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# Somewhat Tautological Bundles and their Degeneracy Loci

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# Abstract

We define a family of coherent sheaves indexed by the natural numbers on the Hilbert scheme of points on a surface S, and study some of their properties. The first two sheaves are bundles whose degeneracy loci parametrize subsets of singular loci of curves on S, that are members of general linear systems of appropriate dimension.

We verify that one degeneracy locus is of expected dimension and compute the total Chern class of the first bundle, as well as relate the Chern classes of the second bundle to the Chern classes of two tautological bundles.

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# Contents

Chapter 0

# A Short Anecdote

I believe it was in the $11^{\text{th}}$ lecture of the commutative algebra course I took last autumn that professor Kristian Ranestad turned thoughtfully to regard the view from the $11^{\text{th}}$ floor of Niels Henrik Abels Hus and muttered: *"there must be something more to this"*. I do not remember the topic of the lecture, but these words remained with me, fore I found them to be the perfect summary of any small investigation into the unknown.

Specifically, writing a master's thesis in mathematics is such an investigation. That is how I have experienced it, at least. It seems to me I have wandered through a cluster of crossroads marked with road signs written in some semi-familiar language.

The road-cluster has been charted by explorers preceding me, of course, so the journey mostly consists of me deciphering the atlas they handed down. It seems then, that I should not have encountered any surprises. That is not the case, however. Sometimes I wandered down deciphered roads of the atlas, coming upon paths not mentioned in the charts. In those moments I could not help but think: *there must be something more to this.*

# Chapter 1

# Introduction

Enumerative geometry is an ancient mathematical discipline. An early example of a problem within enumerative geometry is the problem of Apollonius. The problem bears the name of the ancient greek geometer Apollonius of Tyana, who stated and solved it. Apollonius asks: Given three general circles in the plane (e.g. three circles with distinct radii and centra that are not on a line), how many circles are tangent to all three circles?

The questions studied by enumerative geometers are usually stated in a similar manner to Apollonius' problem. Such questions are generally of the form: "how many geometric objects satisfy a given set of conditions?". This thesis is not an exception to these sort of inquiries. We pick up the threads of a series of results that attempt to answer the question: *"Given a general collection of curves on a surface, how many curves in the collection will be nodal?"*.

Steiner commented in 1848 in the paper[1] [Ste54], that the answer to the question: *"How many reduced degree $d$ curves on $\mathbb{P}^2$ in a general pencil are nodal?"* was $3(d-1)^2$. Kleiman, Piene suggest in [KP04, Remark 3.7] that this was probably known before then. In 1863, Cayley computes the number of reduced 2-nodal curves on $\mathbb{P}^2$ in a general pencil to be $\frac{3}{2}(d-1)(d-2)(3d^2-3d-11)$ [KP04], by constructing a 'discriminant of 2-nodal curves'.

Much later, Severi – a geometer part of the Italian school of the early $20^{\text{th}}$ century – asserted that the discriminant is irreducible, making it a variety. His intuition was right, but the proof was wrong, as Fulton states in [Ful83]. Three years after Fulton remarks the importance of proving the irreducibility of the discriminant, Harris provides a correct proof in [Har86].

Twelve years after Harris' proof, Göttsche conjectures the theorem on which this thesis is based. In 1999, the problem of counting degree $d$ nodal curves on $\mathbb{P}^2$ in general linear systems had been solved[2]. The drive for generalization motivated the generalization to the problem of counting nodal curves on 'well behaved smooth surfaces', such as a K3 surface. It did not take long before mathematicians started working on a generalization to arbitrary smooth projective surfaces. In rough terms, the Göttsche conjecture counts the number of $\delta$-nodal curves on a smooth projective surface in a general linear system,

[1] I could only access an 1854 edition. See [KP04] for the exact reference.

[2] Recursive formulas for computing the number of nodal curves in the linear systems were first given by Ran [Ran89] and later by Harris-Caporaso [CH98].

provided that the ambient complete linear system is 'sufficiently ample'. More accurately, it states the following.

**Theorem 1.1.** (The Göttsche Conjecture, [KST11, Theorem 4.1])
*Suppose that $\mathcal{L}$ is a $\delta$-very ample line bundle on a smooth projective surface $S$. If $\mathbb{P}^\delta$ is a general linear system of curves on $S$, the number of $\delta$-nodal curves in $\mathbb{P}^\delta$ is a polynomial of degree $\delta$ in the Chern numbers $L^2, L.K_S, K_S^2$ and $c_2(S)$.*

By now, Göttsche's conjecture has many proofs (see for instance [KST11] or [Tze12]). It also has a vast generalization provided by Rennemo in [Ren17], under whose supervision I pick up the torch which illuminates the study of linear systems of nodal curves on smooth surfaces. However, instead of studying the collections of nodal curves, we study the collections of their nodes. It turns out that there is a natural generalization of the question: how many $\delta$-nodal curves are there in a general linear system of dimension $\delta$? Namely, we may ask: what happens when the dimension of the system does not equal the number of nodes our curves have? Let us start by studying the simplest possible case.

**Problem 1.** Let $S$ be any smooth projective surface equipped with a sufficiently ample line bundle $\mathcal{L}$. Letting $|\mathcal{L}| = \mathbb{P}H^0(S, \mathcal{L})$, we shall see that there is a codimension 1 subscheme $|\mathcal{L}|_1 \subset |\mathcal{L}|$ consisting of all nodal curves on $S$. Take $\mathbb{P}^2$ to be a general linear system in $|\mathcal{L}|$. Then $\mathbb{P}^2 \cap |\mathcal{L}|_1$ has dimension 1. Away from the curves with more than 1 node in $|\mathcal{L}|_1$, we may map a curve $C \in |\mathcal{L}|_1$ to its nodal point on $S$. Call this map $\phi_1$. If the image of $\phi_1$ is a scheme, then it has a canonical class in the Chow ring. We ask: are we able to identify the class of this scheme in the Chow ring $A(S)$?

There is no reason for us to limit our study to the 1-nodal curves in $|\mathcal{L}|$. As will be shown later, the set $\mathbb{P}^{\delta+1} \cap |\mathcal{L}|_\delta$ will be of dimension 1. This generalization raises some questions. We may ask: letting $C \in \mathbb{P}^{\delta+1} \cap |\mathcal{L}|_\delta$, what should the codomain of the map $C \mapsto C_{\text{sing}}$ be? We require this codomain to be a projective variety that keeps track of $\delta$ points on $S$. Some obvious candidates for this variety are:

1. The $\delta^{\text{th}}$ order symmetric product of $S$ given as $\operatorname{Sym}^\delta S = S^\delta/\mathfrak{S}_\delta$, where $\mathfrak{S}_\delta$ is the symmetric group on $\delta$ elements.

2. The Blowup of $\operatorname{Sym}^\delta S^\delta$ or $S^\delta$ along a number of diagonals.

3. The Hilbert scheme $S^{[\delta]}$ parametrizing 0-dimensional subschemes of $S$ of length $\delta$.

The third alternative is quite promising, due to the fact that much of the modern algebro-geometric theory of counting singular curves on surfaces takes place in the study of Hilbert schemes of points on surfaces. The second alternative is essential to us in our study of the case $\delta = 2$. Among other varieties, we will work with the blowup of $S \times S$ along the diagonal.

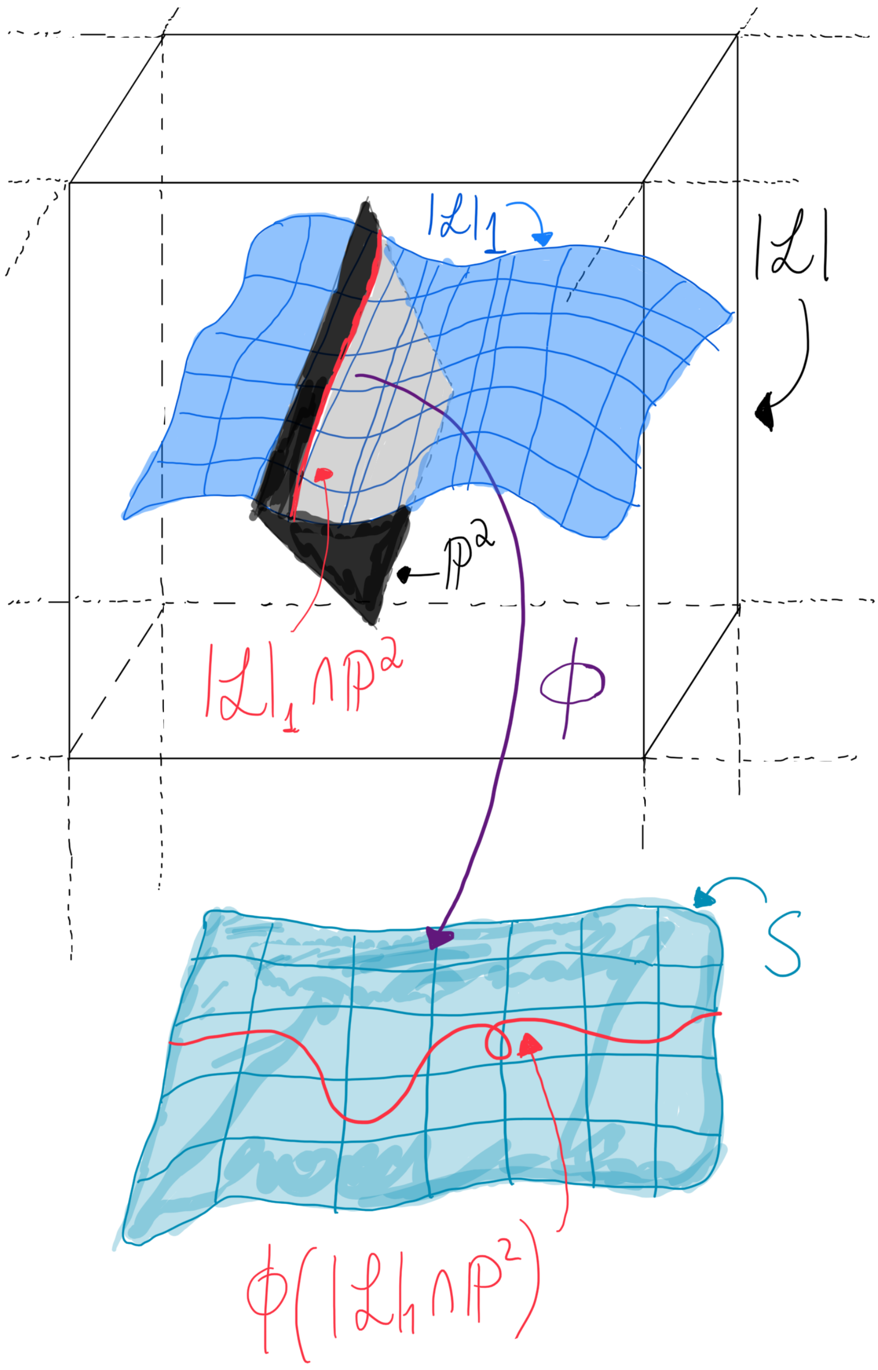
We generalize Problem 1 further.

**Problem 2.** Consider $X = \mathbb{P}^\gamma \cap |\mathcal{L}|_\delta$, where $\gamma \geq \delta$. Under some assumptions, We shall see that $X$ is of dimension $\gamma - \delta$. If it is a scheme, compute the image of $\phi_\delta$ in the Chow ring $A(S^{[\delta]})$, when $\phi_\delta$ is the function defined on the subset of $X$ of curves with exactly $\delta$ nodes, which are mapped to their singular loci, that are closed points in $S^{[\delta]}$.

We have illustrated the situation in Problem 2 for $\gamma = 2$ and $\delta = 1$ in Figure 1.1.

It turns out that our approach to solving problem 2 for the special cases $\delta \in \{1, 2\}$ does not seem to generalize to $\delta > 2$, something we comment on in Section 5.2.1. Note that the case $\mathbb{P}^\delta \cap |\mathcal{L}|_\delta$ is precisely the Göttsche conjecture, something we prove in Section 5.2.3.

Figure 1.1: Illustration of the definitions in Problem 2, in the case where $\gamma = 2$ and $\delta = 1$.



### Results

We solve a readjusted version of Problem 1 in Section 6.1.2. A crucial part of solving the problem is restating it into a manner that is computationally tangible. The restatement is called Problem 3, and it uses the important Observation 5.26, which relates Problem 1 to the Chern classes of the first somewhat tautological bundle $S_T(\mathcal{L}, 1)$. We define the somewhat tautological sheaves $S_T(\mathcal{L}, n)$ in Section 5.2.1, where we also categorize some of their properties. The solution of Problem 3 reads as follows.

**Proposition 1.2.** (Solution of Problem 3)
*Let $(S, \mathcal{L})$ be a smooth projective surface equipped with a 2-very ample line bundle. Then if $\mathbb{P}^2 \subset \mathbb{P}H^0(S, \mathcal{L})$ is a general linear system, the associated class of the locus $V$ of singular points of curves in $\mathbb{P}^2$ in $A(S)$ is*

$$[V] = c_1(S_T(\mathcal{L}, 1)) = 3c_1(\mathcal{L}) + c_1(\Omega_S),$$

*where $S_T(\mathcal{L}, 1)$ is the first somewhat tautological bundle.*

The proposition and proof is given in full detail as Proposition 6.5.

Although Problem 2 remains open, we make some headway on the special case where $\delta = 2$. This special case of Problem 2 is also readjusted using Observation 5.26, which gives us the liberty of working a more direct problem. We use the observation to relate Problem 2 to the Chern classes of the second somewhat tautological bundle $S_T(\mathcal{L}, 2)$. We have dubbed this readjustment Problem 4. We have dedicated Chapter 7 to computing the Chern classes of $S_T(\mathcal{L}, 2)$. The main result of the chapter is the following theorem, wherein we detail the total Chern class of $S_T(\mathcal{L}, 2)$.

**Theorem 1.3.** (Partial solution of Problem 4)
*Let $S$ be a smooth projective surface equipped with a line bundle $\mathcal{L}$. We have that*

$$c_\bullet(S_T(\mathcal{L}, 2)) = c_\bullet(\mathcal{L}_{[2]})c_\bullet(\mathcal{F}) = c_\bullet(\mathcal{L}_{[2]})c_\bullet(q_{B*}(p_B^*\mathcal{L} \otimes \mathcal{I}_B/\mathcal{I}_B^2)) \tag{1.1}$$

*where*

$$c_\bullet(\mathcal{F}) = \frac{c_\bullet((\Omega_S \otimes \mathcal{L})_{[2]})}{c_\bullet(q_{B*}(k_*\mathcal{O}_E(-E) \otimes p_B^*\mathcal{L}))} \tag{1.2}$$

*and*

$$c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2) = \frac{q_B^*\left[c_\bullet(\Omega_{S^{[2]}})\right]}{p_1^*\left[c_\bullet(\Omega_S)\right] c_\bullet(k_*(\Omega_{E/\Delta}))}. \tag{1.3}$$

See Theorem 7.1 for a complete description of the theorem and its setting.

## 1.1 Outline

Chapter 2 consists of a small collection of results that are essential for any algebraic geometer working with locally free sheaves. The chapter is aimed towards new geometers, giving them a frame of reference for important facts that are used sporadically throughout the thesis.

In Chapter 3 we recall the core concepts, results and tools of intersection theory, notably the presentation of the Chow ring of a projective space and the generalized Bezout's theorem, the fundamental theorem for Chern classes and the splitting principle. We move on to discussing characteristics of nodal curves and continue by presenting some common theory surrounding jet bundles, which we will encounter again in Chapter 6. Finally, we briefly discuss some aspects of singularities and state properties of blowups. These topics are essential for the thesis, and we will present the core theory surrounding them.

Linear systems are the topic of Chapter 4. Among the prerequisites of the Göttsche conjecture we find assumptions on the complete linear system $|\mathcal{L}|$ as well as assumptions of generality on the linear systems $\mathbb{P}^\delta$. The chapter revolves around defining and providing examples of these notions, as well as providing results from the literature that are necessary to the thesis.

The fifth chapter introduces the notions of Hilbert schemes of points on surfaces that are necessary to us. We review how the theory of Hilbert schemes of points on surfaces is connected to counting $\delta$-nodal curves (via the Göttsche conjecture) and look into some of the core theory in the literature on tautological bundles. We continue by defining and examining the behavior of the somewhat tautological sheaves $S_T(\mathcal{L}, n)$ by means of examples and propositions, describing some of their most central properties. Among other facts, we assert that the first somewhat tautological bundle is the first jet bundle $J^1\mathcal{L}$. A crucial observation at the end of the chapter allows us to readjust Problem 1 to be more tangible. We also reformulate a special case of Problem 2, using this observation. Finally, we prove that the special case of Problem 2 where one studies $|\mathcal{L}|_\delta \cap \mathbb{P}^\delta$ is precisely the Göttsche conjecture.

We compute the total Chern class of the first somewhat tautological bundle on a smooth projective surface in Chapter 6. Incidentally, this requires us to compute the classes $c_\bullet(\operatorname{Sym}^n \Omega_S)$ for all $n \in \mathbb{N}$ which we successfully do. We go on to solve the readjusted Problem 1, which requires a dimension count of a specific degeneracy locus. Finally, we illustrate how not to generalize the notion of a jet bundle to encompass data on curves with at least 2 singularities.

In Chapter 7 we compute the total Chern class of the second somewhat tautological bundle $S_T(\mathcal{L}, 2)$. This requires a fair amount of bookkeeping; among other things, we prove short exactness of a sequence, as well as asserting commutativity of necessary diagrams. In particular, we relate the Chern classes of the bundle $S_T(\mathcal{L}, 2)$ to two tautological bundles.

The appendix consists of two sections. The first sections consists of proofs of preliminary facts which we considered too tedious, or of too little relevance for the preliminaries. The second section contains a simile, in Dutch.

## 1.2 Notation and Conventions

We use vector bundle and locally free sheaf interchangeably whenever the meaning is clear from context. We let $V(f_1, ..., f_r)$ denote the common zero-set of homogeneous polynomials $f_1, ..., f_r$. Unless otherwise specified, a *scheme* $X$ will be a separated scheme of finite type over $\mathbb{C}$. By *variety* we will mean an integral scheme, i.e. a reduced, irreducible scheme. By *surface* we will mean a 2-dimensional variety, and by a *curve* we will mean 1-dimensional reduced scheme. If no auxiliary information is given, a *point* of a scheme will mean a closed point. We follow Fulton's convention that the projectivization $\mathbb{P}\mathcal{W}$ of vector space $\mathcal{W}$ is defined as $\mathbb{P}\mathcal{W} := \operatorname{Proj} \operatorname{Sym} \mathcal{W}^\vee$, where $\mathcal{W}^\vee$ is the dual vector space of $\mathcal{W}$. When nothing else is specified, $(S, \mathcal{L})$ is a smooth projective surface equipped with a line bundle.

# Part I

# Preliminaries

Chapter 2

# Background on Quasi-Coherent Sheaves

This thesis is about exactly two things: a few handpicked locally free sheaves on smooth projective varieties, and their Chern classes. It is intended for this chapter to be a small collection of propositions and theorems which constitute necessary tools for anybody wishing to work with sheaves in algebraic geometry. Throughout the thesis, I have taken the freedom of taking these results for granted by referencing them only by their slogan (and not their number or name). I hope that the appending of these facts will help readers who are new to algebraic geometry understand the thesis. The experienced geometer may safely skip this chapter.

## 2.1 Exact Functors & Locally Free Sheaves

Tensoring by a locally free sheaf in the category of $\mathcal{O}_X$-modules is exact.

**Proposition 2.1.** (Tensoring by a locally free sheaf is xact)
*Let $X$ be a scheme, let $\mathcal{E}$ be locally free on $X$ and suppose $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ are $\mathcal{O}_X$-modules that fit into an exact sequence*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0 \ .$$

*Then the sequence*

$$0 \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}_1 \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}_2 \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}_3 \longrightarrow 0$$

*is exact.*

The pushforward of quasi-coherent sheaves along a finite morphism is exact.

**Proposition 2.2.** (Pushforward along a finite morphism is exact)
*Let $f : X \to Y$ be a finite morphism of schemes, and let $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ be $\mathcal{O}_X$-modules that fit into an exact sequence*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0 \ .$$

*Then the sequence*

$$0 \longrightarrow f_*\mathcal{F}_1 \longrightarrow f_*\mathcal{F}_2 \longrightarrow f_*\mathcal{F}_3 \longrightarrow 0 \ .$$

*is an exact sequence of $\mathcal{O}_Y$-modules.*

The last proposition of the section provides a practical way of proving an $\mathcal{O}_X$-module is locally free.

**Proposition 2.3.**
*Let $X$ be a scheme and let $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ be coherent $\mathcal{O}_X$-modules that fit into an exact sequence*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0\ .$$

*If $\mathcal{F}_2$ and $\mathcal{F}_3$ are locally free of finite rank, then so is $\mathcal{F}_1$. We also have that if $\mathcal{F}_1$ and $\mathcal{F}_3$ are locally free of finite rank, then so is $\mathcal{F}_2$. Furthermore, if the $\mathcal{F}_i$ all are locally free of finite rank, then they satisfy*

$$\operatorname{rank} \mathcal{F}_2 = \operatorname{rank} \mathcal{F}_1 + \operatorname{rank} F_3.$$

## 2.2 Adjoint Functors & Stalks

The pullback and pushforward of sheaves along morphisms of locally ringed spaces are both functors. In fact, the two functors are adjoint.

**Proposition 2.4.** (Adjoint Property of Push & Pull: [Har77, Page 110])
*Let $f : X \to Y$ be a morphism of schemes, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module and $\mathcal{G}$ an $\mathcal{O}_Y$-module. There is a natural isomorphism of groups*

$$\operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

In the interest of further relating the pushforward and pullback along a morphism, we state the projection formula.

**Proposition 2.5.** (Projection Formula [Har77, Exercise II.5.1d])
*Suppose $f : X \to Y$ is a morphism of schemes, $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank. Then there is an isomorphism*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E}$$

*of $\mathcal{O}_Y$-modules.*

The stalks of the pullback of a quasi-coherent sheaf are well behaved, whereas the stalks of the pushforward of any sheaf are not. The latter is remediable when one pushes forward along a closed immersion.

**Proposition 2.6.** (Stalks of pullbacks of quasi-coherent sheaves [Stacks, Tag 0098])
*Let $f : X \to Y$ be a morphism of schemes, and let $\mathcal{F}$ be an $\mathcal{O}_Y$ module. Let $x \in X$. Then*

$$(f^*\mathcal{F})_x = \mathcal{F}_{f(x)} \otimes_{\mathcal{O}_{Y,x}} \mathcal{O}_{X,x},$$

*as $\mathcal{O}_{X,x}$-modules.*

## 2.3 The Grothendieck Ring

The Grothendieck ring $K(X)$ of a variety $X$ is (in the context of algebraic geometry) a ring of formal sums of isomorphism classes of locally free sheaves modulo some relation. There exists a ring-homomorphism relating the Grothendieck ring to the Chow ring, which explains our interest in $K(X)$.

It is easiest to describe the Grothendieck ring by constructing it. For any scheme $X$, let $L(X)$ denote the free abelian group on the set of isomorphism classes of vector bundles on $X$. As a group we can describe $K(X)$ as a quotient of $L(X)$, where we mod $L(X)$ out with the relation $[\mathcal{F}] = [\mathcal{E}] + [\mathcal{H}]$ whenever there exists an exact sequence of locally free sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow 0 \,.$$

The multiplicative operation making $K(X)$ into a ring is the tensor product $[\mathcal{E}] \cdot [\mathcal{F}] = [\mathcal{E} \otimes \mathcal{F}]$, which we extend linearly. Multiplication distributes over addition, since tensoring by a locally free sheaf is exact.

We introduce $K(X)$ mostly as a referential framework for applying the Grothendieck-Riemann-Roch theorem, as well as for easing the notation in Chapter 7. For a thorough introduction, see [Ful98, Chapter 15]. Let us finish off the chapter by committing a felony: introducing theory that is incomprehensible without the definitions of Section 3.2, a later section.

