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Hermitian K-theory of Finite Fields

A Computation Using the Very Effective Slice Spectral Sequence

Arne Sandrib

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Supervisors:
Håkon Kolderup
Paul Arne Østvær

Abstract

We give an explicit calculation of the Hermitian K-theory groups of finite fields. This is done via the spectral sequence associated to the very effective slice filtration of Spitzweck and Østvær in [SØ12]. This is possible due to the computation of the very effective slices of the Hermitian K-theory spectrum by Bachmann, cf. [Bac17]. The results coincide with those of Friedlander in [Fri76].

Abstract

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Contents

Introduction

Hermitian K-theory is a verison of algebraic K-theory, which studies symmetric bilinear forms over a ring. A more general notion of Hermitan K-theory is that over schemes, classifying vector bundles with symmetric bilinear forms. It is often said that Hermitian K-theory is the algebraic counterpart to real topological K-theory. This thesis will compute the Hermitian K-theory of finite fields, first done by Friedlander in [Fri76]. Letting $\mathbf{KQ}_s(\mathbb{F}_q)$ denote the Hermitian K-groups of finite fields, we recover the following results using the *very effective slice spectral sequence*:

s modulo 8	0	1	2	3	4	5	6	7
$\widetilde{\mathbf{KQ}}_s(\mathbb{F}_q)$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$\frac{\mathbb{Z}}{q^{(s+1)/2}-1}$	0	0	0	$\frac{\mathbb{Z}}{q^{(s+1)/2}-1}$

Table 1: Hermitian K-theory of finite fields.

Motivic Homotopy Theory

Motivic homotopy theory introduces classical homotopy theoretic tools aiding the study of smooth schemes over a field. The stable motivic homotopy category, introduced by Morel and Voevodsky in [MV99], has many spectra representing familiar cohomology theories. In particular, it is the home of $\mathbf{KQ}$, representing Hermitian K-theory. We will obtain the results of table 1 by computing the motivic homotopy groups $\pi_{s,0}\mathbf{KQ}$.

There is a motivic analogue to the Atiyah-Hirzebruch spectral sequence, known as the slice spectral sequence. It takes as input the homotopy groups of so-called slices of a spectrum E, and converges under certain conditions to the motivic homotopy groups of E. However, it relies on the so-called *slice filtration*, which can have a complicated E^1-page, and complicated convergence properties. This is indeed the case for the Hermitan K-theory spectrum. To remedy this, we invoke the *very effective* slice filtration, introduced by Spitzweck and Østvær in their work on twisted motivic K-theory, cf. [SØ12]. The associated spectral sequence has the advantage of always being strongly convergent. However, the very effective slices are in general difficult to identify. Thanks to the work of Bachmann in [Bac17], the very effective slices of Hermitian K-theory are known, and they are given by motivic cohomology and Milnor-Witt motivic cohomology. The papers [SØ12] and [Bac17] are foundational for our computation.

Computation of $\pi_\star\mathrm{KQ}$

This thesis aims to compute the motivic homotopy groups of $\mathbf{KQ}$ for a finite field $\mathbb{F}_q$ as the base scheme, for powers of odd primes q. After reviewing the background material, we compute the E^1-page of the very effective slice spectral sequence in chapter 4. As expected, this gives an 8-periodic spectral sequence, which collapses immediately. Except for one extension problem, the results are instantly read off. The extension problem is resolved by considering a determinant map from $\mathbf{KQ}_1$ into the multiplicative group $\{\pm 1\}$.

Outline of the Thesis

In chapter 1 the following ingredients in the calculation are introduced: Algebraic K-theory, Milnor K-theory, Milnor-Witt K-theory and finally Hermitian K-theory. For each of these, relevant results and computations are included. Chapter 2 introduces motivic homotopy theory, and in particular the stable motivic homotopy category. We give a presentation of motivic cohomology and its spectrum, the algebraic K-theory spectrum, the Hermitian K-theory spectrum and the Milnor-Witt motivic cohomology spectrum. In chapter 3, we introduce spectral sequences, and discuss the ideas of convergence and algebra sepctral sequences. Then a short interlude with an example computational with the Atiyah-Hirzebruch spectral sequence follows, before we define the slice spectral sequences – our most important computational tool. The section concludes with an overview of the very effective slices of $\mathbf{KQ}$. The thesis concludes with chapter 4, where we do the necessary computations to fill out the E^1-page of the very effective slice spectral sequence, before we retrieve the Hermitian K-theory of finite fields.

The content of chapter 1, apart from the definitions of higher algebraic and Hermitian K-theory, should be accessible to anyone with a basic grasp of commutative algebra. We note that the definitions in section 1.4 are brief, but tailored to our needs. A good understanding of algebraic topology and categorical language is definitely an advantage when reading chapter 2. Chapter 3 is readable for those with some knowledge about algebraic topology, and the first three sections stand on their own. The last three are readable after having read section 2.1. Chapter 4 relies heavily on the last two sections of chapter 3. It can be read before the first two chapters, as we provide references every time we use results from previous chapters.

Chapter 1

K-theory

The study of K-theory or K-theories is fairly new in the history of mathematics. It started with Grothendieck in the 1950s, who introduced the functor K_0, where the K came from the German word 'Klassen' meaning 'classes.' This already hints at what K-theory is concerned about – identifying classes of some sort. The study of K-theory has indeed become wide, and one can see this mysterious letter show up everywhere from algebraic geometry and topology, to number theory and functional analysis [Kar10]. This chapter will introduce some K-theories, all of which are important for the computation carried out in chapter 4.

1.1 Algebraic K-theory

The zeroth algebraic K-group was introduced by Grothendieck in 1957 as an attempt to reformulate the famous Riemann-Roch theorem by looking at vector bundles on algebraic varieties. Even though varieties were the objects of study, the interest for algebraic K-theory grew, and was defined for general rings. It is a strong invariant, but famously hard to compute. Indeed, to this day, the K-groups of the ring $\mathbb{Z}$ are not known!

As we will see, algebraic K-theory has a rather strong connection to motivic cohomology (see section 3.4), something that will prove important in chapter 2.

1.1.1 The Grothendieck Construction

From the category CMon of commutative monoids, there is a functor

$$G\colon \mathrm{CMon} \to \mathrm{AbGrp}$$

to the category of abelian groups which formally adds inverses to a monoid M. The resulting group is denoted $G(M)$ or M^+. It is sometimes called the *group completion* of M or the *Grothendieck group* of M. The construction is characterised by the following universal property:

Given a commutative monoid M, the Grothendieck group of M is the group M^+ together with a map $i\colon M \to M^+$, such that given any monoid map $M \to G$ to an abelian group, there is a unique map $\tilde{f}\colon M^+ \to G$ such that $f = \tilde{f} \circ i$, i.e., we have the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\ i\ } & M^+ \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \exists!\tilde{f}} & \\ G & & \end{array}$$

There is a constructive way of obtaining M^+ from M. If we let M be a commutative additive monoid, define the relation $\sim$ on the product $M \times M$ by

$$(x, y) \sim (x', y') \iff x + y' = x' + y \in M.$$

We define $M^+ := M \times M/\sim$ with pointwise addition $(x, y) + (x', y') := (x + x', y + y')$. It is easily checked that this is a group satisfying the universal property. Morover, there is a natural interpretation of the symbols $(x, y) \in M^+$ as $x - y$. Thus, we can think of $i(x) = (x, 0)$, having the additive inverse $(0, x)$. As we will see, this interpretation of i does not work under every circumstance:

Example 1.1.1. The above relation $\sim$ is an equivalence relation if M is a commutative cancellation monoid. That is, a monoid for which $ac = bc$ implies that $a = b$ for all $a, b, c \in M$. Indeed, if $(a, b) \sim (c, d) \sim (e, f)$, then

$$ad = bc \text{ and } cf = de \iff adf = bcf = bde \iff af = eb \iff (a, b) \sim (e, f).$$

Furthermore, if there is a zero-element $0 \in M$ s.t. $0x = 0$ for all $x \in M$, then in $G(M)$, we have

$$x = (0^{-1}0)x = 0^{-1}(0x) = 0^{-1}0 = 0$$

for all x. This $G(M)$ does not contain M, and we see that the map i above need not be injective. ♣

More generally, this process can be carried out for certain symmetric monoidal categories $(\mathscr{C}, \Box)$ (see [Mac98, p. 184] for definition). Assume that the object class of the category $\mathscr{C}^{\text{iso}}$ of isomorphism classes of objects in $\mathscr{C}$ forms a set. Then $\mathscr{C}^{\text{iso}}$ is a commutative monoid with monoid operation given by

$$[C] + [C'] := [C \Box C'].$$

We define the Grothendieck group of $\mathscr{C}$ to be

$$K_0^{\Box}(\mathscr{C}) := (\mathscr{C}^{\text{iso}})^+.$$

To give the definition of the zeroth algebraic K-group, let $\mathscr{P}(R)$ be the category of finitely generated projective modules over R. Then $\mathscr{P}(R)^{\text{iso}}$ becomes symmetric monoidal under direct sum.

Definition 1.1.2. The *zeroth K-group* of the ring R, is defined to be

$$K_0(R) := K_0^{\oplus}(\mathscr{P}(R)).$$

1.1.2 The functors K_1 and K_2

We proceed the story by introducing the first and second K-groups. Consider the general linear group $\mathrm{GL}_n(R)$ over the ring R, and form the directed system

$$\mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}_2(R) \hookrightarrow \cdots$$

where the inclusion map is given by

$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}.$$

We denote the colimit of this system by $\mathrm{GL}(R)$. Let $e_{ij}(a)$, defined for every $i \neq j$, be *elementary matrix* identical to the identity except at the (i, j)-entry, which takes the value a. For every n, these elementary matrices generate a subgroup $E_n(R)$ of $\mathrm{GL}_n(R)$. By the above inclusion map, they form a directed system by $E_n(R) \hookrightarrow E_{n+1}(R)$, whose colimit is denoted by $E(R)$. The following lemma is classic.

Lemma 1.1.3 ([Whi50])**.** *The subgroup $E(R)$ is the commutator of* $\mathrm{GL}_n(R)$*.*

This tells us that $\mathrm{GL}(R)_{\mathrm{ab}} = \mathrm{GL}(R)/E(R)$.

Definition 1.1.4. Given the ring R, we define the *first algebraic K-group* of R to be

$$K_1(R) \coloneqq \mathrm{GL}(R)/E(R).$$

To motivate the definition of the second K-groups of a ring R, consider again the elementary matrices of $\mathrm{GL}(R)$. An easy computation shows that $e_{ij}(a)e_{ij}(b) = e_{ij}(a+b)$. Furthermore, the commutator satisfies the following:

$$[e_{ij}(a), e_{kl}(b)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq l; \\ e_{il}(ab) & \text{if } j = k \text{ and } i \neq l; \\ e_{kj}(-ab) & \text{if } j \neq k \text{ and } i = l. \end{cases}$$

We want the *Steinberg group* to be generated by symbols satisfying the same relations:

Definition 1.1.5. Let R be a ring. We define the *Steinberg group* $\mathrm{St}_n(R)$ of order $n \geq 3$ over R to be the free group generated by symbols $x_{ij}(a)$ for which $0 \leq i, j \leq n$, $i \neq j$ and $a \in R$, subject to the following relations:

1. $x_{ij}(a)x_{ij}(b) = x_{ij}(a + b)$;
2. $[x_{ij}(a), x_{kl}(b)] = x_{il}(ab)$ if $i \neq l$;
3. $[x_{ij}(a), x_{kl}(b)] = 1$ if $j \neq k$ and $i \neq l$.

There are natural maps $\mathrm{St}_n(R) \to \mathrm{St}_{n+1}(R)$ for each n. Let the colimit of the associated directed system be denoted $\mathrm{St}(R)$. Moreover, for any n, define the map $\phi_n \colon \mathrm{St}(R) \to \mathrm{GL}_n(R)$ by the assignment $x_{ij}(x) \mapsto e_{ij}(x)$. The image is precisely the elementary matrix group $E_n(R)$. Taking the colimit gives the map $\phi \colon \mathrm{St}(R) \to E(R)$.

Definition 1.1.6. The *second algebraic K-group* of the ring R is defined to be

$$K_2(R) \coloneqq \ker(\mathrm{St}(R) \to E(R)).$$

For fields, there is a particularly nice description of the algebraic K-theory group, attributed to Matsumoto. A proof can be found in [Ros94, Theorem 4.3.15]:

Theorem 1.1.7 (Matsumoto)**.** *The second K-group $K_2(F)$ of a field F is the free abelian group on generators $\{x, y\}$ with $x, y \in F^\times$, subject only to the following relations:*

1. $\{x, 1 - x\} = 0$ *for* $x \in F - \{0, 1\}$ *(Steinberg relation);*
2. $\{xy, z\} = \{x, z\} + \{y, z\}$*;*
3. $\{x, yz\} = \{x, y\} + \{x, z\}$*.*

Example 1.1.8. Let $F = \mathbb{F}_q$ be a finite field. Then, by the exact same argument we will see in proposition 1.2.4, we have $K_2(F) = 0$. ♣

For the reader familiar with the notion of classifying space and the Quillen plus construction, the following definition of higher algebraic K-groups will make sense:

Definition 1.1.9. For $n > 0$, we define the *nth algebraic K-group* of the ring R to be

$$K_n(R) \coloneqq \pi_n(B\mathrm{GL}(R)^+)$$

Remark 1.1.10. Definition 1.1.9 is the definition for K-groups of an arbitrary ring. It is also possible to define algebraic K-theory of a scheme (see e.g. [Wei13, Ch. IV]). Note that these definitions of higher algebraic K-theory coincide with the previous definitions of K_0, K_1 and K_2.

As our interests are closely related to finite fields, we include this highly nontrivial computation done by Quillen:

Theorem 1.1.11 ([Qui72])**.** *The K-groups of finite fields are given by*

$$K_n(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0; \\ \mathbb{Z}/q^i - 1 & \text{for } n = 2i - 1; \\ 0 & \text{for } n > 0 \text{ even.} \end{cases}$$

1.2 Milnor K-theory

Recall the second algebraic K-group of fields. Due to Matsumoto (Theorem 1.1.7), they have a rather nice description in terms of generators and relations. In order to study the higher algebraic K-groups of fields, Milnor [Mil70] introduced the now called *Milnor K-theory*, extending these relations to every degree. Despite his calling the definition 'purely ad hoc' for $n \geq 3$ [Mil70, p. 319], there are some remarkable connections to the motivic world, as we will see in the next chapter.

For the entire section, let F be a field. We write the n-fold tensor product $(F^\times)^{\otimes n} \coloneqq F^\times \otimes \cdots \otimes F^\times$, where $(F^\times)^{\otimes 0} \coloneqq \mathbb{Z}$.

Definition 1.2.1. Let $n \geq 0$. The *Milnor K-theory of a field F* is the graded ring $K_*^M(F)$ where the nth component is given by

$$K_n^M(F) = \frac{(F^\times)^{\otimes n}}{(a_1 \otimes \cdots \otimes a_n : a_i + a_{i+1} = 1 \text{ for some } i)}.$$

We denote the image of $a_1 \otimes \cdots \otimes a_n \in (F^\times)^{\otimes n}$ in $K_n^M(F)$ by the canonical map $l \colon F^\times \to K_n^M$ on each component by $\{a_1, \ldots, a_n\}$.

Remark 1.2.2. The expressions $\{x_1, \ldots, x_n\} \in K_n^M(F)$ are called *Steinberg symbols*, satisfying the *Steinberg relation* $\{x_1, \ldots, x_n\} = 0$ if $x_i + x_{i+1} = 1$ for some i.

Lemma 1.2.3. *Some basic properties of the Steinberg symbols are*

(a) $\{a^{-1}\} = -\{a\}$*;*

(b) $\{a^{-1}, b\} = -\{a, b\} = \{a, b^{-1}\}$*;*

(c) $\{x, -x\} = 0$ *for all* x*;*

(d) $\{x, y\} = -\{y, x\}$.

Proof. For (a), observe that $\{1\} = \{1\} + \{1\}$, so $\{1\} = 0$. Then we can compute $0 = \{1\} = \{aa^{-1}\} = \{a\} + \{a^{-1}\}$.

For (b), observe that $\{a^{-1}, b\} + \{a, b\} = \{a^{-1}a, b\} = \{1, b\} = 0$. A similar computation gives the second equality.

For (c), we utilize that F is a field, and get

$$\begin{aligned}\{x, -x\} = \left\{x, (1-x)(1-x^{-1})^{-1}\right\} &= \{x, 1-x\} - \left\{x, 1-x^{-1}\right\} \\ &= \{x, 1-x\} + \left\{x^{-1}, 1-x^{-1}\right\} \\ &= 0.\end{aligned}$$

Thus we can compute (d):

$$\begin{aligned}\{x, y\} + \{y, x\} &= \{x, y\} + \{x, -x\} + \{y, -y\} + \{y, x\} \\ &= \{x, -xy\} + \{y, -xy\} \\ &= \{xy, -xy\} \\ &= 0.\end{aligned}$$

□

Proposition 1.2.4. *The Milnor K-theory of a finite field is concentrated in degrees 0 and 1, taking values*

$$K_n^M(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0; \\ \mathbb{F}_q^\times & \text{if } n = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $F = \mathbb{F}_q$. The result for $n = 0$ and $n = 1$ by definition. For $n = 2$, consider the following:

Let $u \in F^\times$ be a generator of the cyclic group of units in F. Then, given any $\{x, y\} \in K_2^M(F)$, we have

$$\{x, y\} = \{u^m, u^n\} = mn\,\{u, u\}.$$

It is enough to show that $\{u, u\} = 0$.

Case: char $F = 2$. Then $x = -x \in F$ and $\{u, u\} = \{u, -u\} = -\{u, u\} = 0$.

Case: char $F \neq 2$. Then we construct the set $A := F^\times - (F^\times)^2$ of non-squares, and the set $B := \{1 - v \mid v \in A\}$. Note that there is an obvious bijection between these sets, and we claim that they have nonempty intersection. To see this, assume that they do not. Then, as both sets have cardinality $(q-1)/2$, their union is exactly $F^\times$. Now, $1 = 1^2 \notin A$, and $0 \notin A \subseteq F^\times$, so $1 \notin B$, which is a contradiction. Thus, A and B have nonempty intersection, and there exist odd m, n such that $u^m = 1 - u^n$. But then

$$0 = \{u^n, u^n - 1\} = \{u^n, u^m\} = mn\,\{u, u\} = (2k+1)\,\{u, u\} = \{u, u\}$$

as we wanted. □

The Milnor K-theory modulo 2 appears in our calculations (e.g. in definition 2.2.5), so we include it here:

Definition 1.2.5. Milnor K-theory modulo 2 is the quotient

$$k_*^M(F) := K_*^M(F)/2K_*^M(F).$$

A discussion on Milnor K-theory is hardly complete without a mention of the famous *Milnor conjecture*, proposed by John Milnor in [Mil70], connecting Milnor K-theory and étale cohomology. It is now a theorem, due to Voevodsky [Voe96].

