120a_3 + 12a_4 + 36a_5 - 4a_7 = 0, \tag{4.24}$$
$$-24a_4 + 12a_6 + 8a_7 - 4a_9 = 0, \tag{4.25}$$
$$-18a_5 + 2a_7 + 24a_8 - 6a_{10} = 0. \tag{4.26}$$

Let now $g_3 = 2d_1d_2d_3 - d_1^3d_3 - d_3^2$. Then

$$\begin{aligned} f(g_3) =& 8a_4c_1^3 + 8a_7c_1c_2 + 4a_9c_3 \\ &- 120a_2c_1^3 - 24a_4c_1c_2 - 12a_6c_3 \\ &- 2a_6c_1^3 - 2a_9c_1c_2 - 6a_{11}c_3 \end{aligned}$$

Gathering the coefficients of each of the monomials c_1^3, c_1c_2 and c_3 gives the following relations between the coefficients for g_3 to be apolar to f.

$$-120a_2 + 8a_4 - 2a_6 = 0 \tag{4.27}$$
$$-24a_4 + 8a_7 - 2a_9 = 0 \tag{4.28}$$
$$-12a_6 + 4a_9 - 6a_{11} = 0 \tag{4.29}$$

We make the coefficient matrix of the system of equations from from the relations 4.18-4.29.

$$\begin{pmatrix} -3024 & 126 & -12 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -840 & 0 & 120 & -8 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -360 & 0 & 36 & 0 & -12 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -120 & 0 & 54 & 0 & -4 & -12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -24 & 0 & 0 & 12 & 0 & -4 & -6 & 0 \\ 0 & -210 & 30 & 20 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -120 & 12 & 36 & 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -24 & 0 & 12 & 8 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -18 & 0 & 2 & 24 & 0 & -6 & 0 \\ 0 & 0 & -120 & 0 & 8 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -24 & 0 & 0 & 8 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 & 0 & 0 & 4 & 0 & -6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \end{pmatrix} = 0.$$

Row reducing this matrix and setting $a_{11} = \frac{1}{6}$ gives the solutions

$$a_0 = \frac{42}{9!} \qquad a_1 = \frac{21}{7!} \qquad a_2 = \frac{5}{6!} \qquad a_3 = \frac{11}{5!2!} \qquad a_4 = \frac{3}{4!} \qquad a_5 = \frac{6}{3!3!}$$

$$a_6 = \frac{1}{3!2!} \qquad a_7 = \frac{2}{2!2!} \qquad a_8 = \frac{3}{4!} \qquad a_9 = \frac{1}{2!} \qquad a_{10} = \frac{1}{3!}$$

We recognize the numerator of the coefficients as the number of points we get when intersecting the given Chern classes, and the denominator as the factorial of the degree of each variable c_i. This means that differentiating the polynomial with respect to a variable $c_1^i c_2^j c_3^k$ where $i + j + k = 9$, gives the number of intersection points of $c_1^i c_2^j c_3^k$.

4.4.3 Skew Symmetric Matrices of Dimension 9

The result in dimension 9 is the following

Theorem 4.4.7. *Let W be a 3-dimensional vector space of 9×9 skew symmetric matrices. Then there exists a surface of degree 38 of 4-dimensional isotropic subspaces to W.*

We will now prove this theorem, but first we need four lemmas.

Lemma 4.4.8. *Let W be a 3-dimensional vector space of 9×9 skew symmetric matrix and let U be an isotropic subspace for W. Then we can choose a row basis for W such that $a_{67} = a_{68} = a_{69} = a_{78} = a_{79} = a_{89} = 0$ for every matrix i W.*

Proof. Let the row basis be such that for $u, v, w, t \in U$ we have that

$$u = (0, 0, 0, 0, 0, 1, 0, 0, 0),$$
$$v = (0, 0, 0, 0, 0, 0, 1, 0, 0),$$
$$w = (0, 0, 0, 0, 0, 0, 0, 1, 0),$$
$$t = (0, 0, 0, 0, 0, 0, 0, 0, 1).$$

Pick $A \in W$. Then

$$\begin{aligned} uAv^T &= a_{67}, \\ uAw^T &= a_{68}, \\ uAt^T &= a_{69}, \\ vAw^T &= a_{78}, \\ vAt^T &= a_{79}, \\ wAt^T &= a_{89}. \end{aligned}$$

As a consequence, $a_{67} = a_{68} = a_{69} = a_{78} = a_{79} = a_{89} = 0$ if $uAv^T = 0$ for all $u, v \in U$. ■

By Proposition 4.3.13, the equivalence class of the set of all 4-dimensional isotropic subspaces to a 9×9 skew symmetric matrix is $c_6(\wedge^2 \mathcal{U}^*)$. We now express $c_6(\wedge^2 \mathcal{U}^*)$ in terms of the Chern classes of $\mathcal{U}^*$ by using the following

Lemma 4.4.9. *Let $\mathcal{E}$ be a rank 4 vector bundle. Let c_i be the Chern classes for $\mathcal{E}$ and d_i the Chern classes for $\wedge^2 \mathcal{E}$. Then $d_6 = c_1c_2c_3 - c_1^2c_4 - c_3^2$.*

Proof. We use the strategy we used in Lemma 4.3.16. Since $\mathcal{U}$ is a rank 4 vector bundle, we get that

$$\begin{aligned} c_1 &= \alpha + \beta + \gamma + \delta, \\ c_2 &= \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta \\ c_3 &= \alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta, \\ c_4 &= \alpha\beta\gamma\delta, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are the Chern roots of $\mathcal{U}$. We have that

$$d_6 = (\alpha + \beta)(\alpha + \gamma)(\alpha + \delta)(\beta + \gamma)(\beta + \delta)(\gamma + \delta).$$

We have used Macaulay2 to check that d_6 and $c_1c_2c_3 - c_1^2c_4 - c_3^2$ are equal. ■

Next we compute the number of points the 0-dimensional. First we prove a lemma about the relation between the Chern classes of $\mathcal{U}^*$.

Lemma 4.4.10. *Let $G = G(4, 9)$ and let c_1, c_2, c_3, c_4 be the Chern classes for $\mathcal{U}^*$. We then have the following relations.*

$$\begin{aligned} 3c_1^2c_4 - 2c_2c_4 + 6c_1c_2c_3 - 4c_1^3c_3 - c_3^2 - 6c_1^2c_2^2 + c_2^3 + 5c_1^4c_2 - c_1^6 &= 0 \\ 4c_1c_2c_4 - c_1^3c_4 - 2c_3c_4 + 2c_1c_3^2 - 6c_1^2c_2c_3 + 3c_2^2c_3 + c_1^4c_3 - 3c_1c_2^3 + 4c_1^3c_2^2 - c_1^5c_2 &= 0 \\ 4c_1c_3c_4 - c_4^2 - 3c_1^2c_2c_4 + c_2^2c_4 + c_1^4c_4 - 3c_1^2c_3^2 + 2c_2c_3^2 - 3c_1c_2^2c_3 + 4c_1^3c_2c_3 - c_1^5c_3 &= 0 \\ 2c_1c_4^2 - 3c_1c_3c_4 + 2c_2c_3c_4 - 3c_1c_2^2c_4 + 4c_1^3c_2c_4 - c_1^5c_4 &= 0 \end{aligned}$$

Proof. We use the strategy described above and get:

$$\begin{aligned} c_t(\mathcal{Q})c_t(\mathcal{U}^*) = 1 + (c_1+d_1)t \\ +(d_2+c_1d_1+c_2)t^2 \\ +(d_3+c_1d_2+c_2d_1+c_3)t^3 \\ +(d_4+c_1d_3+c_2d_2+c_3d_1+c_4)t^4 \\ +(d_5+c_1d_4+c_2d_3+c_3d_2+c_4d_1)t^5 \\ +(c_1d_5+c_2d_4+c_3d_3+c_4d_2)t^6 \\ +(c_2d_5+c_3d_4+c_4d_3)t^7 \\ +(c_3d_5+c_4d_4)t^8 \\ +c_4d_5t^9 = 1, \end{aligned}$$

where d_i are the Chern classes of $\mathcal{Q}$. The equality holds if and only if the following equalities hold:

$$\begin{aligned} c_1+d_1 &= 0, \\ d_2+c_1d_1+c_2 &= 0, \\ d_3+c_1d_2+c_2d_1+c_3 &= 0, \\ d_4+c_1d_3+c_2d_2+c_3d_1+c_4 &= 0, \\ d_5+c_1d_4+c_2d_3+c_3d_2+c_4d_1 &= 0, \\ c_1d_5+c_2d_4+c_3d_3+c_4d_2 &= 0, \\ c_2d_5+c_3d_4+c_4d_3 &= 0, \\ c_3d_5+c_4d_4 &= 0, \\ c_4d_5 &= 0. \end{aligned}$$

Solving these equation recursively gives the relations in the lemma. ∎

Proposition 4.4.11. *Let $G = G(4,9)$ and let c_1, c_2, c_3, c_4 be the Chern classes for $\mathcal{U}^*$. We then have the following number of intersection points.*

$$\begin{aligned} c_1^8c_4^3 &= 14 & c_1^5c_2c_3^3c_4 &= 59 \\ c_1^7c_2c_3c_4^2 &= 49 & c_1^6c_3^2c_4^2 &= 19 \\ c_1^6c_2^2c_3^2c_4 &= 164 & c_1^3c_2c_3^5 &= 59 \\ c_1^5c_2^3c_3^3 &= 539 & c_1^4c_3^4c_4 &= 24 \\ c_1^4c_2^2c_3^4 &= 175 & c_1^2c_3^6 &= 19 \end{aligned}$$

Proof. We follow the same strategy as in the proof of Proposition 4.4.5, that is we multiply the relations in Lemma 4.4.10 with all monomials of degree $14, 13, 12$ and 11, respectively. We use the program in Listing 10.1. There are 47 monomials of degree 14, 38 of degree 13, 34 of degree 12 and 27 of degree 11. This gives 146 relations. There are 134 monomials of degree 20, so in summary we get a matrix of dimension 146×134. After row reducing this matrix we pick out the relations

$$\begin{aligned} c_1^8c_4^3 &= 14c_4^5 & c_1^5c_2c_3^3c_4 &= 59c_4^5 \\ c_1^7c_2c_3c_4^2 &= 49c_4^5 & c_1^6c_3^2c_4^2 &= 19c_4^5 \\ c_1^6c_2^2c_3^2c_4 &= 164c_4^5 & c_1^3c_2c_3^5 &= 59c_4^5 \\ c_1^5c_2^3c_3^3 &= 539c_4^5 & c_1^4c_3^4c_4 &= 24c_4^5 \\ c_1^4c_2^2c_3^4 &= 175c_4^5 & c_1^2c_3^6 &= 19c_4^5 \end{aligned}$$

Since $c_4^5 = 1$ by Corollary 4.3.11 we get the relations in the propositions. ∎

We are now ready to give a proof of Theorem 4.4.7 on page 52.

Proof of Theorem 4.4.1. We want to prove that there is a common 3-dimensional isotropic subspace for a basis of matrices in W. Let A_1, A_2 and A_3 be a basis for W. The basis matrices A_1, A_2 and A_3 has a common isotropic subspace if the intersection

$$c_3(\wedge^2(\mathcal{U}^*))^3$$

is non empty, which is the case. Indeed,

$$\begin{aligned}(c_1c_2c_3 - c1^2c_4 - c_3^2)^3 =& c_1^3c_2^3c_3^3 - 3c_1^4c_2^2c_3^2c_4 + 3c_1^5c_2c_3c_4^2 - c_1^6c_4^3 - 3c_1^2c_2^2c_3^4 \\ &+ 6c_1^3c_2c_3^3c_4 - 3c_1^4c_3^2c_4^2 + 3c_1c_2c_3^5 - 3c_1^2c_3^4c_4 - c_3^6\end{aligned}$$

The degree is

$$\begin{aligned}c_1^2(c_1c_2c_3 - c1^2c_4 - c_3^2)^3 =& c_1^5c_2^3c_3^3 - 3c_1^6c_2^2c_3^2c_4 + 3c_1^7c_2c_3c_4^2 - c_1^8c_4^3 - 3c_1^4c_2^2c_3^4 \\ &+ 6c_1^5c_2c_3^3c_4 - 3c_1^6c_3^2c_4^2 + 3c_1^3c_2c_3^5 - 3c_1^4c_3^4c_4 - c_1^2c_3^6 \\ =& 539 - 3\cdot 164 + 3\cdot 49 - 14 - 3\cdot 175 \\ &+ 6\cdot 59 - 3\ \ \cdot 19 + 3\cdot 59 - 3\dot{2}4 - 19 \\ =& 38\end{aligned}$$

This means that A_1, A_2 and A_3 have a surface of degree 38 of common 4-dimensional isotropic subspaces. ■

5 | Finite Schemes in $\mathbb{P}^2$

In this chapter we explain how we can relate a zero-dimensional ideal I_Γ to the apolar ideal $F^\perp$ of a ternary sextic form F. To find the cactus rank of F, we are looking for minimal schemes Γ such that $I_\Gamma \subset F^\perp$. In particular, we are interested in the ideals I_Γ that are generated by a subset of the generators of $F^\perp$.

The chapter is organized as follows. First, in Section 5.1, we introduce a structure theorem for zero-dimensional ideals I_Γ of schemes Γ $\mathbb{P}^2$ and investigate some properties of the set of schemes. Then we use the structure theorem to associate a *degree matrix* to some specially chosen schemes. Lastly, in Section 5.2, we go through each of the Betti strata $\mathcal{F}_B$ (recall Definition 1.1.6 on page 3) and prove which type of I_Γ that will appear as a subideal of $F^\perp$ for an $[F] \in \mathcal{F}_B$. In Chapter 6, we prove that the subideal we find, actually will by a minimal subideal of $F^\perp$ for an F.

5.1 Hilbert–Burch Matrices of some Finite Schemes in $\mathbb{P}^2$

We have the following structure theorem for finite schemes in $\mathbb{P}^2$.

Theorem 5.1.1. *[Eis06, Theorem 20.15 and p. 503] Let $\Gamma \subset \mathbb{P}^2$ be a finite scheme. Then $I_\Gamma \subset T = \mathbb{C}[y_0, y_1, y_2]$ is generated by the $(\beta-1)\times(\beta-1)$ minors of a $(\beta-1)\times\beta$ matrix, A_Γ. The resolution of I_Γ is*

$$0 \longrightarrow T^{\beta-1} \xrightarrow{A_\Gamma} T^\beta \longrightarrow I_\Gamma \longrightarrow 0.$$

If A_Γ is a $(\beta-1)\times\beta$ matrix where the $(\beta-1)\times(\beta-1)$ minors have no common factor, then the minors generate the ideal of a finite scheme.

The finite schemes of length d in $\mathbb{P}^2$ are parameterized by the *Hilbert scheme*

$$\operatorname{Hilb}_d \mathbb{P}^2 = \{\Gamma \subset \mathbb{P}^2 : \Gamma \text{ has length } d\},$$

[Gro61]. Now, we describe a subscheme of $\operatorname{Hilb}_d \mathbb{P}^2$. Let $P = (p_{ij})$ be a $(\beta-1)\times\beta$ matrix where $p_{ij} \in \mathbb{C}[y_0, y_1, y_2]$ is a homogeneous polynomial. Let the *degree matrix* be the matrix $M_C = (m_{ij})$, where $m_{ij} = \deg p_{ij}$. Since the Hilbert polynomial of I_Γ is determined by the degree matrix of A_Γ, every scheme Γ of an ideal I_Γ that is generated by a matrix P with degree matrix M_C has the same length. Therefore, we define

$$\operatorname{Hilb}_d^C = \{\Gamma \in \operatorname{Hilb}_d \mathbb{P}^2 : A_\Gamma \text{ has degree matrix } M_C\} \subset \operatorname{Hilb}_d \mathbb{P}^2.$$

We will prove that Hilb_d^C is irreducible. In order to do that we need some definitions and results.

Let $\alpha = (\alpha_0, \dots, \alpha_n)$, where $|\alpha| = \alpha_0 + \cdots + \alpha_n$ and $x^\alpha = x_0^{\alpha_0} \dots x_n^{\alpha_n}$. We write $F_i = \sum_{|\alpha|=d_i} c_{i,\alpha} x^\alpha$ for a homogeneous polynomial of degree d_i. Given a polynomial $P \in \mathbb{C}[u_{i,\alpha}]$ we let $P(F_1, \dots, F_n)$ denote the number obtained by replacing the variable $u_{i,\alpha}$ by the corresponding coefficient $c_{i,\alpha}$. We say that P is a *polynomial in the coefficients of the F_i*.

Theorem 5.1.2. *[CLO05, (2.3) Theorem] Let $F_0, \dots, F_n \in \mathbb{C}[x_0, \dots, x_n]$ be homogeneous polynomials. Then the exists a unique polynomial* $\mathrm{Res} \in \mathbb{Z}[u_{i,\alpha}]$ *such that $F_0 = \dots = F_n = 0$ has a nontrivial solution over $\mathbb{C}$ if and only if* $\mathrm{Res}(F_0, \dots, F_n) = 0$.

Lemma 5.1.3. *Fix a degree matrix M_C and let V_C be the vector space of $(\beta - 1) \times \beta$ matrices $P = (p_{ij})$, where $p_{ij} \in T = \mathbb{C}[y_0, y_1, y_2]$ are homogeneous, with degree matrix M_C. Let V_C^0 be the subset of V_C consisting of the matrices where the $(\beta - 1) \times (\beta - 1)$ minors has no common factor. Then V_C^0 is irreducible.*

Proof. Since the degree of the polynomials in the entries of $P \in V_C$ are fixed, the total number N of coefficients of the polynomials are also fixed. Then there is a $1-1$-correspondence between points in $\mathbb{A}^N$ and matrices in V_C. We will prove that V_C^0 is open subset, and thus irreducible, by proving that the complement is closed.

Fix P and let I_P be the ideal of the $(\beta - 1) \times (\beta - 1)$ minors of P. If there is a common factor between the $(\beta - 1) \times (\beta - 1)$ minors, there is also a common factor between the generators of every 2-dimensional subspace of I_P. Let L be a general linear form and let F_1 and F_2 be generators for a 2-dimensional subspace of I_P. Then $F_1 = F_2 = L = 0$ has a non-trivial solution if and only if F_1 and F_2 has a common factor. By Theorem 5.1.2, this is fulfilled if and only if $\mathrm{Res}(F_1, F_2, L) = 0$. Since $\mathrm{Res}(F_1, F_2, L)$ is a polynomial in the coefficients of F_i and L, and F_i are determined by the coefficients of p_{ij}, we get one polynomial R in the coefficients of p_{ij} for each 2-dimensional subspace and for each linear form. There is a common factor between the generators of I_P if and only if the ideal spanned by the polynomials R vanish. The subset of V_C where the coefficients satisfies this condition is closed. This shows that the complement, V_C^0, is open. Since an open subset of $\mathbb{A}^1$ is irreducible, and $V_C^0 \subset V_C \simeq \mathbb{A}^1$, V_C^0 is irreducible. ∎

Proposition 5.1.4. *The subscheme* $\mathrm{Hilb}_d^C \subset \mathrm{Hilb}_d \mathbb{P}^2$ *is irreducible.*

Proof. Consider the surjective map ϕ given by

$$\begin{aligned} \phi : V_C^0 &\to \mathrm{Hilb}_d^C \\ P &\mapsto \Gamma_P, \end{aligned}$$

where Γ_P is the scheme corresponding to I_P generated by the $(\beta - 1) \times (\beta - 1)$ minors of P. The map is well-defined since, by assumption, the $(\beta - 1) \times (\beta - 1)$ minors have no common factor, and by Theorem 5.1.1, I_P is the ideal of a finite scheme. The map is indeed surjective, since every finite scheme has a corresponding Hilbert Burch matrix, Theorem 5.1.1. Since Hilb_d^C is the image under a map and V_C^0 is irreducible, then Hilb_d^C is irreducible. ∎

5.1.1 Finite Schemes in $\mathbb{P}^2$ of Length 4 and 5

In this section we describe and classify all matrices that is a degree matrix of a Hilbert–Burch matrix of I_Γ where Γ is of length four or five.

We recall a special case of Bezout's theorem.

Theorem 5.1.5. *[BE77, Corollary 7.8] Let $F_1, F_2 \subset T = \mathbb{C}[y_0, y_1, y_2]$ and let $C_1 = V(F_1)$ and $C_2 = V(F_2)$ be two curves in $\mathbb{P}^2$ degree d and d', where F_1 and F_2 have no common factor. Assume that C_1 and C_2 intersect transversely. Then the Hilbert polynomial of $T/\langle F_1, F_2\rangle$ is dd'.*

When two curves fulfill the condition in Bezout's theorem we say that they intersect in a complete intersection (CI).

Now, we state and prove some lemmas about ideal of finite schemes that we will use to prove the main result about the degree matrices.

Let I and J be two ideals. Then the set $(I : J) = \{r \in T : rs \in I \text{ for all } s \in J\}$ is called the *colon ideal* of I and J. The scheme defined by the colon ideal is called the *residual scheme*.

Lemma 5.1.6. *Let $I_{\Gamma_j} \subset I_{\Gamma_i}$ be ideals of schemes of length i and j, respectively. Then $(I_{\Gamma_j} : I_{\Gamma_i}) = I_{\Gamma_{j-i}}$. If* $\mathrm{Supp}(I_{\Gamma_{j-i}} \cap I_{\Gamma_i}) = \emptyset$*, then $I_{\Gamma_{j-i}}$ is the ideal of a subscheme of Γ_j not containing Γ_i.*

Proof. Let $p \in I_{\Gamma_{j-i}}$ and $q \in I_{\Gamma_i}$. Then $pq \in I_{\Gamma_j}$. For the other implication, let $r \in T$ be such that $r \in (I_{\Gamma_j} : I_{\Gamma_i})$ and $s \in I_{\Gamma_i}$, while $s \notin I_{\Gamma_{j-i}}$. Then $rs \in I_{\Gamma_j}$, thus $r \in I_{\Gamma_{j-i}}$. ∎

Lemma 5.1.7. *Let I_{Γ_4} be an ideal of a scheme of length four generated by two quadratic forms with a common linear factor, L, and one cubic form that does not has L as a factor. Then there exists linear forms L_1 and L_2 and quadratic forms Q_1 and Q_2 such that $I_{\Gamma_4} = \langle LL_1, LL_2, Q_2L_1 + L_2Q_1\rangle$.*

Proof. Pick a quadratic form $LL_1 \in I_{\Gamma_4}$ and a cubic form $K \in I_{\Gamma_4}$ such that LL_1 and K has no common factor. Consider $I_{\Gamma_6} = \langle LL_1, K\rangle$, which is the ideal of a scheme of length six containing Γ_4. Then there exists a scheme $\Gamma_2 \subset \Gamma_6$ of length two not containing Γ_4 generated by L_1 and a quadratic form Q_1, i.e. $\langle L_1, Q_1\rangle = I_{\Gamma_2}$. Since $I_{\Gamma_6} \subset I_{\Gamma_2}$, we can find a linear form L_2 and a quadratic form Q_2 such that $K = Q_2L_1 + L_2Q_1$. Consider the matrix

$$\begin{pmatrix} L_1 & 0 & -L_2 \\ Q_1 & L & Q_2 \end{pmatrix}.$$

The minors of the matrix generate I_{Γ_4}. Indeed, two of the minors are $LL_1 \in I_{\Gamma_4}$ and $K \in I_{\Gamma_4}$ and the last minor is LL_2. We want to show that $LL_2 \in (I_{\Gamma_6} : I_{\Gamma_2}) = I_{\Gamma_4}$. We show this by direct computations on the generators L_1 and Q_1. Indeed, we have that $(LL_2)L_1 = (LL_1)L_2 \in I_{\Gamma_6}$. For the product $(LL_2)Q_1$, consider the determinant of the following matrix:

$$\begin{pmatrix} Q_1 & L & Q_2 \\ L_1 & 0 & -L_2 \\ Q_1 & L & Q_2 \end{pmatrix}.$$

Since two rows are equal, the determinant equals zero. We get that

$$\begin{aligned} Q_1LL_2 - LK + Q_2LL_1 &= 0 \\ Q_1LL_2 &= LK - Q_2LL_1, \end{aligned}$$

which gives that $(LL_2)Q_1 \in I_{\Gamma_6}$ and $LL_2 \in (I_{\Gamma_6} : I_{\Gamma_2})$. Since $\langle LL_1, LL_2, Q_2L_1 + L_2Q_1\rangle$ is the ideal of a scheme of length four, we have that $\langle LL_1, LL_2, Q_2L_1 + L_2Q_1\rangle = I_{\Gamma_4}$. ∎

Lemma 5.1.8. *Let I_{Γ_5} be an ideal of a scheme of length five generated by one quadratic form and two cubic forms that have no common factor. Then there exists linear forms L_1, L_2, L_3 and L_4 and quadratic forms Q_1 and Q_2 such that $I_{\Gamma_5} = \langle L_1L_3 + L_2L_4, Q_1L_1 + Q_2L_2, L_3Q_2 - L_4Q_1\rangle$.*

Proof. Pick a quadratic form $Q \in I_{\Gamma_5}$ and a cubic form $K \in I_{\Gamma_5}$ such that Q and K has no common factor. Consider $I_{\Gamma_6} = \langle Q, K\rangle$, which is the ideal of a scheme of length six containing Γ_5. Then there exists a scheme $\Gamma_1 \subset \Gamma_6$ of length one not containing Γ_5 generated by two linear forms L_1 and L_2, i.e. $\langle L_1, L_2\rangle = I_{\Gamma_1}$. Since $I_{\Gamma_6} \subset I_{\Gamma_1}$, we can find linear forms L_3 and L_4, and quadratic forms Q_1 and Q_2 such that

$$\begin{aligned} Q &= L_3L_1 + L_4L_2, \\ K &= Q_1L_1 + Q_2L_2. \end{aligned}$$

Consider the matrix

$$\begin{pmatrix} L_1 & -L_4 & -Q_2 \\ L_2 & L_3 & Q_1 \end{pmatrix}.$$

The minors of the matrix generate I_{Γ_5}. Indeed, two of the minors are $Q \in I_{\Gamma_5}$ and $K \in I_{\Gamma_5}$ and the last minor is $L_3Q_2 - L_4Q_1$. We want to show that $L_3Q_2 - L_4Q_1 \in (I_{\Gamma_6} : I_{\Gamma_1}) = I_{\Gamma_5}$. Indeed,

$$\begin{aligned} Q_2Q - L_4K &= Q_2(L_3L_1 + L_4L_2) - L_4(Q_1L_1 + Q_2L_2), \\ &= Q_2L_3L_1 - L_4Q_1L_1, \\ &= (Q_2L_3 - L_4Q_1)L_1, \end{aligned}$$

so $(Q_2L_3 - L_4Q_1)L_1 \in I_{\Gamma_6}$. To show that $(Q_2L_3 - L_4Q_1)L_2 \in I_{\Gamma_6}$, consider the determinant of the following matrix:

$$\begin{pmatrix} L_2 & L_3 & Q_1 \\ L_1 & -L_4 & -Q_2 \\ L_2 & L_3 & Q_1 \end{pmatrix}.$$

Since two rows are equal, the determinant equals zero. We get that

$$\begin{aligned} L_2(Q_2L_3 - Q_1L_4) - L_3K + Q_1Q = 0 \\ L_2(Q_2L_3 - Q_1L_4) = L_3K - Q_1Q, \end{aligned}$$

which gives that $(Q_2L_3 - Q_1L_4)L_2 \in I_{\Gamma_6}$ and $(Q_2L_3 - Q_1L_4) \in (I_{\Gamma_6} : I_{\Gamma_1})$. Since $\langle L_1L_3 + L_2L_4, Q_1L_1 + Q_2L_2, L_3Q_2 - L_4Q_1\rangle$ is the ideal of a scheme of length five, we have that $\langle L_1L_3 + L_2L_4, Q_1L_1 + Q_2L_2, L_3Q_2 - L_4Q_1\rangle = I_{\Gamma_5}$. ■

Lemma 5.1.9. *Let I_{Γ_5} be an ideal of a scheme of length five generated by two quadratic forms with a common linear factor L and one quartic form which does not has L as a factor. Then there exists linear forms L_1 and L_2 and cubic forms K_1 and K_2 such that $I_{\Gamma_5} = \langle LL_1, LL_2, K_1L_2 + K_2L_1\rangle$.*

Proof. Pick a quadratic form $LL_1 \in I_{\Gamma_5}$ and a cubic form $P \in I_{\Gamma_5}$ such that LL_1 and P has no common factor. Consider $I_{\Gamma_8} = \langle LL_1, P\rangle$, which is the ideal of a scheme of length eight containing Γ_5. Then there exists a scheme $\Gamma_3 \subset \Gamma_8$ of length three not containing Γ_5 generated by L_1 and a cubic form K_1, i.e. $\langle L_1, K_1\rangle = I_{\Gamma_3}$. Since $I_{\Gamma_8} \subset I_{\Gamma_3}$, we can find a linear form L_2 and a cubic form K_2 such that $P = K_2L_1 + L_2K_1$. Consider the matrix

$$\begin{pmatrix} L_1 & 0 & -L_2 \\ K_1 & L & K_2 \end{pmatrix}.$$

Now, we show that the minors of the matrix generate I_{Γ_5}. Two of the minors are $LL_1 \in I_{\Gamma_5}$ and $K \in I_{\Gamma_5}$ and the last minor is LL_2. We want to show that $LL_2 \in (I_{\Gamma_8} : I_{\Gamma_3}) = I_{\Gamma_5}$. We show this by direct computations on the generators L_1 and K_1. Indeed, we have that $(LL_2)L_1 = (LL_1)L_2 \in I_{\Gamma_8}$. For the product $(LL_2)K_1$, consider the determinant of the following matrix

$$\begin{pmatrix} K_1 & L & K_2 \\ L_1 & 0 & -L_2 \\ K_1 & L & K_2 \end{pmatrix}.$$

Since two rows are equal, the determinant equals zero. We get that

$$\begin{aligned} K_1LL_2 - LP + K_2LL_1 = 0 \\ K_1LL_2 = LP - K_2LL_1, \end{aligned}$$

which gives that $(LL_2)K_1 \in I_{\Gamma_8}$ and $LL_2 \in (I_{\Gamma_8} : I_{\Gamma_3})$. Since $\langle LL_1, LL_2, K_2L_1 + L_2K_1\rangle$ is the ideal of a scheme of length five, we have that $\langle LL_1, LL_2, K_2L_1 + L_2K_1\rangle = I_{\Gamma_5}$. ■

Now, we are ready to state the two results where we classify every degree matrix of a Hilbert–Burch matrix of an ideal of a finite scheme of length four and five.