Let $\mathcal{E}$ be a locally free sheaf of finite rank on $X$. Factor the total Chern class $c_\bullet(\mathcal{E}) = \Pi(1 + \alpha_i)$, and call $\alpha_i$ the $i^{\text{th}}$ Chern root of $\mathcal{E}$. The *Chern character* $\operatorname{ch}(-)$ *of* $\mathcal{E}$ is defined as

$$\operatorname{ch}(\mathcal{E}) = \sum_i e^{\alpha_i} =: \sum_i \operatorname{ch}_i(\mathcal{E}),$$

where

$$e^{\alpha_i} = \sum_{k=0}^{\operatorname{rank}\mathcal{E}} \frac{{\alpha_i}^k}{k!}.$$

We can express the Chern character of $\mathcal{E}$ in terms of the Chern classes $c_i(\mathcal{E})$ (encountered in Section 3.2). The first terms of the Chern character are

$$\begin{aligned}
\operatorname{ch}_0(\mathcal{E}) &= \operatorname{rank}\mathcal{E};\\
\operatorname{ch}_1(\mathcal{E}) &= c_1(\mathcal{E});\\
\operatorname{ch}_2(\mathcal{E}) &= \frac{c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})}{2};\\
\operatorname{ch}_3(\mathcal{E}) &= \frac{c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}) + 3c_3(\mathcal{E})}{6}.
\end{aligned}$$

The Chern character is quite useful in our pursuit. Under certain assumptions it is an isomorphism that relates the Grothendieck ring to the *Chow ring* (see Section 3.1) via the following.

**Theorem 2.7.** [EH16, Theorem 14.3]
*If $X$ is a smooth projective variety, then the map*

$$\operatorname{ch} : K(X) \otimes \mathbb{Q} \to A(X) \otimes \mathbb{Q}$$

*is an isomorphism of rings.*

It is constructive for the thesis to also introduce the *Todd classes of* $\mathcal{E}$, which are defined in a similar manner to the Chern character. We define the Todd class of $\mathcal{E}$ to be

$$\mathrm{td}(\mathcal{E}) = \prod_{i=1}^{n} \frac{\alpha_i}{1 - e^{-\alpha_i}} =: \sum_i \mathrm{td}_i(\mathcal{E}).$$

The Todd class can also be expressed in terms of the Chern classes $c_i(\mathcal{E})$. The first terms of the Todd classes of a locally free sheaf $\mathcal{E}$ are

$$\begin{aligned} \mathrm{td}_0(\mathcal{E}) &= 1; \\ \mathrm{td}_1(\mathcal{E}) &= \frac{c_1(\mathcal{E})}{2}; \\ \mathrm{td}_2(\mathcal{E}) &= \frac{c_1(\mathcal{E})^2 + c_2(\mathcal{E})}{12}; \\ \mathrm{td}_3(\mathcal{E}) &= \frac{c_1(\mathcal{E})c_2(\mathcal{E})}{24}. \end{aligned}$$

We will make use of the Todd classes when we apply the Grothendieck-Riemann-Roch theorem (that we sometimes abbreviate to GRR). The theorem is stated in Section 3.2.

## 2.4 Operations on Locally Free Sheaves

Given a locally free sheaf $\mathcal{F}$, there are operations that map $\mathcal{F}$ to a different locally free sheaf. In this thesis we make use of three such operations. Firstly, the *n-fold tensor product of* $\mathcal{F}$, denoted $\mathrm{T}^n(\mathcal{F})$. Secondly, the $n^{\text{th}}$ *symmetric algebra of* $\mathcal{F}$, denoted $\mathrm{Sym}^n(\mathcal{F})$ in this thesis, but sometimes $\mathrm{S}^n(\mathcal{F})$ in the literature. Finally, we have the $n^{\text{th}}$ *wedge product of* $\mathcal{F}$, which we denote $\bigwedge^n \mathcal{F}$. We state a theorem which asserts that these sheaves are indeed locally free. The proposition also informs us of their rank; this information will be used multiple times throughout the thesis. We cite [EO22a, Proposition 11.22].

**Theorem 2.8.** (Facts about locally free sheaves)
*Let $X$ be a scheme. The set of locally free sheaves is closed under direct sums, tensor products, symmetric products, exterior products, duals and pullbacks.*
*If $\mathcal{F}$ is locally free of rank $r$ with $m_1, ..., m_r$ a basis of $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module, then*

*i)* *$T^n(\mathcal{F})$ is locally free of rank $r^n$, and the set of sections of the form $m_1 \otimes ... \otimes m_n$ where $i_k \in \{1, ..., r\}$ is a basis of $T^n(\mathcal{F})_x$ as an $\mathcal{O}_{X,x}$-module;*

*ii)* *$\mathrm{Sym}^n(\mathcal{F})$ is locally free of rank $\binom{n+r-1}{r-1}$ and the set of sections of the form $m_1^{n_1} \cdots m_r^{n_r}$ where $\sum n_i = n$ is a basis of $\mathrm{Sym}^n(\mathcal{F})_x$ as an $\mathcal{O}_{X,x}$-module;*

*iii)* *$\bigwedge^n(\mathcal{F})$ is locally free of rank $\binom{r}{n}$, and the set of sections of the form $m_{i_1} \wedge \cdots \wedge m_{i_n}$, where $i_1 < i_2 < ... < i_n$, is a basis of $\bigwedge^n(\mathcal{F})_x$ as an $\mathcal{O}_{X,x}$-module;*

*iv)* *If*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

*is an exact sequence of locally free sheaves of ranks $n', n$ and $n''$ respectively. Then*

$$\bigwedge^{n} \mathcal{F} \cong \bigwedge^{n'} \mathcal{F}' \otimes_{\mathcal{O}_X} \bigwedge^{n''} \mathcal{F}''.$$

Chapter 3

# Intersection Theory

In this chapter we recall some fundamental tools of intersection theory and stockpile relevant facts about singularities and blowups.

## 3.1 The Chow Ring

### Conventions

We denote the Chow ring of a smooth, quasi-projective variety $X$ by $A(X)$. We let it be clear from context when we consider a subvariety of $X$ as an element in the ring, as opposed to a variety in the usual sense.

### The Chow Ring of Projective Space

The Chow ring of a projective space is quite important for this thesis, as the linear systems we study are projective spaces parametrizing curves on a surface.

**Theorem 3.1.** [EH16, Theorem 2.1]
*We have that*

$$A(\mathbb{P}^n_{\mathbb{C}}) = \mathbb{Z}[H]/H^{n+1},$$

*where $H \in A^1(\mathbb{P}^n_{\mathbb{C}})$ is the class of a hyperplane. Furthermore, the class of a subvariety $\mathbb{P}^n_{\mathbb{C}}$ of codimension $k$ and degree $d$ is $dH^k$.*

Having complete knowledge of the Chow ring of $\mathbb{P}^n$ allows for the proof of a more general version of Bezout's theorem. This version is better suited for intersection theoretical applications.

**Theorem 3.2.** (Bezout's Theorem [EH16])
*If $X_1, ..., X_k \subset \mathbb{P}^n$ are subvarieties of codimension $c_1, ..., c_k$ with $\sum c_i \leq n$, and the $X_i$ intersect generically transversely, i.e. $X_1 \cap ... \cap X_k$ is reduced of $\operatorname{codim} c = \sum_{i=1}^{k} c_i$, then*

$$\deg(X_1 \cap ... \cap X_k) = \prod \deg(X_i).$$

*In particular, two subvarieties $X, Y \subset \mathbb{P}^n$ having complementary dimension that intersect transversely will intersect in exactly $\deg(X) \cdot \deg(Y)$ points.*

### The Class of a Subscheme of a Smooth Variety

There is a natural way of associating to a subscheme of a smooth variety $X$ an element in the Chow ring $A(X)$ of $X$. Let $Y \hookrightarrow X$ be a subscheme of $X$, and let $Y_1, ..., Y_k$ be the irreducible components of the reduced scheme $Y_{\text{red}}$. The Jordan-Hölder theorem gives that all composition series in $\mathcal{O}_{Y,Y_i}$ are of the same (finite) length $\ell_i$, since our schemes are Noetherian and the ground field is $\mathbb{C}$. The associated class of $Y$, which henceforth will be referred to as *the class of $Y$ in $A(X)$*, is

$$[Y] := \sum_i \ell_i [Y_i],$$

where $[Y_i]$ denotes the class of the subvariety $Y_i \hookrightarrow X$.

### Functoriality

Let $f : Y \to X$ be a map of schemes. If $f$ is proper, it will induce a group homomorphism $f_*$ between Chow groups $A(Y)$ and $A(X)$. The homomorphism is defined by extending a homomorphism $f_* : Z(Y) \to Z(X)$ between the group of cycles on $Y$ and $X$ respectively. Since this construction is somewhat technical, we refer to [EH16, Chapter 1.3.6] or [Ful98, Chapter 1.4] for a detailed construction, allowing us to only state the theorem we need.

**Theorem 3.3.** [EH16, Theorem 1.20]
*If $f : Y \to X$ be a proper map of schemes, then the map $f_* : Z(Y) \to Z(X)$ detailed in Chapter 1.3.6 in [EH16] induces a map of groups $f_* : A_k(Y) \to A_k(X)$ for each $k$.*

The definition of the pushforward is needed to express the Grothendieck-Riemann-Roch theorem, which we will encounter in the next section.

If $f : Y \to X$ is a flat morphism of smooth projective varieties, then it induces a map between Chow rings.

**Theorem 3.4.** [EH16, Theorem 1.25]
*Suppose $f : Y \to X$ is a flat morphism of smooth projective varieties. Then the map $f^* : A(X) \to A(Y)$ defined on cycles by*

$$f^*(\langle A \rangle) := \langle f^{-1}(A) \rangle \qquad \text{for every subvariety } A \subset X$$

*induces a ring homomorphism.*

This result is useful when computing Chern classes, as we will see in the main theorem of the next section.

## 3.2 Chern classes

### Conventions

We denote the $k^{\text{th}}$ Chern class of a vector bundle $\mathcal{E}$ by $c_k(\mathcal{E}) \in A^k(X)$ and define $c_\bullet(\mathcal{E}) := 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + ... + c_{\operatorname{rank} \mathcal{E}}(\mathcal{E})$ to be the *total Chern class* of $\mathcal{E}$.

### Foundational Theory of Chern Classes

The Chern classes of a vector bundle $\mathcal{E}$ on a smooth, projective variety $X$ are classes in the Chow ring $A(X)$ that are associated to the vector bundle. The Chern classes are uniquely determined by a few natural properties, which may be summarised in one theorem.

**Theorem 3.5.** [EH16, Theorem 5.3]
*Let $X$ be a smooth quasi-projective variety, and let $\mathcal{E}$ be a vector bundle on $X$. There is a unique way of assigning to each vector bundle $\mathcal{E}$ a class $c_\bullet(\mathcal{E}) = c_0 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \cdots \in A(X)$ in such a manner that:*

a. *(Line Bundles) If $\mathcal{L}$ is a line bundle on $X$ then the Chern class of $\mathcal{L}$ is $1 + c_1(\mathcal{L})$.*

b. *(Bundles with enough sections) If $\tau_0, \ldots, \tau_{r-i}$ are global sections of $\mathcal{E}$, and the degeneracy locus $D = V(\tau_0 \wedge \tau_1 \wedge \ldots \wedge \tau_{r-i})$ where the sections are (linearly) dependent has codimension $i$, then $c_i(\mathcal{E}) = [D] \in A^i(X)$.*

c. *(Whitney's formula) If*
$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$
*is a short exact sequence of vector bundles on $X$ then*
$$c_\bullet(\mathcal{F}) = c_\bullet(\mathcal{E})c_\bullet(\mathcal{G}) \in A(X).$$

d. *(Functoriality) If $\phi : Y \to X$ is a morphism of smooth varieties, then*
$$\phi^*(c_\bullet(\mathcal{E})) = c_\bullet(\phi^*(\mathcal{E})).$$

It is possible to extend the definition of the total Chern class to coherent sheaves. Indeed, any coherent sheaf $\mathcal{F}$ on a smooth projective variety $X$ can be resolved by locally free sheaves, giving an exact sequence

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0 \tag{3.1}$$

where the $\mathcal{E}_i$ are locally free. This allows for a canonical way to define the total Chern class of $\mathcal{F}$ in a manner which makes Whitney's formula hold in general.

**Definition 3.6.** (Total Chern class of a coherent sheaf)
Let $\mathcal{F}$ be a coherent sheaf on a smooth projective variety $X$ with a resolution of locally free sheaves as in (3.1). Then we define

$$c_\bullet(\mathcal{F}) := \prod_{i=0}^{n} c_\bullet(\mathcal{E}_i)^{(-1)^i},$$

the *total Chern class of $\mathcal{F}$*.

This definition will be relevant in Chapter 7, where the final computation of the Chern classes of $S_T(\mathcal{L}, 2)$, the second somewhat tautological bundle, will be partially expressed in the Chern classes of some coherent sheaves.

In addition to uniquely determining the Chern classes associated to a vector bundle, the properties in Theorem 3.5 also serve as tools for computing the classes. Pairing the theorem with the *splitting principle* provides us with a powerful combination of tools for proving identities between Chern classes of vector bundles.

**Principle 3.7.** ***The Splitting Principle*** [EH16, Theorem 5.11]
*Any identity among Chern classes of vector bundles that is true for vector bundles that are direct sums of line bundles is true in general.*

This formulation of the splitting principle is somewhat vague; it is not a priori clear what is meant by 'identity'. However, instead of giving an accurate description of the theorem the above principle attempts to describe, we instead illustrate the utility of the splitting principle via an example.

**Example 3.8.** (An Application of the Splitting Principle)
Let $\mathcal{E}$ be a rank 2 vector bundle on a smooth projective variety $X$. We desire to compute $c_\bullet(\operatorname{Sym}^2 \mathcal{E})$, where in this setting, 'compute' means 'express the Chern classes of $\operatorname{Sym}^2 \mathcal{E}$ in terms of the Chern classes of $\mathcal{E}$'. Assume $\mathcal{E} = \mathcal{L} \oplus \mathcal{M}$, where $\mathcal{L}$, $\mathcal{M}$ are line bundles with $c_1(\mathcal{L}) =: \ell$ and $c_1(\mathcal{M}) =: m$. By Whitney's sum formula, we have that

$$c_\bullet(\mathcal{E}) = c_\bullet(\mathcal{L})c_\bullet(\mathcal{M}) = (1+\ell)(1+m) = 1 + (m+\ell) + m\ell.$$

Since the Chow ring is a graded ring, the product $m\ell$ will be in the second graded component of $A(X)$. We may thus conclude that $c_1 := c_1(\mathcal{E}) = m + \ell$ and $c_2 := c_2(\mathcal{E}) = m\ell$.

Now for the main computation, we have that

$$\begin{aligned} c_\bullet(\operatorname{Sym}^2 \mathcal{E}) &= c_\bullet(\operatorname{Sym}^2(\mathcal{L} \otimes \mathcal{M}) \\ &= c_\bullet(\mathcal{L}^{\otimes 2} \oplus (\mathcal{L} \otimes \mathcal{M}) \oplus \mathcal{M}^{\otimes 2}). \end{aligned}$$

By applying Whitney's formula to the right hand side, we obtain

$$c_\bullet(\operatorname{Sym}^2 \mathcal{E}) = c_\bullet(\mathcal{L}^{\otimes 2})c_\bullet(\mathcal{L} \otimes \mathcal{M})c_\bullet(\mathcal{M}^{\otimes 2}).$$

Furthermore, since $c_1 : \operatorname{Pic}(X) \to A^1(X)$ is a homomorphism, the above product may be expanded to

$$\begin{aligned} c_\bullet(\operatorname{Sym}^2 \mathcal{E}) &= (1+2\ell)(1+\ell+m)(1+2m) \qquad (*) \\ &= 1 + 3(m+\ell) + 4m\ell + 2(m+\ell)^2 + 4m\ell(m+\ell). \end{aligned}$$

Substituting for $c_1$ and $c_2$, we obtain the expression

$$c_\bullet(\operatorname{Sym}^2 \mathcal{E}) = 1 + 3c_1 + (4c_2 + 2c_1^2) + 4c_1c_2.$$

Since this expression holds for all rank 2 vector bundles on $X$ that split, it will hold for all rank 2 vector bundles on $X$, by the splitting principle. △

We remark that Eisenbud & Harris provide the same example in [EH16], but make a small error in expanding the product $(*)$, that we have corrected.

A particular application of Whitney's formula in combination with the splitting principle will be applied multiple times in Chapters 6 & 7. We are referring to the proposition below, whose proof follows along the same lines as the method in Example 3.8.

**Proposition 3.9.** [EH16, Proposition 5.17]
*If $\mathcal{E}$ is a vector bundle of rank $r$ and $\mathcal{L}$ is a line bundle, then*

$$c_k(\mathcal{E} \otimes \mathcal{L}) = \sum_{l=0}^{k} \binom{r-l}{k-l} c_1(\mathcal{L})^{k-l} c_l(\mathcal{E}) = \sum_{i=0}^{k} \binom{r-k+i}{i} c_1(\mathcal{L})^i c_{k-i}(\mathcal{E}).$$

**Remark 3.10.** More generally, there are explicit formulas that express the Chern classes of a tensor product of locally free sheaves $c_\bullet(\mathcal{E} \otimes \mathcal{F})$ when $\mathcal{E}$ and $\mathcal{F}$ are of finite rank. The proofs in this thesis only require the application of Proposition 3.9, but we nonetheless give references that state formulae for $c_\bullet(\mathcal{E} \otimes \mathcal{F})$, as this more general case is a useful preliminary in computations similar to those that are present in this thesis. A recent article is [Szi22], which provides two formulas, and also comments on formulae in [Man16]. △

Varieties in general have many affiliated bundles. Computing the Chern classes of such a bundle often comes down to applying Whitney's formula to a short exact sequence the bundle is part of. Since our linear systems are projective spaces, it is worth keeping track of the Chern classes of bundles associated to $\mathbb{P}^n$. We state some in the table below.

Table 3.1: Chern classes of bundles associated to $\mathbb{P}^n$.

| Total Chern class of bundles | Class in $A(\mathbb{P}^n)$ |
|---|---|
| $c_\bullet(\mathcal{T}_{\mathbb{P}^n})$ | $(1+H)^{n+1}$ |
| $c_\bullet(\omega_{\mathbb{P}^n})$ | $1-(n+1)\cdot H$ |
| $c_\bullet(\Omega_{\mathbb{P}^n})$ | $(1-H)^{n+1}$ |
| $c_\bullet(\mathcal{N}_{Y/\mathbb{P}^n})$ | $1+dH'$ |

In the table, $H$ denotes the class of a hypersurface in $A(\mathbb{P}^n)$, $H'$ denotes the inverse image of the inclusion $Y \subset \mathbb{P}^n$, where $Y$ is a smooth degree $d$ hypersurface. Furthermore we have that $\mathcal{T}_{\mathbb{P}^n}$ is the tangent sheaf of $\mathbb{P}^n$, $\omega_{\mathbb{P}^n}$ the canonical sheaf on $\mathbb{P}^n$, $\Omega_{\mathbb{P}^n}$ the sheaf of Kähler differentials on $\mathbb{P}^n$ and $\mathcal{N}_{Y/\mathbb{P}^n}$ the normal bundle of a closed projective subscheme of $\mathbb{P}^n$.

### Grothendieck-Riemann Roch

In simple terms, the Grothendieck-Riemann-Roch formula explains the relation between the Chern classes of a vector bundle to the Chern classes of its pushforward along a proper morphism. We recall from Secion 3.1 that a proper morphism $f : Y \to X$ of schemes induces a group homomorphism $f_* : A(Y) \to A(X)$ of Chow groups. Under the same assumption on $f$, it also induces a map

$$f_* : K(Y) \to K(X)$$

between the Grothendieck groups of $Y$ and $X$. Since the definition of the pushforward $f_*$ is a little technical, and we will not use it explicitly in this thesis, we refer curious readers to [Ful98, Chapter 15.1], where a comprehensive introduction is given.

Having covered its prerequisites, we move on to stating the Grothendieck-Riemann-Roch theorem.

**Theorem 3.11.** (Grothendieck-Riemann-Roch, [Ful98, Theorem 15.2])
*Let $f : X \to Y$ be a proper morphism of non-singular varieties. Then for all $\alpha \in K(X)$,*

$$\mathrm{ch}(f_*\alpha) \cdot \mathrm{td}_Y = f_*(\mathrm{ch}(\alpha) \cdot \mathrm{td}_X)$$

*in $A(Y) \otimes \mathbb{Q}$, where we denote $\mathrm{td}_Z := \mathrm{td}(\mathcal{T}_Z)$ if $Z$ is a smooth variety. In other words, the diagram*

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\mathrm{ch}(-)\cdot \mathrm{td}_X} & A(X) \\ \downarrow f_* & & \downarrow f_* \\ K_0(Y) & \xrightarrow{\mathrm{ch}(-)\cdot \mathrm{td}_Y} & A(Y) \end{array}$$

*commutes.*

We shall make use of the theorem in Chapter 7.

## 3.3 Jet Bundles

Given a smooth projective variety $Y$ equipped with a line bundle $\mathcal{L}$, we denote by $J^n\mathcal{L}$ the $n^{\text{th}}$ *jet bundle* of a line bundle $\mathcal{L}$[1]. The jet bundle is defined to be the locally free sheaf

$$J^n\mathcal{L} = \pi_{1*}(\mathcal{O}_{Y\times Y}/\mathcal{I}_\Delta^{n+1} \otimes \pi_2^*\mathcal{L}), \tag{3.2}$$

where $\pi_1 : Y \times Y \to Y$ is first projection, $\pi_2 : Y \times Y \to Y$ is second projection and where $\mathcal{I}_\Delta$ denotes the ideal of the diagonal $\Delta \subset Y \times Y$.

A lot of information about the jet bundles is made available by the following theorem. I cite [EH16], but I have made a few edits.

**Theorem 3.12.** (The Jet Bundle Theorem)
*Let $Y$ be a smooth projective variety equipped with a line bundle $\mathcal{L}$. The sheaves $J^m\mathcal{L}$ have the following properties:*

a. *If $p \in Y$ is a closed point, then there is a canonical identification of the fiber $J^m\mathcal{L} \otimes \kappa(p)$ of $J^m\mathcal{L}$ at $p$ with the sections of the restriction of $\mathcal{L}$ to the $m^{th}$-order neighborhood of $p$; that is,*

$$J^m(\mathcal{L}) \otimes \kappa(p) = H^0(\mathcal{L} \otimes (\mathcal{O}_{X,p}/\mathfrak{m}_{X,p}^{m+1}))$$

*as vector spaces over $\kappa(p) = \mathcal{O}_{X,p}/\mathfrak{m}_{X,p} = \mathbb{C}$. In other words,*

$$J^m\mathcal{L} \otimes \kappa(p) = \frac{\{\text{germs of sections of } \mathcal{L} \text{ at } p\}}{\{\text{germs vanishing to order} \geq m+1 \text{ at } p\}}.$$

b. *If $s \in H^0(\mathcal{L})$ is a global section, then there is a global section $\widetilde{s} \in H^0(J^m(\mathcal{L}))$ whose value at $p$ is the class of $s$ in $H^0(\mathcal{L} \otimes (\mathcal{O}_{X,p}/\mathfrak{m}_{X,p}^{m+1}))$.*

c. *For each $m > 0$ there is a short exact sequence of $\mathcal{O}_Y$-modules*

$$0 \longrightarrow \mathcal{L} \otimes \mathrm{Sym}^m(\Omega_Y) \longrightarrow J^m\mathcal{L} \longrightarrow J^{m-1}\mathcal{L} \longrightarrow 0$$

*where $\Omega_Y$ denotes the sheaf of $\mathbb{C}$-linear differential forms on $Y$.*

The short exact sequences in *c*) in the theorem are of great use when computing the Chern classes of $J^n\mathcal{L}$, as we may apply Whitney's formula recursively. The challenge in this lies in computing the Chern classes of $\mathrm{Sym}^n(\Omega_Y)$. We make this computation in Section 6.1.1, in the special case where $Y$ is a surface.

[1] In the literature, such as [Ful98] and [EH16], a jet bundle is often referred to as *a bundle of principle parts*. We prefer the term jet bundle.