Theorem 1.2.6. *Let F be a field of characteristic not 2. The norm residue homomorphisms*

$$K_n^M(F)/2 \longrightarrow H_{et}^n(F, \mathbb{Z}/2)$$

are isomorphisms for all $n \geq 0$.

1.3 Milnor-Witt K-theory

Milnor-Witt K-theory is defined similarly to Milnor K-theory, but we introduce a symbol carrying negative degree. This gives rise to a richer structure. We mention Milnor-Witt K-theory because of applications to computations of Hermitian K-theory. The following definition is by Morel in collaboration with Hopkins:

Definition 1.3.1 ([Mor12, Definition 3.1])**.** Let F be a field, and consider the graded free associative algebra generated by symbols $[u]$ of units $u \in F^\times$ of degree 1, and the symbol η of degree -1. The *Milnor-Witt K-theory* of F, $K_*^{\mathrm{MW}}(F)$, is defined as this algebra subject to the following relations

1. $[ab] = [a] + [b] + \eta[a][b]$ for each $a, b \in F^\times$ (η-twisted logarithm rule).
2. $[a][1-a] = 0$ for all $a \in F - \{0, 1\}$ (Steinberg relation).
3. $[a]\eta = \eta[a]$ for every $u \in F^\times$.
4. $(2 + [-1]\eta)\eta = 0$ (Witt relation).

Remark 1.3.2. Taking the quotient $K_*^{\mathrm{MW}}(F)/(\eta)$, recovers the definition of Milnor K-theory.

In order to state some elementary relations for $K_*^{\mathrm{MW}}(F)$, we introduce some notation: $\langle a \rangle \coloneqq 1 + [a]\eta$ and $h \coloneqq \langle 1 \rangle + \langle -1 \rangle$.

Lemma 1.3.3 ([Mor12, Lemma 3.5])**.** *In $K_*^{\mathrm{MW}}(F)$, the following relations hold for every $a, b \in F^\times$:*

1. $[ab] = [a] + \langle a \rangle [b] = [a]\langle b \rangle + [b]$,
2. $\langle ab \rangle = \langle a \rangle \langle b \rangle$,
3. $\langle 1 \rangle = 1 \in K_0^{\mathrm{MW}}(F)$ *and* $[1] = 0 \in K_0^{\mathrm{MW}}$, *and*
4. $[a/b] = [a] - \langle a/b \rangle [b]$.

Proof. All 3 first cases are proven in [Mor12]. The fourth can be shown like this: Note by (a) that $\langle b^{-1} \rangle [b] = [b^{-1}b] - [b^{-1}] = -[b^{-1}]$. Using this, we get

$$\begin{aligned} [a/b] = [ab^{-1}] &= [a] + \langle a \rangle [b^{-1}] \\ &= [a] - \langle a \rangle \langle b^{-1} \rangle [b] \\ &= [a] - \langle a/b \rangle [b]. \end{aligned}$$

□

1.3.1 Symmetric Bilinear Forms and $\mathrm{GW}(F)$

In order to compute the Milnor-Witt K-theory of specific fields, we introduce the Grothendieck-Witt and Witt rings of fields. The following will be based on [Mil73] and [Lam05].

Let R be a commutative ring with unity, and M be an R-module.

Definition 1.3.4. A *bilinear form* on M is a pairing $\beta\colon M \times M \to R$ which is R-linear in the first and the second argument. If the homomorphims $x_0 \mapsto \beta(x_0, -)$ and $y_0 \mapsto \beta(-, y_0)$ are both bijective, then we say that β is an *inner product*. If M is also finitely generated projective, we call (M, β) an *inner product space over* R. Finally, if $\beta(m, m') = \beta(m', m)$, we say that β is a *symmetric bilinear form*.

We define a map of inner product spaces $f\colon (M, \beta) \to (M', \beta')$ to be a map such that the diagram

$$\begin{array}{ccc} M \times M & \xrightarrow{f\times f} & M' \times M' \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\beta'} & \\ & A & \end{array}$$

commutes, and such maps are called *isometries* if they are isomorphisms.

Symmetric inner product spaces over R form a category $\mathrm{SBil}(R)$. We can define the category $\mathrm{SBil}^{\mathrm{iso}}(R)$ of isomorphism classes of inner product spaces. Giving it an additive structure under the orthogonal sum, we get a commutative monoid:

$$(M, \beta) \oplus (M', \beta') \coloneqq (M \oplus M', \beta \oplus \beta'),$$

where $(\beta \oplus \beta')(m_1 \oplus m_1', m_2 \oplus m_2') \coloneqq \beta(m_1, m_2) + \beta'(m_1', m_2')$. Furthermore, we can give $\mathrm{SBil}^{\mathrm{iso}}(R)$ a commutative multiplicative structure under the tensor product over R, where we let

$$(M, \beta) \otimes (M', \beta') \coloneqq (M \otimes M', \beta \otimes \beta').$$

This gives our category a semiring structure. In order to make it a ring, we would like to use the Grothendieck construction from section 1.1.1. This requires Witt's cancellation theorem:

Theorem 1.3.5 ([Mil73, pp. I, 4.4])**.** *If X, Y and Z are inner product spaces, then*

$$X \oplus Y \cong X \oplus Z \implies Y \cong Z.$$

The above theorem implies that $\mathrm{SBil}^{\mathrm{iso}}(R)$ is a cancellation monoid. Applying the K_0-functor adds inverses to the direct sum operation. We define the *Grothendieck-Witt ring* of R to be

$$\mathrm{GW}(R) \coloneqq K_0\mathrm{SBil}^{\mathrm{iso}}(R).$$

When F is a field and $u \in F^\times$, we introduce the symbol $\langle u \rangle$ to be the inner product space (F^2, B_u) with the form $B_u(x, y) = uxy$. This allows us to define the *hyperbolic plane* to be $h \coloneqq \langle 1 \rangle + \langle -1 \rangle$. We get the *Witt ring* $W(R)$ by taking the quotient of $\mathrm{GW}(R)/(h)$, where the ideal (h) is the ideal in $\mathrm{GW}(R)$ generated by h. In fact, one can show that (h) coincides with $\mathbb{Z}h$ [Lam05, Ch. II].

There are maps $\dim\colon \mathrm{GW}(R) \to \mathbb{Z}$ and $\dim_0\colon \mathrm{W}(R) \to \mathbb{Z}/2$. They are the dimension and the dimension mod two maps, respectively. We denote the kernel $GI(R) \coloneqq \ker \dim$

and $I(R) := \ker \dim_0$, and call it the *fundamental ideal.* A snake lemma argument shows that $GI(R) \cong I(R)$, and we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & GI & \longrightarrow & \mathrm{GW}(A) & \xrightarrow{\dim} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\cong} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & \mathrm{W}(A) & \xrightarrow{\dim_0} & \mathbb{Z}/2 & \longrightarrow & 0
\end{array}$$

The Grothendieck-Witt and Witt rings have rather nice presentations with relations making them easier to work with. In order to prove the relationship between Milnor-Witt K-theory and these rings, one needs the following:

Theorem 1.3.6 ([Mil73, p. 84])**.** *Let F be a field. The additive group $\mathrm{W}(F)$ is generated by the elements $\langle u \rangle$ with $u \in F^\times$, subject to the following relations:*

1. $\langle u \rangle = \langle uv^2 \rangle$ *for* $v \neq 0$*;*
2. $\langle u \rangle + \langle -u \rangle = 0$*;*
3. $\langle u \rangle + \langle v \rangle = \langle u + v \rangle + \langle uv(u + v) \rangle$*.*

1.3.2 Comparison with K_*^{MW}

For $n \geq 0$, we construct homomorphisms ϕ_{-n} into $K_{-n}^{\mathrm{MW}}(F)$ as follows: Define $\phi_0 \colon \mathrm{GW}(F) \to K_0^{\mathrm{MW}}(F)$ by $\langle u \rangle \mapsto \langle u \rangle$ and $\phi_{-n} \colon \mathrm{W}(F) \to K_{-n}^{\mathrm{MW}}(F)$ by $\langle u \rangle \mapsto \langle u \rangle$. The left hand side vertical map is the quotient map, and the right hand side vertical maps are multiplication by η maps. Then we get the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{GW}(F) & \xrightarrow{\phi_0} & K_0^{\mathrm{MW}}(F) \\
\downarrow & & \downarrow{\eta} \\
\mathrm{W}(F) & \xrightarrow{\phi_{-1}} & K_{-1}^{\mathrm{MW}}(F) \\
 & \searrow^{\phi_{-2}} & \downarrow{\eta} \\
 & & K_{-2}^{\mathrm{MW}}(F) \\
 & \searrow^{\phi_{-3}} & \downarrow{\eta} \\
 & & K_{-3}^{\mathrm{MW}}(F) \\
 & & \downarrow{\eta} \\
 & & \vdots
\end{array}$$

Theorem 1.3.7 ([Mor12, Lemma 2.10])**.** *The homomorphism ϕ_{-n} is an isomorphism for every $n \geq 0$.*

For this result to be useful to us, we need to know the structure of the Grothendieck-Witt and Witt rings over finite fields. The following is a blend of II.3.5 and II.3.6 in [Lam05]:

Theorem 1.3.8. *Let $F = \mathbb{F}_q$ with q odd. Then we have ring isomorphisms*

1. $\mathrm{GW}(F) \cong \mathbb{Z} \oplus \mathbb{Z}/2$,

2. *if* $q \equiv 1 \mod 4$*, then* $\mathrm{W}(F) \cong \mathbb{Z}/2[F^\times/(F^\times)^2]$,

3. *if* $q \equiv 3 \mod 4$*, then* $\mathrm{W}(F) \cong \mathbb{Z}/4$,

where in case 2, $\mathbb{Z}/2[F^\times/(F^\times)^2]$ *denotes the group ring on* $F^\times/(F^\times)^2$*.*

In order to compute the Milnor-Witt K-theory of $\mathbb{F}_q$ for positive n, we require the following theorem:

Theorem 1.3.9 ([GSZ15])**.** *For any field* F *and every* $n \geq 0$ *, there is a short exact sequence*

$$0 \longrightarrow I^{n+1}(F) \longrightarrow K_n^{\mathrm{MW}}(F) \longrightarrow K_n^{\mathrm{M}}(F) \longrightarrow 0,$$

where $I^{n+1}(F)$ *is the* $(n+1)$*th power of the fundamental ideal.*

It follows from Theorem 1.3.9 that $I^k(\mathbb{F}_q) = 0$ for $k \geq 2$. Thus, so there is an isomorphism $K_n^{\mathrm{MW}}(\mathbb{F}_q) \cong K_n^{\mathrm{M}}(\mathbb{F}_q)$ whenever $n \geq 1$.

Now, we can give a complete description of the Milnor-Witt K-theory of finite fields:

Corollary 1.3.10. *Let* q *be odd. Then the Milnor-Witt* K*-theory of finite fields is given by*

$$K_n^{\mathrm{MW}}(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z}/2[\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2] & \text{if } n < 0 \text{ and } q \equiv 1 \mod 4; \\ \mathbb{Z}/4 & \text{if } n < 0 \text{ and } q \equiv 3 \mod 4; \\ \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } n = 0; \\ \mathbb{F}_q^\times & \text{if } n = 1; \\ 0 & \text{otherwise.} \end{cases}$$

1.4 Hermitian K-theory

Hermitian K-theory is in some sense the algebraic counterpart to real topological K-theory. This section will give definitions for the target of the computation carried out in chapter 4. The construction of the zeroth Hermitian K-ring coincides, as we will see, with the construction of the Grothendieck-Witt ring.

We follow the construction given in [Bak81], tailored to our needs. Instead of considering commutative rings with involution, we consider fields with trivial involution. Throughout this section, let V be a vector space over a field F of characteristic different from 2.

Definition 1.4.1. A *sesqulinear form* B on a vector space V over a field F is a map

$$B : V \to \mathrm{Hom}_F(V, F)$$

If B is injective, it is called *nonsingular*, *regular*, or *non-degenerate*.

Note that there is an adjunction between tensor product and $\mathrm{Hom}(-, F)$, such that $\mathrm{Hom}_F(V \otimes V, F) \cong \mathrm{Hom}_F(V, \mathrm{Hom}_F(V, F))$. Thus, every map $B : V \to \mathrm{Hom}_F(V, F)$ corresponds to a map $B' : V \otimes V \to F$.

We can define the *Hermitian evaluation map* for a finitely generated projective module M, $\mathrm{ev}_M \colon M \to M^{**}$, which is given by $m \mapsto (f \mapsto f(m))$. The following lemma is a classic. We state it for projective modules over rings, but we only need it for vector feilds:

Lemma 1.4.2. *The map* $\mathrm{ev}_M\colon M \to M^{**}$ *with* M *in* $\mathscr{P}(R)$ *is a natural isomorphism of finitely generated projective modules.*

Proof. This is an outline of the proof. Firstly, one checks that the dual of a projective module is indeed projective. Then, note that a fundamental characterization of projective modules is the existence of dual generating sets. In particular, for generating sets $\{x_i\}_{i\in I} \subseteq P$ and $\{f_i\}_{i\in I} \subseteq P^*$, every $x \in P$ can be written $x = \sum_{i\in I} f_i(x)x_i$. Then, for any projective module P, letting $x \mapsto (f \mapsto f(x)) = 0$ in P^{**} is the same as saying $f(x) = 0$ for all $f \in P^*$. In particular, that goes for all f_i in the generating set of P^*. Thus, $x = \sum_{i\in I} f_i(x)(x_i) = 0$, i.e. the map is injective.

For surjectivity, one can show that P is finitely generated projective if and only if there are $x_1, \ldots, x_n \in P$ and $f_1, \ldots f_n \in P^*$, both generating sets, such that every $x \in P$ can be expressed $x = \sum_{i=1}^n f_i(x)x_i$. Let $x_1, \ldots x_n \in P$ and $f_1, \ldots, f_n \in P^*$ be generators, and denote by $\hat{x} \coloneqq ev_M(x)$. Take the generating set $\hat{x}_1, \ldots \hat{x}_n$ of P^{**}, and note that for any $x \in P$, $x = \sum_{i=1}^n f_i(x)x_i$. Applying any $f \in P^*$ on both sides,

$$f(x) = \sum_{i=1}^{n} f_i(x)f(x_i) = \sum_{i=1}^{n} f_i(x)\hat{x}_i(f),$$

i.e., $f = \sum_{i=1}^n \hat{x}_i(f)f_i$. Hence the $\hat{x}_i$ generate P^{**}, and we are done. $\square$

Definition 1.4.3. Given $\lambda \in F$, we define a λ-*Hermitian form* to be a nonsingular sesqulinear form B on V such that the composite $V \xrightarrow{\mathrm{ev_V}} V^{**} \xrightarrow{\lambda B^*} V^*$ is equal to B.

Note that by the adjunction mentioned above, one could equally well have defined the λ-Hermitian form to a nonsingular sesqulinear form B satisfying the relation $B(x, y) = \lambda B(y, x)$.

Now we can define the category $\mathscr{P}^\lambda(F)$ of λ-Hermitian forms on F. Objects are pairs (V, B) of a vector spaces V over F and λ-Hermitian forms $B\colon V \to V^*$. Morphisms are maps $f\colon (V, B) \to (V', B')$ which preserve λ-Hermitian forms, i.e. $f^*B'f = B$, meaning that the following commutes

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ \downarrow{\scriptstyle B} & & \downarrow{\scriptstyle B'} \\ V^* & \xleftarrow{f^*} & (V')^* \end{array}$$

We note that since Hermitian forms are non-singular, all the morphisms are necessarily injective.

Consider the subcategory $\mathscr{P}(F)^{\mathrm{iso}}$ of $\mathscr{P}(F)$ where the objects are the same, but the only morphisms are isomorphisms. We define the so-called *hyperbolization functor* $H^\lambda\colon \mathscr{P}(F)^{\mathrm{iso}} \to \mathscr{P}^\lambda(F)$, sending a vector space $V \mapsto (V \oplus V^*, B)$, where B is the symmetric bilinear form defined by $B((x \oplus f), (y \oplus g)) = f(y) + \lambda g(x)$, or equivalently by the matrix

$$\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}.$$

Morphisms $f\colon V \to W$ are sent to $f \oplus (f^*)^{-1}\colon V \oplus V^* \to W \oplus W^*$.

Definition 1.4.4. Modules in the image of H^λ are called *hyperbolic*, and we call $H^\lambda(F)$ the *hyperbolic plane*.

Similarly to what we did with symmetric bilinear forms in section 1.3.1, we can give the category $\mathscr{P}^\lambda(F)$ a symmetric monoidal structure. We equip it with direct sum as addition and tensor product as multiplication to get

$$(V,B)\oplus(V',B') \coloneqq (V\oplus V', B\oplus B')$$

and

$$(V,B)\otimes(V',B') \coloneqq (V\otimes V', B\otimes B'),$$

giving it the familiar semiring structure. This allows for the use of the Grothendieck construction.

Definition 1.4.5. For a field F, the *zeroth λ-Hermitian* K-theory ring of F is defined to be

$$K_0^\lambda(F) \coloneqq K_0(\mathscr{P}^\lambda(F)).$$

The *symmetric* and *anti-symmetric* forms play significant roles, so define

$$K_0^h(F) \coloneqq K_0^1(F)$$
$$K_0^{\mathrm{sp}}(F) \coloneqq K_0^{-1}(F),$$

where K_0^h is called the *zeroth Hermitian K-theory ring* of F.

Remark 1.4.6. We have that $\mathscr{P}^1(F)$ is just the symmetric bilinear forms on vector spaces over F. So the construction of $K_0^h(F)$ makes it coincide with the familiar Grothendieck-Witt ring of F, so $\mathrm{GW}(F)\cong K_0^hF$.