Proposition 5.1.10. *A Hilbert–Burch matrix of an ideal of a scheme of length four in $\mathbb{P}^2$ has one of the following degree matrices*

$$i)\ \begin{pmatrix}2 & 2\end{pmatrix} \qquad ii)\ \begin{pmatrix}2 & 2 & 1\\ 1 & 1 & 0\end{pmatrix} \qquad iii)\ \begin{pmatrix}4 & 1\end{pmatrix}$$

Proof. A scheme Γ of length four gives four linear conditions on the coordinates of the quadratic forms, thus the dimension of the vector space of quadratic forms in the ideal of the scheme is at least $6-4=2$. Assume first that there are two quadratic forms with no common factor. Then, by Theorem 5.1.1, the two quadratic forms generate a complete intersection, the ideal of a scheme of length four that contain Γ, thus they generate the ideal of Γ. The degree matrix of a Hilbert Burch matrix in this case is $\begin{pmatrix}2 & 2\end{pmatrix}$. Assume now that the two quadratic forms have a common factor, i.e. they have a common linear factor. The number of cubic forms in the ideal is then at least $10-4-2\cdot 3+1=1$. Assume first that there is no common factor between the two quadratic forms and the cubic form. Then, by Lemma 5.1.7, the degree matrix of a Hilbert–Burch matrix for this ideal is

$$\begin{pmatrix}2 & 2 & 1\\ 1 & 1 & 0\end{pmatrix}.$$

Now, we assume that there is a common factor between the two quadratic forms and the cubic form. The number of quartic forms in the ideal is then at least $15-4-2\cdot 6-1\cdot 3+1\cdot 6-1=1$. Thus, the ideal of the scheme is generated by a quartic form and a linear form. We get the degree matrix $\begin{pmatrix}4 & 1\end{pmatrix}$. ■

Corollary 5.1.11. *The ideal of a scheme of length four in $\mathbb{P}^2$ has one of the following Betti tables.*

1	-	-
-	2	-
-	-	1

1	-	-
-	2	1
-	1	1

1	1	-
-	-	-
-	-	-
-	1	1

Proposition 5.1.12. *A Hilbert–Burch matrix of an ideal of a scheme of length five in $\mathbb{P}^2$ has one of the following degree matrices.*

$$i)\ \begin{pmatrix}5 & 1\end{pmatrix} \qquad ii)\ \begin{pmatrix}2 & 1 & 1\\ 2 & 1 & 1\end{pmatrix} \qquad iii)\ \begin{pmatrix}3 & 3 & 1\\ 1 & 1 & 0\end{pmatrix}$$

Proof. A scheme of length five gives five linear conditions on the coordinates of the quadratic forms and the cubic forms, thus the dimension of the vector space of quadratic forms is at least $6-5=1$ and the dimension of the vector space of cubic forms is at least $10-5-1\cdot 3=2$. Assume first that there is no common factor between the quadratic form and the two cubic forms. Then, by Lemma 5.1.8 the degree matrix of a Hilbert–Burch matrix of this ideal is

$$\begin{pmatrix}2 & 1 & 1\\ 2 & 1 & 1\end{pmatrix}.$$

Assume now that there is a common factor between the quadratic form and the two cubic forms. The common factor is a linear form, and if the corresponding line is contained in the scheme, then this linear form is in the ideal and the ideal is generated by a linear form and a quintic form. The degree matrix is then $\begin{pmatrix}5 & 1\end{pmatrix}$. If the line is not contained in the scheme, the ideal of the scheme is generated by two quadratic forms both having the linear form as a factor and a quartic form not having the linear form as a factor. Then, by Lemma 5.1.9 the degree matrix of a Hilbert–Burch matrix of this ideal is

$$\begin{pmatrix} 3 & 3 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

■

Corollary 5.1.13. *The ideal of a scheme of length five in $\mathbb{P}^2$ has one of the following Betti tables.*

1	-	-
-	2	1
-	-	-
-	1	1

1	-	-
-	1	-
-	2	2

1	1	-
-	-	-
-	-	-
-	-	-
-	1	1

5.1.2 Other Finite Schemes in $\mathbb{P}^2$

In this section we describe the degree matrices of the Hilbert–Burch matrices that will appear as submatrix of a Buchsbaum–Eisenbud matrix. Our results are summarized in Figure 5.1 on the following page. When we say that a scheme Γ consists of d general points, we mean that Γ a point in an open subset of $\mathrm{Hilb}_d\,\mathbb{P}^2$.

Lemma 5.1.14. *Let each of the $(\beta-1)\times\beta$ matrices in Figure 5.1 be a degree matrix M_C. Assume P is a matrix with M_C as degree matrix and such that the $(\beta-1)\times(\beta-1)$ minors have no common factor. Then the $(\beta-1)\times(\beta-1)$ minors of P generate the ideal of scheme Γ of the length given in the table and a general element in Hilb_d^C is smooth. If Γ is a set of distinct points, the configuration of points is given i the table. For each set of points we describe the dimension of the family.*

Proof. For the dimension, we will use the same strategy in all cases, expect in case (10). The dimension of the family of points in $\mathbb{P}^2$ is 2, so the dimension of the family of n points in $\mathbb{P}^2$ is $2n$. Since a line $a_0x_0 + a_1x_1 + a_2x_2$ is determined by a point $(a_0 : a_1 : a_2) \in \mathbb{P}^2$, the dimension of the family of lines in $\mathbb{P}^2$ is 2. The dimension of a family of n points, m contained in a line is therefore $2(n-m)+2+m$. Since a conic $a_0x_0^2 + a_1x_0x_1 + a_2x_0x_2 + a_3x_1^2 + a_4x_1x_2 + a_5x_2^2$ is determined by a point $(a_0 : a_1 : a_2 : a_3 : a_4 : a_5) \in \mathbb{P}^5$, the dimension of the family of conics in $\mathbb{P}^2$ is 5. The dimension of a family of n points, m contained in a conic is then $2(n-m)+5+m$. In case (10), we use that the dimension of the family of cubics in $\mathbb{P}^2$ are ten, thus two cubics is determined by eight points. Therefore, the dimension of the family of a complete intersection of two cubic is $2\dot{8} = 16$.

If I_Γ is the ideal of Γ, the length of Γ can be found by computing the Hilbert polynomial of T/I_Γ. In this proof, we will find the length directly by computation on the Hilbert–Burch matrix.

Since being smooth is an open condition, we prove that a general element in Hilb_d^C i smooth for every C by finding one examples of a matrix P that is the Hilbert–Burch matrix of an ideal of a smooth scheme.

Let $L_i, Q_i, K_i, R_i \in \mathbb{C}[x_0, x_1, x_2]$ be general linear, quadratic, cubic and quartic forms, respectively. We will also use the same letters without subscript in some cases. By abuse of notation, we use the same notation for the zero set of the forms.

For almost every case, we have included a picture of curves that intersect in the given configuration of points.

We work through each case:

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ 1) Three general points, dim 6	$\begin{pmatrix} 2 & 2 \end{pmatrix}$ 2) Four points in a CI, dim 8
$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ 3) Four points, three on a line, dim 7	$\begin{pmatrix} 3 & 3 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ 4) Five points, four on a line, dim 8
$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ 5) Five general points, dim 10	$\begin{pmatrix} 2 & 3 \end{pmatrix}$ 6) Six points on a conic, dim 11
$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ 7) Six points, four on a line, dim 10	$\begin{pmatrix} 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ 8) Seven points on a conic, dim 12
$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ 9) Six general points, dim 12	$\begin{pmatrix} 3 & 3 \end{pmatrix}$ 10) Nine points in CI, dim 16
$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ 11) Seven general points, dim 14	$\begin{pmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ 12) Seven points, four on a line, dim 12
$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix}$ 13a) Eight general points, dim 16 13b) Eight points, four on a line, dim 14	$\begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$ 14) Eight points, seven on a conic, dim 14
$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix}$ 15a) Nine general points, dim 18 15b) Nine general points, four on a line, dim 16 15c) Nine general points, seven on a conic, dim 16	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ 16) Ten points, dim 20

Figure 5.1: Degree matrices of Hilbert–Burch matrices of ideal of finite schemes

(1) Let $L_i \in \mathbb{C}[x_0, x_1, x_2]$ be general linear forms and consider the matrix

$$\begin{pmatrix} L_1 & L_2 & L_3 \\ L_4 & L_5 & L_6 \end{pmatrix}.$$

Then this is the Hilbert–Burch matrix of an ideal of a scheme consisting of three general points. Indeed, from the three 2×2 minors we get the quadratic forms

$$\begin{aligned} Q_1 &= L_1 L_6 - L_3 L_4 \\ Q_2 &= L_2 L_6 - L_3 L_5 \\ Q_3 &= L_1 L_5 - L_2 L_4 \end{aligned}$$

Let $p \in \mathbb{P}^2$ be a point on Q_1. This means that when L_1, L_6, L_3 and L_4 are evaluated in p, then $L_1 L_6 - L_3 L_4 = 0$. In other words,

$$\det \begin{pmatrix} L_1 & L_3 \\ L_4 & L_6 \end{pmatrix}_p = 0, \tag{5.1}$$

where the subscript p means that L_i are evaluated in p. In the same we get that if $p' \in \mathbb{P}^2$ is a point on Q_2, then

$$\det \begin{pmatrix} L_2 & L_3 \\ L_5 & L_6 \end{pmatrix}_{p'} = 0. \tag{5.2}$$

We have from Bezout's theorem that Q_1 and Q_2 intersect in four points. We see from 5.1 and 5.2 that one intersection point is where $L_3 = L_6 = 0$. We want to show that the three other points are intersection points of Q_1, Q_2 and Q_3. Indeed, we first observe that the point where $L_3 = L_6 = 0$ is not on Q_3. Second, let $p'' \in Q_1 \cap Q_2$, where $p'' \notin L_3 \cap L_6$. Since the rows of a 2×2 matrix are proportional the determinant of a is zero, we get the relations

$$\begin{aligned} {L_4}_{p''} &= \lambda {L_1}_{p''}, & \qquad {L_5}_{p''} &= \lambda' {L_2}_{p''}, \\ {L_6}_{p''} &= \lambda {L_3}_{p''}, & \qquad {L_6}_{p''} &= \lambda' {L_3}_{p''}, \end{aligned}$$

where $\lambda, \lambda' \in \mathbb{C}$. Since we have assumed ${L_3}_{p''}, {L_6}_{p''} \neq 0$ we get $\lambda = \lambda'$. This gives the relations

$$\begin{aligned} {L_4}_{p''} &= \lambda {L_1}_{p''}, \\ {L_6}_{p''} &= \lambda {L_3}_{p''}, \\ {L_5}_{p''} &= \lambda {L_2}_{p''}. \end{aligned}$$

This gives that

$$\det \begin{pmatrix} L_1 & L_2 \\ L_4 & L_5 \end{pmatrix}_{p''} = \det \begin{pmatrix} L_1 & L_2 \\ \lambda L_1 & \lambda L_2 \end{pmatrix}_{p''} = 0.$$

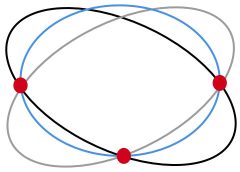
This show that $p'' \in C_3$ and in particular that $p'' \in Q_1 \cap Q_2 \cap Q_3$.

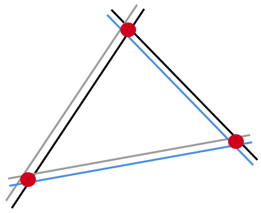
A special case of this type is when the following matrix.

$$\begin{pmatrix} L_1 & 0 & -L_2 \\ -L_1 & L_3 & 0 \end{pmatrix}$$

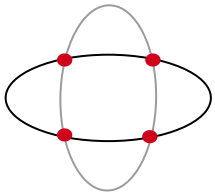
Then we get $Q_1 = L_1 L_2, Q_2 = L_2 L_3$ and $Q_3 = L_1 L_3$. This is three degenerate quadratic forms where each pair of share a common linear factor.

The computation of a smooth scheme can be found in Listing 10.2.



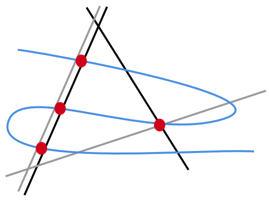


(2) Two general quadrics intersect in four points in a complete intersection by Bezout's theorem. The computation of a smooth scheme can be found in Listing 10.3.



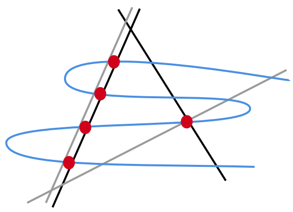
(3) As described in Lemma 5.1.7, the ideal is of the form $\langle LL_1, LL_2, Q_2L_1 + L_2Q_1\rangle$. The two quadratic forms has a common linear form L. Further the quadrics intersect in the intersection point p of L_1 and L_2. The cubic intersect the two quadrics in three points on L and in p. We therefore get four points, three on the line L.

The computation of a smooth scheme can be found in Listing 10.4.



(4) As described in Lemma 5.1.9, the ideal is of the form $\langle LL_1, LL_2, K_1L_2 + K_2L_1\rangle$. The two quadric forms has a common line L. Further the quadrics intersect in the intersection point p of L_1 and L_2. The quartic intersect the two quadrics in four points on L and in p. We therefore get five points, four on the line L.

The computation of a smooth scheme can be found in Listing 10.5.

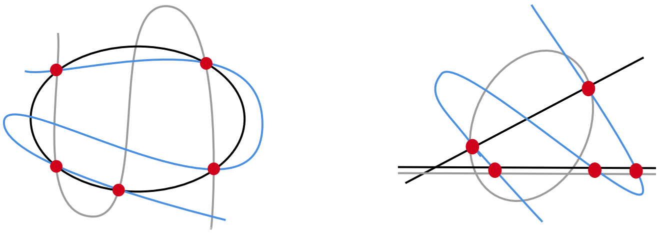


(5) As described in Lemma 5.1.8, the ideal is of the form $\langle L_1L_3 + L_2L_4, Q_1L_1 + Q_2L_2, L_3Q_2 - L_4Q_1\rangle$. Since there are no common components between any pair of the quadric and the cubics, they intersect in five general points. We observe that if $L_2 = 0$ we get $Q = L_1L_3, K_1 = Q_1L_1$ and $K_2 = L_3Q_2 - L_4Q_1$. Then Q and K_1 has a common line L_1 and intersect in two points outside the line. Since K_2 does not contain L_1, K_2 intersect Q and K_1 in L_1 in three points and in two points in the intersection points of L_3 and Q_1.

$$\begin{pmatrix} L_1 & -L_4 & -Q_2 \\ L_2 & L_3 & Q_1 \end{pmatrix}$$

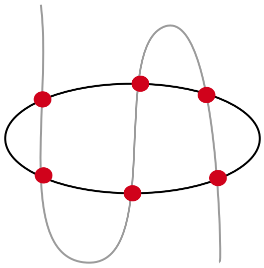
intersect in five points. Since the are no common components the, the scheme consist of five general points.

The computation of a smooth scheme can be found in Listing 10.6.



(6) A general quadric and a general cubic intersects in six points in a complete intersection by Bezout's theorem.

The computation of a smooth scheme can be found in Listing 10.7.



(7) Consider the matrix

$$\begin{pmatrix} K_1 & Q_1 & L_1 \\ Q_2 & L_2 & 0 \end{pmatrix}$$

We follow the strategy from (1). From the 2×2 minors we get

$$\begin{aligned} R &= K_1 L_2 - Q_1 Q_2 \\ K &= Q_2 L_1 \\ Q &= L_1 L_2 \end{aligned}$$

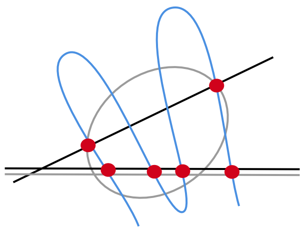
Then Bezout's theorem gives that R and K intersect in 12 points. In other words, there are 12 points $p \in \mathbb{P}^2$ such that

$$\det \begin{pmatrix} K_1 & Q_1 \\ Q_2 & L_2 \end{pmatrix}_p = 0 \qquad \text{and} \qquad \det \begin{pmatrix} K_1 & L_1 \\ Q_2 & 0 \end{pmatrix}_p = 0. \tag{5.3}$$

We see that six of the points are when $K_1 = Q_2 = 0$ and that these points is not on Q. We want to show that the remaining six points is on Q. Indeed, let $p' \in R \cap K$ such that $K_1, Q_2 \neq 0$. Since the rows in a 2 matrix are proportional when the determinant is zero, we get in particular that ${L_1}_{p'} = 0$. This shows that $p' \in Q = L_1 L_2$.

We now show that four of the points are on the line L_1 and that the two remaining points are the intersection points of Q_2 and L_2. Indeed, since L_1 is not a component of R all six points cannot lie on L_1. Further, since R and L_1 has no common component they intersect in four points by Bezout's theorem. Since both K and Q has L_1 as a component, these four points are in the intersection of R, K and Q. For the last two points, we have that Q_2 and L_2 intersect in a complete intersection. To get a total of six points in $R \cap K \cap Q$, R has to intersect the two points in $Q_2 \cap L_2$.

The computation of a smooth scheme can be found in Listing 10.8.



(8) Consider the matrix

$$\begin{pmatrix} K_1 & L_1 & L_2 \\ K_2 & L_3 & L_4 \end{pmatrix}$$

From the 2×2 minors we get

$$\begin{aligned} R_1 &= K_1L_3 - K_2L_1, \\ R_2 &= K_1L_4 - K_2L_2, \\ Q &= L_1L_4 - L_2L_3. \end{aligned}$$

Bezout's theorem gives that R_1 and R_2 intersect in 16 points. In other words, there are 16 points $p \in \mathbb{P}^2$ such that

$$\det \begin{pmatrix} K_1 & L_1 \\ K_2 & L_3 \end{pmatrix}_p = 0 \qquad \text{and} \qquad \det \begin{pmatrix} K_1 & L_2 \\ K_2 & L_4 \end{pmatrix}_p = 0. \tag{5.4}$$

Wee see that 9 of these points are when $K_1 = K_2 = 0$ and that these points are not on Q. We want to show that the remaining 7 points also lie on Q. Indeed, let $p' \in R_1 \cap R_2$ such that $K_1, K_2 \neq 0$. Since the rows in a 2 matrix are proportional when the determinant is zero, we get the relations

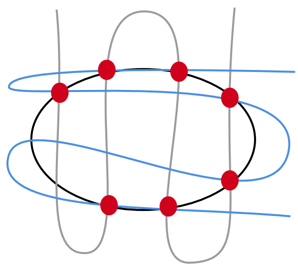
$$\begin{aligned} K_{2_{p'}} &= \lambda K_{1_{p'}}, \\ L_{3_{p'}} &= \lambda L_{1_{p'}}, \\ L_{4_{p'}} &= \lambda L_{2_{p'}}, \end{aligned}$$

where $\lambda \in \mathbb{C}$ and we have used that $K_{1'_p}, K_{2'_p} \neq 0$. This gives that

$$\det \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}_{p'} = \det \begin{pmatrix} L_1 & L_2 \\ \lambda L_1 & \lambda L_2 \end{pmatrix}_{p'} = 0.$$

This shows that $p' \in Q$ and in particular that $p' \in R_1 \cap R_2 \cap Q$. Since the Q is general, we get seven points on the conic Q.

The computation of a smooth scheme can be found in Listing 10.9.



(9) Consider the matrix

$$\begin{pmatrix} L_1 & L_2 & L_3 & L_4 \\ L_5 & L_6 & L_7 & L_8 \\ L_9 & L_{10} & L_{11} & L_{12} \end{pmatrix}.$$

Let K_1 and K_2 be the cubics obtained from the 3×3 minors of

$$\begin{pmatrix} L_1 & L_3 & L_4 \\ L_5 & L_7 & L_8 \\ L_9 & L_{11} & L_{12} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} L_2 & L_3 & L_4 \\ L_6 & L_7 & L_8 \\ L_{10} & L_{11} & L_{12} \end{pmatrix}, \tag{5.5}$$

respectively. From Bezout's theorem we have that K_1 and K_2 intersect in 9 points. From 5.5 we see that three of the points are the points $p \in \mathbb{P}^2$ such that each 2×2 minor of

$$\begin{pmatrix} L_3 & L_7 & L_{11} \\ L_4 & L_8 & L_{12} \end{pmatrix}_p \tag{5.6}$$

vanish. Indeed, if every 2×2 minor of 5.6 vanish, then the 3×3 minor of each matrix in 5.5 vanish when evaluated in p. We have from (1) that there exists three points that vanish on every 2×2 minor of 5.6.

Let now K_3 and K_4 be the cubics obtained from the 3×3 minor of

$$\begin{pmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} L_1 & L_2 & L_4 \\ L_5 & L_6 & L_8 \\ L_9 & L_{10} & L_{12} \end{pmatrix}, \tag{5.7}$$

respectively. We see that the points three points p do not lie on K_3 and K_4, since the fact that the 2×2 minors of 5.6 vanish does not imply that the 3×3 minors of the matrices in 5.7 vanish when evaluated in p. We want to show that the remaining six points in the intersection of K_1 and K_2 also lie on K_3 and K_4. Indeed, let $p' \in K_1 \cap K_2$ such that $p' \neq p$. Then there exists $\lambda, \lambda', \lambda'', \lambda''' \in \mathbb{C}$ such that

$$\begin{aligned} L_{1_{p'}} &= \lambda L_{5_{p'}} + \lambda' L_{9_{p'}}, & L_{2_{p'}} &= \lambda'' L_{6_{p'}} + \lambda''' L_{10_{p'}}, \\ L_{3_{p'}} &= \lambda L_{7_{p'}} + \lambda' L_{11_{p'}}, & L_{3_{p'}} &= \lambda'' L_{7_{p'}} + \lambda''' L_{11_{p'}}, \\ L_{4_{p'}} &= \lambda L_{8_{p'}} + \lambda' L_{12_{p'}}, & L_{4_{p'}} &= \lambda'' L_{8_{p'}} + \lambda''' L_{12_{p'}}. \end{aligned}$$

Since we assumed $p' \neq p$ we must have $\lambda = \lambda''$ and $\lambda' = \lambda'''$. This gives

$$\begin{aligned} L_{1_{p'}} &= \lambda L_{5_{p'}} + \lambda' L_{9_{p'}}, \\ L_{3_{p'}} &= \lambda L_{7_{p'}} + \lambda' L_{11_{p'}}, \\ L_{4_{p'}} &= \lambda L_{8_{p'}} + \lambda' L_{12_{p'}}, \\ L_{2_{p'}} &= \lambda L_{6_{p'}} + \lambda' L_{10_{p'}}. \end{aligned}$$

This gives that

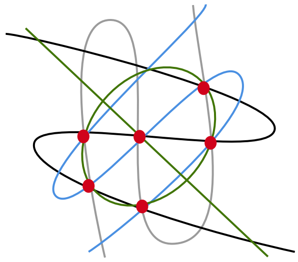
$$\det \begin{pmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{pmatrix}_{p'} = \det \begin{pmatrix} \lambda L_5 + \lambda' L_9 & \lambda L_6 + \lambda' L_{10} & \lambda L_7 + \lambda' L_{11} \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{pmatrix}_{p'} = 0$$

and

$$\det \begin{pmatrix} L_1 & L_2 & L_4 \\ L_5 & L_6 & L_8 \\ L_9 & L_{10} & L_{12} \end{pmatrix}_{p'} = \det \begin{pmatrix} \lambda L_5 + \lambda' L_9 & \lambda L_6 + \lambda' L_{10} & \lambda L_8 + \lambda' L_{12} \\ L_5 & L_6 & L_8 \\ L_9 & L_{10} & L_{12} \end{pmatrix}_{p'} = 0.$$

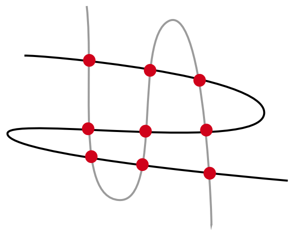
This shows that p' lie on K_3 and K_4. In particular, we have that $p' \in K_1 \cap K_2 \cap K_3 \cap K_4$. Since there are no common components between the cubics, the six points in the intersection are in general position.

