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Nonstandard Stochastic Analysis in Infinite Dimensions

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This master's thesis is submitted under the master's programme *Mathematics*, with programme option *Mathematics for applications*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 60 credits.

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

In this thesis, we define an infinite-dimensional stochastic integral $\int XdM$ where M is a hyperfinite martingale. In order to make sense of this integral we have developed some new concepts in nonstandard functional analysis. We also introduce hyperfinite cylindrical processes and show that $\int XdM$ is cylindrical.

Acknowledgements

I would like to express my gratitude to Professor Tom Lindstrøm for being an outstanding advisor. I thank him for his clever ideas shaping this thesis, for responding to emails quickly with well-written Latex documents, and for guiding me into the nonstandard world. With his intellect and wittiness, it is not a surprise that there is a fan page dedicated to him.

Contents

CHAPTER 1

Introduction

1.1 Outline of Thesis

Chapter 1 provides an introduction to nonstandard analysis.

Chapter 2 gives a short summary of cylindrical measures and cylindrical Lévy processes in Banach spaces. The purpose of introducing this standard theory is to compare it with Chapter 7 where we define hyperfinite cylindrical Lévy processes.

Chapter 3 reviews important one- or finite-dimensional concepts from nonstandard stochastic analysis. We introduce hyperfinite stochastic processes and related terminology and results in order to give meaning to the stochastic integral $\int XdY$ where X, Y are one-dimensional hyperfinite processes which were done in [Lin80]. The last part of this chapter is a quick review of finite-dimensional hyperfinite Lévy processes from [Lin04].

Chapter 4 starts with the well-known theory of internal normed linear spaces and hyperfinite-dimensional linear spaces, and linear operators on these spaces. Secondly, we define hyperfinite-dimensional dual spaces and give a Riesz representation theorem for these spaces. Using our Riesz representation theorem we establish the notion of an adjoint operator on hyperfinite-dimensional inner product spaces. Thirdly, we present nearstandard operators and *strictly nearstandard operators*. We show that the adjoint T^* of a strictly nearstandard operator T is strictly nearstandard. Lastly, we consider Hilbert-Schmidt operators on hyperfinite-dimensional inner product spaces and show that it shares many properties with standard Hilbert-Schmidt operators. In the end, we provide a new concept of operators, namely *strictly Hilbert-Schmidt operators* and show that the adjoint of such map is strictly Hilbert-Schmidt as well.

Chapter 5 is a short chapter where we generalize the concepts of hyperfinite processes, martingales, and hyperfinite random walks in infinite dimensions. In [Lin04] the covariance matrix of a finite-dimensional hyperfinite Lèvy process was introduced. Inspired by this, we define the covariance operator of a hyperfinite random walk. It turns out that it is a key element in order to prove several results regarding our infinite-dimensional stochastic integral in Chapter 6.

Chapter 6 begins with our definition of an infinite-dimensional stochastic integral $\int XdM$. Secondly, we give an example from [Lin83] which is a stochastic integral $\int XdW$ where W is an infinite-dimensional Anderson's process (nonstandard Brownian motion). In the first approach, we present the notion of *weak second moments in the strong sense* and show that $\int XdM$ is finite whenever X is strictly Hilbert-Schmidt and M is a martingale with weak second moments in the strong sense. In the second approach, we show that $\int XdM$ is finite, a martingale, and nearstandard when M is a martingale with covariance operator $C^M = RR$ where R is bounded and X is Hilbert-Schmidt.

Chapter 7 suggests a definition of a *hyperfinite cylindrical process*. We show that the infinite-dimensional Anderson process is such a process. We also show that the integral $\int XdM$ is cylindrical. The second part of this chapter defines *hyperfinite cylindrical Lèvy processes*.

1.2 My contributions

In this thesis, you will find the symbol ★ spread around. It is meant to highlight what I have done myself in this thesis. Much of what I have done is influenced by Tom Lindstrøm.

Here is a list of my contributions:

My result and proof:

- Proposition 5.1.3
- Corollary 5.1.4
- Theorem 5.2.1
- Corollary 5.2.2
- Lemma 5.2.3
- Proposition 5.3.7
- Lemma 5.4.8
- Proposition 6.2.4
- Corollary 6.2.8
- Proposition 7.2.3
- Proposition 7.2.4
- Proposition 7.2.5
- Lemma 7.3.2
- Proposition 7.3.5
- Corollary 7.3.6
- Proposition 7.4.2
- Corollary 7.4.3
- Proposition 7.4.4
- Proposition 8.1.3
- Proposition 8.1.7

Others (or Tom Lindstrøm's) result but my proof:

- Lemma 2.1.5
- Lemma 2.1.6
- Proposition 2.2.7
- Lemma 2.7.2
- Proposition 2.8.3 (b)
- Proposition 2.8.4
- Proposition 2.10.6
- Corollary 2.10.7

- Proposition 5.1.4
- Parts of Proposition 5.1.12
- Lemma 5.3.3
- Proposition 5.4.4
- Lemma 5.4.6
- Proposition 5.4.7
- Lemma 7.1.1
- Proposition 7.2.1
- Proposition 7.2.2
- Lemma 7.3.3
- Lemma 7.3.4

Other contributions:

- Remark 2.2.1
- Remark 2.2.2
- Example 2.2.9
- Example 2.3.6
- Example 2.4.3
- Remark 5.3.2
- Observation 5.3.9
- Observation 5.4.2
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- Remark 6.1.1
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- Definition 8.1.2
- Example 8.1.6
- Example 8.2.2

PART I

Preliminaries

CHAPTER 2

Crash Course in Nonstandard Analysis

2.1 Filters

In this section, we follow [Gol98].

Definition 2.1.1. Let $\mathcal{I}$ be any nonempty index set. A **filter** on $\mathcal{I}$ is a collection $\mathcal{F}$ of subsets of $\mathcal{I}$ satisfying the following axioms

(a) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$,

(b) If $A \subseteq B \subseteq \mathcal{I}$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

Furthermore, we call $\mathcal{F}$ a **proper** filter if $\emptyset \notin \mathcal{F}$, or equivalently if $\mathcal{F}$ is not equal to the power set of $\mathcal{I}$.

An **ultrafilter** is a proper filter that moreover satisfies

(c) For *any* $A \subseteq \mathcal{I}$ either $A \in \mathcal{F}$ or $A^c = \mathcal{I} \setminus A \in \mathcal{F}$ (but not both).

Note that an ultrafilter $\mathcal{F}$ is proper and hence by a) we must have that if $A \in \mathcal{F}$ then $A^c \notin \mathcal{F}$.

Example 2.1.2. Let $i \in \mathcal{I}$ and consider

$$\mathcal{F}^i := \{A \subseteq \mathcal{I} \mid i \in A\}.$$

It is easily verified that a)-c) from Definition 2.1.1 hold as well as $\emptyset \notin \mathcal{F}^i$. Hence it is an ultrafilter. It is called the **principal** ultrafilter generated by i. If a filter is not a principal filter it is called **nonprincipal.**

The goal of this section is to prove that for any infinite set $\mathcal{I}$ there exists a nonprincipal ultrafilter on it.

Remark 2.1.3. For $\mathcal{S} \subset \mathcal{P}(\mathcal{I})$ we denote by $\mathcal{F}^{\mathcal{S}}$ the smallest filter on $\mathcal{I}$ containing $\mathcal{S}$. If $\mathcal{S} = \emptyset$ then $\mathcal{F} = \{\mathcal{I}\}$. We say that $\mathcal{S}$ has the f.i.p. (finite intersection property) provided

$$A_{i_1} \cap \cdots \cap A_{i_n} \neq \emptyset$$

for every $i_1, \ldots i_n \in \mathcal{I}$ and $A_{i_1}, \ldots, A_{i_n} \in \mathcal{S}$. Now suppose that $\mathcal{S} \neq \emptyset$. Then

$$\mathcal{F}^{\mathcal{S}} = \{A \subseteq \mathcal{I} \mid A \supseteq B_1 \cap \cdots \cap B_i \text{ for some } i \in \mathbb{N}, B_1, \ldots, B_i \in \mathcal{S}\}.$$

Thus, clearly, if $\mathcal{S}$ has the f.i.p. then $\mathcal{F}^{\mathcal{S}}$ also has the f.i.p. and hence $\emptyset \notin \mathcal{F}^{\mathcal{S}}$ and $\mathcal{F}^{\mathcal{S}}$ must be proper.

Example 2.1.4. Consider the filter

$$\mathcal{F}^{co} := \{ A \subseteq \mathcal{I} \mid A^c = \mathcal{I} \setminus A \text{ finite } \}.$$

It is called the **cofinite** filter. If $\mathcal{I}$ is infinite then clearly $\emptyset \notin \mathcal{F}^{co}$, and hence proper. If $\mathcal{I}$ is infinite then $\mathcal{F}^{co}$ has the finite intersection property. Indeed, let $N \in \mathbb{N}$ and $A_1, \ldots, A_N \in \mathcal{F}^{co}$ be arbitrary. Since $\mathcal{F}^{co}$ is a filter we have that $\bigcap_{i=1}^N A_i \in \mathcal{F}^{co}$. But $\mathcal{F}^{co}$ is proper and hence $\bigcap_{i=1}^N A_i \neq \emptyset$. But $\mathcal{F}^{co}$ is not an ultrafilter (both A and A^c can be infinite). But later we will see that it can be extended to an ultrafilter if $\mathcal{I}$ is infinite.

Lemma 2.1.5. *Let $(\mathcal{F}_i)_{i\in\mathcal{I}}$ be a collection of filters on $\mathcal{I}$ which is totally ordered by inclusion, i.e., $\mathcal{F}_i \subseteq \mathcal{F}_j$ or $\mathcal{F}_j \subseteq \mathcal{F}_i$ for all $i, j \in \mathcal{I}$. Then*

$$\mathcal{F} := \bigcup_{i\in\mathcal{I}} \mathcal{F}_i$$

is a filter on $\mathcal{I}$.

Proof. ★ This was given as an example in [Gol98] but here we provide a proof. Suppose $A, B \in \mathcal{F}$. Since the collection is *totally* ordered by inclusion there exists $i \in \mathcal{I}$ such that $A, B \in \mathcal{F}_i$. But then since $\mathcal{F}_i$ is a filter we get that $A \cap B \in \mathcal{F}_i \subseteq \mathcal{F}$. Secondly, assume that $A \in \mathcal{F}$ and that $A \subseteq B \subseteq \mathcal{I}$. Then for some $i \in \mathcal{I}$ we have that $A, B \in \mathcal{F}_i$. Thus $B \in \mathcal{F}$. Hence the union $\mathcal{F}$ is a filter on $\mathcal{I}$. ■

In order to prove the existence of a nonprincipal ultrafilter we need to use Zorn's Lemma. Zorn's lemma states that if $(P, \subseteq)$ is a partially ordered set in which every totally ordered subset has an upper bound in P, then P contains a maximal element with respect to $\subseteq$. In our case, if P is a collection of proper filters on $\mathcal{I}$, then P contains a maximal element which is a maximal proper filter. A maximal proper filter is a proper filter that cannot be extended to a larger proper filter.

Lemma 2.1.6. *Let $\mathcal{F}$ be a proper filter on a set $\mathcal{I}$. Then $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F}$ is a maximal proper filter.*

Proof. ★ This was given as an exercise in [Gol98]. Observe that if $\mathcal{F}$ is a proper filter then $\mathcal{F} \cup \{A\}$ has the f.i.p. if and only if $A^c \notin \mathcal{F}$. First, assume that $\mathcal{F}$ is an ultrafilter and $A \notin \mathcal{F}$. Then, since $\mathcal{F}$ is an ultrafilter, we have that $A^c \in \mathcal{F}$. Thus

$$\mathcal{F} \cup \{A\}$$

is no longer contained in a proper filter. Hence $\mathcal{F}$ cannot be extended to a larger proper filter. In other words, $\mathcal{F}$ is a maximal proper filter. Now suppose that $\mathcal{F}$ is a maximal proper filter. Then since $\mathcal{F}$ cannot be extended to a larger proper filter we have for any $A \subseteq \mathcal{I}$ that either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. But then by definition, $\mathcal{F}$ is an ultrafilter. ■

Theorem 2.1.7. *Any infinite set $\mathcal{I}$ has a nonprincipal ultrafilter on it.*

Proof. This was proven in two results in [Gol98] but here we merge it into one proof. We consider the cofinite filter

$$\mathcal{F}^{co} := \{A \subseteq \mathcal{I} \mid A^c = \mathcal{I} \setminus A \text{ finite } \}.$$

Since $\mathcal{I}$ is infinite we have that $\mathcal{F}^{co}$ is proper and has the f.i.p. See Example 2.1.4. First, we will show that the cofinite filter can be extended to an ultrafilter. Let $(P, \subseteq)$ be the collection of all proper filters on $\mathcal{I}$ that contains the cofinite filter $\mathcal{F}^{co}$. By Lemma 2.1.5 and zornification we have that P contains a maximal element denoted by $\mathcal{F}$, i.e., a maximal proper filter and thus an ultrafilter by Lemma 2.1.6. Now we show that $\mathcal{F}$ is not a principal ultrafilter. Let $i \in \mathcal{I}$ be arbitrary. By definition from Example 2.1.2 we have that $\{i\} \in \mathcal{F}^i$. But $\mathcal{I} \setminus \{i\} \in \mathcal{F}^{co} \subseteq \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter $\{i\} \notin \mathcal{F}$ and thus $\mathcal{F} \neq \mathcal{F}^i$. As $i \in \mathcal{I}$ was arbitrary we can conclude. ■

2.2 Nonstandard extension of a set

Here we follow [Cut88].

Let $\mathcal{I}$ be an infinite index set, e.g. $\mathbb{N}$ or $\mathbb{R}$. As $\mathcal{I}$ is infinite we know from the previous section that there exists a nonprincipal ultrafilter on it. Let us fix a nonprincipal ultrafilter and denote it by $\mathcal{F}$. The reason we need ultrafilters will now become more clear. We define a finitely additive measure m on $\mathcal{P}(\mathcal{I})$ by

$$m(A) = 1 \text{ if } A \in \mathcal{F}$$
$$m(A) = 0 \text{ if } A \notin \mathcal{F}.$$

Remark 2.2.1. ★The reason why it is only finitely additive and not necessarily countably additive, is because of the finite intersection property of $\mathcal{F}$. The f.i.p. ensures that any finite collection of sets from the ultrafilter $\mathcal{F}$ cannot be disjoint. Thus if we have a finite collection $(A_{i_n})_{n \leq N}$ of pairwise disjoint sets in $\mathcal{P}(\mathcal{I})$ then at most one of the A_{i_n}'s belongs to the ultrafilter $\mathcal{F}$ and hence

$$m(A_{i_1} \cup \cdots \cup A_{i_N}) = \sum_{n=1}^{N} m(A_{i_n}) = \begin{cases} 1 & \text{if } A_{i_{n'}} \in \mathcal{F} \text{ for some } n' \leq N \\ 0 & \text{if } A_{i_n} \notin \mathcal{F} \text{ for all } n \leq N \end{cases}.$$

From the above, we see that m is finitely additive on $\mathcal{P}(\mathcal{I})$.

Let S be any set, e.g., a vector space or $\mathbb{R}$. Consider the sequence set $S^{\mathcal{I}} = \{s = (s_i)_{i \in \mathcal{I}} \mid s_i \in S \ \forall i \in \mathcal{I}\}$. Sometimes we just write (s_i) as an element of $S^{\mathcal{I}}$. We define an equivalence relation $\equiv$ on $S^{\mathcal{I}}$ by

$$s \equiv r \iff \{i \in \mathcal{I} \mid s_i = r_i\} \in \mathcal{F},$$

or equivalently,

$$s \equiv r \iff m(s_i = r_i) = 1.$$

We denote the set of equivalence classes by ${}^*S = S^{\mathcal{I}}/\equiv$ and we write $[s] = [s_i] = [(s_i)]$ when we wish to emphasize that we work with equivalence classes. We call ${}^*S$ the **nonstandard extension** of S.

Remark 2.2.2. ★ The underlying set S is embedded in ${}^*S$ by the map

$$s \mapsto {}^*s := [s^{\mathcal{I}}]$$

which maps an element in S to an (equivalence class of) constant sequence. Moreover, if S carries operations such as multiplication, addition, and/or scalar multiplication, then we can define these operations to the set of equivalence classes elementwise. As an example, we define addition by

$$[(s_i)] + [(r_i)] := [(s_i + r_i)].$$

It is independent of the choice of representative. Indeed, if $(s_i) \equiv (s_i')$, then $(s_i + r_i) \equiv (s_i' + r_i)$ as we get that

$$\{i \in \mathcal{I} \mid s_i + r_i = s_i' + r_i\} = \{i \in \mathcal{I} \mid s_i = s_i'\} \in \mathcal{F}.$$

Similarly one can show that multiplication and scalar multiplication also are well-defined.

Example 2.2.3.

(a) Let $\mathcal{I} = S = \mathbb{N}$. Then we call ${}^*\mathbb{N} = \mathbb{N}^{\mathbb{N}}/\equiv$ the set of **hyperintegers**.

(b) Let $\mathcal{I} = \mathbb{N}$ and let $S = \mathbb{R}$. We then call ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\equiv$ the **hyperreals**. Later we will show that ${}^*\mathbb{R}$ is an ordered field.

(c) Let $\mathcal{I} = \mathbb{N}$ and $S = V$ be a vector space. Then ${}^*V = V^{\mathbb{N}}/\equiv$ can be viewed as a real vector space when the scalar field is $\mathbb{R}$, and we call ${}^*V$ a **hyperreal vector space** when the scalar field is ${}^*\mathbb{R}$.

Hyperreal numbers

Consider the hyperreals ${}^*\mathbb{R}$. We define addition and multiplication on ${}^*\mathbb{R}$ elementwise, i.e., for $x = [(x_n)], y = [(y_n)] \in {}^*\mathbb{R}$ we define

$$[(x_n)] + [(y_n)] := [(x_n + y_n)],$$

and

$$[(x_n)] \cdot [(y_n)] := [(x_n \cdot y_n)].$$

Similar argument as in Remark 2.2.2 shows that these operations are well-defined.

Proposition 2.2.4. $({}^*\mathbb{R}, +, \cdot)$ *is a field.*

Proof. This proof follows the proof of Theorem 3.6.1 in [Gol98]. It is easily seen that $[\mathbf{0}]$ and $[\mathbf{1}]$ are the additive and multiplicative unities, respectively. One can check that ${}^*\mathbb{R}$ satisfies the axioms of a ring with additive inverse given by

$$-x = -[(x_n)] = [(-x_n)].$$

Thus the last thing to check is that it has multiplicative inverses. Let $[x] \neq [\mathbf{0}]$. Then

$$F = \{n \in \mathbb{N} \mid x_n \neq 0\} \in \mathcal{F}.$$

Define $(y_n) \in \mathbb{R}^{\mathbb{N}}$ by

$$y_n = \begin{cases} \frac{1}{x_n} & ; n \in \mathcal{F} \\ 0 & ; n \notin \mathcal{F} \end{cases}$$

and consider the equivalence class $y = [(y_n)]$. Then

$$\{n \in \mathbb{N} \mid x_n \cdot y_n = 1\} = F \in \mathcal{F}$$

and hence

$$[(x_n)] \cdot [(y_n)] = [(x_n \cdot y_n)] = [\mathbf{1}].$$

■

Let $x, y \in {}^*\mathbb{R}$ and m be a fixed finitely additive measure on $\mathcal{P}(\mathbb{N})$ as described in the previous section. We write $x < y$ when

$$m(\{n \in \mathbb{N} \mid x_n < y_n\}) = 1,$$

or equivalently

$$\{n \in \mathbb{N} \mid x_n < y_n\} \in \mathcal{F}.$$

If $a \in \mathbb{R}$ then we write $x < a$ when

$$m(\{n \in \mathbb{N} \mid x_n < a\}) = 1,$$

and correspondingly for $\leq$, $\geq$ and $>$.

We will show that $({}^*\mathbb{R}, +, \cdot, <)$ is an ordered field but first we recall some definitions.

Definition 2.2.5. A **strict total order** on a set X is a binary relation $<$ on X satisfying for all $x, y, z \in X$:

(a) Not $x < x$ (irreflexive).

(b) If $x < y$ then not $y < x$ (asymmetric).

(c) If $x < y$ and $y < z$ then $x < z$ (transitive).

(d) If $x \neq y$ then $x < y$ or $y < x$ (connected).

Definition 2.2.6. A field $(X, +, \cdot)$ together with a strict total order $<$ on X is called an **ordered field** if for all $x, y, z \in X$ we have:

(a) If $x < y$ then $x + z < y + z$.

(b) If $0 < x$ and $0 < y$ then $0 < x \cdot y$.

Proposition 2.2.7. $({}^*\mathbb{R}, +, \cdot, <)$ *is a an ordered field.*

Proof. ★ First, we need to show that $<$ is a strict total order. We have that

$$\{n \in \mathbb{N} \mid x_n < x_n\} = \emptyset \notin \mathcal{F}$$

hence $x < x$ does *not* hold. Thus $<$ is irreflexive. Now we show it is asymmetric. Let $x < y$, i.e., $F = \{n \in \mathbb{N} \mid x_n < y_n\} \in \mathcal{F}$. But we have that

$$F \cap \{n \in \mathbb{N} \mid x_n > y_n\} = \emptyset \notin \mathcal{F}.$$

Since $F \in \mathcal{F}$ and $\mathcal{F}$ is an ultrafilter we must have $\{n \in \mathbb{N} \mid x_n > y_n\} \notin \mathcal{F}$. Thus $x > y$ does *not* hold. Now we show that it is transitive. Assume $x < y$ and $y < z$. Then we have that

$$F = \{n \in \mathbb{N} \mid x_n < z_n\} \supseteq \{n \in \mathbb{N} \mid x_n < y_n\} = G.$$

Since $G \in \mathcal{F}$ and $G \subseteq F$ we have that $F \in \mathcal{F}$. Now we show that it is connected. First observe that an ultrafilter satisfies the following: $F \cup G \in \mathcal{F}$ iff $F \in \mathcal{F}$ or $G \in \mathcal{F}$. Assume that $x \neq y$. Then

$$\{n \in \mathbb{N} \mid x_n < y_n\} \cup \{n \in \mathbb{N} \mid x_n > y_n\} = \{n \in \mathbb{N} \mid x_n \neq y_n\} \in \mathcal{F}.$$

Hence either $x < y$ or $y < x$.

Now we show that $({}^*\mathbb{R}, +, \cdot, <)$ is a an ordered field. First, we show that $x < y \implies x + z < y + z$. But this clear as

$$\{n \in \mathbb{N} \mid x_n + z_n < y_n + z_n\} = \{n \in \mathbb{N} \mid x_n < y_n\} \in \mathcal{F}.$$

Lastly we show that if $[\mathbf{0}] < x, [\mathbf{0}] < y$ then $[\mathbf{0}] < x \cdot y$. Since filters are closed under intersections and

$$\{n \in \mathbb{N} \mid 0 < x_n \cdot y_n\} = \{n \in \mathbb{N} \mid 0 < x_n\} \cap \{n \in \mathbb{N} \mid 0 < y_n\}$$

we can conclude. ■

Definition 2.2.8.

(a) A hyperreal number $x \in {}^*\mathbb{R}$ is **positive infinite** if for every $\alpha \in \mathbb{R}_+$ we have $x > \alpha$.

(b) A hyperreal number $x \in {}^*\mathbb{R}$ is **negative infinite** if for every $\alpha \in \mathbb{R}_+$ we have $x < -\alpha$.

(c) A hyperreal number $x \in {}^*\mathbb{R}$ is called an **infinitesimal** if for every $\alpha \in \mathbb{R}_+$ we have $-\alpha < x < a$.

(d) A hyperreal number $x \in {}^*\mathbb{R}$ is **limited** or **finite** if for some $\alpha \in \mathbb{R}_+$ we have $-\alpha < x < \alpha$.

Example 2.2.9. ★

(a) We have that $\epsilon = \left(\frac{1}{n}\right)$ is an infinitesimal. If $a > 1$ then

$$\left\{n \in \mathbb{N} \mid -a < \frac{1}{n} < a\right\} = \mathbb{N},$$

and $\mathbb{N} \in \mathcal{F}$ hence

$$m(-a < \epsilon < a) = 1.$$

Now if $0 < a < 1$ then there exists an $n_0 \in \mathbb{N}$ such that $a > \frac{1}{n}$ for all $n \geq n_0$. Since our ultrafilter is nonprincipal we get that

$$\left\{n \in \mathbb{N} \mid -a < \frac{1}{n} < a\right\} = \mathbb{N} \setminus \{1, \ldots, n_0 - 1\}$$

belongs to $\mathcal{F}$.

(b) We have that $\omega = \frac{1}{\epsilon} = (n)_{n\in\mathbb{N}}$ is positive infinite. The argument is similar to the first example.

(c) we have that ${}^*0 = [\mathbf{0}] = [(0,0,0,\dots)]$ is an infinitesimal

(d) Let $s \in \mathbb{R}_+$. Then ${}^*s = [(s,s,\dots)]$ is finite, but not infinitesimal. For $a > s$ we get

$$\{n \in \mathbb{N} \mid -a < s < a\} = \mathbb{N} \in \mathcal{F},$$

but for $a < s$ we have

$$\{n \in \mathbb{N} \mid -a < s < a\} = \emptyset \notin \mathcal{F}.$$

Remark 2.2.10. For $n \in \mathbb{N}$ we have that the set $\{1,\dots,n\} = \{k \in \mathbb{N} \mid k \leq n\}$ is finite. However, if $N \in {}^*\mathbb{N}$ is an unlimited hypernatural number then

$$\{1,\dots,N\} = \{k \in {}^*\mathbb{N} \mid k \leq N\}$$

is infinite in a set-theoretical point of view, but as we shall see later it shares many properties of finite sets. We call such a set *hyperfinite*.