We recall the construction of K_1 in algebraic K-theory from section 1.1.2, as a quotient of $\mathrm{GL}(F)$. We construct $K_1^h(F)$ analaguously:

Consider the group $\mathrm{Aut}(H^\lambda(F^n))$ of automorphisms, as a subgroup of $\mathrm{GL}_{2n}(F)$. Define the *λ-Hermitian general linear groups* $\mathrm{GL}_{2n}^\lambda(F) \coloneqq \mathrm{Aut}(H^\lambda(F^n))$. Further, define $\mathrm{GL}^\lambda(F) \coloneqq \mathrm{colim}_n\, \mathrm{GL}_{2n}^\lambda(F)$ where we take the colimit as a subgroup of $\mathrm{GL}(F) = \mathrm{colim}_n\, \mathrm{GL}_n(F)$. Following the construction of algebraic K-theory, we introduce the subgroup of *λ-Hermitian elementary matrices*, $\mathrm{E}_{2n}^\lambda(F) \subset \mathrm{GL}_{2n}^\lambda(F)$, generated by matrices on the forms

$$\begin{pmatrix} E & 0\\ 0 & E^{-1}\end{pmatrix}, \begin{pmatrix} I & B\\ 0 & I\end{pmatrix}, \begin{pmatrix} I & 0\\ B & I\end{pmatrix},$$

where E is an elementary matrix of $\mathrm{GL}_{2n}(F)$ and B is a λ-Hermitian form. Taking the colimit, we define $E^\lambda(F) \coloneqq \mathrm{colim}_n\, E_{2n}^\lambda(F)$. Furthermore, by [Bak81, corollary 3.9], $E^\lambda(F)$ is equal to $[\mathrm{GL}^\lambda(F), \mathrm{GL}^\lambda(F)]$, the commutator subgroup.

Definition 1.4.7. The *first λ-Hermitian K-group* of F is defined as $K_1^\lambda(F) \coloneqq \mathrm{GL}^\lambda(F)/E^\lambda(F)$, i.e., the abelianization of $\mathrm{GL}^\lambda(F)$.

The determinant map $\det\colon \mathrm{GL}_{2n}^1(F)\to F^\times$ gives an induced determinant map from $K_1^1(F)$. The next lemma is of great importance to the computation carried out in chapter 4.

Lemma 1.4.8. *The induced determinant map from* $\mathrm{GL}_{2n}^1(F)$ *on* $K_1^h(F)$ *is split surjective onto the multiplicative group* $\{\pm 1\}$.

Proof. From [FG05, p. 130], the group of automorphisms on $H^\lambda(F^n)$ can be identified with ${}_\lambda O_n(F)$, defined to be the group consisting of exactly the $2n \times 2n$ block matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $M^{-1}M = MM^{-1} = I_{2n}$ and the inverse is given by

$$M^{-1} = \Gamma(M) := \begin{pmatrix} d^T & \lambda b^T \\ \lambda c^T & a^T \end{pmatrix}.$$

In our case, $\lambda = 1$. Then it is easy to see that Γ is determinant-preserving, as it performs an even number of permutations of columns and rows. Thus $M\Gamma(M) = I$ implies

$$\det(M\Gamma(M)) = \det(M)\det(\Gamma(M)) = 1.$$

Thus, any $M \in O_n(F)$ has determinant ± 1.

There is an inclusion map $O_n(F) \hookrightarrow O_{n+1}(F)$ given by the determinant-preserving map

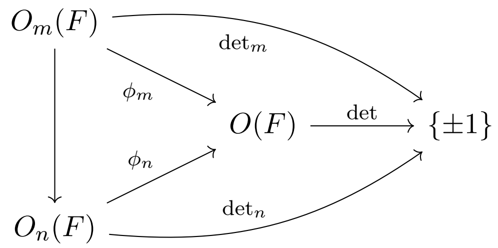
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The fact that this inclusion is determinant preserving, can easily be seen by realizing that the total number of permuations of rows and columns that are needed to get to the matrix with a, b, c and d on the upper left corner is even. Furthermore, there is a directed system

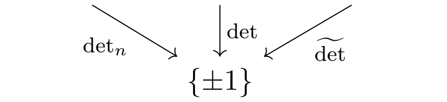
$$O_1(F) \hookrightarrow O_2(F) \hookrightarrow \cdots \hookrightarrow O_n(F) \hookrightarrow \cdots$$

whose colimit, the *orthogonal group*, is deonted by $O(F)$. Recall that $K_1^h(F) \cong O(F)_{\mathrm{ab}}$. Since the abelianization is a quotient, there is a surjective group homomorphism $\pi\colon O(F) \to O(F)_{\mathrm{ab}}$.

We claim that there is a well defined induced determinant map on $O(n)$. Using the universal property of the colimit, there exists a unique map $\det\colon O(F) \to \{\pm 1\}$ through which every determinant map $\det_n\colon O_n(F) \to \{\pm 1\}$ factors. In other words, for all $m < n$, the following commutes:



Consider now the map $\pi\colon O(F) \to O(F)_{\mathrm{ab}}$ to the abelianization of $O(F)$. Recall that the universal property for the abelianization implies that every map $O(F) \to G$ to an abelian group G factors uniquely through the abelianization $O(F)_{\mathrm{ab}}$. Thus, there exists a unique map $\widetilde{\det}\colon O(F)_{\mathrm{ab}} \to \{\pm 1\}$, making the following diagram commute:



(1.1)

To see that $\widetilde{\det}$ is indeed surjective observe that one can choose

$$I_2 \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in O_1(F) \quad \text{and} \quad J_2 \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in O_1(F)$$

whose respective images in $O(F)_{\text{ab}}$ are denoted by I and J. As (1.1) commutes, we have that

$$\widetilde{\det}(I) = \det(I_2) = 1 \quad \text{and} \quad \widetilde{\det}(J) = \det(J_2) = -1.$$

This concludes the part of the proof showing that the induced determinant map on $K_1^h(F) \cong O(F)_{\text{ab}}$ is surjective.

To show that $\widetilde{\det}$ is split surjective, we search for a retraction $r\colon \{\pm 1\} \to O(F)_{\text{ab}}$ such that the composition $\widetilde{\det} \circ r = \text{id}_{\{\pm 1\}}$. We claim that the assignments $1 \mapsto \pi(\phi_1(I_2))$ and $-1 \mapsto \pi(\phi_1(J_2))$ give a homomorphism satisfying the retraction property. To see that r is a homomorphism, note that it factors as $r\colon \{\pm 1\} \to O_1(F) \to O(F) \to O(F)_{\text{ab}}$ which is a homomorphism if the first arrow is, as it would follow that r is a composition of homomorphisms. Routine checks give that

$$\begin{aligned} 1 \cdot 1 = 1 \quad &\mapsto I_2 = I_2 I_2, \\ (-1) \cdot (-1) = 1 \quad &\mapsto I_2 = J_2 J_2, \\ (-1) \cdot 1 = 1 \cdot (-1) = -1 &\mapsto J_2 = I_1 J_2 = J_2 I_1, \end{aligned}$$

and r is indeed a homomorphism. When one realises that composing r with $\widetilde{\det}$ gives the identity on $\{\pm 1\}$, the proof is done. □

Defining the second Hermitian K-group can be done analogously to the algebraic case, by introducing the λ-Hermitian Steinberg group $St^\lambda(F)$. This is done in [Bak81, Lemma 3.16], and resembles the construction of the Steinberg group in the algebraic case. There is a canonical map $St^\lambda(F) \to E^\lambda(F)$ defined in [Bak81, Theorem 3.17], and we can use this to define the second Hermitian K-group:

Definition 1.4.9. The *second λ-Hermitian K-group* is given as $K_2^\lambda \coloneqq \ker(St^\lambda(F) \to E^\lambda(F))$.

Inserting $\lambda = 1$ in the definitions above gives the Hermitian $K_i^h(F)$-groups, known as *Hermitian K-theory*, and inserting $\lambda = -1$ gives the corresponding K-groups for the anti-symmetric forms, $K_i^{\text{sp}}(F)$.

1.4.1 Higher Hermitian K-theory

This presentation of the higher Hermitian K-groups is based on [Hor05]. Before giving the main definition, there are some preliminaries.

Remark 1.4.10. Hornbostel [Hor05] defines the Hermitian K-groups for so-called *additive categories with duality*. This is slightly more general than what we need. Note that $\text{Vect}(X)$, which we introduce further down, is a category with duality in the sense of [Hor05, Definition 1.1].

Definition 1.4.11. Given any small category $\mathcal{C}$, the *nerve* of $\mathcal{C}$ is the simplicial set $N\mathcal{C}$ defined in the following way: Its n-simplices are diagrams $c\colon [n+1] \to \mathcal{C}$ of the form

$$c_0 \to c_1 \to \cdots \to c_n.$$

It comes with the face maps $\partial_i(c)$, which forgets the ith face in the obvious way, and the coface (or degeneracy) map $\sigma_i(c)$, replacing c_i with $c_i \xrightarrow{=} c_i$.

Given a simplicial set, one can consider the geometric realization, a construction detailed in e.g. [May99, ch. 16]. The geometric realization of a simplicial set S is denoted $|S|$, and we define $B\mathcal{C} := |N\mathcal{C}|$.

Denote by $i\mathcal{C}$ the category whose objects are the objects of $\mathcal{C}$ and morphisms are the isomorphisms of $\mathcal{C}$.

Definition 1.4.12. Let $(\mathcal{C}, \square)$ be a symmetric monoidal category for which $i\mathcal{C} = \mathcal{C}$. Construct $\mathcal{C}^+ := \mathcal{C}^{-1}\mathcal{C}$ the following way: Objects are pairs (m, n) of objects in $\mathcal{C}$. Morphisms are equivalence classes of composites

$$(m_1, m_2) \xrightarrow{s\square} (s\square m_1, s\square m_2) \xrightarrow{(f,g)} (n_1, n_2)$$

for s an object of $\mathcal{C}$. We let the above composite be equivalent to the composite

$$(m_1, m_2) \xrightarrow{t\square} (t\square m_1, t\square m_2) \xrightarrow{(f',g')} (n_1, n_2)$$

if and only if there is an isomorphism $\eta\colon s \to t$ in $\mathcal{C}$.

There is an inclusion functor $\mathcal{C} \to \mathcal{C}^{-1}\mathcal{C}$ defined by sending $m \mapsto (m, e)$. This induces a natural map $B\mathcal{C} \to B(\mathcal{C}^{-1}\mathcal{C})$ [Wei13, Remark 4.2.2]. Next we can define $\mathcal{C}^+ := \mathcal{C}^{-1}\mathcal{C}$ with a functor $\mathcal{C} \to \mathcal{C}^+$ such that $B\mathcal{C} \to B\mathcal{C}^{-1}\mathcal{C}$ is a group completion (at least under very mild hypotheses [Hor05, p. 665]).

Let X be a scheme, and $\mathrm{Vect}(X)$ the category of locally free $\mathcal{O}_X$ sheaves of finite rank. Note that this is a symmetric monoidal category with respect to direct sum, and it has a duality functor $* = \mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$ s.t. $*(M) =: M^*$ [Har77, Ch. II.5].

Definition 1.4.13. For a scheme X, define the *Hermitian category* of $\mathrm{Vect}(X)$, denoted $\mathrm{Vect}(X)_h$, to be the category with objects pairs

$$(M, \phi\colon M \xrightarrow{\cong} M^*)$$

for M an object of $\mathrm{Vect}(X)$ and ϕ an isomorphism. Morphisms are maps $f\colon (M, \phi) \to (M', \psi)$ for which the following diagram comutes:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M^* \\ \downarrow f & & \uparrow f^* \\ M' & \xrightarrow{\psi} & (M')^* \end{array}$$

Definition 1.4.14. The *Hermitian K-theory space of a scheme X* is defined to be

$$K^h(X) := B(i\mathrm{Vect}(X)_h^+),$$

and the *Hermitian K-theory* of X is defined to be

$$K_n^h(X) := \pi_n(K^h(X)).$$

Chapter 2

Motivic Homotopy Theory

Motivic homotopy theory aims to study smooth schemes by the means of abstract homotopy theory. Despite its young age, it has already seen spectacular applications, for instance in the proofs of the Milnor and Bloch-Kato conjectures. The overarching idea is to let the affine line $\mathbb{A}^1$ play the role of the unit interval in classical topology. Indeed, the unit interval is not an algebraic variety, but the affine line is, and this would result in a purely algebraic way of doing homotopy theory.

The focus of this chapter is to give a general introduction to motivic homotopy theory, introduce the stable motivic homotopy category along with some of its properties, and some cohomology theories and spectra representing these. Most of these will be important to the computation in chapter 4, as they show up as very effective slices of the Hermitian K-theory spectrum.

2.1 The Construction of $\mathcal{SH}(F)$

The following is based on [Voe98], and is a sketch of Voevodsky's construction. In order to arrive at the stable motivic homotopy category, we need model structures along the way. Few details will be given when it comes to model structures, and we refer to [Hov99] for the curious reader.

The starting point is the category of smooth separated schemes of finite type over a field F, denoted Sm_F, which we will embed into a larger category with nice categorical properties, making it suitable for doing abstract homotopy theory. As mentioned, the basic idea is to let the affine line $\mathbb{A}^1$ play the role of the unit interval, and do homotopy theory on smooth schemes from there. Doing this directly is no good idea, however. Indeed, it is not obvious how one would make the affine line contractible. Another issue is that Sm_F does not have all small colimits, and in particular it does not even have all quotients.

Given a category $\mathcal{C}$, the standard way of formally adding the colimits of all small diagrams is to consider the category of presheaves on $\mathcal{C}$. That is, the category of all contravariant functors from $\mathcal{C}$ to the category Set of sets: $\mathrm{Pre}(\mathcal{C}) := [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$. Every object $X \in \mathcal{C}$ gives rise to a presheaf $R_X : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, sending Y to the set $\mathrm{Hom}_{\mathcal{C}}(Y, X)$. By the Yoneda lemma, we embed $\mathcal{C}$ into $\mathrm{Pre}(\mathcal{C})$, identifying $\mathcal{C}$ it with the full subcategory of its image. The morphisms in this new category are the natural transformations. The resulting category contains all small limits and colimits i.e., it is bicomplete, as wanted. In the same way, we embed $\mathrm{Sm}_F \hookrightarrow \mathrm{Pre}(\mathrm{Sm}_F) = [\mathrm{Sm}_F^{\mathrm{op}}, \mathrm{Set}]$.

To make easier classical homotopical constructions such as taking homotopy (co)limits, we make a further embedding. Let Δ be the simplicial category of finite ordered sets $[n] \coloneqq \{1 < \cdots < n\}$ and morphisms nondecreasing maps of such sets. Let $[\Delta^{\mathrm{op}}, \mathrm{Set}]$ be the category of simplicial sets, that is, contravariant functors $\Delta^{\mathrm{op}} \to \mathrm{Set}$. We embed

$$\mathrm{Pre}(\mathrm{Sm}_F) \hookrightarrow \Delta^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}_F) \coloneqq [\Delta^{\mathrm{op}}, \mathrm{Pre}(\mathrm{Sm}_F)]$$

by mapping $X \mapsto ([n] \mapsto X)$, i.e., mapping any presheaf to a constant simplicial presheaf. We embed the simplicial sets $\Delta^{\mathrm{op}}\mathrm{Set} \hookrightarrow \Delta^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}_F)$ via the mapping $X \mapsto X$, evaluating all objects of Sm_F to X.

Now the category $\Delta^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}_F)$ has all the nice categorical properties we want for doing abstract homotopy theory, and we name it the category of *motivic spaces*, denoted $\mathcal{MS}(F)$. We note that $\mathrm{Spec}\, F$ is the terminal object of this category.

In order to endow $\mathcal{MS}(F)$ with a model structure, one needs to define classes of fibrations, cofibrations and weak equivalences. We will not spend time on that here, and refer to [Voe98] for details. To make this this category useful, we consider Sm_F with the *Nisnevich topology*, which is finer than the Zariski topology, but coarser than the étale topology - exactly what is needed to prove interesting results like the homotopy purity theorem and descent theorems for algebraic K-theory.

The category $\mathcal{MS}(F)$ is now endowed with a model structure, taking the Nisnevich topology on Sm_F into account. We localize this with respect to $\mathbb{A}^1 \to *$, to give a (motivic) model structure which is both proper and closed [Voe98]. The resulting homotopy category will be denoted $\mathcal{H}(F)$, and we denote the morphism classes $\mathrm{Hom}_{\mathcal{H}(F)}(X, Y)$ as $[X, Y]_{\mathcal{H}(F)}$. The category of pointed motivic spaces $\mathcal{MS}_\bullet(F)$, is defined by a canonical functor from an unpointed category via the assignment $X \mapsto X_+ \coloneqq (X \coprod \mathrm{Spec}\, F, \mathrm{Spec}\, F)$. By localizing $\mathcal{MS}_\bullet(F)$, we get the pointed motivic homotopy category $\mathcal{H}_\bullet(F)$.

The category $\mathcal{MS}_\bullet(F)$ is bicomplete, and has in particular all quotients. This means that many of the familiar constructions from topology are available to us. Quotients of spaces X by Y is given by the familiar push-out squares

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & X/Y \end{array}$$

where X/Y is pointed by the image of Y. For an arbitrary collection of spaces pointed spaces $(X_i, x_i)_{i \in I}$, the wedge product $\bigvee_{i \in I}(X_i, x_i)$ is given by

$$\begin{array}{ccc} \coprod_{i \in I} x_i & \longrightarrow & \coprod_{i \in I} X_i \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & \bigvee_{i \in I} X_i \end{array}$$

These operations suffice to define the smash product, $X \wedge Y \coloneqq (X \times Y)/(X \vee Y)$.

From the embeddings of Sm_F and $\Delta^{\mathrm{op}}(\mathrm{Set})$ into $\mathcal{MS}(F)$, we get two different kinds of spheres: The usual one from topology, i.e. the simplicial sphere

$$S^{1,0} \coloneqq \Delta^1/\partial\Delta^1,$$

pointed by Δ^0, and, originating from Sm_F, the so-called *Tate-circle*

$$S^{1,1} \coloneqq \mathbb{A}^1 - \{0\},$$

pointed by 1. Note that the latter often is denoted $\mathbb{G}_m$. Now, as we have a notion of smash product in our category, we can form spheres of arbitrary degree by smashing it with itself:

$$S^{p+q,q} := (S^{1,0})^{\wedge p} \wedge (S^{1,1})^{\wedge q}.$$

Furhtermore, this gives a suspension functor $\Sigma^{p,q}X := S^{p,q} \wedge X$.