The computation of a smooth scheme can be found in Listing 10.10.



(10) We have from Bezout's theorem that two cubics with no common component intersect in 9 points.

The computation of a smooth scheme can be found in Listing 10.11.



(11) Consider the matrix

$$\begin{pmatrix} Q_1 & Q_2 & Q_3 \\ L_1 & L_2 & L_3 \end{pmatrix}$$

From the 2×2 minors we get

$$\begin{aligned} K_1 &= Q_1L_2 - Q_2L_1, \\ K_2 &= Q_1L_3 - Q_3L_1, \\ K_3 &= Q_2L_3 - Q_3L_2. \end{aligned}$$

Bezout's theorem gives that K_1 and K_2 intersect in 9 points. In other words, there are 9 points $p \in \mathbb{P}^2$ such that

$$\det \begin{pmatrix} Q_1 & Q_2 \\ L_1 & L_2 \end{pmatrix}_p = 0 \qquad \text{and} \qquad \det \begin{pmatrix} Q_1 & Q_3 \\ L_1 & L_3 \end{pmatrix}_p = 0. \tag{5.8}$$

Two of the points are the intersection between Q_1 and L_1, and these two points do not lie on K_3. We want to show that the remaining 7 points also lie on K_3. By same arguments as in (1), there exists a $\lambda \in \mathbb{C}$ such that

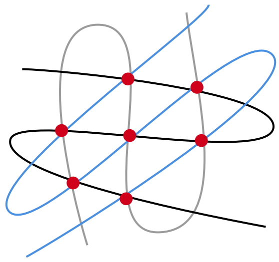
$$\begin{aligned} {L_1}_{p'} &= \lambda {Q_1}_{p'}, \\ {L_2}_{p'} &= \lambda {Q_2}_{p'}, \\ {L_3}_{p'} &= \lambda {Q_3}_{p'}, \end{aligned}$$

where $p' \neq p$ and $p \in K_1 \cap K_2$. This gives that

$$\det \begin{pmatrix} Q_2 & Q_3 \\ L_2 & L_3 \end{pmatrix}_{p'} = \det \begin{pmatrix} Q_2 & Q_3 \\ \lambda Q_2 & \lambda Q_3 \end{pmatrix}_{p'} = 0.$$

This show that $p' \in K_1 \cap K_2 \cap K_3$. Since there are no common components between the cubics, the seven points in the intersection are in general position.

The computation of a smooth scheme can be found in Listing 10.12.



(12) Consider the matrix

$$\begin{pmatrix} Q_1 & Q_2 & Q_3 & L_1 \\ L_2 & L_3 & L_4 & 0 \\ L_5 & L_6 & L_7 & 0 \end{pmatrix}$$

Let R and K_1 be the two cubics obtained from the 3×3 minors of

$$\begin{pmatrix} Q_1 & Q_2 & Q_3 \\ L_2 & L_3 & L_4 \\ L_5 & L_6 & L_7 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} Q_1 & Q_2 & L_1 \\ L_2 & L_3 & 0 \\ L_5 & L_6 & 0 \end{pmatrix}, \tag{5.9}$$

respectively. From Bezout's theorem we have that R and K_1 intersects in 12 points. By the same arguments as in (9), five of the points are points $p \in \mathbb{P}^2$ such that each 2×2 minor of

$$\begin{pmatrix} Q_1 & L_2 & L_5 \\ Q_2 & L_3 & L_6 \end{pmatrix}_p$$

vanish. Let now K_2 and K_3 be the cubics obtained from the 3×3 minors of

$$\begin{pmatrix} Q_1 & Q_3 & L_1 \\ L_2 & L_4 & 0 \\ L_5 & L_7 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} Q_2 & Q_3 & L_1 \\ L_3 & L_4 & 0 \\ L_6 & L_7 & 0 \end{pmatrix}, \tag{5.10}$$

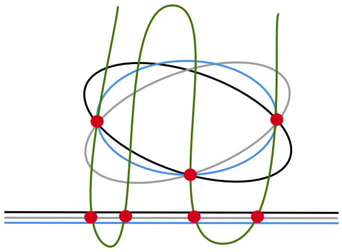
respectively. By the same arguments as in (9) we get that the five points p do not lie on K_2 and K_3 and that the remaining seven points lie on K_2 and K_3.

We now show that of the seven points, four is on the line L_1 and the rest are three general points. Indeed, we get from the matrices in 5.9 and 5.10 that L_1 is a common line of K_1, K_2 and K_3, and that the other component of each cubic is one of the 2×2 minors of

$$\begin{pmatrix} L_2 & L_3 & L_4 \\ L_5 & L_6 & L_7 \end{pmatrix}. \tag{5.11}$$

We have from (1) that the three quadrics obtained from the 2×2 minors of 5.11 intersect in three general points. In summary, the quartic R intersect K_1, K_2, K_3 on the line L_1 in four points and in the three intersection points of their quadric components.

The computation of a smooth scheme can be found in Listing 10.13.



(13) Consider the matrix

$$\begin{pmatrix} Q_1 & Q_2 & L_1 \\ Q_3 & Q_4 & L_2 \end{pmatrix}$$

From the 2×2 minors we get

$$R = Q_1Q_4 - Q_2Q_3$$
$$K_1 = Q_1L_2 - Q_3L_1$$
$$K_2 = Q_2L_2 - Q_4L_1$$

Bezout's theorem gives that R and K_1 intersect in 12 points, where four points are the intersection between Q_1 and Q_3. Let $p \in \mathbb{P}^2$ be the eight remaining points in the intersection between R and K_1. By the same arguments as above, there exists a $\lambda \in \mathbb{C}$ such that

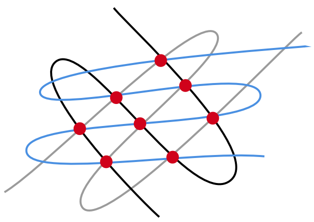
$$Q_{3_p} = \lambda Q_{1_p},$$
$$Q_{4_p} = \lambda Q_{2_p},$$
$$L_{2_p} = \lambda L_{1_p}.$$

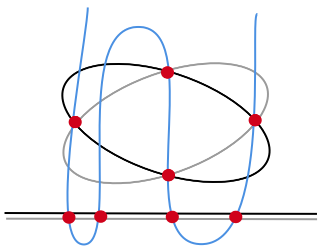
This gives that

$$\det \begin{pmatrix} Q_2 & L_1 \\ Q_4 & L_2 \end{pmatrix}_p = \det \begin{pmatrix} Q_2 & L_1 \\ \lambda Q_2 & \lambda L_1 \end{pmatrix}_p = 0.$$

This shows that $p \in R \cap K_1 \cap K_2$. Since there are no common component between R, K_1 and K_2, the eight points are in general position.

If $L_1 = 1$, then K_1 and K_2 will have L_2 as a common component. Then Q_1 and Q_2 intersect in four point in a complete intersection and R intersect K_1 and K_2 in these four points and in four point on the line L_2.





The computation of a smooth scheme can be found in Listing 10.14 and Listing 10.15.

(14) Consider the matrix

$$\begin{pmatrix} Q_1 & Q_2 & L_1 & L_2 \\ Q_3 & Q_4 & L_3 & L_4 \\ L_5 & L_6 & 0 & 0 \end{pmatrix}$$

Let R_1 and R_2 be the quartics obtained from the 3×3 minors of

$$\begin{pmatrix} Q_1 & Q_2 & L_1 \\ Q_3 & Q_4 & L_3 \\ L_5 & L_6 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Q_1 & Q_2 & L_2 \\ Q_3 & Q_4 & L_4 \\ L_5 & L_6 & 0 \end{pmatrix}, \tag{5.12}$$

respectively. From Bezout's therem we have that R_1 and R_2 intersect in 16 points. By the same arguments as in (9), eight of the points are points $p \in \mathbb{P}^2$ such that each 2×2 minor of

$$\begin{pmatrix} Q_1 & Q_3 & L_5 \\ Q_2 & Q_4 & L_6 \end{pmatrix}_p$$

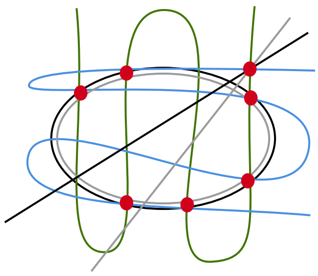
vanish. Let now K_1 and K_2 be the cubics obtained from the 3×3 minors of

$$\begin{pmatrix} Q_1 & L_1 & L_2 \\ Q_3 & L_3 & L_4 \\ L_5 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Q_2 & L_1 & L_2 \\ Q_4 & L_3 & L_4 \\ L_6 & 0 & 0 \end{pmatrix}, \tag{5.13}$$

respectively. By the same arguments as in (9) we get that the eight points p do not lie on K_1 and K_2 and that the remaining eight points lie on K_1 and K_2.

We now show that of the eight points, seven lie on a conic. We see from 5.13 that K_1 and K_2 has the quadric $L_1L_4 - L_2L_3$ as a common component. The other component is the line L_5 and L_6, respectively. Since the quartics R_1 and R_2 are irreducible, that is, they do not contain $L_1L_4 - L_2L_3$, they have seven point on $L_1L_4 - L_2L_3$ and intersect L_5 and L_6 in one point.

The computation of a smooth scheme can be found in Listing 10.16.



(15) Consider the matrix

$$\begin{pmatrix} Q_1 & L_1 & L_2 & L_3 \\ Q_2 & L_4 & L_5 & L_6 \\ Q_3 & L_7 & L_8 & L_9 \end{pmatrix}$$

Let R_1 and R_2 the quartics obtained from the 3×3 minors of

$$\begin{pmatrix} Q_1 & L_1 & L_2 \\ Q_2 & L_4 & L_5 \\ Q_3 & L_7 & L_8 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Q_1 & L_1 & L_3 \\ Q_2 & L_4 & L_6 \\ Q_3 & L_7 & L_9 \end{pmatrix},$$

respectively. From Bezout's theorem we have that R_1 and R_2 intersect in 16 points. By the same arguments as in (9), seven of the points are points $p \in \mathbb{P}^2$ such that each 2×2 minor of

$$\begin{pmatrix} Q_1 & Q_2 & Q_3 \\ L_1 & L_4 & L_7 \end{pmatrix}_p$$

vanish. Let now R_3 and K_1 be the quartic and the cubic obtained from the 3×3 minors of

$$\begin{pmatrix} Q_1 & L_2 & L_3 \\ Q_2 & L_5 & L_6 \\ Q_3 & L_8 & L_9 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} L_1 & L_2 & L_3 \\ L_4 & L_5 & L_6 \\ L_7 & L_8 & L_9 \end{pmatrix},$$

respectively. By the same arguments as in (9) we get that the seven points p do not lie on R_3 or K_1, and that the remaining nine points lie on R_3 and K_1. Since R_1, R_2, R_3 and K_1 has no common component, the nine points are in general position.

If $L_6 = L_9 = 0$ we get

$$\begin{aligned} R_1 &= Q_1(L_4L_8 - L_5L_7) - Q_2(L_1L_8 - L_2L_7) + Q_3(L_1L_5 - L_2L_4), \\ R_2 &= L_3(Q_2L_7 - Q_3L_4), \\ R_3 &= L_3(Q_2L_8 - Q_3L_5), \\ R_4 &= L_3(L_4L_8 - L_5L_7), \end{aligned}$$

that is, R_2, R_3 and K_1 share the line L_3. This gives nine points, four on the line L_3.

If $L_8 = L_9 = 0$ we get

$$\begin{aligned} R_1 &= Q_1L_5L_7 - L_2L_7 - Q_3(L_1L_5 - L_2L_4), \\ R_2 &= -Q_1L_6L_7 + Q_2L_6L_7 + Q_3(L_1L_6 - L_3L_4), \\ R_3 &= Q_3(L_2L_6 - L_3L_5), \\ K_1 &= L_7(L_2L_6 - L_3L_5), \end{aligned}$$

that is, R_3 and K_1 share the quadric $(L_2L_6 - L_3L_5)$. This gives nine points, seven on $(L_2L_6 - L_3L_5)$ and the two last in the intersection of Q_3 and L_7.

The computation of a smooth scheme can be found in Listing 10.17, Listing 10.18 and Listing 10.19.

(16) Consider the matrix

$$\begin{pmatrix} L_1 & L_2 & L_3 & L_4 & L_5 \\ L_6 & L_7 & L_8 & L_9 & L_{10} \\ L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{16} & L_{17} & L_{18} & L_{19} & L_{20} \end{pmatrix} \tag{5.14}$$

Let R_1 and R_2 be the quartics obtained form the 4×4 minors of

$$\begin{pmatrix} L_1 & L_2 & L_3 & L_4 \\ L_6 & L_7 & L_8 & L_9 \\ L_{11} & L_{12} & L_{13} & L_{14} \\ L_{16} & L_{17} & L_{18} & L_{19} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} L_1 & L_2 & L_3 & L_5 \\ L_6 & L_7 & L_8 & L_{10} \\ L_{11} & L_{12} & L_{13} & L_{15} \\ L_{16} & L_{17} & L_{18} & L_{20} \end{pmatrix},$$

respectively. From Bezout's theorem we have that R_1 and R_2 intersect in 16 points. By the same arguments as in (9), six of the points are points $p \in \mathbb{P}^2$ such that each 2×2 minor of

$$\begin{pmatrix} L_1 & L_6 & L_{11} & L_{16} \\ L_2 & L_7 & L_{12} & L_{17} \\ L_3 & L_8 & L_{13} & L_{18} \end{pmatrix}_p$$

vanish. By the same arguments as in (9) we get that the six points do not lie on the three quartics obtained from the remaining 3×3 minors of 5.14 R_3, R_4 or R_5 and that the remaining ten points do lie on all five quartics. Since the quartics have no common component, the ten points are in general position.

The computation of a smooth scheme can be found in Listing 10.20. ■

5.2 Zero-Dimensional Subideals of $F^\perp$

In this section we describe which ideals of finite schemes Γ that appear as subideal of $F^\perp$. First, we prove that if I_Γ is generated by some of the generators of $F^\perp$, then a Hilbert–Burch matrix of I_Γ is a submatrix of a Buchsbaum–Eisenbud matrix of $F^\perp$. We are specially interested in the subideals I_Γ that are generated by some of the generators of $F^\perp$, because the minimal subideals of $F^\perp$ will be ones that are generated by some of the generators of $F^\perp$. Furthermore, we prove which submatrices of a Buchsbaum–Eisenbud matrix of $F^\perp$ that are Hilbert–Burch matrices that actually generates a subideal of $F^\perp$. Lastly, we go through each of the Betti strata $\mathcal{F}_B$, and prove that for each $[F] \in \mathcal{F}_B$ there exists a subideal I_Γ with a Hilbert–Burch matrix such that the degree matrix is the same for every $[F] \in \mathcal{F}_B$. In Chapter 6, we prove that the ideals we find actually are minimal subideal $F^\perp$ for some F.

Let $P' = (p_{ij})$ be an $n \times n$ matrix where $p_{ij} \in \mathbb{C}[y_0, y_1, y_2]$ is a homogeneous polynomial. Let the degree matrix $M_B = (m_{ij})$, where $m_{ij} = \deg p_{ij}$.

Lemma 5.2.1. *Let $F^\perp \subset T$ and let Γ be a finite scheme. Let the $(\beta-1)\times\beta$ matrix P be a Hilbert–Burch matrix of I_Γ and let the $n \times n$ matrix P' be a Buchsbaum–Eisenbud matrix of $F^\perp$. If I_Γ is generated by some of the generators of $F^\perp$, then P is a submatrix of P'.*

Proof. If $I_\Gamma \subset F^\perp$, there exists maps ϕ and ψ such that the following diagram commutes.

$$\begin{array}{ccccccccccc}
 & & 0 & \longrightarrow & T^{\beta-1} & \xrightarrow{P} & T^\beta & \longrightarrow & I_\Gamma & \longrightarrow & 0\\
 & & & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \downarrow & & \\
0 & \longrightarrow & T & \longrightarrow & T^n & \xrightarrow{P'} & T^n & \longrightarrow & F^\perp & \longrightarrow & 0
\end{array}$$

If the generators of I_Γ is a linear combination of the generators of $F^\perp$, $\phi : T^\beta \to T^\alpha$ takes the generators of I_Γ to this linear combination in T^n. The map $\psi : T^{\beta-1} \to T^n$ does the same for the syzygies, thus ϕ and ψ are inclusion maps. Since the columns of P' and P are generators for the syzygies of $F^\perp$ and I_Γ, by Theorem 2.3.3 and Theorem 5.1.1, respectively, P has to be a submatrix of P'. ∎

Lemma 5.2.2. *Let $F^\perp \subset T$ and Assume $F^\perp$ is minimally generated by $n = 2k+1$ elements. Let the $n \times n$ matrix P' be a Buchsbaum–Eisenbud matrix of $F^\perp$. If there exists a basis such that*

$$P' = \begin{pmatrix}
0 & c_{01} & \cdots & c_{0k} & c_{0,k+1} & \cdots & c_{0,n-1}\\
-c_{01} & 0 & & \vdots & \vdots & & \vdots\\
\vdots & & \ddots & \vdots & \vdots & & \vdots\\
-c_{0k} & \cdots & \cdots & 0 & c_{k,k+1} & \cdots & c_{k,n-1}\\
-c_{0,k+1} & \cdots & \cdots & -c_{k,k+1} & 0 & \cdots & 0\\
\vdots & & & \vdots & \vdots & \ddots & \vdots\\
-c_{0,n-1} & \cdots & \cdots & -c_{k,n-1} & 0 & \cdots & 0
\end{pmatrix},$$

then the ideal generated by the $k \times k$ minors of the $k \times (k+1)$-submatrix

$$P = \begin{pmatrix}
-c_{0,k+1} & \cdots & \cdots & -c_{k,k+1}\\
\vdots & & & \vdots\\
-c_{0,n-1} & \cdots & \cdots & -c_{k,n-1}
\end{pmatrix}.$$

is a subideal of $F^\perp$.

Proof. Indeed, we observe that P' consists of four blocks, that is P, $-(P)^T$, the $k \times k$ zero block and the last $(k+1) \times (k+1)$ skew symmetric block. To compute the first $(n-1)$th order pfaffian we delete the first row and first column of P' and compute the determinant of the remaining matrix. After deleting the first row and the first column the remaining submatrix of P' consists of four block, each of dimension $k \times k$. We use the rule for computing the determinant of a block matrix consisting of square blocks, that is $\det M = \det A \det D - \det B \det C$, where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We get the following computation.

$$\det \begin{pmatrix} 0 & c_{12} & \cdots & c_{1k} & c_{1,k+1} & \cdots & c_{1,n-1} \\ -c_{12} & 0 & & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ -c_{1k} & \cdots & \cdots & 0 & c_{k,k+1} & \cdots & c_{k,n-1} \\ -c_{1,k+1} & \cdots & \cdots & -c_{k,k+1} & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & \ddots & \vdots \\ -c_{1,n-1} & \cdots & \cdots & -c_{k,n-1} & 0 & \cdots & 0 \end{pmatrix}$$

$$= \det \begin{pmatrix} 0 & c_{12} & \cdots & c_{1k} \\ -c_{12} & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ -c_{1k} & \cdots & \cdots & 0 \end{pmatrix} \det \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

$$- \det \begin{pmatrix} -c_{1,k+1} & \cdots & \cdots & -c_{k,k+1} \\ \vdots & & & \vdots \\ -c_{1,n-1} & \cdots & \cdots & -c_{k,n-1} \end{pmatrix} \det \begin{pmatrix} c_{1,k+1} & \cdots & c_{1,n-1} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ c_{k,k+1} & \cdots & c_{k,n-1} \end{pmatrix}$$

$$= \det \begin{pmatrix} -c_{1,k+1} & \cdots & \cdots & -c_{k,k+1} \\ \vdots & & & \vdots \\ -c_{1,n-1} & \cdots & \cdots & -c_{k,n-1} \end{pmatrix}^2 ,$$

where we have used that $\det A = \det A^T$ for a general matrix A. Since the pfaffian is the square root of the determinant, we get that the first $(n-1)$th order pfaffian is the determinant of the submatrix of P obtained by deleting the first row. This is indeed the first $k \times k$ minor of P'. The case for the $k-1$ next $(n-1)$th order pfaffians are the same. ∎

Given an $\mathcal{F}_B$, the Buchsbaum–Eisenbud matrix of $F^\perp$ for an $[F] \in \mathcal{F}_B$ is given by the Betti table $B_{b_{12}b_{13}b_{14}}$. Thus, the degree matrix is the same for every Buchsbaum–Eisenbud matrix of $F^\perp$ for an $[F] \in \mathcal{F}_B$. Therefore, we find for each type F_B which degree matrix of Hilbert–Burch matrices that is a submatrix of the degree matrix of a Buchsbaum–Eisenbud of $F^\perp$ for an $[F] \in \mathcal{F}_B$. By Lemma 5.2.2, we then have that there exists a subideal of $F^\perp$ generated by some of the generators of $F^\perp$.

Proposition 5.2.3. *Let $M_{[300]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^\perp$ for an $[F] \in \mathcal{F}_{[300]}$. Then there is a subideal of $F^\perp$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Proof. We have

$$M_{[300]} = \begin{pmatrix} 0 & 5 & 5 & 1 & 1 \\ 5 & 0 & 5 & 1 & 1 \\ 5 & 5 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

■

Proposition 5.2.4. *Let $M_{[200]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^\perp$ for an $[F] \in \mathcal{F}_{[200]}$. Then there is a subideal of $F^\perp$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 2 & 2 \end{pmatrix}$$

Proof. We have

$$M_{[200]} = \begin{pmatrix} 0 & 5 & 2 \\ 5 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

Then two of the generators of $F^\perp$ are the 1×1 minors of the submatrix. ■

Proposition 5.2.5. *Let $M_{[210]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^\perp$ for an $[F] \in \mathcal{F}_{[210]}$. Then there is a subideal of $F^\perp$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Proof. We have

$$M_{[210]} = \begin{pmatrix} 0 & 5 & 4 & 2 & 1 \\ 5 & 0 & 4 & 2 & 1 \\ 4 & 4 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

■

Proposition 5.2.6. *Let $M_{[202]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^\perp$ for an $[F] \in \mathcal{F}_{[202]}$. Then there is a subideal of $F^\perp$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 3 & 3 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Proof. We have

$$M_{[202]} = \begin{pmatrix} 0 & 5 & 3 & 3 & 1 \\ 5 & 0 & 3 & 3 & 1 \\ 3 & 3 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

■

Proposition 5.2.7. *Let $M_{[120]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^\perp$ for an $[F] \in \mathcal{F}_{[120]}$. Then there is a subideal of $F^\perp$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Proof. We have

$$M_{[120]} = \begin{pmatrix} 0 & 4 & 4 & 2 & 2 \\ 4 & 0 & 3 & 1 & 1 \\ 4 & 3 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \end{pmatrix}$$

■

Proposition 5.2.8. *Let $M_{[111]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^\perp$ for an $[F] \in \mathcal{F}_{[111]}$. Then there is a subideal of $F^\perp$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 3 & 2 \end{pmatrix}$$

Proof. We have

$$M_{[111]} = \begin{pmatrix} 0 & 4 & 3 \\ 4 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}$$

■

Proposition 5.2.9. *Let $M_{[112]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^\perp$ for an $[F] \in \mathcal{F}_{[112]}$. Then there is a subideal of $F^\perp$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

Proof. We have

$$M_{[112]} = \begin{pmatrix} 0 & 4 & 3 & 3 & 2 \\ 4 & 0 & 2 & 2 & 1 \\ 3 & 2 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \end{pmatrix}$$

■

Proposition 5.2.10. *Let $M_{[104]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^\perp$ for an $[F] \in \mathcal{F}_{[104]}$. Then there is a subideal of $F^\perp$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Proof. We have

$$M_{[104]} = \begin{pmatrix} 0 & 3 & 3 & 3 & 3 \\ 3 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 \end{pmatrix}$$

By Lemma 4.2.7 and Theorem 4.2.6, there exists a row basis such that the 4×4 submatrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad \text{is of the form} \qquad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

We use this row basis such that $M_{[104]}$ is of the form

$$\begin{pmatrix} 0 & 3 & 3 & 3 & 3 \\ 3 & 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

■

Proposition 5.2.11. *Let $M_{[040]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^\perp$ for an $[F] \in \mathcal{F}_{[040]}$. Then there is a subideal of $F^\perp$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Proof. We have

$$M_{[040]} = \begin{pmatrix} 0 & 3 & 3 & 3 & 1 & 1 & 1 \\ 3 & 0 & 3 & 3 & 1 & 1 & 1 \\ 3 & 3 & 0 & 3 & 1 & 1 & 1 \\ 3 & 3 & 3 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

■

Proposition 5.2.12. *Let $M_{[300]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^\perp$ for an $[F] \in \mathcal{F}_{[300]}$. Then there is a subideal of $F^\perp$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 3 & 3 \end{pmatrix}$$

Proof. We have

$$M_{[030]} = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix}.$$

■

Proposition 5.2.13. *Let $M_{[031]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^\perp$ for an $[F] \in \mathcal{F}_{[031]}$. Then there is a subideal of $F^\perp$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Proof. We have

$$M_{[031]} = \begin{pmatrix} 0 & 3 & 3 & 2 & 1 \\ 3 & 0 & 3 & 2 & 1 \\ 3 & 3 & 0 & 2 & 1 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

■

Proposition 5.2.14. *Let $M_{[032]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^{\perp}$ for an $[F] \in \mathcal{F}_{[032]}$. Then there is a subideal of $F^{\perp}$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Proof. We have

$$M_{[032]} = \begin{pmatrix} 0 & 3 & 3 & 2 & 2 & 1 & 1 \\ 3 & 0 & 3 & 2 & 2 & 1 & 1 \\ 3 & 3 & 0 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 0 & 1 & 0 & 0 \\ 2 & 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