Definition 2.2.11. A hyperreal number $x \in {}^*\mathbb{R}$ is said to be **infinitely close** to a hyperreal y if $x - y$ is an infinitesimal. We write $x \approx y$. The **monad** of an hyperreal x is the equivalence class determined by $\approx$, denoted by

$$\mu(x) \coloneqq \{y \in {}^*\mathbb{R} \mid y \approx x\}$$

Remark 2.2.12. Hence the set of all infinitesimals is equal to $\mu(0)$.

2.3 Internal sets and functions

Definition 2.3.1. A collection $(A_i)_{i\in\mathcal{I}}$ of subsets of S defines a subset $[A_i]$ of ${}^*S$ by

$$\begin{aligned} x = (x_i) \in [A_i] &\iff m(\{i \in \mathcal{I} \mid x_i \in A_i\}) = 1 \\ &\iff \{i \in \mathcal{I} \mid x_i \in A_i\} \in \mathcal{F}. \end{aligned}$$

Sets of ${}^*S$ which can be obtained this way are called **internal sets**

Definition 2.3.2. Let $(f_i)_{i\in\mathcal{I}}$ be a collection of functions $f_i : S_1 \to S_2$, where S_1, S_2 are sets. We can then define a function

$$[f_i] : {}^*S_1 \to {}^*S_2$$

by

$$[f_i]([x_i]) \coloneqq [f_i(x_i)].$$

A function on ${}^*S_1$ defined this way is called **internal.**

Example 2.3.3.

(a) If $\mathcal{I} = \mathbb{N}$ and $A \subseteq S$ then ${}^*A = [A^{\mathcal{I}}] = [(A,A,A,\dots)]$ is an internal set.

(b) Any finite set $X = \{r^1, \dots, r^k\}$ of hyperelements $r^i = [r_n^i]$ is internal because we can construct $X = [A_n]$ from $A_n = \{r_n^1, \dots, r_n^k\}$.

(c) Suppose that $(S, <)$ is an ordered set and for $x = [x_i], y = [y_i]$ in ${}^*S$ we have $x < y$, i.e.,

$$\{i \in \mathcal{I} \mid x_i < y_i\} \in \mathcal{F}.$$

Then the open interval of hyperelements

$$(x, y) = \{z \in {}^*S \mid x < z < y\}$$

is internal. This is because we can construct $(x, y) = [(x_i, y_i)]$ where

$$[z_i] \in [(x_i, y_i)] \iff \{i \in \mathcal{I} \mid z_i \in (x_i, y_i)\} \in \mathcal{F}.$$

Definition 2.3.4. An internal set $A = [(A_i)_{i\in\mathcal{I}}]$ is called **hyperfinite** if

$$\{i \in \mathcal{I} \mid A_i \text{ is finite }\} \in \mathcal{F}$$

Since we have that

$$[A_i] = [B_i] \iff \{i \in \mathcal{I} \mid A_i = B_i\} \in \mathcal{F}$$

we can assume that all the $A_i's$ are finite

Definition 2.3.5. The **internal cardinality** of a hyperfinite set A is the hyperinteger

$$|A| \coloneqq [(|A_i|)_{i\in\mathcal{I}}].$$

Example 2.3.6. ★

(a) Consider a countable collection $(x_{i_n})_{n\in\mathbb{N}} \subseteq S$. For $A_n \coloneqq \{x_{i_1}, \dots x_{i_n}\}$ we obtain a hyperfinite set $A \coloneqq [A_n]$. The internal cardinality of A is then

$$|A| = [(|A_1|, |A_2|, |A_3|, \dots)] = [(1, 2, 3, \dots)]$$

(b) Let $N = [N_n] \in {}^*\mathbb{N}$ and consider

$$A_n = \left\{0, \frac{1}{N_n}, \dots, \frac{N_n - 1}{N_n}, 1\right\}.$$

Note that $|A_n| = N_n + 1$. Then the *timeline*

$$\mathbb{T} = [A_n] = \left\{0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N}, 1\right\}$$

is hyperfinite with internal cardinality $N + 1 = [N_n + 1]$.

(c) Let $N = [N_n] \in {}^*\mathbb{N}$ and consider

$$A_n = \left\{0, \frac{1}{N_n}, \dots, \frac{N_n^2 - 1}{N_n}, N_n\right\}.$$

Note that $|A_n| = N_n^2 + 1$. Then the *timeline*

$$\mathbb{T} = [A_n] = \left\{0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N - 1^2}{N}, N\right\}$$

is hyperfinite with internal cardinality $N^2 + 1 = [N_n^2 + 1]$.

2.4 Superstructures

Definition 2.4.1. Let S be a set. Inductively we define

$$U_0(S) = S,$$
$$U_{n+1} = U_n \cup \mathcal{P}(U_n(S)).$$

A **superstructure** over S is the union

$$U(S) = \bigcup_{n\in\mathbb{N}} U_n(S).$$

Definition 2.4.2. Let $x \in U(S)$. We say that x has **rank zero** if $x \in S$. We say that x has **rank n** if $x \in U_n \setminus U_{n-1}$.

Example 2.4.3. ★ Let $S = \{1, 2\}$. Then

$$U_1 = \{\{1\}, \{2\}, S, \emptyset\}$$

and

$$U_2 = \{\{S\}, \{\{1\}\}, \{\{2\}\}, \{\{1\}, \{2\}\}, \{\{1\}, S\}, \{\{2\}, S\}, U_1, \emptyset\}.$$

We have that 1 has rank zero, while $\{1\}$ is of rank one and $\{\{1\}\}$ is of rank two.

If we recall the set-theoretic definition of a function, we have that $f : S \to S$ can be viewed as a *subset* of $U_2(S)$, but *belongs* to $U_3(S)$. Any topology on S is a *subset* of U_2, but *belongs* to U_3, and the collection of all topologies on S belongs to U_4. The point with superstructures is that it contains everything we need, e.g., a Banach space X, scalar field $\mathbb{F}$, functions and functionals, measure space, topologies, etc. As an example, if we want to consider a function $f : X \to Y$ then we would choose $S = X \cup Y$. Thus the underlying set S of our superstructure might be $S = X \cup \mathbb{F} \cup \Omega \dots$ depending on what we shall work with.

Our generalized internal entities will be elements of $U({}^*S)$ arising from sequences (A_n) of elements in $U(S)$.

2.5 Saturation

The term saturation tells us that that the nonstandard extension of a set ${}^*S$ is full of elements compared to S. This will become more clear from Example 2.5.2.

Theorem 2.5.1 (Countable Saturation)**.** *Let $A_1 \supseteq A_2 \supseteq \dots$ be a decreasing sequence of nonempty internal sets. Then*

$$\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset.$$

Proof. See [Gol98] page 138. ■

Example 2.5.2. Consider the hyperreal interval

$$A_n = \left(0, \frac{1}{n}\right) = \left\{x \in {}^*\mathbb{R} \mid 0 < x < \frac{1}{n}\right\}.$$

We have that $(A_n)_{n\in\mathbb{N}}$ is decreasing and each A_n is nonempty. By countable saturation we get that the intersection $\bigcap_{n\in\mathbb{N}} A_n$ is nonempty. In fact, it is exactly the set of positive infinitesimals. On the other hand, if we let $B_n = (0, 1/n) \subset \mathbb{R}$ be the *real* interval, then we get that the intersection $\bigcap_{n\in\mathbb{N}} B_n$ is empty.

Definition 2.5.3. Let κ be a cardinal. We say that the nonstandard model $U({}^*S)$ is **κ-saturated** if whenever $(A_\gamma)_{\gamma\in\Gamma}$ is a collection of internal sets with the finite intersection property and $\mathrm{card}(\Gamma) < \kappa$ then

$$\bigcap_{\gamma\in\Gamma} A_\gamma \neq \emptyset.$$

Definition 2.5.4. We call $U({}^*(S))$ **polysaturated** if it is κ-saturated for some $\kappa \geq \mathrm{card}(U(S))$.

In [Cut88] it is shown that we can always construct a polysaturated model $U({}^*S)$ whenever $\mathrm{card}(U({}^*S))$ is infinite.

2.6 The transfer principle

The transfer principle says that a statement ϕ is true in the language $L(U(S))$ if and only if the *-version ${}^*\phi$ is true in the language $L(U({}^*S))$. But first, we need to give precise meaning to the word statement.

The languages $L(U(S))$ and $L(U({}^*S))$

In this section, we fix a superstructure $U(S)$ and its nonstandard version $U({}^*S)$.

Definition 2.6.1. A function $F : U(S)^k \to U(S)$ is **tame** if for each $n \in \mathbb{N}$ there is a $m \in \mathbb{N}$ such that

$$F(a_1, \ldots, a_k) \in U_m(S)$$

whenever $a^1, \ldots, a^k \in U_n(S)$.

Suppose F is tame and $a_i^1, \ldots a_i^k$ are internal. Then

$${}^*F(a^1, \ldots, a^k) = \langle F(a_i^1, \ldots, a_i^k) \rangle$$

defines an internal set. But we wish to work with functions

$${}^*F : U({}^*S)^k \to U({}^*S)$$

so we extend ${}^*F$ by assigning arbitrary values when some of the input variables $a_1, \ldots a_k$ are external.

We now specify the alphabet $\mathcal{A}(U(S))$ of the language $L(U(S))$. The alphabet consists of symbols such as variables, constant symbols, relation symbols, function symbols, connectives, quantifiers, and parentheses. Let us be more specific:

- variables: $x_1, x_2, \ldots$
- constant symbols: a symbol $\bar{a}$ for each $a \in U(S)$,
- relation symbols: $=, \in$,

- function symbols: A symbol $\bar{F}$ for each tame function $F : U(S)^k \to U(S)$,
- connectives: $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\implies$ (implies), $\iff$ (iff),
- quantifiers: $\exists$ (there exists), $\forall$ (for all),
- parentheses: (,).

Definition 2.6.2. A **string** over $\mathcal{A}(U(S))$ is a finite sequence $s_1 s_2 \dots s_n$ of symbols.

The arbitrary composition of symbols does not necessarily make sense. Hence we introduce subclasses of strings called terms and formulas.

Definition 2.6.3. The class of terms $\mathcal{T}$ is the smallest class Γ of strings such that

(a) If a string t consists of a single variable or of a single constant symbol, then $t \in \Gamma$.

(b) If F is a tame function of k variables and $t_1, \dots t_k \in \Gamma$, then $F(t_1, \dots, t_k) \in \Gamma$.

Definition 2.6.4. The class $\mathcal{F}$ of formulas is the smallest set Φ of strings such that

(a) If t_1 and t_2 are terms, then the strings $t_1 = t_2$ and $t_1 \in t_2$ belong to Φ.

(b) if $\phi \in \Phi$, then $\neg\phi \in \Phi$.

(c) if $\phi, \psi \in \Phi$, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $\phi \implies \psi$, and $(\phi \iff \psi)$ belongs to Φ.

(d) if $\phi \in \Phi$, x is a variable, and t is a term which does not contain x, then $\exists x \in t\ \phi$ and $\forall x \in t\ \phi$ belong to Φ.

Terms and formulas make sense, but they need not be true. The alphabet $\mathcal{A}(U(^*S))$ of $L(U(^*S))$ is the same except that we denote each constant symbol $\bar{a}$ with $^*a$ and each function symbol $\bar{F}$ with $^*F$

Definition 2.6.5. Let t be a term and ϕ a formula. The ***-transform** of the term t, is the term $^*t$ obtained by replacing $\bar{a}$ and $\bar{F}$ in t with $^*a$ and $^*F$, respectively. Similarly, $^*\phi$, the ***-transform** of the formula ϕ, is the formula obtained by replacing $\bar{a}$ and $\bar{F}$ in ϕ with $^*a$ and $^*F$, respectively.

Example 2.6.6. The formula

$$\forall x_2 \in \bar{a}\ \exists x_1 \in \bar{b}\ \bar{F}(x_1) = x_2$$

tells us that for every x_2 in a there is an element x_1 in b such that $F(x_2) = x_1$. The $*$-transform of this formula is

$$\forall x_2 \in {}^*a\ \exists x_1 \in {}^*b\ {}^*F(x_1) = x_2.$$

Definition 2.6.7. Let y be either a constant or a variable. A variable x is **bound** in a formula ϕ if it occurs in ϕ and every occurrence takes the form $(\forall x \in y)\theta$ or $(\exists x \in y)\theta$ and the occurrence can also be in θ. A variable occurring in a formula ϕ but not bound in ϕ is called a **free** variable in ϕ. A **sentence** in $L(U(S))$ is a formula where all variables are bound.

Example 2.6.8. Suppose $\bar{a}$ denotes some set $A \in U(S)$ and $\phi(x)$ a formula where x is the only free variable. Then the following is a sentence

$$(\forall x \in \bar{a})\phi(x).$$

It says that for every a ranging in A we have that $\phi(a)$ holds.

Theorem 2.6.9 (The Transfer Principle)**.** *A sentence ϕ in $L(U(S))$ is true if and only if its *-transform, ${}^*\phi$, is true in $L(U({}^*S))$.*

Proof. See [Cut88]. ∎

Theorem 2.6.10 (Overflow)**.** *Let $\phi(x)$ be a formula in $L(U({}^*S))$ with only the variable x free. If there exists a $k \in \mathbb{N}$ such that $\phi(n)$ is true for all $n \in \mathbb{N}$ with $k \leq n$, then there exists $K \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $\phi(n)$ is true for all $k \leq n \leq K$.*

Proof. See [Gol98] page 192. ∎

2.7 Standard, internal, and external entities

Here we follow [Gol98]. Let S be a set and consider the superstructures $U(S)$ and $U({}^*S)$. We call elements of $U({}^*S)$ of the form ${}^*A$ where $A \in U(S)$ **standard entities.** The other members of $U({}^*S)$ are called **nonstandard**.

Definition 2.7.1. An element $A \in U({}^*S)$ is **internal** if it belongs to some standard set, i.e., we have $A \in {}^*B$ for some $B \in U(S)$.

If $\mathcal{A}$ is *finite* then ${}^*\mathcal{A} = \{{}^*A \mid A \in \mathcal{A}\}$. Hence, if ${}^*A \in U({}^*S)$ is a standard entity then it is internal since ${}^*A \in \{{}^*A\} = {}^*\{A\}$. An entity in $U({}^*S)$ which is not internal is called **external**.

Now we will show that this definition of internal sets is in line with how we defined internal sets in Section 2.3. Recall that if $A = [A_n]$ then

$$[x_n] \in [A_n] \iff \{n \in \mathbb{N} \mid x_n \in A_n\} \in \mathcal{F}$$

where (A_n) is a sequence of subsets of $B \in U(S)$ and for some nonprincipal ultrafilter $\mathcal{F}$ on $\mathbb{N}$.

Lemma 2.7.2. *An element $A \in U({}^*S)$ is internal if and only if it is of the form $A = [A_n]$.*

Proof. ★ An elegant proof of this is given in [Cut88] page 24. Here we give a different proof. Let $B \in U(S)$. If we recall how we constructed the nonstandard extension of a set, we have ${}^*B = B^{\mathbb{N}} \setminus \equiv$. Hence any element A in ${}^*B$ is of the form $A = [A_n]$. Conversely, for $B \in U(S)$ and (A_n) be any sequence of subsets of B, let $A = [A_n]$. But then $A \in {}^*\mathcal{P}(B)$, where $\mathcal{P}(B) \in U(S)$. Hence A is internal. ∎

2.8 Topology in nonstantard outfit

Let $(X, \mathcal{T})$ be a topological space and let S be a set that contains $X, \mathbb{R}$, and all other entities we need. We will work with a polysaturated model $U({}^*S)$ of $U(S)$.

Definition 2.8.1. Given $a \in X$ the **monad** of a is a subset of ${}^*X$ defined by

$$\mu(a) = \bigcap_{a \in O \in \mathcal{T}} {}^*O.$$

Definition 2.8.2. We say that $x \in {}^*X$ is **nearstandard** to $a \in X$ provided $x \in \mu(a)$. We write $x \approx a$. We denote $\mathrm{Ns}({}^*X)$ the set of all nearstandard elements of ${}^*X$.

Thus for $a \in X$ we can write

$$\mu(a) = \{x \in {}^*X \mid x \approx a\}$$

Proposition 2.8.3. *Let $A \subseteq X$. Then*

(a) *A is open if and only if for all $a \in A$, $\mu(a) \subseteq {}^*A$.*

(b) *A is closed if and only if whenever $x \in {}^*A$ is nearstandard to some $a \in X$, then $a \in A$.*

(c) *A is compact if and only if all $x \in {}^*A$ is nearstandard to some $a \in A$.*

Proof.

(a) See [Cut88]

(b) ★ Let $x \in {}^*A$ be nearstandard to $a \in X$. We have that A is closed if and only if A^c is open. This is equivalent to that for all $a \in A^c$ $\mu(a) \subseteq {}^*A^c$. Hence if $a \in A^c$ then $x \in \mu(a) \subseteq {}^*A^c$. But this is not possible since we assumed $x \in {}^*A$. Thus a must belong to A.

(c) See [Cut88]

■

Proposition 2.8.4. *If X is Hausdorff then each $x \in Ns({}^*X)$ is nearstandard to exactly one $a \in X$.*

Proof. ★ This is usually stated in most of the literature as it is quite easy to see, but let us quickly prove it anyway. Since X is Hausdorff we know that for each distinct $a, b \in X$ that there exists $a \in O_a \in \mathcal{T}$ and $b \in O_b \in \mathcal{T}$ such that $O_a \cap O_b = \emptyset$. But then by definition of monads $\mu(a) \cap \mu(b) = \emptyset$. Hence no $x \in \mathrm{Ns}({}^*X)$ can belong to both $\mu(a)$ and $\mu(b)$ when $a \neq b$. ■

Definition 2.8.5. Let X be Hausdorff and $x \in \mathrm{Ns}({}^*X)$ be nearstandard to $a \in X$. We call a the **standard part** of x. We write $a = {}^\circ x = \mathrm{st}(x)$

Definition 2.8.6. Let $x, y \in \mathrm{Ns}({}^*X)$. We say that x and y are **infinitely close** if ${}^\circ x = {}^\circ y$.

Tychonov's theorem is a well-known result and the proof of this theorem is usually very long. But using nonstandard analysis the proof becomes very short and simple. Consider the product space $X = \prod_{i \in \mathcal{I}} X_i$. Then an element $f \in X$ can be viewed as a function $f : \mathcal{I} \to X_i$. An element $g \in {}^*X$ is an internal function $g : {}^*\mathcal{I} \to {}^*X_i$

Theorem 2.8.7 (Tychonov's Theorem)**.** *Let $(X_i)_{i\in\mathcal{I}}$ be a collection of compact spaces. Then*

$$X = \prod_{i\in\mathcal{I}} X_i$$

is compact.

Proof. This proof follows Theorem 4.1.19 in [Rob96]. Let $g \in {}^*X$. Since each X_i is compact we have for each $i \in \mathcal{I}$ that $g(i)$ is nearstandard to some $y_i \in X_i$. Let $f(i) = y_i$ so $f \in X$. But then since $g(i) \in \mu(f(i))$ for every $i \in \mathcal{I}$ we have $g \in \mu(f)$ and thus nearstandard to f. Since g was arbitrary we get that X is compact. ∎

2.9 Topologies on $^*X$

Here we follow [Rob96]. First, we remind the reader of the concept of a basis for a topology. Let X be a set. A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a **basis** for a topology on X if

1. For each $x \in X$ there is a $B \in \mathcal{B}$ such that $x \in B$.
2. If $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Given a basis $\mathcal{B}$ we can define a topology $\mathcal{T}_{\mathcal{B}}$ on X by

$$\mathcal{T}_{\mathcal{B}} = \{U \subseteq X \mid \text{for each } x \in U \text{ there is a } B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U\}.$$

We call this topology **the topology generated by $\mathcal{B}$.** As it turns out, if $\mathcal{T}_{\mathcal{B}}$ is a topology generated by a basis $\mathcal{B}$, then $\mathcal{T}_{\mathcal{B}}$ equals the collection of all unions of elements in $\mathcal{B}$.

Conversely, If we are given a topological space $(X, \mathcal{T})$ and $\mathcal{B}$ is a collection of open sets such that for each $U \in \mathcal{T}$ and each $x \in U$ there is a $B \in \mathcal{B}$ such that $x \in B \subseteq U$, then $\mathcal{B}$ is a basis for $\mathcal{T}$, i.e., $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$.

Let $(X, \mathcal{T})$ be a topological space and $^*X$ the nonstandard extension of X. Observe that $^*\mathcal{T}$ is a standard entity and hence every $U \in {}^*\mathcal{T}$ is internal. Moreover, we have that $^*\emptyset = \emptyset$ and $^*X$ belongs to $^*\mathcal{T}$. By transfer we get that $U_1 \cap U_2 \in {}^*\mathcal{T}$ for any $U_1, U_2 \in {}^*\mathcal{T}$. But $^*\mathcal{T}$ is not closed under arbitrary unions *unless* we take the union of a collection $\Gamma \subset {}^*\mathcal{T}$ which is *internal*. Thus $^*\mathcal{T}$ is not a topology.

Q-topology

Since $^*X \in {}^*\mathcal{T}$, and $^*\mathcal{T}$ is closed under finite intersections, it is easy to see that $^*\mathcal{T}$ is a base for a topology $\mathcal{T}^Q$ where $\mathcal{T}^Q$ is the topology generated by $^*\mathcal{T}$. We call this topology the **Q-topology**. Q-open sets can be external, but we have the following result from [Rob96] on page 99:

Theorem 2.9.1. *Let $\mathcal{I} := \{B \in \mathcal{T}^Q \mid B$ internal.$\}$. Then $^*\mathcal{T} = \mathcal{I}$.*

Proof. First, assume that $B \in {}^*\mathcal{T}$, i.e., internal open. Since $^*\mathcal{T} \subset \mathcal{T}^Q$, it is obvous that B is Q-open.

Suppose that B is Q-open and internal. Let $\Gamma = \{U \subset B \mid U \in {}^*\mathcal{T}\}$. Since B is internal it can be written on the form $B = [B_n]$. Hence $\Gamma = [\Gamma_n]$ where $\Gamma_n = \{U_n \subseteq B_n \mid U_n \in \mathcal{T}\}$. Thus Γ is internal. Because Γ is internal we get that $B = \bigcup_{U \in \Gamma} U$ is internal open. ■

Now suppose that $(X, \|\cdot\|)$ is a normed linear space where $\mathcal{T}$ is the topology generated by the norm, i.e., the topology generated by the basis $\mathcal{A} = \{B_r(x) \mid x \in X, r \in \mathbb{R}_+\}$ where $B_r(x) = \{y \in X \mid \|x - y\| < r\}$.

We will now show that the collection

$$\mathcal{B} = \{B_r(x) \mid r \in {}^*\mathbb{R}_+, x \in {}^*X\}$$

of internal open balls is a basis for the Q-topology. If $U \in {}^*\mathcal{T}$ then, by transfer, for every $x \in U$ there exists $r \in {}^*\mathbb{R}_+$ such that $B_r(x) \subseteq U$. We have that $\Gamma = \{B \subseteq U \mid B \in \mathcal{B}\}$ can be written as $\Gamma = [\Gamma_n]$ where $\Gamma_n = \{B_n \subseteq U_n \mid B_n \in \mathcal{A}\}$ for $U = [U_n]$, hence Γ is internal, and therefore $U = \bigcup_{B \in \Gamma} B$ is internal open. Now if $O \in \mathcal{T}^Q$ then $O = \bigcup_{\alpha \in \mathcal{I}} U_\alpha$ where $U_\alpha \in {}^*\mathcal{T}$ for $\alpha \in \mathcal{I}$. If $x \in O$ then $x \in U_\alpha$ for some $\alpha \in \mathcal{I}$. Hence there exists a $B_r(x) \subseteq U_\alpha \subseteq O$. Hence $\mathcal{B}$ is a basis for $\mathcal{T}^Q$.

If $x, y \in {}^*X$ are distinct put $r = \|x - y\|$ and let $s \leq \frac{r}{2}$. Then we can find two internal open balls which are disjoint, namely $B_s(x)$ and $B_s(y)$. Consequently, the Q-topology is Hausdorff.

S-topology

Given a normed linear space $(X, \|\cdot\|)$, We define an S-ball in ${}^*X$ to be of the form

$$S_r(x) = \{y \in {}^*X \mid {}^\circ\|x - y\| < r\}$$

for any *standard* positive r and $x \in {}^*X$.

We will show that the collection of S-balls, $\mathcal{S} = \{S_r(x) \mid x \in {}^*X, r \in \mathbb{R}_+\}$ is a basis for a topology. Obviously, for any $x \in {}^*X$ we can find a $S_r(x) \in \mathcal{S}$ containing x. Let $S_r(x)$ and $S_{r'}(y)$ be two S-balls with nonempty intersection. Let $z \in S_r(x) \cap S_{r'}(y)$ and put $s = {}^\circ\|x - z\|$ and $s' = {}^\circ\|z - y\|$ so that $s < r$ and $s' < r'$. Further choose $t = \min\{r - s, r' - s'\} \in \mathbb{R}_+$, then $S_t(z) \subseteq S_r(x) \cap S_{r'}(y)$. Thus $\mathcal{S}$ is a basis for a topology, denoted by $\mathcal{T}^S$, and we call this topology the **S-topology**.

Observe that $\mathcal{T}^Q$ is finer than $\mathcal{T}^S$, i.e., $\mathcal{T}^S \subseteq \mathcal{T}^Q$. Indeed, let $x \in {}^*X$ be arbitrary and let $r \in \mathbb{R}_+$ and $y \in {}^*X$ be such that

$$x \in S_r(y) = \{z \in {}^*X \mid {}^\circ\|z - y\| < r\}$$

hence $S_r(y)$ is any basis element in $\mathcal{S}$ containing x. Set $s = {}^\circ\|x - y\| < r$ and let $t \in \mathbb{R}_+$ be such that $t \leq r - s$. Then $B_t(x) \subseteq S_r(y)$.

Note that $\mathcal{T}^S$ is not Hausdorff: for $x \in {}^*X$ let $\mathcal{S}^x = \{S^x \in \mathcal{S} \mid x \in S^x\}$ denote the collection of all basis elements containing x. Then we have that every $y \in \mu(x)$ also belongs to every $S^x \in \mathcal{S}^x$. The lack of Hausdorff property yields that if $(x_\lambda)_{\lambda \in \Lambda}$ is a net converging to $x \in {}^*X$ then the net will also converge to every $y \in \mu(x)$.