**Example 3.13.**
The $0^{\text{th}}$ jet Bundle of $\mathcal{L}$ is $\mathcal{L}$. We have worked out a proof and appended it to A.1.1. $\triangle$

A corollary to the theorem is $J^n\mathcal{L}$ being locally free of rank $\binom{n+k}{k}$. The proof is suggested by Vakil in an unpublished note. We have written it out and appended it to A.1.2

**Corollary 3.14.** *If* $\dim Y = k$*, then* $J^n(\mathcal{L})$ *is a locally free sheaf of rank* $\binom{n+k}{k}$*.*

## 3.4 On Singularities and Blowing Up

The Göttsche conjecture counts the number of nodal curves in a general linear system of a sufficiently very-ample line bundle. Heuristically, we say a curve is nodal if at some point, the curve looks like $\{xy = 0\}$ at the point. Rigorously we say a curve $C$ is nodal at a point $p$ if it is analytically isomorphic to the origin in the scheme $Y = \operatorname{Spec}(\mathbb{C}[x,y]/(xy))$. That is, the completion of the local ring of $p \in C$ is isomorphic to the completion of the local ring of $\mathcal{O}_{Y,0}$, where $0 \in Y$ is the origin.

**Example 3.15.** (A slightly altered version of [Har77, Example I.5.6.3])
The origin of the plane curve $C$ given by $y^2 - x^2(x+1) = 0$ has a node at the origin. This becomes apparent when we study the LHS of the equation in the ring $\mathbb{C}[[x,y]]$, where we can factor $y^2 - x^2(x+1)$. We have that

$$y^2 - x^2(x+1) = (y - x\sqrt{1+x})(y + x\sqrt{1+x}),$$

the term $\sqrt{1+x}$ being an element in $\mathbb{C}[[x,y]]$ as it is a binomial series. Indeed, it equals the formal power series

$$(1+x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k},$$

where

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

The crux of the idea is that we have factored $y^2 - x^2(x+1) = gh$, where $g, h \in \mathbb{C}[[x,y]]$. Hartshorne then shows that $\mathcal{O}_{C,0} \cong \mathbb{C}[[x,y]]/(gh)$. He continues by applying an automorphism mapping $g$ and $h$ to $x$ and $y$ respectively, granting us the isomorphism $\mathcal{O}_{C,0} \cong \mathbb{C}[[x,y]]/(xy)$. See Figure 3.1 for a plot of the curves, as well as Figure 3.2 for a plot of the *folium of Descartes.* $\triangle$

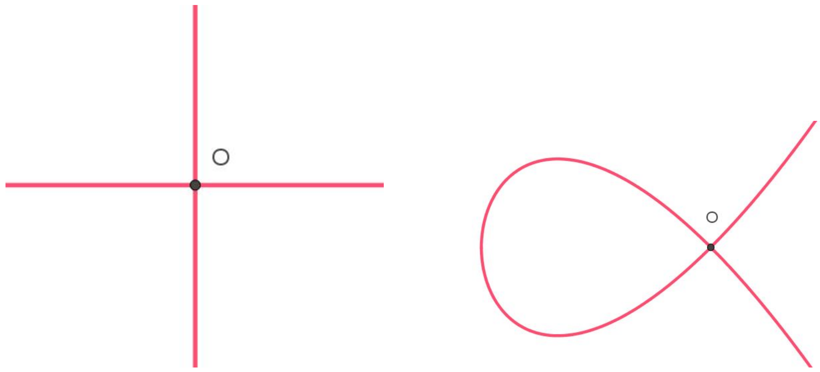
**Definition 3.16.** We say a curve is $\delta$-nodal if it has $\delta$ nodes.

Let us introduce some tools that will give insight into the geometry of singular curves. We usually focus on reduced curves. A nice property of reduced curves is that their singular locus is empty or zero-dimensional.

**Remark 3.17.** For a reduced curve $C$, we have $\dim C_{\text{sing}} \leq 0$. $\triangle$

We will use this fact in the proof of Proposition 6.5.

A convenient way of asserting whether a point is singular (on any projective variety) is by applying the projective Jacobian criterion. We cite the proposition from [EO22b].



(a) The curve $xy = 0$. (b) The curve $y^2 - x^2(x + 1) = 0$.

Figure 3.1: Plot of the plane curves in example 3.15.

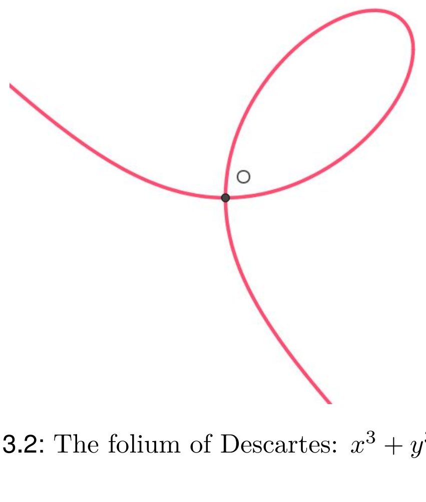


Figure 3.2: The folium of Descartes: $x^3 + y^3 - 3xy = 0$

**Proposition 3.18.** (The projective Jacobian Criterion)
*Let $X = V(F_1, F_2, ..., F_r) \subset \mathbb{P}^n$ be a closed irreducible projective algebraic set, and let*

$$\mathcal{J} = \left( \frac{\partial F_i}{\partial z_j}(p) \right).$$

*Then the rank of $\mathcal{J}$ does not depend on the choice of representative for $p$. Moreover $X$ is non-singular at $p$ if and only if* $\operatorname{rank} \mathcal{J} = n - \dim X$.

### Blowups and the Resolving of Singularities

Some singularity types are possible to 'resolve' by performing a series of blowups at the singular locus. The idea is that, to any projective variety $X$, we may associate a smooth projective variety $\tilde{X}$ such that there is a birational map $\tilde{X} \xrightarrow{\Phi} X$. This was proved by Hironaka in a famous paper.

**Theorem 3.19.** (Hironaka's theorem on resolutions of singularities [Hir64])
*Let $B$ be a field of characteristic zero. If $X$ is an algebraic $B$-scheme, say reduced and irreducible, then there exists an algebraic subscheme $D$ of $X$ such that*

*(i) the set of points of $D$ is exactly the singular locus of $X$, and*

*(ii) if $f : \tilde{X} \to X$ is the blowup of $X$ along $D$, then $\tilde{X}$ is non-singular.*

The restriction of $\Phi$ is an isomorphism from $\Phi|_{\tilde{X} \setminus E} : \tilde{X} \setminus E \to X \setminus Z$, where $Z$ is the singular locus on $X$.

This interpretation of blowups is geometric and allows for a lot of intuition. It is however not hands-on, in the sense that we obtain tools to work with the blowup of a variety. This is remedied by the following identification of blowups with fibre products.

**Theorem 3.20.**
*Let $Y, Z$ be smooth projective varieties such that $Z \xhookrightarrow{closed} Y$. Then the following are true:*

a. *$E$ is the fibre product in the diagram:*

$$\begin{array}{ccc} E & \xrightarrow{k} & Bl_Z(Y) \\ \downarrow \phi & \ulcorner & \downarrow \pi \\ Z & \xhookrightarrow{closed} & Y \end{array}$$

*where $\pi : Bl_Z(Y) \to Y$ is the blow-up morphism and $k$ is the inclusion of the exceptional divisor into the blow-up.*

b. *Let the setting be that of a. Then $E = \mathbb{P}(\mathcal{N}_{Z/Y})$.*

This theorem will come in handy in Section 7.5, wherein we compute the total Chern class of the second somewhat tautological bundle $S_T(\mathcal{L}, 2)$.

# Chapter 4

# Linear Systems

Linear systems, also known as linear series, are among the simplest examples of moduli spaces. They are projectivizations of vector subspaces of $V = H^0(S, \mathcal{L})$, meaning that they take on the form of projective spaces. In our setting, the linear systems parameterize curves on a smooth projective surface. In this chapter, we briefly recall the common[1] theory about linear systems, before introducing conditions on the systems that are vital for the general setting of our thesis.

## 4.1 Common Theory of Linear Systems

### Conventions

Let $S$ be a smooth projective surface equipped with a line bundle $\mathcal{L}$. We write $|\mathcal{L}|$ for the complete linear system $\mathbb{P}(H^0(S, \mathcal{L}))$. By a linear sub-system of $|\mathcal{L}|$, we will always refer to a linear system $\mathbb{P}^\delta \subset |\mathcal{L}|$ of curves on $S$. We will refer to linear sub-systems of $|\mathcal{L}|$ for some line bundle $\mathcal{L}$ as *linear systems of* $|\mathcal{L}|$ or just linear systems, if there can be no confusion about their ambient space. We let the dimension of a linear system be its dimension as a scheme. A linear system of dimension 1, 2 or 3 is respectively called a *pencil*, a *net* or a *web*.

### Definitions and Common Theory

Let us recall the definition of a linear system. Given a divisor $D$ on a smooth projective variety $X$ (we may assume it is a Cartier divisor), we can define the complete linear system $|D|$:

$$\begin{aligned} |D| &= \{ D_0 \mid D_0 \text{ is a divisor linearly equivalent to } D \} \\ &= \{ D_0 \mid D_0 - D = \operatorname{div}(s), \text{where } s \in H^0(S, \mathcal{L}(D)) \}, \end{aligned}$$

where $\mathcal{L}(D)$ is the line bundle associated to the divisor $D$ and $\operatorname{div}(s)$ is the divisor induced by a rational function on $X$. By the following proposition, the set $|D|$ corresponds to the projectivization of the global sections of the line bundle associated to the divisor $D$.

**Proposition 4.1.** [Har77, Proposition II.7.7]
*Let $X$ be a smooth projective variety over the algebraically closed field $k$. Let $D$ be a divisor on $X$ and let $\mathcal{L} \cong \mathcal{L}(D)$ be the corresponding invertible sheaf. Then:*

[1] Here we mean common in the sense that it is covered in [Har77], allowing it to be considered 'general knowledge'.

a. *for each nonzero* $s \in \Gamma(X, \mathcal{L})$*, the divisor of zeros div*$(s)$ *is an effective divisor linearly equivalent to* $D$*;*

b. *every effective divisor linearly equivalent to* $D$ *is div*$(s)$ *for some* $s \in \Gamma(X, \mathcal{L})$*; and,*

c. *two sections* $s, s' \in \Gamma(X, \mathcal{L})$ *have the same divisor of zeros if and only if there is a* $\lambda \in k^*$ *such that* $s' = \lambda s$*.*

The Göttsche conjecture loosely states that under some constraints on our linear systems, the amount of $\delta$ nodal curves in said linear systems are finite and there is a formula for the amount of $\delta$ nodal curves in the linear system. The first of these constraints is a notion of generality of our choice of linear system. This is a common theme in intersection theory; the developed machinery is only applicable to 'good' cases. Picking a general linear system with regard to a property $\mathcal{P}$ in $|\mathcal{L}|$ is defined as follows.

**Definition 4.2.** (General linear systems w.r.t. a property)
Let $\dim |\mathcal{L}| = N - 1$, i.e. $|\mathcal{L}| = \mathbb{P}^{N-1}$. We say that *a property* $\mathcal{P}$ *holds for a general linear system* $\mathfrak{d} = \mathbb{P}^\delta$ if there exists a nonempty Zariski open $U \subset Gr(\delta + 1, N) = \{\mathfrak{d} \subset |\mathcal{L}| : \dim \mathfrak{d} = \delta\}$ where $\mathcal{P}$ holds for all $\mathbb{P}^\delta \in U$.

Here $Gr(k, n)$ denotes the Grassmannian whose points parameterize $k$-dimensional vector subspaces of $\mathbb{C}^n$.

## 4.2 Theory of Linear Systems Special to the Thesis Setting

We require the notion of $\delta$-very ampleness of a line bundle to impose the desired criterion on our linear systems that many results in this thesis rely upon.

### 4.2.1 $\delta$-Very Ample Linear Systems

In order to count the amount of singular curves in a linear system of curves on a smooth surface, one needs to impose a criterion on the ambient complete linear system $|\mathcal{L}|$. We recall that, for a non-singular projective variety $X$, *the length of a 0-dimensional subscheme* $Z \subset X$ is defined to be the dimension of $H^0(Z, \mathcal{O}_Z)$ as a $\mathbb{C}$-vector space. The criterion used by (but not invented by) Göttsche in [Got98] uses the notion of *$\delta$-very ampleness* of the line bundle $\mathcal{L}$.

**Definition 4.3.** ($\delta$-very ample line bundles & linear systems)
We say that a line bundle $\mathcal{L}$ on a smooth, projective variety $X$ is *$\delta$-very ample* if for every length $(\delta + 1)$-subscheme $Z \subset X$, the restriction map

$$H^0(X, \mathcal{L}) \to H^0(Z, \mathcal{L}|_Z) \overset{\text{def}}{=} H^0(Z, \iota^*(\mathcal{L}))$$

is surjective, where $Z \overset{\iota}{\hookrightarrow} X$ is the inclusion.
We say the complete linear system $|\mathcal{L}|$ *is $\delta$-very ample* whenever $\mathcal{L}$ is $\delta$-very ample.

**Remark 4.4.** An $n$-very ample line bundle is $k$-very ample whenever $n \geq k$. This follows from the fact that $(\delta + 1)$-very ample line bundles are $\delta$-very ample. Indeed, for a length $\delta$-subscheme $W$ of $X$ and all length $(\delta + 1)$ subschemes $Z$ containing it, the induced restriction $H^0(Z, \mathcal{L}|_Z) \to H^0(W, \mathcal{L}|_W)$ is surjective. This follows from the fact that $Z$ and $W$ may be realized as affine schemes $Z = \operatorname{Spec}(A)$ and $W = \operatorname{Spec}(A/I)$. The restriction map $H^0(Z, \mathcal{L}|_Z) = H^0(Z, \mathcal{O}_Z) \to H^0(Z, \mathcal{O}_W) = H^0(W, \mathcal{L}|_W)$ will therefore be the natural map $A \to A/I$, which is a surjection. Since the restriction map $H^0(S, \mathcal{L}) \to H^0(W, \mathcal{L}|_W)$ factors through all vector spaces $H^0(Z, \mathcal{L}|_Z)$ (the composition of two pullback maps is the pullback map of the composition), and the respective composition of these maps are surjective, it follows that the map $H^0(S, \mathcal{L}) \to H^0(W, \mathcal{L}|_W)$ is surjective. △

**Remark 4.5.** Being 0-very ample is the same as being globally generated, and being 1-very ample is the same as being very-ample. If $\mathcal{L}$ is $\ell$-very ample and $\mathcal{M}$ is $m$-very ample, then $\mathcal{L} \otimes \mathcal{M}$ is $(\ell + m)$-very ample [HTT05]. We have appended a neat little proof of the equivalence of the notions of 0-very ampleness and being globally generated, in A.2. △

Let us study some examples before explaining why the notion of $\delta$-very ampleness is crucial to our thesis.

**Example 4.6.** (Partial categorization of $\delta$-very ample line bundles of $\mathbb{P}^2$)
Let $S = \mathbb{P}^2$. If $\mathcal{L}$ is a line bundle on $S$, then $\mathcal{L} = \mathcal{O}_S(d)$ for some $d \in \mathbb{Z}$. The line bundle $\mathcal{O}_S(1)$ is clearly very ample, hence it is 1-very ample. By Remark 4.5, we get that $\mathcal{O}_S(d)$ is $d$-very ample. Thus for all $d \geq \delta$ the bundles $\mathcal{O}_{\mathbb{P}^2}(d)$ are $\delta$-very ample, by remark 4.4. △

**Example 4.7.** (A collection of canonical examples of $\delta$-very ample line bundles)
Let $Z \overset{\iota}{\hookrightarrow} \mathbb{P}^n$ be a closed immersion that corresponds to a line bundle $\mathcal{L}$ on $Z$. Then $\mathcal{L} = \iota^*(\mathcal{O}_{\mathbb{P}^n}(1))$, which by definition is 1-very ample. If $\mathcal{M}$ is any globally generated line bundle on $Z$, we have that the line bundle $\mathcal{L}^{\otimes \delta} \otimes \mathcal{M}^{\otimes k}$, for $\delta, k \in \mathbb{N}$, is $\delta$-very ample. The point is that the problem of finding a $\delta$-very ample line bundle is reducible to the problem of finding a very ample line bundle. △

The condition of $\mathcal{L}$ being $\delta$-very ample provides us with a description of curves in $|\mathcal{L}|$. This description presupposes a notion of generality on the linear systems we study, which we specified in Definition 4.2.

**Proposition 4.8.** [KST11, Prop 2.1]
*If $\mathcal{L}$ is $\delta$-very ample then the general $\delta$-dimensional linear system $\mathbb{P}^\delta \subset \mathbb{P}(H^0(\mathcal{L}))$ contains a finite number of $\delta$-nodal curves appearing with multiplicity 1. All other curves are reduced with geometric genus $\bar{g} > g - \delta$.*

Here $g$ denotes the arithmetic genus of a curve $C \subset S$, i.e. $g := g(C) = \dim_{\mathbb{C}} H^1(C, \mathcal{O}_C)$ and $\bar{g}$, the geometric genus of $C$, denotes the genus of its normalization $\bar{C} \to C$.

This an important prerequisite for the Göttsche conjecture, as it details the dimension of the linear systems of $|\mathcal{L}|$ that are of interest, as well as describing some properties the curves in the systems have. Kool, Shende & Thomas were by no means first in proving such a result, see for instance [Got98, Prop 5.2] or [KP99, Chapter 3], but they lowered the necessary bound of very-ampleness of $\mathcal{L}$ from $(5\delta - 1)$-very ample to $\delta$-very ample.

The proposition inspires the slogan: *"the ampler the line bundle $\mathcal{L}$ is, the more well behaved a general linear system in $|\mathcal{L}|$ is"*. We are not interested in computing lower bounds for the required degree $\delta$ of $\delta$-very ampleness that make our proofs work. To this end, we adapt the following convention.

**Convention 4.9.** (Sufficiently ample line bundle)
When a statement holds for a $\delta$-very ample line bundle (for some $\delta \in \mathbb{N}$), we might preface the statement by specifying $\mathcal{L}$ to be *sufficiently ample with regard to* the criteria or the assertion of the statement.

To illustrate this convention, we could have rewritten Proposition 4.8 to read "If $\mathcal{L}$ is sufficiently ample, then the general $\delta$-dimensional (...)".

It is convenient for us to define the subsets of $|\mathcal{L}|$ which consist of curves having have a specific number of nodes.

**Definition 4.10.** We define

$$|\mathcal{L}|_n = \{C \in |\mathcal{L}| \,:\, \text{C has } n \text{ or more nodes}\}.$$

It is possible to give $|\mathcal{L}|_n$ a natural scheme structure, whose singularities have been identified.

**Proposition 4.11.** (Some characteristics of $|\mathcal{L}|_n$ as a scheme)
*For sufficiently ample $|\mathcal{L}|$, we have that $|\mathcal{L}|_n$ is a scheme of codimension $n$ in $|\mathcal{L}|$. It is smooth at exactly the points that are curves with exactly n nodes (counted with multiplicity).*

*Proof.* The proof of this proposition is contained in the proof of [KST11, Proposition 2.1.]. □

We conclude this section by adapting a semantical convention regarding the choice of *general* global sections of a line bundle.

**Convention 4.12.** (General global sections of a line bundle)
When we define a general linear system $\mathbb{P}^\delta = \mathbb{P}\mathcal{W} \subset |\mathcal{L}|$ by giving sections of its corresponding vector space, we start by choosing $\mathcal{W} \subset H^0(S, \mathcal{L})$ so that $\mathbb{P}\mathcal{W}$ is general, before we select a set of sections which span $\mathcal{W}$. We will write a general linear system of dimension $\delta$ as $\mathbb{P}^\delta := \mathbb{P}\langle s_0, ..., s_\delta \rangle_{\mathbb{C}} = \mathbb{P}\mathcal{W}$.

**Remark 4.13.** The projectivization of the vector space in the above remark gives an equality of sets

$$\mathbb{P}\langle s_0, ..., s_k \rangle_{\mathbb{C}} = \{a_0 s_0 + a_1 s_1 + ... + a_k s_k \,|\, (a_0 : ... : a_k) \in \mathbb{P}^k_{\mathbb{C}}\}.$$

We use this equality for multiple elementary, albeit important arguments that regard the interpretation of the results of the thesis. △

### 4.2.2 $\Sigma_{\mathcal{W}}$, the Discriminant and the Incriminant

In studying a linear system $|\mathcal{L}| = \mathbb{P}^N$ of degree $d$ hypersurfaces of $\mathbb{P}^n$, it is convenient to work with a projective variety $\Sigma_{n,d}$ called the *universal singular point*. It is defined as

$$\Sigma_{n,d} = \{(Y, p) \in \mathbb{P}^N \times \mathbb{P}^n \,|\, p \in Y_{\text{sing}}\}$$

and has a natural scheme structure which is described, for instance, in [EH16, Chapter 7]. Let $\pi_1 : \Sigma_{n,d} \to \mathbb{P}^N$ and $\pi_2 : \Sigma_{n,d} \to \mathbb{P}^n$ be projections. The scheme $\mathcal{D} = \pi_1(\Sigma_{n,d})$ is called the *discriminant*, and is the subset of curves in $|\mathcal{L}|$ which are singular. The scheme $\mathcal{D}$ is in fact a variety.

**Example 4.14.** (Counting singular curves in a general pencil)
Let $S = \mathbb{P}^2_{\mathbb{C}}$ be equipped with a line bundle $\mathcal{L}$. The amount of singular curves on $S$ in a general pencil $\mathbb{P}^1 \subset |\mathcal{L}|$ equals the degree of the variety $\mathcal{D}$, by Bezout's theorem. Irreducibility of $\mathcal{D}$ was proven by Harris in [Har86]. △

Whereas the discriminant has been extensively studied, the image of the projection $\pi_2$ has not. We shall study the image of a morphism similar to $\pi_2$, in Chapter 6. To this end, we define a scheme similar to $\Sigma_{n,d}$ that better fits our setting.

**Definition 4.15.** Let $S$ be a smooth projective surface equipped with a line bundle $\mathcal{L}$. Let $\mathcal{W} \subset H^0(S, \mathcal{L})$ be a vector subspace. We define

$$\Sigma_{\mathcal{W}} = \{(C, p) \in \mathbb{P}\mathcal{W} \times S \,|\, p \in C_{\text{sing}}\}.$$

We let $\pi_1$ and $\pi_2$ be the natural projection maps, and denote by $\mathcal{D}_{\mathcal{W}} = \pi_1(\Sigma_{\mathcal{W}})$ the *discriminant of* $\mathcal{W}$, or simply the *discriminant* $\mathcal{D}$ when it is clear which linear system $\mathbb{P}\mathcal{W}$ we are referring to. We also introduce the *incriminant of* $\mathcal{W}$ as the scheme $\mathcal{I}_{\mathcal{W}} := \pi_2(\Sigma_{\mathcal{W}})$, limiting ourselves to just the *incriminant* $\mathcal{I}$ if the context allows for it. The term 'incriminant' is our own, and is not (to our knowledge) used in other literature.

**Remark 4.16.** (The incriminant as a set)
Let $\mathbb{P}\mathcal{W} \subset |\mathcal{L}|$ be a linear system. As a set, we have that

$$\mathcal{I}_{\mathcal{W}} = \{p \in S \,|\, \exists C \in \mathbb{P}\mathcal{W} \text{ such that } p \in C_{\text{sing}}\}.$$

△

This remark will prove useful in Chapter 6, wherein we study the classes of the first somewhat tautological bundle $S_T(\mathcal{L}, 1)$.

# Part II

# Hilbert Schemes

# Chapter 5

# Hilbert Schemes of Points on Surfaces

The theory of Hilbert schemes of points on surfaces is widely and actively studied. The construction of Hilbert schemes in general is quite explicit, allowing us great insight into their scheme structure. It is therefore unfortunate that most Hilbert schemes are difficult to work with. There are, however, some Hilbert schemes which are quite well-behaved; the Hilbert schemes of points on a surface being among the prime examples.

Hilbert schemes of points on surfaces are smooth varieties. The Hilbert scheme $S^{[n]}$ of points parametrizes 0-dimensional subschemes of $S$ of length $n$, i.e. subschemes $Z \subset S$ such that $\dim_{\mathbb{C}} H^0(Z, \mathcal{O}_Z) = \sum_{p \in \mathrm{Supp}(Z)} \dim(\mathcal{O}_{Z,p}) = n$. The variety $S^{[n]}$ is a resolution of singularities of $S^{(n)}$, which is the variety $S^n/\mathfrak{S}_n$, or in other words, the $n^{\text{th}}$ order product of $S$ modded out by the group action of the symmetric group on $n$ elements.

We are primarily interested in the Hilbert scheme $S^{[2]}$. The points of $S^{[2]}$ generically parametrize pairs of points, and constitute a convenient manner of storing the data of the singular locus of curves with two singularities.