■

Proposition 5.2.15. *Let $M_{[023]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^{\perp}$ for an $[F] \in \mathcal{F}_{[023]}$. Then F is either of type $[023a]$ and there is a subideal of $F^{\perp}$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 3 & 3 \end{pmatrix}$$

or F is of type $[023b]$ and there is a subideal of $F^{\perp}$ with a Hilbert–Burch matrix with degree matrix

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix}$$

or F is of type $[023c]$ and there is a subideal of $F^{\perp}$ with a Hilbert–Burch matrix with degree matrix

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 0 \end{pmatrix}$$

Proof. We have

$$M_{[023]} = M_{[023a]} = \begin{pmatrix} 0 & 3 & 2 & 2 & 2 \\ 3 & 0 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 & 1 \\ 2 & 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 1 & 0 \end{pmatrix}$$

Let K_1 and K_2 be the two cubics in $F^{\perp}$. Assume K_1 and K_2 has none common component. Then K_1 and K_2 intersect in a complete intersection where a corresponding Hilbert–Burch matrix has degree matrix

$$\begin{pmatrix} 3 & 3 \end{pmatrix}$$

Assume that the linear forms in the matrix are dependent. Then we say that F is of type $[023b]$ and there exists a row basis such that

$$M_{[023b]} = \begin{pmatrix} 0 & 3 & 2 & 2 & 2 \\ 3 & 0 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 & 1 \\ 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \end{pmatrix}.$$

Assume that the linear forms are proportional. Then we say that F is of type $[023b]$ and there exists a row basis such that

$$M_{[023c]} \begin{pmatrix} 0 & 3 & 2 & 2 & 2 \\ 3 & 0 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \end{pmatrix},$$

■

Proposition 5.2.16. *Let $M_{[024]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^\perp$ for an $[F] \in \mathcal{F}_{[024]}$. Then there is a subideal of $F^\perp$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Proof. We have

$$M_{[024]} = \begin{pmatrix} 0 & 3 & 2 & 2 & 2 & 2 & 1 \\ 3 & 0 & 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 0 & 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 & 1 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By Theorem 4.2.6 and Lemma 4.2.7, there exists a row basis such that the submatrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad \text{is of the form} \qquad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

We use this row basis such that $M_{[024]}$ is of the form

$$\mathcal{M}_{[024]} = \begin{pmatrix} 0 & 3 & 2 & 2 & 2 & 2 & 1 \\ 3 & 0 & 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 0 & 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 & 1 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

■

Proposition 5.2.17. *Let $M_{[016]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^{[perp]}$ for an $[F] \in \mathcal{F}_{[016]}$. Then F is either of type $[016a]$ and there is a subideal of $F^{\perp}$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix}$$

or F is of type $[016b]$ and there is a subideal of $F^{\perp}$ with a Hilbert–Burch matrix with degree matrix

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 \end{pmatrix}$$

or F is of type $[016c]$ and there is a subideal of $F^{\perp}$ with a Hilbert–Burch matrix with degree matrix

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

Proof. We have

$$M_{[016]} = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

By Theorem 4.4.1 there exists a row basis such that the submatrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{is of the form} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We use this row basis such that $M_{[016]}$ is of the form

$$\begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

We say that F is of type $[016a]$ if the linear forms in general are not zero. If there exists are row basis such that two of the linear form are zero, we say that F is of type $[016b]$ or $[016c]$. We distinguish the two types in the following way:

$$M_{[016b]} = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_{[016c]} = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

∎

Proposition 5.2.18. *Let $M_{[009]}$ be the degree matrix of a Buchsbaum–Eisenbud matrix of $F^{\perp}$ for an $[F] \in \mathcal{F}_{[009]}$. Then there is a subideal of $F^{\perp}$ with a Hilbert–Burch matrix with degree matrix*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Proof. We have

$$M_{[009]} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

By Theorem 4.4.7 the exists a row basis such that $M_{[009]}$ can be written in the form

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

∎

Observe that the results above show that for every $\mathcal{F}_B$ except for the types [030] and [023a], we have found a unique degree matrix of a Hilbert–Burch matrix as a submatrix of the degree matrix of a Buchsbaum–Eisenbud matrix. In the next chapter, we will prove that the Hilbert–Burch matrix we have found indeed generated a minimal subideal of the apolar ideal of an F with the corresponding Buchsbaum–Eisenbud matrix. For this reason we make the following definition

Definition 5.2.19.

$$\mathcal{G}_B = \{\Gamma \subset S_1 : I_\Gamma \subset F^{\perp} \text{ is minimal for some } F \in \mathcal{F}_B\}.$$

For the cases [030] and [023a] case we write $\mathcal{G}_{[030]/[023a]}$.

6 | Rank and Power Sum Representations

In this chapter we state and prove one of the main results in this thesis. For a general $[F]$ in each $\mathcal{F}_B$, we find the rank and the VSP. The chapter is organized as follows. First, we state our results. Then, we prove that every Betti strata $\mathcal{F}_B$ is irreducible. Thereafter, we introduce some theory about binary forms, before we finally prove our result.

Proposition 6.0.1. *Let F be a non-degenerate ternary sextic forms. Then F belongs to one of the 20 irreducible sets in Figure 6.1 with the given dimension. For a general form F in each set, the table give the* $\mathrm{r}(F)$, $\mathrm{VSP}(F, \mathrm{r}(F))$ *and the minimal configuration of points in Γ apolar to F.*

Betti table $B_{[b_{12}b_{13}b_{14}]}$	$\mathrm{r}(F)$	$\mathrm{VSP}(F,r)$	Γ	$\dim(\mathcal{F}_B)$
[300]	3	one point	three points	8
[210]	4	one point	four points, three on a line	10
[200]	4	one point	four points	11
[202]	5	$\mathbb{P}^1$	five points, four on a line	11
[120]	5	one point	five points	14
[112]	6	$\mathbb{P}^1$	six points, four on a line	14
[111]	6	one point	six points on a conic	16
[104]	7	$\mathbb{P}^1$	seven points on a conic	17
[040]	6	one point	six points	17
[032]	7	$\mathbb{P}^1$	seven points, four on a line	17
[031]	7	one point	seven points	20
[023c]	8	$\mathbb{P}^1$	eight points, four on a line	20
[024]	8	$\mathbb{P}^1$	eight points, seven on a conic	20
[023b]	8	one point	eight points	23
[030]	9	$\mathbb{P}^2$	nine points in a CI	21
[016b]	9	$\mathbb{P}^1$	nine points, four on a line	23
[016c]	9	$\mathbb{P}^1$	nine points, seven on a conic	23
[023a]	9	one point	nine points in a CI	24
[016a]	9	two points	nine points	26
[009]	10	K3 surface	ten points	27

Figure 6.1: Rank and VSP for the Betti strata

To prove that each Betti strata in Figure 6.1 is irreducible, we prove the following lemma.

Lemma 6.0.2. *Fix a degree matrix $M_B = (m_{ij})$ and let V_B be the vector space of skew symmetric $n \times n$ matrices $P' = (p_{ij})$ with homogeneous polynomials as entries, where the polynomial in the entry p_{ij} has degree m_{ij}. Let V_B^0 be the subset of V_B consisting of the matrices where the ideal I'_P generated by the $(n-1)$th order pfaffians of P' is of codimension 3. Then V_B^0 is irreducible.*

Proof. Since the degree of the polynomials in the entries of $P' \in V_B$ are fixed, the total number N of coefficients of the polynomials are also fixed. Then there is a $1-1$-correspondence between points in $\mathbb{A}^N$ and matrices in V_B. We prove that V_B^0 is open by proving that the complement is closed. Indeed, fix P' and consider I'_P. If $\operatorname{codim} I'_P < 3$, then every 3-dimensional subspace I_3 of I'_P has codimension less than 3. For every $I_3 \subset I_P$ we can find F_1, F_2, F_3 such that $I_3 = \langle F_1, F_2, F_3 \rangle$. By Theorem 5.1.2, $F_0 = F_1 = F_2 = 0$ if and only if $\operatorname{Res}(F_0, F_1, F_2) = 0$. Since $\operatorname{Res}(F_0, F_1, F_2)$ is a polynomial in the coefficients of F_i, and F_i are determined by the coefficients of p_{ij}, we get one polynomial R in the coefficients of the p_{ij} for each 3-dimensional subspace of I'_P. Then every 3-dimensional subspace has codimension less than three if and only if the ideal spanned by the polynomials R vanish. The subset of V_B where the coefficients satisfies this condition is closed. This shows that V_B^0 is open. Every open subset of $\mathbb{A}^N$ is irreducible, thus V_B^0 is irreducible. ∎

Proposition 6.0.3. *The* 20 *Betti strata* $\mathcal{F}_B$ *in the table in Figure 6.1 are irreducible.*

Proof. Consider the map

$$\begin{aligned} \phi : V_B^0 &\to \mathcal{F}_B \\ P' &\mapsto [F'_P], \end{aligned}$$

where F'_P is the polynomial with the apolar ideal $F_P'^{\perp}$ generated by the $(n-1)$th order pfaffians of P'. Then map is well-defined since, by assumption, the $(n-1)$th order pfaffians of P' generates an ideal I'_P of codimension 3, and by Theorem 2.3.3 I'_P is Artinian Gorenstein. By Definition-Proposition 2.2.1, $I'_P = F_P'^{\perp}$ for an F, and by [Eis95] F'_P is unique up to scalar. To prove that ϕ is surjective, observe that to every F there is a corresponding apolar ideal which is Artinian Gorenstein by Definition-Proposition 2.2.1, and by Theorem 2.3.3 every Artinian Gorenstein ideal of codimension 3 arises as $(n-1)$th order pfaffians of a skew symmetric matrix. Since $\mathcal{F}_B$ is the image under a map and V_B^0 is irreducible, then $\mathcal{F}_B$ is irreducible. ∎

Now, we introduce some theory about binary forms of even degree that we will use to prove Proposition 6.0.1.

Theorem 6.0.4 (Sylvester)**.** *Let* $d = 2k$*. For a general* $F \in \mathbb{C}[x_0, x_1]_d$*, then* $\mathrm{r}(F) = k + 1$ *and* $\mathrm{VSP}(F, k+1) = \mathbb{P}^1$.

Definition 6.0.5. We say that a scheme Γ is *apolar to* F if $I_\Gamma \subset F^\perp$.

Lemma 6.0.6. *Let* $F \in \mathbb{C}[x_0, x_1, x_2]_6$*. Assume* Γ *is apolar to an* F *and that* Γ *is contained in a curve* C.

1. *If* C *is a line* L*, then* $\langle v_6(C) \rangle = \mathbb{P}^6$ *and* F *can be considered as a general binary sextic form.*
2. *If* C *is a conic* Q*, then* $\langle v_6(C) \rangle = \mathbb{P}^{12}$ *and* F *can be considered as a general binary form of degree 12.*

Proof. (1) By Lemma 2.1.5, $[F] \in \langle v_6(\Gamma) \rangle \subset \langle v_6(L) \rangle \subset \langle v_6(\mathbb{P}^2) \rangle$. Let $H_G = \{[F] \in \mathbb{P}(S_6) : G(F) = 0\}$. Then $\langle v_6(L) \rangle = \bigcap_{H_G \supset v_6(L)} H_G$. Further, we have that $v_6(L) \subset H_G$ if and only if $G \in I_{L,6}$. The dimension of the space of ternary quintic forms is 21, thus $I_{L,6}$ is 21-dimensional. Since $H_G \subset \mathbb{P}^{27}$, we have that $\langle v_6(L) \rangle = \bigcap_{H_G \supset v_6(L)} H_G = \mathbb{P}^6$. A point in $\mathbb{P}^6$ can be considered as a binary sextic, that is

$$(a_0 : a_1 : a_2 : a_3 : a_4 : a_5) \longleftrightarrow a_0 x_0^6 + a_1 x_0^5 x_1 + a_2 x_0^4 x_1^2 + a_3 x_0^3 x_1^3 + a_4 x_0^2 x_1^4 + a_5 x_0 x_1^5 + a_6 x_1^6.$$

Since $[F] \in \langle v_6(C) \rangle$, F can be considered as a general binary sextic form.

(2) By Lemma 2.1.5, $[F] \in \langle v_6(\Gamma) \rangle \subset \langle v_6(Q) \rangle \subset \langle v_6(\mathbb{P}^2) \rangle$. Let $H_G = \{[F] \in \mathbb{P}(S_6) : G(F) = 0\}$. Then $\langle v_6(Q) \rangle = \bigcap_{H_G \supset v_6(Q)} H_G$. Further, we have that $v_6(Q) \subset H_G$ if and only if $G \in I_{Q,6}$. The dimension of the space of ternary quartic forms is 15, thus $I_{Q,6}$ is 15-dimensional. Since $H_G \subset \mathbb{P}^{27}$, we have that $\langle v_6(Q) \rangle = \bigcap_{H_G \supset v_6(Q)} H_G = \mathbb{P}^1 2$. In the same way as above a point in

$\mathbb{P}^{12}$ can be considered as a general binary polynomial of degree 12. Since $[F] \in \langle v_6(Q) \rangle$, F can be considered as a binary polynomial of degree 12. ■

Lemma 6.0.7. *Let $F \in \mathbb{C}[x_0, x_1, x_2]_6$. Assume $I_\Gamma \subset F^\perp$ is the ideal of a scheme Γ which is contained in a curve C.*

1. *If C is a line L containing a scheme of length 4 and F is not apolar to a scheme of length less than 4, then $\mathrm{r}(F) = 4$ and $\mathrm{VSP}(F, 4) = \mathbb{P}^1$.*
2. *If C is a conic Q containing a scheme of length 7 and F is not apolar to a scheme of length less than 7, then $\mathrm{r}(F) = 7$ and $\mathrm{VSP}(F, 7) = \mathbb{P}^1$.*

Proof. (1) By Lemma 6.0.6, F can be considered as a binary form of degree 6. By Theorem 6.0.4, and since F is not apolar to a scheme of length less than 4, $\mathrm{r}(F) = 4$ and $\mathrm{VSP}(F, 4) = \mathbb{P}^1$.

(2) By Lemma 6.0.6, F can be considered as a binary form of degree 12. By Theorem 6.0.4, and since F is not apolar to a scheme of length less than 7, $\mathrm{r}(F) = 7$ and $\mathrm{VSP}(F, 7) = \mathbb{P}^1$. ■

Now, give lemma about of the rank of a ternary sextic form F. Thereafter, we prove Proposition 6.0.1.

Lemma 6.0.8. *Let b_{12} and b_{13} be equal the number of, respectively, quadratic and cubic generators of $F^\perp$ for a ternary sextic form F. Let $\mathrm{r}(F)$ be the rank of F. Then $\mathrm{r}(F) \geq 6 - b_{12}$ and, if $b_{12} = 0$, then $\mathrm{r}(F) \geq 10 - b_{13}$.*

Proof. Assume for contradiction that $\mathrm{r}(F) < 6 - b_{12}$. Then there exists a $I_\Gamma \subset F^\perp$ such that Γ has length $\mathrm{r}(F)$. Then the dimension of the space of quadratic forms in I_Γ is at least $6 - \mathrm{r}(F)$. Since $6 - \mathrm{r}(F) > b_{12}$ we have a contradiction. The proof for the second inequality is similar. ■

Proof of Proposition 6.0.1. We go through each type:

[300] By Proposition 5.2.3 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length three. For a general F, the scheme consists of three points, so $\mathrm{r}(F) \leq 3$. Since the space of quadratic forms in $F^\perp$ is 3-dimensional, $\mathrm{r}(F) \geq 3$ by Lemma 6.0.8, thus $\mathrm{r}(F) = 3$. The ideal I_Γ is generated by a 3-dimensional space of quadratic forms, that is the whole space of quadratic forms in $F^\perp$, hence the $\mathrm{VSP}(F, 3)$ is one point. The dimension of the family of 3-tuples in $\mathbb{P}^2$ is 6, by Lemma 5.1.14. Three points in $\mathbb{P}^{27}$ span a $\mathbb{P}^2$, so the dimension of $\mathcal{F}_{[300]}$ is $6 + 2 = 8$.

[210] By Proposition 5.2.5 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length four, with a subscheme of length three contained in a line. For a general F, the scheme consists of four points, thus $\mathrm{r}(F) \leq 4$. Since the space of quadratic forms in $F^\perp$ is 2-dimensional, $\mathrm{r}(F) \geq 4$ by Lemma 6.0.8, thus $\mathrm{r}(F) = 4$. The ideal I_Γ is generated by a 2-dimensional space of quadratic forms and a 1-dimensional space of cubic forms, that is the whole space of quadratic forms and cubic forms in $F^\perp$, hence the $\mathrm{VSP}(F, 4)$ is one point. The dimension of the family of 4-tuples, three on a line in $\mathbb{P}^2$ is 7, by Lemma 5.1.14. Each 4-tuple spans a $\mathbb{P}^3$ in $\mathbb{P}^{27}$, so the dimension of $\mathcal{F}_{[210]}$ is $7 + 3 = 10$.

[200] By Proposition 5.2.4 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length four. For a general F, the scheme consists of four points, so $\mathrm{r}(F) \leq 4$. Since the space of quadratic forms in $F^\perp$ is 2-dimensional, $\mathrm{r}(F) \geq 4$ by Lemma 6.0.8, thus $\mathrm{r}(F) = 4$. The ideal I_Γ is generated by a 2-dimensional space of quadratic forms, that is the whole space of quadratic forms in $F^\perp$, hence the $\mathrm{VSP}(F, 4)$ is one point. The dimension of the family of 4-tuples in $\mathbb{P}^2$ is 8, by Lemma 5.1.14. Four points in $\mathbb{P}^{27}$ span a $\mathbb{P}^3$, thus the dimension of $\mathcal{F}_{[200]}$ is $8 + 3 = 11$.

[202] By Proposition 5.2.6 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length five, with a subscheme of length four contained in a line. For a general F, the scheme consists of five points, so $\mathrm{r}(F) \leq 5$. By comparing the Betti tables for a the ideal of a scheme of length four given in Corollary 5.1.11 with the Betti table for [202] we see that there are no subideals of $F^{\perp}$ of a scheme of length four. Thus, $\mathrm{r}(F) = 5$.

We claim that $\mathrm{VSP}(F, 5) = \mathbb{P}^1$. Indeed, let L be the line containing the subscheme of Γ of length four and Γ_1 the remaining point. We have that $\langle v_6(L), v_6(\Gamma_1)\rangle = \mathbb{P}^7$ and further that $\langle v_6(L)\rangle = \mathbb{P}^6$ and $v_6(\Gamma_1) = [F_1]$. Consider $\langle [F_1], [F]\rangle = \mathbb{P}^1$. Since $[F], [F_1] \in \langle v_6(L), v_6(\Gamma_1)\rangle$ and $\langle v_6(L)\rangle \subset \langle v_6(L), v_6(\Gamma_1)\rangle)$, $\dim(\langle [F_1], [F]\rangle \cap \langle v_6(L)\rangle) = 0$. Let

$$[F_2] = \langle v_6(L)\rangle \cap \langle [F_1], [F]\rangle.$$

Of this reason, $[F_2] \in \langle [F], [F_1]\rangle$ and $[F] \in \langle [F_1], [F_2]\rangle$. This shows that there exists a unique decomposition $F = F_1 + F_2$, where F_2 is apolar to L and F_1 is apolar to a point. By Lemma 6.0.7, $\mathrm{r}(F_2) = 4$ and $\mathrm{VSP}(F_2, 4) = \mathbb{P}^1$. Since $\mathrm{VSP}(F_1, 1)$ is one point, we get that $\mathrm{VSP}(F, 5) = \mathbb{P}^1$.

The dimension of the family of 5-tuples, four on a line in $\mathbb{P}^2$ is 8, by Lemma 5.1.14. Each 5-tuple in $\mathbb{P}^{27}$ span a $\mathbb{P}^4$. Since $\mathrm{VSP}(F, 5) = \mathbb{P}^1$, the dimension of $\mathcal{F}_{[202]}$ is $8 + 4 - 1 = 11$.

[120] By Proposition 5.2.7 there exists a subideal I_Γ of $F^{\perp}$ which is the ideal of a scheme of length five. For a general F, the scheme consists of five points, so $\mathrm{r}(F) \leq 5$. Since the space of quadratic forms in $F^{\perp}$ is 1-dimensional, $\mathrm{r}(F) \geq 5$ by Lemma 6.0.8, thus $\mathrm{r}(F) = 5$. The ideal I_Γ is generated by a 1-dimensional space of quadratic forms and a one 2-dimensional space of cubic forms, that is the whole space of quadratic forms and cubic forms in $F^{\perp}$, hence the $\mathrm{VSP}(F, 5)$ is one point. The dimension of the family of 5-tuples $\mathbb{P}^2$ is 10, by Lemma 5.1.14. Five points in $\mathbb{P}^{27}$ span a $\mathbb{P}^4$, so the dimension of $\mathcal{F}_{[120]}$ is $10 + 4 = 14$.

[111] By Proposition 5.2.8 there exists a subideal I_Γ of $F^{\perp}$ which is the ideal of a scheme of length six contained in a conic. For a general F, the scheme consists of six points, so $\mathrm{r}(F) \leq 6$. By comparing the Betti tables for a the ideal of a scheme of length five given in Corollary 5.1.13 with the Betti table for [111] we see that there are no subideals of $F^{\perp}$ of a scheme of length five. Thus, $\mathrm{r}(F) = 6$. The ideal I_Γ is generated by a 1-dimensional space of quadratic forms and a one 1-dimensional space of cubic forms, that is the whole space of quadratic forms and cubic forms in $F^{\perp}$, hence the $\mathrm{VSP}(F, 6)$ is one point. The dimension of the family of 6-tuples $\mathbb{P}^2$ on a conic is 11, by Lemma 5.1.14. Each 6-tuple in $\mathbb{P}^{27}$ span a $\mathbb{P}^5$, so the dimension of $\mathcal{F}_{[111]}$ is $11 + 5 = 16$.

[112] By Proposition 5.2.9 there exists a subideal I_Γ of $F^{\perp}$ which is the ideal of a scheme of length six, with a subscheme of length four contained in a line. For a general F, the scheme consists of six points, so $\mathrm{r}(F) \leq 6$. By comparing the Betti tables for a the ideal of a scheme of length five given in Corollary 5.1.13 with the Betti table for [112] we see that there are no subideals of $F^{\perp}$ of a scheme of length five. Thus, $\mathrm{r}(F) = 6$.

We claim that $\mathrm{VSP}(F, 6) = \mathbb{P}^1$. Indeed, let Γ_4 be the subscheme of Γ for length four contained in a line and Γ_2 the scheme of the remaining two points. We have that $\langle v_6(\Gamma)\rangle = \mathbb{P}^5$ and further that $\langle v_6(\Gamma_4)\rangle = \mathbb{P}^3$ and $v_6(\Gamma_2) = \mathbb{P}^1$. Consider $\langle v_6(\Gamma_2), [F]\rangle = \mathbb{P}^2$. Since $[F], v_6(\Gamma_2) \in \langle v_6(\Gamma)\rangle$ and $\langle v_6(\Gamma_4\rangle \subset \langle v_6(\Gamma)\rangle)$, $\dim(\langle v_6(\Gamma_2), [F]\rangle \cap \langle v_6(\Gamma_4)\rangle) = 0$. Let

$$[F_2] = \langle v_6(\Gamma_4)\rangle \cap \langle v_6(\Gamma_2), [F]\rangle.$$

By the same arguments, there exists an

$$[F_1] = \langle v_6(\Gamma_4), [F]\rangle \cap \langle v_6(\Gamma_2)\rangle.$$

Of this reason, $[F_1], [F_2], [F] \in \langle v_6(\Gamma_4), v_6(\Gamma_2)\rangle = \mathbb{P}^1$ so $[F_1], [F_2], [F]$ are colinear. As a consequence, $[F] \in \langle [F_1], [F_2]\rangle$. This shows that there exists a decomposition $F = F_1 + F_2$, where F_2 is apolar to a line and F_1 is apolar to two points. By Lemma 6.0.7 $\mathrm{VSP}(F_2, 4) = \mathbb{P}^1$. Since $\mathrm{VSP}(F_1, 2)$ is one point, we get that $\mathrm{VSP}(F, 5) = \mathbb{P}^1$. The dimension of the family of 6-tuples, four on a line in $\mathbb{P}^2$ is 10, by Lemma 5.1.14. Each 6-tuple in $\mathbb{P}^{27}$ span a $\mathbb{P}^5$. Since $\mathrm{VSP}(F,6) = \mathbb{P}^1$, the dimension of $\mathcal{F}_{[112]}$ is $10 + 5 - 1 = 14$.

[104] By Proposition 5.2.10 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length seven contained in a conic. For a general F, the scheme consists of seven points, so $\mathrm{r}(F) \leq 7$. By comparing the Betti tables for a the ideal of a scheme of length five given in Corollary 5.1.13 with the Betti table for [112] we see that there are no subideals of $F^\perp$ of a scheme of length five. There are no subscheme of length six either. Indeed, assume for contradiction the I_Γ is an ideal of a scheme of length six. If the quadric in $\mathcal{F}^\perp$ is in I_Γ then we are in the cases $\mathcal{G}_{(6)}$ and $\mathcal{G}_{(8)}$, but then I_Γ contained one cubic, which is a contradiction since there are no cubic forms in $F^\perp$. If the quadric is not in I_Γ, then there are at least two cubic forms in I_Γ, which also is a contradiction. Thus, $\mathrm{r}(F) = 7$. By the same arguments as in [112], $\mathrm{VSP}(F,7) = \mathbb{P}^1$. The dimension of the family of 7-tuples on a conic in $\mathbb{P}^2$ is 12, by Lemma 5.1.14. Each 7-tuple in $\mathbb{P}^{27}$ span a $\mathbb{P}^6$. Since $\mathrm{VSP}(F,7) = \mathbb{P}^1$, the dimension of $\mathcal{F}_{[104]}$ is $12 + 6 - 1 = 17$.

[040] By Proposition 5.2.11 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length six. For a general F, the scheme consists of six points, so $\mathrm{r}(F) \leq 6$. Since the space of cubic forms in $F^\perp$ is 4-dimensional, $\mathrm{r}(F) \geq 6$ by Lemma 6.0.8, thus $\mathrm{r}(F) = 6$. The ideal I_Γ is generated by a 4-dimensional space of cubic forms, that is the whole space of cubic forms in $F^\perp$, hence the $\mathrm{VSP}(F,6)$ is one point. The dimension of the family of 6-tuples $\mathbb{P}^2$ is 12, by Lemma 5.1.14. Six points in $\mathbb{P}^{27}$ span a $\mathbb{P}^5$, so the dimension of $\mathcal{F}_{[040]}$ is $12 + 5 = 17$.

[032] By Proposition 5.2.14 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length seven, with a subscheme of length four contained in a line. For a general F, the scheme consists of seven points, so $\mathrm{r}(F) \leq 7$. Since the space of cubic forms in $F^\perp$ is 3-dimensional, $\mathrm{r}(F) \geq 7$ by Lemma 6.0.8, thus $\mathrm{r}(F) = 7$. By the same arguments as is [112], we have that $\mathrm{VSP}(F,7) = \mathbb{P}^1$. The dimension of the family of 7-tuples, four on a line in $\mathbb{P}^2$ is 12, by Lemma 5.1.14. Each 7-tuple in $\mathbb{P}^{27}$ span a $\mathbb{P}^6$. Since $\mathrm{VSP}(F,7) = \mathbb{P}^1$, the dimension of $\mathcal{F}_{[032]}$ is $12 + 6 - 1 = 17$.