Consider a function $f : D \subseteq {}^*X \to {}^*Y$. Let x_0 belong to the S-closure of D. We call $y_0 \in {}^*Y$ an **S-limit** of $f(x)$ as x tends to x_0 in D if

$$f(\mu(x_0) \cap (D \setminus \{x_0\}) \subseteq \mu(y_0).$$

We call f **S-continuous** at x_0 in D if $f(x_0)$ is an S-limit of $f(x)$ as x tends to x_0. In other words, f is S-continuous at a point $x_0 \in D$ if and only if

$$f(\mu(x_0) \cap D) \subseteq \mu(f(x_0)).$$

A third way of of describing S-continuity: $f : D \subseteq {}^*X \to {}^*Y$ is S-continuous if and only if $x, y \in D$, $x \approx y$ implies $f(x) \approx f(y)$.

2.10 Loeb measure

Here we are going to construct the Loeb measure where we turn an internal finitely additive measure with values in ${}^*\mathbb{R}_+ \cup \{0, \infty\}$ into a premeasure with values in $\mathbb{R} \cup \{0, \infty\}$ by taking standard parts. And then we consider the outer measure extension of this premeasure arriving at a complete measure by Carathéodory's Extension Theorem.

Caratheodory's Theorem

Let $\bar{\mathbb{R}}_+ \coloneqq \mathbb{R}_+ \cup \{0, \infty\}$ and Ω be a set.

Definition 2.10.1. An **algebra** $\mathcal{A}$ of Ω is a collection of subsets of Ω satisfying

(a) $\emptyset, \Omega \in \mathcal{A}$,

(b) if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$

(c) if $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$.

Definition 2.10.2. We call $\mu : \mathcal{A} \to \bar{\mathbb{R}}_+$ a **premeasure** on an algebra $\mathcal{A}$ if

$$\mu(\emptyset) = 0$$

and

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

whenever $A_n \in \mathcal{A}$ are disjoint and $\bigcup_{n=1}^{\infty} A_n$ happens to be in $\mathcal{A}$.

The *outer measure* μ^* of μ is defined by

$$\mu^*(B) = \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_n) \mid B \subseteq \bigcup_{n \in \mathbb{N}} A_n,\ A_n \in \mathcal{A} \right\}$$

where μ^* takes values from the σ-algebra of μ^*-measurable sets. A subset $E \subseteq \Omega$ is called μ^*-measurable if

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A)$$

for all $A \subseteq \Omega$. See e.g. [Lin17] for details.

Theorem 2.10.3 (Carathéodory's Extension Theorem)**.** *Assume $\mathcal{A}$ is an algebra and that μ is a premeasure on $\mathcal{A}$. Then the measure μ^* is a complete measure extending μ.*

Proof. See [Lin17]. ■

Loeb Measure

Let Ω be an internal set in some superstructure $U(^*S)$. Let $^*\bar{\mathbb{R}}_+ = {^*\mathbb{R}_+} \cup \{0, \infty\}$

Definition 2.10.4. An **internal algebra** on Ω is an internal set $\mathcal{A}$ which is an algebra on Ω.

Definition 2.10.5. An **internal, finitely additive measure** on $\mathcal{A}$ is an internal function $\mu : \mathcal{A} \to {^*\bar{\mathbb{R}}_+}$ satisfying

(a) $\mu(\emptyset) = 0$

(b) $\mu(A \cup B) = \mu(A) + \mu(B)$, for all disjoint $A, B \in \mathcal{A}$.

We can make an internal finitely additive measure to a real-valued measure by taking its standard parts: we define $^\circ\mu : \mathcal{A} \to \bar{\mathbb{R}}_+$ by

$$^\circ\mu(A) = {^\circ(\mu(A))}.$$

And if $\mu(A)$ is infinite we define $^\circ\mu(A) = \infty$. We will now show that $^\circ\mu$ is a premeasure and hence we can apply Caratheodory's theorem.

Proposition 2.10.6. *If $A_1, A_2, \dots \in \mathcal{A}$ and $\bigcup_{n=1}^\infty A_n$ belongs to the internal algebra $\mathcal{A}$ then there exists a $k \in \mathbb{N}$ such that*

$$\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^k A_n.$$

Proof. ★ Suppose that there does *not* exists $k \in \mathbb{N}$ satisfying $\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^k A_n$. Set $B = \bigcup_{n=1}^\infty A_n$ and define $B_k = \bigcup_{n=1}^k A_n$. Let us define $C_k = B \setminus B_k$. By assumption we have that each $C_k = B \setminus B_k \neq \emptyset$ is internal and $C_0 \subseteq C_2 \subseteq \dots$. But by construction

$$\bigcap_{k=1}^\infty (B \setminus B_k) = \emptyset.$$

But this contradicts countable saturation Theorem 2.5.1. ■

Corollary 2.10.7. *We have that $^\circ\mu$ is a premeasure on $\mathcal{A}$.*

Proof. ★ Suppose that $A_1, A_2, \dots \in \mathcal{A}$ and that $\bigcup_{n=1}^\infty A_n$ happens to be in $\mathcal{A}$. Then there exists $k \in \mathbb{N}$ such that $\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^k A_n$. Hence, by finite additivity

$$\mu(\bigcup_{n=1}^\infty A_n) = \mu(\bigcup_{n=1}^k A_n) = \sum_{n=1}^k \mu(A_n) = \sum_{n=1}^\infty \mu(A_n).$$

And thus also

$$^\circ\mu(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty {^\circ\mu(A_n)}.$$

■

The Loeb measure μ_L is the outer measure extension of ${}^\circ\mu$, i.e.,

$$\mu_L := ({}^\circ\mu)^*(B) = \inf\left\{\sum_{n\in\mathbb{N}} {}^\circ\mu(A_n) \mid B \subseteq \bigcup_{n\in\mathbb{N}} A_n,\, A_n \in \mathcal{A}\right\}.$$

We denote by $\mathcal{A}_L$ the completion of $\sigma(\mathcal{A})$ with respect to μ_L. We call $(\Omega, \mathcal{A}_L, \mu_L)$ the **Loeb measure space** of $(\Omega, \mathcal{A}, \mu)$. If Ω is hyperfinite and $\mu(\Omega) = 1$ we call $(\Omega, \mathcal{F}, \mu)$ a **hyperfinite probability space**.

Let $(E, \|\cdot\|)$ be a Banach space. We call a function $f : \Omega \to E$ **$\mathcal{A}_L$-measurable** (or Loeb-measurable) if $f^{-1}(U) \in \mathcal{A}_L$ for all open $U \in E$. A function $F : \Omega \to {}^*E$ is called **$\mathcal{F}$-measurable** if $F^{-1}(U) \in \mathcal{F}$ for all *open sets $U \in {}^*\mathcal{T}$.

CHAPTER 3

Probability in Banach spaces

Later in this thesis, we will discuss *hyperfinite cylindrical processes* and in particular *hyperfinite cylindrical Lévy processes*. Thus it is convenient to be familiar with the standard analytic concept. We start this section by introducing probability measures and random variables in Banach spaces, secondly cylindrical measures, and lastly cylindrical processes.

3.1 Radon probability measures and Banach space valued random variables

Here we follow [Lin86].

For a topological space $(X, \mathcal{T})$ we define the **Borel σ-algebra on X**, denoted by $\mathcal{B}(X)$, to be the σ-algebra generated by $\mathcal{T}$, i.e., $\mathcal{B}(X) \coloneqq \sigma(\mathcal{T})$.

Let E be a Banach space.

Definition 3.1.1. A *finite* measure $\mu : \mathcal{B}(E) \to \mathbb{R}$ is called a **Radon measure** if

$$\mu(B) = \sup\{\mu(K) \mid K \subseteq B,\ K \text{ compact}\}$$

for every $B \in \mathcal{B}(E)$.

Proposition 3.1.2. *Let E be a Banach space and let μ be a Borel measure on E. Then μ is Radon if and only if there is a closed separable subset $S \subseteq E$ such that $\mu(E \setminus S) = 0$.*

Proof. This was proved for complete metric spaces in [Par05]. ■

Consequently, Radon measures on $\mathcal{B}(E)$ live on separable subsets. Moreover, if E is separable we let $\mathcal{P}(E)$ denote the set of Borel probability measures on $(E, \mathcal{B}(E))$, otherwise $\mathcal{P}(E)$ will denote the set of Radon probability measures. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For a measurable map $X : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ we denote its distribution by

$$\mathbb{P}_X(B) \coloneqq \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in B\}), \quad B \in \mathcal{B}(E).$$

Definition 3.1.3. Let $X : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$. We call X a **random variable** if it is measurable and its distribution is Radon, i.e., $\mathbb{P}_X \in \mathcal{P}(E)$.

Remark 3.1.4. Let X be a random variable. Since $\mathbb{P}_X \in \mathcal{P}(E)$ we have from Proposition 3.1.2 that there exists a closed separable subset $S \subseteq E$ such that

$$\mathbb{P}_X(S) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in S\}) = 1.$$

Hence X is separably valued almost surely. In light of this, one can see that an E-valued random variable is the same as a *Bochner-measurable* map if that terminology is more familiar.

For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Banach space E we define

$$L_0(\Omega, \mathcal{F}, \mathbb{P}; E) = \{Y : \Omega \to E \mid Y \text{ is an } E\text{-valued r.v.}\}$$

and when $E = \mathbb{R}$ we just write $L_0(\Omega, \mathcal{F}, \mathbb{P}) := L_0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$.

The characteristic function of $\mu \in \mathcal{P}(E)$ is defined by $\hat{\mu} : E^* \to \mathbb{C}$

$$\hat{\mu}(\phi) := \int_E e^{i\phi(x)} d\mu(x).$$ [1]

Let E^* denote the continuous dual of E. Do not be confused with ${}^*E$, the nonstandard extension of E. The characteristic function for an E-valued random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ is the characteristic function of its distribution $\mathbb{P}_X$, i.e., the map $E^* \ni \phi \mapsto \widehat{\mathbb{P}}_X(\phi) \in \mathbb{C}$ given by

$$\begin{aligned} \widehat{\mathbb{P}}_X(\phi) &= \int_E e^{i\phi(x)} d\mathbb{P}_X(x) \\ &= \int_\Omega e^{i\phi(X(\omega))} d\mathbb{P}(\omega) \\ &= \mathbb{E}\left(e^{i\phi(X(\omega))}\right). \end{aligned}$$

3.2 Cylindrical measures

Here we follow [Lin86]. We call $C \subset E$ a **cylindrical set** if

$$C = \{x \in E \mid (\phi_1(x), \dots, \phi_n(x)) \in B\}.$$

for $\phi_1, \dots, \phi_n \in E^*$ and $B \in \mathcal{B}(\mathbb{R}^n)$. Given $S = (\phi_1, \dots, \phi_n) \subseteq E^*$, we let $\sigma(S)$ denote the σ-algebra of cylindrical sets generated by $S = (\phi_1, \dots, \phi_n)$. We define $\mathcal{C}(E)$ to be the collection of all cylindrical sets, i.e.,

$$\mathcal{C}(E) = \bigcup_{S \subseteq E^*} \sigma(S).$$

Definition 3.2.1. Let $\mu : \mathcal{C}(E) \to [0, 1]$. We say that μ is a **cylindrical measure** if for every *finite* subset $S \subseteq E^*$ we have that $\mu \mid_{\sigma(S)}$, the restriction of μ to the σ-algebra $\sigma(S)$, is a probability measure on $(E, \sigma(S))$.

[1] Some use the notation $\langle x, \phi \rangle := \phi(x)$ for the duality pairing. In the case $E = H$ is a Hilbert space with innerproduct $\langle \cdot, \cdot \rangle$, one can view the characteristic function as a function $\hat{\mu} : H \to \mathbb{C}$

$$\hat{\mu}(y) = \int_E e^{i\langle x, y \rangle} d\mu(x)$$

using that $H \simeq H^*$.

When dim $E = \infty$ we have that $\mathcal{C}(E)$ is only an algebra. In general a cylindrical measure μ is only finitely additive. Because of the lack of σ-additivity, we want to know when there exists a Radon probability measure $\bar{\mu} \in \mathcal{P}(E)$ which extends μ, i.e., $\bar{\mu} \mid_{\mathcal{C}(E)} = \mu$. This extension is unique when it exists. Further, we have that μ admits a Radon extension if and only if for every $\epsilon > 0$ there exists a compact set $K_\epsilon \subseteq E$ such that for all $K_\epsilon \subseteq C \in \mathcal{C}(E)$ we have that

$$\mu(C) \geq 1 - \epsilon.$$

Now consider a linear map $X : E^* \to L_0(\Omega, \mathcal{F}, \mathbb{P})$. Then for a cylindrical set

$$C = \{x \in E \mid (\phi_1(x), \dots, \phi_n(x)) \in B\}$$

where $B \in \mathcal{B}(\mathbb{R}^n)$ we have that

$$\begin{aligned} \mu^X(C) &:= \mathbb{P}_{X(\phi_1),\dots,X(\phi_n)}(B) \\ &= \mathbb{P}(\{\omega \in \Omega \mid (X(\phi_1)(\omega), \dots, X(\phi_n)(\omega)) \in B\}) \end{aligned}$$

defines a cylindrical measure μ^X. Conversely, in [Sch73] it was shown that for any cylindrical measure μ there exists a linear map $X : E^* \to L_0(\Omega, \mathbb{P})$ such that $\mu = \mu^X$, where μ^X is the cylindrical measure defined by

$$\mu^X(\{x \in E \mid (\phi_1(x), \dots, \phi_n(x)) \in B\}) = \mathbb{P}_{X(\phi_1),\dots,X(\phi_n)}(B)$$

for all $(\phi_i)_{i=1}^n \subseteq E^*$. Thus we give the following definition.

Definition 3.2.2. We call a *linear* map $X : E^* \to L_0(\Omega, \mathcal{F}, \mathbb{P})$ a **cylindrical random variable.**

Moreover, in [Sch73] it was proved that μ^X admits a Radon extension if and only if there exists an E-valued random variable $Y \in L_0(\Omega, \mathcal{F}, \mathbb{P}; E)$ such that for all $\phi \in E^*$ and $\omega \in \Omega$ we have that

$$X(\phi)(\omega) = \langle Y(\omega), \phi \rangle = \phi(Y(\omega)) \quad \mathbb{P}\text{-a.e.}$$

We define the characteristic function of a cylindrical measure μ by $\hat{\mu} : E^* \to \mathbb{C}$ by

$$\hat{\mu}(\phi) := \widehat{\mu_\phi}(1) = \int_{\mathbb{R}} e^{it} d\mu_\phi(t) = \int_E e^{i\langle x, \phi \rangle} d\mu(x)$$

where μ_ϕ is a probability measure on $\mathbb{R}$ for every $\phi \in E^*$ hence $\widehat{\mu_\phi}(t)$ is well-defined. Lastly, we have that a cylindrical measure μ admits a Radon extension if and only if $\hat{\mu}$ is the characteristic function of a Radon probability measure.

3.3 Cylindrical processes

Here we follow [AR10].

First, we introduce cylindrical Wiener processes and secondly, we define cylindrical Lévy processes.

A **cylindrical process** is a collection $(X_t)_{t \geq 0}$ of cylindrical random variables $X_t : E^* \to L_0(\Omega, \mathcal{F}, \mathbb{P})$. The filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by a cylindrical process $(X_t)_{t \geq 0}$ is given by

$$\mathcal{F}_t = \sigma(\{X_s^{-1}(\phi) \mid \phi \in E^*,\ s \in [0, t]\}).$$

We call an adapted $\mathbb{R}^n$-valued process $(X_t)_{t\geq 0}$ a Wiener process if it starts in zero a.s. and if the increments $X_t - X_s$ are independent, stationary, and normally distributed with expectation $E[X_t - X_s] = 0$ and variance $Var(X_t - X_s) = |t-s|C$ where C is a non-negative, definite symmetric matrix.

Definition 3.3.1. A cylindrical process $W : [0,\infty) \times E^* \to L_0(\Omega, \mathcal{F}, \mathbb{P})$ is called a **cylindrical Wiener process** if for every $\phi_1, \dots, \phi_n$ and $n \in \mathbb{N}$ the stochastic process

$$(W_t(\phi_1), \dots, W_t(\phi_n))_{t\geq 0}$$

is a Wiener process in $\mathbb{R}^n$.

The characteristic function of a cylindrical Wiener process, $\widehat{\mathbb{P}}_{W(t)} : E^* \to \mathbb{C}$, is given by

$$\widehat{\mathbb{P}}_{W(t)}(\phi) = e^{-\frac{1}{2}t\langle Q(\phi),\phi\rangle},$$

where $Q \in L(E^*, E)$ is positive and symmetric, i.e., for every $\phi, \psi \in E^*$ we have $\langle Q(\phi), \phi\rangle = \phi(Q(\phi)) \geq 0$ and $\langle Q(\phi), \psi\rangle = \langle Q(\psi), \phi\rangle$, respectively. Such an operator Q is called the covariance operator of the cylindrical Wiener process W.

Remark 3.3.2. Let $E = H$ be a Hilbert space and by Riesz representation theorem, we identify H with H^*. Suppose that $Q \in L(H)$ is positive and symmetric. Since Q is positive there exists a self-adjoint square root $R \in L(H)$ such that $RR = Q$. We have that for $x \in H$

$$\widehat{\mathbb{P}}_{W(t)}(x) = e^{-\frac{1}{2}t\langle Qx,x\rangle} = e^{-\frac{1}{2}t\|Rx\|^2}$$

is the characteristic function of a proper H-valued (Wiener) process if and only if R is Hilbert-Schmidt (iff Q is trace class). In other words, R is Hilbert-Schmidt if and only if $\mathbb{P}_{W(t)}$ is Radon for all $t \geq 0$. This was shown in [Lin86] page 55 in a more general setting where R is required to be *Gauss-Radonifying* - a generalization of Hilbert-Schmidt maps.

Definition 3.3.3. Let H be a Banach space isomorphic to a Hilbert space. A σ-finite Radon measure ν is called a **Lèvy measure** if

(a) $\nu(0) = 0$,

(b) $\int_H \min(1, \|x\|^2) d\nu(x) < \infty$.

Definition 3.3.4. Let ν be a cylindrical measure on $\mathcal{C}(E)$. We call ν a **cylindrical Lévy measure** if for all $\phi \in E^*$ we have ν_ϕ is a Lévy measure on $\mathcal{B}(\mathbb{R})$, where

$$\nu_\phi(B) = \nu(\{x \in E \mid \phi(x) \in B\})$$

for $B \in \mathcal{B}(\mathbb{R})$.

An adapted $\mathbb{R}^n$-valued process $(X_t)_{t\geq 0}$ is called a Lévy process if $X_0 = 0$ a.s., the increments are independent and stationary, and is continuous in probability, i.e., for every $t \geq 0$ and $\epsilon > 0$ we have $\lim_{s\to t} \mathbb{P}[|X_s - X_t| > \epsilon] = 0$.

Definition 3.3.5. A cylindrical process $L : [0,\infty) \times E^* \to L_0(\Omega, \mathcal{F}, \mathbb{P})$ is called a **cylindrical Lévy process** if for every $\phi_1, \dots, \phi_n$ and $n \in \mathbb{N}$ the stochastic process

$$(L_t(\phi_1), \dots, L_t(\phi_n))_{t\geq 0}$$

is a Lèvy process in $\mathbb{R}^n$.

As shown in [AR10], we get that $\widehat{\mathbb{P}}_{L(t)} : E^* \to \mathbb{C}$, the characteristic function of $L(t)$, is given by

$$\begin{aligned}&\hat{\mathbb{P}}_{L(t)}(\phi)\\&= \exp\left[t\left(im(\phi) - \frac{1}{2}\langle Q(\phi), \phi\rangle + \int_E \left(e^{i\phi(x)} - 1 - i\phi(x)1_{B_{\mathbb{R}}}(\phi(x)\right)\nu(dx)\right)\right].\end{aligned}$$

where $m : E^* \to \mathbb{R}$ is continuous, $Q \in L(E^*, E)$ is positive and symmetric, and ν is a cylindrical Lévy measure.

PART II

Main

CHAPTER 4

Nonstandard Stochastic Analysis in Finite Dimensions

4.1 Hyperfinite processes

Here we follow [Lin80]. Let S be a set including all entities we need such as $\mathbb{R}$ and relevant measure spaces. Let $U(S)$ be the superstructure over S and we let ${}^*U(S)$ be our polysaturated model. See Definition 2.5.4. Let $(\Omega, \mathcal{F}, P)$ be a hyperfinite probability space. A **hyperfinite timeline** $\mathbb{T}$ is a hyperfinite subset of ${}^*\mathbb{R}$ such that $0 \in \mathbb{T}$ and for each $x \in \mathbb{R}$ there is a $t \in \mathbb{T}$ such that $x \approx t$.

Example 4.1.1. Let $N \in {}^*\mathbb{N}$ be an infinite integer. Then the following is a timeline

$$\mathbb{T} = \left\{0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N^2-1}{N}, N\right\}.$$

Recall that $X : \Omega \to {}^*\mathbb{R}$ is $\mathcal{F}$-measurable if $X^{-1}(B) \in \mathcal{F}$ for every *open set $B \in {}^*\mathcal{T}$ where $\mathcal{T}$ is the topology on $\mathbb{R}$.

Definition 4.1.2. Let $(\Omega, \mathcal{F}, P)$ be a hyperfinite probability space and $\mathbb{T}$ a timeline. A **hyperfinite stochastic process** is an internal map $X : \mathbb{T} \times \Omega \to {}^*\mathbb{R}$ such that

$$\omega \mapsto X(t, \omega)$$

is $\mathcal{F}$-measurable.

Fix $t \in \mathbb{T}$. The expectation of X_t is given by

$$E[X_t] = \sum_{\omega \in \Omega} X_t(\omega) P(\{\omega\}).$$

Let $0 = t_0 < t_1 < \cdots < t_N = t_\infty$ be the elements of $\mathbb{T}$. An increment of a process X is of the form $\triangle X_{t_i} = X_{t_{i+1}} - X_{t_i}$. And if $s = t_i,\ t = t_j$ for $i < j$ we write

$$\sum_{r=s}^{t} X_r$$

for the sum

$$\sum_{n=i}^{j-1} X_{t_n},$$

and note that $X_t = X_{t_j}$ is not included in the sum.

Definition 4.1.3. $X : \mathbb{T} \times \Omega \to {}^*\mathbb{R}$ be a hyperfinite process. Its **quadratic variation** $[X] : \mathbb{T} \times \Omega \to {}^*\mathbb{R}^d$ is defined to be

$$[X](t,\omega) = \sum_{s=0}^{t} (\triangle X(s,\omega))^2.$$

Definition 4.1.4. Let $\mathbb{T}$ be a hyperfinite timeline, Ω a hyperfinite set, $(\mathcal{F}_t)_{t\in\mathbb{T}}$ an increasing sequence of algebras of subsets of Ω, and let P be a probability measure on $\mathcal{F}_{t_\infty}$ where t_∞ is the largest element in $\mathbb{T}$. We then call $(\Omega, (\mathcal{F})_{t\in\mathbb{T}}, P)$ an **internal basis**.

We say that a hyperfinite process $X : \mathbb{T} \times \Omega \to {}^*\mathbb{R}$ is **adapted** to an internal basis $(\Omega, (\mathcal{F})_{t\in\mathbb{T}}, P)$ if for each $t \in \mathbb{T}$ we have that

$$\omega \mapsto X(t,\omega)$$

is $\mathcal{F}_t$-measurable.

Definition 4.1.5. A hyperfinite process $M : \mathbb{T} \times \Omega \to {}^*\mathbb{R}$ is called a **hyperfinite martingale** with respect to $(\Omega, (\mathcal{F})_{t\in\mathbb{T}}, P)$ if M is adapted to $(\Omega, (\mathcal{F})_{t\in\mathbb{T}}, P)$ and for all $s, t \in \mathbb{T}$, $s < t$ and all $A \in \mathcal{F}_s$ we have

$$E\left[1_A(M_t - M_s)\right] = 0.$$

Equivalently, one can define martingales to be adapted and that

$$E[M_t \mid \mathcal{F}_s] = M_s$$

for all $t \geq s$. If we instead have that $E\left[1_A(M_t - M_s)\right] \geq 0$ we call M a **submartingale**, and with the opposite inequality, $E\left[1_A(M_t - M_s)\right] \leq 0$, we then call M a **supermartingale**.

Definition 4.1.6. Let $M : \mathbb{T} \times \Omega \to {}^*\mathbb{R}$ be a hyperfinite martingale with respect to $(\Omega, (\mathcal{F})_{t\in\mathbb{T}}, P)$. We call M a λ^2**-martingale** if for every $t \in \mathbb{T}$ $E[M_t^2]$ is finite.