## 5.1 Preliminaries

### 5.1.1 Introductory Theory

We state one of the central theorems for Hilbert schemes of points on smooth surfaces.

**Theorem 5.1.** [Fog68, Theorem 2.4]
*Let $S$ be a smooth surface over the field $k$. Then $S^{[n]}$ is a smooth scheme of dimension $2n$.*

The theorem is vital to the thesis, as it ensures that the Chow group $A(S^{[n]})$ has a well defined ring-structure. The final computations of Chapters 6 and 7 are all performed in these rings.

All Hilbert schemes come equipped with a so called *universal family.* This is a scheme $\mathcal{Z}_n(S) \xhookrightarrow{\textit{closed}} S \times \mathrm{Hilb}_{S/\operatorname{Spec}\mathbb{C}}$ that is flat over $\mathrm{Hilb}_{S/\operatorname{Spec}\mathbb{C}}$. In fact, any scheme $X$ over $\operatorname{Spec}\mathbb{C}$ that maps into $\mathrm{Hilb}_{S/\operatorname{Spec}\mathbb{C}}$, or more specifically, into $S^{[n]}$, will have a flat family over it.

**Remark 5.2.** There is a bijection between the sets

$$\mathrm{Hom}_{\mathrm{sch}/\mathbb{C}}(X, S^{[n]}) \xleftrightarrow{\text{1-1}} \{Z \xhookrightarrow{\text{closed}} X \times S \mid Z \text{ is flat over } X \text{ and } \forall x \in X,$$
$$q^{-1}(x) \subset X \times S \text{ is of length } n \text{ and } q \text{ is the projection map}\}.$$

In practice, this means that giving a morphism $f : X \to S^{[n]}$ from an arbitrary scheme $X$ into the Hilbert scheme $S^{[n]}$ is equivalent to giving a scheme $Z_f$ that fits into the diagram immediately below.

$$\begin{array}{ccc} Z_f & \xhookrightarrow{closed} & X \times S \\ {\scriptstyle q}\big\downarrow {\scriptstyle flat} & & \\ X & & \end{array} \tag{5.1}$$

It is well known that, given such a morphism $f$, this flat scheme equals the fibre product $Z_f := \mathcal{Z}_n(S) \times_{S^{[n]}} X$ in the below diagram.

$$\begin{array}{ccc} Z_f & \xrightarrow{f'} & \mathcal{Z}_n(S) \\ \big\downarrow {\scriptstyle q'} & \ulcorner & \big\downarrow {\scriptstyle q} \\ X & \xrightarrow{f} & S^{[n]} \end{array} \tag{5.2}$$

We remark that this is just a reformulation of the fact that the Hilbert scheme $S^{[n]}$ represents the functor on the right hand side of the bijection above. See [Göt05, Chapter 5] or [Ric23, Chapter 5.2] for a comprehensive explanation of these facts, as well as a general introduction to Hilbert schemes. △

**Remark 5.3.** The universal family can, as a set, be described as

$$\mathcal{Z}_n(S) = \{(Z, x) \mid x \in Z\} \subset S^{[n]} \times S,$$

or equivalently, by the fact that $Z \subset S$ is the fiber of $Z \in S^{[n]}$. △

### 5.1.2 The Connection Between Hilbert Schemes and the Göttsche Conjecture

The connection of Hilbert schemes of points with the Göttsche conjecture stems from Göttsche's article [Got98]. Göttsche studies the bundle[1] $\mathcal{L}_{[\delta]}$ (a so called *tautological bundle*) associated to a sufficiently ample line bundle $\mathcal{L}$, and shows that the integral $d_n(\mathcal{L})$ of a particular Chern class of $\mathcal{L}_{[\delta]}$ equals the amount $t^S_\delta(\mathcal{L})$ of $\delta$-nodal curves on $S$ in a general $\delta$-dimensional linear system of $|\mathcal{L}|$. It is useful to do some bookkeeping before we showcase how Göttsche connects the theory of Hilbert schemes of points on surfaces to the conjecture that bears his name.

Define the projections $p_n : \mathcal{Z}_n(S) \to S$, $q_n : \mathcal{Z}_n(S) \to S^{[n]}$ giving us the below diagram.

$$\begin{array}{ccc} \mathcal{Z}_n(S) & \xrightarrow{p_n} & S \\ \big\downarrow {\scriptstyle q_n} & & \\ S^{[n]} & & \end{array} \tag{5.3}$$

The importance of the tautological bundles merits a definition environment.

[1]The first construction of this vector bundle precedes Göttsche's article by many years. It is for instance mentioned in [Ful84, Example 2.5.6]. Fulton also cites earlier articles where the bundle is studied.

**Definition 5.4.** (The tautological bundle on $S^{[n]}$ associated to a line bundle)
Let $\mathcal{L}$ be a line bundle on a smooth projective surface $S$. We define the *tautological bundle* $\mathcal{L}_{[n]}$ on $S^{[n]}$ to be

$$\mathcal{L}_{[n]} := (q_n)_*(p_n)^*(\mathcal{L}),$$

where the morphisms are given in Diagram 5.3.

Let us verify that the sheaves $\mathcal{L}_{[n]}$ indeed are locally free.

**Remark 5.5.** $\left(\mathcal{L}_{[n]} \text{ is locally free of rank } n\right)$
The bundle $\mathcal{L}_{[n]}$ is locally free of rank $n$. Indeed, the pullback $p_n^*(\mathcal{L})$ of $\mathcal{L}$ along a morphism $p_n$ is a line bundle. The morphism $q_n$ being finite and flat implies $(q_n)_*(p_n)^*(\mathcal{L})$ is locally free. We have appended a proof in A.1 for the fact that the fibre of $\mathcal{L}_{[n]}$ at a point $Z \in X^{[n]}$ is the vector space $H^0(Z, \mathcal{L}|_Z)$, which implies $\mathcal{L}_{[n]}$ has rank $n$, since $H^0(Z, \mathcal{L}|_Z) \cong H^0(Z, \mathcal{O}_Z)$, and $Z \subset S$ is of length $n$. △

The Chern classes of tautological bundles associated to line bundles are known.

**Remark 5.6.** $\left(\text{The Chern classes of } \mathcal{L}_{[n]}\right)$
The Chern classes of $\mathcal{L}_{[n]}$ were computed by Manfred Lehn in [Leh99, Theorem 4.6]. As we shall see in Chapter 7, the Chern classes of the second somewhat tautological bundle can be expressed as a product that includes the total Chern class $c_\bullet(\mathcal{L}_{[n]})$ as a term, making the article by Lehn a crucial reference for this thesis. △

Definition 5.4 is clearly generalizable from the line bundle case to the case of locally free sheaves of finite rank.

**Remark 5.7.** (Tautological Bundles in General)
Suppose $\mathcal{E}$ is a locally free sheaf of rank $r$ on $S$, then $\mathcal{E}_{[n]} = (q_n)_*(p_n)^*\mathcal{E}$ is a locally free sheaf, as $q_n$ is finite and flat. In the article [EGL99], Ellingsrud, Göttsche and Lehn give an algorithm that express the Chern classes of the bundle $\mathcal{E}_{[n]}$ in terms of the classes of $\mathcal{E}$. Thus, whenever one knows the total Chern class of $\mathcal{E}$, one theoretically knows the classes of $\mathcal{E}_{[n]}$. An unfortunate fact is that the algorithm is somewhat inefficient, making it impractical to compute the classes of $\mathcal{E}_{[n]}$ for locally free sheaves in general. Nonetheless, we will assume the classes of $\mathcal{E}_{[n]}$ to be known whenever $c_\bullet(\mathcal{E})$ is known. In the main computation of Chapter 7, we derive an expression of $c_\bullet(S_T(\mathcal{L}, 2))$ that contains two tautological bundles. △

We are now ready to describe the setting in which Göttsche asserts the connection between Hilbert schemes of points on surfaces and his conjecture.

**Definition 5.8.** [Got98, Def 5.19]
Let $S_2^\delta \subset S^{[3\delta]}$ be the closure (with the reduced induced structure) of the locally closed subset $S_{2,0}^\delta$ which parametrizes subschemes of the form

$$\amalg_{i=1}^{\delta} \operatorname{Spec}(\mathcal{O}_{S,x_i}/m_{S,x_i}^2),$$

where $x_1, ..., x_\delta$ are distinct points in $S$. It is easy to see that $S_2^\delta$ is birational to $S^{[\delta]}$. We put $d_n(\mathcal{L}) := \int_{{S_2}^\delta} c_{2\delta}(\mathcal{L}_{[3\delta]})$.

Göttsche then goes on to prove the proposition which connects the theory of punctual Hilbert schemes of smooth surfaces to the question of counting $\delta$-nodal curves in a general $\delta$-dimensional linear system.

**Proposition 5.9.** [Got98, Prop 5.2]
*Assume $\mathcal{L}$ is $(3\delta - 1)$-very ample. Then a general $\delta$-dimensional sub-linear system $V \subset |\mathcal{L}|$ contains precisely $d_n(\mathcal{L})$ curves with $C_i$ with $\geq \delta$ singularities. If furthermore $\mathcal{L}$ is $(5\delta - 1)$-very ample (5-very ample if $\delta = 1$), then the $C_i$ have precisely $\delta$ nodes as singularities, i.e. $d_n(\mathcal{L}) = t^S_\delta(\mathcal{L})$.*

Since the publication of [Got98], recursive formulae have been found to compute the aforementioned polynomials. See for instance [KST11, Chapter 4].

## 5.2 Applications

The terminology ‘somewhat tautological’ is our own, as are the proposition and examples below. The name was chosen to reflect the close relationship the sheaves have with the tautological bundles. The construction of the somewhat tautological sheaves only differs by pulling back to and pushing forward from the *doubling* of the universal family, rather than the universal family itself.

### 5.2.1 Somewhat Tautological Sheaves

Let us give life to the words *somewhat tautological* by defining the sheaves our thesis revolves around. Defining the sheaves requires the notion of *doubling*[2] a closed subscheme.

**Definition 5.10.** (The doubling of a closed subscheme)
Let $X$ be a scheme and let $Z \subset X$ be the closed subscheme associated with the ideal sheaf $\mathcal{I}_Z$. We define the *doubling of $Z$* to be the subscheme $Z_2 \subset X$ associated with the ideal sheaf $\mathcal{I}_Z^2$.

The notion of doubling is intrinsically related to singularities of curves.

**Lemma 5.11.** *Let $p \in C$ be a closed point of the curve $C \subset S$. Let $\mathcal{I}_C \subset \mathcal{O}_S$ be the ideal sheaf associated to $C$, and let $\mathcal{I}_p \subset \mathcal{O}_S$ be the ideal sheaf associated to the point $p$. Then*

$$p \in C_{sing} \iff \mathcal{I}_C \subset \mathcal{I}_p^2 \subset \mathcal{O}_S$$

*Proof.* Let $\mathfrak{m}_{p,S}$ and $\mathfrak{m}_{p,C}$ denote the maximal ideals of the local rings of $p$ in $C$ and $S$ respectively, i.e. $\mathfrak{m}_{p,S}$ is the maximal ideal of the stalk of $\mathcal{I}_p$ at $p$ and $\mathfrak{m}_{p,C}$ the maximal ideal of the stalk of $\mathcal{I}_p/\mathcal{I}_p\mathcal{I}_C$ at $p$. Since $j_*\mathcal{O}_C = \mathcal{O}_S/\mathcal{I}_C$, where $j : C \hookrightarrow S$ is the inclusion, we have that the maximal ideal in the local ring of $p$ in $C$ is $\mathfrak{m}_{p,C} = \mathfrak{m}_{p,S}/\mathcal{I}_{C,p}$ and its square is $\mathfrak{m}_{p,C}^2 = (\mathfrak{m}_{p,S}^2 + \mathcal{I}_{C,p})/\mathcal{I}_{C,p}$. We have the following isomorphisms of $\mathbb{C}$-vector spaces

$$\mathfrak{m}_{p,C}/\mathfrak{m}_{p,C}^2 \cong (\mathfrak{m}_{p,S}/\mathcal{I}_{C,p})/((\mathfrak{m}_{p,S}^2 + \mathcal{I}_{C,p})/\mathcal{I}_{C,p}) \cong \mathfrak{m}_{p,S}/(\mathfrak{m}_{p,S}^2 + \mathcal{I}_{C,p}). \tag{5.4}$$

Suppose $\mathcal{I}_C \subset \mathcal{I}_p^2$. Then $\mathcal{I}_{C,p} \subset \mathfrak{m}_p^2$, so equation (5.4) becomes

$$\mathfrak{m}_{p,C}/\mathfrak{m}_{p,C}^2 \cong \mathfrak{m}_{p,S}/\mathfrak{m}_{p,S}^2.$$

Since $\mathfrak{m}_{p,S}/\mathfrak{m}_{p,S}^2$ is the Zariski cotangent space of $S$, it is of dimension 2. This implies the Zariski cotangent space of $p$ in $C$ has dimension 2, so $C$ is singular at $p$.

[2] I have not encountered this terminology in the literature, but it seems too intuitive to be novel.

Suppose $\mathcal{I}_C \not\subset \mathcal{I}_p^2$. Then $\mathcal{I}_{C,p} \not\subset \mathfrak{m}_{p,C}^2$. Hence we must have

$$\mathfrak{m}_{p,S}/(\mathfrak{m}_{p,S}^2 + \mathcal{I}_{C,p}) \subsetneq \mathfrak{m}_{p,S}/(\mathfrak{m}_{p,S}^2)$$

implying a strict inequality of their dimension as $\mathbb{C}$-vector spaces. Therefore,

$$1 \leq \dim_{\mathbb{C}}(\mathfrak{m}_{p,C}/\mathfrak{m}_{p,C}^2) = \dim_{\mathbb{C}} \mathfrak{m}_{p,S}/(\mathfrak{m}_{p,S}^2 + \mathcal{I}_{C,p}) < \dim_{\mathbb{C}} \mathfrak{m}_{p,S}/(\mathfrak{m}_{p,S}^2) = 2$$

which implies $C$ is not singular at $p$. □

In geometrical terms, the lemma tells us the following.

**Corollary 5.12.** *Let $C$ be a curve on $S$, and $p \in C$. Then $C$ is singular at $p$ if and only if $C$ contains the doubling $p_2$ of the point $p$.*

We shall require an appropriate generalization of this corollary to points of the Hilbert scheme $S^{[n]}$ in order to prove a central property the somewhat tautological sheaves $S_T(\mathcal{L}, n)$ exhibit. To this end, we investigate what happens when one doubles a point $Z \in S^{[n]}$ that is supported on $n$ points of $S$.

**Lemma 5.13.** *Suppose $Z = p_1 \sqcup ... \sqcup p_r \subset S$ is a disjoint union of $r \in \mathbb{N}$ closed points on a smooth projective surface $S$. Then the doubling $Z_2$ of $Z$ is the disjoint union of the doubling of points.*

*Proof.* Let $S$ be a smooth projective surface. We shall use the following fact. If $X, Y \subset S$ are disjoint closed subschemes, then their ideal sheaves satisfy the identity $\mathcal{I}_{Y \sqcup W} = \mathcal{I}_Y \mathcal{I}_W$. Denoting the doubling of the point $p_i$ as $p_i^2$ (breaking the notational convention), we have that the ideal sheaf of the disjoint union of doubled points satisfies

$$\mathcal{I}_{p_1^2 \sqcup ... \sqcup p_r^2} = \mathcal{I}_{p_1^2} \cdots \mathcal{I}_{p_1^2} \stackrel{\text{def}}{=} (\mathcal{I}_{p_1}^2 \cdots \mathcal{I}_{p_r}^2) = (\mathcal{I}_{p_1} \cdots \mathcal{I}_{p_r})^2 = \mathcal{I}_{p_1 \sqcup ... \sqcup p_r}^2 = \mathcal{I}_Z^2 \stackrel{\text{def}}{=} \mathcal{I}_{Z_2},$$

where $\stackrel{\text{def}}{=}$ signifies use of the definition of doubling. □

Applying Corollary 5.12 to the scheme $Z$ in Lemma 5.13 gives us the appropriate generalization of the former.

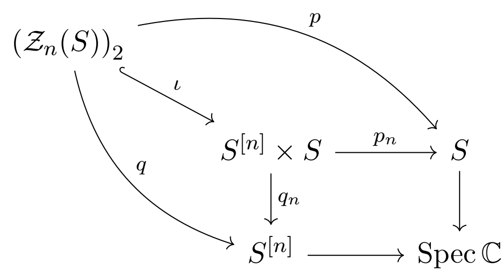
**Corollary 5.14.** *Let $C$ be a curve on $S$, and let $p_1, ..., p_r \in C$ be closed points. Then $C$ is singular at the points $p_1, ..., p_r$ if and only if $C$ contains the scheme $Z_2 = p_1^2 \sqcup ... \sqcup p_r^2$, where $p_i^2$ denotes the doubling of the point $p_i$.*

*Proof.* The curve $C$ containing $Z_2$ is the same as $C$ containing the doubling of the points of $p_1, ..., p_r$, which is equivalent to being singular at all the points $p_i$, by Corollary 5.12. □

It is instructive to get a notion of how doubling affects the length of 0-dimensional subschemes.

**Example 5.15.** (The doubling of a reduced point has length 3)
Let $p \in S$ be a point. Since $S$ is smooth, its local rings are regular. Let $\mathfrak{m}_p$ denote the maximal ideal in the local ring $\mathcal{O}_{S,p}$. By [Stacks, Tag 00NO] there is an isomorphism $\bigoplus_{i=0} \mathfrak{m}_p^i/\mathfrak{m}_p^{i+1} \stackrel{\phi}{\cong} \mathcal{O}_{S,p}/\mathfrak{m}_p[x,y] = \mathbb{C}[x,y]$ that allows us to choose $x', y' \in \mathfrak{m}_p/\mathfrak{m}_p^2$ such that $\phi(x') = x$ and $\phi(y') = y$. The doubling of $p$ is given by the scheme $\operatorname{Spec} \mathcal{O}_{S,p}/\mathfrak{m}_p^2$, which by the above equals the vector space $\mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}y$. Hence $\dim_{\mathbb{C}} H^0(p_2, \mathcal{O}_{p_2}) = 3$, meaning $p_2$ is of length 3. △

We shall use the setting of Diagram 5.5 to define the somewhat tautological sheaves associated to $\mathcal{L}$. The square in the diagram is the fibre product over $\operatorname{Spec}\mathbb{C}$, with $\iota$ being the inclusion and $p_n, q_n$ are the natural projections with $p, q$ being their respective restrictions.



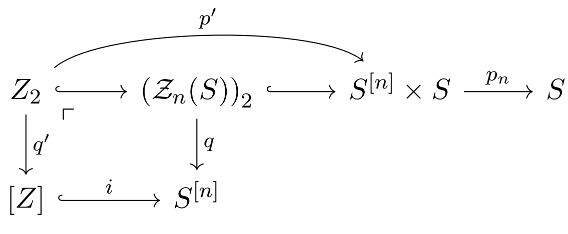

(5.5)

**Definition 5.16.** (Somewhat tautological sheaves associated to a line bundle $\mathcal{L}$)
Let $S$ be a smooth projective surface equipped with a line bundle $\mathcal{L}$, and let the setting be that of Diagram 5.5. The $n^{th}$ *somewhat tautological sheaf* $S_T(\mathcal{L}, n)$ associated to $\mathcal{L}$ is defined as $S_T(\mathcal{L}, n) := q_* p^* \mathcal{L}$.

Naturally, the first question we must ask ourselves is: why are these sheaves interesting; what information do they possess? One answer that is fruitful for our endeavour is that the sheaf $S_T(\mathcal{L}, n)$ contains the data of whether the curve $C = V(s)$ corresponding to a section $s \in H^0(S, \mathcal{L})$ is singular at $n$ distinct points of $S$.

**Proposition 5.17.**
*Let $S$ be a smooth projective surface with a line bundle $\mathcal{L}$. Let $s \in H^0(S, \mathcal{L})$ and define $C = V(s)$. The curve $C$ is singular at $n$ distinct points $p_1, ..., p_n$ of $S$ if and only if its induced section $\hat{s} \in H^0(S^{[n]}, S_T(\mathcal{L}, n))$ vanishes when restricted to the point $Z = p_1 \sqcup \cdots \sqcup p_n \in S^{[n]}$.*

*Proof.* We compute the fibres of the sheaf $S_T(\mathcal{L}, n)$. Consider Diagram 5.6.




(5.6)

We start by showing that the fibre of $S_T(\mathcal{L}, n)$ at a point $[Z] \in S^{[n]}$ equals $H^0(Z_2, \mathcal{L}|_{Z_2})$, where $Z_2$ is the doubling of $Z$ as a subscheme of $S$. The fibre of $S_T(\mathcal{L}, n)$ at $Z$ is its pullback to $[Z]$ along $i$. We claim that $i^* S_T(\mathcal{L}, n) = q'_* p'^* (p_n^* \mathcal{L})$. Since the diagram commutes and $q$ is finite (as we shall demonstrate in Remark 5.18), we may reduce the claim to proving it on affine opens. Let $\operatorname{Spec} A \subset S^{[n]}$ be an open containing the point $[Z]$. Since $q$ is finite, hence affine, there is a $\mathbb{C}$-algebra $B$ such that $q^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B$, giving us the diagram

$$\begin{array}{ccccc} \operatorname{Spec}(A_{\mathfrak{m}_Z}/\mathfrak{m} \otimes_A B) & \xrightarrow{j} & \operatorname{Spec} B & \xrightarrow{p} & \operatorname{Spec} R \\ \downarrow{\scriptstyle q'} & \ulcorner & \downarrow{\scriptstyle q} & & \\ \operatorname{Spec}(A_{\mathfrak{m}_Z}/\mathfrak{m}) & \xhookrightarrow{i} & \operatorname{Spec} A & & \end{array} \tag{5.7}$$

where $\operatorname{Spec} R \subset S$ and $A_{\mathfrak{m}_Z}/\mathfrak{m}$ is the local ring of $[Z]$ in $\operatorname{Spec} A$ modded out with its maximal ideal $\mathfrak{m}$. Let $L$ be the $R$-module such that $\mathcal{L} = \widetilde{L}$. Then $p^*\mathcal{L} = \widetilde{L \otimes_R B}$. By the properties of the '$\sim$'-functor w.r.t pullback and pushforward of quasi-coherent sheaves on Noetherian schemes, we obtain the expressions

$$i^* q_* p^* \mathcal{L} = (L \otimes_R \widetilde{B)_A \otimes_A} A_{\mathfrak{m}_Z}/\mathfrak{m}; \tag{5.8}$$

$$q'_* j^* p^* \mathcal{L} = (L \otimes_R A_{\mathfrak{m}_Z} \widetilde{/\mathfrak{m} \otimes_A} B)_{A_{\mathfrak{m}_Z}/\mathfrak{m}}. \tag{5.9}$$

But since $B$ is an $A$-algebra, and $A_{\mathfrak{m}_Z}/\mathfrak{m} \cong \mathbb{C}$ and all of these rings are $\mathbb{C}$-algebras, we have that

$$q'_* j^* p^* \mathcal{L} = L \otimes_R \widetilde{B \otimes_A} \mathbb{C} = i^* q_* p^* \mathcal{L},$$

confirming the claim. Since $i^* S_T(\mathcal{L}, n) = q'_* p'^*(p_n^* \mathcal{L})$, we have following chain of equalities

$$\begin{aligned} q'_* p'^*(p_n^* \mathcal{L}) &= q'_* p'^*(\mathcal{L}|_{S^{[2]} \times S}) \\ &= q'_*(\mathcal{L}|_{Z_2}) = H^0(Z_2, \mathcal{L}|_{Z_2}), \end{aligned}$$

where the final equality follows from the fact that pushing forward to a point corresponds to taking global sections. Identifying the fibres of $S_T(\mathcal{L}, n)$ reveals what happens when one restrict a global section of $S_T(\mathcal{L}, n)$ to a point of $S^{[n]}$, which we shall demonstrate now. Recall first that

$$\mathcal{L}|_{Z_2} = \mathcal{L} \otimes \mathcal{O}_S / \mathcal{I}_{Z_2} = \mathcal{L}/\mathcal{I}_{Z_2}\mathcal{L}.$$

If the induced section $\widehat{s} \in H^0(S, \mathcal{L}/\mathcal{I}_{Z_2}\mathcal{L})$ of a section $s \in H^0(S, \mathcal{L})$ vanishes, then it comes from $\mathcal{I}_{Z_2}\mathcal{L}$. But the section $s$ comes from from the sheaf $\mathcal{I}_{Z_2}\mathcal{L}$ if and only if the curve $C = V(s)$ contains the scheme $Z_2$. It then follows from Corollary 5.14 that $C$ is singular at the points in $\operatorname{Supp} Z$. This completes the proof. □

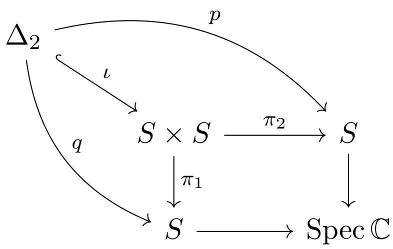
**Remark 5.18.** The morphism $q$ in diagrams (5.5) & (5.6) is finite for all $n$. Indeed, since $q$ is projective, we must only verify that its fibres are finite sets. The morphism $\alpha$ (which is just the restriction of the projection $q_n$) is finite, by Remark 5.3. Furthermore, since $\mathcal{Z}_n(S)$ and $(\mathcal{Z}_n(S))_2$ are schemes defined on the same underlying set, the map $\beta$ must be bijective, since it is an immersion.

$$\begin{array}{ccc} (\mathcal{Z}_n(S))_2 & & \\ {\scriptstyle \text{closed}}\uparrow{\scriptstyle \beta} & \searrow^{\iota} & \\ \mathcal{Z}_n(S) & \xhookrightarrow{\text{closed}} & S^{[2]} \times S \\ \downarrow{\scriptstyle \alpha} & \swarrow_{q_n} & \\ S^{[n]} & & \end{array} \tag{5.10}$$

A diagram chase of the commutative diagram (5.10) then reveals that the fibres of $q = q_n \circ \iota$ are finite. △

The observant reader will have realized that Proposition 5.17 is remarkably similar to point $b$ in Theorem 3.12 (the jet bundle theorem). This is by no means surprising; it turns out that the first jet bundle associated to $\mathcal{L}$ is a somewhat tautological sheaf.