[031] By Proposition 5.2.11 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length seven. For a general F, the scheme consists of seven points, so $\mathrm{r}(F) \leq 7$. Since the space of cubic forms in $F^\perp$ is 3-dimensional, $\mathrm{r}(F) \geq 7$ by Lemma 6.0.8, thus $\mathrm{r}(F) = 7$. The ideal I_Γ is generated a 3-dimensional space of cubic forms, that is the whole space of cubic forms in $F^\perp$, hence the $\mathrm{VSP}(F,7)$ is one point. The dimension of the family of 7-tuples $\mathbb{P}^2$ is 14, by Lemma 5.1.14. Seven points in $\mathbb{P}^{27}$ span a $\mathbb{P}^6$, so the dimension of $\mathcal{F}_{[031]}$ is $14 + 6 = 20$.

[023c] By Proposition 5.2.15 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length eight, with a subscheme of length four contained in a line. For a general F, the scheme consists of eight points, so $\mathrm{r}(F) \leq 8$. Since the space of cubic forms in $F^\perp$ is 2-dimensional, $\mathrm{r}(F) \geq 8$ by Lemma 6.0.8, thus $\mathrm{r}(F) = 8$. By the same arguments as is [112], we have that $\mathrm{VSP}(F,8) = \mathbb{P}^1$. The dimension of the family of 8-tuples, four on a line in $\mathbb{P}^2$ is 14, by Lemma 5.1.14. Each 8-tuple in $\mathbb{P}^{27}$ span a $\mathbb{P}^7$. Since $\mathrm{VSP}(F,8) = \mathbb{P}^1$, the dimension of $\mathcal{F}_{[023c]}$ is $14 + 7 - 1 = 20$.

[024] By Proposition 5.2.16 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length eight, with a subscheme of length seven contained in a conic. For a general F, the scheme consists of eight points, so $\mathrm{r}(F) \leq 8$. Since the space of cubic forms in $F^\perp$ is 2-dimensional, $\mathrm{r}(F) \geq 8$ by Lemma 6.0.8, thus $\mathrm{r}(F) = 8$. By the same arguments as is [112], we have that $\mathrm{VSP}(F,8) = \mathbb{P}^1$. The dimension of the family of 8-tuples, seven on a conic in $\mathbb{P}^2$ is 14, by Lemma 5.1.14. Each 8-tuple in $\mathbb{P}^{27}$ span a $\mathbb{P}^7$. Since $\mathrm{VSP}(F,8) = \mathbb{P}^1$, the dimension of $\mathcal{F}_{[024]}$ is $14 + 7 - 1 = 20$.

[023b] By Proposition 5.2.15 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length eight. For a general F, the scheme consists of eight points, so $\mathrm{r}(F) \leq 8$. Since the space of cubic forms in $F^\perp$ is 2-dimensional, $\mathrm{r}(F) \geq 8$ by Lemma 6.0.8, thus $\mathrm{r}(F) = 8$. We see from the degree matrix in Proposition 5.2.15 that there are only one row basis such that a Hilbert–Burch matrix of I_Γ is a submatrix of $M_{[023b]}$, hence the $\mathrm{VSP}(F, 8)$ is one point. The dimension of the family of 8-tuples $\mathbb{P}^2$ is 16, by Lemma 5.1.14. Eight points in $\mathbb{P}^{27}$ span a $\mathbb{P}^7$, so the dimension of $\mathcal{F}_{[023b]}$ is $16 + 7 = 23$.

[030] By Proposition 5.2.12 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length nine. For a general F, the scheme consists of nine points, so $\mathrm{r}(F) \leq 9$. We claim that $\mathrm{r}(F) = 0$. Indeed, $F^\perp$ is generated by a 3-dimension space of cubics forms, with no linear or quadratic syzygies. Since $\mathcal{F}^\perp$ is of codimension 3, the only subideals of finite scheme are therefore generated by a 2-dimensional space of cubic forms that intersect in a CI. Thus, $\mathrm{r}(F) = 9$. Since the dimension of 2-dimensional subspaces of a 3-dimensional space is 2, $\mathrm{VSP}(F, 9) = \mathbb{P}^2$.

An $[F] \in \mathcal{F}_B$ is determined by the 3-dimensional space of cubic forms in $F^\perp$. Therefore, the dimension of $\mathcal{F}_B$ is equal to the dimension of 3-dimensional subspaces of the 10-dimensional space of cubic forms, or $\dim G(3, 10)$. We have that $\dim G(3, 10) = 3\dot{7} = 21$, thus the dimension of $\mathcal{F}_{[030]}$ is 21.

[016b] By Proposition 5.2.17 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length nine, with a subscheme of length four contained in a line. For a general F, the scheme consists of nine points, so $\mathrm{r}(F) \leq 9$. Since the space of cubic forms in $F^\perp$ is 1-dimensional, $\mathrm{r}(F) \geq 9$ by Lemma 6.0.8, thus $\mathrm{r}(F) = 9$. By the same arguments as in [112], we have that $\mathrm{VSP}(F, 9) = \mathbb{P}^1$. The dimension of the family of 9-tuples, four on a line in $\mathbb{P}^2$ is 16, by Lemma 5.1.14. Each 9-tuple in $\mathbb{P}^{27}$ span a $\mathbb{P}^8$. Since $\mathrm{VSP}(F, 9) = \mathbb{P}^1$, the dimension of $\mathcal{F}_{[024]}$ is $16 + 8 - 1 = 23$.

[016c] By Proposition 5.2.17 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length nine, with a subscheme of length seven contained in a conic. For a general F, the scheme consists of nine points, so $\mathrm{r}(F) \leq 9$. Since the space of cubic forms in $F^\perp$ is 1-dimensional, $\mathrm{r}(F) \geq 9$ by Lemma 6.0.8, thus $\mathrm{r}(F) = 9$. By the same arguments as in [112], we have that $\mathrm{VSP}(F, 9) = \mathbb{P}^1$. The dimension of the family of 9-tuples, seven on a conic in $\mathbb{P}^2$ is 16, by Lemma 5.1.14. Each 9-tuple in $\mathbb{P}^{27}$ span a $\mathbb{P}^8$. Since $\mathrm{VSP}(F, 9) = \mathbb{P}^1$, the dimension of $\mathcal{F}_{[024]}$ is $16 + 8 - 1 = 23$.

[023a] By Proposition 5.2.15 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length nine. For a general F, the scheme consists of nine points, so $\mathrm{r}(F) \leq 9$. If the rank is less than 9, then we are in one of the cases [023b] and [023c]. Since we have assumed that we are in case [023a], $\mathrm{r}(F) = 9$. The ideal I_Γ is generated a 2-dimensional space of cubic forms, that is the whole space of cubic forms in $F^\perp$, hence the $\mathrm{VSP}(F, 9)$ is one point. The dimension of the family of 9-tuples $\mathbb{P}^2$ in a complete intersection is 16, by Lemma 5.1.14. Nine points in $\mathbb{P}^{27}$ span a $\mathbb{P}^8$, so the dimension of $\mathcal{F}_{[023a]}$ is $16 + 8 = 24$.

[016a] By Proposition 5.2.17 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length nine. For a general F, the scheme consists of nine points, so $\mathrm{r}(F) \leq 9$. Since the space of cubic forms in $F^\perp$ is 1-dimensional, $\mathrm{r}(F) \geq 9$ by Lemma 6.0.8, thus $\mathrm{r}(F) = 9$. From Theorem 4.4.1 we have that there exists two row bases such that we get the zero block, hence $\mathrm{VSP}(F, 9)$ is two points. The dimension of the family of 9-tuples $\mathbb{P}^2$ is 18, by Lemma 5.1.14. Nine points in $\mathbb{P}^{27}$ span a $\mathbb{P}^8$, so the dimension of $\mathcal{F}_{[016]}$ is $18 + 8 = 26$.

[009] By Proposition 5.2.17 there exists a subideal I_Γ of $F^\perp$ which is the ideal of a scheme of length nine. For a general F, the scheme consists of nine points, so $\mathrm{r}(F) \leq 9$. Since the space of cubic forms in $F^\perp$ is 1-dimensional, $\mathrm{r}(F) \geq 9$ by Lemma 6.0.8, thus $\mathrm{r}(F) = 9$. From

Theorem 4.4.7 we have that there exists a surface of row bases such that we get the zero block, hence $\mathrm{VSP}(F, 10)$ is a surface. By [Muk09], the surface is a $K3$ surface. The dimension of the family of 10-tuples $\mathbb{P}^2$ is 20, by Lemma 5.1.14. Ten points in $\mathbb{P}^{27}$ span a $\mathbb{P}^9$. Since $\mathrm{VSP}(F, 10)$ is a surface, the dimension of $\mathcal{F}_{[009]}$ is $20 + 9 - 2 = 27$. ■

7 | Stratification

In this chapter we give a stratification $\mathbb{P}^{27} = \mathbb{P}(S_6)$ in terms of the Betti strata

$$\mathcal{F}_B = \{F \in \mathbb{P}(S_6) : S/F^{\perp} \text{ has Betti table B}\}.$$

In other words, $\mathbb{P}^{27} = \bigsqcup \mathcal{F}_B$ and that $\overline{\mathcal{F}_B} = \bigsqcup_{\mathcal{F}' \subset \overline{\mathcal{F}}} \mathcal{F}_{B'}$. Our approach is first, in Section 7.1, to investigate two relations between the

$$\mathcal{G}_B = \{\Gamma \subset S_1 : I_\Gamma \subset F^{\perp} \text{ is minimal for some } F \in \mathcal{F}_B\}.$$

The first is a closure relation between the $\mathcal{G}_B$ consisting of schemes of equal length, and the second is a containment relation between the $\mathcal{G}_B$ consisting of schemes of different length. Then, in Section 7.2, we use the two relations between the $\mathcal{G}_B$ to prove relations between most of the $\mathcal{F}_B$. For the remaining relations between the $\mathcal{F}_B$ and prove the remaining relations with a direct argument. We also prove some non-containments. Thereafter, we include some geometric description of the relation between some of the Betti strata. Lastly, in Section 7.3, we include the Betti strata such that there is a linear form in $F^{\perp}$ and explain how these strata fit into our stratification.

7.1 Relations Between the $\mathcal{G}_B$s

7.1.1 Closure Relations Between some of the $\mathcal{G}_B$s

In this section we will consider a closure relation between schemes of equal length. We will do this by proving when $\mathcal{G}_B \subset \overline{\mathcal{G}'_B}$, by proving the existence of a deformation of subschemes from $\mathcal{G}'_B$ to $\mathcal{G}_B$. $X = \bigcup(\Gamma_t, t)$ is a deformation of $X_0 = (\Gamma_0, 0)$.

Let $X \subset \mathbb{P}^2 \times \mathbb{A}^1$ and $f : X \to \mathbb{A}^1$. We call f a *deformation* if $f^{-1}(t)$ has the same Hilbert polynomial for every $t \in \mathbb{A}^1$. We are now ready to state the main result of this section.

Proposition 7.1.1. *We have the following closure relations:*

1. $\mathcal{G}_{[210]}$ *(four points, three on a line)* $\subset \overline{\mathcal{G}_{[200]}}$ *(four general points),*
2. $\mathcal{G}_{[111]}$ *(six points on a conic)* $\subset \overline{\mathcal{G}_{[040]}}$ *(six general points),*
3. $\mathcal{G}_{[023a/030]}$ *(nine points in CI)* $\subset \overline{\mathcal{G}_{[016a]}}$ *(nine general points),*
4. $\mathcal{G}_{[202]}$ *(five points, four in a line)* $\subset \overline{\mathcal{G}_{[120]}}$ *(five general points) ,*
5. $\mathcal{G}_{[112]}$ *(six points, four on a line)* $\subset \overline{\mathcal{G}_{[040]}}$ *(six general points),*
6. $\mathcal{G}_{[112]}$ *(six points, four on a line)* $\subset \overline{\mathcal{G}_{[111]}}$ *(six points on a conic),*
7. $\mathcal{G}_{[032]}$ *(seven points, four on a line)* $\subset \overline{\mathcal{G}_{[031]}}$ *(seven general points),*
8. $\mathcal{G}_{[104]}$ *(seven points on a conic)* $\subset \overline{\mathcal{G}_{[031]}}$ *(seven general points),*
9. $\mathcal{G}_{[024]}$ *(eight points, seven on a conic)* $\subset \overline{\mathcal{G}_{[023b]}}$ *(eight general points),*

10. $\mathcal{G}_{[023c]}$ *(eight points, four on a line)* $\subset \overline{\mathcal{G}_{[023b]}}$ *(eight general points),*
11. $\mathcal{G}_{[016c]}$ *(nine points, seven on a conic)* $\subset \overline{\mathcal{G}_{[016a]}}$ *(nine general points),*
12. $\mathcal{G}_{[026b]}$ *(nine points, four on a line)* $\subset \overline{\mathcal{G}_{[016a]}}$ *(nine general points).*

Proof. Since each $\mathcal{G}_B$ is irreducible, we can show the relation for a general element. We will give an explicit construction for (1) and (4). The remaining relations can be shown in a similar way.

(1) Let $\Gamma \in \mathcal{G}_{[210]}$. Without loss of generality, assume that $\Gamma = \{p_0, p_1, p_{2,3}\}$, where

$$p_0 = (1:1:0), \qquad p_2 = (0:1:0),$$
$$p_1 = (1:0:0), \qquad p_3 = (0:0:1).$$

That is, Γ consists of four point, three contained in the line $x_2 = 0$. Let further $\Gamma_0 = (p_1, p_2, p_3)$. Let $X_0 = (\{p_1, p_2, p_3, p_0\}, 0)$ and $X_t = (\{p_1, p_2, p_3, p_t\}, t)$, where $p_t = (1:1:t)$. Then $f : X \to \mathbb{A}^1$ is a deformation of X_0 and Γ is the limit of $\Gamma_t = \Gamma \oplus p_t$. Since every Γ_t is in $\mathcal{G}_{[200]}$, we are done.

For the next two inclusion, we do as above. That is, let I_0 be the ideal of a scheme in $\Gamma_0 \in \mathcal{G}$ and let Γ be a subscheme of Γ_0 of length $d-1$, where the points in Γ are general. Let p_0 be the remaining point in Γ_0. Let I_t be a family of ideals of schemes $\Gamma_t \in \mathcal{G}'$, where $\Gamma_t = \Gamma \oplus p_t$ such that p_t has p_0 as its limit.

(4) Let $\Gamma = (p_0, p_{0'}, p_1, p_2, p_3)$, where

$$p_0 = (1:1:0), \qquad p_{0'} = (1:2:0), \qquad p_1 = (1:0:0),$$
$$p_2 = (0:1:0), \qquad p_3 = (0:0:1)$$

That is, Γ consists of five points, four contained in the line $x_2 = 0$, thus $\Gamma \in \mathcal{G}_{[202]}$. Let further $\Gamma_0 = (p_1, p_2, p_3)$. Let $X_0 = (\{p_1, p_2, p_3, p_0, p_{0'}\}, 0)$ and $X_t = (\{p_1, p_2, p_3, p_t, p_{t'}\}, t)$, where $p_t = (1:1:t)$ and $p_{t'} = (1:2:t)$. Then $f : \mathbb{P}^2 \times \mathbb{A}^1 \to \mathbb{A}^1$ is a deformation of X_t to X_0 and Γ is the limit of $\Gamma_t = \Gamma \oplus p_t \oplus p_{t'}$. Since every Γ_t is in $\mathcal{G}_{[120]}$, we are done.

For the remaining inclusion, we do as in above. That is, let I_0 be the ideal of a scheme in $\Gamma_0 \in \mathcal{G}$ and let Γ be a subscheme of Γ_0 of length $d-2$, where the points in Γ are general. Let p_0 and $p_{0'}$ be the remaining points in Γ_0. Let I_t be a family of ideals of schemes $\Gamma_t \in \mathcal{G}'$, where $\Gamma_t = \Gamma \oplus p_t \oplus p_{t'}$ such that p_t has p_0 as its limit and $p_{t'}$ has $p_{0'}$ as its limit. ■

7.1.2 Containment Relations Between some of the $\mathcal{G}_B$s

In this section we will prove a containment relation between $\mathcal{G}_B$ consisting of schemes of different length.

Definition 7.1.2. We say that $\mathcal{G}_{B_1} \sqsubset \mathcal{G}_{B_2}$ if for a general $\Gamma_2 \in \mathcal{G}_{B_2}$, there exists a $\Gamma_1 \subset \Gamma_2$ such that $\Gamma_1 \in \mathcal{G}_{B_1}$.

We observe that if $\mathcal{G}_{B_1} \sqsubset \mathcal{G}_{B_2}$ then for each $\Gamma_1 \in \mathcal{G}_{B_1}$, we can find a $\Gamma_2 \in \mathcal{G}_{B_2}$ such that $\Gamma_1 \subset \Gamma_2$ by adding an appropriate point p to Γ_1. We show the strategy in an example.

We are now ready to state the first proposition in this section.

Proposition 7.1.3. *We have the following relations of the kind* $\sqsubset$*:*

1. $\mathcal{G}_{[200]}$ *(four points)* $\sqsubset \mathcal{G}_{[120]}$ *(five points)* $\sqsubset \mathcal{G}_{[040]}$ *(six points)* $\sqsubset \mathcal{G}_{[031]}$ *(seven points)* $\sqsubset \mathcal{G}_{[023b]}$ *(eight points)* $\sqsubset \mathcal{G}_{[016a]}$ *(nine points)* $\sqsubset \mathcal{G}_{[009]}$ *(ten points)*
2. $\mathcal{G}_{[104]}$ *(seven points on a conic)* $\sqsubset \mathcal{G}_{[024]}$ *(eight points, seven on a conic)* $\sqsubset \mathcal{G}_{[016c]}$ *(nine points, seven on a conic)*

3. $\mathcal{G}_{[300]}$ *(three points)* $\sqsubset \mathcal{G}_{[210]}$ *(four points, three on a line)* $\sqsubset \mathcal{G}_{[202]}$ *(five points, four on a line)* $\sqsubset \mathcal{G}_{[112]}$ *(six points, four on a line)* $\sqsubset \mathcal{G}_{[032]}$ *(seven points, four on a line)* $\sqsubset \mathcal{G}_{[023c]}$ *(eight points, four on a line)* $\sqsubset \mathcal{G}_{[016b]}$ *(nine points, four on a line)*
4. $\mathcal{G}_{[120]}$ *(five general points)* $\sqsubset \mathcal{G}_{[111]}$ *(six points on a conic)* $\sqsubset \mathcal{G}_{[104]}$ *(seven points on a conic)*
5. $\mathcal{G}_{[023b]}$ *(eight points)* $\sqsubset \mathcal{G}_{[016a/030]}$ *(nine points in CI)*

Proof. Since each $\mathcal{G}_B$ is irreducible, we can show the relation for a general element.

For a given $\Gamma \in \mathcal{G}_B$ we can choose a point $p \in \mathbb{P}^2$ such that $\Gamma \oplus p \in \mathcal{G}_{B'}$. For the first and second case, we add a general point. For the third case we add a point on a line contained in the scheme, the forth a point on a conic contained in the scheme and the fifth a point on two cubics contained in the scheme. ■

The relation from Proposition 7.1.1 and Proposition 7.1.3 are depicted in Figure 7.1. A dashed arrow represents a containment relation $\sqsubset$ and a regular arrow represents a closure relation.

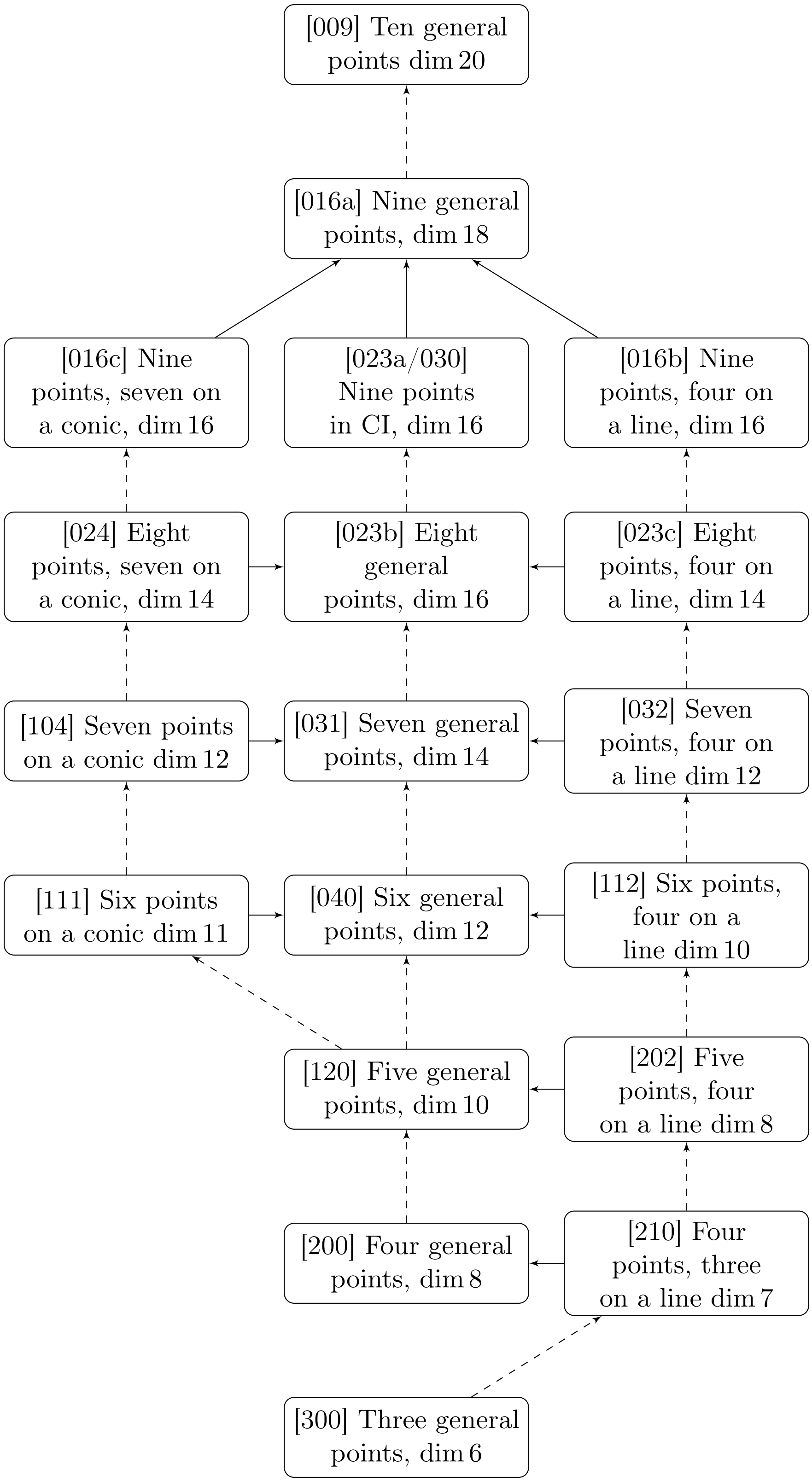


Figure 7.1: Relations between the $\mathcal{G}_B$s. A dashed arrow represents a containment relation $\sqsubset$ and a regular arrow represents a closure relation

7.2 Relations between the $\mathcal{F}_B$s

In this section we will prove the closure relations between the $\mathcal{F}_B$. First, we prove how we can use the relations between the $\mathcal{G}_B$ to get relations between most of the $\mathcal{F}_B$. Since $\mathcal{G}_[023a/030]$ consist of schemes such that I_Γ is minimal for either $\mathcal{F}_{[023a]}$ or $\mathcal{F}_{[030]}$, we need additional arguments to prove the relation between $\mathcal{F}_{[023a]}$ and $\mathcal{F}_{[030]}$. Thereafter, we prove the remaining relations using the concept of catalecticants and secant varieties. Lastly, we prove some non-containments. Recall that we say that Γ is minimal for F if the length of Γ is equal to the Cactus rank of F.

Now, we are ready to explain how we use the relations between the $\mathcal{G}_B$s to the $\mathcal{F}_B$s.