Remark 2.1.1. Note that another common grading convention for motivic spheres is the so-called $(m+\alpha n)$-grading, where $S^1 := S^{1,0}$ and $S^\alpha = S^{1,1}$. We can translate betwwen them by

$$S^{p,q} = S^{(p-q)+q\alpha},$$

but we will primarily be using the (p,q)-grading in this thesis. The α in the definition of the Tate circle is nothing else than a 'basis vector' to keep track of how many of what sphere we have smashed with. Thus, the bigrading carries the exact same meaning.

Proposition 2.1.2 ([Voe98, Lemma 4.1])**.** *In the pointed motivic homotopy category $\mathcal{H}_\bullet(F)$, we have the canonical isomorphisms*

$$(\mathbb{A}^n - \{1\}) \cong S^{2n-1,n-1} \quad \text{and} \quad \mathbb{P}^n/\mathbb{P}^{n-1} \cong \mathbb{A}^n/\mathbb{A}^n - \{0\} \cong S^{2n,n}.$$

Proof. The following is the proof the case $n = 1$ of the latter claim. See e.g. [MV99, pp. 110-113] for a proof of the general case. Begin by computing the homotopy colimit of the diagram $* \longleftarrow (\mathbb{A}^1 - \{0\}) \longrightarrow \mathbb{A}^1$ in the category of motivic spaces. By [Str11, p. 153], this is equal to the simplicial suspension of $\mathbb{A}^1 - \{0\}$, giving

$$\operatorname{hocolim}(* \longleftarrow (\mathbb{A}^1 - \{0\}) \longrightarrow \mathbb{A}^1) \simeq \Sigma^{1,0}(\mathbb{A}^1 - \{0\}) = S^{2,1}.$$

We compute this homotopy colimit a different way, keeping in mind that the affine line is weakly equivalent to $*$. Consider the diagram

$$\begin{array}{ccc} \mathbb{A}^1 - \{0\} & \rightarrowtail & \mathbb{A}^1 \\ \downarrow & \ulcorner & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

which is a pushout by gluing. We now have the equivalences

$$S^{2,1} \simeq \operatorname{hocolim}(* \longleftarrow (\mathbb{A}^1 - \{0\}) \longrightarrow *) \simeq \mathbb{P}^1,$$

which become canonical isomorphisms in $\mathcal{H}_\bullet(F)$. □

To construct the motivic stable homotopy category, we need the notion of a spectrum. Denote the $\mathbb{P}^1$-suspension functor $X \mapsto \mathbb{P}^1 \wedge X$ by $\Sigma_{\mathbb{P}^1}$. Inductively, define $\Sigma^n_{\mathbb{P}^1} := \Sigma_{\mathbb{P}^1} \wedge \Sigma^{n-1}_{\mathbb{P}^1}$.

Definition 2.1.3. A $\mathbb{P}^1$-*spectrum* is a sequence of pointed spaces $E = \{E_n\}_{n\in\mathbb{N}}$ for $X_n \in \mathcal{MS}_\bullet(F)$ and structure maps $\sigma^E_n : \Sigma_{\mathbb{P}^1}E_n \to E_{n+1}$. A *map of* $\mathbb{P}^1$-*spectra* $f\colon E \to F$ is a sequence of maps $\{f_n\colon X_n \to Y_n\}_{n\in\mathbb{N}}$ for which the square

$$\begin{array}{ccc} \Sigma_{\mathbb{P}^1}E_n & \xrightarrow{\sigma^E_n} & E_{n+1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma_{\mathbb{P}^1}F_n & \xrightarrow{\sigma^F_n} & F_{n+1} \end{array}$$

commutes for every n. These objects and morphisms constitute the category of $\mathbb{P}^1$-spectra, denoted $\operatorname{Spt}(F)$.

As an analogue to the suspension spectrum in topology, we define the stabilization functor $\Sigma^\infty_{\mathbb{P}^1}\colon \mathcal{MS}_\bullet(F) \to \mathrm{Spt}(F)$ mapping a pointed space X to the spectrum $\{\Sigma^n_{\mathbb{P}^1}X, \mathrm{id}\}_{n\in\mathbb{N}}$, where $\Sigma^n_{\mathbb{P}^1}$ is the 'smashing with $\mathbb{P}^1$ n times-functor'. The previously discussed canonical map $\mathcal{MS}(F) \to \mathcal{MS}_\bullet(F)$ now gives a functor $\mathcal{MS}(F) \to \mathrm{Spt}(F)$. It takes a scheme $X \in \mathrm{Sm}_F$ to the spectrum $\Sigma^\infty_{\mathbb{P}^1}X_+$.

Definition 2.1.4. We define the *sphere spectrum* to be $S \coloneqq \Sigma^\infty_{\mathbb{P}^1}\mathrm{Spec}\,F_+$.

We mention briefly how we give $\mathrm{Spt}(F)$ a stable model structure and refer to [Jar00] for the detail-hungry reader. Let the category of spectra $\mathrm{Spt}(F)$ inherit a levelwise model structure from the category $\mathcal{MS}_\bullet(F)$. I.e., a map $X \to Y$ in $\mathrm{Spt}(F)$ is a weak equivalence (resp. (co)fibration) if the component maps $X_n \to Y_n$ are weak equivalences (resp. (co)fibrations) in $\mathcal{MS}_\bullet(F)$. Having this levelwise model structure on $\mathrm{Spt}(F)$, one proceeds to construct a *stable model structure* on $\mathrm{Spt}(F)$ [Voe98] . The associated homotopy category is the *stable motivic homotopy category* $\mathcal{SH}(F)$. We denote the morphism classes $\mathrm{Hom}_{\mathcal{SH}(F)}(X,Y) \eqqcolon [X,Y]_{\mathcal{SH}(F)}$.

Theorem 2.1.5 ([Voe98])**.** *The following statements hold for the category $\mathcal{SH}(F)$:*

- *$\mathcal{SH}(F)$ is a triangulated category, and the shift functor is given by the simplicial suspension: $\Sigma^{1,0} = S^{1,0} \wedge -$.*
- *$\mathcal{SH}(F)$ is symmetric monoidal with unit the sphere spectrum.*
- *$\mathcal{SH}(F)$ is an additive category.*
- *The smash product and homotopy colimits in $\mathcal{SH}(F)$ commute.*

We proceed to define the stable motivic homotopy groups.

Definition 2.1.6. Let $m, n \in \mathbb{N}$, $X \in \mathrm{Spt}(F)$ and $U \in \mathrm{Sm}_F$. There is a directed system

$$[S^{m,n} \wedge U_+, X_0] \to [S^{m+2,n+1} \wedge U_+, X_1] \to \cdots .$$

Define $\underline{\pi}_{m,n}X(U)$ to be the colimit of the above system, defining a presheaf. We define the bigraded *stable homotopy groups* of X as $\pi_{m,n}X \coloneqq \underline{\pi}_{m,n}(X)(\mathrm{Spec}\,F)$.

Note that for a spectrum X of $\mathcal{SH}(F)$, the stable homotopy groups are given as

$$\pi_{p,q}X = [S^{p,q}, X]_{\mathcal{SH}(F)}.$$

Remark 2.1.7. Instead of writing $\pi_{*,*}$ we will emphasize the bigrading of the homotopy groups (and later (co)homology groups) by $\pi_\star$. For motivic homotopy groups $\pi_{m,n}(X)$, we will refer to m as the *topological degree* and n as the *weight*.

Similarly, there are motivic (co)homology theories associated with motivic spectra:

Definition 2.1.8. For an object $E \in \mathcal{SH}(F)$, we can assign a cohomology theory $E^{p,q}(-)$ and a homology theory $E_{p,q}(-)$ on $\mathcal{MS}_\bullet(F)$, as functors $\mathcal{MS}_\bullet(F) \to \mathrm{Ab}$, given by

$$E^{p,q}(X) \coloneqq [\Sigma^\infty_{\mathbb{P}^1}X, S^{p,q} \wedge E]$$
$$E_{p,q}(X) \coloneqq [S^{p,q}, E \wedge \Sigma^\infty_{\mathbb{P}^1}X].$$

Among the most fundamental objects to study in stable motivic homotopy theory are the homotopy groups of sphere, $\pi_{m,n}S$, and in fact, most of these are still unknown. By necessity, they are more difficult to compute than the topological ones. This stems from the fact that the complex realization of the sphere spectrum is just the topological one, while the motivic spheres, of course, are more complicated.

Computations done by Morel show the following:

Theorem 2.1.9 ([Mor06]). *For a perfect field F with $\operatorname{char} F \neq 2$ and $n \in \mathbb{Z}$, there are isomorphisms*

$$\pi_{n,n}S \xrightarrow{\cong} K^{\mathrm{MW}}_{-n}(F)$$

and for $m < n$,

$$\pi_{m,n}S = 0.$$

We end the section with some words on Brown representability.

Definition 2.1.10 ([Wei94, Definition 10.2.7]). Let $\mathcal{K}$ be a triangulated category with shift functor T, and $\mathcal{A}$ an abelian category. A functor $H\colon \mathcal{K} \to \mathcal{A}$ is called covariant cohomological (or homological) if the exact triangle (u, v, w) of (X, Y, Z) induces the long exact sequence

$$\cdots \xrightarrow{w^*} H(T^iX) \xrightarrow{u^*} H(T^iY) \xrightarrow{v^*} H(T^iZ) \xrightarrow{w^*} H(T^{i+t}X) \xrightarrow{u^*} \cdots .$$

Define a contravariant cohomological functor H to be a homological functor $H\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{A}$.

Remark 2.1.11. The cohomology theory functors of definition 2.1.8 are (co)homological functors.

An important feature of $\mathcal{SH}(F)$ is that it satisfies Brown representability:

Theorem 2.1.12 (Brown representability, [NS11]). *Let F be a countable field and $H\colon \mathcal{SH}(F) \to \mathcal{A}$ a (co)homological functor. Then there exists a spectrum $E \in \mathcal{SH}(F)$ such that $E^\star = H$. In other words, every cohomology theory is representable by some spectrum.*

2.2 Motivic Cohomology

One of the immediate questions after constructing the stable motivic homotopy category, is how we can define motivic cohomology. As motivic cohomology shows up on E^1-page of the very effective slice spectral sequence, it will be of great use to us. This section introduces the concept and provides some relevant calculations and results.

Definition 2.2.1. Let $X, U \in \mathrm{Sm}_F$. An *elemantary finite correspondence* of $U \times X$ is a closed irreducible subset $Z \subset U \times X$ which is surjective and finite over U. Let $\mathrm{Cor}_F(U, X)$ denote the free abelian group generated by such finite correspondences of $U \times X$. Elements of $\mathrm{Cor}_F(U, X)$ are commonly referred to as *finite correspondences.*

Example 2.2.2. For a morphism $f\colon U \to X$, the graph $\Gamma_f = \{(u, f(u)) \,|\, u \in U\}$ is an example of a cycle in $\mathrm{Cor}_F(U, X)$. ♣

Denote by Cor_F the categroy whose objects coincide with those in Sm_F and with morphisms $\mathrm{Cor}_F(X, Y)$.

Definition 2.2.3 ([MVW06, definition 2.1]). A *presheaf with transfers* is a contravariant additive functor $F\colon \mathrm{Cor}_F^{\mathrm{op}} \to \mathrm{Ab}$. The category of presheaves with transfers is denoted $\mathrm{PST}(F)$.

Let $\mathbb{Z}_{\mathrm{tr}}$ denote the class of presheaves with transfers provided by representable functors.

Definition 2.2.4 ([MVW06, definition 3.1]). For an integer $q \geq 0$, define a *motivic complex* $\mathbb{Z}(q)$ to be the following complex of presheaves with transfers:

$$\mathbb{Z}(q) \coloneqq C_*\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\wedge q})[-q]$$

where $\mathbb{Z}(q)$ is considered bounded above – the shifting convention for $[-q]$ implies that the terms $\mathbb{Z}(q)^i = C_{q-i}\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\wedge q})$ vanish whenever $i > q$, and C_* means taking the *Suslin complex* (see e.g. [MVW06, p. 16]). For any abelian group A, define

$$A(q) \coloneqq \mathbb{Z}(q) \otimes A.$$

Definition 2.2.5 ([MVW06, definition 3.4]). For $p, q \in \mathbb{Z}$, the *motivic cohomology groups* $H^{p,q}(X, \mathbb{Z})$ are defined to be the hypercohomology of the motivic complexes $\mathbb{Z}(q)$ with respect to the Zariski topology:

$$H^{p,q}(X, \mathbb{Z}) \coloneqq \mathbb{H}^p_{\mathrm{Zar}}(X, \mathbb{Z}(q)).$$

Furthermore, for any abelian group A, we define

$$H^{p,q}(X, A) \coloneqq \mathbb{H}^p_{\mathrm{Zar}}(X, A(q)).$$

To aid computations, we mention some of the vanishing properties of motivic cohomology. Here is a summary of some of these.

Theorem 2.2.6 ([MVW06, Theorem 3.6, 19.3, Corollary 4.2 and p.viii]). *Let X be a smooth scheme, and A an abelian group.*

- *Then $H^{p,q}(X, A) = 0$ when $p > q + \dim X$, where $\dim X$ is the dimension in the Zariski topology.*
- *$H^{p,q}(X, A) = 0$ for $p > 2q$,*
- *$H^{p,q}(X, A) = 0$ for $q < 0$,*
- *For X connected,*
$$H^{p,0}(X, A) = \begin{cases} A & \text{for } p = 0, \\ 0 & \text{else,} \end{cases}$$
- *and*
$$H^{p,1}(X, \mathbb{Z}) = \begin{cases} \mathcal{O}^*(X) & \text{for } p = 1, \\ \mathrm{Pic}(X) & \text{for } p = 2, \\ 0 & \text{for } p \neq 1, 2, \end{cases}$$
where $\mathcal{O}^*$ is the sheaf of invertible elements in $\mathcal{O}$ and $\mathrm{Pic}(X)$ is the Picard group of X [Har77, p. 141, 143].

Of particular interest to us is the motivic cohomology of a point:

Theorem 2.2.7 ([MVW06, p. viii, Theorem 5.1]). *For any field F and any n, the motivic cohomology along the diaognal is given by*

$$H^{n,n}(\operatorname{Spec} F, A) \cong K_n^M(F) \otimes A,$$

where $K_n^M(F)$ is the nth Milnor K-group of F. Furthermore, from [DI10], we have that

$$H^\star(\operatorname{Spec} F, \mathbb{Z}/2) = k_*^M(F)[\tau],$$

where k_*^M is mod 2 Milnor K-theory (as in definition 1.2.5) and $|\tau| = (0,1)$.

The following proposition will be essential to the main computation of this thesis:

Proposition 2.2.8. *The integer coefficient motivic cohomology of finite fields of order q is given by*

$$H^{m,n}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & m = n = 0, \\ \mathbb{Z}/q^n - 1 & m = 1,\, n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For coefficients in $\mathbb{Z}/2$, the motivic cohomology is given by $H^{*,*}(\operatorname{Spec} \mathbb{F}, \mathbb{Z}/2) = k_*^{\mathrm{M}}(\mathbb{F}_q)[\tau]$ for $|\tau| = (0,1)$. In particular, we get degreewise

$$H^{m,n}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & m = 0,\, n \geq 0 \text{ or } m = 1, n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The case for mod-2 coefficients follows immediately from theorem 2.2.7.

For the case of integer coefficients, we use the slice spectral sequence defined in section 3.4, which is strongly convergent in this case by theorem 3.4.5. The E_2-page has input in motivic cohomology, and it converges to algebraic K-theory. The abutment is the algebraic K-theory of finite fields, given by theorem 1.1.11, and one can try to reconstruct the E_2-page from there. Using the vanishing results of theorem 2.2.6, we immediately see that the first and second quadrants are zero everywhere except at the origin. The fourth quadrant is zero everywhere except for at $p = 0$. Futhermore, the cohomological dimension of finite fields is 1 [Mil13, ch. 15], and thus $H^{p,q}(\mathbb{F}_q) = 0$ when $p > 1$, and also when $p < 0$. Thus, we have the following E_2-page. The slots without marked input are all zero, and the sequence of (possibly) nonzero motivic cohomology continues infinitely far down to the left.

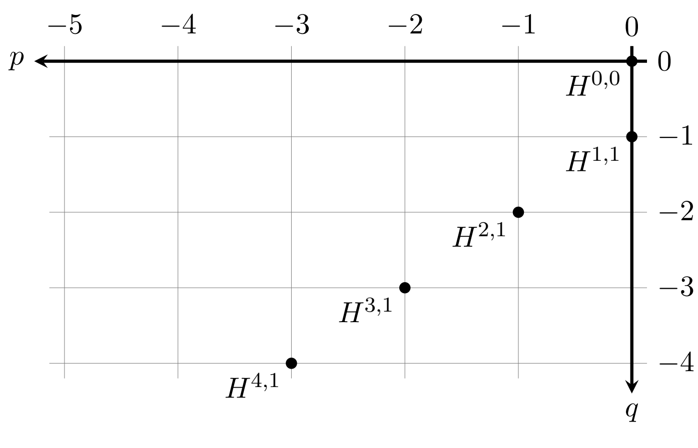


Figure 2.1: $E^1_{*,*,0}$.

Now, the spectral sequence converges to algebraic K-theory. Since every diagonal $-p-q=n$ has one nonzero group at most, this group has be isomorphic to the algebraic K-group $K_n(\mathbb{F}_q)$. Now the result follows from theorem 1.1.11. □

2.3 The Motivic Cohomology Spectrum

Finding a spectrum representing motivic cohomology is done by constructing a motivic analogue to the Eilenberg-Maclane spectrum in classical homotopy theory. This presentation is a sketch of the one given in [Voe98, section 6.1].

For any X in Sm_F, we define the functor $L(X)\colon \mathrm{Sm}_F^{\mathrm{op}} \to \mathrm{Ab}$ by the assignment $U \mapsto \mathrm{Cor}_F(U,X)$. In other words, $L(X)=\mathrm{Cor}_F(-,X)$. Voevodsky proves that this is a Nisnevich sheaf. He proceeds to show that $L(-)$ extends to a functor from $\mathcal{MS}_\bullet(F)$ to Nisnevich sheaves with values in simplicial abelian groups. This is due to the fact that L preserves colimits, and every pointed motivic space indeed is the colimit of representable functors. Hence it suffices to describe L for smooth schemes.

We are finally ready to define the motivic analogues to Eilenberg-Maclane spaces. In what follows, consider $\mathbb{P}^1$ pointed by ∞.