**Example 5.19.** (The first jet bundle is a somewhat tautological sheaf)
We work over the 1st Hilbert scheme of points on $S$, which is just $S$ itself. The universal family lying flat over $S$ is $\Delta \subset S \times S$, the diagonal. This fact is well known, and is for instance stated in [Leh99, Chapter 1.2]. Denote by $\Delta_2$ the doubling of the diagonal. Then, with notation as in Diagram 5.11, we have that $J^1\mathcal{L} = q_* p^* \mathcal{L} \overset{\text{def}}{=} S_T(\mathcal{L}, 1)$, which we shall now verify.



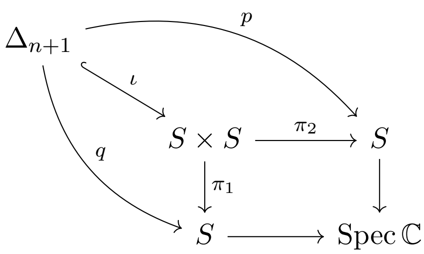

(5.11)

Diagram 5.11 commutes, since we are taking the fibre product of $S$ with itself. We must show that $q_* p^* \mathcal{L} = \pi_{1_*}(\mathcal{O}_{Y\times Y}/\mathcal{I}_\Delta^2 \otimes \pi_2^*\mathcal{L}) \overset{\text{def}}{=} J^1\mathcal{L}$. This follows from the set of equalities below, where the second equality is an application of the projection formula.

$$\begin{aligned}
\pi_{1_*}(\mathcal{O}_{Y\times Y}/\mathcal{I}_\Delta^2 \otimes \pi_2^*\mathcal{L}) &= \pi_{1_*}(\iota_* \mathcal{O}_{\Delta_2} \otimes \pi_2^*\mathcal{L}) \\
&= \pi_{1_*}\iota_*(\mathcal{O}_{\Delta_2} \otimes \iota^*\pi_2^*\mathcal{L}) \\
&= \pi_{1_*}\iota_*(\iota^*\pi_2^*\mathcal{L}) \\
&= (\pi_1 \circ \iota)_*(\pi_2 \circ \iota)^*\mathcal{L} = q_* p^* \mathcal{L}.
\end{aligned}$$

△

**Remark 5.20.**
We remark that all jet bundles may be expressed in this form. Substitute Diagram 5.11 with Diagram 5.12 and observe that the computation in Example 5.19 is equally valid when we substitute $n+1$ for 2.




(5.12)

What makes the case $n+1 = 2$ stand out is that we are interested in the somewhat tautological sheaves, which are defined via doubling and not '$n+1$-tupling'. △

The identification of the first somewhat tautological sheaf with a vector bundle raises a question: are all of the sheaves $S_T(\mathcal{L}, n)$ locally free? The answer is no. It turns out to be true for $n \in \{1, 2\}$. The crux of this fact is that the doubling of the universal families $\mathcal{Z}_n(S)$ are not flat over $S^{[n]}$ in general. Thus pushing a locally free sheaf forward from these schemes does not necessarily give a locally free sheaf. We verify that $(\mathcal{Z}_3(S))_2$ is not flat over $S^{[3]}$ in example 5.23. Let us verify that $S_T(\mathcal{L}, 2)$ is locally free.

**Corollary 5.21.**
*$S_T(\mathcal{L}, n)$ is a locally free sheaf of rank $3n$ for $n \in \{1, 2\}$.*

*Proof.* We have already shown that the corollary holds for $n = 1$, since $S_T(\mathcal{L}, 1) = J^1\mathcal{L}$. Let therefore $n = 2$. The pullback of a line bundle is a line bundle, implying $p^*\mathcal{L}$ is locally free of rank one. We argue that $q$ is flat and finite, which would imply that $q_*(p^*\mathcal{L})$ is locally free. The rank is then determined by studying the dimension of the fibres of $S_T(\mathcal{L}, 2)$, which by the proof of Proposition 5.17, equals $\dim H^0(Z_2, \mathcal{L}|_{Z_2}) = \dim H^0(Z_2, \mathcal{O}_{Z_2})$.

The morphism $q$ is finite by Remark 5.18. We now argue that $q$ is flat. By [Har77, Theorem III.9.9] it suffices to show that the fibres $((\mathcal{Z}_2(S))_2|_p$ for all $p \in S^{[2]}$ have the same length. We claim that $Z_2$ is of length 6, whenever $Z$ is of length 2. Proving this claim will assert that the scheme $(\mathcal{Z}_2(S))_2$ corresponds to a morphism $h : S^{[2]} \to S^{[6]}$. Indeed, the morphism $h$ will map a point $Z \in S^{[2]}$ to its doubling $Z_2$.

Assume $Z \in S^{[2]}$. There are two cases: either $Z = P \sqcup Q$ for $P, Q \in S$ or $\operatorname{Supp} Z = \{P\}$. In the first case, we have that $Z_2 = P_2 \sqcup Q_2$ by Lemma 5.13. Since this is a disjoint union of affine schemes, we have that $H^0(Z_2, \mathcal{O}_{Z_2}) = H^0(P_2, \mathcal{O}_{P_2}) \oplus H^0(Q_2, \mathcal{O}_{Q_2})$. This vector space has dimension 6, by example 5.15.

Now suppose $\operatorname{Supp} Z = \{P\}$. In this case, $Z \subset S$ is a non-reduced point, meaning it satisfies $\mathfrak{m}_P^2 \subsetneq I(Z) \subsetneq \mathfrak{m}_P$, where $I(Z)$ is the ideal that corresponds to $Z$. We are done if we manage to show that $\dim_{\mathbb{C}}(\mathcal{O}_{S^{[2]},P}/I(Z)^2) = 6$. Since $S^{[2]}$ is smooth, it is regular. Hence we may choose a basis $\{x', y'\}$ of the vector space $\mathfrak{m}_p/\mathfrak{m}_p^2 \cong \mathbb{C}^2$, where $x', y'$ generate $\mathfrak{m}_p$. Since the inclusion relations $\mathfrak{m}_P^2 \subsetneq I(Z) \subsetneq \mathfrak{m}_P$ are strict, we can write $I(Z) = \mathfrak{m}_p^2 + (f)$, where $f \in (x', y')$, but with an appropriate choice of $x', y'$ we can express $I(Z) = \mathfrak{m}_p^2 + (x')$. Furthermore we have the short exact sequence of vector spaces

$$0 \longrightarrow I(Z)^2/\mathfrak{m}_P^4 \longrightarrow \mathcal{O}_{S^{[2]},P}/\mathfrak{m}_P^4 \longrightarrow \mathcal{O}_{S^{[2]},P}/I(Z)^2 \longrightarrow 0$$

which reduces the problem of finding the length of $Z$ to computing the lengths of $R/\mathfrak{m}_p^4$ and $I(Z)^2/\mathfrak{m}_P^4$. By [Stacks, Tag 00NO], regularity of $S^{[n]}$ implies that $\mathcal{O}_{S^{[2]},P}/\mathfrak{m}_P^4 \cong \mathbb{C}[x, y]/(x, y)^4$. A basis of this $\mathbb{C}$-vector space is $\{1, x, y, ..., , xy^2, y^3\}$, meaning it is of dimension 10. Now consider $I(Z)^2/\mathfrak{m}_P^4$. Since $\mathfrak{m}_P = (x', y')$, we may express $I(Z) = (x'^2, x'y', y'^2) + (x') = (x', y'^2)$. These identifications give us

$$I(Z)^2/\mathfrak{m}_P^4 = (x', y'^2)^2/(x', y')^4 = \mathbb{C}x'^2 + \mathbb{C}x'y'^2 + \mathbb{C}x'^2y' + \mathbb{C}x'^3,$$

which is a four dimensional vector space. Hence, $\mathcal{O}_{S^{[2]},P}/I(Z)^2$ is of dimension six, meaning $Z_2$ has length 6. □

**Convention 5.22.** (Somewhat Tautological Bundles)
When referring to the somewhat tautological bundles (associated to a line bundle $\mathcal{L}$), we refer to the sheaves $S_T(\mathcal{L}, 1) = J^1\mathcal{L}$ and $S_T(\mathcal{L}, 2)$.

**Example 5.23.** $\Big((\mathcal{Z}_3(S))_2$ is not flat over $S^{[3]}\Big)$
Fix $S = \mathbb{P}^2_{\mathbb{C}} = \operatorname{Proj}\mathbb{C}[x_0, x_1, x_2]$ and choose coordinates $D_+(x_2) = \operatorname{Spec}\mathbb{C}[x, y]$. Let $P \subset D_+(x_2)$ be the length 3 point $P = \operatorname{Spec}(\mathbb{C}[x, y]/I)$, where $I = (x, y)^2$, meaning $P \in S^{[3]}$. We have that the doubling of $P$ satisfies

$$P_2 = \operatorname{Spec}(\mathbb{C}[x, y]/I^2).$$

The length of $P_2$ will then be

$$\begin{aligned}\operatorname{len}(P_2) &\stackrel{\text{def}}{=} \dim_{\mathbb{C}} H^0(P_2, \mathcal{O}_{P_2}) \\ &= \dim_{\mathbb{C}} \mathbb{C}[x, y]/I^2 = \dim_C \langle 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\rangle_{\mathbb{C}} = 10.\end{aligned}$$

By a similar argument as in the proof of Corollary 5.21, the doubling of a point $Z \in S^{[3]}$ that is supported on three points of $S$ will have length 9. Since $Z_2$ is the fibre of $q : (\mathcal{Z}_3(S))_2 \to S^{[3]}$ over $Z$, and the length of the fibres of $q$ are not constant, [Har77, Theorem III.9.9] gives that $(\mathcal{Z}_3(S))_2$ is not flat over $S^{[3]}$. △

We contend that a similar argument showcases that the doubling of the higher universal families $(\mathcal{Z}_{>3}(S))_2$ are not flat over $S^{[n]}$ either. However, since the morphism $q : (\mathcal{Z}_n(S))_2 \to S^{[n]}$ is finite, we do have the following.

**Remark 5.24.** (The sheaves $S_T(\mathcal{L}, n)$ are coherent)
The sheaves $S_T(\mathcal{L}, n) = q_*p^*\mathcal{L}$ are coherent for all $n$. Indeed, pushing a locally free sheaf forward with a finite morphism gives a coherent sheaf. △

**Remark 5.25.** (The difficulties in working with $S_T(\mathcal{L}, n)$ for $n > 2$)
A consideration that should be made apparent is that our understanding of the somewhat tautological sheaves relies partly on our understanding of the universal families $\mathcal{Z}_n(S)$ that are flat over $S^{[n]}$. The universal families for $n > 2$ are not as well behaved as when $n \in \{1, 2\}$. Indeed, see for instance [Son16] for a reference stating the higher $\mathcal{Z}_n(S)$ are singular, Cohen-Macaulay and irreducible. This makes working with the sheaves $S_T(\mathcal{L}, n)$ for $n > 2$ significantly more difficult than studying the somewhat tautological bundles. △

We have nothing more to say about the sheaves $S_T(\mathcal{L}, n)$ for $n > 2$, and we will restrict the remainder of the thesis to working with the bundles $S_T(\mathcal{L}, 1)$ and $S_T(\mathcal{L}, 2)$. The beauty of these bundles lies in that their degeneracy loci have a crystal clear geometric interpretation. This unlocks a tangible reformulation of Problem 1, and a palpable formulation of a special case of Problem 2, as will be shown in the next section.

### 5.2.2 The Defining Rationale: Degeneracy Loci of Somewhat Tautological Bundles

We recall that the $i^{\text{th}}$ Chern class of a rank $r$ vector bundle is equal to the class of the degeneracy locus $V := V(\tau_0 \wedge \tau_1 \wedge ... \wedge \tau_{r-i})$, given that the locus is of codimension $i$ (Theorem 3.5). We shall see how the interpretation of Chern classes as classes of degeneracy loci unlock a natural correspondence between the Chern classes of the somewhat tautological bundles, and the schemes $\mathbb{P}^\delta \cap |\mathcal{L}|_n$, that were introduced in the introduction. This will allow for a more tangible reformulation of Problem 1, as well as a palpable formulation of a special case of Problem 2. For the remainder of the section, let $\widehat{s} \in H^0(S^{[n]}, S_T(\mathcal{L}, n))$ denote the induced section of $s \in H^0(S, \mathcal{L})$.

The bundles $S_T(\mathcal{L}, n)$ contain the binary data of whether a curve $C = v(s)$ associated to the section $s \in H^0(S, \mathcal{L})$ is singular at all points in $\mathrm{Supp}(Z)$ on $S$, when $Z \in S^{[n]}$ is supported on $n$ points. Our interest lies in studying specific global sections of $S_T(\mathcal{L}, n)$. Namely, given sections $s_0, s_1, ..., s_j \in H^0(S, \mathcal{L})$, general in the sense of Remark 4.13, we want to study the section $\widehat{f_k} = k_0\widehat{s_0} + k_1\widehat{s_1} + ... + k_j\widehat{s_j}$, where $k = (k_0, ..., k_j) \in \mathbb{C}^j$ is non-zero. Specifically, we want to determine at which points on $S$ the curve $V(f_k)$ is singular, where $f_k = k_0 s_0 + k_1 s_1 + ... + k_j s_j$. This may be done by investigating at which point $Z \in S^{[n]}$ the induced section $\widehat{f_k}$ vanishes.

Our interest in sections of this particular form stems from the following observations. Asking where $\widehat{f_k}$ vanishes for some nonzero $k \in \mathbb{C}^j$ is equivalent to asking where the wedge sum $\widehat{s_0} \wedge ... \wedge \widehat{s_j}$ vanishes. Indeed, the relations that define the exterior algebra dictate that the section $\widehat{s_0} \wedge ... \wedge \widehat{s_j} \in H^0(S^{[n]}, \bigwedge^{j+1} S_T(\mathcal{L}, n))$ vanishes precisely when the collection of $k_i\widehat{s_i}$ are linearly dependent, or in symbols, when there is a choice of k such that

$$\widehat{f_k} = \sum k_i \widehat{s_i} = 0.$$

These observations are of importance to us, since the general linear systems we work with are precisely described by a section of the form $f_k = k_0 s_0 + k_1 s_1 + ... + k_j s_j$ (Remark 4.13)! This motivates the following observation.

**Observation 5.26.** (Interpretation of the degeneracy loci of $S_T(\mathcal{L}, 1)$ and $S_T(\mathcal{L}, 2)$) *Let $\mathcal{L}$ be a sufficiently ample line bundle on a smooth projective surface $S$ such that $|\mathcal{L}|_n$ is of expected dimension. Suppose $3n \geq \delta > n$ where $n \in \{1, 2\}$ and suppose $\mathbb{P}\langle s_0, ..., s_\delta\rangle_{\mathbb{C}} = \mathbb{P}^\delta$ is a general linear system in $|\mathcal{L}|$. Consider the map*

$$\mathbb{P}^\delta \cap |\mathcal{L}|_n \setminus |\mathcal{L}|_{n+1} \xrightarrow{\phi_n} V(\widehat{s_0} \wedge ... \wedge \widehat{s_\delta}) \subset S^{[n]}$$

*of Problem 2, which maps $C \overset{\phi_n}{\mapsto} Z \in S^{[n]}$, where* $\mathrm{Supp}(Z) = C_{sing}$. *We have that* $\mathrm{im}(\phi_n) \subset V$*, where $V$ is the degeneracy locus $V(\widehat{s_0} \wedge ... \wedge \widehat{s_\delta})$. Furthermore, the degeneracy locus $V$ consists of exactly the schemes $Z \in S^{[n]}$ that are supported on $n$ points in $S$, such that there exists a curve $C \in \mathbb{P}^\delta$ which is singular at the points in* $\mathrm{Supp}(Z)$.

*Proof.* We are interested in studying the cases where $3n \geq \delta > n$ and $n \in \{1, 2\}$. Rephrasing this, we let $\delta = 3n - i$, where $i \in \{0, 1, ..., 2n - 1\}$. Let $s_0, ..., s_\delta \in H^0(S, \mathcal{L})$ be general sections in the sense of Remark 4.13, and denote their induced sections by $\widehat{s_i} \in H^0(S^{[n]}, S_T(\mathcal{L}, n))$. We have the following chain of equalities.

$$\begin{aligned}
V(\widehat{s_0} \wedge ... \wedge \widehat{s_\delta}) &= \{Z \in S^{[n]} \mid \widehat{s_0} \wedge ... \wedge \widehat{s_\delta}(Z) = 0\}; \\
&= \{Z \in S^{[n]} \mid c_0\widehat{s_0}(Z) + ... + c_\delta\widehat{s_\delta}(Z) = 0, \text{ for some } c = (c_0 : ... : c_\delta) \in \mathbb{P}^\delta\}; \\
&= \{Z \in S^{[n]} \mid \widehat{f_c}(Z) = 0 \text{ for some } c \in \mathbb{P}^\delta\}; \\
&= \{Z \in S^{[n]} \mid f_c \text{ is singular at all } p \in \mathrm{Supp}(Z) \text{ for some } c \in \mathbb{P}^\delta\}; \\
&\supset \{Z \in S^{[n]} \mid f_c \text{ has a node at precisely the } p \in \mathrm{Supp}(Z) \text{ for some } c \in \mathbb{P}^\delta\}; \\
&= \phi_n(\mathbb{P}^\delta \cap |\mathcal{L}|_n \setminus |\mathcal{L}|_{n+1}) = \mathrm{im}\, \phi_n,
\end{aligned}$$

where $\widehat{f_c}(Z) = \sum_{i=0}^{\delta} c_i\widehat{s_i}(Z)$ and $f_c(p) = \sum_{i=0}^{\delta} c_i s_i(p)$. The fourth equality is an application of Proposition 5.17 □

We remark that the choice of $\delta = 3n - i$ is not arbitrary. Theorem 3.5 states that for this choice of $\delta$, the class $[V] \in A(S^{[n]})$ equals $c_i(S_T(\mathcal{L}, n))$, if $V$ is of codimension $i$ in $S^{[n]}$.

A first step to solving problems 1 and 2, wherein we ask about the image of $\phi_n$, is perhaps to understand the degeneracy loci $V$ in the observation. This motivates the following readjustment of Problem 1.

**Problem 3.** (Readjustment of Problem 1)
Compute the class of $V(\widehat{s_0} \wedge \widehat{s_1} \wedge \widehat{s_2})$ in $A(S)$, where $\mathbb{P}\mathcal{W} = \mathbb{P}\langle s_0, s_1, s_2\rangle_{\mathbb{C}}$ is a general net in a sufficiently ample linear system $|\mathcal{L}|$.

We successfully solve this problem in Section 6.1.2 by proving $V(\widehat{s_0} \wedge \widehat{s_1} \wedge \widehat{s_2})$ is of appropriate dimension and by computing the Chern class $c_1(S_T(\mathcal{L}, 1))$, which equals the class of $V$ in $A(S)$.

Problem 2 is left open, but we attempt to solve the problem for the cases $\mathbb{P}^\delta \cap |\mathcal{L}|_2$, where $5 \geq \delta > 2$. By the observation, these special cases of Problem 2 are solved if we prove that the appropriate degeneracy loci are of expected dimension as well as compute the total Chern class $c_\bullet(S_T(\mathcal{L}, 2))$. We have dedicated Chapter 7 to the latter. We shall call this endeavour: 'Problem 4'.

**Problem 4.** (Special cases of Problem 2)
Compute the total Chern class $c_\bullet(S_T(\mathcal{L}, 2))$ and investigate whether the degeneracy loci $V_\delta := V(\widehat{s_0} \wedge ... \wedge \widehat{s_\delta})$ are of expected dimension, when $\delta = 6 - i$ for $i \in \{0, 1, 2, 3\}$ and where $\mathbb{P}\mathcal{W} = \mathbb{P}\langle s_0, ..., s_\delta\rangle_{\mathbb{C}} \subset |\mathcal{L}|$ is a general linear system.

An interesting question, that we will not further touch upon in this thesis, is the following.

**Question 1.** Let $n, \delta$ and $V$ be as in the observation. Is it true that $\overline{\operatorname{im}\phi_n} = V$?

We have not been able to come up with any counterexamples. Should the answer be yes, it would strengthen the connection between problems 1 & 2 and problems 3 & 4.

### 5.2.3 The Case $|\mathcal{L}|_\delta \cap \mathbb{P}^\delta$ Corresponds to the Göttsche Conjecture

We have yet to comment on the omitting of some cases of Problem 2. Given the scheme $|\mathcal{L}|_n \cap \mathbb{P}^\delta$, how do the cases $\delta = n$ and $\delta < n$ relate to the problem? The latter is trivial: if $\delta < n$, then $|\mathcal{L}|_n \cap \mathbb{P}^\delta = \emptyset$, since $\operatorname{codim}|\mathcal{L}|_n > \dim \mathbb{P}^\delta$, and the scheme $|\mathcal{L}|_n \cap \mathbb{P}^\delta$ is a transversal intersection. The case where $\delta = n$, on the other hand, is worth commenting on.

Proposition 5.9 states that if $\mathcal{L}$ is sufficiently ample, then all $\delta$-nodal curves in a linear system $\mathbb{P}^\delta \subset |\mathcal{L}|$ will have precisely $\delta$ nodes as singularities. Furthermore, by Theorem 1.1, the amount $t^S_\delta(\mathcal{L})$ of $\delta$-nodal curves in a general linear system $\mathbb{P}^\delta$, is finite. We shall prove that $|\mathbb{P}^\delta \cap |\mathcal{L}|_\delta| = t^S_\delta(\mathcal{L})$. But before we do so, we remark two useful facts.