Lemma 7.2.1. *Let $\mathcal{F}_B$ be one of the irreducible Betti strata. Let $F \in \mathcal{F}_B$ be general assume Γ minimal for F. Then $\langle v_6(\Gamma)\rangle \subset \overline{\mathcal{F}_B}$.*

Proof. Let $\mathcal{F}_\Gamma = \{F \in \mathbb{P}(S_6) : \Gamma \text{ is minimal for } F\}$. We obviously have $\mathcal{F}_\Gamma \subset \langle v_6(\Gamma)\rangle$, and we claim that $\overline{\mathcal{F}_\Gamma} = \langle v_6(\Gamma)\rangle$. Indeed, let $n+1$ be the cardinality of points in Γ. Since Γ is minimal for an F, then $\langle v_6(\Gamma)\rangle = \mathbb{P}^n$. Further, we have that $\mathcal{F}_\Gamma$ is open in $\langle v_6(\Gamma)\rangle$, hence $\overline{\mathcal{F}_\Gamma} = \langle v_6(\Gamma)\rangle$. To complete the proof, observe that $\mathcal{F}_\Gamma \subset \mathcal{F}_B$. Thus $\langle v_6(\Gamma)\rangle = \overline{\mathcal{F}_\Gamma} \subset \overline{\mathcal{F}_B}$. ■

Proposition 7.2.2. *Assume $F_i \in \mathcal{F}_{\mathcal{B}_i}$ is apolar to a scheme $\Gamma_i \in \mathcal{G}_{B_i}$ and that Γ_i is minimal for F_i for $i \in \{1,2\}$. If $\mathcal{G}_{B_1} \sqsubset \mathcal{G}_{B_2}$, then $\mathcal{F}_{\mathcal{B}_1} \subset \overline{\mathcal{F}_{\mathcal{B}_2}}$.*

Proof. Since the Betti strata $\mathcal{F}_B$ are irreducible we only need to prove the result for a general F. Let $F_1 \in \mathcal{F}_{\mathcal{B}_1}$ be general. By assumption, there exists an $\Gamma_2 \in \mathcal{G}_{B_2}$ such that $\Gamma_1 \subset \Gamma_2$. Since $F_1 \in \langle v_6(\Gamma_1)\rangle \subset \langle v_6(\Gamma_2)\rangle \subset \overline{\mathcal{F}_{B_2}}$ by Lemma 7.2.1, we are done. ■

Proposition 7.2.3. *Assume $F_i \in \mathcal{F}_{\mathcal{B}_i}$ is apolar to a scheme $\Gamma_i \in \mathcal{G}_{B_i}$ and that Γ_i is minimal for F_i for $i \in \{1,2\}$. If $\mathcal{G}_{B_1} \subset \overline{\mathcal{G}_{B_2}}$, then $\mathcal{F}_{\mathcal{B}_1} \subset \overline{\mathcal{F}_{\mathcal{B}_2}}$.*

Proof. Since the Betti strata $\mathcal{F}_B$ are irreducible we only need to prove the result for a general F. Let $F_1 \in \mathcal{F}_{\mathcal{B}_1}$ be general. By assumption there exists $(\Gamma_2^t)_{t\in\mathbb{A}^1-0}$ such that $\Gamma_2^0 = \Gamma_1 \in \mathcal{G}_{B_1}$ is the limit and $\Gamma_2^t \in \mathcal{G}_{B_2}$. Then there exists F_t such that Γ_2^t i apolar to F_t and $F_t \in \langle v_6(\Gamma_2^t)\rangle \subset \overline{\mathcal{F}_{\mathcal{B}_2}}$ by Lemma 7.2.1. Since $F_1 \in \langle v_6(\Gamma_1)\rangle$, F_1 is the limit of $(F_t)_{t\ \in\mathbb{A}^1-0}$ and $F_1 \in \overline{\mathcal{F}_{B_2}}$. ■

By using Proposition 7.1.1 together with Proposition 7.2.3 and Proposition 7.1.3 together with Proposition 7.2.2, we get the following

Proposition 7.2.4. *There are* 20 *irreducible Betti strata of non-degenerate ternary quartic forms and they satisfy the following closure relations:*

1. $\mathcal{F}_{[210]}$ *(four points, three on a line)* $\subset \overline{\mathcal{F}_{[200]}}$ *(four general points),*
2. $\mathcal{F}_{[111]}$ *(six points on a conic)* $\subset \overline{\mathcal{F}_{[040]}}$ *(six general points),*
3. $\mathcal{F}_{[023a/030]}$ *(nine points in CI)* $\subset \overline{\mathcal{F}_{[016a]}}$ *(nine general points),*
4. $\mathcal{F}_{[202]}$ *(five points, four in a line)* $\subset \overline{\mathcal{F}_{[120]}}$ *(five general points) ,*
5. $\mathcal{F}_{[112]}$ *(six points, four on a line)* $\subset \overline{\mathcal{F}_{[040]}}$ *(six general points),*
6. $\mathcal{G}_{[112]}$ *(six points, four on a line)* $\subset \overline{\mathcal{G}_{[111]}}$ *(six points on a conic),*
7. $\mathcal{F}_{[032]}$ *(seven points, four on a line)* $\subset \overline{\mathcal{F}_{[031]}}$ *(seven general points),*
8. $\mathcal{F}_{[104]}$ *(seven points on a conic)* $\subset \overline{\mathcal{F}_{[031]}}$ *(seven general points),*
9. $\mathcal{F}_{[024]}$ *(eight points, seven on a conic)* $\subset \overline{\mathcal{F}_{[023b]}}$ *(eight general points),*
10. $\mathcal{F}_{[023c]}$ *(eight points, four on a line)* $\subset \overline{\mathcal{F}_{[023b]}}$ *(eight general points),*
11. $\mathcal{F}_{[016c]}$ *(nine points, seven on a conic)* $\subset \overline{\mathcal{F}_{[016a]}}$ *(nine general points),*
12. $\mathcal{F}_{[026b]}$ *(nine points, four on a line)* $\subset \overline{\mathcal{F}_{[016a]}}$ *(nine general points).*

13. $\mathcal{F}_{[200]}$ *(four points)* $\subset \overline{\mathcal{F}_{[120]}}$ *(five points)* $\subset \overline{\mathcal{F}_{[040]}}$ *(six points)* $\subset \overline{\mathcal{F}_{[031]}}$ *(seven points)* $\subset \overline{\mathcal{F}_{[023b]}}$ *(eight points)* $\subset \overline{\mathcal{F}_{[016a]}}$ *(nine points)* $\subset \overline{\mathcal{F}_{[009]}}$ *(ten points)*
14. $\mathcal{F}_{[104]}$ *(seven points on a conic)* $\subset \overline{\mathcal{F}_{[024]}}$ *(eight points, seven on a conic)* $\subset \overline{\mathcal{F}_{[016c]}}$ *(nine points, seven on a conic)*
15. $\mathcal{F}_{[300]}$ *(three points)* $\subset \overline{\mathcal{F}_{[210]}}$ *(four points, three on a line)* $\subset \overline{\mathcal{F}_{[202]}}$ *(five points, four on a line)* $\subset \overline{\mathcal{F}_{[112]}}$ *(six points, four on a line)* $\subset \overline{\mathcal{F}_{[032]}}$ *(seven points, four on a line)* $\subset \overline{\mathcal{F}_{[023c]}}$ *(eight points, four on a line)* $\overline{\subset \mathcal{F}_{[016b]}}$ *(nine points, four on a line)*
16. $\mathcal{F}_{[120]}$ *(five general points)* $\subset \overline{\mathcal{F}_{[111]}}$ *(six points on a conic)* $\subset \overline{\mathcal{F}_{[104]}}$ *(seven points on a conic)*
17. $\mathcal{F}_{[023b]}$ *(eight points)* $\subset \overline{\mathcal{F}_{[016a/030]}}$ *(nine points in CI)*

Proposition 7.2.5. $\mathcal{F}_{[031]} \subset \overline{\mathcal{F}_{[030]}} \subset \overline{\mathcal{F}_{[023a]}}$

Proof. We let $B_1 = [030]$ and $B_2 = [023a]$. Let $\Gamma \in \mathcal{G}_{[030]/[023a]}$ be general. Then there exists $F_1 \in \mathcal{F}_{B_1}$ and $F_2 \in \mathcal{F}_{B_2}$ such that Γ is minimal for F_1 and F_2. Let $\mathcal{F}_{\Gamma_i} = \{F \in \langle v_6(\Gamma)\rangle : F \in F_{B_i}\}$ for $i \in \{1,2\}$. We claim that $\overline{\mathcal{F}_{\Gamma_2}} = \langle v_6(\Gamma)\rangle$. Indeed, since $F^\perp$ for $F \in \mathcal{F}_{B_1}$ is generated by three cubic forms, we have that each 3-dimensional subspace of the vector space of cubic forms containing I_Γ gives a point $F \in \mathcal{F}_{\Gamma_1}$. Since the space of 3-dimensional subspaces of the vector space of cubic forms is a $\mathbb{P}^7$, $\mathcal{F}_{\Gamma_1} = \mathbb{P}^7$. Further, we have that $\mathcal{F}_{\Gamma_2}$ is open in $\langle v_6(\Gamma)\rangle$, hence $\overline{\mathcal{F}_{\Gamma_2}} = \langle v_6(\Gamma)\rangle$. To complete the proof, observe that $\mathcal{F}_{\Gamma_1} \subset \langle v_6(\Gamma)\rangle$ and $\mathcal{F}_{\Gamma_2} \subset \mathcal{F}_{\mathcal{B}_2}$. Thus, $F_1 \in \mathcal{F}_{\Gamma_1} \subset \overline{\mathcal{F}_{\Gamma_2}} \subset \overline{\mathcal{F}_{\mathcal{B}_2}}$. ■

Lemma 7.2.6. *Let h_{ij} be the dimension of the space of forms of degree j in $F_i^\perp$ for $F_i \in \mathcal{F}_i$. If $\mathcal{F}_1 \subset \overline{\mathcal{F}_2}$, then $h_{1j} \geq h_{2j}$.*

Proof. Let $F \in \mathcal{F}_1$ be the limit of $(F_t)_{t\in\mathbb{A}-0}$ where $F_t \in \mathcal{F}_2$. Since $F_t^\perp$ contains a h_{2j}-dimensional space of forms of degree j, then also $F_0^\perp$ contains a h_{2j}-dimensional space of forms of degree j. This shows that $h_{1j} \geq h_{2j}$ for every j. ■

Proposition 7.2.7. *We have the following non-containments*

1. $\mathcal{F}_{[024]} \not\subseteq \overline{\mathcal{F}_{[030]}}$
2. $\mathcal{F}_{[023c]} \not\subseteq \overline{\mathcal{F}_{[030]}}$
3. $\mathcal{F}_{[016c]} \not\subseteq \overline{\mathcal{F}_{[023a]}}$
4. $\mathcal{F}_{[016b]} \not\subseteq \overline{\mathcal{F}_{[023a]}}$
5. $\mathcal{F}_{[030]} \not\subseteq \overline{\mathcal{F}_{[016c]}}$
6. $\mathcal{F}_{[030]} \not\subseteq \overline{\mathcal{F}_{[016b]}}$

Proof. The first four cases follows directly from Lemma 7.2.6. For case (5), observe that a general $F \in \mathcal{F}_{[016c]}$ is apolar to a scheme of length nine with a subscheme of length seven contained in a conic. This must also be the case for an $F \in \overline{\mathcal{F}_{[016c]}}$. Since an $F \in \mathcal{F}_{[030]}$ is apolar to a scheme which is a complete intersection of two cubics, and there are no subscheme of seven points contained in a conic in the complete intersection of two cubics, $\mathcal{F}_{[030]} \not\subseteq \overline{\mathcal{F}_{[016c]}}$. For case (6), observe that, by similar arguments, an $F \in \overline{\mathcal{F}_{[016b]}}$ is apolar to a scheme of length nine containing a subscheme of length four contained in a line. There are no subscheme of length four contained in a line in a complete intersection of two cubics, hence $\mathcal{F}_{[030]} \not\subseteq \mathcal{F}_{[016b]}$. ■

The relation from Proposition 7.2.4, Proposition 7.2.5 and Proposition 7.2.7 are depicted in Figure 7.2. An arrow represents a closure relation.

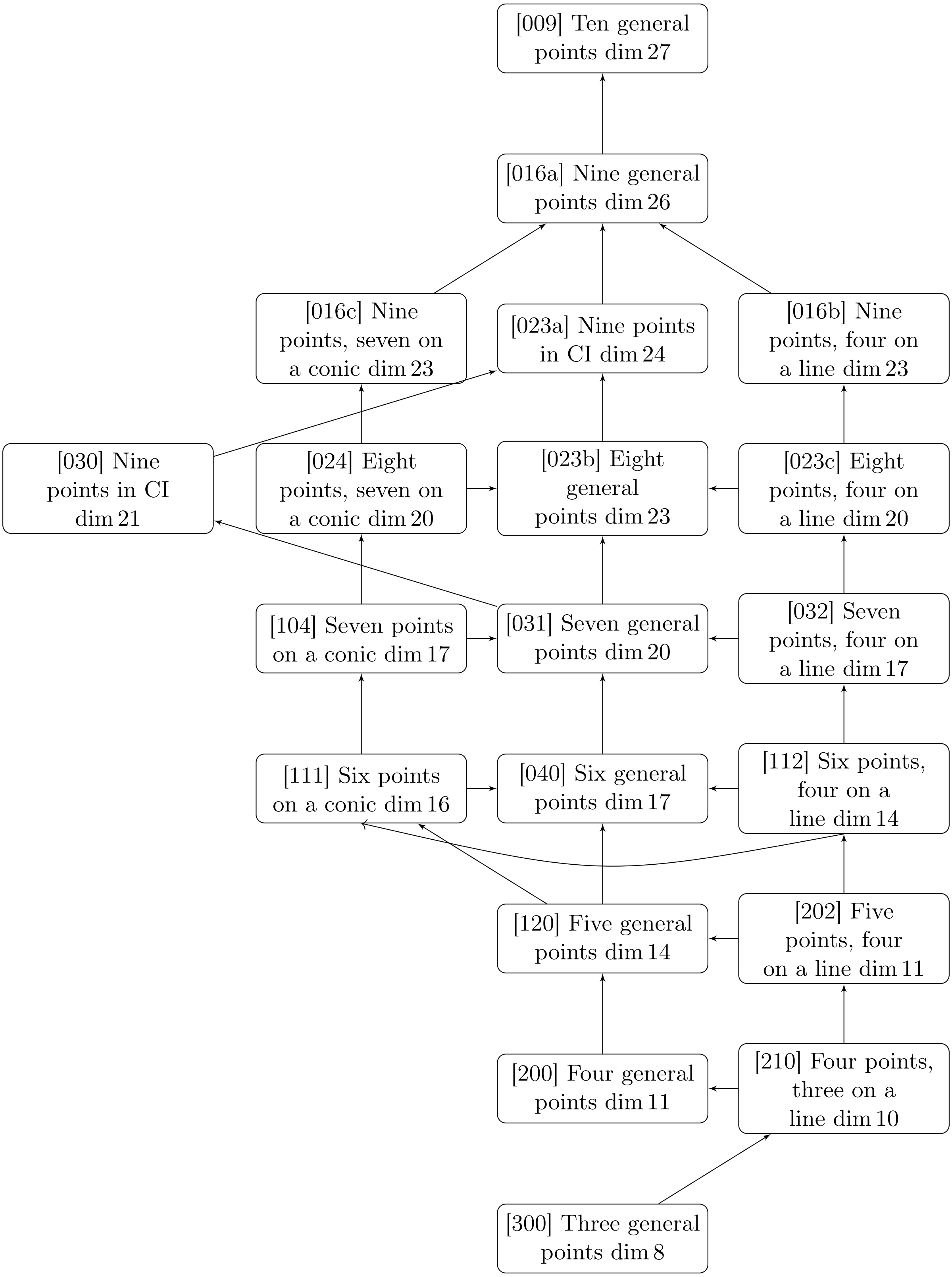


Figure 7.2: Stratification of $\mathbb{P}(S_6) = \mathbb{P}^{27}$. An arrow represents a closure relation

7.2.1 Geometric Interpretation

The sth secant variety to the Veronese variety is

$$\mathrm{Sec}_s(v_d(\mathbb{P}(S_1))) = \overline{\{[F] \in \mathbb{P}(S_d) : F = L_1^d + \cdots + L_s^d \text{ for some } L_1, \ldots, L_s \in S_1\}}$$

Definition 7.2.8. Let $F \in S_d = \mathbb{C}[x_0, \ldots, x_n]$ and $G \in T_{d-u}$ and consider the map

$$\begin{aligned} T_{d-u} &\to S_u \\ G &\mapsto F(G). \end{aligned}$$

The matrix associated to the map is called the *catalecticant* and is written $\mathrm{Cat}(u, d-u, n+1)(F)$.

We will use the case $n = 1$, $d = 2k$ and $u = k$, so let $F = \sum_{j=0}^{d-1} \binom{d}{j} a_j x_0^{d-j} x_1^d$. Then, by [Dol12, Example 1.4.1],

$$\mathrm{Cat}(k, k, 2)(F) = \begin{pmatrix} a_0 & a_1 & \ldots & a_k \\ a_1 & a_2 & \ldots & a_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_k & a_{k+1} & \ldots & a_{2k} \end{pmatrix}.$$

We have the following relationship between the catalecticant and the secant varieties.

Lemma 7.2.9. *[Iar99, Theorem 1.45] Let $C = \mathbb{P}^1$ be a curve and let $F = v_d(C) \subset \mathbb{P}(S_d)$. Let $I_{s+1}(\mathrm{Cat}(k, k, 2)(F))$ be the ideal generated by the $(s+1) \times (s+1)$-minors of $\mathrm{Cat}(k, k, 2)(F)$ and let $s \geq 1$. Then*

$$I_{s+1}(\mathrm{Cat}(k, k, 2)(F)) = I(\mathrm{Sec}_s(F)).$$

We give give two examples. In the first example, C is a line and in the second example, C is a conic.

Example 7.2.10. Let C be a line L. Then $F = v_6(L) \subset \mathbb{P}^{27}$ is a sextic. Recall that an $[F_L] \in \langle v_6(L) \rangle$ can be considered as a general binary form. By Theorem 6.0.4, $\mathrm{r}(F) = 4$, thus we have the following strict inclusions:

$$v_6(L) \subsetneq \mathrm{Sec}_2(v_6(L)) \subsetneq \mathrm{Sec}_3(v_6(L)) \subsetneq eq\, \mathrm{Sec}_4(v_6(L)) = \langle v_6(L) \rangle.$$

We write $F = \sum_{j=0}^{5} \binom{6}{j} a_j x_0^{6-j} x_1^6$ and consider $(a_0 : \cdots : a_6)$ as coordinates in $\langle v_6(L) \rangle = \mathbb{P}^6$. This gives

$$\mathrm{Cat}(3, 3, 2)(F) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{pmatrix}.$$

For an $[F_L] \in \mathbb{P}^6$, we have by Lemma 7.2.9, that $[F_L] \in \mathrm{Sec}_3(F)$ if and only if $\det(\mathrm{Cat}(3, 3, 2)(F))$ vanish in $[F_L]$, $[F_L] \in \mathrm{Sec}_2(F)$ if and only if every 3×3 minor of $\mathrm{Cat}(3, 3, 2)(F)$ vanish in $[F_L]$, and $[F_L] \in F$ if and only if every 2×2 minor of $\mathrm{Cat}(3, 3, 2)(F)$ vanish in $[F_L]$.

Example 7.2.11. Let C be a line Q. Then $F = v_6(Q) \subset \mathbb{P}^{27}$ is a degree twelve polynomial. Recall that an $[F_Q] \in \langle v_6(Q) \rangle$ can be considered as a general binary form. By Theorem 6.0.4, $\mathrm{r}(F) = 7$, thus we have the following strict inclusions:

$$v_6(Q) \subsetneq \mathrm{Sec}_2(v_6(Q)) \subsetneq \mathrm{Sec}_3(v_6(Q)) \subseteq \mathrm{Sec}_4(v_6(Q)) = \langle v_6(Q) \rangle.$$

We write $F = \sum_{j=0}^{11} \binom{12}{j} a_j x_0^{12-j} x_1^{12}$ and consider $(a_0 : \cdots : a_{12})$ as coordinates in $\langle v_6(Q) \rangle = \mathbb{P}^{12}$. This gives

$$\operatorname{Cat}(6,6,2)(F) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} \\ a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix}.$$

For an $[F_Q] \in \mathbb{P}^{12}$, we have by Lemma 7.2.9, that $[F_Q] \in \operatorname{Sec}_s(F)$ if and only if every $(s-1)\times(s-1)$ minor of $\operatorname{Cat}(3,3,2)(F)$ vanish in $[F_Q]$ for $3 \leq s \leq 7$, and $[F_Q] \in F$ if and only if every 2×2 minor of $\operatorname{Cat}(3,3,2)(F)$ vanish in $[F_Q]$.

Now, we introduce some lemmas we will use in the discussion below.

Lemma 7.2.12. *Let A be an $n \times n$ matrix. Then A has rank r if and only if A can be written as a minimal sum of r matrices of rank 1.*

Proof. Let $A = (v_1, \ldots, v_n)$, that is v_i are the columns of A. Since A has rank r, there exists a basis $\{b_1, \ldots, b_r\}$ for the column space of A. In other word, there exists a_{ij} such that

$$\begin{aligned} v_1 &= a_{11}b_1 + \cdots + a_{1r}b_r, \\ &\vdots \\ v_n &= a_{n1}b_1 + \cdots + a_{nr}b_r. \end{aligned}$$

This gives that

$$\begin{aligned} A &= (a_{11}b_1 + \cdots + a_{1r}b_r, \ldots, a_{n1}b_1 + \cdots + a_{nr}b_r) \\ &= (a_{11}b_1, \ldots, a_{n1}b_1) + \cdots + (a_{1r}b_r + \cdots + a_{nr}b_r). \end{aligned}$$

Since $A_i = (a_{1i}b_i, \ldots, a_{ni}b_i)$ is a matrix where every column is a multiple of b_i, A_i has rank 1.

For the other implication, assume that

$$\begin{aligned} A &= (a_{11}b_1, \ldots, a_{n1}b_1) + \cdots + (a_{1r}b_r + \cdots + a_{nr}b_r) \\ &= (a_{11}b_1 + \cdots + a_{1r}b_r, \ldots, a_{n1}b_1 + \cdots + a_{nr}b_r). \end{aligned}$$

Since r is minimal, $\{b_1, \ldots, b_r\}$ is linearly independent. Since the columns of A are spanned by $\{b_1, \ldots, b_r\}$, the column space as dimension n, which implies that A has rank n. ∎

Let the multiplicity of p in $\det A$ be

$$m_p(\det A) = \max\{m : D(\det A)(p) = 0 \text{ for all } D \in \mathbb{C}[\frac{\partial}{\partial a_{ij}}]_{m-1}\}$$

Lemma 7.2.13. *Let $A = (a_{ij})$ be an $n \times n$ matrix and let $p \in \mathbb{P}^{n^2-1}$ Assume $m = m_p(\det A)$. Then every $(n-m+1) \times (n-m+1)$ minors vanish in p.*

Proof. Let $A_{i,j}$ be the $(n-1) \times (n-1)$ minor obtained by deleting the ith row and jth column, Generally, let $\det A_{i_1 \ldots i_{m-1}, j_1 \ldots j_{m-1}}$ be the $(n-m+1) \times (n-m+1)$ minor obtained by deleting the rows $i_1, \ldots, i_{m-1}$ and columns $j_1, \ldots, j_{m-1}$. Then $\det A = \sum_{j=0}^{n-1} a_{ij} \det A_{ij}$. Thus,

$$\frac{\partial \det A}{\partial a_{ij}} = \det A_{ij}.$$

In the same way, we let $0 \leq i_1 < \cdots < i_{m-1} \leq n-1$ and $0 \leq j_1 < \cdots < j_{m-1} \leq n-1$ be given. Then

$$\frac{\partial^m \det A}{\partial a_{i_1 j_1} \ldots a_{i_{m-1} j_{m-1}}} = \det A_{i_1 \ldots i_{m-1}, j_1 \ldots j_{m-1}}.$$

Evaluating in p, we get

$$\frac{\partial^m \det A}{\partial a_{i_1 j_1} \ldots a_{i_m j_m}}(p) = \det A_{i_1 \ldots i_m, j_1 \ldots j_m}(p).$$

Since $\frac{\partial \det A}{\partial a_{i_1 j_1} \ldots a_{i_m j_m}}(p) = 0$ by assumption, we are done. ∎

Lemma 7.2.14. *Let* $\mathrm{Cat}(k,k,2)(F)$ *be given and consider the hypersurface* $H = V(\det(\mathrm{Cat}(k,k,2)(F)))$. *If* $p \subset \mathrm{Sec}_s(F)$, *but* $p \not\subseteq \mathrm{Sec}_{s-1}(F)$, *then* $m_p = k+1-s$.

Proof. This follows from Lemma 7.2.13, Lemma 7.2.12 and Lemma 7.2.9. ∎

Proposition 7.2.15. $\mathcal{F}_{[016c]} \neq \mathcal{F}_{[016b]}$

Proof. Since $\dim \mathcal{F}_{[016c]} = \dim \mathcal{F}_{[016b]}$ it is enough to show that for a general element $F \in \mathcal{F}_{[016b]}$ then $F \notin \mathcal{F}_{[016c]}$. Assume therefore $F \in \mathcal{F}_{[016b]}$, that is F is apolar to a scheme Γ where a subscheme of length four is contained in a line L. Let Q be the conic containing the remaining five points. We have that $\langle v_6(L) \rangle = \mathbb{P}^6$, $\langle v_6(Q) \rangle = \mathbb{P}^{12}$. Let $p_1, p_2 \in L \cap Q$. Then $\langle v_6(p_1), v_6(p_2) \rangle \subset \langle v_6(L) \rangle \cap \langle v_6(Q) \rangle$, which gives that $\langle v_6(L), v_6(Q) \rangle = \mathbb{P}^{17}$. Let

$$\hat{L} = \langle v_6(L) \rangle \cap \langle v_6(Q), F \rangle = \mathbb{P}^2 \subset \mathbb{P}^6,$$
$$\hat{Q} = \langle v_6(Q) \rangle \cap \langle v_6(L), F \rangle = \mathbb{P}^2 \subset \mathbb{P}^{12}.$$

A general point in $\hat{L}$ has rank 4 and a general point in $\hat{Q}$ has rank 7. If we prove that there exists an $F_1 \in \hat{L}$ such that F_1 has rank 2, then we can find a unique $F_2 \in \hat{Q}$ such that $F_2 \in \langle F_1, F \rangle$, so F would be of type $[016b]$. We claim that such an F_1 does not exists. Indeed, let

$$H = V(I_3(\mathrm{Cat}(3,3,2)(v_6(L)))),$$

and consider $H \cap \hat{L}$. If there exists an $F_1 \in H \cap \hat{L}$, then $F_1 \in \mathrm{Sec}_2(v_6(L))$. Since $\dim S_2(v_6(L)) = 3$ and both $H, \hat{L} \subset \mathbb{P}^6$, we have $H \cap \hat{L} = \emptyset$ in general. Consequently, there does not exists an $F_1 \in \hat{L}$ of rank 2, so $F \notin \mathcal{F}_{[016c]}$. ∎

Remark 7.2.16. The relationship between the catalecticant and the secant varieties to a curve can be used to give a geometric description of the $\mathrm{VSP}(F,9)$ for an $F \in \mathcal{F}_{[016a]}$. Let $F \in \mathcal{F}_{[016a]}$ and recall that $\mathrm{VSP}(F,9)$ is two points, that is F can be written as a power sum representation of nine linear forms in two ways. By apolarity, there exists a $I_\Gamma \subset F^\perp$ and we have shown that I_Γ is generated by a cubic K and a 3-dimensional subset of quadrics. In particular, Γ is contained in K. Assume that K is the union of a line L and a quadric Q. Then there exists a subscheme $\Gamma_L \subset \Gamma$ of length three contained in L and a subscheme $\Gamma_Q \subset \Gamma$ of length six contained in Q.