Definition 2.3.1. For $n \geq 0$, define the nth *motivic Eilenberg-Maclane space* to be

$$K(\mathbb{Z}(n),2n) := L((\mathbb{P}^1)^{\wedge n}).$$

If we let X and Y be in Sm_F, then there is a bilinear morphism $L(X)\times L(Y) \to L(X\times Y)$ induced by the external product of cycles. This construction also extends to pointed motivic spaces. Thus there are the following canonical maps:

$$\mu_{m,n}\colon K(\mathbb{Z}(m),2m)\wedge K(\mathbb{Z}(n),2n) \to K(\mathbb{Z}(m+n),2(m+n)).$$

There is a canonical map $i\colon \mathbb{P}^1 \to L(\mathbb{P}^1)$ defined in [Voe98, section 6]. Composing $\mu_{1,n}$ with $i\wedge \mathrm{id}$ gives the structure maps:

$$e_n\colon \mathbb{P}^1 \wedge K(\mathbb{Z}(n),2n) \to K(\mathbb{Z}(n+1),2(n+1)).$$

Definition 2.3.2. The *Eilenberg-Maclane spectrum* $H\mathbb{Z}$ consists of the spaces $K(\mathbb{Z}(n),2n)$ where the structure maps $e_n\colon \mathbb{P}^1\wedge K(\mathbb{Z}(n),2n) \to K(\mathbb{Z}(n+1),2(n+1))$ between them are given by the following composition

$$\mathbb{P}^1\wedge K(\mathbb{Z}(n),2n) \xrightarrow{i\wedge\mathrm{id}} K(\mathbb{Z}(1),2)\wedge K(\mathbb{Z}(n),2n) \xrightarrow{\mu_{1,n}} K(\mathbb{Z}(m+n),2(m+n)).$$

Remark 2.3.3. For a prime number p, one can construct the *Eilenberg MacLane-spectrum with mod-p coefficients.* In the category of abelian sheaves, it is defined by taking the reduction of $L(X)$ modulo p.

Example 2.3.4. For a spectrum E in $\mathcal{SH}(F)$, the motivic cohomology of E is given by $H^{p,q}(E,\mathbb{Z}) = H\mathbb{Z}^{p,q}(E) = [E, S^{p,q}\wedge H\mathbb{Z}]_{\mathcal{SH}(F)}$. ♣

Theorem 2.3.5 ([Voe98, Theorem 6.2])**.** *The structure maps of the motivic Eilenberg-Maclane spectrum have $\mathbb{A}^1$-weak equivalences as adjoints. In other words, $H\mathbb{Z}$ is an $\Omega_{\mathbb{P}^1}$-spectrum.*

2.4 The Algebraic K-Theory Spectrum

This section outlines the construction of the $\mathbb{P}^1$-spectrum **KGL**, representing algebraic K-theory (as defined in section 1.1). Once again, the presentation follows [Voe98], despite his calling the construction 'rather ugly'.

Consider the Grassmanian $\mathrm{Gr}_k(\mathbb{A}^n)$ of k-planes in $\mathbb{A}^n$, and consider the canonical inclusions

$$\mathrm{Gr}_k(\mathbb{A}^n) \hookrightarrow \mathrm{Gr}_k(\mathbb{A}^{n+1})$$
$$\mathrm{Gr}_k(\mathbb{A}^n) \hookrightarrow \mathrm{Gr}_{n+1}(\mathbb{A}^n),$$

giving rise to the inclusion

$$\mathrm{Gr}_k(\mathbb{A}^n) \hookrightarrow \mathrm{Gr}_{k+1}(\mathbb{A}^{n+1})$$
$$L \mapsto L \oplus \{0\}\,.$$

Now, define

$$\mathrm{BGL}_n \coloneqq \operatorname*{colim}_k \mathrm{Gr}_k(\mathbb{A}^{n+k})$$

and

$$\mathrm{BGL} \coloneqq \operatorname*{colim}_k \mathrm{BGL}_k.$$

The constituent spaces of the spectrum **KGL** consist of copies of the space KGL, which is defined as a fibrant replacement of the space

$$\mathbb{Z} \times \mathrm{BGL} \coloneqq \coprod_{i \in \mathbb{Z}} \mathrm{BGL}.$$

The structure maps $\mathbb{P}^1 \wedge \mathrm{KGL} \to \mathrm{KGL}$ are defined in the following way: By [Voe98, p. 600], there is an isomorphism

$$\mathrm{Hom}_{\mathcal{H}_\bullet}(\mathbb{P}^1 \wedge (\mathbb{Z} \times \mathrm{BGL}), \mathbb{Z} \times \mathrm{BGL}) \cong \mathrm{Hom}_{\mathcal{H}_\bullet}(\mathbb{Z} \times \mathrm{BGL}, \mathbb{Z} \times \mathrm{BGL}), \tag{2.1}$$

and we define the map $\bar{e}\colon \mathbb{P}^1 \wedge (\mathbb{Z} \times \mathrm{BGL}) \to \mathbb{Z} \times \mathrm{BGL}$ to be the map corresponding to the identity on $\mathbb{Z} \times \mathrm{BGL}$ under the isomorphism (2.1). Since KGL is fibrant, $\bar{e}$ lifts to the map

$$e\colon \mathbb{P}^1 \wedge \mathrm{KGL} \to \mathrm{KGL}.$$

Definition 2.4.1. The *algebraic K-theory spectrum* **KGL** consists of spaces KGL and structure maps $e\colon \mathbb{P}^1 \wedge \mathrm{KGL} \to \mathrm{KGL}$.

Indeed, **KGL** represents algebraic K-theory, as we have the following theorem:

Theorem 2.4.2 ([Voe98, Theorem 6.9])**.** *For any $X \in \mathrm{Sm}_F$, one has canonical isomorphisms*

$$K_{m-n}(X) \cong \mathrm{Hom}_{\mathcal{SH}(F)}(S^{m+n,n} \wedge X_+, \mathbf{KGL}).$$

In particular, given $X = \operatorname{Spec} F$, we have $K_{m-n}(F) \cong \pi_{m+n,n}(\mathbf{KGL})$.

We conclude this section by mentioning the Bott periodicity for the algebraic K-theory spectrum:

Theorem 2.4.3 ([Voe98, Theorem 6.8])**.** *There is a canonical isomorphism* $\mathbf{KGL} \cong \mathbb{P}^1 \wedge \mathbf{KGL}$.

2.5 The Hermitian K-Theory Spectrum

This section outlines the construction of the spectrum **KQ** representing Hermitian K-theory. The representability of Hermitian K-theory as a spectrum in the motivic stable homotopy category was first proved by Hornbostel [Hor05], but we follow the presentation by Röndigs and Østvær [RØ16].

Define the functor $KO\colon \mathrm{Sm}_F \to \mathcal{MS}_\bullet(F)$ sending some smooth F-scheme to the space representing Hermitian K-groups for trivial involution (as in section 1.4), $\lambda = 1$. For $\lambda = -1$, take instead the functor $KSp\colon \mathrm{Sm}_F \to \mathcal{MS}_\bullet(F)$. The forgetful functor $F\colon \mathscr{P}(F)_h \to \mathscr{P}(F)$ of [Hor05, Definition 4.1] induces the natural maps $f_0\colon KO \to KGL$ and $f_2\colon KSp \to KGL$. The homotopy fibers VQ and VSp (resp.) yield the following homotopy fiber sequences. Using the fact that $\mathcal{SH}(F)$ is triangulated, we rotate the triangle and take the adjoint map on the left:

$$\Omega^{1,0}K \xrightarrow{h_3'} VQ \xrightarrow{\text{can}} KO \xrightarrow{f_0} \text{KGL},$$
$$\Omega^{1,0}K \xrightarrow{h_1'} VSp \xrightarrow{\text{can}} KSp \xrightarrow{f_2} \text{KGL}.$$

There exist natural maps $h_0\colon K \to KO$ and $h_2\colon K \to KSp$ which are induced by the hyperbolic functor $H\colon i\mathscr{P}(F) \to i\mathscr{P}(F)_h$ of [Hor05]. Taking the homotopy fiber of these and rotating the triangle give the homotopy fiber sequences:

$$\Omega^{1,0}KO \xrightarrow{\text{can}} UQ \xrightarrow{f_3} \text{KGL} \xrightarrow{h_0} KO,$$
$$\Omega^{1,0}KSp \xrightarrow{\text{can}} USp \xrightarrow{f_1} \text{KGL} \xrightarrow{h_2} KSp.$$

We proceed by stating the fundamental theorem of Hermitian K-theory.

Theorem 2.5.1 ([Kar80])**.** *There are natural weak equivalences*

$$\Omega^{1,0}USp \xrightarrow{\sim} VQ \text{ and } \Omega^{1,0}UQ \xrightarrow{\sim} VSp.$$

Furthermore, there is a theorem by Hornbostel and Sclichting:

Theorem 2.5.2 ([HS04, Section 1.8])**.** *The homotopy cofiber of the maps*

$$KO \to KO(\mathbb{A}^1 - \{0\} \times -) \text{ and } KSp \to KSp(\mathbb{A}^1 - \{0\} \times -)$$

induced by the map $\mathbb{A}^1 - \{0\} \to *$ is naturaly weakly equivalent to $\Sigma^{1,0}UQ$ and $\Sigma^{1,0}USp$, respectively.

Using these theorems, one can show that there exist weak equivalences

$$UQ \xrightarrow{\sim} \Omega^{1,0}\Sigma^{1,0}UQ \sim \Omega^{2,1}KO$$

and

$$USp \xrightarrow{\sim} \Omega^{1,0}\Sigma^{1,0}USp \sim \Omega^{2,1}KSp.$$

Concluding the construction, we refer to the paper [Hor05], in which Hornbostel shows that Hermitian K-theory is representable in the motivic unstable and stable homotopy category. He shows that it can be represented by the motivic spectrum

$$\mathbf{KQ} := (KO, USp, KSp, UQ, KO, USp, \ldots)$$

with structure maps the adjoints of the weak equivalences:

$$\begin{aligned} KO &\xrightarrow{\sim} \Omega^{2,1}USp, \\ USp &\xrightarrow{\sim} \Omega^{2,1}KSp, \\ KSp &\xrightarrow{\sim} \Omega^{2,1}UQ, \\ UQ &\xrightarrow{\sim} \Omega^{2,1}KO. \end{aligned}$$

Example 2.5.3. The homotopy groups of $\mathbf{KQ}$ have the isomorphisms $\pi_{0,0}\mathbf{KQ} \cong \mathrm{GW}(F)$ and $\pi_{1,0}\mathbf{KQ} \cong K_1^h(F)$. In general, $\pi_{n,0}\mathbf{KQ} = \pi_n(K^h(X))$ for a scheme X, where $K^h(X)$ denotes the Hermitian K-theory space of definition 1.4.14. ♣

We conclude the section with a result describing Bott periodicity for Hermitian K-theory, a phenomenon of which the computation in chapter 4 is an example.

Proposition 2.5.4 ([Hor05, Section 5])**.** *Hermitian K-theory is $(8,4)$-periodic so that* $\pi_{p,q}\mathbf{KQ} = \pi_{p+8,q+4}\mathbf{KQ}$.

2.6 Milnor-Witt Motivic Cohomology

In this section we define Milnor-Witt motivic cohomology. We follow [Bac+22, Ch. 2]. Throughout this section, denote $\mathbb{G}_{m,1} \coloneqq (S^{1,1}, 1)$ as the Tate circle pointed by 1. We note that the definition is similar to motivic cohomology, but the complexes are more complex. We warn that this is a brief presentation.

Definition 2.6.1. Let $q \in \mathbb{Z}$. Define the *Zariski sheaf* $\widetilde{\mathbb{Z}}\{q\}$ to be

$$\widetilde{\mathbb{Z}}\{q\} \coloneqq \begin{cases} \widetilde{c}(\mathbb{G}_{m,1}^{\wedge q}) & \text{if } q > 0; \\ \widetilde{c}(F) & \text{if } q = 0; \\ \underline{\mathrm{Hom}}(\mathbb{G}_{m,1}^{\wedge q}, \widetilde{c}(F)) & \text{if } q < 0. \end{cases}$$

In the above definition, $\widetilde{c}(X)$ means the presheaf $\widetilde{\mathrm{Cor}}_F(-, X)$. To explain what this means, recall the correspondences defined in section 2.3. For schemes U and X, definition 2.2.1 coincides with the construction

$$\mathrm{Cor}_F(U, X) \coloneqq \operatorname*{colim}_{T \subseteq U \times X} H_T^{\dim X}(U \times X, \underline{K}^M_{\dim X}).$$

Here the T in the colimit is finite surjective on U, but not necessarily irreducible, $\underline{K}_*^{\mathrm{MW}}$ is the sheaf of Milnor K-groups, and we take cohomology compactly supported on T. Elements of $\mathrm{Cor}_F(U, X)$ are finite linear combinations of the form $\sum_i n_i Z_i$ where $Z_i \subseteq U \times X$ and $n_i \in \mathbb{Z} = K_0^M(F)$. The analogous concept in the current context is

$$\widetilde{\mathrm{Cor}}(U, X) \coloneqq \operatorname*{colim}_{T \subseteq U \times X} H_T^{\dim X}(U \times X, \underline{K}^{\mathrm{MW}}_{\dim X}, \omega)$$

where we take cohomology with coefficients in the sheaf Milnor-Witt K-groups $\underline{K}_*^{\mathrm{MW}}$. The ω denotes the *twist* and details can be found in [Bac+22, Ch. 1].

Definition 2.6.2. Let $q \in \mathbb{Z}$. Define $\widetilde{\mathbb{Z}}(q)$ to be the Suslin complex of Zariski sheaves of $K_0^{\mathrm{MW}}(F)$-modules $C_*(\widetilde{\mathbb{Z}}\{q\})[-q]$.

Now we arrive at the definition of the Milnor-Witt motivic cohomology groups:

Definition 2.6.3. The *Milnor-Witt motivic cohomology groups* of a smooth scheme X with integer coefficients is defined to be the hypercohomology groups

$$H^{p,q}_{\mathrm{MW}}(X,\mathbb{Z}) := \mathbb{H}^p_{\mathrm{Zar}}(X,\widetilde{\mathbb{Z}}(q)).$$

Remark 2.6.4. There is a spectrum $H\widetilde{\mathbb{Z}}$ representing Milnor-Witt motivic cohomology. The construcion is analgous to the construction of $H\mathbb{Z}$ in terms of EilenbergMaclane spaces built from $\widetilde{\mathbb{Z}}(q)$, and we refer to [Bac+22, Ch. 1] for the details.

We will need the following basic results about the cohomology theory in question:

Lemma 2.6.5 ([Bac17, Lemma 18])**.** *The homotopy group of $H\widetilde{\mathbb{Z}}$ are given by*

$$\pi_{p,q}H\widetilde{\mathbb{Z}} = \begin{cases} K^{\mathrm{MW}}_{-p}(F) & \textit{if } p = q; \\ \pi_{p,q}H\mathbb{Z} & \textit{otherwise.} \end{cases}$$

Chapter 3

Spectral Sequences

Spectral sequences are incredibly useful mathematical tools for calculation in several areas of mathematics. They were invented by Leray in 1945 while a prisoner of war, and quickly became a hit. As early as 1955, Massey famously wrote ‘It is now abundantly clear that the spectral sequence is one of the fundamental algebraic structures needed for dealing with topological problems’ [Mas55, p. 329]. Indeed, they are important to us, as we want to use the slice spectral sequence to compute Hermitian K-groups.

In this chapter, we introduce spectral sequences and convergence properties of them. Then we proceed to see an example of algebra spectral sequences, namely the Atiyah-Hirzebruch spectral sequence, before we move on to the spectral sequences most important to this thesis: The slice spectral sequences.

3.1 The Basics

To give the basic definitions needed to grasp the fundamentals of spectral sequences, we follow the guide of McCleary [McC01].

Definition 3.1.1 ([McC01, definition 2.1])**.** A *differential bigraded module* over a ring R, is a collection of R-modules, $\{E^{p,q}\}$, $p, q \in \mathbb{Z}$, together with an R-linear mapping $d\colon E^{p,q} \to E^{p+s,q+t}$ of *bidegree* (s,t), the *differential* satisfying $d \circ d = 0$ (for composable d).

A differential bigraded module usually has bidegree $(-r, r-1)$, in which case we say that it is of *homological type*, or bidegree $(r, 1-r)$, in which case we say that it is of *cohomological type*

Definition 3.1.2 ([McC01, definition 2.2])**.** A *spectral sequence* is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}$ called *pages*, where $r \geq 1$. If the differentials are all of bidegree $(-r, r-1)$, it is called a *homology type spectral sequence*, and if all differentials are of bidegree $(r, 1-r)$ it is called a *cohomology type spectral sequence.* For every p, q, r, we define $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r) = \ker d_r / \operatorname{Im} d_r$.

Note that all pages of a spectral sequence can be considered quotients (and thus submodules) of E_1. One can consider the d_r-boundaries $Z_r := \ker d_r$ and the d_r-cycles $B_r := \operatorname{Im} d_r$ of the E_r-page. These modules give the following tower of inclusions:

$$0 = B_0 \subset B_1 \subset \cdots \subset Z_1 \subset Z_0 = E_1.$$

We let $Z_\infty = \cap Z_r$ and $B_\infty = \cup B_r$, and define $E_\infty \coloneqq Z_\infty / B_\infty$. We call this page the *infinity-page of the spectral sequence*.

If there is some $n \in \mathbb{Z}$ such that $E_r = E_\infty$ whenever $r \geq n$, then we say that the spectral sequence *collapses on the nth page*. This means that $E_n = E_{n+1} = \cdots = E_\infty$. Note that this happens if and only if $d_r = 0$ for all $r \geq n$.

The spectral sequences we consider collapse, making computations way easier. We are not interested only in the collapsing, however. We care about the notion of *convergence*, which tells us something about whether we can extract useful information from the infinity-page of the spectral sequence.

3.1.1 Convergence of a Spectral Sequence

Definition 3.1.3. Let M be an R-module. A *filtration* F^* on M is a sequence of submodules $\{F^p M\}$ for $p \in \mathbb{Z}$ such that

$$\cdots \subset F^{p+1}M \subset F^p M \subset F^{p-1}M \subset \cdots M \text{ or}$$
$$\cdots \subset F^{p-1}M \subset F^p M \subset F^{p+1}M \subset \cdots M.$$

The filtrations are called *decreasing* or *increasing*, respectively.