Lemma 4.1.7. *For a hyperfinite process $X : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ we have*

$$[X](t) = X(t)^2 - X(0) - 2\int_0^t X dX.$$

Proof. This proof is similar to the proof of Proposition 17. in [Lin80]. First note that we have the following equalities

$$X_{t_k} = X_0 + \sum_{i=0}^{k-1} \triangle X_{t_k}$$

and

$$\begin{aligned} 2\sum_{i=0}^{k-1}\sum_{j=0}^{i-1} \triangle X_{t_i} \triangle X_{t_j} &= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \triangle X_{t_i} \triangle X_{t_j} - \sum_{i=0}^{k-1} (\triangle X_{t_i})^2 \\ &= \left(\sum_{i=0}^{k-1} \triangle X_{t_i}\right)^2 - \sum_{i=0}^{k-1} (\triangle X_{t_i})^2 \\ &= \left(\sum_{i=0}^{k-1} \triangle X_{t_i}\right)^2 - [X](t_k) \end{aligned}$$

Hence we get

$$\begin{aligned}
X_{t_k}^2 - X_0^2 - 2\int_0^{t_i} XdX &= \left(X_0 + \sum_{i=0}^{k-1} \triangle X_{t_k}\right)^2 - X_0^2 - 2\sum_{i=0}^{k-1} X_{t_i}\triangle X_{t_i} \\
&= X_0^2 + 2X_0\sum_{i=0}^{k-1}\triangle X_{t_i} + \left(\sum_{i=0}^{k-1}\triangle X_{t_i}\right)^2 - X_0^2 \\
&\quad - 2\sum_{i=0}^{k-1}\left(X_0 + \sum_{j=0}^{i-1}\triangle X_{t_j}\right)\triangle X_{t_i} \\
&= \left(\sum_{i=0}^{k-1}\triangle X_{t_i}\right)^2 - 2\sum_{i=0}^{k-1}\sum_{j=0}^{i-1}\triangle X_{t_j}\triangle X_{t_i} \\
&= \left(\sum_{i=0}^{k-1}\triangle X_{t_i}\right)^2 - \left(\sum_{i=0}^{k-1}\triangle X_{t_i}\right)^2 + [X](t_k) \\
&= [X](t_k).
\end{aligned}$$

■

Remark 4.1.8. If X is a martingale then $\int XdX$ is a martingale starting at zero and hence has zero expectation, thus using the formula we just proved we get

$$E[X_t^2] = E[X_0^2 + [X](t)].$$

Consequently, if X is a martingale and $E[X_0^2 + [X](t)]$ is finite for every $t \in \mathbb{T}$ then X is moreover a λ^2-martingale.

Proposition 4.1.9 (Doob's Inequality)**.** *If* $X : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ *is a positive submartingale then for all* $p > 1$ *and all* $t \in \mathbb{T}$

$$\|\sup_{s\leq t} X_s\|_p \leq \frac{p}{p-1}\|X_t\|_p$$

where $\|\cdot\|_p$ *denotes the* L^p *norm.*

Proof. See [Alb+86]. ■

4.2 Martingale integration

Let $(X, \mathcal{A}, \nu)$ be an internal measure space and let $(X, \mathcal{A}_L, \nu_L)$ be the corresponding Loeb space.

Definition 4.2.1. A function $f : X \to {}^*\mathbb{R}$ is **S-integrable** if it is $\mathcal{A}$-measurable and

(a) $\int |f|d\nu$ is finite

(b) If $A \in \mathcal{A}$ and $\nu(A) \approx 0$, then $\int_A |f|d\nu \approx 0$

(c) If $A \in \mathcal{A}$ and $f(A) \subseteq \mu(0)$, then $\int_A |f|d\nu \approx 0$

In [And76] Anderson proved that f is S-integrable if and only if $^\circ f$ is ν_L-integrable and

$$^\circ\left(\int |f| d\nu\right) = \int |^\circ f| d\nu_L.$$

Definition 4.2.2. For $1 \le p < \infty$ let $SL^p(X, \mathcal{A}, \nu)$ be the collection of equivalence classes of all $f : X \to {}^*\mathbb{R}$ such that f is $\mathcal{A}$-measurable and $|f|^p$ is S-integrable under the equivalence relation

$$f_1 \sim f_2 \iff \left(\int |f_1 - f_2|^p d\nu\right)^p \approx 0.$$

We can define a norm on $SL^p(X, \mathcal{A}, \nu)$ by

$$\|f\| = {}^\circ\left(\int |f|^p d\nu\right)^{1/p}.$$

We say that $f : X \to {}^*\mathbb{R}$ is **square S-integrable** if $f \in SL^2(X, \mathcal{A}, \nu)$. In [And76] it was shown that

(a) if $f : X \to {}^*\mathbb{R}$ is $\mathcal{A}$-measurable, then

$$f \in SL^p(X, \mathcal{A}, \nu) \iff {}^\circ f \in L^p(X, \mathcal{A}_L, \nu_L) \text{ and } \|f\|_p = \|{}^\circ f\|_p.$$

(b) If $g : X \to \mathbb{R}$ belongs to $L^p(X, \mathcal{A}_L, \nu_L)$ then there exists a unique $f \in SL^p(X, \mathcal{A}, \nu)$ such that $^\circ f = g$ ν_L-a.e.

By (b) we get that the standard part map $^\circ : SL^p(X, \mathcal{A}, \nu) \to L^p(X, \mathcal{A}_L, \nu_L)$ is surjective. By (a) we see that the standard part map is an isometry. Consequently, $SL^p(X, \mathcal{A}, \nu)$ and $L^p(X, \mathcal{A}_L, \nu_L)$ are isometrically isomorphic.

We are now ready to introduce martingale integration which was done in [Lin80].

Let $X, Y : \mathbb{T} \times \Omega \to {}^*\mathbb{R}$ be two hyperfinite processes. We define $\int X dY$ the stochastic integral of X with respect to Y by

$$(t, \omega) \mapsto \sum_{s=0}^{t} X(s, \omega) \triangle Y(s, \omega)$$

We will look at stochastic integrals $\int X dM$ where M is a hyperfinite martingale with respect to $(\Omega, (\mathcal{F}_t)_t, P)$ and where X is adapted to the same basis.

For $t \in \mathbb{T}$ let us define $\mathbb{T}_t = \mathbb{T} \cap {}^*[0, t)$ and let $\mathcal{T}_t$ be the internal power set of $\mathbb{T}_t$. Suppose $M : \mathbb{T} \times \Omega \to {}^*\mathbb{R}$ is a λ^2-martingale with respect to $(\Omega, (\mathcal{F}_t), P)$. We define an internal measure ν_{M_t} on $(\mathbb{T}_t \times \Omega, \mathcal{T}_t \times \mathcal{F}_{t_\infty})$ by

$$\nu_{M_t}((s, \omega)) = \triangle M(s, \omega)^2 P(\omega).$$

Note that

$$\nu_{M_t}(\mathbb{T}_t \times \Omega) = \sum_{\omega \in \Omega} \sum_{s=0}^{t} \triangle M(s, \omega)^2 P(\{\omega\}) = E[[M](t)].$$

Now let $X|_t = X|_{\mathbb{T}_t \times \Omega}$ denote the restriction of X to $\mathbb{T}_t \times \Omega$. We make the following definition

Definition 4.2.3. Let $M : \mathbb{T} \times \Omega \to {}^*\mathbb{R}$ be a λ^2-martingale with respect to $(\Omega, (\mathcal{F}_t)P)$ and let X be a process adapted to the same basis. We say that $X \in SL^2(M)$ if $X|_t \in SL^2(\mathbb{T}_t \times \Omega, \mathcal{T}_t \times \mathcal{F}_{t_\infty}, \nu_{M_t})$ for all finite $t \in \mathbb{T}$.

Hence the set $SL^2(M)$ is the *right* collection of integrands when we integrate with respect to a λ^2-martingale.

4.3 Hyperfinite Lévy processes

In this section we define Hyperfinite Lévy processes as in [Lin04].

Let $N \in {}^*\mathbb{N}$ be an infinite integer. We will work with a timeline

$$\mathbb{T} = \left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N^2-1}{N}, N\right\}$$

which replaces the interval $[0, \infty)$ if N is infinite. We will write $\triangle t := \frac{1}{N}$ to denote the positive infinitesimal to emphasize it being a time step. We have that $\mathbb{T}$ is hyperfinite with internal cardinality equal to $|\mathbb{T}| = N^2 + 1$. See Example 2.3.6 for details. For a hyperfinite set $A = \{a_i\}_{i \in I} \subseteq {}^*\mathbb{R}^n$ we choose some probabilities $\{p_a\}_{a \in A}$ in ${}^*\mathbb{R}$ satisfying $\sum_{a \in A} p_a = 1$ and $p_a > 0$. We let $\Omega = A^{\mathbb{T}}$ which is a hyperfinite set with internal cardinality $|A|^{|\mathbb{T}|} = |A|^{N^2+1}$. We wish to consider internal processes $X : \Omega \times \mathbb{T} \to {}^*\mathbb{R}^n$ and we denote its increments by $\triangle X(t) := X(\triangle t + t) - X(t)$ for $t \in \mathbb{T}$.

Definition 4.3.1. A **hyperfinite random walk** with increments A and transition probabilities $\{p_a\}_{a \in A}$ is an internal map $X : \Omega \times \mathbb{T} \to {}^*\mathbb{R}^n$ such that

(a) $X(0) = 0$.

(b) The increments $\{\triangle X(t)\}_{t \in \mathbb{T}}$ are *-independent.

(c) All increments $\triangle X(t)$ have distribution

$$\mathbb{P}[\{\omega \mid \triangle X(\omega, t) = a\}] = p_a$$

for all $t \in \mathbb{T}$ and all $a \in A$.

Example 4.3.2.

(a) Let $A = \{-\sqrt{\triangle t}, \sqrt{\triangle t}\}$ and assign probabilities $p_{-\sqrt{\triangle t}} = p_{\sqrt{\triangle t}} = \frac{1}{2}$. This is called the Anderson random walk.

(b) Let α be a real number, let $A = \{0, 1\}$ and put $p_0 = 1 - \alpha \triangle t$ and $p_1 = \alpha \triangle t$. This is a hyperfinite Poisson process with rate α.

Definition 4.3.3. A hyperfinite random walk $L : \Omega \times \mathbb{T} \to {}^*\mathbb{R}^n$ is a **hyperfinite Lévy process** if the set

$$\{\omega \mid L(\omega, t) \text{ is finite for all } t \in \mathbb{T}\}$$

has Loeb measure 1.

Before we give the characterization result from [Lin04] we need to define the following hyperreal number

$$q_k = \frac{1}{\triangle t} \sum_{|a|>k} p_a.$$

Theorem 4.3.4. *A hyperfinite random walk $L : \Omega \times \mathbb{T} \to {}^*\mathbb{R}^n$ is a Lévy process if and only if the following three conditions are satisfied:*

(a) $\frac{1}{\triangle t} \sum_{|a|\leq k} ap_a$ *is finite for all finite and noninfinitesimal $k \in {}^*\mathbb{R}$.*

(b) $\frac{1}{\triangle t} \sum_{|a|\leq k} |a|^2 p_a$ *is finite for all finite $k \in {}^*\mathbb{R}$.*

(c) $\lim_{k\to\infty} {}^\circ q_k = 0$ *in the sense that for every $\epsilon \in \mathbb{R}_+$ there is an $N \in \mathbb{N}$ such that $q_k < \epsilon$ when $k \geq N$.*

Proof. See [Lin04]. ∎

CHAPTER 5

Nonstandard Functional Analysis

5.1 Linear spaces and operators

Here we follow a combination of [Alb+86], [LW15] and [Gol14].

In this section let $U(X)$ be the superstructure where X contains the normed linear spaces we want to consider and the scalar field $\mathbb{F}$ equal to $\mathbb{R}$ or $\mathbb{C}$. Recall that $G \in U({}^*(X))$ is *internal* if $G \in {}^*F$ for some $F \in U(X)$.

Now suppose $\mathcal{F} \subset U_n(X)$ for some $n \geq 1$ and assume that all $F \in \mathcal{F}$ are normed linear spaces. By definition, we have that every $G \in {}^*\mathcal{F}$ is internal. By the transfer principle we get that G is an internal vector space over ${}^*\mathbb{F}$ equipped with an internal function $|| \cdot || : G \to {}^*\mathbb{R}$ satisfying the axioms of a norm. Note that it is not a proper norm as $\| \cdot \|$ takes values in ${}^*\mathbb{R}$ and not $\mathbb{R}$. If $\mathcal{F}$ is a collection of Banach spaces then, by transfer again, G is complete in the following sense:

If $(x_n)_{n \in {}^*\mathbb{N}}$ is an internal Cauchy sequence then there exists $x \in G$ such that for each $\epsilon > 0$ in ${}^*\mathbb{R}$ there is an n_ϵ in ${}^*\mathbb{N}$ such that

$$||x - x_n|| < \epsilon \quad \textit{for } n \in {}^*\mathbb{N}, n \geq n_\epsilon.$$

Let $(E, || \cdot ||)$ be a normed linear space over a scalar field $\mathbb{F}$. For $x = [x_n]_n \in {}^*E$ we mean $[||x_n||]_n$ when we write $||x||$. We obtain these corresponding definitions with respect to the norm.

Definition 5.1.1.

(a) We call $x \in {}^*E$ **nearstandard** to $y \in E$ if $||x - y|| \approx 0$. We write $\mathrm{Ns}({}^*E)$ for the set of nearstandard elements in ${}^*E$.

(b) We call $x \in {}^*E$ **(norm)finite** if $||x||$ is a finite hyperreal. We write $\mathrm{Fin}({}^*E)$ for the set of finite elements in ${}^*E$.

(c) We call $x \in {}^*E$ **infinitesimal** if $||x|| \approx 0$.

One can show that if $E = \mathbb{F}$ then $\mathrm{Fin}({}^*\mathbb{F}) = \mathrm{Ns}({}^*\mathbb{F})$. Actually, this holds for every finite-dimensional normed space. But in infinite dimensions, this does not hold in general. Let us prove this, but first, we start with a lemma.

Lemma 5.1.2. *Suppose $x_1, \ldots, x_n \in E$ are linearly independent and $\alpha_1, \ldots, \alpha_n \in {}^*\mathbb{F}$ are such that $\alpha_1 x_1 + \ldots, \alpha_n x_n \in Fin({}^*E)$. Then $\alpha_i \in Fin({}^*\mathbb{F})$ for every $i = 1, \ldots, n$.*

Proof. This proof is from [Gol14] Without loss of generality we may assume that $\max(|\alpha_1|, \cdots, |\alpha_n|) = |\alpha_1|$. Assume for contradiction that α_1 is *not* finite. Since $\frac{1}{|\alpha_1|} \approx 0$ and $\|\alpha_1 x_1 + \cdots + \alpha_n x_n\|$ is finite we get that

$$\left\| x_1 + \frac{\alpha_2}{\alpha_1} x_2 + \dots \frac{\alpha_n}{\alpha_1} x_n \right\| = \frac{1}{|\alpha_1|} \|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \approx 0.$$

Since $|\alpha_i| \leq |\alpha_1|$ for $i = 1, \dots, n$ we have that ${}^\circ(\alpha_i/\alpha_1)$ exists and

$$\begin{aligned}
&\left\| x_1 + \frac{\alpha_2}{\alpha_1} x_2 + \cdots + \frac{\alpha_n}{\alpha_1} x_n - \left(x_1 + {}^\circ\!\left(\frac{\alpha_2}{\alpha_2}\right) x_2 + \cdots + {}^\circ\!\left(\frac{\alpha_n}{\alpha_1}\right) x_n \right) \right\| \\
&\leq \left| \frac{\alpha_2}{\alpha_1} - {}^\circ\!\left(\frac{\alpha_2}{\alpha_1}\right) \right| \|x_2\| + \cdots + \left| \frac{\alpha_n}{\alpha_1} - {}^\circ\!\left(\frac{\alpha_n}{\alpha_1}\right) \right| \|x_n\| \\
&\approx 0
\end{aligned}$$

Hence

$$0 \approx x_1 + \frac{\alpha_2}{\alpha_1} x_2 + \dots \frac{\alpha_n}{\alpha_1} x_n \approx x_1 + {}^\circ\!\left(\frac{\alpha_2}{\alpha_2}\right) x_2 + \cdots + {}^\circ\!\left(\frac{\alpha_n}{\alpha_1}\right) x_n$$

But then we must have

$$x_1 + {}^\circ\!\left(\frac{\alpha_2}{\alpha_2}\right) x_2 + \cdots + {}^\circ\!\left(\frac{\alpha_n}{\alpha_1}\right) x_n = 0$$

which contradicts that $x_1, \dots, x_n$ are linearly independent. ■

Proposition 5.1.3. *If E is finite dimensional then* $\mathrm{Fin}(^*E) = \mathrm{Ns}(^*E)$.

Proof. This proof is from [Gol14]. Let $(x_1, \dots, x_n)$ be a basis for E. Clearly $\mathrm{Ns}(^*E) \subseteq \mathrm{Fin}(^*E)$. By transfer every element $x \in {}^*E$ is of the form $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$ where $\alpha_i \in {}^*\mathbb{F}$ for $i = 1, \dots, n$. Now if $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$ is finite then by previous lemma we have that $\alpha_i \in \mathrm{Fin}(^*\mathbb{F}) = \mathrm{Ns}(^*\mathbb{F})$. Let

$$x' = {}^\circ\alpha_1 x_1 + \cdots + {}^\circ\alpha_n x_n.$$

Then

$$\begin{aligned}
||x - x'|| &= ||\alpha_1 x_1 + \dots \alpha_n x_n - ({}^\circ\alpha_1 x_1 + \cdots + {}^\circ\alpha_n x_n)|| \\
&\leq |\alpha_1 - {}^\circ\alpha_1| ||x_1|| + \cdots + |\alpha_n - {}^\circ\alpha_n| ||x_n|| \\
&\approx 0.
\end{aligned}$$

Hence ${}^\circ x = x'$ and thus $x \in \mathrm{Ns}(^*E)$. ■

Proposition 5.1.4. $\mathrm{Fin}(^*E)$ *is a vector space over* $\mathbb{F}$.

Proof. ★ Let $x, y \in \mathrm{Fin}(^*E)$. Then there are finite hyperreals K_1, K_2 such that $||x|| = K_1$ and $||y|| = K_2$. But then

$$||x + y|| \leq ||x|| + ||y|| = K_1 + K_2$$

is finite as well hence $x + y \in \mathrm{Fin}(^*E)$. Now let $\alpha \in \mathbb{F}$. Then $||\alpha x|| = |\alpha| K_1$ which is finite and hence $\alpha x \in \mathrm{Fin}(^*E)$. ■

We call the quotient space $\tilde{E} := \mathrm{Fin}({}^*E)/\approx$ the **nonstandard hull** of E. On page 54 in [Alb+86] it is proved that a nonstandard hull is complete; hence, $\hat{E}$ is a Banach space over the scalar field $\mathbb{F}$.

Now let E be a normed linear space in our superstructure and let $\mathcal{F}$ denote the class of all finite-dimensional subspaces of E. We have a map $dim : \mathcal{F} \to \mathbb{N}$ where $dim\ F$ denotes the dimension of $F \in \mathcal{F}$. By transfer we obtain an object ${}^*\mathcal{F} \in U({}^*X)$ and a map ${}^*dim : {}^*\mathcal{F} \to {}^*\mathbb{N}$. If $F \in {}^*\mathcal{F}$ and ${}^*dim(F) = \gamma \in {}^*\mathbb{N}$ it follows by transfer that there is an internal sequence $(e_n)_{n\leq\gamma} \subseteq {}^*E$ such that every $x \in F$ can be written as

$$x = \sum_{n=1}^{\gamma} \alpha_n e_n$$

for some $(\alpha_n)_{n\leq\gamma} \subset {}^*\mathbb{F}$. We call the space F a **hyperfinite-dimensional** linear space.

Proposition 5.1.5. *Let E be a normed linear space. Then there is an $F \in {}^*\mathcal{F}$ such that*

$$E \subseteq F \subseteq {}^*E.$$

When we write $E \subseteq F$ we mean ${}^*x \in F$ for all $x \in E$.

Proof. This is a detailed version of the proof in given in [Alb+86] page 55. For each $x \in E$ define

$$A^x = \{F \in {}^*\mathcal{F} \mid {}^*x \in F\}.$$

By Definition 2.7.1 we have that each $F \in {}^*\mathcal{F}$ is internal and thus $F = [F_n]$. Hence we can write $A^x = [A^x_n]$ where $A^x_n = \{F_n \in \mathcal{F} \mid x \in F_n\}$. Consequently, $(A_x)_{x\in E}$ is a collection of internal sets. Moreover, the sets have the finite intersection property. Indeed, the intersection of $A_{x_1}, \ldots, A_{x_n}$ will at least contain the subspace generated by ${}^*x_1, \ldots, {}^*x_n$. Hence by saturation Definition 2.5.3 there is some $F \in {}^*\mathcal{F}$ such that ${}^*x \in F$ for every $x \in E$. ■

Suppose that $E = H$ is a separable Hilbert space with an orthonormal basis $(e_n)_{n\in\mathbb{N}}$. We can then extend this basis to an internal orthonormal basis $(e_n)_{n\in{}^*\mathbb{N}}$ in ${}^*H$ where $e_n \in {}^*H$ is nearstandard to $e_n \in H$ for $n \in \mathbb{N}$. In particular, if $\gamma \in {}^*\mathbb{N}$ is infinite then the ${}^*\mathbb{F}$-linear span of $\{e_1, \ldots, e_\gamma\}$ becomes a hyperfinite-dimensional space $H \subset U \subset {}^*H$.

Proposition 5.1.6. *For a separable Hilbert space H, let $H \subseteq U \subseteq {}^*H$ be a hyperfinite dimensional linear space with internal orthonormal basis $(e_n)_{n=1}^{\gamma}$. If $x = \sum_{n=1}^{\gamma} x_n e_n$ then $x \in \mathrm{Ns}(F)$ if and only if $\|x\| \in \mathrm{Fin}(\mathbb{F}) = \mathrm{Ns}(\mathbb{F})$ and*

$$\sum_{n=\theta}^{\gamma} x_n^2 \approx 0$$

for every infinite $\theta \leq \gamma$. In this case

$$^\circ x = \sum_{n\in\mathbb{N}} {}^\circ x_n e_n.$$

Proof. [Gol14] page 76. ■

Definition 5.1.7. An element $x = \sum_{n=1}^{\gamma} x_n e_n \in U$ is called **remote** if

$$\sum_{n=1}^{K} x_n^2 \approx 0$$

for every *finite* K. We denote the space of remote elements by $r(U)$.

By overflow (c.f. Theorem 2.6.10) we have that $\sum_{n=1}^{K} x_n^2$ will be an infinitesimal for an infinite K as well.

Lemma 5.1.8 (Tom Lindstrøm)**.** *An element $x \in U$ is infinitesimal if and only if $x \in Ns(U) \cap r(U)$.*

Proof. First assume that $x \in \mu(0)$. Then $\|x\| \approx 0$. Using Bessel's inequality we get that

$$\sum_{n=K}^{\gamma} x_n^2 \leq \|x\|^2 \approx 0$$

and

$$\sum_{n=1}^{K} x_n^2 \leq \|x\|^2 \approx 0$$

for every K. Thus x is both nearstanadard and remote.

Now assume that x is both nearstandard and remote. Since $x \in r(U)$ we have that

$$\left\{ K \in {}^*\mathbb{N} \mid \sum_{n=1}^{K} x_n^2 \leq \frac{1}{K} \right\}$$

contains every finite K and hence by overflow it also contains an infinite K. For such infinite K we have

$$\|x\|^2 = \sum_{n=1}^{K} x_n^2 + \sum_{n=K+1}^{\gamma} x_n^2 \approx 0$$

since both terms are infinitesimal by assumption. ■

We now introduce a decomposition of normfinite elements

Lemma 5.1.9 (Tom Lindstrøm)**.** *Let $x \in U$ and $\|x\| \in Fin({}^*\mathbb{R})$. Then there exist $x_1 \in Ns(U)$ and $x_2 \in r(U)$ such that $x_1 \perp x_2$ and $x = x_1 + x_2$. If $x = \tilde{x}_1 + \tilde{x}_2$ for another such decomposition then $x_1 \approx \tilde{x}_1$ and $x_2 \approx \tilde{x}_2$.*

Proof. Let $x = \sum_{n=1}^{\gamma} x_n e_n$ and set

$$a = {}^{\circ}\lim_{K \to \infty} \left(\sum_{n=1}^{K} x_n^2 \right) \leq \|x\|^2.$$

The Internal set

$$\left\{ K \in {}^*\mathbb{N} \mid \sum_{n=1}^{K} x_n^2 \leq a + \frac{1}{K} \right\}$$

contains every standard K and thus by overflow also an infinite K. For such K we define $x_1 = \sum_{n=1}^{K} x_n e_n$ and $x_2 = \sum_{n=K+1}^{\gamma} x_n e_n$. Then obviously $x_1 \perp x_2$,

$x = x_1 + x_2$ and $x_2 \in r(U)$. Now we show that $x_1 \in \mathrm{Ns}(U)$. Note that for every infinite $N \leq K$ we have that

$$^\circ\left(\sum_{n=1}^{N} x_n^2\right) = a,$$

and thus

$$\sum_{n=N}^{K} x_n^2 = \sum_{n=1}^{K} x_n^2 - \sum_{n=1}^{N} x_n^2 \approx 0$$

since both terms are infinitely close to a.

Now suppose $x = \tilde{x}_1 + \tilde{x}_2$ for another such decomposition. Since $x_1 + x_2 = \tilde{x}_1 + \tilde{x}_2$ put $s = x_1 - \tilde{x}_1 = x_2 - \tilde{x}_2$. Since $x_1 - \tilde{x}_1$ is nearstandard and $x_2 - \tilde{x}_2$ is remote we get that s is both nearstandard and remote and hence by Lemma 5.1.8 it must be an infinitesimal. ■

Linear operators

Now let F and G be two internal normed spaces. We denote the set of all internal ${}^*\mathbb{F}$-linear maps $T : F \to G$ by $L(F, G)$ and $L(F) \coloneqq L(F, F)$.

Definition 5.1.10. We call $T \in L(F, G)$ **S-bounded** if

$$||T|| = \sup\{||Tx|| \mid ||x|| \leq 1,\, x \in F\}$$

belongs to $\mathrm{Fin}(\mathbb{R}) = \mathrm{Ns}(\mathbb{R})$.

Note that the supremum of an internal bounded subset of ${}^*\mathbb{R}$ exists by transfer.

Definition 5.1.11. We call $T \in L(F, G)$ **S-continuous** if $||x - y|| \approx 0$ implies $||T(x) - T(y)|| \approx 0$.

We will now present a result that can be found as Proposition 4.2.22 in [LW15] which shows that S-boundedness is equivalent to other nice properties.

Proposition 5.1.12. *For $T \in L(F, G)$ the following are equivalent:*

(a) *T is S-continuous*

(b) *T is S-bounded*

(c) *For some standard M we have $||Tx|| \leq M||x||$ for all $x \in F$.*

(d) *$T(Fin(F)) \subseteq Fin(G)$*

Proof.