**Remark 5.27.** (All curves in a pencil are singular at a point, if two curves are)
Consider the pencil $\mathbb{P}\mathcal{W} = \{f_p = p_0 s_0 + p_1 s_1 \mid p = (p_0 : p_1) \in \mathbb{P}^1\}$ where $s_0, s_1 \in H^0(S, \mathcal{L})$. If $f_q, f_{q'}$ are singular at the point $x \in S$, whenever $q \neq q'$, then $f_p$ is singular at $x$ for all $p \in \mathbb{P}^1$. Indeed, assume two such points $q, q'$ exist, and fix representatives $(q_0 : q_1)$ and $(q'_0 : q'_1)$ for these points. Then the induced global sections $\widetilde{s_0}$ and $\widetilde{s_1}$ of the jet bundle $J^1\mathcal{L}$ satisfy

$$\begin{pmatrix} q_0 & q_1 \\ q'_0 & q'_1 \end{pmatrix} \begin{pmatrix} \widetilde{s_0}(x) \\ \widetilde{s_1}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

But since the points $q, q'$ are distinct, the matrix

$$\begin{pmatrix} q_0 & q_1 \\ q_0' & q_1' \end{pmatrix}$$

is invertible, which implies

$$\begin{pmatrix} \widetilde{s_0}(x) \\ \widetilde{s_1}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus $p_0\widetilde{s_0}(x) + p_1\widetilde{s_1}(x) = 0$ for all $(p_0 : p_1) \in \mathbb{P}^1$, meaning that $f_p$ is singular at $x \in S$ for all $p \in \mathbb{P}^1$. △

This remark can clearly be generalized to the case of somewhat tautological sheaves.

**Remark 5.28.** If two general sections $s_0, s_1 \in H^0(S, \mathcal{L})$ that define the pencil $\mathbb{P}^1 = \mathbb{P}\langle s_0, s_1\rangle_{\mathbb{C}}$ induce sections $\widehat{s_0}, \widehat{s_1} \in H^0(S^{[\delta]}, S_T(\mathcal{L}, \delta))$ that both vanish at $Z = P_1 \sqcup ... \sqcup P_\delta \in S^{[\delta]}$, then all curves in the pencil $\mathbb{P}\langle s_0, s_1\rangle_{\mathbb{C}}$ are singular at the points in $\operatorname{Supp} Z$. Indeed, if $C_0 = V(s_0)$ and $C_1 = V(s_1)$ are both singular at $P_1, ..., P_\delta$, then the induced sections $\widehat{s_0}, \widehat{s_1}$ vanish when restricted to $Z$, by Proposition 5.17. Since all curves in $\mathbb{P}\langle s_0, s_1\rangle_{\mathbb{C}}$ are of the form $\widehat{s} := \alpha\widehat{s_0} + \beta\widehat{s_1}$ with $(\alpha : \beta) \in \mathbb{P}^1$, all of their induced sections vanish when restricted to $Z$, meaning all curves $C \in \mathbb{P}\langle s_0, s_1\rangle_{\mathbb{C}}$ are singular at the points in $\operatorname{Supp} Z$. △

**Claim 5.29.** (The Göttsche conjecture corresponds to the case $\delta = n$)
Let $\delta \in \mathbb{N}$, and let $\mathbb{P}\mathcal{W} = \mathbb{P}^\delta \subset |\mathcal{L}|$ be a general linear system, where $\mathcal{L}$ is a sufficiently ample such that $\mathbb{P}\mathcal{W}$ only contains a finite amount of $\delta$-nodal curves, and no curves with $\delta + 1$ nodes. Then we have

$$\deg([\phi_\delta(|\mathcal{L}|_\delta \cap \mathbb{P}^\delta)]) = t_\delta^S(\mathcal{L}),$$

where $\phi_\delta$ maps a curve to the point $Z \in S^{[n]}$ such that $\operatorname{Supp} Z = C_{\text{sing}}$, and $[\phi_\delta(|\mathcal{L}|_\delta \cap \mathbb{P}^\delta)]$ is the class of $\phi_\delta(|\mathcal{L}|_\delta \cap \mathbb{P}^\delta)$ in the Chow ring $A(S^{[\delta]})$.

*Proof.* No two curves in $\mathbb{P}\mathcal{W} = \mathbb{P}^\delta$ have the same set of $\delta$ nodes. Indeed, assume $V(s_1) = C_1 \in \mathbb{P}\mathcal{W} = \mathbb{P}^\delta$ and $V(s_2) = C_2 \in \mathbb{P}\mathcal{W}$ both have the points $P_1, ..., P_\delta$ as nodes. By applying Remark 5.28 to the pencil $\mathbb{P}^1 = \mathbb{P}\langle s_1, s_2\rangle_{\mathbb{C}} \subset \mathbb{P}\mathcal{W}$, we have that every single curve in $\mathbb{P}^1$ is nodal at the $\delta$ points. This contradicts the Göttsche conjecture, since the amount of $\delta$-nodal curves in $\mathbb{P}\mathcal{W}$ is finite. We must therefore have $C_1 = C_2$, meaning no two curves have all their nodes in common. This induces a natural injection $\phi_\delta$ that maps each curve in

$$\mathbb{P}\mathcal{W} \cap |\mathcal{L}|_\delta = \{C \in \mathbb{P}\mathcal{W} \mid \text{C is } \delta\text{-nodal}\}$$

to a unique point $Z \in S^{[\delta]}$, namely the point $Z$ such that $\operatorname{Supp} Z = C_{\text{sing}}$. The image of $\phi_\delta$ injects by the above discussion, meaning $Y := \operatorname{im} \phi$ is a disjoint union of exactly $t_\delta^S(\mathcal{L})$ points. The class $[Y] \in A(S^{[\delta]})$ therefore equals $[Y] = t_\delta^S(\mathcal{L})[P]$, where $[P]$ is the class of a closed point $P \in S^{[\delta]}$ in the Chow ring $A(S^{[\delta]})$. Applying deg to the class $[Y]$ grants us the result. □

The claim illustrates that we are indeed studying a generalization of the Göttsche conjecture.

# Part III

# Somewhat Tautological Bundles

# Chapter 6

# Jet Bundles

The jet bundles $J^n\mathcal{L}$ associated to a line bundle $\mathcal{L}$ on a smooth projective variety $S$ keep track of whether a curve on $S$ is 'severely singular' at a point. The severity of the singularities the bundles $J^n\mathcal{L}$ keep track of are dependent on $n$. The larger $n$ is, the more severe are the singularities $J^n\mathcal{L}$ keep track of. This formulation might make it seem like $J^n\mathcal{L}$ contains strictly more information than $J^k\mathcal{L}$, whenever $n > k$, but this is not true. The bundle $J^n\mathcal{L}$ is incapable of detecting singularities that are not sufficiently severe. The bundle $J^2\mathcal{L}$, for instance, is not able to discern whether a curve $C = V(s)$ corresponding to a section $s \in H^0(S, \mathcal{L})$, is nodal at a point. This leads us to remark that $J^1\mathcal{L}$ is the only jet bundle that is suited for the purpose of studying the degeneracy loci of the previous chapter, as these loci roughly parametrize points at which a curve in a linear system is singular, completely disregarding the severity of said singularities. To this end, we shall study $S_T(\mathcal{L}, 1) = J^1\mathcal{L}$ in order to solve Problem 3, which we do in Section 6.1.2.

Other than solving this problem, we compute the Chern classes of all jet bundles $J^n\mathcal{L}$ on a smooth projective surface $S$, as the generalization from $n = 1$ to any $n \in \mathbb{N}$ is relatively straightforward. In Section 6.2 we showcase a generalization of $J^1\mathcal{L}$ that fails to keep track of singular loci of curves in a proper manner.

## 6.1 Computation of the Classes $c_\bullet(J^n\mathcal{L})$ and a Degeneracy Locus

#### Conventions

For the entirety of this section, we denote $\lambda := c_1(\mathcal{L})$ and $c_k := c_k(\Omega_S)$. Note that $\Omega_S$ is a bundle of rank 2, so its higher Chern classes $c_k$ for $k > 2$ disappear.

### 6.1.1 The Total Chern Class of $J^n\mathcal{L}$

To solve Problem 3, we need to compute a specific degeneracy locus of global sections of $J^1\mathcal{L}$. We shall prove that this locus is of expected codimension, meaning its class in the Chow ring equals an appropriate Chern class of $J^1\mathcal{L}$. To this end, we begin by computing the Chern classes of $J^n\mathcal{L}$.

We showed that $J^0\mathcal{L} = \mathcal{L}$ in example 3.13, meaning the case where $n = 0$ is known. Let us instead investigate the case $n = 1$. Eisenbud & Harris use a trick to compute this efficiently in [EH16, Page 259]. I will deviate slightly from this computation in order to obtain an expression which is somewhat nicer to generalize.

Applying Whitney's formula to the sequence

$$0 \longrightarrow \mathcal{L} \otimes \Omega_S \longrightarrow J^1\mathcal{L} \longrightarrow \mathcal{L} \longrightarrow 0$$

grants us the expression

$$c_\bullet(J^1\mathcal{L}) = c_\bullet(\mathcal{L})c_\bullet(\mathcal{L} \otimes \Omega_S) = (1+\lambda)c_\bullet(\mathcal{L} \otimes \Omega_S).$$

By Proposition 3.9, we have that

$$(1+\lambda)c_\bullet(\mathcal{L} \otimes \Omega_S) = (1+\lambda)\left[\sum_{k=0}^{2}\sum_{i=0}^{k}\binom{2-i}{k-i}\lambda^{k-i}c_i(\Omega_S)\right],$$

which gives us the equality

$$c_\bullet(J^1\mathcal{L}) = (1+\lambda)\left[\sum_{k=0}^{2}\sum_{i=0}^{k}\binom{2-i}{k-i}\lambda^{k-i}c_i(\Omega_S)\right].$$

Applying Whitney's formula to

$$0 \longrightarrow \mathcal{L} \otimes \mathrm{Sym}^2(\Omega_S) \longrightarrow J^2\mathcal{L} \longrightarrow J^1\mathcal{L} \longrightarrow 0$$

gives us

$$c_\bullet(J^2\mathcal{L}) = c_\bullet(J^1\mathcal{L})c_\bullet(\mathcal{L} \otimes \mathrm{Sym}^2(\Omega_S)).$$

We need only compute the rightmost factor, which we do by applying proposition 3.9 again, leaving us with

$$c_\bullet(\mathcal{L} \otimes \mathrm{Sym}^2(\Omega_S)) = \sum_{m=0}^{2}\sum_{j=0}^{m}\binom{3-j}{m-j}\lambda^{m-j}c_j(\mathrm{Sym}^2(\Omega_S)).$$

Hence, we have

$$\begin{aligned}
c_\bullet(J^2\mathcal{L}) &= c_\bullet(J^1\mathcal{L})c_\bullet(\mathcal{L} \otimes \mathrm{Sym}^2(\Omega_S)) \\
&= (1+\lambda)\left[\sum_{k=0}^{2}\sum_{i=0}^{k}\binom{2-i}{k-i}\lambda^{k-i}c_i(\Omega_S)\right]\left(\sum_{m=0}^{2}\sum_{j=0}^{m}\binom{3-j}{m-j}\lambda^{m-j}c_j(\mathrm{Sym}^2(\Omega_S))\right).
\end{aligned}$$

We generalize this computation to the $n^{\text{th}}$ jet bundle.

**Proposition 6.1.**
*For $n > 1$, we have*

$$c_\bullet(J^n\mathcal{L}) = (1+\lambda)\prod_{y=1}^{n}\left[\sum_{k=0}^{2}\sum_{i=0}^{k}\binom{y+1-i}{k-i}\lambda^{k-i}c_i\left(\mathrm{Sym}^y(\Omega_S)\right)\right]. \tag{6.1}$$

*Proof.* We perform induction on $n$, the power of jet bundle. We let the inductive hypothesis be the equation in the proposition. The hypothesis holds for $n = 2$, by the above.

Suppose the hypothesis holds for some $n$. Applying Whitney's formula to the short exact sequence

$$0 \longrightarrow \mathcal{L} \otimes \mathrm{Sym}^{n+1}(\Omega_S) \longrightarrow J^{n+1}\mathcal{L} \longrightarrow J^n\mathcal{L} \longrightarrow 0$$

yields

$$c_\bullet(J^{n+1}\mathcal{L}) = c_\bullet(J^n\mathcal{L})c_\bullet(\mathcal{L} \otimes \mathrm{Sym}^{n+1}(\Omega_S)).$$

By applying Proposition 3.9 to $c_\bullet(\mathcal{L}\otimes\mathrm{Sym}^{n+1}(\Omega_S))$ and using that the rank of $\mathrm{Sym}^n(\Omega_S)$ is $n+1$,[1] we obtain

$$c_\bullet(\mathcal{L} \otimes \mathrm{Sym}^{n+1}(\Omega_S)) = \sum_{m=0}^{2}\sum_{j=0}^{m}\binom{n+2-j}{m-j}\lambda^{m-j}c_j(\mathrm{Sym}^{n+1}(\Omega_S)).$$

Multiplying this by the hypothesized expression for $J^n\mathcal{L}$ gives us the desired formula. □

Before this equation is of any use to us, we must compute the classes $c_i\left(\mathrm{Sym}^y(\Omega_S)\right)$. An equivalent piece of datum is the total Chern class $c_\bullet\left(\mathrm{Sym}^y(\Omega_S)\right)$, which we compute in the lemma below.

**Lemma 6.2.**
*Denote $c_i := c_i(\Omega_S)$. We have that*

$$c_\bullet(\mathrm{Sym}^n(\Omega_S)) = \begin{cases} \prod_{k=0}^{\lfloor \frac{n}{2} \rfloor}\left[1 + n\cdot c_1 + (kn-k^2)c_1^2 + (n^2+4(k^2-kn))c_2\right] & \text{if } n \text{ is odd} \\ \frac{1}{1+\frac{n}{2}c_1}\prod_{k=0}^{\frac{n}{2}}\left[1 + n\cdot c_1 + (kn-k^2)c_1^2 + (n^2+4(k^2-kn))c_2\right] & \text{if } n \text{ is even} \end{cases}$$

*Proof.* We shall make use of the splitting principle. Assume $\Omega_S = \mathcal{L} \oplus \mathcal{M}$ where the latter are line bundles with Chern classes $c_1(\mathcal{L}) = x$ and $c_1(\mathcal{M}) = y$. By Whitney's formula we have

$$c_\bullet(\Omega_S) = c_\bullet(\mathcal{L})c_\bullet(\mathcal{M}) = (1+x)(1+y) = 1 + (x+y) + xy,$$

i.e. we have that $c_1 = c_1(\Omega_S) = x + y$ and $c_2 = c_2(\Omega_S) = xy$. Furthermore, we have that

$$\mathrm{Sym}^n(\Omega_S) = \mathrm{Sym}^n(\mathcal{L}\oplus\mathcal{M}) = \mathcal{L}^{\otimes n} \oplus (\mathcal{L}^{\otimes n-1}\otimes\mathcal{M}) \oplus ... \oplus \mathcal{M}^{\otimes n} = \bigoplus_{k=0}^{n}\left[\mathcal{L}^{\otimes k}\otimes\mathcal{M}^{\otimes n-k}\right].$$

By Whitney's formula, we obtain

$$c_\bullet(\mathrm{Sym}^n(\Omega_S)) = \prod_{k=0}^{n}\left[1 + kx + (n-k)y\right]. \tag{6.2}$$

We want to express this product in terms of the Chern classes $c_i$. We achieve this by writing out the product in (6.2) in a convenient manner. Consider the following product.

$$[1 + kx + (n-k)y][1 + (n-k)x + ny]. \tag{6.3}$$

Writing it out gives us

$$\begin{aligned} &(1 + kx + (n-k)y)(1 + (n-k)x + ky) \\ &= 1 + n(x+y) + k^2xy - k^2x^2 + k^2xy - k^2y^2 + knx^2 - 2knxy + kny^2 + n^2xy \\ &= 1 + n(x+y) + (kn-k^2)(x-y)^2 + n^2xy \\ &= 1 + n(x+y) + (kn-k^2)[(x-y)^2 + 4xy] - (kn-k^2)4xy + n^2xy \\ &= 1 + n(x+y) + (kn-k^2)(x+y)^2 + (n^2 + 4(k^2-kn))xy \end{aligned}$$

[1] This follows from the fact that $\operatorname{rank}\mathcal{L}\otimes\mathrm{Sym}^n(\Omega_S) = \operatorname{rank}\mathrm{Sym}^n(\Omega_S) = \binom{n+2-1}{2-1} = \binom{n+1}{1} = n+1$.

Substituting for $c_1$ and $c_2$, we obtain

$$= 1 + n \cdot c_1 + (kn - k^2)c_1^2 + (n^2 + 4(k^2 - kn))c_2.$$

We acknowledge that if $n$ is odd in the expression of (6.2), we may rewrite it to

$$\begin{aligned} c_\bullet(\operatorname{Sym}^n \Omega_S) &= \prod_{k=0}^{n} [1 + kx + (n-k)y] \\ &= \prod_{k=0}^{\lfloor \frac{n}{2} \rfloor} [1 + kx + (n-k)y]\,[1 + (n-k)x + ky] \\ &= \prod_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Big[1 + n \cdot c_1 + (kn - k^2)c_1^2 + (n^2 + 4(k^2 - kn))c_2\Big]. \end{aligned}$$

If $n$ is even, there is a $k$ such that $n/2 = k$. In this case, the product above will have one too many terms, namely a multiple of $(1 + \frac{n}{2}x + \frac{n}{2}y)$ too many. We remedy this by dividing the product by this term. This gives us the expression in the lemma. □

We do not provide an expression for the individual Chern classes of $\operatorname{Sym}^n(\Omega_S)$, as it was not entirely clear how one would construct a formula for these by using the method in the proof of the lemma. It seems reasonable to assume it is possible to write a script which computes the total Chern class using the expression in the lemma. The script can then sort the individual Chern classes $c_i(\operatorname{Sym}^n(\Omega_S))$ after it has computed the total Chern class. We therefore consider the problem of computing $c_\bullet(J^n\mathcal{L})$ on a smooth projective surface solved and celebrate by appending a noteworthy remark.

**Remark 6.3.**
The Chern polynomial $c_\bullet(J^n\mathcal{L})$ is a polynomial in the ring

$$\mathbb{Z}[\lambda, c_1, c_2]/(\lambda^3, \lambda^2 c_1, \lambda^1 c_1^2, c_1^3, c_2\lambda, c_2 c_1, c_2^2).$$

△

For the sake of future convenience, we have calculated[2] the total Chern classes of the first two non-trivial jet bundles using the formulae above.

**Example 6.4.**
Letting $\lambda = c_1(\mathcal{L})$ and $c_i = c_i(\Omega_S)$, we have

$$c_\bullet(J^1\mathcal{L}) = 1 + (3\lambda + c_1) + (3\lambda^2 + 2\lambda c_1 + c_2) \tag{6.4}$$
$$c_\bullet(J^2\mathcal{L}) = 1 + (6\lambda + 4c_1) + (15\lambda^2 + 20\lambda c_1 + 5c_1^2 + 2c_2). \tag{6.5}$$

△

### 6.1.2 Solution of Problem 3.

Recall that $s \in H^0(S, \mathcal{L})$ induces a global section of $J^1\mathcal{L}$, which we denote by $\widetilde{s}$. Choose $\mathcal{W} = \langle s_0, s_1, s_2 \rangle_\mathbb{C}$ such that $\mathbb{P}\mathcal{W}$ is a general net in $|\mathcal{L}|$. We are interested in computing $V := V(\widetilde{s_0} \wedge \widetilde{s_1} \wedge \widetilde{s_2})$, as described in Problem 3. Since $V$ is a degeneracy locus of three sections (and $J^1\mathcal{L}$ has rank 3), we have (by Theorem 3.5) that $[V] = c_1(J^1\mathcal{L})$ if $V \subset S$

[2] By hand.

is of codimension 1. The below proposition verifies that $V$ is indeed of the expected dimension. Recall that $\mathcal{I}_\mathcal{W}$ denotes the incriminant of Section 4.2.2, which as a set is described by

$$\mathcal{I}_\mathcal{W} = \{p \in S \mid \exists C \in \mathbb{P}\mathcal{W} \text{ such that } p \in C_{\text{sing}}\},$$

as stated in Remark 4.16. Recall further that, as a set,

$$\Sigma_\mathcal{W} = \{(C,p) \in \mathbb{P}\mathcal{W} \times S \mid p \in C_{\text{sing}}\}.$$

We are finally ready to solve Problem 3.

**Proposition 6.5.** (Solution of Problem 3)
*Let $(S,\mathcal{L})$ be a smooth projective surface equipped with a 2-very ample line bundle. Then if $\mathbb{P}\mathcal{W} = \mathbb{P}\langle s_0, ..., s_2\rangle_\mathbb{C} \subset |\mathcal{L}|$ is a general net in $|\mathcal{L}|$, we have*

$$[\mathcal{I}_\mathcal{W}] = [V] = c_1(J^1\mathcal{L}) \in A(S),$$

*where $V := V(\tilde{s_0} \wedge \tilde{s_1} \wedge \tilde{s_2})$.*

*Proof.* Let $\mathbb{P}\mathcal{W} = \{f_p \mid p \in \mathbb{P}^2\}$, where $p = (p_0 : p_1 : p_2)$ and $f_p = p_0 s_0 + p_1 s_1 + p_2 s_2$. We want to show that

$$[\mathcal{I}_\mathcal{W}] = [V(\tilde{s_0} \wedge \tilde{s_1} \wedge \tilde{s_2})] = c_1(J^1\mathcal{L}).$$

Choose three general sections $s_0, s_1, s_2 \in H^0(S,\mathcal{L})$ in the sense of Remark 4.13. By Theorem 3.5 we have that

$$c_1(J^1\mathcal{L}) = [V(\tilde{s_0} \wedge \tilde{s_1} \wedge \tilde{s_2})] \in A^1(S),$$

if the degeneracy locus $V(\tilde{s_0} \wedge \tilde{s_1} \wedge \tilde{s_2})$ is of expected dimension. It is therefore sufficient to assert the equality $V = \mathcal{I}_\mathcal{W}$ of sets, in addition to showing that the degeneracy locus $V$ is of codimension 1. We start by proving the equality.

Denoting $\tilde{f_p} := p_0\tilde{s_0} + p_1\tilde{s_1} + p_2\tilde{s_2}$, we have the following equalities of sets:

$$\begin{aligned} V(\tilde{s_0} \wedge \tilde{s_1} \wedge \tilde{s_2}) &= \{p \in S \mid \tilde{s_0} \wedge \tilde{s_1} \wedge \tilde{s_2}(p) = 0\} \\ &= \{p \in S \mid c_0\tilde{s_0}(p) + c_1\tilde{s_1}(p) + c_2\tilde{s_2}(p) = 0, \text{ for some } c = (c_0 : c_1 : c_2) \in \mathbb{P}^2\} \\ &= \{p \in S \mid \tilde{f_c}(p) = 0 \text{ for some } c \in \mathbb{P}^2\} \\ &= \{p \in S \mid f_c \text{ is singular at } p \text{ for some } c \in \mathbb{P}^2\} \\ &= I_\mathcal{W}. \end{aligned}$$

The penultimate equality follows from Proposition 5.17, and the last equality is just Remark 4.16.

We now argue that $\mathcal{I}_\mathcal{W}$ is of dimension 1 in $S$. It suffices to show that $\dim \Sigma_\mathcal{W} = 1$. Indeed, since $\mathcal{I}_\mathcal{W}$ is the image of $\Sigma_\mathcal{W}$ by a morphism, its dimension can not exceed the dimension of $\Sigma_\mathcal{W}$. Also, we have that $V = \mathcal{I}_\mathcal{W}$ has dimension $\geq 1$ by Krull's principal ideal theorem, implying $\dim V = 1$. We shall therefore prove that $\dim \Sigma_\mathcal{W} = 1$.

Since $\mathcal{L}$ is 2-very ample, all curves $C \in \mathbb{P}\mathcal{W}$ are reduced by Proposition 4.8. The fibres of the map

$$\pi_1 : \Sigma_\mathcal{W} \to |\mathcal{L}|_1 \cap \mathbb{P}\mathcal{W},$$

over a curve $C$ equal $C_{\text{sing}}$. Since the curve $C$ is reduced and singular, the fibre $\pi_1^{-1}(C)$ is of dimension 0. It is true that $|\mathcal{L}|_1 \cap \mathbb{P}\mathcal{W}$ has dimension 1 in $|\mathcal{L}|$, since $\operatorname{codim} |\mathcal{L}|_\delta = \delta$, by Proposition 4.11. From a well known result on dimension of fibres of dominant morphisms, it follows that

$$\dim \Sigma_{\mathcal{W}} \leq \dim \pi_1^{-1}(C) + \dim |\mathcal{L}|_1 \cap \mathbb{P}\mathcal{W} = 0 + 1.$$

Since the case $\dim \Sigma_{\mathcal{W}} = 0$ does not occur by assumption of generality on $\mathcal{W}$, we are done. □

## 6.2 Failure of a Naive Generalization of $J^1\mathcal{L}$

It is inconvenient to work over $S^{\times n}$ when one intends to keep track of the singularities of curves. To illustrate this, we provide a generalization of the first jet bundle $J^1\mathcal{L}$ that seems deceptively natural and showcase its subsequent failure to store data on the pairs of points where a curve $C \in |\mathcal{L}|$ might be singular. As we shall see, the points in the diagonal of $S \times S$ are troublesome.