A filtered module can be collapsed in the following way:

Definition 3.1.4. Let M be an R-module and F^* a filtration on M. Then the *associated graded module* of F^*, denoted $E_0^*(R)$, is given by

$$E_0^p = F^p M / F^{p+1} M$$

for decreasing filtrations and

$$E_0^p = F^p M / F^{p-1} M$$

for increasing filtrations.

Reconstructing some filtered module is often the last step of a computation done with spectral sequences. This often turns out to be difficult, and sometimes even impossible to uniquely determine without any extra information about the target of the spectral sequence. This can be illustrated by an example.

Example 3.1.5. We are given an R-module M with a decreasing filtration F^* which is *bounded*, that is, there exists an n such that $F^k M = 0$ for $k > n$ and an m such that $F^k M = M$ for $k < m$. Assume without loss of generality that it is bounded below by 0. Then the filtration looks like this:

$$\{0\} = F^{n+1}M \subset F^n M \subset F^{n-1}M \subset \cdots \subset F^1 M \subset F^0 M \subset F^{-1} M = M.$$

The associated graded module in degree p is $E_0^p(M) = F^p M / F^{p+1} M$, thus there are short exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & F^n M & \xrightarrow{=} & E_0^n(M) & \longrightarrow & 0 \\
0 & \longrightarrow & F^n M & \longrightarrow & F^{n-1} M & \longrightarrow & E_0^{n-1}(M) & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & F^1 M & \longrightarrow & F^0 M & \longrightarrow & E_0^0(M) & \longrightarrow & 0 \\
0 & \longrightarrow & F^0 M & \longrightarrow & F^{-1} M = M & \longrightarrow & E_0^{-1}(M) & \longrightarrow & 0.
\end{array}$$

Note that E_0^n determines F^n, while F^{n-1} is determined only up to some choice of extension of F^nM by $E_0^{n-1}(M)$. In general, F^kM is determined only up to some choice of extension of $F^{k+1}M$ by $E_0^k(M)$. ♣

Definition 3.1.6. Given a filtration F^* on a graded R-module H^*. For each degree n, define $F^pH^n := F^pH^* \cap H^n$. Then define the *associated bigraded module* for this filtration to be

$$E_0^{p,q}(H^*, F) := \begin{cases} F^pH^{p+q}/F^{p+1}H^{p+q} & \text{if } F^* \text{ is decreasing and} \\ F^pH^{p+q}/F^{p-1}H^{p+q} & \text{if } F^* \text{ is increasing.} \end{cases}$$

Most spectral sequence computations aim to determine the H^* of definition 3.1.6. This, however, as illustrated by example 3.1.5, can be difficult. To quote McCleary, 'If there is a spectral sequence converging to H^* *and* if it converges uniquely to H^* *and* if all of the extension problems can be settled, then H^* is determined (a lot of *ifs*)'. [McC01, p. 33]

To speak of different types of convergence, we need to be able to speak of different kinds of filtrations:

Definition 3.1.7 ([Boa99, Definition 2.1])**.** Given a decreasing filtration F^*G of the group G. The filtration is

- *exhaustive* if $G = \cup_s F^sG$;
- *Hausdorff* if $\cap_s F^sG = 0$;
- *complete* if every Cauchy sequence in G converges.

For these notions to make sense, topologize G by letting cosets $x + F^sG$ of F^sG play the role of basic open sets for arbitrary s and x. The following proposition is easily verifiable, and shown in [Boa99].

Proposition 3.1.8. *The topological space G is Hausdorff if and only if the filtration F^s is Hausdorff.*

By *Cauchy sequence* in definition 3.1.7, we mean a sequence $n \mapsto x_n$ for which $x_n - x_m \to 0$ when $m, n \to \infty$ in the topological sense of convergence. Next we introduce 3 different ways a spectral sequence can converge.

Definition 3.1.9 ([Boa99, p. 63])**.** Given a spectral sequence $r \mapsto (E_r, d_r)$, and a target group G with (decreasing) filtration F^*G, we say the spectral sequence:

1. *converges weakly* to G if the filtration exhausts G and we have isomorphisms $E_\infty^s \cong F^s/F^{s+1}$ and for all s;
2. *converges to* G if (1) holds and the filtration of G is Hausdorff;
3. *converges strongly to* G if (1) holds and the filtration of G is complete Hausdorff.

3.1.2 Algebra Spectral Sequences

Many algebraic structures are richer than having merely one operation. Singular cohomolgy, for example, has both addition of cochains and multiplication via the cup product. Many spectral sequences converge not only as a group or module, but as an algebra. We will see how this can help us both to compute the abutment of the spectral sequence, as well as computing the structure of the target.

Definition 3.1.10. Let (M, d) and (M', d') be differential bigraded modules over R. The *tensor product of differential graded modules* over R, written $(M \otimes_R M', d_\otimes)$ is defined to be

$$(M \otimes_R M')^{p,q} := \bigoplus_{\substack{s+u=p \\ r+t=q}} M^{r,s} \otimes_R (M')^{t,u}$$

where $d_\otimes(m \otimes m') = d(m) \otimes m' + (-1)^{r+s} m \otimes d'(m')$. The differential bigraded module (M, d) together with morphisms $\psi\colon (M \otimes M)^{*,*} \to M^{*,*}$ is a *differential bigraded algebra* over R if the product is associative, i.e., if the following commutes:

$$\begin{array}{ccc} M^* \otimes M^* \otimes M^* & \xrightarrow{\psi \otimes 1} & M^* \otimes M^* \\ \big\downarrow{\scriptstyle 1 \otimes \psi} & & \big\downarrow{\scriptstyle \psi} \\ M^* \otimes M^* & \xrightarrow{\quad \psi \quad} & M^* \end{array}$$

Definition 3.1.11. A spectral sequence $\{E_r, d_r\}$ is a *spectral sequence of algebras* if it has maps $\psi_r\colon E_r \otimes E_r \to E_r$ for every r and the induced map ψ_{r+1} can be written as the composite

$$\psi_{r+1}\colon E_{r+1} \otimes_R E_{r+1} \overset{\cong}{\to} H(E_r) \otimes H(E_r) \overset{p}{\to} H(E_r \otimes E_r) \xrightarrow{H(\psi_r)} H(E_r) \cong E_{r+1}$$

where $p([a] \otimes [b]) = [a \otimes b]$. Furthermore, $\{E_r\}$ is said to *converge as an algebra* if it converges as a spectral sequence in the ordinary sense, and if the algebra structure of $E_\infty^{*,*}$ is isomorphic to the induced algebra structure on the associated bigraded object.

3.2 The Atiyah-Hirzebruch Spectral Sequence

In this section, we present the Atiyah-Hirzebruch spectral sequence, and give an example on how it can be used to compute the complex K-theory of complex projective n-space, as well as its algebra structure. For the reader unfamiliar with complex K-theory, the book of May [May99, ch. 24] gives an introduction.

Theorem 3.2.1 ([McC01, Theorem 11.16])**.** *Given a spectrum E and a space X homotopic to some CW-complex, there are half-plane spectral sequences*

$$E_2^{p,q} \cong H^p(X; E_q(*)) \text{ and } E^2_{p,q} \cong H_p(X; E_q(*))$$

converging conditionally (in the sense of Boardman [Boa99]) to E^* and strongly to E_*, respectively.

Theorem 3.2.2 ([Dug14, p. 209])**.** *Let E be a spectrum with a procuct $E \wedge E \to E$. Then there is a pairing of Atiyah-Hirzebruch spectral sequences where the product on E_2-terms $H^p(X; E^q) \otimes H^{p'}(Y; E^{q'}) \to H^{p+p'}(X \times Y; E^{q+q'})$ is equal to $(-1)^{p'q}$ times the cup product.*

To demonstrate the power of the Atiyah-Hirzebruch spectral sequence, we give an example from Dugger's book:

Example 3.2.3 ([Dug14, Example 29.15])**.** In this example, we compute the complex K-theory of $\mathbb{C}P^n$ for any n. We begin by noting that the cohomology of $\mathbb{C}P^n$ is given by $H^i(\mathbb{C}P^n) \cong \mathbb{Z}$ if $i \leq 2n$ even, and $H^i(\mathbb{C}P^n) = 0$ otherwise. Thus we have the necessary ingredients to fill out the E_2-page, which looks like fig. 3.1. Note that this gives rise to a *half-plane spectral sequence*, and the only nonzero algebras on the E_2-page have degrees (p, q) where both p and q are even and positive. Since the bidegree $(r, 1-r)$ of the differential always has odd total, all differentials from nonzero modules land on a zero algebra. Thus, there are only trivial differentials on every page, and the spectral sequence collapses at $E_2 = E_\infty$. The infinity-page has the following generators:

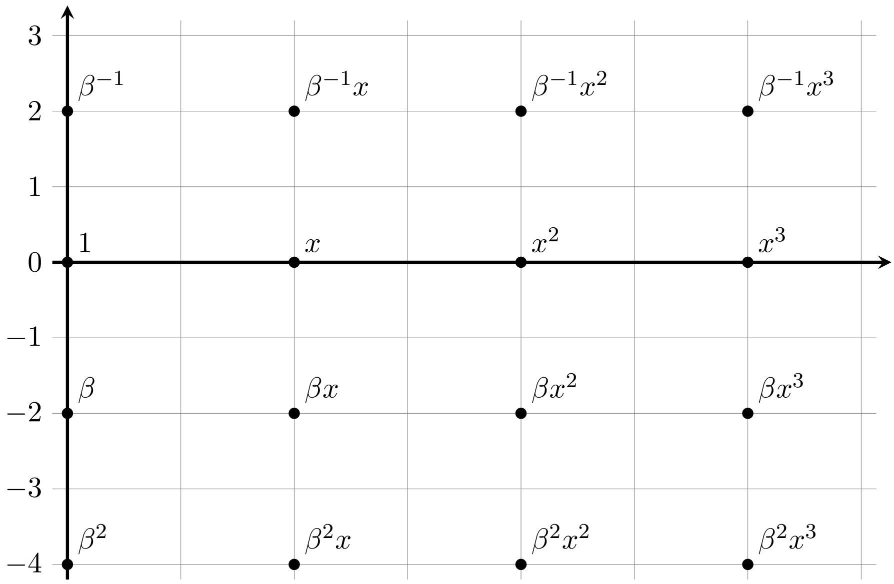


Figure 3.1: $E_2 = E_\infty$.

Note that the diagonals $p+q = 2k$ consist of something isomorphic to $\mathbb{Z}$ every other entry. As $\mathbb{Z}$ is free all extension problems are solvable, and we get

$$K^i(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z}^{n+1} & \text{for } i \text{ even;} \\ 0 & \text{otherwise.} \end{cases}$$

But the spectral sequence also gives information about the multiplicative structure, and we focus on $K^0(\mathbb{C}P^n)$. Notice that we have the decreasing filtration

$$0 = F^{2n+1} \subset F^{2n} = F^{2n-1} \subset F^{2n-2} = \cdots \subset F^1 = F^0 \subset K^0(\mathbb{C}P^n).$$

Now, there is a short exact sequence

$$0 \to F^3 \to F^2 \to E_\infty^{2,-2} \to 0,$$

and thus there is a canonical map $F^2/F^3 \xrightarrow{\cong} E_\infty^{2,-2} = \mathbb{Z}\langle x\beta \rangle$. Let $\alpha \in F^2/F^3$ denote the inverse of βx under the isomorphism. The multiplicativity of the spectral sequence implies that $\alpha^k \mapsto \beta^k x^k \in E_\infty^{2k,-2k}$. Furthermore, if $k < n+1$, then α^k is nonzero, but note that $\alpha^{n+1} \in E_\infty^{2n+2,-2n-2}$, which is just trivial.

To show that $\mathbb{Z}[\alpha]/(\alpha^{n+1}) \to K^0(\mathbb{C}P^n)$ is indeed an isomorphism, note that it is a map of filtered rings: The domain is filtered by powers of the ideal (α) and the codomain is filtered by the filtration F^*. This is an isomorphism in each degree, and there are only finitely many nonzero degrees. Hence, the map is indeed an isomorphism. ♣

3.3 The Construction of a Spectral Sequence

This section will briefly explain the basic construction of a spectral sequence as found in [Boa99, Section 0]. In this section, we take A^s and E^s to be graded abelian groups (i.e. we supress one of the gradings).

The method we will use to construct spectral sequences, is through *unrolled exact couples*. This is a commutative diagram of graded abelian groups and homomorphisms between them of the form

$$\begin{array}{ccccccccccc} \cdots \rightarrowtail & A^{s+2} & \xrightarrow{i} & A^{s+1} & \xrightarrow{i} & A^{s} & \xrightarrow{i} & A^{s-1} & \rightarrowtail \cdots \\ & \nwarrow^{k} & \swarrow_{j} & \nwarrow^{k} & \swarrow_{j} & \nwarrow^{k} & \swarrow_{j} & & \\ \cdots & & E^{s+1} & & E^{s} & & E^{s-1} & & \cdots \end{array} \tag{3.1}$$

in such a way that each triangle, i.e. $\cdots \to A^{s+1} \to A^s \to E^s \to A^{s+1} \to \cdots$, is a long exact sequence. We proceed to show how this can give rise to a spectral sequence.

Given an unrolled exact couple as in (3.1), let the E^s be the components of the E_1-page of the spectral sequence we construct. I.e., $E_1^s = E^s$. For notational ease, define for $s \in \mathbb{Z}$ and $r \geq 1$:

- $Z_r^s = k^{-1}(\operatorname{Im}[i^{(r-1)} \colon A^{s+r} \to A^{s+1}])$, to be the rth cycle subgroup of E_1^s,
- $B_r^s = j \ker[i^{(r-1)} \colon A^s \to A^{s-r+1}]$ to be the rth boundary of E_1^s,
- $E_r^s = Z_r^s / B_r^s$,
- $\operatorname{Im}^r A^s := \operatorname{Im}[i^{(r)} \colon A^{s+r} \to A^s]$,

where $i^{(n)}$ denotes the nth composition of i. Now there is a tower of subgroups of E^s:

$$0 = B_1^s \subset B_2^s \subset \cdots \subset \operatorname{Im} j = \ker k \subset \cdots \subset Z_2^s \subset Z_1^s = E^s.$$

Consider

$$\begin{array}{ccccccccc} A^{s+r+1} & \xrightarrow{i} & A^{s+r} & \xrightarrow{\quad i^{(r-1)} \quad} & A^{s+1} & \xrightarrow{i} & A^{s} \\ \nwarrow^{k} & & \swarrow_{j} & & \nwarrow^{k} & & \swarrow_{j} \\ & E^{s+r} & & & & E^{s} & \end{array}$$

and extract the short exact sequences

$$0 \longrightarrow \frac{Z_r^s}{\ker k} \xrightarrow{k} \operatorname{Im}^{r-1} A^{s+1} \xrightarrow{i} \operatorname{Im}^r A^s \longrightarrow 0 \tag{3.2}$$

and

$$0 \longrightarrow \operatorname{Im}^r A^{s+1} \hookrightarrow \operatorname{Im}^{r-1} A^{s+1} \longrightarrow \frac{\operatorname{Im} j}{B_r^{s+r}} \longrightarrow 0, \tag{3.3}$$

where the last arrow can be defined by lifting by $i^{(r-1)}$ and applying j. Note that $\operatorname{Im} j = \ker k$ to splice these short exact sequences together for various r and s, forming the $(r-1)$th *derived exact couple* of (3.1). It consists of long exact sequences

$$\cdots \to \operatorname{Im}^{(r-1)} A^{s-r+2} \xrightarrow{i} \operatorname{Im}^{r-1} A^{s-r+1} \to E_r^s \xrightarrow{k} \operatorname{Im}^{r-1} A^{s+1} \xrightarrow{i} \operatorname{Im}^{r-1} A^s \to \cdots .$$

Furthermore, we define the *differential* of this derived exact couple to be

$$d_r \colon E_r^s \xrightarrow{k} \operatorname{Im}^{r-1} A^{s+1} \longrightarrow E_r^{s+r}.$$

The degree of d_r is $\deg j + \deg k$ (at least when we assume $\deg i = 0$). Now we define the cycles of d_r to be Z_{r+1}^s / B_r^s, and the boundaries to be

$$\operatorname{Im}\left[d_r \colon E_r^s \to E_r^{s+r}\right] = \frac{B_{r+1}^{s+r}}{B_r^{s+r}} \cong \frac{Z_r^s}{Z_{r+1}^s} \cong \frac{\ker[\operatorname{Im}^{r-1} A^{s+1} \to A^s]}{\ker[\operatorname{Im}^{r} A^{s+1} \to A^s]},$$

and take homology to get the next page, i.e., $E_{r+1}^s \coloneqq Z_{r+1}^s / B_{r+1}^s$.

3.4 The Slice Spectral Sequence

There is a motivic analogue to the Atiyah-Hirzebruch spectral sequence, relating motivic cohomology and algebraic K-theory.

We follow Voevodsky's Nordfjordeid lectures [VRØ07, section 4]. Note that he writes $\Sigma_s = \Sigma^{1,0}$ for suspension with the simplicial circle sphere and $\Sigma_t = \Sigma^{1,1}$ for suspension with the Tate circle.

Definition 3.4.1. We define $\mathcal{SH}^{\text{eff}}(F)$ to be the smallest triangulated subcategory of $\mathcal{SH}(F)$ being closed under arbitrary direct sums, and having all $\Sigma^\infty_{\mathbb{P}^1} X_+$ for $X \in \mathrm{Sm}_X$.

Note that even though $\mathcal{SH}^{\text{eff}}(F)$ is closed under certain spectra, if $n \geq 1$, $\mathcal{SH}^{\text{eff}}(F)$ does not have the desuspension spectrum $\Sigma^{-n,-n}\Sigma^\infty_{\mathbb{P}^1} X_+$.

Definition 3.4.2. Consider the sequence of full embeddings of categories,

$$\cdots \hookrightarrow \Sigma^1_{\mathbb{P}^1}\mathcal{SH}^{\text{eff}}(F) \hookrightarrow \mathcal{SH}^{\text{eff}}(F) \hookrightarrow \Sigma^{-1}_{\mathbb{P}^1}\mathcal{SH}^{\text{eff}}(F) \hookrightarrow \cdots \hookrightarrow \mathcal{SH}(F) \tag{3.4}$$

which we call the *slice filtration*, or the *effective slice filtration*.