1. (a) $\implies$ (b): Suppose that T is not S-bounded. Then there exists $x \in F$ with $\|x\| = 1$ such that $\|Tx\|$ is infinite. Define $y = \frac{x}{\|Tx\|}$. Then $\|y\| = \frac{1}{\|Tx\|} \approx 0$. But $\|Ty\| = \frac{\|Tx\|}{\|Tx\|} = 1$. Hence T is not S-continuous.

2. (b) $\implies$ (d): Assume T is S-bounded and let $x \in \mathrm{Fin}(F)$. We have that $||Tx|| \leq ||T||||x||$ and since $||T||||x||$ is a finite hyperreal number we get that $Tx \in \mathrm{Fin}(G)$.

3. ★(d) $\implies$ (b): Suppose T is not bounded. Then there exists $x \in \mathrm{Fin}(F)$ with $\|x\| = 1$ such that $\|Tx\|$ is infinite. But then $Tx \notin \mathrm{Fin}(G)$.

4. ★(c) $\implies$ (a): Let $M \in \mathbb{R}$ be such that $\|Tx\| \leq M\|x\|$ for all $x \in F$. Then for any $\|x - y\| \approx 0$ we get $\|T(x) - T(y)\| \leq M\|x - y\| \approx 0$. Thus T is S-continuous.

5. ★(b) $\implies$ (c): Since $\|T\| \in \mathrm{Fin}({}^*\mathbb{R}) = \mathrm{Ns}({}^*\mathbb{R})$ we can choose some standard positive $M \geq \|T\|$. Hence $\|Tx\| \leq M\|x\|$.

■

Let E and F be *standard* Banach spaces and we denote by $\mathcal{B}(E, F)$ the space of bounded linear operators from E to F. Note that

$$T = [T_n] \in {}^*\mathcal{B}(E, F) \iff \{n \in \mathbb{N} \mid T_n \in \mathcal{B}(E, F)\} \in \mathcal{F}$$

where $\mathcal{F}$ is a nonprincipal ultrafilter on $\mathbb{N}$.

Proposition 5.1.13. ★ *Let $T : {}^*E \to {}^*F$ be internal and linear. If T is S-bounded then $T \in {}^*\mathcal{B}(E, F)$.*

Proof. ★ Since T is internal we have that $T = [T_n]$ for some sequence $T_n : E \to F$. As T is linear we must have that each T_n is linear. Since T is S-bounded there exists ${}^*M = [(M, M, \dots)]$ for $M \in \mathbb{R}$ such that $\|T(x)\| \leq {}^*M\|x\|$ for all $x = [x_n] \in {}^*E$. But then we have that

$$\{n \in \mathbb{N} \mid \|T_n(x_n)\| \leq M\|x_n\|\} \in \mathcal{F}.$$

And since x was arbitrary we have that x_n is arbitrary. Consequently, $T \in {}^*\mathcal{B}(E, F)$. ■

If we consider the hyperfinite-dimensional spaces $H_1 \subset U \subset {}^*H_1$ and $H_2 \subset V \subset {}^*H_2$ for some Hilbert spaces H_1, H_2 we have that:

Corollary 5.1.14. ★ *Let $T : U \to V$ be internal, linear, and S-bounded. Then T has an extension $\overline{T} \in {}^*\mathcal{B}(H_1, H_2)$.*

Proof. ★ Let us define $\overline{T} : {}^*H_1 \to {}^*H_2$ by letting $\bar{T}$ to be equal T on U and letting $\overline{T}$ be zero on the orthogonal complement

$$U^\perp = \{x \in {}^*H_1^* \mid \langle x, u\rangle = 0 \; \forall u \in U\}.$$

Note that $\overline{T}$ is linear and internal. It is also S-bounded since

$$\begin{aligned} \|\overline{T}\| &= \sup\{\|\overline{T}(x)\| \mid \|x\| = 1, x \in {}^*H_1\} \\ &= \sup\{\|T(u)\| \mid \|u\| = 1, u \in U\} \\ &= \|T\|. \end{aligned}$$

Then it follows by the proposition above. ■

5.2 Dual space, Riesz representation theorem, and adjoint operators

Let E be a Banach space and E^* denote its continuous dual, i.e., every linear and bounded $\phi : E \to \mathbb{F}$. Do not be confused with ${}^*E$, the nonstandard extension of E. We have that the nonstandard extension of E^* is

$$^*(E^*) = \{\phi : {}^*E \to {}^*\mathbb{F} \mid \phi = [\phi_i]_{i\in\mathbb{N}}, \phi_i \in E^*\}.$$

Consequently, each $\phi \in {}^*E^*$ is linear. Let

$$\mathrm{Fin}(E^*) = \{\phi \in {}^*E^* \mid \|\phi\| \in \mathrm{Fin}({}^*\mathbb{R}) = \mathrm{Ns}({}^*\mathbb{R})\},$$

i.e., the S-bounded maps in ${}^*E^*$. Consider $E \subset F \subset {}^*E$. For F we define F^* as a subset of ${}^*E^*$ in the following way:

$$F^* := \{\phi|_F \mid \phi \in {}^*E^*\}.$$

Let us call F^* the **hyperfinite-dimensional dual space** of F. Similarly, we let $\mathrm{Fin}(F^*) = \{\phi \in F^* \mid \phi \text{ S-bounded}\}$.

Now consider $H \subset U \subset {}^*H$, a hyperfinite dimensional space of an underlying Hilbert space H with inner product $\langle\cdot,\cdot\rangle$.

Theorem 5.2.1 (Riesz Representation Theorem)**.** ★ *Let U be a hyperfinite dimensional inner product space. Then for each $\phi \in U^*$ there exists a unique $y \in U$ such that*

$$\phi(x) = \phi_y(x) := \langle x, y\rangle$$

for all $x \in U$ satisfying $\|\phi_y\| = \|y\|$.

Proof. ★ For the underlying Hilbert space H we have by the standard Riesz representation theorem that there exists a unique $y \in H$ for each $\phi \in H^*$ such that $\phi = \phi_y$ and $\|\phi\| = \|y\|$. By the Transfer principle we have that for each $\phi \in {}^*H^*$ there exists a unique $y \in {}^*H$ such that $\phi = \phi_y$ and $\|\phi_y\| = \|y\|$. Now if $\phi \in U^*$ let us define it on all of ${}^*H^*$ by letting $\phi = 0$ on $U^\perp$, the orthogonal complement of U defined by

$$U^\perp = \{x \in {}^*H \mid \langle x, u\rangle = 0 \text{ for all } u \in U\}.^1$$

If $\phi \equiv 0$ then $\phi = \phi_0 = \langle\cdot, 0\rangle$. Now suppose $\phi \neq 0$. Let y be such that $\phi = \phi_y$. Then we can make the following decomposition $y = y^U + y^{U^\perp}$ where $y^U \in U$ and $y^{U^\perp} \in U^\perp$. We will show that the restriction on U satisfies $\phi_y|_U = \phi_{y^U}(x)$. If $x \in U$ then we get that

$$\begin{aligned}\phi_y(x) &= \langle x, y^U\rangle + \langle x, y^{U^\perp}\rangle \\ &= \langle x, y^U\rangle \\ &= \phi_{y^U}(x).\end{aligned}$$

■

[1] As explained in [SL76] page 271, hyperfinite dimensional spaces are closed.

Corollary 5.2.2. ★ *For a hyperfinite dimensional inner product space U we have*

$$U \simeq U^*.$$

Proof. ★ By similar arguments as in the standard case, we have that the map

$$\Phi : U \to U^*, \quad \Phi(y) = \phi_y = \langle \cdot, y \rangle$$

is an ${}^*\mathbb{F}$-*linear* isometry. The map Φ is obviously (complex) linear and from Riesz representation theorem we have that $\|\phi_y\| = \|y\|$ hence $\|\Phi(y)\| = \|y\|$. Note that since Φ is an isometry we get that Φ is injective and S-continuous: Suppose $x, y \in U$ and $\Phi(x) = \Phi(y)$. By linearity, we get that

$$0 = \|\Phi(x - y)\| = \|x - y\|.$$

Hence $x = y$. It is S-continuous since if $x \approx y$ then we get that

$$\|\Phi(x) - \Phi(y)\| = \|x - y\| \approx 0.$$

Riesz representation theorem gives us surjectivity of Φ. Let us define $\Phi^{-1} : U^* \to U$ by

$$\Phi^{-1}(\phi_y) = y.$$

Since we have that $\|\Phi^{-1}(\phi_y)\| = \|y\| = \|\phi_y\|$ it is easy to see that Φ^{-1} is S-continuous. Thus Φ is an isometric isomorphism (w.r.t. the S-topology) and consequently, we have $U \simeq U^*$. ■

Using our Riesz representation theorem we can establish the existence and uniqueness of an adjoint operator.

Lemma 5.2.3. ★ *Let $T \in L(U, V)$ be S-bounded. Then there exists a unique map $T^* \in L(V, U)$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x \in U, y \in V$. Moreover we have $\|T^*\| = \|T\|$ and consequently T^* is S-bounded.*

Proof. ★ For an S-bounded map $T \in L(U, V)$ we have from Corollary 5.1.14 that there is an extension $\overline{T} \in {}^*\mathcal{B}(H_1, H_2)$. For $x \in {}^*H_1^*$ and $v \in V$ define $\phi_v(x) = \langle \overline{T}(x), v \rangle$ and we see that $\phi_v \in {}^*\mathcal{B}(H_1, \mathbb{R}) = {}^*H_1^*$. Note that $\phi_v|_U = \langle T(u), v \rangle \in U^*$. Hence we can apply our nonstandard Riesz representation theorem Theorem 5.2.1. Thus we have for each $v \in V$ that there exists a unique $t(v) \in U$ such that

$$\phi_v(u) = \langle u, t(v) \rangle.$$

Let us define $T^* : V \to U$ by $T^*(v) = t(v)$. Then T^* is the unique map such that

$$\langle T(u), v \rangle = \langle u, T^*(v) \rangle$$

for all $u \in U, v \in V$. Note that since T is internal we have that ϕ_v is internal and hence T^* must be internal.

Let us show that it is linear. For $\alpha, \beta \in {}^*\mathbb{F}$ and $x \in U, y, z \in V$ we have

$$\begin{aligned}
\langle x, T^*(\alpha y + \beta z) \rangle &= \langle Tx, \alpha y + \beta z \rangle \\
&= \bar{\alpha} \langle Tx, y \rangle + \bar{\beta} \langle Tx, x \rangle \\
&= \langle x, \alpha T^* y \rangle + \langle x, \beta T^* z \rangle \\
&= \langle x, \alpha T^* y + \beta T^* z \rangle.
\end{aligned}$$

By uniqueness, we must have $T^*(\alpha y + \beta z) = \alpha T^* y + \beta T^* z$.

Now we show that T^* is bounded by showing $\|T\| = \|T^*\|$. The proof is exactly the same as in the standard case: We have that

$$\|T\|^2 = \sup_{\|x\|=1} \langle Tx, Tx \rangle = \sup_{\|x\|=1} \|T^* T x\| \leq \|T^*\| \|T\|,$$

hence $\|T\| \leq \|T^*\|$. The same argument with T^* instead we get $\|T^*\| \leq \|T\|$. Thus $\|T\| = \|T^*\|$. ■

5.3 Nearstandard operators

Now let $(e_n)_{n \leq \gamma}$ and $(f_n)_{n \leq \eta}$ be internal orthonormal bases in hyperfinite dimensional inner product spaces U and V, respectively.

Definition 5.3.1. We call $T \in L(U, V)$ **nearstandard** if $\|T\|$ is finite and $T(e_n)$ is nearstandard for every $n \in \mathbb{N}$.

Remark 5.3.2. ★ In light of 5.1.6 this means that, in addition to finite operator norm, for every $n \in \mathbb{N}$ we have that

$$\sum_{n=\theta}^{\gamma} \langle T(e_n), f_n \rangle^2 \approx 0$$

for every infinite $\theta \leq \gamma$.

We will show that a nearstandard map maps nearstandard points to nearstandard points, but first, we need the following lemma:

Lemma 5.3.3. *If* (x_n) *is a sequence of nearstandard elements and* $\lim_{n\to\infty} {}^\circ\|x - x_n\| = 0$*, then* x *is nearstandard.*

Proof. ★ Suppose that x is not nearstandard. Then there exists a standard $\epsilon > 0$ such that $\|x - y\| > \epsilon$ for every $y \in \mathrm{Ns}(U)$. In particular $\|x - x_n\| > \epsilon$ for every $n \in \mathbb{N}$. But this contradicts $\lim_{n\to\infty} {}^\circ\|x - x_n\| = 0$. ■

Lemma 5.3.4 (Tom Lindstrøm)**.** *If* T *is nearstandard then* $T(\mathrm{Ns}(U)) \subseteq \mathrm{Ns}(V)$.

Proof. Suppose $x = \sum_{n=1}^{\gamma} x_n e_n \in \mathrm{Ns}(U)$. We then have for every finite K that $T\left(\sum_{n=1}^{K} x_n e_n\right) = \sum_{n=1}^{K} x_n T(e_n)$ is nearstandard. Since x is nearstandard we have

$$\lim_{K\to\infty} {}^\circ\left\| x - \sum_{n=1}^{K} x_n e_n \right\| = 0.$$

But then, since T has finite norm it follows that

$$\lim_{K\to\infty} {}^\circ\left\| T(x) - \sum_{n=1}^{K} x_n T(e_n) \right\| \leq {}^\circ\|T\| \lim_{K\to\infty} {}^\circ\left\| x - \sum_{n=1}^{K} x_n e_n \right\| = 0.$$

Thus by Lemma 5.3.3 above $T(x)$ is nearstandard.

■

Example 5.3.5. Let $T \in L(U,V)$ be defined by $T(e_\gamma) = f_1$ and $T(e_n) = 0$ for $n \neq \gamma$. The adjoint $T^* \in L(V,U)$ becomes

$$T^*(f_n) = \begin{cases} e_\gamma & n = 1 \\ 0 & n \neq 1 \end{cases}.$$

From this example it is easy to see that T has finite operator norm equal to one and that $T(e_n)$ is nearstandard for every $n = 1, \ldots, \gamma$. So T is nearstandard. But T^* is not nearstandard (e_γ is remote and not an infinitesimal). Therefore we will give a stricter definition of nearstandardness in order to solve this asymmetry.

Definition 5.3.6 (Tom Lindstrøm)**.** We call $T \in L(U,V)$ **strictly nearstandard** if

(a) $\|T\| \in \mathrm{Fin}(\mathbb{R})$

(b) $T(x) \in \mathrm{Ns}(V)$ for every $x \in \mathrm{Ns}(U)$

(c) $T(x) \approx 0$ for every $x \in r(U)$ with finite norm.

Proposition 5.3.7. ★ *Let $T \in L(U,V)$ be strictly nearstandard. Then $T(Fin(U)) \subseteq Ns(V)$.*

Proof. ★ Let $u \in \mathrm{Fin}(U)$. Then by Lemma 5.1.9 we have that there exists $u_1 \in \mathrm{Ns}(U)$ and $u_2 \in r(U)$ such that $u = u_1 + u_2$ and $u_1 \perp u_2$. By orthogonality we have that $\|u\|^2 = \|u_1\|^2 + \|u_2\|^2$. Hence $\|u_2\|$ must be finite. Since u_2 is a remote point with finite norm we get that $T(u) = T(u_1) + T(u_2) \approx T(u_1)$. But $T(u_1)$ is nearstandard since u_1 is nearstandard. ■

Proposition 5.3.8 (Tom Lindstrøm)**.** *If $T \in L(U,V)$ is strictly nearstandard then the adjoint $T^* \in L(V,U)$ is strictly nearstandard.*

Proof. Assume that T is strictly nearstandard. The first item follows from the fact that $\|T\| = \|T^*\|$. To check (b) we assume for contradiction that $y \in \mathrm{Ns}(V)$ but $T^*(y) \notin \mathrm{Ns}(U)$. Then there exists a remote $x \in U$ such that $\langle x, T^*(y)\rangle$ is not an infinitesimal. But this is impossible since by assumption of T we have $\|T(x)\| \approx 0$ and hence

$$\begin{aligned} |\langle x, T^*(y)\rangle| &= |\langle T(x), y\rangle| \\ &\leq \|T(x)\|\|y\| \\ &\approx 0. \end{aligned}$$

To show (c) assume that $y \in V$ is remote with finite norm and consider $x = T^*(y)$. The norm of $\|x\|$ is finite since T^* and y have a finite norm. Therefore we can decompose $x = x_1 + x_2$ where $x_1 \perp x_2$, $x_1 \in \mathrm{Ns}(U)$ and $x_2 \in r(U)$. Then since T is strictly nearstandard we have that $T(x_1) \in \mathrm{Ns}(V)$ and $T(x_2) \approx 0$. Since x_1 and x_2 are orthogonal we have that $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2$. Moreover, we have that

$$\begin{aligned}\|x_1\|^2 &= \langle x_1, x_1\rangle \\ &= \langle x_1, x\rangle \\ &= \langle x_1, T^*(y)\rangle \\ &= \langle T(x_1), y\rangle \\ &\approx 0,\end{aligned}$$

since $T(x_1)$ is nearstandard and y is remote. But we also have that

$$\begin{aligned}\|x_2\|^2 &= \langle x_2, x_2\rangle \\ &= \langle x_2, x\rangle \\ &= \langle x_2, T^*(y)\rangle \\ &= \langle T(x_2), y\rangle \\ &\approx 0,\end{aligned}$$

since $T(x_2) \approx 0$. Hence $\|T^*(y)\|^2 = \|x_1\|^2 + \|x_2\|^2 \approx 0$. ■

Observation 5.3.9. ★ *Let $U = V$. First note that the set S-bounded maps in $L(U)$ is a normed $\mathbb{F}$-algebra with composition as multiplication. For $\alpha \in \mathbb{F}$, and S-bounded $R, S, T \in L(U)$ we have:*

- $\alpha R + ST \in L(U)$,
- $\|TS\| \leq \|T\|\|S\|$,
- $\|\alpha R + ST\| \leq |\alpha|\|R\| + \|S\|\|T\|$ *is finite,*
- $\alpha(TS) = (\alpha T)S = T(\alpha S)$

*Moreover, The algebra of S-bounded maps is *-closed with respect to the involution $T \mapsto T^*$ since $\|T^*\| = \|T\|$ and $T^* \in L(U)$.*

Let $\mathcal{NS}_s(U) := \{T \in L(U) \mid T$ is strictly nearstandard$\}$ be the subset of S-bounded maps of strictly nearstandard maps. $\mathcal{NS}_s(U)$ is a normed $\mathbb{F}$-subalgebra. For $\alpha \in \mathbb{F}$, $R, S, T \in \mathcal{NS}_s(U)$ we have:

- *for $x \in Ns(U)$ we have $(\alpha R + ST)(x) = \alpha R(x) + S(T(x)) \in Ns(U)$ since $\alpha \in \mathbb{F}$ and $T(x) \in Ns(U)$,*
- *For $x \in r(u)$ with $\|x\|$ finite we have $(\alpha R + ST)(x) = \alpha R(x) + S(T(x)) \approx S(0) = 0$.*

*Moreover, by Proposition 5.3.8 we have that $\mathcal{NS}_s(U)$ is *-closed, and hence $\mathcal{NS}_s(U)$ is a normed $*$-subalgebra.*

5.4 Internal Hilbert-Schmidt operators

Let U and V be two hyperfinite dimensional spaces equipped with an inner product $\langle\cdot,\cdot\rangle$ and with internal orthonormal bases $(e_n)_{n\leq\gamma}$ and $(f_m)_{m\leq\eta}$ for U and V, respectively. As before $L(U, V)$ denotes the set of internal and linear maps $T : U \to V$ and $L(U) := L(U,U)$. We denote by T^* the adjoint of T - when it exists. For $T \in L(U)$ let us define $tr_U : L(U) \to {}^*\mathbb{R}$ by

$$tr_U(T) = \sum_{n \leq \gamma} \langle T(e_n), e_n \rangle$$

and similarly when $T \in L(V)$ we define $tr_V : L(V) \to {}^*\mathbb{R}$ by

$$tr_V(T) = \sum_{m \leq \eta} \langle T(f_m), f_m \rangle.$$

Note that if $T \in L(U,V)$ and the adjoint T^* exists then $T^*T \in L(U)$ and we get that

$$\begin{aligned} tr_U(T^*T) &= \sum_{n \leq \gamma} \langle T^*T(e_n), e_n \rangle \\ &= \sum_{n \leq \gamma} \langle T(e_n), T(e_n) \rangle \\ &= \sum_{n=1}^{\gamma} \|Te_n\|^2 \end{aligned}$$

Definition 5.4.1. A map $T \in L(U,V)$ is called **Hilbert-Schmidt** if

(a) $tr(T^*T) = \sum_{n=1}^{\gamma} \|Te_n\|^2$ is finite

(b) $\sum_{n=\theta}^{\eta} \|T^*(f_n)\|^2$ is an infinitesimal for every *infinite* $\theta \leq \eta$.

We denote the set of Hilbert-Schmidt operators by $\mathcal{HS}(U,V)$. We shall write

$$||T||_{\mathcal{HS}} = tr_U(T^*T)^{1/2}.$$

Observation 5.4.2. ★ *Note that we have not argued for the existence of the adjoint T^*. But if we have that $\|T\|_{\mathcal{HS}}^2 = \sum_{n=1}^{\gamma} \|Te_n\|^2$ is finite, then from Proposition 5.4.3 below we have that T is S-bounded. Hence from Lemma 5.2.3, we have that T^* exists so the definition of internal Hilbert-Schmidt operators is well-defined.*

Proposition 5.4.3. *For a Hilbert-Schmidt map $T \in \mathcal{HS}(U,V)$ we have that*

(a) $\|T\| \leq \|T\|_{\mathcal{HS}}$,

(b) *T is S-bounded.*

Proof. First let us prove that $\|T\| \leq \|T\|_{\mathcal{HS}}$. Let x have norm equal to 1. By using that $x = \sum_{n=1}^{\gamma} \langle x, e_n \rangle e_n$ and Cauchy-Schwartz inequality we obtain the inequality

$$\begin{aligned} ||Tx|| &= \| \sum_{n=1}^{\gamma} \langle x, e_n \rangle T e_n \| \\ &\leq \sum_{n=1}^{\gamma} |\langle x, e_n \rangle| \|T e_n\| \\ &\leq \left(\sum_{n=1}^{\gamma} |\langle x, e_n \rangle|^2 \right)^{1/2} \left(\sum_{n=1}^{\gamma} \|T e_n\|^2 \right)^{1/2} \\ &= \|x\| \|T\|_{\mathcal{HS}} \\ &= \|T\|_{\mathcal{HS}} \end{aligned}$$

Hence $\|T\| = \sup\{\|Tx\| \mid \|x\| = 1\} \leq \|T\|_{\mathcal{HS}}$.

That T is S-bounded is immediate from (a) since $\|T\|_{\mathcal{HS}}$ is finite and $\|T\| \leq \|T\|_{\mathcal{HS}}$. ■

Proposition 5.4.4. *We have that $tr_U(T^*T) = tr_V(TT^*)$.*

Proof. ★ By Parseval's identity, we have

$$\begin{aligned} ||Te_n||^2 &= \sum_{m \leq \eta} \langle Te_n, f_m \rangle^2 \\ &= \sum_{m \leq \eta} \langle e_n, T^* f_m \rangle^2. \end{aligned}$$

and hence

$$\begin{aligned} tr_U(T^*T) &= \sum_{n \leq \gamma} ||Te_n||^2 \\ &= \sum_{n \leq \gamma} \sum_{m \leq \eta} \langle Te_n, f_m \rangle^2 \\ &= \sum_{m \leq \eta} \sum_{n \leq \gamma} \langle e_n, T^* f_m \rangle^2 \\ &= \sum_{m \leq \eta} ||T^* f_m||^2 \\ &= \sum_{m \leq \eta} \langle TT^* f_m, f_m \rangle^2 \\ &= tr_V(TT^*). \end{aligned}$$

■

Remark 5.4.5. ★ Suppose that $T \in \mathcal{HS}(U, V)$. Then by definition we have $\|T\|_{\mathcal{HS}}^2 = tr_U(T^*T)$ is finite. But then we have that

$$\|T^*\|_{\mathcal{HS}} = tr_V(TT^*) = tr_U(T^*T)$$

is finite as well. In standard analysis we have that Hilbert-Schmidt maps are *-closed, but in nonstandard analysis, this is not the case as we do not necessarily have the other part of the definition, i.e.,

$$\sum_{n=\theta}^{\gamma} \|(T^*)^*(e_n)\|^2 = \sum_{n=\theta}^{\gamma} \|T(e_n)\|^2 \approx 0$$

for infinite $\theta \leq \gamma$. Later in this section, we will introduce *strictly Hilbert-Schmidt* maps and show that the adjoint of such a map is strictly Hilbert-Schmidt as well.