Let $p_1, p_2 \in L \cap Q$ and let $L' = \langle v_6(p_1), v_6(p_2) \rangle$. By the same arguments as in the proof of Proposition 7.2.15, we can find two planes

$$\hat{L} = \langle v_6(L) \rangle \cap \langle v_6(Q), F \rangle = \mathbb{P}^2 \subset \mathbb{P}^6,$$
$$\hat{Q} = \langle v_6(Q) \rangle \cap \langle v_6(L), F \rangle = \mathbb{P}^2 \subset \mathbb{P}^{12}.$$

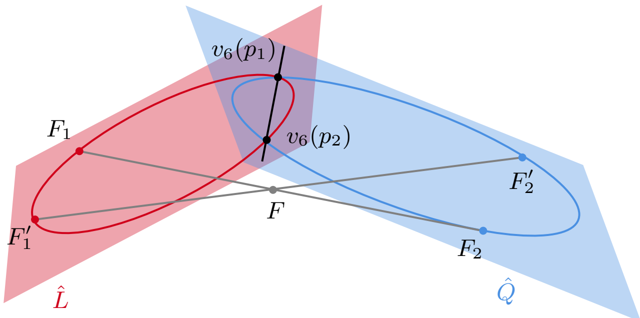
Let

$$H_L = V(\det(\mathrm{Cat}(3,3,2)(v_6(L)))),$$

$$H_Q = V(\det(\mathrm{Cat}(6,6,2)(v_6(Q)))).$$

We first consider $H_L \cap \hat{L} \in \mathbb{P}^6$. Since $\dim H_L = 5$ and $\dim \hat{L} = 2$, $\dim(H_L \cap \hat{L}) = 1$. For every $p \in L'$, we have that $p \in \mathrm{Sec}_2(v_6(L))$, thus, by Lemma 7.2.14, $L'^2 Q' = H_L \cap \hat{L}$, where Q' is a conic. Since $v_6(p_1), v_6(p_2) \in v_6(L)$, the multiplicity of $H_L \cap \hat{L}$ in these point is 3, thus $v_6(p_1), v_6(p_2) \in Q'$. Secondly, we consider $H_Q \cap \hat{Q} \in \mathbb{P}^{12}$. Since $\dim H_Q = 11$ and $\dim \hat{Q} = 2$, $\dim(H_Q \ \cap \hat{Q}) = 1$. For every $p \in L'$, we have that $p \in \mathrm{Sec}_2(v_6(Q))$, thus, by Lemma 7.2.14 $L'^5 Q'' = H_Q \cap \hat{Q}$, where Q'' is a conic. Since $v_6(p_1), v_6(p_2) \in v_6(Q)$, the multiplicity of $H_Q \cap \hat{Q}$ in these points i 6, thus $v_6(p_1), v_6(p_2) \in Q''$. To summarize, each point on Q' has rank 3 and each point on Q'' as rank 6.

We will show that there are two pairs of points on Q' and Q'' such that F lies in the span of each pair. Pick a point $F_L \in Q'$ and let F_Q be the corresponding point in $\hat{Q}$ such that $F \in \langle F_L, F_Q \rangle$. Let F_L variate along Q' and consider the conic $Q_{\hat{Q}}$ in $\hat{Q}$ parameterizing the corresponding F_Q. Then $Q_{\hat{Q}} \in \hat{Q} = \mathbb{P}^2$ and $Q'' \in \hat{Q} = \mathbb{P}^2$ intersect in four points, where two of the points are $v_6(p_1), v_6(p_2)$. Let F_2, F_2' be the remaining two points and let F_1, F_1' be the corresponding points on Q'. Then $F \in \langle F_1, F_2 \rangle$ and $F \in \langle F_1', F_2' \rangle$. See the figure.



We now investigate how a general element of $\mathcal{F}_{[016b]}$ or $\mathcal{F}_{[016c]}$ fits into the figure. Let $F \in \mathcal{F}_{[016c]}$. Thus F is apolar to a scheme Γ of length nine, where a subscheme of length two is contained in a line L and the remaining subscheme of length seven in a conic Q. Let $L', \hat{L}, \hat{Q}, H_L$ and H_Q be as above. We consider $H_L \cap \hat{L} \in \mathbb{P}^6$. As in the case $[016a]$, $L'Q' = H_L \cap \hat{L}$, where Q' is a conic. By assumption, there exists an $F_{(2)} \in \hat{L}$ of rank 2. Thus, $m_{F_{(2)}}(H_L) = 2$, so Q' is the union of two lines intersecting in $F_{(2)}$. A general point in $\hat{Q}$ has rank 7, thus we can find an $F_{(7)} \in \hat{Q}$ of rank 7 such that $F \in \langle F_{(2)}, F_{(7)} \rangle$. Since $\mathrm{VSP}(F_{(7)}, 7) = \mathbb{P}^1$, there is pencil of ways F can be decomposed as a sum of $F_{(2)}$ and $F_{(7)}$. Consider now $H_Q \cap \hat{Q}$. The situation is an in the case $[016a]$, so $L'^5 Q'' = H_Q \cap \hat{Q}$, where Q'' in general is a non-degenerate conic. We use the same strategy to find F_1, F_2 and F_1', F_2' such that $F \in \langle F_1, F_2 \rangle$ and $F \in \langle F_1', F_2' \rangle$ and F_1, F_1' has rank 3 and F_2, F_2' has rank 6. Since F_1 lies on the line spanned by $v_6(p_1)$ and $F_{(2)}$ and $v_6(p_1)$, F_2 lies on the line spanned by $v_6(p_2)$ and $F_{(7)}$, the decomposition $F = F_1 + F_2$ is one of the decompositions of in $\mathrm{VSP}(F, 7) = \mathbb{P}^1$.

Let $F \in \mathcal{F}_{[016b]}$. We can do a similar description as in the case $[016c]$, where we note that Q'' and not Q' is degenerate, since we assume that there exists a point of rank 5 in $\hat{Q}$.

7.3 Binary Forms

In this section we explain how the F where $F^{\perp}$ contains a linear form fits into the stratification just given. Recall that if $F^{\perp}$ contains a linear form, then F can be considered as a binary form. The power sum decomposition of binary forms are completely understood and are stated in the following

Theorem 7.3.1 (Sylvester)**.** *Let $d = 2k$ and $F \in \mathbb{C}[x_0, x_1]_d$. Then $F^\perp = \langle G_1, G_2\rangle$, where $\deg G_1 = d_1$ and $\deg G_2 = d_2$ and $d_1 + d_2 = d + 2$. Let $d_1 \leq d_2$. Then $\mathrm{r}(F) = d_2$ $\mathrm{VSP}(F, \mathrm{r}(F))$ is one point for $\mathrm{r}(F) \leq k$ and $\mathrm{VSP}(F, k+1) = \mathbb{P}^1$.*

Corollary 7.3.2. *Let $d = 6$ and $F \in \mathbb{C}[x_0, x_1, x_2]_6$ and assume that F is apolar to a line. Then $\mathrm{r}(F) \leq 4$ and $\mathrm{VSP}(F, \mathrm{r}(F))$ is one points for $\mathrm{r}(F) \leq 3$ and $\mathrm{VSP}(F, 4) = \mathbb{P}^1$.*

By the corollary we can find the Betti tables for each case.

1	2	1	-
-	-	-	-
-	-	-	-
-	-	-	-
-	-	-	-
-	-	-	-
-	2	1	1

1	1	-	-
-	1	1	-
-	-	-	-
-	-	-	-
-	-	-	-
-	1	1	-
-	-	1	1

1	1	-	-
-	-	-	-
-	1	1	-
-	-	-	-
-	1	1	-
-	-	-	-
-	-	1	1

1	-	-	-
-	-	-	-
-	-	-	-
-	2	2	-
-	-	-	-
-	-	-	-
-	-	-	1

We summarize the corollary in a table.

B	$\mathrm{r}(F)$	$\mathrm{VSP}(F, r)$	Γ	$\dim(\mathcal{F}_\mathcal{B})$
[000]	1	one point	one point	2
[100]	2	one point	two points	5
[010]	3	one point	three points on a line	7
[002]	4	$\mathbb{P}^1$	four points on a line	8

Observe that the type [000] is $v_6(\mathbb{P}^2) \subset \mathbb{P}^{27}$. We show how these Betti strata fit into our stratification in Figure 7.3.

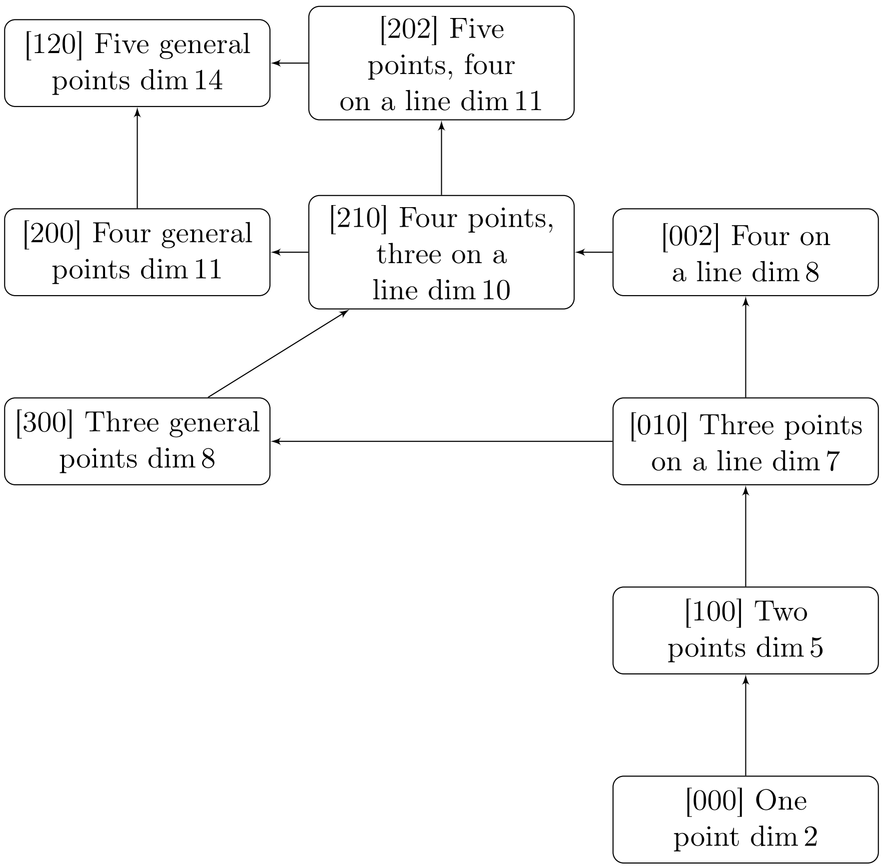


Figure 7.3: Stratification including the binary forms

8 | Double Cubic Forms

In this chapter we do explicit computation on the apolar ideal of double cubic forms. In particular, we investigate the relation between a given form $F = Q^2$, where Q is a ternary cubic, and a cubic apolar to F.

We begin with a familiar characterization of ternary cubics. The family of ternary cubics of the form $Q = x_0^3 + x_1^3 + x_2^3 + \lambda x_0 x_1 x_2 = 0$ for $\lambda \in \mathbb{A}^1$ is called *the Hesse pencil.*

Lemma 8.0.1. *[AD08] Let $Q = 0$ be a smooth ternary cubic. Then $Q = 0$ is a member of the Hesse pencil. The only singular member of the Hesse pencil is the following*

1. $x_0 x_1 x_2 = 0$
2. $(x_0 + x_1 + x_2)(x_0 + \epsilon x_1 + \epsilon^2 x_2)(x_0 + \epsilon^2 x_1 + \epsilon x_2) = 0$
3. $(x_0 + \epsilon x_1 + x_2)(x_0 + \epsilon_1^x + \epsilon_2^x)(x_0 + x_1 + \epsilon x_2) = 0$
4. $(x_0 + \epsilon^2 x_1 + x_2)(x_0 + \epsilon x_1 + \epsilon x_2)(x_0 + x_1 + \epsilon^2 x_2) = 0$*, where* $\epsilon^3 = 1$.

A cubic of the form $Q = x_0^3 + x_1^2 x_2 - x_0^2 x_2 = 0$ is called a *nodal cubic* and a cubic of the form $Q = x_0^3 - x_1^2 x_2 = 0$ is called a cuspidal cubic.

Lemma 8.0.2. *Let $Q = 0$ med a non-degenerate ternary cubic not in the Hesse pencil. Then $Q = 0$ is either a nodal cubic or a cuspidal cubic.*

Now, we give a result that gives a correspondence between a double cubic form F and a cubic in the apolar ideal.

Proposition 8.0.3. *Let $Q = x_0^3 + x_1^3 + x_2^3 + \lambda x_0 x_1 x_2$ and let $Q' = y_0^3 + y_1^3 + y_2^3 + \lambda' y_0 y_1 y_2$. Let $Q^2 = F$. If $\lambda\lambda' = -18$, then $Q' \in F^\perp$.*

Proof. We use the program in Listing 10.21 and get the following output:

$$\begin{aligned} Q'(Q^2) =& (8\lambda\lambda' + 144)x_0^3 \\ &+ (8\lambda\lambda' + 144)x_1^3 \\ &+ (8\lambda\lambda' + 144)x_2^3 \\ &+ (8\lambda\lambda' + 144)\lambda x_0 x_1 x_2 \end{aligned}$$

Since $18 \cdot 8 = 144$, we get that if $\lambda\lambda' = -18$ then $Q'(Q^2) = 0$. ■

Corollary 8.0.4. *Let $Q = x_0^3 + x_1^3 + x_2^3 + \lambda x_0 x_1 x_2$ and let $Q' = y_0^3 + y_1^3 + y_2^3 + \lambda' y_0 y_1 y_2$ and assume $\lambda\lambda' = -18$. Then $Q \in (Q'^2)^\perp$ and $Q' \in (Q^2)^\perp$*

Remark 8.0.5. By, [Fri02, Proposition 2.16], we have that the j-invariant of a ternary cubic form in Hesse form $E_\lambda : x_0^3 + x_1^3 + x_2^3 + \lambda x_0 x_1 x_2$ is given by

$$j(E_\lambda) = \frac{\lambda^3(\lambda^3 - 216)^3}{(\lambda + 3)^3(\lambda + 3\epsilon)^3(\lambda + 3\epsilon^2)^3}$$

We use the program in Listing 10.25 to show that when $\lambda\lambda' = -18$, for $\lambda = 1$, then $j(E_\lambda) \neq j(E_\lambda)$, which means that the two general curves $Q = 0$ and $Q' = 0$ in Corollary 8.0.4 are not isomorphic.

Proposition 8.0.6. *Let $Q = x_0^3 - x_1^2 x_2$. Let $F = Q^2$ and $Q' = y_2^3$ and $Q'' = y_0 y_2^2$. Then $Q', Q'' \in F^\perp$ and Q', Q'' are the only cubic forms in the ideal.*

It is easy to check that $Q', Q'' \in F^\perp$. For the uniqueness, let

$$\hat{Q} = b_0 y_0^3 + b_1 y_1^3 + b_2 y_2^3 + b_3 y_0^2 y_1 + b_4 y_0 y_1^2 + b_5 y_0^2 y_2 + b_6 y_0 y_2^2 + b_7 y_1^2 y_2 + b_8 y_1 y_2^2 + b_9 y_0 y_1 y_2$$

be a general cubic. Then, by the program in Listing 10.23, we have

$$\begin{aligned}\hat{Q}(Q^2) = (120b_0 - 4b_7)x_0^3 \\ -(12b_0 - 24b_7)x_1^2 x_2 \\ -12b_9 x_0^2 x_1 - 12b_5 x_0 x_1^2 \\ +8b_8 x_1^3 - 12b_4 x_0^2 x_2 \\ -24b_3 x_0 x_1 x_2 + 24b_1 x_1 x_2^2\end{aligned}$$

We have that $Q' \in F^\perp$ if every coefficient of each monomial is zero. We have immediately that $b_1 = b_3 = b_4 = b_5 = b_8 = b_9 = 0$. Further, we get the equations

$$120b_0 - 4b_7 = 0, \tag{8.1}$$
$$12b_0 - 24b_7 = 0. \tag{8.2}$$

This system of equations is true only if $b_0 = b_7 = 0$. To summarize, we have that b_2 and b_6 are free variables, so $Q' = y_2^3$ and $Q'' = y_0 y_2^2$ are the only cubics in $F^\perp$.

Proposition 8.0.7. *Let $Q = x_0^3 + x_1^2 x_2 - x_0^2 x_2$. Let $F = Q^2$ and $Q' = y_2^3$. Then $Q' \in F^\perp$ and Q' are the only cubic forms in the ideal.*

Proof. It is easy to check that $Q' \in F^\perp$. For the uniqueness, let

$$\hat{Q} = b_0 y_0^3 + b_1 y_1^3 + b_2 y_2^3 + b_3 y_0^2 y_1 + b_4 y_0 y_1^2 + b_5 y_0^2 y_2 + b_6 y_0 y_2^2 + b_7 y_1^2 y_2 + b_8 y_1 y_2^2 + b_9 y_0 y_1 y_2$$

be a general cubic. Then, by the program in Listing 10.24

$$\begin{aligned}\hat{Q}(Q^2) = (120b_0 - 40b_5 + 8b_6 + 4b_7)x_0^3 \\ -(8b_8 - 12b_9)x_0^2 x_1 \\ +(12b_5 - 8b_6)x_0 x_1^2 \\ +8b_8 x_1^3 \\ -(120b_0 - 12b_4 - 24b_5 + 8b_7)x_0^2 x_2 \\ +(24b_3 - 16b_9)x_0 x_1 x_2 \\ +(12b_0 - 8b_5 + 24b_7)x_1^2 x_2 \\ +(24b_0 - 8b_4)x_0 x_2^2 \\ (24b_1 - 8b_3)x_1 x_2^2\end{aligned}$$

We have that $Q' \in F^\perp$ if every coefficient of each monomial is zero. We first see that $b_8 = 0$, which implies that $b_9 = b_3 = b_1 = 0$. The remaining equations are:

$$\begin{aligned}120b_0 - 40b_5 + 8b_6 + 4b_7 = 0 \\ 120b_0 - 12b_4 - 24b_5 + 8b_7 = 0 \\ 12b_0 - 8b_5 + 24b_7 = 0 \\ 24b_0 - 8b_4 = 0\end{aligned}$$

$$12b_5 - 8b_6 = 0$$

The corresponding matrix equation is

$$\begin{pmatrix} 120 & 0 & -40 & 8 & 4 \\ 120 & -12 & -24 & 0 & 8 \\ 12 & 0 & -8 & 0 & 24 \\ 24 & -8 & 0 & 0 & 0 \\ 0 & 0 & 12 & -8 & 0 \end{pmatrix} \begin{pmatrix} b_0 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{pmatrix}$$

By row reduction, we get that the only solution is $b_0 = b_4 = b_5 = b_6 = b_7 = 0$. To summarize, $b_0 = b_1 = b_3 = b_4 = b_5 = b_6 = b_7 = b_8 = b_9 = 0$, and b_2 is a free variable. This implies $\hat{Q} = b_2 y_2^3$, which shows that the only cubic in the apolar ideal of F is y_2^3. ■

Theorem 8.0.8. *Let Q be a irreducible ternary cubic form and let $F = Q^2$. Then $F^\perp$ contains at least one cubic form.*

Proof. By Lemma 8.0.1 and Lemma 8.0.2, the non-degenerate cubics are either in the Hesse pencil or a nodal or a cuspidal cubic. By Proposition 8.0.3, an element in the Hesse pencil has at least one cubic form in the apolar ideal, from Proposition 8.0.7 that a nodal cubic has exactly one cubic form in the apolar ideal, and from Proposition 8.0.6 that a cuspidal cubic has exactly two cubic forms in the apolar ideal. ■

Remark 8.0.9. Since the irreducible ternary cubics are open in the space of ternary cubics, the result holds for a reducible ternary cubic as well.

Now, we show that if Q is the Fermat cubic and $F = Q^2$ then $F^\perp$ contains exactly one cubic form.

Proposition 8.0.10. *Let $Q = x_0^3 + x_1^3 + x_2^3$. Let $F = Q^2$ and $Q' = y_0y_1y_2$. Then $Q \in F^\perp$ and Q' is the only cubic in the ideal.*

Proof. We obviously have $Q'(Q^2) = 0$. For the proof of uniqueness, let

$$Q'' = b_0y_0^3 + b_1y_1^3 + b_2y_2^3 + b_3y_0^2y_1 + b_4y_0y_1^2 + b_5y_0^2y_2 + b_6y_0y_2^2 + b_7y_1^2y_2 + b_8y_1y_2^2 + b_9y_0y_1y_2$$

be a general cubic. Then, by the program in Listing 10.22, we have

$$\begin{aligned} Q''(Q^2) = &(120b_0 + 12b_1 + 12b_2)x_0^3 \\ &+(12b_0 + 12b_1 + 120b_2)x_2^3 \\ &+(12b_0 + 120b_1 + 12b_2)x_1^3 \\ &+36b_4x_0^2x_1 + 36b_3x_0x_1^2 \\ &+36b_6x_0^2x_2 + 36b_8x_1^2x_2 \\ &+36b_5x_0x_2^2 + 36b_7x_1x_2^2 \end{aligned}$$

We have that $Q' \in F^\perp$ if every coefficient of each monomial is zero. We have immediately that $b_3 = b_4 = b_5 = b_6 = b_7 = b_8 = 0$. Further, we have the following equations:

$$120b_0 + 12b_1 + 12b_2 = 0 \tag{8.3}$$
$$12b_0 + 12b_1 + 120b_2 = 0 \tag{8.4}$$
$$12b_0 + 120b_1 + 12b_2 = 0 \tag{8.5}$$

To solve 8.3 - 8.5, we consider the coefficient matrix

$$\begin{pmatrix} 120 & 12 & 12 \\ 12 & 12 & 120 \\ 12 & 120 & 12 \end{pmatrix}.$$

Row reducing gives the identity matrix, which implies that the only solution is $b_0 = b_1 = b_2 = 0$. To summarize, $b_0 = b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = b_7 = b_8 = 0$, and b_9 is a free variable. This implies $Q'' = b_9 y_0 y_1 y_2$, which shows that the only cubic in the apolar ideal of F is $y_0 y_1 y_2$. ■

Corollary 8.0.11. *Let Q be a general ternary cubic and let $F = Q^2$. Then $F^\perp$ contains exactly one cubic form.*

Proof. By Theorem 8.0.8, $F^\perp$ contains at least one cubic form for a general Q. To contained more than one cubic form in the ideal is a closed condition, so if we can find an element that contain only one cubic, we are done. In Proposition 8.0.10 we showed that if Q is the Fermat cubic has only one cubic, then $F^\perp$ contain only one cubic form. ■

Now, we investigate the case when the cubic form in the apolar ideal of a double cubic is a triple line.

Proposition 8.0.12. *Assume $F = Q^2$ for a cubic Q such that $y_2^3 \in F^\perp$. Then*

$$Q = a_0x_0^3 + a_1x_1^3 + a_3x_0^2x_1 + a_4x_0x_1^2 + a_5x_0^2x_2 + a_7x_1^2x_2 + a_9x_0x_1x_2.$$

In particular, Q is singular.

Proof. Let $Q = a_0x_0^3+a_1x_1^3+a_2x_2^3+a_3x_0^2x_1+a_4x_0x_1^2+a_5x_0^2x_2+a_6x_0x_2^2+a_7x_1^2x_2+a_8x_1x_2^2+a_9x_0x_1x_2$ be a general cubic. Then

$$\begin{aligned} y_2^3(Q^2) = (12a_0a_2 + 12a_5a_6)x_0^3 \\ +(12a_2a_3 + 12a_5a_8 + 12a_6a_9)x_0^2x_1 \\ +(12a_2a_4 + 12a_6a_7 + 12a_8a_9)x_0x_1^2 \\ +(12a_1a_2 + 12a_7a_8)x_1^3 \\ +(48a_2a_5 + 24a_6^2)x_0^2x_2 \\ +(48a_6a_8 + 48a_2a_9)x_0x_1x_2 \\ +(48a_2a_7 + 24a_8^2)x_1^2x_2 \\ +120a_2a_6x_0x_2^2 \\ +120a_2a_8x_1x_2^2 \\ +120a_2^2x_2^3 \end{aligned}$$

We have that y_2^3 is apolar to F if $y_2^3(Q)$. We get that $a_2 = 0$, which implies that $a_6 = a_8 = 0$. Since every term of $y_2^3(Q^2)$ contains either a_2, a_6 or a_8, we are done.

To show that Q is singular, let $x_2 = 1$. Then $Q = a_0x_0^3 + a_1x_1^3 + a_3x_0^2x_1 + a_4x_0x_1^2 + a_5x_0^2 + a_7x_1^2 + a_9x_0x_1$. The cubic Q is singular if there is a point on the curve where both partial derivatives vanish.

$$\begin{aligned} F_{x_0} &= 3a_0x_0^2 + +2a_3x_0x_1 + a_4x_1^2 + 2a_5x_0 + a_9x_1, \\ F_{x_1} &= 3a_1x_1^2 + a_3x_0^2 + 2a_4x_0x_1 + 2a_7x_1 + a_9x_0. \end{aligned}$$

When $x_0 = x_1 = 0$ we get $F_{x_0} = F_{x_1} = 0$. The corresponding projective coordinate $(0 : 0 : 1)$ is a point on the Q and is therefore a singular point. ■

9 | Conclusion

9.1 Comparison of the Secant and the Catelecticant Varieties

In this section we compare the secant varieties and the catelecticant varieties to the Veronese embedding. We have that $\mathrm{Sec}_r(v_6(\mathbb{P}(S_1))) \subset \mathrm{Cat}_r(v_6(\mathbb{P}(S_1)))$. The equations defining the point in $\mathrm{Cat}_r(v_6(\mathbb{P}(S_1)))$ are defined by the catalecticant matrix. The equations defining $\mathrm{Sec}_r(v_6(\mathbb{P}(S_1)))$ are, however, not known in general. Therefore, we compare the catalecticant and secant varieties by using our stratification. In the cases where they coincide, we have the equations for the secant variety. We have that $\mathrm{rank}(\mathrm{Cat}(F)) = 10 - \dim(F^{\perp})_3$.

Theorem 9.1.1. *We have the following*

1. $\mathrm{Sec}_r(v_6(\mathbb{P}(S_1))) = \mathrm{Cat}_r(v_6(\mathbb{P}(S_1)))$, *for* $r \leq 6$,
2. $\mathrm{Sec}_7(v_6(\mathbb{P}(S_1))) = \mathrm{Cat}_7(v_6(\mathbb{P}(S_1))) - \mathcal{F}_{[030]}$
3. $\mathrm{Sec}_8(v_6(\mathbb{P}(S_1))) = \mathrm{Cat}_8(v_6(\mathbb{P}(S_1))) - (\mathcal{F}_{[023a]} \cup \mathcal{F}_{[030]})$
4. $\mathrm{Sec}_9(v_6(\mathbb{P}(S_1))) = \mathrm{Cat}_9(v_6(\mathbb{P}(S_1)))$
5. $\mathrm{Sec}_{10}(v_6(\mathbb{P}(S_1))) = \mathrm{Cat}_{10}(v_6(\mathbb{P}(S_1)))$

Proof. We have that $\mathrm{rank}(\mathrm{Cat}(F)) = 10 - \dim(F^{\perp})_3$. Since $\mathrm{Sec}_r(v_6(\mathbb{P}(S_1)))$ is irreducible, [Har92], we only need to compare the rank of a general element $F \in \mathcal{F}_B$ with the $\mathrm{rank}(v_6(\mathbb{P}(S_1)))$. We get that for r different from 8 and 9, then F is the limit of an F_t, where $\mathrm{r}(F_t) = \mathrm{rank}(\mathrm{Cat}(F))$. We now consider the cases [030] and [023a].