Remark 3.4.3. An equivalent notion to the slice filtration of (3.4) is the filtration

$$\cdots \hookrightarrow \Sigma^{0,1}\mathcal{SH}^{\text{eff}}(F) \hookrightarrow \mathcal{SH}^{\text{eff}}(F) \hookrightarrow \Sigma^{0,-1}\mathcal{SH}^{\text{eff}}(F) \hookrightarrow \cdots \hookrightarrow \mathcal{SH}(F).$$

This is a consequence of the fact that the category $\mathcal{SH}^{\text{eff}}(F)$ is triangulated, and thus closed under simplicial desuspension.

As a consequence of [Nee96, Theorem 4.1], the full inclusion functor $i_q \colon \Sigma^q_{\mathbb{P}^1}\mathcal{SH}^{\text{eff}}(F) \hookrightarrow \mathcal{SH}(F)$ has a right adjoint $r_q \colon \mathcal{SH}(F) \to \Sigma^q_{\mathbb{P}^1}\mathcal{SH}^{\text{eff}}(F)$ with the property that the unit of the adjunction $\eta \colon \mathrm{id} \to r_q \circ i_q$ is a natural isomorphism. Composing in the other direction, gives

$$\mathsf{f}_q \colon \mathcal{SH}(F) \xrightarrow{r_q} \Sigma^q_{\mathbb{P}^1}\mathcal{SH}^{\text{eff}}(F) \xrightarrow{i_q} \mathcal{SH}(F).$$

The counit of the adjunction $i_{q+1} \colon \Sigma^{q+1}_{\mathbb{P}^1}\mathcal{SH}^{\text{eff}}(F) \rightleftarrows \mathcal{SH}(F) \colon r_{q+1}$ is $\mathsf{f}_{q+1} \to \mathrm{id}$. Applying this to f_q determines a natural transformation

$$\mathsf{f}_{q+1} = \mathsf{f}_{q+1} \circ \mathsf{f}_q \to \mathsf{f}_q.$$

The image $\mathsf{f}_q E = \mathsf{f}_q(E)$ is often called the *q-effective cover* of the spectrum E.

Definition 3.4.4. Let E be a motivic spectrum and $q \in \mathbb{Z}$. The *slice tower of E* is the distinguished triangles

$$\mathsf{f}_{q+1}E \to \mathsf{f}_q E \to s_q E \to \Sigma^{1,0}\mathsf{f}_{q+1}E$$

where the functor $\mathsf{s}_q \colon \mathcal{SH}(F) \to \mathcal{SH}(F)$ is the *qth slice of E*.

Letting q vary in the above construction, consider the induced long exact sequence in homotopy groups, giving rise to an unrolled exact couple:

$$\begin{array}{ccccccc} \cdots \to \pi_\star f_{q+1}E & \longrightarrow & \pi_\star f_q E & \longrightarrow & \pi_\star f_{q-1}E \to \cdots \\ \nwarrow & \swarrow & \nwarrow & \swarrow & \\ & \pi_\star s_q E & & \pi_\star s_{q-1}E & \end{array} \tag{3.5}$$

which in turn gives rise to the *slice spectral sequence* [Voe02b]. In fact, there is a spectral sequence for any $n \in \mathbb{Z}$ with initial groups $\pi_{p,n}(\mathsf{s}_q(E))$. For the rth differential on the nth spectral sequence, the differentials go from $\pi_{p,n}(\mathsf{s}_q(E))$ to $\pi_{p-1,n}(\mathsf{s}_{q+r}(E))$. It is visualized like the Adams spectral sequence with p on the horizontal axis and q on the vertical. Note that this grading is different from that of e.g. the Atiyah-Hirzebruch spectral sequence seen in section 3.2.

The convergence properties for the slice spectral sequence are complicated, and in many circumstances, we do not have strong convergence. For our purposes, the following suffices:

Theorem 3.4.5 ([Voe02a, Section 5])**.** *The slice spectral sequence associated to* (3.5) *with $E = \mathbf{KGL}$ and $X \in \mathrm{Sm}_F$,*

$$E_2^{p,q} = H^{p-q,q}(X, \mathbb{Z}) \Longrightarrow K_{-p-q}(X),$$

is strongly convergent the algebraic K-theory of X.

Example 3.4.6. In [RSØ19], Röndigs, Spitzweck and Østvær use the slice spectral sequence to compute some instances of stable homotopy groups of the motivic sphere spectrum S. They prove that for fields F of charactereistic 0, there is a short-exact sequence of groups

$$0 \to K_2^{\mathrm{M}}(F)/24 \to \pi_{1,0}S \to F^\times/2 \oplus \mathbb{Z}/2 \to 0$$

and, for every $n \in \mathbb{Z}$, an exact sequence of Nisnevich sheaves on smooth schemes of finite type

$$0 \to \underline{K}^M_{2-n}/24 \to \pi_{n+1,n}S \to \pi_{n+1,n}\mathsf{f}_0(\mathbf{KQ}).$$

Rather surprisingly, these computations relate Milnor K-theory, Hermitan K-theory and the stable homotopy groups of the sphere spectrum. ♣

We conclude the section by mentioning what the slices of Algebraic and Hermitian K-theory are.

Theorem 3.4.7 ([RØ16, Theorem 4.1])**.** *There is an isomorphism* $\mathsf{s}_q(\mathbf{KGL}) \cong \Sigma^{2q,q}H\mathbb{Z}$ *for all $q \in \mathbb{Z}$.*

Theorem 3.4.8 ([RØ16, Theorem 4.18]). *The (effective) slices of the Hermitian K-theory spectrum* $\mathbf{KQ}$ *are given by*

$$\mathsf{s}_q(\mathbf{KQ}) = \begin{cases} \Sigma^{2q,q} H\mathbb{Z} \vee \bigvee_{i<\frac{q}{2}} \Sigma^{2i+q,q} H\mathbb{Z}/2 & q \equiv 0 \mod 2 \\ \bigvee_{i<\frac{q+1}{2}} \Sigma^{2i+q,q} H\mathbb{Z}/2 & q \equiv 1 \mod 2. \end{cases}$$

Needless to say, doing computations of Hermitian K-theory with the slice filtration might involve a lot of difficulty. The next section descirbes a filtration giving slices that are much nicer to work with - at least for our purposes.

3.5 The Very Effective Slice Spectral Sequence

Despite the success of the slice filtration defined in the previous section, Bachmann points out that 'there are some indications that this filtration is not quite right in certain situations' [Bac17, p. 2]. The most imoportant of these, might be that the slice filtration does not always converge. As a solution, Spitzweck and Østvær introduced the the *very effective slice filtration.*

Let F be a field.

Definition 3.5.1 ([SØ12, Definition 5.5]). The *very effective motivic stable homotopy category* $\mathcal{SH}^{\mathrm{veff}}(F)$ is the smallest subcategory of $\mathcal{SH}(F)$ containing all suspension spectra of shooth schemes of finite type, and is closed under extensions and homotopy colimits. It is also known as the category of *very effective spectra.*

In the above definition, being *closed under extensions* is taken to mean that for cofiber sequences $X \to Y \to Z$ where X and Z belong to $\mathcal{SH}^{\mathrm{veff}}(F)$, then Y also belongs to $\mathcal{SH}^{\mathrm{veff}}(F)$.

Remark 3.5.2. The category $\mathcal{SH}^{\mathrm{veff}}(F)$ is contained in $\mathcal{SH}^{\mathrm{eff}}(F)$, but it is not triangulated as it is not closed under simplicial desuspension.

The *very effective slice filtration* looks like the following

$$\cdots \subset \Sigma^{2q+2,q+1}\mathcal{SH}^{\mathrm{eff}}(F) \subset \Sigma^{2q,q}\mathcal{SH}^{\mathrm{eff}}(F) \subset \Sigma^{2q-2,q-1}\mathcal{SH}^{\mathrm{eff}}(F) \subset \cdots .$$

Again, using [Nee96, Theorem 4.1], obtain the full inclusion functor $\widetilde{i}_q \colon \Sigma^q_{\mathbb{P}^1}\mathcal{SH}(F) \hookrightarrow \mathcal{SH}(F)$ with the right adjoint $\widetilde{r}_q \colon \mathcal{SH}(F) \to \Sigma^q_{\mathbb{P}^1}\mathcal{SH}(F)$, with unit the natural isomorphism $\widetilde{\eta} \colon \mathrm{id} \to \widetilde{r}_q \circ \widetilde{i}_q$. Composing the other direction gives

$$\widetilde{\mathsf{f}}_q \colon \mathcal{SH}(F) \xrightarrow{\widetilde{r}_q} \Sigma^q_{\mathbb{P}^1}\mathcal{SH}^{\mathrm{eff}}(F) \xrightarrow{\widetilde{i}_q} \mathcal{SH}(F).$$

The counit of the adjunction

$$\widetilde{i}_{q+1} \colon \Sigma^{q+1}_{\mathbb{P}^1}\mathcal{SH}^{\mathrm{eff}}(F) \rightleftarrows \mathcal{SH}(F) \colon \widetilde{r}_{q+1}$$

is given by $\widetilde{\mathsf{f}}_{q+1} \to \mathrm{id}$. We now obtain the natural transformation

$$\widetilde{\mathsf{f}}_{q+1} = \widetilde{\mathsf{f}}_{q+1} \circ \widetilde{\mathsf{f}}_q \to \widetilde{\mathsf{f}}_q$$

where $\widetilde{\mathsf{f}}_q E$ often is called the *very q-effective cover* of the spectrum E of $\mathcal{SH}(F)^{\mathrm{veff}}$. The cofibers

$$\widetilde{\mathsf{f}}_{q+1}E \longrightarrow \widetilde{\mathsf{f}}_q E \longrightarrow \widetilde{\mathsf{s}}_q E \longrightarrow \Sigma^{1,0}\widetilde{\mathsf{f}}_{q+1}E$$

determine the *very effective slices* $\widetilde{\mathsf{s}}_q$.

Lemma 3.5.3 ([SØ12, Lemma 5.10]). *For a spectrum $E \in \mathcal{SH}^{\text{veff}}(F)$ and F a perfect field, we have $\pi_{m,n}E = 0$ for $m < n$.*

Example 3.5.4. Consider the spectrum **KGL** as defined in section 2.4. Then the very effective slices coincide with the effective slices: $\widetilde{\mathsf{s}}_q\mathbf{KGL} = \mathsf{s}_q\mathbf{KGL}$. ♣

Due to lemma 3.5.3, the spectral sequence associated to the very effective slice filtration is always strongly convergent.

Theorem 3.5.5. *For a spectrum E in $\mathcal{SH}^{\text{veff}}(E)$ and an $n \in \mathbb{Z}$, there is a spectral sequence*

$$E^1_{p,q,n} = \pi_{p,n}(\widetilde{\mathsf{s}}_q(E)) \implies \pi_{p,n}(E)$$

converging strongly to the homotopy groups of E.

3.6 The Very Effective Slices of Hermitian K-Theory

One of the challenges associated to the very effective slice filtration is the difficulty of identifying the very effective slices. As proven by Bachmann, the very effective slices of Hermitian K-theory can be expressed in terms of motivic cohomology and Milnor-Witt motivic cohomology [Bac17]:

Theorem 3.6.1 ([Bac17, Theorem 16]). *The very effective slices of Hermitian K-theory are given by*

$$\widetilde{\mathsf{s}}_n\mathbf{KQ} \simeq S^{2q,q} \wedge \begin{cases} \widetilde{\mathsf{s}}_0\mathbf{KQ} & n \equiv 0 \mod 4 \\ H\mathbb{Z}/2 & n \equiv 1 \mod 4 \\ H\mathbb{Z} & n \equiv 2 \mod 4 \\ 0 & n \equiv 3 \mod 4 \end{cases}$$

and there is a cofiber sequence

$$S^{1,0} \wedge H\mathbb{Z}/2 \longrightarrow \widetilde{\mathsf{s}}_0\mathbf{KQ} \longrightarrow H\widetilde{\mathbb{Z}}.$$

Recall that $H\widetilde{\mathbb{Z}}$ denotes the Milnor-Witt motivic cohomology spectrum (remark 2.6.4). For finite fields, this is easily computable. Using the induced long exact sequence in homotopy groups, the zeroth very effective slice is computable as an extension of $\pi_\star S^{1,0} \wedge H\mathbb{Z}/2$ by $\pi_\star H\widetilde{\mathbb{Z}}$. We now have all the necessary ingredients to compute the Hermitian K-groups of finite fields.

Chapter 4

The Hermitian K-Groups of $\mathbb{F}_q$

In this chapter, we use the very effective slice spectral sequence of section 3.5 to compute the weight zero homotopy groups of **KQ** over a finite base field. The entails computing the E^1-page, using the tools of section 3.6, and then determining the abutment of the spectral sequence. The computation concludes when the extension problems are resolved. The chapter concludes by discussing some further topics of research.

4.1 The Computation

Before we compute the E^1-page, we make note some notational conventions:

Notation 4.1.1. In the following proof, we denote by $H^\star$ the motivic cohomolgy groups of $\operatorname{Spec}\mathbb{F}_q$ with integer coefficients, and by $h^\star$ the motivic cohomology groups of $\operatorname{Spec}\mathbb{F}_q$ with $\mathbb{Z}/2$-coefficients. Moreover, we denote morphism classes $\operatorname{Hom}(X, Y)$ in $\mathcal{SH}(F)$ by $[X, Y]$.

Let $q > 0$ be odd. We are finally ready to prove the main theorem of this thesis:

Theorem 4.1.2. *The spectral sequence*

$$E^1_{s,t,0}(\mathbf{KQ}) = \pi_{s,0}(\tilde{\mathsf{s}}_t\mathbf{KQ}) \Longrightarrow \pi_{s,0}\mathbf{KQ}$$

converges strongly to the (unreduced) Hermitian K-groups of $\mathbb{F}_q$. Furthermore, all nonzero entries of $E^1_{\star,0}$ are given by fig. 4.1, except for $E^1_{0,0,0} \cong \mathrm{GW}(F_q)$, corresponding to the unreduced 0th Hermitian K-group of $\mathbb{F}_q$.

t \ s	$8m$	$8m+1$	$8m+2$	$8m+3$	$8m+4$	$8m+5$	$8m+6$	$8m+7$
$4m+4$								$H^{1,(s+1)/2}$
$4m+3$								
$4m+2$				$H^{1,(s+1)/2}$				
$4m+1$		$h^{1,4m}$	$h^{0,4m}$					
$4m$	$h^{1,4m}$	$h^{0,4m}$						

Figure 4.1: $E^1_{\star,0}(\mathbf{KQ})$.

Before proving Theorem 4.1.2 we recall some results which we make heavy use of. By proposition 1.2.4, Milnor K-theory is concentrated in degrees zero and one:

$$K_n^M(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0; \\ \mathbb{F}_q^\times & \text{if } n = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 1.3.10 gives a full description of the Milnor-Witt K-theory of finite fields:

$$K_n^{\mathrm{MW}}(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z}/2[\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2] & \text{if } n < 0 \text{ and } q \equiv 1 \mod 4; \\ \mathbb{Z}/4 & \text{if } n < 0 \text{ and } q \equiv 3 \mod 4; \\ \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } n = 0; \\ \mathbb{F}_q^\times & \text{if } n = 1; \\ 0 & \text{otherwise.} \end{cases}$$

From proposition 2.2.8, motivic cohomology of $\operatorname{Spec} F_q$ is given by

$$H^{m,n} := H^{m,n}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & m = n = 0; \\ \mathbb{Z}/q^n - 1 & m = 1,\, n > 0; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$h^{m,n} := H^{m,n}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & m = 0,\, n \geq 0 \text{ or } m = 1, n > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Finally, lemma 2.6.5 says that Milnor-Witt motivic cohomology is given by

$$\pi_{p,q} H\widetilde{\mathbb{Z}} = \begin{cases} K_{-p}^{\mathrm{MW}}(\mathbb{F}_q) & \text{if } p = q; \\ \pi_{p,q} H\mathbb{Z} & \text{otherwise.} \end{cases}$$

Now we give the proof of Theorem 4.1.2:

Proof. To compute $E^1_{s,t,0} = \pi_{s,0}(\widetilde{\mathsf{s}}_t \mathbf{KQ})$, we write s and t as $s = 8m + p$ and $t = 4n + r$ for integers m, n, p, r, where $0 \leq p \leq 7$ and $0 \leq r \leq 3$, and consider different cases for these.

To ease the computation, we begin by finding some general expressions that will be useful. We start by computing a general expression for the long exact sequence induced by the cofiber sequence

$$S^{1,0} \wedge H\mathbb{Z}/2 \to \widetilde{\mathsf{s}}_0 \mathbf{KQ} \to H\widetilde{\mathbb{Z}}. \tag{4.1}$$

Whenever $r = 0$, we want to compute the group

$$\begin{aligned} E^1_{s,t,0} = \pi_{8m+p,0} \widetilde{\mathsf{s}}_{4n} \mathbf{KQ} &= \pi_{8m+p,0} S^{8n,4n} \wedge \widetilde{\mathsf{s}}_0 \mathbf{KQ} \\ &= \left[S^{8m+p,0}, S^{8n,4n} \wedge \widetilde{\mathsf{s}}_0 \mathbf{KQ}\right] \\ &= \left[S^{8(m-n)+p,-4n}, \widetilde{\mathsf{s}}_0 \mathbf{KQ}\right] \\ &= \pi_{8(m-n)+p,-4n} \widetilde{\mathsf{s}}_0 \mathbf{KQ}. \end{aligned}$$

We can compute this by inserting it into the long exact sequence induced by (4.1). The following part will be relevant to this computation:

$$\begin{aligned} \cdots \to \pi_{8(m-n)+(p+1),-4n} H\widetilde{\mathbb{Z}} \to \pi_{8(m-n)+p,-4n} S^{1,0} \wedge H\mathbb{Z}/2 \to \pi_{8(m-n)+p,-4n} \widetilde{\mathsf{s}}_0 \mathbf{KQ} \\ \to \pi_{8(m-n)+p,-4n} H\widetilde{\mathbb{Z}} \to \pi_{8(m-n)+(p-1),-4n} S^{1,0} \wedge H\mathbb{Z}/2 \to \cdots . \end{aligned} \tag{4.2}$$

Furthermore, for integers p, we have the formulas

$$\pi_{8m+p,0}\widetilde{\mathsf{s}}_{4n+1}\mathbf{KQ} = \pi_{8m+p,0}S^{8n+2,4n+1} \wedge H\mathbb{Z}/2 = h^{8(n-m)+(2-p),4n+1} \tag{4.3}$$

$$\pi_{8m+p,0}\widetilde{\mathsf{s}}_{4n+2}\mathbf{KQ} = \pi_{8m+p,0}S^{8n+4,4n+2} \wedge H\mathbb{Z} \quad = H^{8(n-m)+(4-p),4n+2} \tag{4.4}$$

$$\pi_{8m+p,0}\widetilde{\mathsf{s}}_{4n+3}\mathbf{KQ} = \pi_{8m+p,0}S^{8n+6,4n+3} \wedge 0 \quad = 0. \tag{4.5}$$

We know of Bott periodicity of the Hermitian K-theory spectrum by Proposition 2.5.4. Thus, we divide the computation of fig. 4.1 into eight simple steps. Assume s and t are as stated in the beginning of this proof.