Lemma 5.4.6. *Let $T \in \mathcal{HS}(U,V)$ and $S \in L(U)$ be bounded. Then $TS \in \mathcal{HS}(U,V)$.*

Proof. ★ Recall from the remark above that $\|T^*\|^2_{\mathcal{HS}} = tr(TT^*)$ is finite. Since S is S-bounded we have that S^* is S-bounded as well (c.f. Lemma 5.2.3). Moreover, we have that

$$\begin{aligned}\|TS\|^2_{\mathcal{HS}} &= tr((TS)^*TS)\\ &= tr(TS(TS)^*)\\ &= \sum_{n\leq\eta} \|S^*T^*(f_n)\|^2\\ &\leq \|S^*\|^2 \sum_{n\leq\eta} \|T^*(f_n)\|^2\\ &\leq \|S^*\|^2 tr_V(TT^*)\\ &< \infty.\end{aligned}$$

Now we check the other requirement. Let $\theta \leq \eta$ be infinite. We get that

$$\begin{aligned}\sum_{n=\theta}^{\eta} \|(TS)^*(f_n)\|^2 &= \sum_{n=\theta}^{\eta} \|S^*T^*(f_n)\|^2\\ &\leq \|S^*\| \sum_{n=\theta}^{\eta} \|T^*(f_n)\|^2\\ &\approx 0\end{aligned}$$

since $\|S^*\|$ is finite and by definition of Hilbert-Schmidt maps we have $\sum_{n=\theta}^{\eta} \|T^*(f_n)\|^2 \approx 0$. ■

Proposition 5.4.7. *A Hilbert-Schmidt map $T \in \mathcal{HS}(U,V)$ is nearstandard.*

Proof. ★ We show this as in Remark 5.3.2. We have that

$$\|T\| \leq \|T\|_{\mathcal{HS}}$$

and hence $\|T\|$ is finite since the Hilbert-Schmidt norm is finite by definition. Using Cauchy-Schwartz inequality we get for each infinite $\theta \leq \eta$

$$\begin{aligned}\sum_{n=\theta}^{\eta} \langle Te_k, f_n \rangle^2 &= \sum_{n=\theta}^{\eta} \langle e_k, T^* f_n \rangle^2 \\ &\leq \sum_{n=\theta}^{\eta} \|e_k\|^2 \|T^* f_n\|^2 \\ &= \sum_{n=\theta}^{\eta} \|T^* f_n\|^2\end{aligned}$$

where the last term is infinitesimal by the definition of Hilbert-Schmidt maps. ∎

Lemma 5.4.8. ★ *If $T \in \mathcal{HS}(U, V)$ then $T(\mathrm{Fin}(U)) \subseteq \mathrm{Ns}(V)$, i.e., if $u \in U$ and $\|u\|$ finite, then $T(u)$ is nearstandard.*

Proof. ★ Suppose $u \in U$ is normfinite. Then $\|Tu\| \leq \|T\|_{\mathcal{HS}} \|u\|$ is finite. Moreover, we have

$$\sum_{n=\theta}^{\gamma} \langle Tu, f_n \rangle^2 = \sum_{n=\theta}^{\gamma} \langle u, T^* f_n \rangle^2 \leq \|u\|^2 \sum_{n=\theta}^{\gamma} \|T^* f_n\|^2$$

which is an infinitesimal since u is finite and by definition of Hilbert-Schmidt operators. And hence $T(u)$ is nearstandard. ∎

As mentioned in Remark 5.4.5, we do not necessarily have that the adjoint of a Hilbert-Schmidt operator is Hilbert-Schmidt as well. Therefore we make the following definition.

Definition 5.4.9 (Tom Lindstrøm)**.** We call $T \in L(U, V)$ **strictly Hilbert-Schmidt** if

(a) T is strictly nearstandard.

(b) The Hilbert-Schmidt norm $\|T\|_{\mathcal{HS}} = tr(T^*T)$ is finite.

(c) For every *infinite* $\theta \leq \gamma$ we have $\sum_{n=\theta}^{\gamma} \|T(e_n)\|^2 \approx 0$.

Theorem 5.4.10 (Tom Lindstrøm)**.** *If $T \in L(U, V)$ is strictly Hilbert-Schmidt, then the adjoint T^* is strictly Hilbert-Schmidt.*

Proof. Assume that T is strictly Hilbert-Schmidt. By Proposition 5.3.8 we have that T^* is strictly nearstandard. As already shown in Remark 5.4.5, the Hilbert-Schmidt norm of T^* is finite. Now we show the last property. Let standard $\epsilon > 0$ be arbitrary. Since T is strictly Hilbert-Schmidt there exists a finite M such that

$$\|T\|_{\mathcal{HS}}^2 - \frac{\epsilon}{2} < \sum_{n=1}^{M} \|T(e_n)\|^2.$$

And since $T(e_n) \in \mathrm{Ns}(V)$ there is a finite K such that

$$\|T(e_n)\|^2 \leq \sum_{m=1}^{K} \langle T(e_n), f_m \rangle^2 + \frac{\epsilon}{2M}$$

for $n = 1, \ldots, M$. Hence we get that

$$\begin{aligned}
\|T\|^2_{\mathcal{HS}} - \frac{\epsilon}{2} &< \sum_{n=1}^{M} \|T(e_n)\|^2 \\
&\leq \sum_{n=1}^{M} \sum_{m=1}^{K} \langle T(e_n), f_m\rangle^2 + \frac{\epsilon}{2} \\
&= \sum_{m=1}^{K} \sum_{n=1}^{M} \langle T(e_n), f_m\rangle^2 + \frac{\epsilon}{2} \\
&= \sum_{m=1}^{K} \sum_{n=1}^{M} \langle e_n, T^*(f_m)\rangle^2 + \frac{\epsilon}{2} \\
&\leq \sum_{m=1}^{K} \sum_{n=1}^{\gamma} \langle e_n, T^*(f_m)\rangle^2 + \frac{\epsilon}{2} \\
&= \sum_{m=1}^{K} \|T^*(f_m)\|^2 + \frac{\epsilon}{2}.
\end{aligned}$$

This yields that

$$\sum_{m=1}^{K} \|T^*(f_m)\|^2 > \|T\|^2_{\mathcal{HS}} - \epsilon.$$

Hence by rearranging terms and using that $\|T\|_{\mathcal{HS}} = \|T^*\|_{\mathcal{HS}}$ we have shown that for each $\epsilon > 0$ there is a finite K such that

$$\sum_{m=K}^{\eta} \|T^*(f_m)\|^2 < \epsilon$$

which is what we wanted to show. ■

Remark 5.4.11. ★ As one would expect, if T is strictly Hilbert-Schmidt then T is also Hilbert-Schmidt. But this is because of the theorem we just proved. If T is strictly Hilber-Schmidt then so is T^*. In particular, we have that for every infinite $\theta \leq \eta$ that

$$\sum_{n=\theta}^{\eta} \|T^*(f_n)\|^2 \approx 0.$$

Thus the second property of Hilbert-Schmidt operators is satisfied. The first property, that the Hilbert-Schmidt norm is finite, is obvious.

Observation 5.4.12. ★ *Let $U = V$ and let $\mathcal{HS}_s(U) := \{T \in L(U) \mid T \text{ is strictly Hilbert-Schmidt}\}$. We will show that $\mathcal{HS}_s(U)$ is a *-closed subalgebra of $\mathcal{NS}_s(U)$ (and hence also of the S-bounded maps in $L(U)$). For $\alpha \in \mathbb{F}$, $R, S, T \in \mathcal{HS}_s(U)$ we have:*

- $\|\alpha T + SR\|_{\mathcal{HS}} = \sum_{i=1}^{\gamma} \|(\alpha T + SR)e_n\|^2 \leq |\alpha|\|T\|_{\mathcal{HS}} + \|S\|\|R\|_{\mathcal{HS}}$ *which is finite,*
- *for any infinite $\theta \leq \gamma$ we have $\sum_{i=\theta}^{\gamma} \|(\alpha T + SR)e_n\|^2 \leq |\alpha| \sum_{i=\theta}^{\gamma} \|Te_n\|^2 + \|S\| \sum_{i=\theta}^{\gamma} \|Re_n\|^2 \approx 0$.*

So $\alpha T + SR \in \mathcal{HS}_s(U)$*. Moreover, since we showed that the adjoint of a strictly Hilbert-Schmidt operator is strictly Hilbert-Schmidt, we have that* $\mathcal{HS}_s(U)$ *is *-closed.*

Furthermore, We have that $\mathcal{HS}_s(U)$ *is a two-sided ideal in* $\mathcal{NS}_s(U)$*. For* $T \in \mathcal{HS}_s(U)$ *and* $S \in \mathcal{NS}_s(U)$ *we have:*

- $\|ST\|_{\mathcal{HS}} = tr((ST)^*ST) = \sum_{i=1}^{\gamma} \|STe_n\|^2 \leq \|S\|\|T\|_{\mathcal{HS}}$ *which is finite,*
- *for any infinite* $\theta \leq \gamma$ *we have* $\sum_{i=\theta}^{\gamma} \|STe_n\|^2 \leq \|S\|^2 \sum_{i=\theta}^{\gamma} \|Te_n\|^2 \approx 0$ *since* $T \in \mathcal{HS}(U)$*.*

So we have that $ST \in \mathcal{HS}_s(U)$ *and hence* $\mathcal{HS}_s(U)$ *is a left-sided ideal in* $\mathcal{NS}_s(U)$*.* $\mathcal{HS}_s(U)$ *is also a right-sided ideal in* $\mathcal{NS}_s(U)$ *since*

- $\|TS\|_{\mathcal{HS}} = tr((TS)^*TS) = tr(TS(TS)^*) = \sum_{i=1}^{\gamma} \|S^*T^*e_n\|^2 \leq \|S^*\|\|T^*\|_{\mathcal{HS}}$ *which is finite,*
- *for any infinite* $\theta \leq \gamma$ *we have* $\sum_{i=\theta}^{\gamma} \|TSe_n\|^2 \leq \|T\|^2 \sum_{i=\theta}^{\gamma} \|Se_n\|^2 \approx 0$ *since* $S(x) \approx 0$ *for every* $x \in r(U)$ *with finite norm.*

Observe that for $T \in \mathcal{NS}_s(U)$ *and* $S \in L(U)$ *S-bounded we do* not *necessarily have that* ST *or* TS *maps nearstandard points to nearstandard points. Therefore neither* $\mathcal{HS}_s(U)$ *or* $\mathcal{NS}_s(U)$ *can be ideals of S-bounded maps in* $L(U)$*.*

CHAPTER 6

Nonstandard Stochastic Analysis in Infinite Dimensions

6.1 Stochastic processes

Here we follow [Lin80] but generalize this to Banach spaces. Let S be a set including all entities we need such as a Banach space E and relevant measure spaces. Let $U(S)$ be the superstructure over S and we let ${}^*U(S)$ be our polysaturated model. See Definition 2.5.4. Let $(\Omega, \mathcal{F}, P)$ be a hyperfinite probability space and let $\mathbb{T}$ be a hyperfinite timeline.

A hyperfinite stochastic process is an internal map $X : \mathbb{T} \times \Omega \to {}^*E$ such that

$$\omega \mapsto X(t, \omega)$$

is $\mathcal{F}$-measurable.

Remark 6.1.1. ★ Let $E \subset F \subset {}^*E$ and $\mathcal{T}$ denote the topology on E. The map $Y : \mathbb{T} \times \Omega \to F$ is called $\mathcal{F}$-measurable if $Y^{-1}(O) \in \mathcal{F}$ for all $O \in F \cap {}^*\mathcal{T} = \{F \cap U \mid U \in {}^*\mathcal{T}\}$.

Fix $t \in \mathbb{T}$. Similar to the finite dimensional case, the expectation of X_t is given by

$$E[X_t] = \sum_{\omega \in \Omega} X_t(\omega) P(\{\omega\}).$$

Let $0 = t_0 < t_1 < \cdots < t_N = t_\infty$ be the elements of $\mathbb{T}$. An increment of a process X is of the form $\triangle X_{t_i} = X_{t_{i+1}} - X_{t_i}$. And if $s = t_i,\ t = t_j$ for $i < j$ we write

$$\sum_{r=s}^{t} X_r$$

for the sum

$$\sum_{n=i}^{j-1} X_{t_n},$$

and note that $X_t = X_{t_j}$ is not included in the sum.

For an $(\mathcal{F}_t)_{t \in \mathbb{T}}$ an increasing sequence of algebras of subsets of Ω, and let P be a probability measure on $\mathcal{F}_{t_\infty}$ where t_∞ is the largest element in $\mathbb{T}$. Let $(\Omega, (\mathcal{F})_{t \in \mathbb{T}}, P)$ be an internal basis. As before, we say that a hyperfinite process

$X : \mathbb{T} \times \Omega \to {}^*E$ is adapted to an internal basis $(\Omega, (\mathcal{F})_{t\in\mathbb{T}}, P)$ if for each $t \in \mathbb{T}$ we have that

$$\omega \mapsto X(t, \omega)$$

is $\mathcal{F}_t$-measurable.

A hyperfinite martingale $M : \mathbb{T} \times \Omega \to {}^*E$ is exactly the same as before, i.e., if M is adapted to $(\Omega, (\mathcal{F})_{t\in\mathbb{T}}, P)$ and for all $s, t \in \mathbb{T}$, $s < t$ and all $A \in \mathcal{F}_s$ we have

$$E\left[1_A(M_t - M_s)\right] = 0.$$

Or equivalently, $E[M_t \mid \mathcal{F}_s] = M_s$ for all $t \geq s$

Definition 6.1.2. ★ Let $M : \mathbb{T} \times \Omega \to {}^*E$ be a hyperfinite martingale with respect to $(\Omega, (\mathcal{F})_{t\in\mathbb{T}}, P)$. We call M a **λ^2-martingale** if for every $t \in \mathbb{T}$ $E[\|M_t\|^2]$ is finite.

6.2 Hyperfinite random walk and its covariance operator

In [Lin04] the covariance matrix of a finite-dimensional hyperfinte Lèvy process was introduced. Inspired by this, we will define the covariance operator of a hyperfinite random walk. It turns out that this covariance operator is a key element in order to prove several results regarding our infinite-dimensional stochastic integral in Chapter 6.

For a Banach space E let F be a hyperfinite dimensional linear space such that $E \subset F \subset {}^*E$ with internal basis $(e_n)_{n\leq\eta}$ where $\eta \in {}^*\mathbb{N}$ is a nonstandard integer which can be infinite. See Proposition 5.1.5

As before, let $\mathbb{T}$ be the timeline

$$\mathbb{T} = \left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N^2-1}{N}, N\right\}$$

where $N \in {}^*\mathbb{N}$ is an infinite integer. We will consider a hyperfinite random walk on F, which is defined the same way as in Definition 4.3.1 i.e.,

Definition 6.2.1. We call $X : \Omega \times \mathbb{T} \to F$ a **hyperfinite random walk on** F given by increments $A \subset F$ and probabilities $(p_a)_{a\in A}$ if $L(0) = 0$, and the increments $\triangle L_t$ are independent with distribution $P[\triangle L_t = a] = p_a$.

For a hyperfintie process $L : \Omega \times \mathbb{T} \to F$ we define the covariance operator $C^L \in L(F^*, F)$ to be the map

$$C^L(\phi) = \frac{1}{\triangle t} \sum_{a\in A} \phi(a) a p_a.$$

We now define what it means for an operator $T \in L(F^*, F)$ to be symmetric and positive. We define it the same way as in the standard case. Here we use the dual pairing notation, i.e., $\langle x, \phi\rangle := \phi(x)$ for $x \in F$ and $\phi \in F^*$.

Definition 6.2.2. Let $T \in L(F^*, F)$. We call T **symmetric** if for $\phi, \psi \in F^*$ we have

$$\langle T\phi, \psi\rangle = \langle T\psi, \phi\rangle.$$

Definition 6.2.3. Let $T \in L(F^*, F)$. We call T **positive (nonnegative definite)** if

$$\langle T\phi, \phi\rangle := \phi(T(\phi)) \geq 0$$

for all $\phi \in F^*$.

Proposition 6.2.4. ★ *We have That C^L is symmetric and positive.*

Proof. ★
We have for $\phi, \psi \in F^*$ that

$$\begin{aligned}
\langle C^L\phi, \psi\rangle &= \psi(C^L(\phi)) \\
&= \psi\left(\frac{1}{\triangle t}\sum_{a\in A}\phi(a)ap_a\right) \\
&= \frac{1}{\triangle t}\sum_{a\in A}\phi(a)\psi(a)p_a \\
&= \phi\left(\frac{1}{\triangle t}\sum_{a\in A}a\psi(a)p_a\right) \\
&= \phi(C^L(\psi)) \\
&= \langle C^L\psi, \phi\rangle.
\end{aligned}$$

Hence C^L is symmetric. Since we have that

$$\langle C^L\phi, \phi\rangle = \phi(C^L(\phi)) = \frac{1}{\triangle t}\sum_{a\in A}\phi(a)^2 p_a \geq 0,$$

it is positive as well. ∎

Now consider $H \subset U \subset {}^*H$ where H is a Hilbert space. For $T \in L(U^*, U)$ we identify H^* with H and hence also U^* with U so that $T \in L(U)$, c.f. Corollary 5.2.2. Thus we get the following definitions which are in line with Definition 6.2.2 and Definition 6.2.3 above.

Definition 6.2.5. Let $T \in L(U)$. We call T **symmetric** if $\langle Tx, y\rangle = \langle Ty, x\rangle$ for all $x, y \in U$.

Definition 6.2.6. Let $T \in L(U)$. We call T **positive (nonnegative definite)** if $\langle Tx, x\rangle \geq 0$ for all $x \in U$.

For a hyperfintie random variable $L : \Omega \times \mathbb{T} \to U$ get that the covariance operator $C^L \in L(U^*, U) \simeq L(U)$ becomes

$$C^L(x) = \frac{1}{\triangle t}\sum_{a\in A}\langle a, x\rangle ap_a.$$

Remark 6.2.7. ★ From Proposition 6.2.4 we have that C^L is positive and symmetric. Using our inner product version of the definitions of positive and symmetric operators it is easy to show: let $x, y \in U$

$$\begin{aligned}\langle C^L x, y\rangle &= \left\langle \frac{1}{\triangle t}\sum_{a\in A}\langle a,x\rangle a p_a, y\right\rangle \\ &= \frac{1}{\triangle t}\sum_{a\in A}\langle a,x\rangle\langle a,y\rangle p_a \\ &= \frac{1}{\triangle t}\sum_{a\in A}\langle a,y\rangle\langle a,x\rangle p_a \\ &= \langle C^L y, x\rangle.\end{aligned}$$

Thus we have that C^L is symmetric. In particular, we have that

$$\langle C^L x, x\rangle = \frac{1}{\triangle t}\sum_{a\in A}\langle a,x\rangle^2 p_a,$$

and from this it is easy to see that C^L is positive since

$$\langle C^L x, x\rangle = \frac{1}{\triangle t}\sum_{a\in A}\langle a,x\rangle^2 p_a \geq 0$$

for every $x \in U$

Corollary 6.2.8. ★ *Let $C^L : U \to U$ be defined by*

$$C(x) = \frac{1}{\triangle t}\sum_{a\in A}\langle a,x\rangle a p_a.$$

If C^L is S-bounded, then C^L has a square root.

Proof. ★ In standard analysis, we have that if $T \in \mathcal{B}(H)$ and $\langle Tx, x\rangle \geq 0$ for all $x \in H$, then there exists a self-adjoint $S \in \mathcal{B}(H)$ such that $T = SS$. By the transfer principle, we have that if $T \in {}^*\mathcal{B}(H)$ and $\langle Tx, x\rangle \geq 0$ for all $x \in {}^*H$, then there exist a self-adjoint $S \in {}^*\mathcal{B}(H)$ such that $T = SS$.

We have already shown that C^L is positive hence all we need to show is that C^L can be viewed as an element in ${}^*\mathcal{B}(H)$. First note that C^L is linear. Note also that C^L is internal since for all $x = [x_n] \in U$ we can write $C^L(x) = [C_n(x_n)]$ where

$$C_n(x_n) = \frac{1}{\triangle t}\sum_{a\in A}\langle a_n, x_n\rangle a_n p_a.$$

Thus we have shown that $C^L \in L(U)$ and by assumption C^L is S-bounded. Therefore, by Corollary 5.1.14 we have that there is an extension $\overline{C}^L \in {}^*\mathcal{B}(H)$. The extension $\overline{C}^L$ is also positive and thus there exists a self-adjoint $\overline{R} \in {}^*\mathcal{B}(H)$ such that $\overline{C^L} = \overline{RR}$. But then, letting $R = R|_U$, we have that $C^L = RR$. ■

CHAPTER 7

Stochastic Integration

The article [Rie14] establishes stochastic integration with respect to cylindrical Lévy processes using standard analysis. Inspired by this paper, we wish to define a stochastic integral $\int XdL$ which can be defined as an integral with respect to hyperfinite Lévy processes L. We make this possible but require the process we integrate against, L, to be a martingale - and not all Lévy processes are martingales. Actually, we construct an integral where L is a hyperfinite martingale and we do *not* require the process to be a Lèvy process, but the integral can be applied to Lèvy martingales.

7.1 Definition of the integral

Let U, V be two hyperfinite dimensional inner product spaces with internal orthonormal bases $(e_i)_{i=1}^{\gamma}$ and $(f_i)_{i=1}^{\eta}$, respectively. Let $\triangle t = \frac{1}{N}$ be an infinitesimal and let $\mathbb{T} = \{0, \triangle t, 2\triangle t, \dots\}$ be our timeline. Further let

$$\Omega = \{\omega : \mathbb{T} \to A \mid \omega \text{ is internal}\}$$

where $A \subset U$ is a hyperfinite subset of U. Let $L : \Omega \times \mathbb{T} \to U$ be a hyperfinite random walk on U given by $A \subset U$ and probabilities $(p_a)_{a\in A}$. Recall that this means that $L(0) = 0$, and the increments $\triangle L_t$ are independent with distribution $P[\triangle L_t = a] = p_a$. As before we have

$$\triangle L(\omega, t) = L(\omega, t + \triangle t) - L(\omega, t).$$

Finally, let $(\mathcal{F}_t)_{t\in\mathbb{T}}$ be the filtration generated by L, i.e.,

$$\mathcal{F}_t = \{L_s^{-1}(O) \mid s \in \mathbb{T}, s \leq t, O \in U \cap {}^*\mathcal{T}\},$$

where $\mathcal{T}$ is the topology of our underlying Hilbert space H such that $H \subset U \subset {}^*H$.

Assume further that L is a hyperfinite martingale, i.e., L is adapted to $(\Omega, (\mathcal{F})_{t\in\mathbb{T}}, P)$ and for all $s, t \in \mathbb{T}$, $s < t$ and all $A \in \mathcal{F}_s$ we have

$$E\left[1_A(L_t - L_s)\right] = 0,$$

or equivalently that

$$E[L_t \mid \mathcal{F}_s] = L_s.$$

Lemma 7.1.1. *$L : \Omega \times \mathbb{T} \to U$ is a martingale if and only if $\sum_{a\in A} ap_a = 0$*

Proof. ★ Assume L is a hyperfinite martingale. Let $s \in \mathbb{T}$ and let $t = s + \triangle t$. For $\Omega \in \mathcal{F}_s$ we get

$$\sum_{a \in A} a p_a = E[\triangle L_s] = E[1_\Omega (L_t - L_s)] = 0$$

Now assume that $\sum_{a \in A} a p_a = 0$. Then $E[\triangle L_t \mid \mathcal{F}_t] = 0$. Since $(\mathcal{F}_t)$ is the filtration generated by L it is adapted. For $s, t \in \mathbb{T}$ $s < t$ consider $L_t - L_s = \sum_{r=s}^{t} \triangle L_r$. Then we get by using the tower property that

$$\begin{aligned} E[L_t - L_s \mid \mathcal{F}_s] &= \sum_{r=s}^{t} E[\triangle L_r \mid \mathcal{F}_s] \\ &= \sum_{r=s}^{t} E[E[\triangle L_r \mid \mathcal{F}_r] \mid \mathcal{F}_s] \\ &= \sum_{r=s}^{t} E[0 \mid \mathcal{F}_s] \\ &= 0. \end{aligned}$$

And thus $E[L_t \mid \mathcal{F}_s] = L_s$. ■

Remark 7.1.2. ★ If L is a martingale then since $\sum_{a \in A} a p_a = 0$ we have that $\langle \sum_{a \in A} a p_a, u \rangle = 0$. Thus we get

$$\begin{aligned} \sum_{a \in A} \langle a, u \rangle p_a &= \sum_{a \in A} \langle a p_a, u \rangle \\ &= \left\langle \sum_{a \in A} a p_a, u \right\rangle \\ &= 0 \end{aligned}$$

We now assume that $L : \Omega \times \mathbb{T} \to U$ is a martingale and let $(\mathcal{F}_t)_{t \in \mathbb{T}}$ be the filtration generated by L. Our integrands will be of the form $X : \Omega \times \mathbb{T} \to L(U, V)$. Before we give meaning to X being adapted we need to establish internal open sets in $L(U, V)$. We can define a nonstandard operator norm on $L(U, V)$ by

$$\|T\| = \sup\{\|Tu\|_V \mid \|u\|_U = 1\}.$$

We can define a basis for a topology on $L(U, V)$ by the collection of balls $\mathcal{B} = \{B_r(T) \mid T \in L(U, V), r \in {}^*\mathbb{R}_+\}$ where

$$B_r(T) = \{S \in L(U, V) \mid \|T - S\| < r\}.$$

Hence we call $X : \Omega \times \mathbb{T} \to L(U, V)$ adapted if for all $t \in \mathbb{T}$ we have $X_t^{-1}(B_r(T)) \in \mathcal{F}_t$ for all $B_r(T) \in \mathcal{B}$.

Definition 7.1.3. ★ For $L : \Omega \times \mathbb{T} \to U$ and adapted integrand $X : \Omega \times \mathbb{T} \to L(U, V)$ we define the stochastic integral $\int_s^t X dL$ to be the process with values in V defined by

$$\left(\int X dL \right)(\omega, s, t) = \sum_{r=s}^{t} X(\omega, r)(\triangle L(r, \omega)).$$

We wish to estimate the size of the integral $\int_s^t XdL$ given the size of X and we do this by considering the seminorm given by

$$\|X\|_{[s,t]}^2 \coloneqq E\left[\sum_{r=s}^{t} \|X(\omega,r)\|_{\mathcal{HS}}^2 \triangle t\right].$$

Note that $\|\cdot\|_{[0,t]}$ is not a proper semi-norm as it takes values in ${}^*\mathbb{R}$.

We will show two settings for which the integral makes sense for different integrands. In the first approach, we require the integrand to be strictly Hilbert-Schmidt and put a probabilistic assumption on $L : \Omega \times \mathbb{T} \to U$. In the second approach, we only require the integrand to be Hilbert-Schmidt and put an assumption on the covariance operator for L. In the second approach, we will show some more properties of the integral such as when the integral is a λ^2-martingale and nearstandard. But first, we will give an example.

7.2 An example: Anderson's process

For motivation we introduce Anderson's process, i.e., Brownian motion, on hyperfinite dimensional linear spaces which were done in [Lin83].