### 6.2.1 A Naive Generalization of $J^1\mathcal{L}$

Let us start by considering a heuristic argument. We propose the following naive generalization of $J^1\mathcal{L}$, that will keep track of curves with 2 singularities. Consider the direct sum $J^1\mathcal{L} \oplus J^1\mathcal{L}$ and study the behaviour of the induced global section $\bar{s} = (\widetilde{s}, \widetilde{s})$ of $J^1\mathcal{L} \oplus J^1\mathcal{L}$ on $S \times S$. If the curve $V(s)$, where $s \in H^0(S, \mathcal{L})$, is singular at the points $p$ and $q$, then the section $\bar{s}$ will vanish when restricted to the fibre of $J^1\mathcal{L} \oplus J^1\mathcal{L}$ at the point $(p, q) \in S \times S$. The bundle is thus able to ascertain if a curve $V(s)$ is singular at two distinct points. There is however a problem: the section $\bar{s}$ also vanishes at the point $(p, p)$ when $V(s)$ is singular only at the point $p$. We illustrate below how this fact causes the degeneracy locus of the bundle to not be of the expected dimension. In fact, the degeneracy locus will not even correspond to the set of points in $S \times S$ that we are interested in.

**Example 6.6.** (Failure of a naive generalization of $J^1\mathcal{L}$)
Let the $\pi_i$'s in the diagram below be distinct projection maps.

$$\begin{array}{ccc} S \times S & \xrightarrow{\pi_2} & S \\ \downarrow{\scriptstyle \pi_1} & & \\ S & & \end{array}$$

We define $\mathcal{E} = \pi_1^*(J^1\mathcal{L}) \oplus \pi_2^*(J^1\mathcal{L})$. If $s \in \Gamma(S, \mathcal{L})$, we denote its induced section of $\mathcal{E}$ by $\bar{s} = \pi_1^*(\tilde{s}) \oplus \pi_2^*(\tilde{s})$, where $\tilde{s}$ is the induced global section of $J^1\mathcal{L}$.

It is a slight subtlety that $\mathcal{E}$ does not contain the desired data. After all, if a curve $Z(s) \in \mathcal{L}$ is singular at two points $p_1, p_2 \in S$, then $\bar{s}(p_1, p_2) = (0, 0)$, meaning the section $s$ is singular at $p_1$ and $p_2$.

The supposed appropriate setting for $\mathcal{E}$, in the context of Problem 2, would be to study whether its degeneracy locus relates to the image $\phi_2(\mathbb{P}^3 \cap |\mathcal{L}|_2 \setminus |\mathcal{L}|_3)$ where $\mathbb{P}^3 = \mathbb{P}\mathcal{W}$ is a general linear system, and whether the locus is of the appropriate dimension. We shall show that neither are true. Indeed, the degeneracy locus of four general sections $\bar{s_0}, ..., \bar{s_3} \in \Gamma(S \times S, \mathcal{E})$ induced by $s_0, ..., s_3 \in \Gamma(S, \mathcal{L})$ satisfies the equalities

$$\begin{aligned}
&V(\bar{s_0} \wedge \bar{s_1} \wedge \bar{s_2} \wedge \bar{s_3}) \\
&= \{(P_1, P_2) \in S \times S \mid \textstyle\sum c_i \bar{s_i}(P_1, P_2) = (0,0), \text{ where } c = (c_0 : c_1 : c_2 : c_3) \in \mathbb{P}^3\} \\
&= \{(P_1, P_2) \in S \times S \mid \bar{f_c}(P_1, P_2) = (0,0) \text{ where } c \in \mathbb{P}^3\} \\
&= \{(P_1, P_2) \in S \times S \mid f_c \text{ is singular at } P_1 \text{ and } P_2, \text{ where } c \in \mathbb{P}^3\} \\
&= \{(P_1, P_2) \in S \times S \mid C = V(f_c) \in \mathbb{P}\mathcal{W} \text{ is singular at } P_1 \text{ and } P_2, \text{ where } c \in \mathbb{P}^3\} \\
&=: D,
\end{aligned}$$

where $\bar{f_c} = \sum c_i \bar{s_i}$ and $f_c = \sum c_i s_i$. Since $\mathcal{E}$ is a rank 6 bundle, the expected dimension of $D$ is 1, by Theorem 3.5. But $D$ will have at least dimension 2. Indeed, we have that

$$Y = \{(p,p) \in S \times S \mid \exists C \in \mathbb{P}\mathcal{W} \text{ which is singular at } p\} \subset D.$$

Clearly, $Y$ is in bijection with the set

$$Y' = \{p \in S \mid \exists C \in \mathbb{P}\mathcal{W} \text{ which is singular at } p\},$$

but the latter is exactly a degeneracy locus of 4 sections of $J^1\mathcal{L}$ on $S$, by a similar set of equalities as above. A degeneracy locus of four sections of a rank 3 bundle is, by Theorem 3.5, of codimension 0, meaning $Y' = S$. It follows that

$$Y = \{(p,p) \in S \times S \mid \exists C \in \mathbb{P}\mathcal{W} \text{ which is singular at } p\} = \Delta \subset S \times S,$$

meaning $D$ contains a dimension 2 subscheme.

We remark that the degeneracy locus $D$ is not of any interest to us to begin with, since $\mathbb{P}^3 \cap |\mathcal{L}|_2$ contains curves that have at least two nodes, and the locus $D$ contains curves that only have one singularity. $\triangle$

Chapter 7

# The Second Somewhat Tautological Bundle

This chapter is dedicated to taking on part of Problem 4. We compute the Chern classes of $S_T(\mathcal{L}, 2)$, in the sense that we relate them to the Chern classes of bundles that are known and some of which are likely to be computable. We compute the classes of $S_T(\mathcal{L}, 2)$ in two separate approaches.

## 7.1 Results

This chapter mostly consists of bookkeeping and technical arguments, making it somewhat tedious to follow. We therefore start by presenting our main findings as well as stating the contents of the sections, hoping this provides some overview.

Our strategy for computing $c_\bullet(S_T(\mathcal{L}, 2))$ lies in applying Whitney's sum formula to an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow S_T(\mathcal{L}, 2) \longrightarrow \mathcal{L}_{[2]} \longrightarrow 0$$

granting us $c_\bullet(S_T(\mathcal{L}, 2)) = c_\bullet(\mathcal{F})c_\bullet(\mathcal{L}_{[2]})$. The Chern classes $c_\bullet(\mathcal{L}_{[2]})$ are known, as detailed in Remark 5.6. Hence the computation reduces to computing the Chern classes of the bundle $\mathcal{F}$. We prove exactness of this sequence (with an explicit expression for $\mathcal{F}$) in Section 7.3. Two different approaches of computing $c_\bullet(\mathcal{F})$ are presented in section 7.5. The first approach argues that computing the Chern classes of $\mathcal{F}$ almost reduces to computing the Chern classes of $\mathcal{I}_B/\mathcal{I}_B^2$, and follows up on this by computing $c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2)$. The second approach computes $c_\bullet(\mathcal{F})$ more directly. The approaches differ practically in that we apply Whitney's formula to two distinct short exact sequences. We now present the main result.

**Theorem 7.1.** *Let $S$ be a smooth projective surface equipped with a line bundle $\mathcal{L}$. We have that*

$$c_\bullet(S_T(\mathcal{L}, 2)) = c_\bullet(\mathcal{L}_{[2]})c_\bullet(\mathcal{F}) = c_\bullet(\mathcal{L}_{[2]})c_\bullet(q_{B*}(p_B^*\mathcal{L} \otimes \mathcal{I}_B/\mathcal{I}_B^2)) \tag{7.1}$$

*where*

$$c_\bullet(\mathcal{F}) = \frac{c_\bullet((\Omega_S \otimes \mathcal{L})_{[2]})}{c_\bullet(q_{B*}(k_*\mathcal{O}_E(-E) \otimes p_B^*\mathcal{L}))} \tag{7.2}$$

*and*

$$c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2) = \frac{q_B^*\left[c_\bullet(\Omega_{S^{[2]}})\right]}{p_1^*\left[c_\bullet(\Omega_S)\right] c_\bullet(k_*(\Omega_{E/\Delta}))}, \tag{7.3}$$

*where the morphisms are given in diagrams* (7.4) *&* (7.8)*.*

Section 7.4 is dedicated to proving short exactness of a particular sequence that we use in the computation of (7.3). The purpose of the next section is to introduce diagrams and facts that we will use throughout the remainder of the chapter.

## 7.2 Setting the Stage

Computing the total Chern class of the jet bundles $J^n\mathcal{L}$ was made easy by the existence of the sequences

$$0 \longrightarrow \mathcal{L} \otimes \mathrm{Sym}^m(\Omega_Y) \longrightarrow J^m\mathcal{L} \longrightarrow J^{m-1}\mathcal{L} \longrightarrow 0$$

that consisted of bundles whose Chern classes were either known or they were recursively computable. We are less fortunate in our ambition of computing $c_\bullet(S_T(\mathcal{L}, 2))$. The appropriate sequences available to us consist of sheaves we know less about. As shall be revealed in Section 7.3, our approach of computing $c_\bullet(S_T(\mathcal{L}, 2))$ relies on computing the total Chern class of a bundle $\mathcal{F}$. We have found two sequences that help us in our quest of computing $c_\bullet(\mathcal{F})$, should they be short exact. This section is dedicated to do the bookkeeping required to assert short exactness of these sequences. To that end, we start by repeating an important fact.

**Lemma 7.2.**
*Let $\mathcal{Z}_2(S)$ be the universal family lying flat over $S^{[2]}$ that is closed in $S^{[2]} \times S$. Then we have that*

$$B := Bl_\Delta(S \times S) = \mathcal{Z}_2(S).$$

*Proof.* The result is well known. It is for instance stated in [Leh99, Chapter 1.2]. □

Let $B_2$ be the closed subscheme of $S^{[2]} \times S$ defined by the doubling of $B$, i.e. by the ideal sheaf $\mathcal{I}_B^2$, where $\mathcal{I}_B$ is the ideal sheaf associated to $B$. The doubling of $B$ has the following connection to $S^{[2]}$.

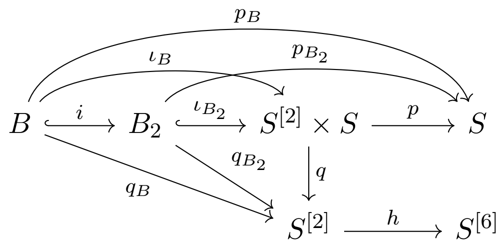
**Lemma 7.3.**
*Let $\mathcal{Z}_6(S)$ be the universal family lying flat over $S^{[6]}$. The morphism $h : S^{[2]} \to S^{[6]}$ in the diagram corresponds to $B_2$ in the sense that $\mathcal{Z}_6(S)$ pulls back to $B_2$. I.e, we have that the diagram*

$$\begin{array}{ccc} B_2 & \longrightarrow & \mathcal{Z}_6(S) \\ \downarrow{\scriptstyle q_B} & \ulcorner & \downarrow{\scriptstyle \pi} \\ S^{[2]} & \xrightarrow{h} & S^{[6]}. \end{array}$$

*commutes.*

*Proof.* We consider the morphism $h$ from the proof of Corollary 5.21 and apply the universal property of $S^{[6]}$ to it. □

**Lemma 7.4.**




(7.4)

*a. All the arrows in Diagram 7.4 are morphisms of schemes, and the diagram commutes wherever possible.*

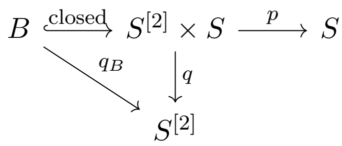
*b. if $\mathcal{L}$ is a line bundle on $S$, it holds that*

$$\mathcal{L}_{[2]} \overset{def}{=} q_{B*}p_B^*(\mathcal{L}) = q_*(\iota_{B*}\mathcal{O}_B \otimes p^*\mathcal{L}) \tag{7.5}$$

*c. if $\mathcal{L}$ is a line bundle on $S$, it holds that*

$$S_T(\mathcal{L}, 2) \overset{def}{=} q_{B_2*}p_{B_2}^*(\mathcal{L}) = q_*(\iota_{B_2*}\mathcal{O}_{B_2} \otimes p^*\mathcal{L}) \tag{7.6}$$

*Proof.* a. By Lemma 7.2, the diagram




(7.7)

commutes. The morphisms $q_B$ and $h$ come from Lemma 7.3. The morphisms $p, q$ are first and second projection respectively and $q_B$ is a projection. The remaining morphisms are the natural ones, or they are given by composition of the aforementioned morphisms.

b. This equality follows from an application of the projection formula, which we use in the first equality below.

$$\begin{aligned} q_*(\iota_{B*}\mathcal{O}_B \otimes p^*\mathcal{L}) &= q_*(\iota_{B*}(\mathcal{O}_B \otimes \iota_B^*p^*\mathcal{L})) \\ &= (q \circ \iota_B)_*(\mathcal{O}_B \otimes (p \circ \iota_B)^*) \\ &= q_{B*}(\mathcal{O}_B \otimes p_B^*\mathcal{L}) = q_{B*}(p_B^*\mathcal{L}) = q_{B*}p_B^*\mathcal{L}. \end{aligned}$$

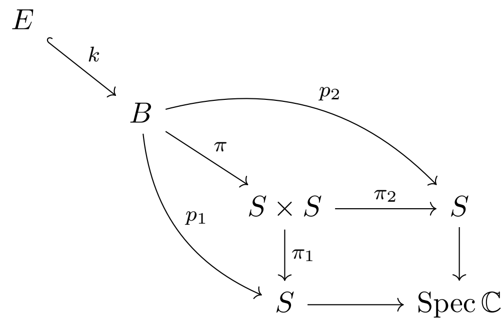
The penultimate equality is a trivial computation in the Picard group of $B$.

c. Exchange "$B$" with "$B_2$" in $b$.

□

Along with Diagram (7.4), we will be needing the following pullback diagram.

**Remark 7.5.** Diagram (7.8) is the fibre product of $S \times S$ along with the data $(E, k)$, which represents the closed embedding of the exceptional divisor into the blowup $B$. The morphism $p_2$ equals the morphism $p_B$ in Diagram 7.4, as follows by the identification $\mathcal{Z}_2(S) = B$. △




(7.8)

**Convention 7.6.**
For the remainder of this chapter, all unspecified morphisms refer to the morphisms in Diagrams (7.4) & (7.8).

## 7.3 Initial Computations of $c_\bullet(S_T(\mathcal{L},2))$

This section is dedicated to reducing the computation of $c_\bullet(S_T(\mathcal{L},2))$ to classes of other sheaves, as well as describing the further complications this brings. Our initial strategy will be to apply Whitney's formula to the following short exact sequence.

**Lemma 7.7.**
*The sequence of $\mathcal{O}_{S^{[2]}}$-modules*

$$0 \longrightarrow q_*(p^*\mathcal{L} \otimes \mathcal{I}_B/\mathcal{I}_B^2) \longrightarrow q_*(p^*\mathcal{L} \otimes \mathcal{O}_{B_2}) \longrightarrow q_*(p^*\mathcal{L} \otimes \mathcal{O}_B) \longrightarrow 0 \tag{7.9}$$

*is an exact sequence of locally free sheaves.*

*Proof.* The sequence of locally free sheaves

$$0 \longrightarrow \mathcal{I}_B/\mathcal{I}_B^2 \longrightarrow \mathcal{O}_{B_2} \longrightarrow \mathcal{O}_B \longrightarrow 0$$

is exact by the fact that $\mathcal{O}_{B_2} = \mathcal{O}_{S^{[2]}\times S}/\mathcal{I}_B^2$ and $\mathcal{O}_B = \mathcal{O}_{S^{[2]}\times S}/\mathcal{I}_B$. Tensoring by $p^*\mathcal{L}$, a line bundle, is exact, giving us the short exact sequence

$$0 \longrightarrow p^*\mathcal{L} \otimes \mathcal{I}_B/\mathcal{I}_B^2 \longrightarrow p^*\mathcal{L} \otimes \mathcal{O}_{B_2} \longrightarrow p^*\mathcal{L} \otimes \mathcal{O}_B \longrightarrow 0\ .$$

Note that this is a sequence of $\mathcal{O}_B$-modules. Pushing the sequence forward by $q$ is exact, since $q$ is finite. This leaves us with the sought after short exact sequence

$$0 \longrightarrow q_*(p^*\mathcal{L} \otimes \mathcal{I}_B/\mathcal{I}_B^2) \longrightarrow q_*(p^*\mathcal{L} \otimes \mathcal{O}_{B_2}) \longrightarrow q_*(p^*\mathcal{L} \otimes \mathcal{O}_B) \longrightarrow 0\ .$$

This is a sequence of locally free sheaves, since the second and third sheaf are locally free by Remark 5.5 and Corollary 5.21. □

Note that $\mathcal{I}_B/\mathcal{I}_B^2$ is supported on $B$. Henceforth, we shall abuse notation slightly by writing $\mathcal{I}_B/\mathcal{I}_B^2$ when we think of it as a sheaf on $B$. With this notation we may appropriately relabel the sheaves (using Lemma 7.4) of the sequence (7.9) to

$$0 \longrightarrow \mathcal{F} \longrightarrow S_T(\mathcal{L},2) \longrightarrow \mathcal{L}_{[2]} \longrightarrow 0 \tag{7.10}$$

wherein we define $\mathcal{F} := q_{B*}(p_B^*\mathcal{L} \otimes \mathcal{I}_B/\mathcal{I}_B^2)$.

Applying Whitney's formula to sequence (7.10) reduces the problem of computing $c_\bullet(S_T(\mathcal{L},2))$ to computing $c_\bullet(\mathcal{F})$ and $c_\bullet(\mathcal{L}_{[2]})$. As stated in Remark 5.6, the Chern classes of the tautological bundles $\mathcal{L}_{[n]}$ are known. The problem of computing $c_\bullet(S_T(\mathcal{L},2))$ therefore further reduces to computing $c_\bullet(\mathcal{F})$. This task is the subject of Section 7.5. Our first approach of computing $c_\bullet(\mathcal{F})$ requires us to do some more bookkeeping, which we dedicate the next section to.

## 7.4 A Blowup Short Exact Sequence

Our first approach of computing $c_\bullet(\mathcal{F})$ starts by computing $c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2)$. As will be shown in Section 7.5, this computation is reducible to the computation of $c_\bullet(\Omega_B)$, which motivates our interest in proving exactness of the following sequence.

**Lemma 7.8.**
*Let the setting be as in Diagram 7.8, where $\pi$ is the blow-up morphism and $k$ is the inclusion of the exceptional divisor. Then the sequence*

$$0 \longrightarrow \pi^*\Omega_{S\times S} \longrightarrow \Omega_B \longrightarrow k_*\Omega_{E/\Delta} \longrightarrow 0 \tag{7.11}$$

*is exact, and*

$$k_*\Omega_{E/\Delta} \cong \Omega_{B/S\times S}.$$

*Proof.* Recall the cotangent exact sequence of $\mathcal{O}_B$-modules (see for instance 22.2.25 in [Vak22]):

$$\pi^*\Omega_{S\times S} \longrightarrow \Omega_B \longrightarrow \Omega_{B/S\times S} \longrightarrow 0$$

We claim that this sequence is short exact. Indeed, the blowup morphism $\pi$ restricted to $U$ is an isomorphism. The morphism $\pi^*\Omega_{S\times S} \to \Omega_B$ is therefore trivially an isomorphism on $U$. But a morphism of locally free sheaves is injective if there exists an open on which its restriction is injective. The morphism

$$\pi^*\Omega_{S\times S} \longrightarrow \Omega_B$$

is thus injective, proving short exactness of the sequence

$$0 \longrightarrow \pi^*\Omega_{S\times S} \longrightarrow \Omega_B \longrightarrow \Omega_{B/S\times S} \longrightarrow 0\ . \tag{7.12}$$

Observe that (7.11) and (7.12) only differ by their rightmost sheaves. This motivates us to prove exactness of (7.12) by proving the isomorphism $k_*\Omega_{E/\Delta} \cong \Omega_{B/S\times S}$, thereby asserting that it is identical to sequence (7.11).

By 22.2.27b) in [Vak22], commutativity of the diagram

$$\begin{array}{ccc} E & \xrightarrow{\;k\;} & \mathrm{Bl}_\Delta(S\times S) \\ \big\downarrow{\scriptstyle \phi} & \ulcorner & \big\downarrow{\scriptstyle \pi} \\ \Delta & \xhookrightarrow{\;closed\;} & S\times S \end{array} \tag{7.13}$$

implies

$$k^*\Omega_{B/S\times S} \overset{\beta}{\cong} \Omega_{E/\Delta}.$$

By adjointness, we obtain a morphism

$$\Omega_{B/S\times S} \xrightarrow{\alpha} k_*\Omega_{E/\Delta}$$

that fits into the diagram

$$\begin{array}{ccc} k^*\Omega_{B/S\times S} & \xrightarrow{\beta} & \Omega_{E/\Delta} \cong k^*k_*\Omega_{E/\Delta} \\ {\scriptstyle k^*}\uparrow & & {\scriptstyle k^*}\uparrow \\ \Omega_{B/S\times S} & \xrightarrow{\alpha} & k_*\Omega_{E/\Delta} \end{array} \tag{7.14}$$

The morphism $\alpha$ is an isomorphism if it induces isomorphisms on stalks, by [Har77, Prop II.1.1]. Recall that if $\iota : Z \hookrightarrow X$ is a closed immersion and $\mathcal{F}$ is an $\mathcal{O}_Z$-module, then $\iota^*\iota_*\mathcal{F} \cong \mathcal{F}$ [Stacks, Tag 04CJ]. Since $k$ is a closed immersion, we obtain the following deconstruction of the map $\beta$:

$$k^*\Omega_{B/S\times S} \xrightarrow{k^*\alpha} k^*k_*\Omega_{E/\Delta} \cong \Omega_{E/\Delta}.$$

We study the stalks of these sheaves. Note that if $x \in B \setminus E$, then both sheaves localize to the 0-ring on the stalk of $x$. Assume therefore $x \in E$. The equalities below follow from the identification of stalks of pullbacks along morphisms of ringed spaces (see e.g. [Stacks, Tag 0098]).

$$\begin{aligned} (k^*\Omega_{B/S\times S})_x &= \Omega_{B/S\times S,x} \otimes_{\mathcal{O}_{B,x}} \mathcal{O}_{E,x} \\ \cong (k^*k_*\Omega_{E/\Delta})_x &= k_*\Omega_{E/\Delta,x} \otimes_{\mathcal{O}_{B,x}} \mathcal{O}_{E,x}. \end{aligned}$$

The isomorphism on stalks is, by adjunction, given by $\alpha_x \otimes \mathrm{id}$. The only remaining step is to prove that this implies $\alpha_x$ is an isomorphism. We may express this desire in a diagram. The morphism $\alpha_x$ factors through $\beta_x$ in the following way:

$$\begin{array}{ccc} \Omega_{B/S\times S,x} \otimes_{B,x} \mathcal{O}_{E,x} & \xrightarrow[\beta_x]{\sim} & k_*\Omega_{E/\Delta,x} \otimes_{B,x} \mathcal{O}_{E,x} \\ {\scriptstyle \sigma}\uparrow & & \downarrow{\scriptstyle \cong} \\ \Omega_{B/S\times S,x} & \xrightarrow{\alpha_x} & k_*\Omega_{E/\Delta,x} \end{array} \tag{7.15}$$

We wish to assert injectivity of the map $\sigma$, as this would make it a bijection. It is convenient to restate this diagram in terms of the definition of the blowup $B$ as a zero locus of a homogeneous ideal. We obtain the diagram

$$\begin{array}{ccc} \Omega_{B/\operatorname{Proj}(A),\,x} \otimes_{\mathcal{O}_{B,x}} (\mathcal{O}_B/\mathcal{I}_E)_x & \xrightarrow[\beta_x]{\sim} & (k_*\Omega_{E/\Delta,\,x})_{B,x} \otimes_{\mathcal{O}_{B,x}} (B/\mathcal{I}_E)_x \\ {\scriptstyle \sigma}\uparrow & & \downarrow{\scriptstyle \cong} \\ \Omega_{B/\operatorname{Proj}(A),\,x} & \xrightarrow{\alpha_x} & k_*\Omega_{E/\Delta,\,x} \end{array} \tag{7.16}$$

where $\operatorname{Proj} A = S \times S$. Restating the fibre product diagram (7.13) with this notation gives us Diagram (7.17)

$$\begin{array}{ccc} B \times_{\operatorname{Proj} A} E & \xrightarrow{k} & B = \mathrm{Bl}_\Delta(\operatorname{Proj} A) \\ \downarrow{\scriptstyle \phi} & \ulcorner & \downarrow{\scriptstyle \pi} \\ \Delta & \xhookrightarrow{\textit{closed}} & \operatorname{Proj} A \end{array} \tag{7.17}$$

We may assume that the blowup $B$ is embedded in $\mathbb{P}^n_A = \mathbb{P}^{n-1} \times \operatorname{Spec} A = \operatorname{Proj} A[x_1, ..., x_n]$ as the zero set of the ideal $I = (I_{i,j})_{i,j}$ generated by the homogeneous forms $I_{i,j} = x_i f_j - x_j f_i$. We may further assume $I_E = (f_1, ..., f_n)$ where $\widetilde{I_E} = \mathcal{I}_E$ and where the $f_i$'s form a regular sequence. Defining $Bl$ such that $B = \operatorname{Proj}(Bl)$, our goal has become to show that $I_E \cdot \Omega_{Bl/A} = (0)$, as this would mean the kernel of $\sigma'$ is zero, in terms of the sequence

$$0 \longrightarrow I_E \cdot \Omega_{Bl/A} \longrightarrow \Omega_{Bl/A} \xrightarrow{\sigma'} \Omega_{Bl/A} \otimes Bl/(I_E \cdot \Omega_{B/A}) \longrightarrow 0 .$$

Since localization commutes with the tensor product, we may reduce to the case where we study the non-localized variation of diagram (7.16).