For a general element $F \in \mathcal{F}_{[030]}$, $\dim(F^{\perp})_3 = 3$, thus $F \in \mathrm{Cat}_7$. The rank of a general element in $\mathcal{F}_{[030]}$ is 9, so $\mathcal{F}_{[030]} \subset \mathrm{Sec}_9(v_6(\mathbb{P}(S_1))) - \mathrm{Sec}_8(v_6(\mathbb{P}(S_1)))$. For a general element $F \in F_{[023a]}$, $\dim(\mathcal{F}^{\perp})_3 = 2$, thus $F \in \mathrm{Cat}_8(v_6(\mathbb{P}(S_1)))$. The rank of a general element in $F_{[023a]}$ is 9, so $\mathcal{F}_{[023a]} \subset \mathrm{Sec}_9(v_6(\mathbb{P}(S_1))) - \mathrm{Sec}_8(v_6(\mathbb{P}(S_1)))$. ■

9.2 Further Questions

In this section we raise some open questions.

The first question is related to the results we found in Chapter 8. Since there are only two ways to find a basis for a general F of cactus rank 9, there might be difficult to find smooth schemes that are apolar to F. For the double cubics, we found explicit equations for the cubic forms in the apolar ideal. Therefore, it should be possible investigate if there are some smooth schemes at all. We raise the following question

Question 9.2.1. Is there a smooth scheme of length nine apolar to a double cubic?

The second question we raise is if our methods work for forms of higher degrees.

Question 9.2.2. Is it possible to use the same methods that we have used for ternary forms of degree larger than six?

10 | Appendix

10.1 Computation for Chapter 4

Listing 10.1: Chern classes for $G(4,9)$

```
    R=QQ[c1,c2,c3,c4]
g6=3*c1^2*c4-2*c2*c4+6*c1*c2*c3-4*c1^3*c3-c3^2-6*c1^2*c2^2+c2^3+5*c1^4*c2-c1^6
g7=4*c1*c2*c4-c1^3*c4-2*c3*c4+2*c1*c3^2-6*c1^2*c2*c3+3*c2^2*c3
+c1^4*c3-3*c1*c2^3+4*c1^3*c2^2-c1^5*c2
g8=4*c1*c3*c4-c4^2-3*c1^2*c2*c4+c2^2*c4+c1^4*c4-
3*c1^2*c3^2+2*c2*c3^2-3*c1*c2^2*c3+4*c1^3*c2*c3-c1^5*c3
g9=2*c1*c4^2-3*c1^2*c3*c4+2*c2*c3*c4-
3*c1*c2^2*c4+4*c1^3*c2*c4-c1^5*c4

e1=g6*c1^14, e2=g6*c1^12*c2,e3=g6*c1^11*c3,e4=g6*c1^10*c2^2,
e5=g6*c1^10*c4,e6=g6*c1^9*c2*c3, e7=g6*c1^8*c2^3, e8=g6*c1^8*c3^2
e9=g6*c1^8*c2*c4,e10=g6*c1^7*c3*c4,
e11=g6*c1^7*c2^2*c3, e12=g6*c1^6*c2^4, e13=g6*c1^6*c2*c3^2, e14=g6*c1^6*c2^2*c4,
e15=g6*c1^6*c4^2
e16=g6*c1^5*c3^3, e17=g6*c1^5*c2^3*c3, e18=g6*c1^5*c2*c3*c4, e19=g6*c1^4*c2^5,
e20=g6*c1^4*c2^3*c4, e21=g6*c1^4*c2^2*c3^2
e22=g6*c1^4*c2*c4^2, e23=g6*c1^4*c3^2*c4,
e24=g6*c1^3*c2^4*c3, e25=g6*c1^3*c2^2*c3*c4,
e26=g6*c1^3*c2*c3^3, e27=g6*c1^3*c3*c4^2
e28=g6*c1^2*c2^6,e29=g6*c1^2*c2^4*c4,
e30=g6*c1^2*c2^3*c3^2, e31=g6*c1^2*c2^2*c4^2,
e32=g6*c1^2*c2*c3^2*c4, e33=g6*c1^2*c3^4
e34=g6*c1^2*c4^3, e35=g6*c1*c2^5*c3,
e36=g6*c1*c2^2*c3*c4, e37=g6*c1*c2^2*c3^3,
e38=g6*c1*c2*c3*c4^2, e39=g6*c1*c3^3*c4,
e40=g6*c2^7
e41=g6*c2^5*c4, e42=g6*c2^4*c3^2,
e43=g6*c2^3*c4^2, e44=g6*c2^2*c3^2*c4,
e45=g6*c2*c3^4, e46=g6*c2*c4^3, e47=g6*c3^2*c4

b1=g7*c1^13, b2=g7*c1^11*c2, b3=g7*c1^10*c3,
b4=g7*c1^9*c2^2, b5=g7*c1^9*c4, b6=g7*c1^8*c2*c3,
b7=g7*c1^7*c2^3, b8=g7*c1^7*c3^2
b9=g7*c1^7*c2*c4, b10=g7*c1^6*c3*c4, b11=g7*c1^6*c2^2*c3,
b12=g7*c1^5*c2^4, b13=g7*c1^5*c2*c3^2, b14=g7*c1^5*c2^2*c4
b15=g7*c1^5*c4^2, b16=g7*c1^4*c3^3, b17=g7*c1^4*c2^3*c3,
b18=g7*c1^4*c2*c3*c4, b18=g7*c1^3*c2^5, b19=g7*c1^3*c2^3*c4
b20=g7*c1^3*c2^2*c3^2, b21=g7*c1^3*c2*c4^2,
b22=g7*c1^3*c3^2*c4, b23=g7*c1^2*c2^4*c3,
b24=g7*c1^2*c2^2*c3*c4, b25=g7*c1^2*c2*c3^3
b26=g7*c1^2*c3*c4^2, b27=g7*c1*c2^6, b28=g7*c1*c2^4*c4,
b29=g7*c1*c2^3*c3^2, b30=g7*c1*c2*c4^2,
b31=g7*c1*c2*c3^2*c4,
b32=g7*c1*c3^4, b33=g7*c1*c4^3, b34=g7*c2^5*c3,
b35=g7*c2^3*c3*c4, b36=g7*c2^2*c3^3, b37=g7*c2*c3*c4^2,
b38=g7*c3^3*c4
```

```
d1=g8*c1^12, d2=g8*c1^10*c2, d3=g8*c1^9*c3,
d4=g8*c1^8*c2^2, d5=g8*c1^8*c4, d6=g8*c1^7*c2*c3,
d7=g8*c1^6*c2^3, d8=g8*c1^6*c3^2
d9=g8*c1^6*c2*c4, d10=g8*c1^5*c3*c4, d11=g8*c1^5*c2^2*c3,
d12=g8*c1^4*c2^4, d13=g8*c1^4*c2*c3^2, d14=g8*c1^4*c2^2*c4
d15=g8*c1^4*c4^2, d16=g8*c1^3*c3^3, d17=g8*c1^3*c2^3*c3,
d18=g8*c1^3*c2*c3*c4, d19=g8*c1^2*c2^5, d20=g8*c1^2*c2^3*c4
d21=g8*c1^2*c2^2*c3^2, d22=g8*c1^2*c2*c4^2,
d23=g8*c1^2*c3^2*c4, d24=g8*c1*c2^4*c3,
d25=g8*c1*c2^2*c3*c4, d26=g8*c1*c2*c3^3
d27=g8*c1*c3*c4^2, d28=g8*c2^6, d29=g8*c2^4*c4,
d30=g8*c2^3*c3^2, d31=g8*c2^2*c4^2, d32=g8*c2*c3^2*c4,
d33=g8*c3^4, d34=g8*c4^3

f1=g9*c1^11, f2=g9*c1^9*c2, f3=g9*c1^8*c3,
f4=g9*c1^7*c2^2, f5=g9*c1^7*c4, f6=g9*c1^6*c2*c3,
f7=g9*c1^5*c2^3, f8=g9*c1^5*c3^2
f9=g9*c1^5*c2*c4, f10=g9*c1^4*c3*c4, f11=g9*c1^4*c2^2*c3,
f12=g9*c1^3*c2^4, f13=g9*c1^3*c2*c3^2, f14=g9*c1^3*c2^2*c4
f15=g9*c1^3*c4^2, f16=g9*c1^2*c3^3, f17=g9*c1^2*c2^3*c3,
f18=g9*c1^2*c2*c3*c4, f19=g9*c1*c2^5, f20=g9*c1*c2^3*c4
f21=g9*c1*c2^2*c3^2, f22=g9*c1*c2*c4^2 ,f23=g9*c1*c3^2*c4,
f24=g9*c2^4*c3, f25=g9*c2^2*c3*c4, f26=g9*c2*c3^3,
f27=g9*c3*c4^2

P=matrix{{e1,e2,e3,e4,e5,e6,e7,e8,e9,e10,e11,e12,e13,e14,e1
5,e16,e17,e18,e19,e20,e21,e22,e23,e24,e25,
e26,e27,e28,e29,e30,e31,e32,e33,e34,e35,e36,e37,e38,e39,e40
,e41,e42,e43,e44,e45,e46,e47,b1,b2,b3,b4,
b5,b6,b7,b8,b9,b10,b11,b12,b13,b14,b15,b16,b17,b18,b19,b20,
b21,b22,b23,b24,b25,b26,b27,b28,b29,b30,
b31,b32,b33,b34,b35,b36,b37,b38,d1,d2,d3,d4,d5,d6,d7,d8,d9,
d10,d11,d12,d13,d14,d15,d16,d17,d18,d19,
d20,d21,d22,d23,d24,d25,d26,d27,d28,d29,d30,d31,d32,d33,d34
,f1,f2,f3,f4,f5,f6,f7,f8,f9,f10,f11,f12,
f13,f14,f15,f16,f17,f18,f19,f20,f21,f22,f23,f24,f25,f26,f27}}
(M,C)=coefficients P
F = transpose(C)

R=QQ
G=lift(F,R)
M
T=reducedRowEchelonForm G
```

10.2 Computation for Chapter 5

Listing 10.2: Three points

```
    i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{random(1,R),random(1,R),random(1,R)},
     {random(1,R),random(1,R),random(1,R)}}

o2 = | 6/7x+6/5y+7/3z  2/3x+2y+7/4z 8x+5/2y+4/3z  |
     | 10/9x+4/5y+9/5z 3x+1/2y+1/2z 4x+10/7y+2/5z |

               2       3
o2 : Matrix R  <--- R

i3 : I = minors(2,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3

i6 : degree variety I

o6 = 3
```

Listing 10.3: Four points

```
  i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{random(2,R),random(2,R)}}

o2 = | 1/4x2+7/9xy+3/2y2+4/7xz+3/8yz+1/2z2 8/3x2+1/4xy+7/9y2+8xz+yz+1/3z2 |

               1       2
o2 : Matrix R  <--- R

i3 : I = minors(1,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3

i6 : degree variety I
```

```
o6 = 4
```

Listing 10.4: Four points, three on a line

```
i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{(x+y)*y,(x+z)*(y+z),x},{z,y,0}}

o2 = | xy+y2 xy+xz+yz+z2 x |
     | z     y           0 |

               2        3
o2 : Matrix R  <--- R

i3 : I = minors(2,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3
```

Listing 10.5: Five points, four on a line

```
i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{(x+y)*y^2,(x+z)*(y+z)*z,x},{y,z,0}}

o2 = | xy2+y3 xyz+xz2+yz2+z3 x |
     | y      z              0 |

               2        3
o2 : Matrix R  <--- R

i3 : I = minors(2,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3
```

Listing 10.6: Five points

```
i1 : R=QQ[x,y,z]
```

```
o1 = R

o1 : PolynomialRing

i2 : M = matrix{{x,x-y,x^2+y^2},{y,z,x^2+z^2}}

o2 = | x x-y x2+y2 |
     | y z   x2+z2 |

              2        3
o2 : Matrix R  <--- R

i3 : I = minors(2,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3
```

Listing 10.7: Six points

```
i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : I = ideal(x^2+y^2+z^2,x*z^2+x^2*y)

             2    2    2   2       2
o2 = ideal (x  + y  + z , x y + x*z )

o2 : Ideal of R

i3 : J=minors(2,jacobian I);

o3 : Ideal of R

i4 : codim variety ideal(I,J)

o4 = 3
```

Listing 10.8: Six points, four on a line

```
i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{(y)*(y-x)*(z-x),(x+z)*(y+z),x},{(y-z)*(x+z),z+y,0}}

o2 = | x2y-xy2-xyz+y2z xy+xz+yz+z2 x |
     | xy-xz+yz-z2     y+z         0 |

              2        3
o2 : Matrix R  <--- R
```

```
i3 : I = minors(2,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3
```

Listing 10.9: Seven points on a conic

```
i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{(y)*(y-x)*(z-x),(x+z),x},{(y-z)*(x+z)*(x+y),z+y,x-y}}

o2 = | x2y-xy2-xyz+y2z         x+z  x   |
     | x2y+xy2-x2z+y2z-xz2-yz2 y+z  x-y |

              2        3
o2 : Matrix R  <--- R

i3 : I = minors(2,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3
```

Listing 10.10: Six points

```
i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{random(1,R),random(1,R),random(1,R),random(1,R)},
     {random(1,R),random(1,R),random(1,R),random(1,R)},
     {random(1,R),random(1,R),random(1,R),random(1,R)}}

o2 = | 7/9x+3/2y+5z    x+3/10y+z      7/4x+1/3y+9/5z  10x+4/3y+5/2z  |
     | 3/10x+3/7y+3/5z 1/4x+1/2y+3z   3/8x+y+7/9z     1/3x+3/7y+5/2z |
     | 1/2x+7/9y+1/5z  7x+5/7y+10/7z  2/3x+1/10y+9/5z 7/9x+3/2y+5/7z |

              3        4
o2 : Matrix R  <--- R

i3 : I = minors(3,M);

o3 : Ideal of R
```

```
i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3

i6 : degree variety I

o6 = 6
```

Listing 10.11: Nine points in a CI

```
i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{x^3+y^3,z^3+x^2*y}}

o2 = | x3+y3 x2y+z3 |

             1       2
o2 : Matrix R  <--- R

i3 : I = minors(1,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3

i6 : degree variety I

o6 = 9
```

Listing 10.12: Seven points

```
i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{(y+x+z)*x,(x+z)*(y-x),(x+z)*(x+y)},{(y-z),(x+z),z+y}}

o2 = | x2+xy+xz -x2+xy-xz+yz x2+xy+xz+yz |
     | y-z      x+z          y+z         |

              2         3
o2 : Matrix R  <--- R

i3 : I = minors(2,M);

o3 : Ideal of R
```

```
i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3
```

Listing 10.13: Seven points, four on a line

```
i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{random(2,R),random(2,R),random(2,R),random(1,R)},
     {random(1,R),random(1,R),random(1,R),0},
     {random(1,R),random(1,R),random(1,R),0}}

o2 = | 5/8x2+10/9xy+4y2+5/2xz+5/2yz+7z2
7/9x2+10xy+6/7y2+9/8xz+8/3yz+3/2z2
x2+9/8xy+7/8y2+1/4xz+2yz+2z2 3/2x+5/8y+3z |

     | 5/9x+8y+7/2z      1/2x+7/6y+1/2z      2/5x+1/2y+3/5z           0
|
     | 7/6x+5/4y+7/5z     3/7x+5/8y+7/3z          2x+y+5/8z           0
|

              3        4
o2 : Matrix R  <--- R

i3 : I = minors(3,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3

i6 : degree variety I

o6 = 7
```

Listing 10.14: Eight points

```
i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{(y+x+z)*(x+y),(x+z)*(y-x)+x^2,(x+y-z)},
{(x+z)*(x+y)+z^2,(x+z+y)*(x-y)-y^2,z+y+x}}

o2 = | x2+2xy+y2+xz+yz xy-xz+yz     x+y-z |
     | x2+xy+xz+yz+z2  x2-2y2+xz-yz x+y+z |
```

```
              2        3
o2 : Matrix R  <--- R

i3 : I = minors(2,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3
```

Listing 10.15: Eight points, four on a line

```
i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{(y+x+z)*(x+y),(x+z)*(y-x)+x^2,(x+y-z)},
      {(x+z)*(x+y)+z^2,(x+z+y)*(x-y)-y^2,0}}

o2 = | x2+2xy+y2+xz+yz xy-xz+yz      x+y-z |
     | x2+xy+xz+yz+z2  x2-2y2+xz-yz  0     |

              2        3
o2 : Matrix R  <--- R

i3 : I = minors(2,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3
```

Listing 10.16: Eight points, seven on a conic

```
i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{random(2,R),random(2,R),random(1,R),random(1,R)},
     {random(2,R),random(2,R),random(1,R),random(1,R)},
     {random(1,R),random(1,R),0,0}}

o2 = | 1/4x2+4xy+3y2+1/3xz+7/4yz+1/5z2
3x2+4/7xy+y2+1/3xz+3/2yz+4/5z2
8x+7/9y+1/10z 1/2x+2y+3z  |

     | x2+1/5xy+2y2+4/3xz+4/5yz+2/5z2
     x2+8/9xy+1/3y2+2/3xz+5yz+z2
```

```
     1/3x+1/2y+2z  3/2x+4/9y+z |
     | 2x+5/3y+1/7z              2x+5/8y+2z                        0              0
|

              3        4
o2 : Matrix R  <--- R

i3 : I = minors(3,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3

i6 : degree variety I

o6 = 8
```

Listing 10.17: Nine points

```
i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{random(2,R),random(1,R),random(1,R),random(1,R)},
     {random(2,R),random(1,R),random(1,R),random(1,R)},
     {random(2,R),random(1,R),random(1,R),random(1,R)}}

o2 = | 3/8x2+5/2xy+1/2y2
+5/2xz+1/6yz+3/2z2 7/8x+3/5y+1/4z 5/4x+9/10y+2z 3/10x+1/10y+7/9z |
     | 10/7x2+3/10xy
     +2/3y2+7/8xz+yz+5/4z2  x+6/5y+5/4z    10x+10y+1/2z  10x+5/2y+9/2z    |
     | 3/4x2+2/3xy
     +2y2+3/2xz+yz+z2         x+9/10y+9/10z  7/5x+3y+1/9z  5x+9/4y+4/5z     |

              3        4
o2 : Matrix R  <--- R

i3 : I = minors(3,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3

i6 : degree variety I

o6 = 9
```

Listing 10.18: Nine points, four on a line

```
    i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{random(2,R),random(1,R),random(1,R),random(1,R)},
     {random(2,R),random(1,R),random(1,R),0},
     {random(2,R),random(1,R),random(1,R),0}}

o2 = | 1/8x2+3/4xy+8/9y2
+8/3xz+yz+1/10z2    3/10x+5/7y+3/7z 3/10x+3y+1/2z  3/4x+5y+z |
     | 1/7x2+xy+8/5y2
     +xz+1/3yz+3/4z2        1/5x+7/2y+1/3z  6/5x+5/7y+9/7z 0          |
     | 8/5x2+8/9xy+10/3y2+5/8xz
     +5/4yz+5/8z2 5/7x+9/7y+1/6z  7/6x+5/3y+2/7z 0          |

              3        4
o2 : Matrix R  <--- R

i3 : I = minors(3,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3

i6 : degree variety I

o6 = 9
```

Listing 10.19: Nine points, seven on a conic

```
    i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{random(2,R),random(1,R),random(1,R),random(1,R)},
     {random(2,R),random(1,R),random(1,R),random(1,R)},
     {random(2,R),random(1,R),0,0}}

o2 = | 5/2x2+9/4xy
+y2+4xz+7/3yz+4/5z2      3x+8/9y+2/5z 9/5x+y+4/5z    3x+3/4y+5z   |
     | 1/2x2+2xy+9/2y2
     +2/3xz+10/7yz+z2      x+1/2y+3/2z  5/3x+3/4y+1/2z 2/3x+1/3y+5z |
     | 9/7x2+8/3xy+9/5y2+3/2xz+1/8yz+2/9z2 x+6/7y+2z    0              0
|

              3        4
o2 : Matrix R  <--- R

i3 : I = minors(3,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);
```

```
o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3

i6 : degree variety I

o6 = 9
```

Listing 10.20: Ten points

```
    i1 : R=QQ[x,y,z]

o1 = R

o1 : PolynomialRing

i2 : M = matrix{{random(1,R),random(1,R),random(1,R),random(1,R),random(1,R)},
     {random(1,R),random(1,R),random(1,R),random(1,R),random(1,R)},
     {random(1,R),random(1,R),random(1,R),random(1,R),random(1,R)},
     {random(1,R),random(1,R),random(1,R),random(1,R),random(1,R)}}

o2 = | 5/3x+2/5y+z  2/5x+4/3y+4/3z 3/2x+4/3y+5/8z 7/2x+9y+1/6z   7x+1/3y+4/5z   |
     | x+9/7y+3/2z  1/5x+5y+1/8z   4/7x+7/6y+4/7z x+2/3y+4/7z    2/7x+8y+8z     |
     | x+9/10y+1/5z 1/4x+1/5y+z    3/5x+2/9y+2z   5/4x+8/5y+4/7z 9/2x+9/5y+2/5z |
     | 10x+y+3/10z  2/3x+4/3y+2/5z 1/4x+9/2y+7/5z 1/4x+5/2y+2/3z 3x+1/6y+5/7z   |

              4        5
o2 : Matrix R  <--- R

i3 : I = minors(4,M);

o3 : Ideal of R

i4 : J=minors(2,jacobian I);

o4 : Ideal of R

i5 : codim variety ideal(I,J)

o5 = 3

i6 : degree variety I

o6 = 10
```

10.3 Computation for Chapter 8

Listing 10.21: Hesse pencil

```
i1 : A=QQ[l,l']

o1 = A

o1 : PolynomialRing

i2 : R=A[x0,x1,x2]

o2 = R

o2 : PolynomialRing

i3 : Q = (x0^3+x1^3+x2^3+l*x0*x1*x2)^2

       6      3  3     6        4                 4       2  2  2  2      3  3
3  3              4     6
o3 = x0  + 2x0 x1  + x1  + 2l*x0 x1*x2 + 2l*x0*x1 x2 + l x0 x1 x2  + 2x0 x2
+ 2x1 x2  + 2l*x0*x1*x2  + x2

o3 : R

i4 : diff(x0^3+x1^3+x2^3,Q)+l'*diff(x0*x1*x2,Q)

                    3                  3      2
3
o4 = (81*l' + 144)x0  + (81*l' + 144)x1  + (81 l' + 144l)x0*x1*x2
+ (81*l' + 144)x2

o4 : R
```

Listing 10.22: Fermat cubic

```
    i1 : A=QQ[b0,b1,b2,b3,b4,b5,b6,b7,b8,b9]

o1 = A

o1 : PolynomialRing

i2 : R=A[x0,x1,x2]

o2 = R

o2 : PolynomialRing

i3 : Q = (x0^3+x1^3+x2^3)^2

       6      3  3     6      3  3      3  3     6
o3 = x0  + 2x0 x1  + x1  + 2x0 x2  + 2x1 x2  + x2

o3 : R

i4 : b0*diff(x0^3,Q)+b1*diff(x1^3,Q)+b2*diff(x2^3,Q)+b3*diff(x0^2*x1,Q)
+b4*diff(x0*x1^2,Q)+b5*diff(x0^2*x2,Q)+b6*diff(x0*x2^2,Q)+b7*diff(x1^2*x2,Q)
+b8*diff(x1*x2^2,Q)+b9*diff(x0*x1*x2,Q)

                           3          2                2
3          2            2                2              2
3
```

```
o4 = (120b0 + 12b1 + 12b2)x0  + 36b4*x0 x1 + 36b3*x0*x1  + (12b0 + 120b1 + 12b2)x1
+ 36b6*x0 x2 + 36b8*x1 x2 + 36b5*x0*x2  + 36b7*x1*x2  + (12b0 + 12b1 + 120b2)x2

o4 : R
```

Listing 10.23: Cuspidal cubic

```
i1 : A=QQ[b0,b1,b2,b3,b4,b5,b6,b7,b8,b9]

o1 = A

o1 : PolynomialRing

i2 : R=A[x0,x1,x2]

o2 = R

o2 : PolynomialRing

i3 : Q = (x0^3-x1^2*x2)^2

       6      3  2       4  2
o3 = x0  - 2x0 x1 x2 + x1 x2

o3 : R

i4 : b0*diff(x0^3,Q)+b1*diff(x1^3,Q)+b2*diff(x2^3,Q)+b3*diff(x0^2*x1,Q)
+b4*diff(x0*x1^2,Q)+b5*diff(x0^2*x2,Q)+b6*diff(x0*x2^2,Q)+b7*diff(x1^2*x2,Q)
+b8*diff(x1*x2^2,Q)+b9*diff(x0*x1*x2,Q)

                    3          2                2         3          2
2               2
o4 = (120b0 - 4b7)x0  - 12b9*x0 x1 - 12b5*x0*x1  + 8b8*x1  - 12b4*x0 x2
- 24b3*x0*x1*x2 + (- 12b0 + 24b7)x1 x2 + 24b1*x1*x2

o4 : R
```

Listing 10.24: Nodal cubic

```
    i1 : A=QQ[b0,b1,b2,b3,b4,b5,b6,b7,b8,b9]

o1 = A

o1 : PolynomialRing

i2 : R=A[x0,x1,x2]

o2 = R

o2 : PolynomialRing

i3 : Q = (x0^3+x1^2*x2-x0^2*x2)^2

       6      5        3  2       4  2       2  2  2      4  2
o3 = x0  - 2x0 x2 + 2x0 x1 x2 + x0 x2  - 2x0 x1 x2  + x1 x2

o3 : R

i4 : b0*diff(x0^3,Q)+b1*diff(x1^3,Q)+b2*diff(x2^3,Q)+b3*diff(x0^2*x1,Q)
+b4*diff(x0*x1^2,Q)+b5*diff(x0^2*x2,Q)+b6*diff(x0*x2^2,Q)+b7*diff(x1^2*x2,Q)
+b8*diff(x1*x2^2,Q)+b9*diff(x0*x1*x2,Q)
```

```
                                         3                       2                           2
3                                          2
o4 = (120b0 - 40b5 + 8b6 + 4b7)x0  +
(- 8b8 + 12b9)x0 x1 + (12b5 - 8b6)x0*x1  + 8b8*x1  +
(- 120b0 + 12b4 + 24b5 - 8b7)x0 x2
+ (24b3 - 16b9)x0*x1*x2
+ (12b0 - 8b5 +
             2                         2                         2
     24b7)x1 x2 + (24b0 - 8b4)x0*x2  + (24b1 - 8b3)x1*x2

o4 : R
```

Listing 10.25: j-invariant

```
i1 : A=QQ[lambda,e]/(e^2+e+1)

o1 = A

o1 : QuotientRing

i2 : R=A[x0,x1,x2]

o2 = R

o2 : PolynomialRing

i3 : jE_lambda = (1^3*(1^3-216)^3)/((1+3)^3*(1+3*e)^3*(1+3*e^2)^3)

     -9938375
o3 = --------
       21952

o3 : frac A

i4 : jE_lambda' = ((-18/1)^3*((-18/1)^3-216)^3)/(((-18/1)+3)^3
*((-18/1)+3*e)^3*((-18/1)+3*e^2)^3)

     -65548320768
o4 = ------------
        9938375

o4 : frac A
```

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