Let U and V be hyperfinite dimensional linear spaces with orthonormal basis $(e_n)_{n\le\gamma}$ and $(f_n)_{n\le\eta}$ and consider the timeline $\mathbb{T} = \{0, \triangle t, 2\triangle t, \dots, 1\}$ where $\triangle t = 1/N$ for some $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. We let Ω be the set of all maps

$$\omega : \{1, \dots, \gamma\} \times \mathbb{T} \to \{-1, 1\}.$$

The Anderson process is the map $W : \Omega \times \mathbb{T} \to U$ given by

$$W_t(\omega) = \sum_{i=1}^{\gamma} \sum_{s=0}^{t} \sqrt{\triangle t}\,\omega_i(s) e_i.$$

Proposition 7.2.1. *The Anderson process $W : \Omega \times \mathbb{T} \to U$ is a martingale.*

Proof. ★ We will show that W is a martingale by showing $\sum_{a\in A} ap_a = 0$, c.f. Lemma 7.1.1. We have that

$$\begin{aligned}
\sum_{a\in A} ap_a &= E[\triangle W_t] \\
&= E\left[\sum_{i=1}^{\gamma} \sqrt{\triangle t}\,\omega_i(t) e_i\right] \\
&= \sum_{i=1}^{\gamma} \sqrt{\triangle t} e_i E[\omega_i(t)] \\
&= \sum_{i=1}^{\gamma} \sqrt{\triangle t} e_i \cdot 0 \\
&= 0.
\end{aligned}$$

■

Proposition 7.2.2. *For Anderson process $W : \Omega \times \mathbb{T} \to U$ we have*

$$E[\|W_t\|^2] = \gamma t.$$

Proof. ★ Using orthonormality and independence we get that

$$\begin{aligned}
E[\|W_t\|^2] &= E\left[\left\|\sum_{i=1}^{\gamma}\sum_{s=0}^{t}\sqrt{\triangle t}\omega_i(s)e_i\right\|^2\right] \\
&= E\left[\left\langle\sum_{i=1}^{\gamma}\sum_{s=0}^{t}\sqrt{\triangle t}\omega_i(s)e_i, \sum_{i=1}^{\gamma}\sum_{s=0}^{t}\sqrt{\triangle t}\omega_i(s)e_i\right\rangle\right] \\
&= \sum_{i=1}^{\gamma}\sum_{s=0}^{t}\sum_{j=1}^{\gamma}\sum_{r=0}^{t} E[\langle\sqrt{\triangle t}\omega_i(s)e_i, \sqrt{\triangle t}\omega_j(r)e_j\rangle] \\
&= \sum_{i=1}^{\gamma}\sum_{s=0}^{t}\sum_{j=1}^{\gamma}\sum_{r=0}^{t} E[\sqrt{\triangle t}\omega_i(s)\sqrt{\triangle t}\omega_j(r)\langle e_i, e_j\rangle] \\
&= \sum_{i=1}^{\gamma}\sum_{s=0}^{t}\sum_{r=0}^{t} E[\triangle t\omega_i(s)\omega_i(r)\langle e_i, e_i\rangle] \\
&= \sum_{i=1}^{\gamma}\sum_{s=0}^{t}\sum_{r=0}^{t} E[\triangle t\omega_i(s)\omega_i(r)] \\
&= \sum_{i=1}^{\gamma}\sum_{s=0}^{t} \triangle t E[\omega_i(s)^2] \\
&= \sum_{i=1}^{\gamma}\sum_{s=0}^{t} \triangle t \\
&= \sum_{i=1}^{\gamma}\frac{t}{\triangle t}\triangle t \\
&= \gamma t.
\end{aligned}$$

■

Let $A \subset U$ be the collection such that $P[\triangle W_t = a] = p_a$ for $a \in A$. Since

$$\triangle W_t = \sum_{i=1}^{\gamma}\sqrt{\triangle t}\omega_i(t)e_i$$

we see that every $a \in A$ is of the form $a = (a_i)_{i=1}^{\gamma}$ where each $a_i = \langle a, e_i\rangle$ is either equal to $-\sqrt{\triangle t}$ or $+\sqrt{\triangle t}$. Note that $p_a = \frac{1}{2^\gamma}$ for all $a \in A$ and clearly $p_a \geq 0$ for all $a \in A$ and $\sum_{a\in A} p_a = 1$. Moreover we have that $a_i^2 = \triangle t$ for any $a \in A$ and when $i \neq j$ we have $a_i a_j$ is either equal to $-\triangle t$ or $\triangle t$.

Proposition 7.2.3. ★ *The covariance operator C^W of W is S-bounded.*

Proof. ★ For any $u \in U$ we have that

$$\begin{aligned}
\|C^W(u)\| &= \|\frac{1}{\triangle t}\sum_{a\in A}\langle a, u\rangle a p_a\| \\
&\leq \frac{1}{\triangle t}\sum_{a\in A}\|a\|^2\|u\|p_a
\end{aligned}$$

$$= \sum_{a\in A} \|1\|^2 p_a \|u\|$$
$$= \|u\|.$$

Consequently, C^W is S-bounded. ■

Proposition 7.2.4. ★ *Let $C^W_{ij} = \langle C^W e_i, e_j\rangle$ where C^W is the covariance operator for W. We get that*

$$C^W_{ij} = \begin{cases} 1 & ; i = j \\ 0 & ; i \neq j \end{cases}.$$

Proof. ★

$$C^W_{ij} = \frac{1}{\triangle t}\sum_{a\in A} \langle a, e_i\rangle\langle a, e_j\rangle p_a$$
$$= \frac{1}{\triangle t}\sum_{a\in A} a_i a_j p_a$$

For $i = j$ we get that

$$C^W_{ii} = \frac{1}{\triangle t}\sum_{a\in A} a_i^2 p_a$$
$$= \frac{1}{\triangle t}\sum_{a\in A} \triangle t p_a$$
$$= \sum_{a\in A} p_a$$
$$= 1.$$

Let $i \neq j$ and consider $A_{ij} = \{a_{ij} = a_i a_j \mid a \in A\}$. Note that there are just as many $a'_{ij}s$ equal to $-\triangle t$ as there are $a'_{ij}s$ equal to $\triangle t$. Hence we get that

$$C^W_{ij} = \frac{1}{\triangle t}\sum_{a\in A} a_i a_j p_a$$
$$= \frac{1}{\triangle t}\sum_{a\in A} a_i a_j \frac{1}{2^\gamma}$$
$$= 0.$$

■

Proposition 7.2.5. ★ *For a ${}^*\mathbb{F}$-linear functional ψ we have that*

$$E[\psi(W_t)^2] = t\sum_{j=1}^{\gamma} \psi(e_j)^2.$$

Moreover, if $\psi \in U^*$ then

$$E[\psi(W_t)^2] = t\|\psi\|^2.$$

Proof. ★ For linear $\psi : U \to {}^*\mathbb{R}$ we get the following calculation:

$$\begin{aligned} E[\psi(W_t)^2] &= E\left[\psi\left(\sum_{i=1}^{\gamma}\sum_{s=0}^{t}\sqrt{\triangle t}\omega_i(s)e_i\right)^2\right] \\ &= E\left[\sum_{i=1}^{\gamma}\sum_{s=0}^{t}\sum_{j=1}^{\gamma}\sum_{r=0}^{t}\triangle t\omega_i(s)\omega_j(r)\psi(e_i)\psi(e_j)\right] \\ &= \sum_{i=1}^{\gamma}\sum_{s=0}^{t}\sum_{j=1}^{\gamma}\sum_{r=0}^{t}\triangle t E\left[\omega_i(s)\omega_j(r)\right]\psi(e_i)\psi(e_j) \end{aligned}$$

Consider when $s \neq r$ and $r > s$. We get that

$$\begin{aligned} E[\omega_i(r)\omega_j(s)] &= E[\omega_i(r)E[\omega_j(s) \mid \mathcal{F}_s]] \\ &= E[\omega_i(r) \cdot 0] \\ &= 0. \end{aligned}$$

Hence we get

$$E[\psi(W_t)^2] = \sum_{i=1}^{\gamma}\sum_{s=0}^{t}\sum_{j=1}^{\gamma}\triangle t\psi(e_i)\psi(e_j)E[\omega_i(s)\omega_j(s)].$$

Now since $\omega_i(s)$ and $\omega_j(s)$ are independent for $i \neq j$ we have that

$$E[\omega_i(s)\omega_j(s)] = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Finally we get that

$$\begin{aligned} E[\psi(W_t)^2] &= \sum_{s=0}^{t}\sum_{j=1}^{\gamma}\triangle t\psi(e_j)^2 \\ &= \frac{t}{\triangle t}\triangle t\sum_{j=1}^{\gamma}\psi(e_j)^2 \\ &= t\sum_{j=1}^{\gamma}\psi(e_j)^2 \end{aligned}$$

If $\psi \in U^*$ then by Riesz representation theorem Theorem 5.2.1 we have that there exists a unique $y \in U$ such that $\psi(x) = \langle x, y\rangle$ for all $x \in U$ and $\|\psi\| = \|y\|$. Moreover we have that

$$\sum_{j=1}^{\gamma}\psi(e_j)^2 = \sum_{j=1}^{\gamma}\langle e_j, y\rangle^2 = \|y\|^2 = \|\psi\|^2.$$

Consequently, we get that

$$E[\psi(W_t)^2] = t\|\psi\|^2.$$

■

Now we want to consider the stochastic integral with respect to W. For $X : \Omega \times \mathbb{T} \to \mathcal{HS}(U, V)$ let

$$M_t(\omega) = \left(\int X dW \right)(\omega, t) = \sum_{s=0}^{t} X_s(\omega)(\triangle W_s(\omega))$$

and let $M_k(t) = \langle M(t), f_k \rangle$ denote the k'th component of M.

Proposition 7.2.6. *For $M = \int X dW$ we have that*

$$E[M_k(t)^2] = E\left[\int_0^t \|X_s^*(f_k)\|^2 ds \right].$$

Proof. This proof is a detailed version of the proof of Theorem 4 in [Lin83].

$$\begin{aligned} E[M_k(t)^2] &= E\left[\left(\sum_{s=0}^{t} \langle X_s(\triangle W(s)), f_k \rangle \right)^2 \right] \\ &= E\left[\left(\sum_{s=0}^{t} \sum_{j=1}^{\gamma} \omega_j(s) \sqrt{\triangle t} \langle X_s(e_j), f_k \rangle \right)^2 \right] \\ &= \triangle t \sum_{s=0}^{t} \sum_{j=1}^{\gamma} \sum_{r=0}^{t} \sum_{i=1}^{\gamma} E\left[\omega_j(s) \omega_i(r) \langle X_s(e_j), f_k \rangle \langle X_r(e_i), f_k \rangle \right] \end{aligned}$$

Now consider the case when $s \neq r$ and $s > r$. Since $\omega_i(r) \langle X_s(e_j), f_k \rangle \langle X_r(e_i), f_k \rangle$ is $\mathcal{F}_s$-measurable and $E[\omega_j(s) \mid \mathcal{F}_s] = 0$ we get using the tower property that

$$\begin{aligned} &E[\omega_j(s)\omega_i(r)\langle X_s(e_j), f_k \rangle \langle X_r(e_i), f_k \rangle] \\ &= E[E[\omega_j(s)\omega_i(r)\langle X_s(e_j), f_k \rangle \langle X_r(e_i), f_k \rangle \mid \mathcal{F}_s]] \\ &= E[\omega_i(r)\langle X_s(e_j), f_k \rangle \langle X_r(e_i), f_k \rangle E[\omega_j(s) \mid \mathcal{F}_s]] \\ &= 0. \end{aligned}$$

Hence,

$$E[M_k(t)^2] = \triangle t \sum_{s=0}^{t} \sum_{j=1}^{\gamma} \sum_{i=1}^{\gamma} E\left[\omega_j(s)\omega_i(s)\langle X_s(e_j), f_k \rangle \langle X_r(e_i), f_k \rangle \right].$$

Now consider the case when $i \neq j$. Then since $E[\omega_i(s)\omega_j(s) \mid \mathcal{F}_s] = 0$ we get that

$$\begin{aligned} &E\left[\omega_j(s)\omega_i(s)\langle X_s(e_j), f_k \rangle \langle X_r(e_i), f_k \rangle\right] \\ &= E\left[E[\omega_j(s)\omega_i(s)\langle X_s(e_j), f_k \rangle \langle X_r(e_i), f_k \rangle \mid \mathcal{F}_s\right] \\ &= E\left[\langle X_s(e_j), f_k \rangle \langle X_r(e_i), f_k \rangle E[\omega_j(s)\omega_i(s) \mid \mathcal{F}_s\right] \\ &= 0. \end{aligned}$$

Thus, since $\omega_i(s)^2 = 1$ we get that

$$E[M_k(t)^2] = E\left[\sum_{s=0}^{t} \sum_{j=1}^{\gamma} \triangle t \langle X_s(e_j), f_k \rangle^2 \right].$$

Finally, we get that

$$\begin{aligned} E[M_k(t)^2] &= E\left[\sum_{s=0}^{t}\sum_{j=1}^{\gamma} \triangle t \langle X_s(e_j), f_k\rangle^2\right] \\ &= E\left[\sum_{s=0}^{t}\sum_{j=1}^{\gamma} \triangle t \langle e_j, X_s^* f_k\rangle^2\right] \\ &= E\left[\sum_{s=0}^{t} \triangle t \|X_s^*(f_k)\|^2\right] \\ &= E\left[\int_0^t \|X_s^*(f_k)\|^2 ds\right] \end{aligned}$$

■

Corollary 7.2.7. *If $X : \Omega \times \mathbb{T} \to \mathcal{HS}(U, V)$ then $M = \int XdW$ is nearstandard almost everywhere.*

Proof. This proof is based on the proof of Theorem 4 in [Lin83]. Using Proposition 7.2.6 and Doob's inequality we get that

$$\begin{aligned} 0 &\leq E\left[\sup_{t\leq 1}\sum_{k=\theta}^{\gamma} M_k(t)^2\right] \\ &\leq 4E\left[\int_0^1 \sum_{k=\theta}^{\gamma} \|X_s^*(f_k)\|^2 ds\right] \end{aligned}$$

where the last term is infinitely close to zero since $X_s(\omega) \in \mathcal{HS}(U, V)$. ■

7.3 First approach

We will first introduce some probabilistic theory which we will impose on $L : \Omega \times \mathbb{T} \to U$ in addition to being a martingale. At the end of this section, we will present the set of suitable integrands in this case.

Definition 7.3.1.

(a) We say that $L : \Omega \times \mathbb{T} \to U$ has **weak second moments** if for every $u \in \mathrm{Ns}(U)$ we have

$$\frac{1}{\triangle t} E[\langle \triangle L_t, u\rangle^2] = \frac{1}{\triangle t}\sum_{a\in A} \langle a, u\rangle^2 p_a = \langle C^L u, u\rangle$$

is finite

(b) We say that $L : \Omega \times \mathbb{T} \to U$ has **weak second moments in the strong sense** if there exists $K \in \mathbb{R}$ such that for each $u \in \mathrm{Ns}(U)$ we have

$$\frac{1}{\triangle t} E[\langle \triangle L_t, u\rangle^2] = \frac{1}{\triangle t}\sum_{a\in A} \langle a, u\rangle^2 p_a \leq K\|u\|^2$$

Lemma 7.3.2. ★ *Assume $L : \Omega \times \mathbb{T} \to U$ is a martingale. Then*

$$E[\langle L(t), u\rangle^2] = \frac{t}{\triangle t} E[\langle \triangle L_t, u\rangle^2].$$

Proof. ★ Since $L(0) = 0$ we have that $\langle L_t, u\rangle = \sum_{r<t} \langle \triangle L_r, u\rangle$. Thus we get that

$$\begin{aligned} E[\langle L_t, u\rangle^2] &= E\left[\sum_{s<t} \langle \triangle L_s, u\rangle \sum_{r<t} \langle \triangle L_r, u\rangle\right] \\ &= \sum_{s<t} E\left[\langle \triangle L_s, u\rangle^2\right] + \sum_{0\le r\neq s<t} E[\langle \triangle L_s, u\rangle\langle \triangle L_r, u\rangle] \end{aligned}$$

Since for $r < s$ we have that $\langle \triangle L_s, u\rangle$ and $\langle \triangle L_r, u\rangle$ are independent and using Remark 7.1.2 we get that

$$\begin{aligned} E[\langle \triangle L_s, u\rangle\langle \triangle L_r, u\rangle] &= E[\langle \triangle L_s, u\rangle]E[\langle \triangle L_r, u\rangle] \\ &= \left(\sum_{a\in A} \langle a, u\rangle p_a\right)^2 \\ &= 0. \end{aligned}$$

Note also that if $t = K\triangle t$ for some $K \in {}^*\mathbb{N}$ then we get for some constant a that

$$\sum_{s<t} a = Ka = \frac{K\triangle t}{\triangle t} a = \frac{t}{\triangle t} a.$$

Hence we further get that

$$\begin{aligned} E[\langle L_t, u\rangle^2] &= \sum_{s<t} E\left[\langle \triangle L_s, u\rangle^2\right] + \sum_{0\le r\neq s<t} E[\langle \triangle L_s, u\rangle\langle \triangle L_r, u\rangle] \\ &= \frac{t}{\triangle t} E\left[\langle \triangle L_s, u\rangle^2\right] + 0. \end{aligned}$$

■

Lemma 7.3.3 (Tom Lindstrøm)**.** *Assume that $L : \Omega \times \mathbb{T} \to U$ is a martingale and that it has weak second moments. Then for all $t \in \mathbb{T}$ and $u \in \mathrm{Ns}(U)$ we have that*

$$E[\langle L(t), u\rangle^2]$$

is finite.

Proof. ★ By Lemma 7.3.2 we have that

$$E[\langle L(t), u\rangle^2] = \frac{t}{\triangle t} E[\langle \triangle L_t, u\rangle^2].$$

Since $t \in \mathbb{T}$ is finite and $\frac{1}{\triangle t} E[\langle \triangle L_t, u\rangle^2]$ is finite for $u \in \mathrm{Ns}(U)$ we get the result. ■

Lemma 7.3.4. *Assume that $L : \Omega \times \mathbb{T} \to U$ is a martingale and L has weak second moments in the strong sense. Then for all $t \in \mathbb{T}$ and $u \in \mathrm{Ns}(U)$ we have that there exists $K \in \mathbb{R}$ such that*

$$E[\langle L(t), u\rangle^2] \le tK\|u\|^2.$$

Proof. ★ Using Lemma 7.3.2 and that L has weak second moments in the strong sense gives us that

$$\begin{aligned} E[\langle L(t), u\rangle^2] &= t\frac{1}{\triangle t}E[\langle \triangle L(t), u\rangle^2] \\ &\leq tK\|u\|^2. \end{aligned}$$

■

Proposition 7.3.5. ★ *Suppose that $L : \Omega \times \mathbb{T} \to U$ is a martingale with weak second moments in the strong sense and that $X : \Omega \times \mathbb{T} \to L(U, V)$ is strictly Hilbert-Schmidt for every (ω, t). Then there exists a real $K \in \mathbb{R}$ such that*

$$E\left[\left\|\int_s^t XdL\right\|^2\right] \leq K\|X\|^2_{[s,t]}$$

Proof. ★ Using the orthonormal basis $(f_n)_{n\leq\eta}$ of V we get that

$$\begin{aligned} E\left[\left\|\int_s^t XdL\right\|^2\right] &= E\left[\left\|\sum_{r=s}^{t} X(\omega, r)(\triangle L_r(\omega))\right\|^2\right] \\ &= E\left[\sum_{n=1}^{\eta}\left\langle\sum_{r=s}^{t} X(\omega, r)(\triangle L_r(\omega)), f_n\right\rangle^2\right] \\ &= \sum_{n=1}^{\eta}\sum_{r=s}^{t}\sum_{k=s}^{t} E\left[\langle X(\omega, r)(\triangle L_r(\omega)), f_n\rangle \langle X(\omega, k)(\triangle L_k(\omega)), f_n\rangle\right] \end{aligned}$$

When $r \neq k$ and $k > r$ we get using the tower property that

$$\begin{aligned} &E[\langle \triangle L_r(\omega), X_r^*(\omega)(f_n)\rangle\langle \triangle L_k(\omega), X_k^*(\omega)(f_n)\rangle] \\ &= E[\langle \triangle L_r(\omega), X_r^*(\omega)(f_n)\rangle E[\langle \triangle L_k(\omega), X_k^*(\omega)(f_n)\rangle \mid \mathcal{F}_k]] \\ &= E[\langle \triangle L_r(\omega), X_r^*(\omega)(f_n)\rangle 0] \\ &= 0 \end{aligned}$$

since L is a martingale. Hence we further get that

$$\begin{aligned} E\left[\left\|\int_s^t XdL\right\|^2\right] &= \sum_{n=1}^{\eta}\sum_{r=s}^{t} E\left[\langle X(\omega, r)(\triangle L_r(\omega)), f_n\rangle^2\right] \\ &= \sum_{n=1}^{\eta}\sum_{r=s}^{t} E\left[\langle \triangle L_r(\omega), X^*(\omega, r)f_n\rangle^2\right] \\ &= \sum_{n=1}^{\eta}\sum_{r=s}^{t} E\left[E\left[\langle \triangle L_r(\omega), X^*(\omega, r)f_n\rangle^2 \mid \mathcal{F}_r\right]\right]. \end{aligned}$$

Since $\triangle L_r$ is independent of $\mathcal{F}_r$ we get that

$$E\left[E\left[\langle \triangle L_r(\omega), X^*(\omega, r)f_n\rangle^2 \mid \mathcal{F}_r\right]\right] = E\left[E\left[\langle \triangle L_r(\omega), X^*(\omega, r)f_n\rangle^2\right]\right].$$

Because X is strictly Hilbert-Schmidt we can use Lemma 5.4.8 on the adjoint X^* and this together with Lemma 7.3.4 we get that

$$E\left[E\left[\langle \triangle L_r(\omega), X^*(\omega,r)f_n\rangle^2\right]\right] \le K\triangle t E\left[\|X^*(\omega,r)(f_k)\|^2\right]$$

for some real K. But then we get that

$$\begin{aligned} E\left[\left\|\int_s^t XdL\right\|^2\right] &\le \sum_{n=1}^{\eta}\sum_{r=s}^{t} K\triangle t E\left[\|X^*(\omega,r)(f_k)\|^2\right] \\ &= K\triangle t\sum_{r=s}^{t} E\left[\|X^*(\omega,r)\|^2_{\mathcal{HS}}\right] \\ &= K\triangle t\sum_{r=s}^{t} E\left[\|X(\omega,r)\|^2_{\mathcal{HS}}\right] \\ &= KE\left[\sum_{r=s}^{t}\|X(\omega,r)\|^2_{\mathcal{HS}}\triangle t\right] \\ &= K\|X\|^2_{[s,t]} \end{aligned}$$

■

Corollary 7.3.6. ★ *Let $M : \Omega\times\mathbb{T}\to V$ be defined by*

$$M_t(\omega) = \int_0^t XdL = \sum_{s=0}^{t} X(\omega,s)(\triangle L(s,\omega)).$$

If L is a martingale with weak second moments in the strong sense, $X(t,\omega)$ is strictly Hilbert-Schmidt for all s and ω and $\|X\|_{[0,t]}$ is finite for every $t\in\mathbb{T}$, then M_t is finite almost surely for all $t\in\mathbb{T}$.

Proof. ★ By the proposition above we have that

$$E\left[\|M_t\|^2\right] \le K\|X\|^2_{[s,t]}$$

for some real K. Since $\|X\|_{[0,t]}$ is finite for all $t\in\mathbb{T}$ we can conclude. ■

Therefore, whenever $L : \Omega\times\mathbb{T}\to U$ is a martingale with weak second moments in the strong sense and we let $\mathcal{HS}_s(U,V)$ denote the space of strictly Hilbert-Schmidt operators, we define the space of suitable integrands to be

$$\mathcal{H}(U,V) := \{X : \Omega\times\mathbb{T}\to\mathcal{HS}_s(U,V) \mid X \text{ adapted}, \|X\|_{[s,t]} < \infty \text{ for } s,t\in\mathbb{T}\}.$$

7.4 Second approach: covariance operator

Let $H\subset U\subset {}^*H$ be a hyperfinite dimensional space of an underlying Hilbert space H with inner product $\langle\cdot,\cdot\rangle$. As before, we let $L : \Omega\times\mathbb{T}\to U$ be a hyperfinite random walk om U which is also a martingale. Recall that $C^L\in L(U)\simeq L(U^*,U)$ the covariance operator of L is given by $C^L(x) = \frac{1}{\triangle t}\sum_{a\in A}\langle a,x\rangle ap_a$.

Remark 7.4.1. ★ Now we will see a connection with the first approach. Since $C^L \in L(U)$ is positive there exists a self-adjoint operator $R \in L(U)$ such that $RR = C^L$. In other words, R is the *square root of* C^L. If R is bounded, i.e., there exists $K \in \mathbb{R}$ such that $\|Ru\| \leq K\|u\|$ for all $u \in U$, then L has weak second moments in the strong sense. Indeed, we have that

$$\begin{aligned}\frac{1}{\triangle t}E[\langle \triangle L_t, u\rangle] &= \frac{1}{\triangle t}\sum_{a\in A}\langle a, u\rangle^2 p_a \\ &= \langle C^L u, u\rangle \\ &= \langle Ru, Ru\rangle \\ &= \|Ru\|^2 \\ &\leq K^2\|u\|^2.\end{aligned}$$

Let $M : \Omega \times \mathbb{T} \to V$ be the integral operator

$$M(\omega, t) = \left(\int_0^t X dL\right)(\omega).$$

We will now again show that under some conditions that $E[\|M_t\|^2]$ is finite. But this time we do *not* require the integrand X to be strictly Hilbert-Schmidt.

Proposition 7.4.2. ★ *Let $X : \Omega \times \mathbb{T} \to L(U, V)$ and suppose $L : \Omega \times \mathbb{T} \to U$ is a martingale with covariance operator $C^L = RR$ such that $R \in L(U)$ is S-bounded. If $\|X\|_{[0,t]}$ is finite, then $E[\|M_t\|^2]$ is finite.*

If X was strictly Hilbert-Schmidt this would follow by Remark 7.4.1 and Proposition 7.3.5. But here we only require $\|X\|_{[0,t]}$ to be finite.