By (4.5), the cases $t = 4n + 3$ will be left out.

Case: $s = 8m$. We begin by considering the case $t = 4n$. In order to compute $\pi_{8m,0}\widetilde{\mathsf{s}}_{4n}\mathbf{KQ} = \pi_{8(m-n),-4n}\widetilde{\mathsf{s}}_0\mathbf{KQ}$, we use the long exact sequence given by (4.2). We compute the following:

- $\pi_{8(m-n)+1,-4n}H\widetilde{\mathbb{Z}} = \pi_{8(m-n)+1,-4n}H\mathbb{Z} = H^{8(n-m)-1,4n} = 0$ for all m, n.

- $\pi_{8(m-n),-4n}S^{1,0} \wedge H\mathbb{Z}/2 = h^{1-8(m-n)-1,4n} = \begin{cases} \mathbb{Z}/2 & \text{for } m = n > 0; \\ 0 & \text{otherwise.} \end{cases}$

- In order to compute $\pi_{8(m-n),-4n}H\widetilde{\mathbb{Z}}$, observe that when $n = 2m$, this equals $K^{\mathrm{MW}}_{-8(m-n)}(\mathbb{F}_q)$, which is zero in every even nonzero degree. Notice for $m = n = 0$, we have $K^{\mathrm{MW}}_{-8(m-n)}(\mathbb{F}_q) = K^{\mathrm{MW}}_0(\mathbb{F}_q) \cong \mathrm{GW}(\mathbb{F}_q) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. If $n \neq 2m$, we have $\pi_{8(m-n),-4n}H\widetilde{\mathbb{Z}} = H^{8(n-m),4n} = 0$. Hence

$$\pi_{8(m-n)+1,-4n}H\widetilde{\mathbb{Z}} \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & \text{for } n = 2m = 0; \\ 0 & \text{otherwise.} \end{cases}$$

- Finally, $\pi_{8(m-n)-1,-4n}S^{1,0} \wedge H\mathbb{Z}/2 = h^{2-8(m-n),4n} = 0$ for all m, n.

Inserting all this (4.2), we obtain isomorphisms

$$\pi_{8m,0}\widetilde{\mathsf{s}}_{4n}\mathbf{KQ} \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & m = n = 0; \\ \mathbb{Z}/2 & m = n > 0; \\ 0 & \text{otherwise;} \end{cases}$$

Case: $s = 8m, t = 4n + 1$. By (4.3), $\pi_{8m,0}\widetilde{\mathsf{s}}_{4n+1}\mathbf{KQ} = h^{8(n-m)+2,4n+1} = 0$ for all m, n.

Case: $s = 8m, t = 4n + 2$. By (4.4), $\pi_{8m,0}\widetilde{\mathsf{s}}_{4n+2}\mathbf{KQ} = H^{8(n-m)+4,4n+2} = 0$ for all m, n.

Case: $s = 8m + 1, t = 4n$. We compute $\pi_{8m+1,0}S^{4n} \wedge \widetilde{\mathsf{s}}_0\mathbf{KQ} = \pi_{8(m-n)+1,-4n}\widetilde{\mathsf{s}}_0\mathbf{KQ}$. Using the (4.2), we observe the following:

- $\pi_{8(m-n)+2,-4n}H\widetilde{\mathbb{Z}} = H^{8(n-m)-2,4n} = 0$ for all m, n.

- $\pi_{8(m-n)+1,-4n}H\widetilde{\mathbb{Z}} = H^{8(n-m)-1,4n} = 0$ for all m, n.

- $\pi_{8(m-n)+1,-4n}S^{1,0} \wedge H\mathbb{Z}/2 = h^{8(n-m),4n} \cong \begin{cases} \mathbb{Z}/2 & \text{for } m = n \geq 0; \\ 0 & \text{otherwise.} \end{cases}$

Inserting into the (4.2), the above computations show that there is an isomorphism

$$\pi_{8(m-n)+1,-4n}\widetilde{\mathsf{s}}_0\mathbf{KQ} \cong h^{8(n-m),4n} \cong \begin{cases} \mathbb{Z}/2 & \text{for } m = n \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Case: $s = 8m+1, t = 4n+1$. Then

$$\begin{aligned} \pi_{8m+1,0}\widetilde{\mathsf{s}}_{4n+2}\mathbf{KQ} &= \pi_{8m+1,0}S^{8n+2,4n+1} \wedge H\mathbb{Z}/2 \\ &= h^{8(n-m)+1,4n+1} \\ &\cong \begin{cases} \mathbb{Z}/2 & \text{for } m = n \geq 0; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Case: $s = 8m+1, t = 4n+1$. Then we have

$$\pi_{8m+1,0}\widetilde{\mathsf{s}}_{4n+1}\mathbf{KQ} = h^{8(n-m)+1,4n+1} = \begin{cases} \mathbb{Z}/2 & \text{for } m = n \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Case: $s = 8m+1, t = 4n+2$. This gives $\pi_{8m+1,0}\widetilde{\mathsf{s}}_{4n+2}\mathbf{KQ} = H^{8(n-m)+3,4n+2} = 0$ for all m, n.

Case: $s = 8m+2, t = 4n$. To compute

$$\pi_{8m+2,0}\widetilde{\mathsf{s}}_{4n}\mathbf{KQ} = \pi_{8(m-n)+2,-4n}S^{8n,4n} \wedge \widetilde{\mathsf{s}}_0\mathbf{KQ},$$

we insert the expression into its (4.2). We compute

- $\pi_{8(m-n)+2,-4n}S^{1,0}H\mathbb{Z}/2 = h^{8(n-m)-1,4n} = 0$ for all m, n.
- $\pi_{8(m-n)+2,-4n}H\widetilde{\mathbb{Z}} = 0$ for all m, n.

By this squeeze, $\pi_{8m+2,0}\widetilde{\mathsf{s}}_{4n}\mathbf{KQ} = 0$.

Case: $s = 8m+2, t = 4n+1$. Then

$$\pi_{8m+2,0}\widetilde{\mathsf{s}}_{4n}\mathbf{KQ} = h^{8(n-m),4n+1} \cong \begin{cases} \mathbb{Z}/2 & \text{for } m = n \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Case: $s = 8m+2, t = 4n+2$. Then $\pi_{8m+2,0}\widetilde{\mathsf{s}}_{4n+2}\mathbf{KQ} = H^{8n-m+2,4n+2} = 0$ for all m, n.

Case: $s = 8m+3, t = 4n$. Then $\pi_{8m+3,0}\widetilde{\mathsf{s}}_{4n}\mathbf{KQ} = \pi_{8(m-n)+3,-4n}\widetilde{\mathsf{s}}_0\mathbf{KQ}$. Consider (4.2) and compute

- $\pi_{8(m-n)+3,-4n}S^{1,0} \wedge H\mathbb{Z}/2 = h^{8(n-m)-2,4n} = 0$ for all m, n.
- $\pi_{8(m-n)+3,-4n}H\widetilde{\mathbb{Z}} = H^{8(n-m)-3,4n} = 0$ for all m, n.

This squeeze ensures that $\pi_{8m+3,0}\widetilde{\mathsf{s}}_{4n}\mathbf{KQ} = 0$ for all m, n.

Case: $s = 8m+3, t = 4n+1$. $\pi_{8m+3,0}\widetilde{\mathsf{s}}_{4n+1}\mathbf{KQ} = h^{8(n-m)-1,4n+1} = 0$ for all m, n.

Case: $s = 8m+3, t = 4n+2$. We compute

$$\pi_{8m+3,0}\widetilde{\mathsf{s}}_{4n+2}\mathbf{KQ} = H^{8(n-m)+1,4n+2} = \begin{cases} H^{1,4n+2} = H^{1,(s+1)/2} = \frac{\mathbb{Z}}{q^{(s+1)/2}-1} & \text{for } m = n; \\ 0 & \text{otherwise;} \end{cases}$$

Case: $s = 8m+4, t = 4n$. We compute

- $\pi_{8(m-n)+4,-4n}S^{1,0} \wedge H\mathbb{Z}/2 = h^{8(n-m)-3,4n} = 0$ for all m, n.

- $\pi_{8(m-n)+4,-4n}H\widetilde{\mathbb{Z}} = \begin{cases} K^{\mathrm{MW}}_{4n}(\mathbb{F}_q) & \text{for } 8(m-n)+4 = -4n; \\ H^{8(n-m)-4,4n} & \text{otherwise.} \end{cases}$

  Observe that $8(m-n)+4 = -4n \iff n = 2m+1$. But $K^{\mathrm{MW}}_k(\mathbb{F}_q) = 0$ for all odd multiples of 4. Thus, $\pi_{8(m-n)+4,-4n}H\widetilde{\mathbb{Z}} = 0$ for all m, n.

By this squeeze, $\pi_{8m+4,0}\widetilde{\mathsf{s}}_{4n}\mathbf{KQ} = 0$ for all m, n.

Case: $s = 8m+4, t = 4n+1$. Then $\pi_{8m+4,0}\widetilde{\mathsf{s}}_{4n+1}\mathbf{KQ} = h^{8(n-m)-2,4n+1} = 0$ for all m, n.

Case: $s = 8m+4, t = 4n+2$. Then $\pi_{8m+4,0}\widetilde{\mathsf{s}}_{4n+2}\mathbf{KQ} = H^{8(n-m),4n+2} = 0$ for all m, n.

Case: $s = 8m+5, t = 4n$. Then for $\pi_{8m+5,0}\widetilde{\mathsf{s}}_{4n}\mathbf{KQ}$, we compute the following parts of (4.2):

- $\pi_{8(m-n)+5,-4n}S^{1,0} \wedge H\mathbb{Z}/2 = h^{8(n-m)-4,4n} = 0$ for all m, n.

- $\pi_{8(m-n)+5,-4n}H\widetilde{\mathbb{Z}} = \pi_{8(m-n)+5,-4n}H\mathbb{Z} = H^{8(n-m)-5,4n} = 0$ for all m, n.

Thus, $\pi_{8m+5,0}\widetilde{\mathsf{s}}_{4n}\mathbf{KQ} = 0$ for all m, n.

Case: $s = 8m+5, t = 4n+1$. $\pi_{8n+5,0}\widetilde{\mathsf{s}}_{4n+1}\mathbf{KQ} = h^{8(n-m)-4,4n+1} = 0$ for all m, n.

Case: $s = 8m+5, t = 4n+2$. $\pi_{8n+5,0}\widetilde{\mathsf{s}}_{4n+2}\mathbf{KQ} = H^{8(n-m)-1,4n+2} = 0$ for all m, n.

Case: $s = 8m+6, t = 4n$. To compute $\pi_{8m+6,0}\widetilde{\mathsf{s}}_{4n}\mathbf{KQ} = \pi_{8m+6,0}S^{8(m-n)+6,4n}$, compute the following terms of (4.2):

- $\pi_{8(m-n)+6,-4n}S^{1,0} \wedge H\mathbb{Z}/2 = h^{8(n-m)-5,4n} = 0$ for all m, n.

- $\pi_{8(m-n)+6,-4n}H\widetilde{\mathbb{Z}} = \pi_{8(m-n)+6,-4n}H\mathbb{Z} = H^{8(n-m)-6,4n} = 0$ for all m, n.

By this squeeze, we get $\pi_{8n+6,0}\widetilde{\mathsf{s}}_{4n}\mathbf{KQ} = 0$ for all m, n.

Case: $s = 8m+6, t = 4n+1$. Then $\pi_{8n+6,0}\widetilde{\mathsf{s}}_{4n+1}\mathbf{KQ} = h^{8(n-m)-4,4n} = 0$ for all m, n.

Case: $s = 8m+6, t = 4n+2$. Then $\pi_{8n+6,0}\widetilde{\mathsf{s}}_{4n+2}\mathbf{KQ} = H^{8(n-m)-2,4n} = 0$ for all m, n.

Case: $s = 8m+7, t = 4n$. Compute

- $\pi_{8(m-n)+7,-4n}S^{1,0} \wedge H\mathbb{Z}/2 = h^{8(n-m)-6,4n} = 0$ for all m, n.

- $\pi_{8(m-n)+6,-4n}S^{1,0} \wedge H\mathbb{Z}/2 = 0$ from the previous case.

Inserting this into (4.2), we get that

$$\begin{aligned} \pi_{8n+7,0}\widetilde{\mathsf{s}}_{4n}\mathbf{KQ} &\cong \pi_{8(m-n)+7,-4n}H\widetilde{\mathbb{Z}} = \pi_{8(m-n)+7,-4n}H\mathbb{Z} \\ &\cong \begin{cases} H^{1,4n} = \frac{\mathbb{Z}}{q^{(s+1)/2}} & \text{for } n = m+1; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Case: $s = 8m+7, t = 4n+1$. Then $\pi_{8m+7,0}\widetilde{\mathsf{s}}_{4n+1}\mathbf{KQ} = h^{8(n-m)-5,4n+1} = 0$ for all m, n.

Case: $s = 8m+7, t = 4n+2$. Then $\pi_{8m+7,0}\widetilde{\mathsf{s}}_{4n+2}\mathbf{KQ} = H^{8(n-m)-3,4n+2} = 0$ for all m, n. This, being the last possible case, concludes the proof and outputs the E^1-page with a periodicity of 8. We summarize the computations in the following chart. Given $s = 8m + p$ and $t = 4n + r$,

$$E^1_{s,t,0} = \begin{cases} \mathrm{GW}(\mathbb{F}_q) & s \equiv 0 \pmod 8 \quad m = n = 0, \quad p = r = 0; \\ h^{1,4n} & s \equiv 0 \pmod 8 \quad m = n > 0, \quad p = r = 0; \\ h^{0,4n} & s \equiv 1 \pmod 8 \quad m = n \geq 0, \quad p = 1, r = 0; \\ h^{1,4n+1} & s \equiv 1 \pmod 8 \quad m = n \geq 0, \quad p = r = 1; \\ h^{0,4n+1} & s \equiv 2 \pmod 8 \quad m = n \geq 0, \quad p = 2, r = 1; \\ H^{1,4n+2} & s \equiv 3 \pmod 8 \quad m = n, \quad p = 3, r = 2; \\ H^{1,4n+2} & s \equiv 7 \pmod 8 \quad m = n - 1, \quad p = 7, r = 0; \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

□

It is visually evident that there are no differentials on any possible page of the spectral sequence: No nonzero entry can be hit by a nontrivial differential from the column on its right hand side. Thus the spectral sequence collapses immediately, and the results can can be read off.

All the h's are isomorphic to $\mathbb{Z}/2$. When $m = n = 0$, we have $E^\infty_{0,0,0} \cong \mathrm{GW}(\mathbb{F}_q) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. Note that taking the reduced Hermitian K-theory eliminates this extraneous $\mathbb{Z}$.

By proposition 2.2.8, the group $H^{1,(s+1)/2}$ is isomorphic to $\mathbb{Z}/q^{(s+1)/2} - 1$, where s denotes its horizontal position on the E^1-page.

All columns contain one group, except for columns with position $s \equiv 1 \mod 8$, which have two. Thus, only one extension problem arises. It is dealt with in the following lemma.

Lemma 4.1.3. *Let $s \equiv 1 \mod 8$. Then the short exact sequence*

$$0 \to \mathbb{Z}/2 \to \mathbf{KQ}_s \to \mathbb{Z}/2 \to 0$$

is split. Hence $\mathbf{KQ}_s = (\mathbb{Z}/2)^2$.

Proof. If one can prove the claim for $s = 1$, all other cases follow from Bott periodicity (proposition 2.5.4). To settle that case, it suffices to prove the existence of a map $\mathbf{KQ}_1 \to \mathbb{Z}/2$ which is split surjective. Indeed, there are only two possible extensions of $\mathbb{Z}/2$ by $\mathbb{Z}/2$: $\mathbf{KQ}_1 \cong \mathbb{Z}/4$ or $\mathbf{KQ}_1 \cong (\mathbb{Z}/2)^2$. It follows that any surjective map $\mathbf{KQ}_1 \to \mathbb{Z}/2$ has kernel of order 2 which is isomorphic to $\mathbb{Z}/2$. Recall lemma 1.4.8, proving the existence of an induced split surjective determinant map from $\mathbf{KQ}_1$ onto the multiplicative group $\{\pm 1\}$, having a retraction $r\colon \{\pm 1\} \to \mathbf{KQ}_1$. All these facts suffice to prove that following commutes and has exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathbf{KQ}_1 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\ & & \downarrow\cong & & \| & & \downarrow\cong & & \\ 0 & \longrightarrow & \ker\det & \longrightarrow & \mathbf{KQ}_1 & \xrightarrow{\det} & \{\pm 1\} & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & \overset{r}{\dashleftarrow} & \downarrow\cong & & \\ 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & (\mathbb{Z}/2)^2 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \end{array}$$

finishing the proof. □

At last, observe that our results agree with those of Friedlander:

Corollary 4.1.4 ([Fri76, p. 90])**.** *Hermitian K-theory of finite fields of order q is given by*

s *modulo* 8	0	1	2	3	4	5	6	7
$\widetilde{\mathbf{KQ}}_s(\mathbb{F}_q)$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$\frac{\mathbb{Z}}{q^{(s+1)/2}-1}$	0	0	0	$\frac{\mathbb{Z}}{q^{(s+1)/2}-1}$

4.2 Further Topics of Research

The very effective slice spectral sequence has yet to see abundant usage. In our case, the E^1-page was indeed very *kind*, and there is hope that Hermitian K-theory of other smooth schemes can be calculated with similar difficulty – or maybe *lack of* difficulty. Further computations can be made for instance with

- projective spaces like $\mathbb{P}^1$, $\mathbb{P}^n$, smooth projective curves or other classic varieties;
- classifying spaces, or lens spaces.

Another challenge is to compute the very effective slices of the sphere spectrum; these are still unknown. A discovery of that kind could improve chances of computing stable homotopy groups of the motivic spheres.

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