Proof. ★ We make the following calculations:

$$\begin{aligned}E[\|M_t\|^2] &= E\left[\left\|\sum_{s=0}^{t} X_s(\omega)(\triangle L_s(\omega))\right\|^2\right] \\ &= E\left[\sum_{n\leq\eta}\left\langle\sum_{s=0}^{t} X_s(\omega)(\triangle L_s(\omega)), f_n\right\rangle^2\right] \\ &= \sum_{n\leq\eta}\sum_{s=0}^{t}\sum_{r=0}^{t} E[\langle \triangle L_s(\omega), X_s^*(\omega)(f_k)\rangle\langle \triangle L_r(\omega), X_r^*(\omega)(f_k)\rangle]\end{aligned}$$

When $r \neq s$ and $s > r$ we get using the tower property that

$$\begin{aligned}&E[\langle \triangle L_r(\omega), X_r^*(\omega)(f_k)\rangle\langle \triangle L_s(\omega), X_s^*(\omega)(f_k)\rangle] \\ &= E[\langle \triangle L_r(\omega), X_r^*(\omega)(f_k)\rangle E[\langle \triangle L_s(\omega), X_s^*(\omega)(f_k)\rangle \mid \mathcal{F}_s]] \\ &= E[\langle \triangle L_r(\omega), X_r^*(\omega)(f_k)\rangle 0] \\ &= 0\end{aligned}$$

since L is a martingale. Thus we further get that

$$
\begin{aligned}
E[\|M_t\|^2] &= \sum_{n\leq\eta}\sum_{s=0}^{t} E[\langle\triangle L_s(\omega), X_s^*(\omega)(f_n)\rangle^2] \\
&= \sum_{n\leq\eta}\sum_{s=0}^{t} E[E[\langle\triangle L_s(\omega), X_s^*(\omega)(f_n)\rangle^2 \mid \mathcal{F}_s]] \\
&= \sum_{n\leq\eta}\sum_{s=0}^{t} E[\sum_{a\in A}\langle a, X_s^*(\omega)f_n\rangle^2 p_a] \\
&= \sum_{n\leq\eta}\sum_{s=0}^{t} \triangle t E[\langle C^L X_s^*(\omega)f_n, X_s^*(\omega)f_n\rangle] \\
&= \sum_{s=0}^{t} \triangle t \sum_{n\leq\eta} E[\langle RX_s^*(\omega)f_n, RX_s^*(\omega)f_n\rangle] \\
&= \sum_{s=0}^{t} \triangle t E[\sum_{n\leq\eta} \|RX_s^*(\omega)(f_n)\|^2] \\
&= \sum_{s=0}^{t} \triangle t E[\|RX_s^*(\omega)\|_{\mathcal{HS}}^2]
\end{aligned}
$$

Using Proposition 5.4.4 and Proposition 5.4.3, we also have that

$$
\|RX_s^*(\omega)\|_{\mathcal{HS}}^2 \leq \|R\|^2\|X_s^*(\omega)\|_{\mathcal{HS}}^2 = \|R\|^2\|X_s(\omega)\|_{\mathcal{HS}}^2
$$

Thus we get

$$
\begin{aligned}
E[\|M_t\|^2] &= \sum_{s=0}^{t} \triangle t E[\|RX_s^*(\omega)\|_{\mathcal{HS}}^2] \\
&\leq \|R\|^2 \sum_{s=0}^{t} \triangle t E[\|X_s(\omega)\|_{\mathcal{HS}}] \\
&= \|R\|^2\|X\|_{[0,t]}^2
\end{aligned}
$$

Hence $E[\|M_t\|^2]$ is finite since R is bounded and $\|X\|_{[0,t]}$ is finite. ■

Corollary 7.4.3. ★ *For $X : \Omega \times \mathbb{T} \to L(U, V)$ and $L : \Omega \times \mathbb{T} \to U$ assume that $\|X\|_{[0,t]}$ is finite, L is a martingale and that $R \in L(U)$ is bounded. Then M is a λ^2-martingale (Definition 6.1.2).*

Proof. ★ We have already shown in the proposition above that $E[\|M_t\|^2]$ is finite for every $t \in \mathbb{T}$. So we only need to show that M is a hyperfinite martingale.

Let $s \leq t$, $s, t \in \mathbb{T}$ and let $A \in \mathcal{F}_s$. We have

$$\begin{aligned} E[1_A(M_t - M_s)] &= E\left[1_A\left(\sum_{r=0}^{t} X_r(\omega)(\triangle L_r(\omega) - \sum_{r=0}^{s} X_r(\omega)(\triangle L_r(\omega)\right)\right] \\ &= E\left[1_A\left(\sum_{r=s}^{t} X_r(\omega)(\triangle L_r(\omega)\right)\right] \\ &= E\left[1_A\left(\sum_{r=s}^{t} X_r(\omega)(0)\right)\right] \\ &= E[0] \\ &= 0. \end{aligned}$$

since L is a martingale and $X_r(\omega)$ is linear. ■

Proposition 7.4.4. ★ *Let $C^L = RR \in L(U)$ be the covariance operator of a hyperfinite random walk $L : \Omega \times \mathbb{T} \to U$. If L is a martingale and R is bounded, then for integrand $X : \Omega \times \mathbb{T} \to \mathcal{HS}(U, V)$ we have that*

$$M(\omega, t) = \left(\int X dL\right)(\omega, t) = \sum_{s=0}^{t} X(\omega, s)(\triangle L(s, \omega))$$

is nearstandard almost everywhere.

Proof. ★ Let $M_k(t)$ denote the k'th component of M. We have that

$$\begin{aligned} E[M_k(t)^2] &= E[\langle M(t), f_k\rangle^2] \\ &= E\left[\left\langle \sum_{s=0}^{t} \triangle M(s), f_k \right\rangle^2\right] \\ &= \sum_{s=0}^{t}\sum_{r=0}^{t} E[\langle \triangle M(s), f_k\rangle\langle \triangle M(r), f_k\rangle] \end{aligned}$$

Now suppose $s \neq r$ and $s > r$ then using the tower property and that M is martingale c.f. Section 7.4 we get that

$$\begin{aligned} E[\langle M(s), f_k\rangle\langle M(r), f_k\rangle] &= E[E[\langle M(s), f_k\rangle\langle M(r), f_k\rangle \mid \mathcal{F}_s]] \\ &= E[\langle M(r), f_k\rangle E[\langle M(s), f_k\rangle \mid \mathcal{F}_s]] \\ &= 0. \end{aligned}$$

Thus we are left with

$$\begin{aligned} E[M_k(t)^2] &= \sum_{s=0}^{t} E[\langle \triangle M(s), f_k\rangle^2] \\ &= \sum_{s=0}^{t} E[\langle X_s(\triangle L(s)), f_k\rangle^2] \\ &= \sum_{s=0}^{t} E[\langle \triangle L(s), X_s^* f_k\rangle^2]. \end{aligned}$$

Further since

$$\begin{aligned} E[\langle \triangle L(s), X_s^* f_k \rangle^2] &= E[E[\langle \triangle L(s), X_s^* f_k \rangle^2 \mid \mathcal{F}_s]] \\ &= \sum_{a \in A} E[\langle a, X_s^* f_k \rangle^2 p_a] \end{aligned}$$

we get that

$$\begin{aligned} E[M_k(t)^2] &= \sum_{s=0}^{t} \sum_{a \in A} E[\langle a, X_s^* f_k \rangle^2 p_a] \\ &= \sum_{s=0}^{t} \triangle t E[\langle C^L X_s^*(f_k), X_s^*(f_k) \rangle] \\ &= \sum_{s=0}^{t} \triangle t E[\|R X_s^*(f_k)\|^2] \\ &= E\left[\int_0^t \|R X_s^*(f_k)\|^2 ds\right] \end{aligned}$$

Finally, using Doob's inequality we get that for infinite $\theta \leq \eta$

$$\begin{aligned} E\left[\sup_t \sum_{k=\theta}^{\eta} M_k(t)^2\right] &\leq 4E\left[\int_0^\infty \sum_{k=\theta}^{\eta} \|R X_s^*(f_k)\|^2 ds\right] \\ &\leq E\left[\int_0^\infty \|R\| \sum_{k=\theta}^{\eta} \|X_s^*(f_k)\|^2 ds\right] \\ &\approx 0 \end{aligned}$$

since we assumed that $X_s(\omega) \in \mathcal{HS}(U, V)$ and R is bounded.[1]

■

In light of these results, if $L : \Omega \times \mathbb{T} \to U$ is a martingale and the square root of C^L, $R \in L(U)$ is bounded, then the set of suitable integrands are

$$\mathcal{H}(U, V) \coloneqq \{X : \Omega \times \mathbb{T} \to \mathcal{HS}(U, V) \mid X \text{ adapted}, \|X\|_{[s,t]} < \infty \text{ for } s, t \in \mathbb{T}\}.$$

[1]Actually, by Lemma 5.4.6 we have that RX is Hilbert-Schmidt and hence

$$\sum_{k=\theta}^{\eta} \|R X_s^*(f_k)\|^2$$

is an infinitesimal for every infinite $\theta \leq \eta$.

CHAPTER 8

Hyperfinite Cylindrical Processes

8.1 Hyperfinite cylindrical processes

Before we define cylindrical processes we need the following definition.

Definition 8.1.1. An internal process $Y : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ is **nearstandard** if there is a set Ω' with Loeb measure 1 such that if $\omega \in \Omega'$ then $Y(\omega, t)$ is finite/nearstandard for all finite $t \in \mathbb{T}$.

Now let $E \subset F \subset {}^*E$ be a hyperfinite dimensional space and recall that $F^* = \{\phi|_F \mid \phi \in {}^*E^*\}$ and $\text{Fin}(F^*) = \{\phi \in F^* \mid \phi \text{ S-bounded}\}$.

Definition 8.1.2. ★ We call an internal stochastic process $X : \Omega \times \mathbb{T} \to F$ for a **hyperfinite cylindrical process** if $\phi(X)$ is nearstandard process in ${}^*\mathbb{R}$ for all $\phi \in \text{Fin}(F^*)$.

Why we define it this way is because of the following proposition.

Proposition 8.1.3. ★ *Let $x \in F$. We have that x is finite if and only if for every $\phi \in \text{Fin}(F^*)$ we have that $\phi(x) \in \text{Fin}({}^*\mathbb{R})$.*

Proof. The proof is almost the same as in the proof of Proposition 4.2.3 in [LW15]. But this was proved for E and ${}^*E^*$ and not F and F^*. Suppose $x \in \text{Fin}(F)$. Then for all $\phi \in \text{Fin}(F^*)$ we have that $|\phi(x)| \leq \|\phi\|\|x\|$ is finite. Hence $\phi(x) \in \text{Fin}({}^*\mathbb{R})$ for all $\phi \in \text{Fin}(F^*)$. For the converse, assume that $x \in F$ is not finite. Then $\|x\|$ is infinite. Let B_{E^*} denote the unit ball in E^*. Using the formula

$$\begin{aligned}\|x\| &= \sup\{|\phi(x)| \mid \phi \in {}^*E^*, \|\phi\| \leq 1\} \\ &= \sup\{|\phi(x)| \mid \phi \in F^*, \|\phi\| \leq 1\}\end{aligned}$$

and the transfer principle, we have for $\epsilon > 0$ that there exists a $\phi \in {}^*B_{E^*}$ such that $\|x\| - \epsilon \leq |\phi(x)| \leq \|x\|$. Consequently, if we consider the restriction $\psi = \phi|_F$ then we have found a $\psi \in \text{Fin}(F^*)$ such that $|\psi(x)|$ is infinite. ■

Hence a process $X : \Omega \times \mathbb{T} \to F$ is cylindrical if and only if there is a set Ω' with probability 1 such that for $\omega \in \Omega'$ we have that $X(\omega, t)$ is finite for all finite $t \in \mathbb{T}$.

Example 8.1.4 (Anderson's Process)**.** We will show that Anderson's process from Section 7.2 is cylindrical. Recall that Anderson's process is given by

$$W_t(\omega) = \sum_{i=1}^{\gamma}\sum_{s=0}^{t} \sqrt{\triangle t}\omega_i(s)e_i.$$

Let $\phi \in \text{Fin}(F^*)$ and consider

$$X_t(\omega) = \phi(W_t(\omega)) = \phi\left(\sum_{i=1}^{\gamma}\sum_{s=0}^{t} \sqrt{\triangle t}\omega_i(s)e_i\right) = \sum_{i=1}^{\gamma}\sum_{s=0}^{t} \sqrt{\triangle t}\omega_i(s)\phi(e_i).$$

We get that

$$\begin{aligned}
E[\triangle X_t(\omega)^2 \mid \mathcal{F}_t] &= E\left[\left(\sum_{i=1}^{\gamma} \sqrt{\triangle t}\omega_i(s)\phi(e_i)\right)^2 \mid \mathcal{F}_t\right] \\
&= E\left[\sum_{i=1}^{\gamma}\sum_{j=1}^{\gamma} \triangle t\omega_i(t)\omega_j(t)\phi(e_i)\phi(e_j) \mid \mathcal{F}_t\right] \\
&= \sum_{i=1}^{\gamma}\sum_{j=1}^{\gamma} \triangle t\phi(e_i)\phi(e_j)E\left[\omega_i(t)\omega_j(t) \mid \mathcal{F}_t\right].
\end{aligned}$$

When $i = j$ we have that $E\left[\omega_i(t)^2 \mid \mathcal{F}_t\right] = 1$ and when $i \neq j$ we have that $\omega_i(t)$ and $\omega_j(t)$ are independent and thus

$$E\left[\omega_i(t)\omega_j(t) \mid \mathcal{F}_t\right] = E\left[\omega_i(t) \mid \mathcal{F}_t\right]E\left[\omega_j(t) \mid \mathcal{F}_t\right] = 0.$$

Hence we get that

$$E[\triangle X_t(\omega)^2 \mid \mathcal{F}_t] = \sum_{i=1}^{\gamma} \phi(e_i)^2\triangle t.$$

By Riesz theorem Theorem 5.2.1, we have that there exists a unique $y \in F$ such that $\phi = \phi_y = \langle \cdot, y\rangle$ and $\|\phi_y\| = \|y\|$. As $y = \sum_{i=1}^{\gamma}\langle e_i, y\rangle e_i = \sum_{i=1}^{\gamma} \phi_y(e_i)e_i$ we get $\|\phi_y\|^2 = \|y\|^2 = \sum_{i=1}^{\gamma} \phi_y(e_i)^2$. Thus we further get that

$$E[\triangle X_t(\omega)^2 \mid \mathcal{F}_t] = \|\phi\|^2\triangle t.$$

Moreover we have that by using Remark 4.1.8

$$\begin{aligned}
E[X_t^2] &= E[[X](t)] \\
&= \sum_{s=0}^{t} E[\triangle X_t(\omega)^2] \\
&= \sum_{s=0}^{t} E[E[\triangle X_t(\omega)^2 \mid \mathcal{F}_t]] \\
&= \sum_{s=0}^{t} \|\phi\|^2\triangle t \\
&= \|\phi\|^2\frac{t}{\triangle t}\triangle t \\
&= \|\phi\|^2 t
\end{aligned}$$

Hence X is even a λ^2-martingale. $X(t)$ is finite almost everywhere for finite $t \in \mathbb{T}$. Indeed, let $t \in \mathbb{T}$ be finite. Since X has finite second moments and using the equality $|x| \leq 1 + |x|^2$ for $x \in \mathbb{R}$ (by transfer this holds for $x \in {}^*\mathbb{R}$ as well), we get that

$$E[|X_t|] \leq E[1 + |X_t|^2] = 1 + \|\phi\|^2 t$$

which is finite. Thus X is nearstandard and since $\phi \in \mathrm{Fin}(F^*)$ was arbitrary we get that W is cylindrical.

Remark 8.1.5. In [Lin83] it was shown that for the infinite-dimensional Anderson process W and $P : F \to F_0$, the projection on the finite dimensional subspace F_0, we have that ${}^\circ P(W_t)$ is a standard Brownian motion on F_0. Hence for any $\phi \in \mathrm{Fin}(F^*)$ we have that ${}^\circ\phi(W_t)$ is a standard one-dimensional Brownian motion.

Example 8.1.6. ★ Let $L : \Omega \times \mathbb{T} \to U$ and $X : \Omega \times \mathbb{T} \to L(U, V)$. Consider the integral with values in V given by $M : \Omega \times \mathbb{T} \to V$

$$M(\omega, t) = \left(\int X dL\right)(\omega, t) = \sum_{s=0}^{t} X(\omega, s)(\triangle L(s, \omega)).$$

If the square root R of the covariance operator C^L of L is bounded, L is a martingale, and $X : \Omega \times \mathbb{T} \to \mathcal{HS}(U, V)$ then we have by Proposition 7.4.4 that M is even nearstandard almost everywhere. Thus by Proposition 8.1.3 we have that $\phi(M) \in \mathrm{Fin}({}^*\mathbb{R})$ almost everywhere for all $\phi \in \mathrm{Fin}(V^*)$. Hence the integral $M(\omega, t) = \int_0^t X dL$ is cylindrical.

Next consider $F = U$ where $H \subset U \subset {}^*H$ is a hyperfinite-dimensional inner product space of some underlying Hilbert space H.

Proposition 8.1.7. ★ *Let $X : \Omega \times \mathbb{T} \to U$ be a hyperfinite random walk with S-bounded covariance operator $C^X = RR$. If R is bounded then X is a hyperfinite cylindrical process.*

Proof. ★ Since R is S-bounded let $K \in \mathbb{R}$ be such that $\|Ru\| \leq K\|u\|$ for all $u \in U$ Let $\phi \in \mathrm{Fin}(U^*)$ be arbitrary. By Theorem 5.2.1 we have that $\phi = \phi_y = \langle \cdot, y\rangle$ for some unique $y \in \mathrm{Fin}(U)$. Then we have that

$$\begin{aligned}
E[\phi(X_t)^2] &= \frac{t}{\triangle t}\sum_{a \in A} \phi(a)^2 p_a \\
&= \frac{t}{\triangle t}\sum_{a \in A} \langle a, y\rangle^2 p_a \\
&= t\langle C^X y, y\rangle \\
&= t\|Ry\|^2 \\
&\leq t\|R\|^2\|y\|^2.
\end{aligned}$$

Moreover, we have that

$$E[|\phi(X_t)|] \leq 1 + E[\phi(X_t)^2] \leq 1 + tK^2\|\phi\|^2,$$

and the right-hand side is finite for every finite $t \in \mathbb{T}$. Hence $\phi(X)$ is a nearstandard process in ${}^*\mathbb{R}$. Since $\phi \in \mathrm{Fin}(U^*)$ was arbitrary we can conclude that X is a hyperfinite cylindrical process.

■

8.2 Hyperfinite cylindrical Lévy processes

Let $L : \Omega \times \mathbb{T} \to F$ be a hyperfinite random walk on a hyperfinite dimensional linear space F given by increments $A \subset F$ and probabilities $(p_a)_{a \in A}$. Recall that $F^* = \{\phi|_F \mid \phi \in {}^*H^*\}$ and let $\overrightarrow{\phi} = (\phi_1, \ldots, \phi_n)$ denote a tuple of elements ϕ_i in F^*.

Definition 8.2.1. We call $L : \Omega \times \mathbb{T} \to F$ a **hyperfinite cylindrical Lévy process** if for every tuple $\overrightarrow{\phi} \subset \mathrm{Fin}(F^*)$ we have that

$$\overrightarrow{\phi}(L_t) = (\phi_1(L_t), \ldots, \phi_n(L_t))$$

is a hyperfinite Lévy process on ${}^*\mathbb{R}^n$.

If you recall the definition of finite dimensional Lévy processes (Definition 4.3.3), it is obvious that a hyperfinite cylindrical Lévy process is cylindrical.

Let $\phi \in F^*$. Note that by linearity we have that

$$\begin{aligned}\phi(\triangle L(t)) &= \phi(L(t + \triangle t) - L(t)) \\ &= \phi(L(t + \triangle t)) - \phi(L(t)) \\ &= \triangle \phi(L(t)).\end{aligned}$$

Suppose that ϕ is injective on $A \subseteq F$, then we get that

$$\mathbb{P}[\{\omega \mid \triangle \phi(L(\omega, t)) = \phi(a)\}] = p_a.$$

If ϕ is not injective on A then there exists a collection $(a^j)_{j \in \mathcal{I}} \subset A$ such that $b = \phi(a^j)$ for all $j \in \mathcal{I}$. Then we get that

$$\mathbb{P}[\{\omega \mid \triangle \phi(L(\omega, t)) = b\}] = \sum_{j \in \mathcal{I}} p_{a^j}.$$

Let us denote this by $p_b = p_{\phi(a^j)} = \sum_{j \in \mathcal{I}} p_{a^j}$. Thus $\phi(L)$ is determined by $\phi(A) \subset {}^*\mathbb{R}$ and probabilities $(p_b)_{b \in \phi(A)} \subset {}^*\mathbb{R}$.

Example 8.2.2. ★ Let us again consider Anderson's process

$$W_t(\omega) = \sum_{s=0}^{t} \sum_{i=1}^{\gamma} \sqrt{\triangle t}\, \omega_i(s) e_i.$$

Recall that $A = \{a = (a_i)_{i=1}^{\gamma} \mid a_i = \pm\sqrt{\triangle t}\}$ and $p_a = \frac{1}{2^\gamma}$ for all $a \in A$. For some $j \in \{1, \cdots, \gamma\}$ let $\phi = \langle \cdot, e_j \rangle$. Then we get that $\phi(A) = \{-\sqrt{\triangle t}, \sqrt{\triangle t}\}$. Further let

$$B = \left\{a \in A \mid a_j = \phi(a) = \sqrt{\triangle t}\right\}$$

which is a set containing $\frac{2^\gamma}{2}$ elements. Hence we get that

$$\begin{aligned}p_{\sqrt{\triangle t}} &= \sum_{a \in B} p_a \\ &= \sum_{a \in B} \frac{1}{2^\gamma} \\ &= \frac{2^\gamma}{2} \frac{1}{2^\gamma} \\ &= \frac{1}{2}.\end{aligned}$$

Similarly, we get that $p_{-\sqrt{\triangle t}} = \frac{1}{2}$. Thus we have that $\phi(W_t) = \langle W_t, e_j\rangle$ is exactly the one-dimensional Anderson random walk defined in Example 4.3.2.

For any $\phi \in \mathrm{Fin}(F^*)$ let $\phi(L)$ be hyperfinite random walk determined by $B = \phi(A)$ and probabilitites $(p_b)_{b\in B}$. Since $\phi(L)$ is a hyperfinite Lévy process we have from Theorem 4.3.4 that:

(a) $\frac{1}{\triangle t}\sum_{|b|\leq k} bp_b$ is finite for every noninfinitesimal $k \in \mathrm{Fin}({}^*\mathbb{R})$,

(b) $\frac{1}{\triangle t}\sum_{|b|\leq k} b^2 p_b$ is finite for every $k \in \mathrm{Fin}({}^*\mathbb{R})$,

(c) For $q_k^\phi = \frac{1}{\triangle t}\sum_{|b|>k} p_b$ we have that for each $\epsilon \in \mathbb{R}_+$ there is an $N \in \mathbb{N}$ such that $p_k^\phi < \epsilon$ when $k \geq N$.

Remark 8.2.3. In [Lin04] it was shown for finite-dimensional Lévy process $l : \Omega \times \mathbb{T} \to {}^*\mathbb{R}^n$ we have that ${}^\circ l$ is a standard n-dimensional Lévy process. Hence for a hyperfinite cylindrical Lèvy process L we have that

$$^\circ\overrightarrow{\phi}(L_t)$$

is an n-dimensional (standard) Lèvy process for any $\overrightarrow{\phi} = (\phi_1, \ldots, \phi_n) \subset \mathrm{Fin}(F^*)$.

CHAPTER 9

Summary

This conclusion will summarize some of our results and explain how these findings are linked together. In this thesis, we have mostly worked with hyperfinite-dimensional inner product spaces U and V with underlying Hilbert spaces H_1 and H_2, respectively. We have shown that if an internal map $T \in L(U, V)$ is S-bounded then we can view T as an element in ${}^*\mathcal{B}(H_1, H_2)$ by considering an extension. This result was important in order to use the transfer principle in several results such as the existence and uniqueness of an adjoint operator T^* and the existence of a square root $R \in L(U)$ of $T \in L(U)$. We also introduced the concept of hyperfinite-dimensional dual spaces U^* and provided a Riesz representation theorem for U^*. The consequences of this theorem were that U is isomorphic to U^* (with respect to the S-topology), and the existence and uniqueness of an adjoint.

We defined nearstandard operators, but the adjoint of a nearstandard operator is not necessarily nearstandard. Therefore we presented strictly nearstandard maps and proved that the adjoint of such a map is also strictly nearstandard. Next, we considered internal Hilbert-Schmidt operators which we proved have a lot of properties in common with standard Hilbert-Schmidt operators. A property that internal Hilbert-Schmidt operators do not share with standard Hilbert-Schmidt operators, is that the adjoint of an internal Hilbert-Schmidt operator is Hilbert-Schmidt. Therefore we introduced a subset of strictly nearstandard operators, namely strictly Hilbert-Schmidt operators which solves this asymmetry. This was important for the main result in section 7.3. We also considered the covariance operator C^X of hyperfinite random walks $X : \Omega \times \mathbb{T} \to U$ and showed that if C^X is S-bounded then there exists a $R \in L(U)$ such that $C^X = RR$.

Everything mentioned above was some of the necessary results in order to show under which conditions our stochastic integral $\int X dM : \Omega \times \mathbb{T} \to V$ is *well-behaved*. When we say well-behaved we mean that the stochastic integral is finite almost surely, nearstandard almost surely, and a hyperfinite martingale.

Lastly, we introduced hyperfinite cylindrical processes. We showed that the Anderson process and our stochastic integral are cylindrical. Furthermore, we also proved that if the covariance operator of a hyperfinite random walk X, has an S-bounded square root, then X is cylindrical.

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