We prove the statement $I_E \cdot \Omega_{Bl/A} = (0)$ by verifying it on the distinguished opens $D_+(x_i) \in \mathbb{P}^n_A$. We start by checking $i = 1$, and remark that the other cases will be similar. Fix $U = D_+(x_1)$ and choose coordinates $y_i = x_i/x_1$. The scheme $B \cap U$ equals the zero locus of the expressions $f_j - y_j f_1$. The module $\Omega_{Bl/A}$ is therefore generated by the elements $dg = \sum_I a_I d \prod y_i^{n_i} = \sum_I a_I \sum_i n_i dy_i$, where $g \in Bl \cap U = \operatorname{Spec}(A\,[y_2, ..., y_n]\,/(f_j - y_j f_1)_{j=2,...,n})$. The module $\Omega_{Bl/A}$ is therefore generated by the elements $dy_i$. Furthermore, by the relations defining $Bl \cap U$, we have that $I_E = (f_1, ..., f_j) = (f_1)$. Hence, for the product $\Omega_{Bl/A}$ to vanish, it suffices to show that $f_1 dy_i = 0$. By the Leibniz rule, and the fact that $f_1 \in A$, we have that

$$f_1 dy_i = d(f_1 y_i).$$

But by the defining relation $f_i = y_i f_1$ of $B \cap U$, we obtain the equality

$$d(f_1 y_i) = d(f_i) = 0.$$

Which is what we wanted to show. We therefore have that $I_E \cdot \Omega_{B/A} = 0$, which implies injectivity of $\sigma'$. The map $\alpha_x$ thus factors through three isomorphisms, meaning it is an isomorphism itself. □

The proof above does not rely explicitly on our varieties, meaning it generalizes to the following more general setting.

**Lemma 7.9.** (Generalization of Lemma 7.8)
*Let the setting be that of the pullback/blowup diagram*

$$\begin{array}{ccc} E & \xhookrightarrow{\iota} & Bl_Y(X) \\ \downarrow{\scriptstyle g} & \ulcorner & \downarrow{\scriptstyle \pi} \\ Y & \longrightarrow & X \end{array}$$

*where $\iota$ is a closed immersion, $\pi$ is the blowup morphism and $X, Y$ are smooth projective varieties. Then the sequence*

$$0 \longrightarrow \pi^*\Omega_X \longrightarrow \Omega_{Bl_Y(X)/X} \longrightarrow \iota_*\Omega_{E/Y} \longrightarrow 0 \tag{7.18}$$

*is exact, and*

$$\iota_*\Omega_{E/Y} \cong \Omega_{Bl_Y(X)/X}.$$

Note that if $X, Y$ are smooth, then so are $E$ and $\operatorname{Bl}_Y(X)$, meaning we do not have to assume the latter to match our setting.

## 7.5 Computing $c_\bullet(\mathcal{F}) = c_\bullet(q_{B*}(p_B^*\mathcal{L} \otimes \mathcal{I}_B/\mathcal{I}_B^2))$

We present two different approaches of computing $c_\bullet(\mathcal{F})$. The approaches differ in that they rely on applying Whitney's formula to different sequences, which in turn lead to different complications in expressing $c_\bullet(\mathcal{F})$ in terms of Chern classes that are known. In particular, the first approach uses the blowup short exact sequence of Section 7.4. The second approach relies on a conjecture wherein we claim short exactness of a sequence, motivated by the claim that $B$ is generically a cyclic double cover of $S^{[2]}$.

The result of the second approach is somewhat better, as it completely reduces the computation of $c_\bullet(\mathcal{F})$ down to computing the Chern classes of a relatively innocent looking sheaf. The first approach is, however, quite insightful, justifying its presence in this thesis.

### 7.5.1 Approach 1: Computing $c_\bullet(\mathcal{F})$ via $c_\bullet(k_*(\Omega_{E/\Delta}))$

Before we compute the Chern classes of $\mathcal{F} = q_{B*}(p_B^*\mathcal{L} \otimes \mathcal{I}_B/\mathcal{I}_B^2)$, it seems natural to explain what we mean by 'computing $c_\bullet(\mathcal{F})$'. In our case, the word compute means to express $c_\bullet(\mathcal{F})$ mostly in terms of the Chern classes of the bundles $p_B^*\mathcal{L}$, and $\mathcal{I}_B/\mathcal{I}_B^2$. Example 3.8, Remark 5.7 and Lemma 6.2 motivate our desire in this endeavour, as they exemplify that the Chern classes of complicated bundles can sometimes be expressed in terms of the Chern classes of the bundles they are made up of. We shall consider the classes $c_\bullet(p_B^*\mathcal{L})$ to be known, since the pullback of the total Chern class is well behaved along flat morphisms. We shall therefore focus our attention on the sheaf $\mathcal{I}_B/\mathcal{I}_B^2$.

The computation of $c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2)$ is performed in three steps. The first step is to prove exactness of a particular sequence that contains $\mathcal{I}_B/\mathcal{I}_B^2$. The second step is to apply Whitney's formula, and reduce the expression of $c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2)$ to bundles whose total Chern class is known. We manage to express $c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2)$ to a product containing only one term, $c_\bullet(k_*(\Omega_{E/\Delta}))$, whose Chern classes we do not know. The chapter finishes off by sketching a possible continuation (in which we assume $c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2)$ to be known) by applying Grothendieck-Riemann-Roch to $\mathcal{F}$.

#### Computing $c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2)$

We finally compute $c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2)$ using, among others, the blowup exact sequence of the previous section.

**Proposition 7.10.** (Computation of $c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2) \in A(B)$)
*Let the setting be that of Diagrams (7.4) and (7.8). We have that*

$$c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2) = \frac{q_B^*\left[c_\bullet(\Omega_{S^{[2]}})\right]}{p_1^*\left[c_\bullet(\Omega_S)\right] c_\bullet(k_*(\Omega_{E/\Delta}))}. \tag{7.19}$$

*Proof.* The sequence

$$0 \longrightarrow \mathcal{I}_B/\mathcal{I}_B^2 \longrightarrow \iota_B^*\Omega_{S^{[2]}\times S} \longrightarrow \Omega_B \longrightarrow 0$$

is exact since $B$ and $S^{[2]} \times S$ are smooth, and $\iota_B$ is a closed immersion. We apply Whitney's formula to it, obtaining

$$c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2) = \frac{c_\bullet(\iota_B^*\Omega_{S^{[2]}\times S})}{c_\bullet(\Omega_B)}. \tag{7.20}$$

Recall that the cotangent sheaves of products split (see [Har77, Exercise II.8.3]) in the sense that

$$\iota_B^*(\Omega_{S^{[2]}\times S}) = \iota_B^* q^*\Omega_{S^{[2]}} \oplus \iota_B^* p^*\Omega_S.$$

Using that diagrams (7.4) and (7.8) commute, Whitney's formula, and the fact that the total Chern class commutes with pullback, we may reformulate (7.20) to

$$c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2) = \frac{q_B^*\left[c_\bullet(\Omega_{S^{[2]}})\right] p_B^*\left[c_\bullet(\Omega_S)\right]}{c_\bullet(\Omega_B)}. \tag{7.21}$$

The cotangent sheaves of $S$ and $S^{[2]}$ are considered variables, as they certainly differ from surface to surface. This reduces the proof to computing $c_\bullet(\Omega_B)$.
By Lemma 7.8, we have that the sequence

$$0 \longrightarrow \pi^*\Omega_{S\times S} \longrightarrow \Omega_B \longrightarrow k_*\Omega_{E/\Delta} \longrightarrow 0$$

is exact. Applying the same tools as before, we obtain the chain of equalities below.

$$\begin{aligned} c_\bullet(\Omega_B) &= c_\bullet(\pi^*(\Omega_{S\times S}))c_\bullet(k_*(\Omega_{E/\Delta})) \\ &= c_\bullet(\pi^*(\pi_1^*(\Omega_S) \oplus \pi_2^*(\Omega_S)))c_\bullet(k_*(\Omega_{E/\Delta})) \\ &= p_1^*\left[c_\bullet(\Omega_S)\right] p_2^*\left[c_\bullet(\Omega_S)\right] c_\bullet(k_*(\Omega_{E/\Delta})). \end{aligned}$$

Inserting this into equation 7.21 yields

$$c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2) = \frac{q_B^*\left[c_\bullet(\Omega_{S^{[2]}})\right] p_B^*\left[c_\bullet(\Omega_S)\right]}{p_1^*\left[c_\bullet(\Omega_S)\right] p_2^*\left[c_\bullet(\Omega_S)\right] c_\bullet(k_*(\Omega_{E/\Delta}))}.$$

Since the morphism $p_2$ in Diagram 7.8 and the morphism $p_B$ in Diagram 7.4 are the same (as mentioned in Remark 7.5), two terms in the above expression cancel. This grants us the expression

$$c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2) = \frac{q_B^*\left[c_\bullet(\Omega_{S^{[2]}})\right]}{p_1^*\left[c_\bullet(\Omega_S)\right] c_\bullet(k_*(\Omega_{E/\Delta}))},$$

which is the desired equality. □

The proof of Proposition 7.10 raises a question: why did we bother proving the isomorphism in Lemma 7.8, when we might as well have expressed $c_\bullet(k_*(\Omega_{E/\Delta}))$ as $c_\bullet(\Omega_{B/S\times S})$? One answer is that the total Chern class of the sheaf of relative differentials is generally not known. Expressing the bundle as $k_*(\Omega_{E/\Delta})$ instead, kindles the hope of allowing an explicit computation of the total Chern class of $c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2)$ using Grothendieck-Riemann-Roch.

### Further Deliberations on Approach 1

Approach 1 leaves us with two problems. Firstly: we do not know the Chern classes $c_\bullet(k_*\Omega_{E/\Delta})$. If we were to learn these classes, we would by Approach 1 know the total Chern class $c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2)$, which in turn would mean we could express the total Chern class of $\mathcal{E} := p_B^*\mathcal{L} \otimes \mathcal{I}_B/\mathcal{I}_B^2$, by an easy application of Proposition 3.9. This is where the second problem comes in. Assuming $c_\bullet(\mathcal{E})$ is known, can we express $c_\bullet(\mathcal{F}) = c_\bullet(q_{B*}\mathcal{E})$ in terms of the Chern classes of $\mathcal{E}$?

We do not have any answers to these problems, but we have given them some thought. Regarding the first problem, we can say the following about its setting. We are going to need the following well-known fact.

**Fact 7.11.** Let $X$ be a smooth projective variety, and let $X \overset{\Delta}{\hookrightarrow} X \times X$ be the diagonal embedding. Then $\mathcal{N}_{\Delta/X\times X} \cong \mathcal{T}_X$.

Combining this fact with Theorem 3.20.$b$), we obtain a result that helps us identify the cotangent sheaf $\Omega_{E/\Delta}$. Namely, we have the isomorphisms

$$\Omega_{E/\Delta} \cong \Omega_{\mathbb{P}(\mathcal{N}_{\Delta/S\times S})/\Delta} \cong \Omega_{\mathbb{P}(\mathcal{T}_S)/\Delta}.$$

If this identification allows for a simpler computation of $c_\bullet(\Omega_{\mathbb{P}(\mathcal{T}_S)/\Delta})$, then odds are it is possible to use GRR to compute the class $c_\bullet(k_*(\Omega_{E/\Delta})) = c_\bullet(k_*(\Omega_{\mathbb{P}(\mathcal{T}_S)/\Delta}))$.

What regards the second problem, let us assume we have been successful in expressing $c_\bullet(\mathcal{F})$ in terms of known classes. We apply Grothendieck-Riemann-Roch to $\mathcal{F}$, but remark that we do not complete the computation. For ease of notation, we denote $\mathrm{td}_X := \mathrm{td}(\mathcal{T}_X)$. Theorem 3.11 (GRR) states that

$$\mathrm{ch}(q_{B*}\mathcal{E}) \cdot \mathrm{td}_{S^{[2]}} = q_{B*}(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}_B),$$

which we further reformulate to

$$\begin{aligned} \mathrm{ch}(q_{B*}\mathcal{E}) &= q_{B*}(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}_B)\,\mathrm{td}_{S^{[2]}}^{-1} \\ &= q_{B*}(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}_B \cdot q_B^*(\mathrm{td}_{S^{[2]}}^{-1})) \\ &= q_{B*}(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}(\mathcal{T}_{B/S^{[2]}})). \end{aligned}$$

The last equality is obtained in the following manner. Recall the exact cotangent sequence, which is short exact since our varieties are smooth and projective

$$0 \longrightarrow q_B^*\Omega_{S^{[2]}} \longrightarrow \Omega_B \longrightarrow \Omega_{B/S^{[2]}} \longrightarrow 0\,.$$

The pullback commutes with dualizing, since the sheaves are locally free of finite rank. By dualizing we obtain

$$0 \longrightarrow \mathcal{T}_{B/S^{[2]}} \longrightarrow \mathcal{T}_B \longrightarrow q_B^*\mathcal{T}_{S^{[2]}} \longrightarrow 0\,.$$

Applying additivity of the Todd classes, we obtain the equality

$$\mathrm{td}(\mathcal{T}_{B/S^{[2]}}) = \frac{\mathrm{td}(\mathcal{T}_B)}{\mathrm{td}(q_B^*\mathcal{T}_{S^{[2]}})}.$$

Hence, it is reasonable to assume that if one manages to compute the Todd class $\mathrm{td}(\mathcal{T}_{B/S^{[2]}})$, then one can ascertain an expression for $\mathrm{ch}(q_{B*}\mathcal{E})$.

### 7.5.2 Approach 2: Computing $c_\bullet(\mathcal{F})$ via $c_\bullet(\mathcal{I}_E/\mathcal{I}_E^2)$

The computation of $c_\bullet(\mathcal{F})$ in this approach is quite simple, but it hinges on a sequence being short exact, which is something we conjecture. We begin by performing the computation before motivating the conjecture.

### The Computation

Recall that the morphisms in this chapter are the ones given in Diagrams (7.4) & (7.8) and recall the short exact sequence

$$0 \longrightarrow q_{B*}(p_B^*\mathcal{L} \otimes \mathcal{I}_B/\mathcal{I}_B^2) \longrightarrow S_T(\mathcal{L}, 2) \longrightarrow \mathcal{L}_{[2]} \longrightarrow 0\ .$$

As in the previous section, we shall begin by computing $c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2)$, which once again requires us to compute $c_\bullet(\Omega_B)$. This is where our two approaches diverge; in this section we compute $c_\bullet(\Omega_B)$ by applying Whitney's formula to a short exact sequence that is different to the one in Lemma 7.8. This novel expression of $c_\bullet(\mathcal{I}_B/\mathcal{I}_B^2)$ makes the final expression of $c_\bullet(\mathcal{F})$ quite simple, and even relates it to the tautological bundles of Remark 5.7.

For the sake of simplicity, assume $\mathcal{L} = \mathcal{O}_S$, making the deriving of an expression somewhat less messy. Consider again, the exact sequence

$$0 \longrightarrow \mathcal{I}_B/\mathcal{I}_B^2 \longrightarrow \iota_B^*\Omega_{S\times S^{[2]}} \longrightarrow \Omega_B \longrightarrow 0\ .$$

The cotangent sheaf of a direct sum splits, giving us

$$\Omega_{S\times S^{[2]}} = p^*\Omega_S \oplus q^*\Omega_{S^{[2]}}.$$

Let us express the class of $\mathcal{I}_B/\mathcal{I}_B^2$ in the Grothendieck ring $K(B)$:

$$\mathcal{I}_B/\mathcal{I}_B^2 = \iota_B^*\Omega_{S\times S^{[2]}} - \Omega_B \tag{7.22}$$
$$= p_B^*\Omega_S + q_B^*\Omega_{S^{[2]}} - \Omega_B, \tag{7.23}$$

where we have used that $(p_B)^* = \iota_B^* \circ p^*$ and $(q_B)^* = \iota_B^* \circ q^*$. We conjecture that the following sequence is short exact.

**Conjecture 7.12.**
The sequence

$$0 \longrightarrow q_B^*\Omega_{S^{[2]}} \longrightarrow \Omega_B \longrightarrow \mathcal{I}_E/\mathcal{I}_E^2 \longrightarrow 0 \tag{7.24}$$

is an exact sequence of locally free sheaves.

We motivate the conjecture by sketching a suggestion of proof in the next section.

By the sequence, we have the relation

$$\Omega_B = q_B^*\Omega_{S^{[2]}} + \mathcal{I}_E/\mathcal{I}_E^2$$

in the Grothendieck ring of $B$. Inserting this into (7.23) gives us

$$\begin{aligned}\mathcal{I}_B/\mathcal{I}_B^2 &= p_B^*\Omega_S + q_B^*\Omega_{S^{[2]}} - \Omega_B \\ &= p_B^*\Omega_S + q_B^*\Omega_{S^{[2]}} - q_B^*\Omega_{S^{[2]}} - \mathcal{I}_E/\mathcal{I}_E^2 \\ &= p_B^*\Omega_S - \mathcal{I}_E/\mathcal{I}_E^2.\end{aligned}$$

Since $E$ is a divisor on $B$, we have the well known-identification:

$$\mathcal{I}_E/\mathcal{I}_E^2 = k_*\mathcal{O}_E(-E), \tag{7.25}$$

where $k : E \hookrightarrow B$ is the inclusion and $\mathcal{O}_E(-E)$ is the line bundle associated to the divisor $E$. Applying the pushforward $q_{B*}$ to both sides of the equality $\mathcal{I}_B/\mathcal{I}_B^2 = p_B^*\Omega_S - \mathcal{I}_E/\mathcal{I}_E^2$ gives us

$$q_{B*}(\mathcal{I}_B/\mathcal{I}_B^2) = q_{B*}(p_B^*\Omega_S) - q_{B*}(k_*\mathcal{O}_E(-E)).$$

By Remark 5.7 we have that $q_{B*}(p_B^*\Omega_S) = (\Omega_S)_{[2]}$. Hence,

$$q_{B*}(\mathcal{I}_B/\mathcal{I}_B^2) = (\Omega_S)_{[2]} - (q_B \circ k)_*(\mathcal{O}_E(-E)).$$

Performing the same sequence of operations without the simplification $\mathcal{L} = \mathcal{O}_S$ gives us the following more general result.

**Proposition 7.13.**
*We have that*

$$\mathcal{F} \overset{def}{=} q_{B*}(\mathcal{I}_B/\mathcal{I}_B^2 \otimes p_B^*\mathcal{L}) = (\Omega_S \otimes \mathcal{L})_{[2]} - q_{B*}(k_*\mathcal{O}_E(-E) \otimes p_B^*\mathcal{L}), \tag{7.26}$$

*where the sheaf $(\Omega_S \otimes \mathcal{L})_{[2]}$ is a tautological bundle, as defined in Remark 5.7. Expressing this in the language of Chern classes, we have*

$$c_\bullet(\mathcal{F}) = \frac{c_\bullet((\Omega_S \otimes \mathcal{L})_{[2]})}{c_\bullet(q_{B*}(k_*\mathcal{O}_E(-E) \otimes p_B^*\mathcal{L}))}.$$

It is clear from proposition 7.13 why we consider Approach 2 to be the more promising approach. Obtaining a closed expression of $c_\bullet(\mathcal{F})$ only requires the computation of $c_\bullet(q_{B*}(k_*\mathcal{O}_E(-E) \otimes p_B^*\mathcal{L}))$, since the total Chern class $c_\bullet((\Omega_S \otimes \mathcal{L})_{[2]})$ is computable with the methods in [EGL99]. It therefore seems likely that an application of Grothendieck-Riemann-Roch to $q_{B*}(k_*\mathcal{O}_E(-E) \otimes p_B^*\mathcal{L})$ will reveal the total Chern class $c_\bullet(\mathcal{F})$.

### Motivating Conjecture 7.12

Recall that $q_B : B \to S^{[2]}$ is a flat finite morphism. We make the following claim.

**Claim 7.14.** The morphism $q_B$ is generically a cyclic double cover. It is ramified over the closed subscheme $D := \{Z \in S^{[2]} \mid \operatorname{Supp} Z = \{p\} \text{ where } p \in S\}$. Letting $U := S^{[2]} \setminus D$, we claim $q_B$ is étale when restricted to $B \times_{S^{[2]}} U$.

We sketch the situation in Diagram (7.27), where we have called $\tilde{D} := D \times_{S^{[2]}} B$.

$$\begin{array}{ccccc} \tilde{D} & \longrightarrow & B & \longleftarrow & B \times_{S^{[2]}} U \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & S^{[2]} & \longleftarrow & U \end{array} \tag{7.27}$$

Consider the exact sequence

$$q_B^*\Omega_{S^{[2]}} \longrightarrow \Omega_B \longrightarrow \Omega_{B/S^{[2]}} \longrightarrow 0\,.$$

Assuming the claim, since derivations pull back and $q_B$ restricted to $B \times_{S^{[2]}} U$ is étale, we have that the sheaves $q_B^*\Omega_{S^{[2]}}|_{B\times_{S^{[2]}}U} \cong \Omega_B|_U$ are isomorphic. In particular the map between them is injective, implying that the unrestricted map of sheaves is also injective. We therefore obtain left-exactness of the sequence

$$0 \longrightarrow q_B^*\Omega_{S^{[2]}} \longrightarrow \Omega_B \longrightarrow \Omega_{B/S^{[2]}} \longrightarrow 0\,.$$

The goal is now to prove that $\Omega_{B/S^{[2]}} \cong \mathcal{I}_E/\mathcal{I}_E^2$. A further approach would be to restrict the bundles to the ramification divisor $q_B^{-1}(D) = E \subset B$, and compare the two.

# Appendices

# Appendix A

# Appendix

## A.1 Assorted Proofs

### A.1.1 Proof of Example 3.13

*Proof.* We have that

$$J^0\mathcal{L} = \pi_{1*}(\mathcal{O}_{Y\times Y}/\mathcal{I}_\Delta \otimes \pi_2^*\mathcal{L}).$$

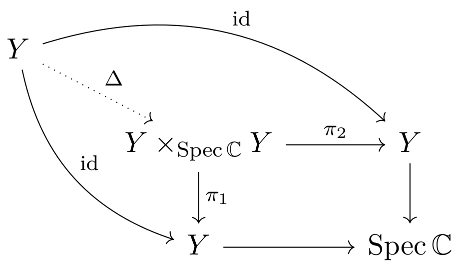
Now recall that as a sheaf over $Y$, we have that $\mathcal{O}_{Y\times Y}/\mathcal{I}_\Delta^1$ simply means $\Delta_*\mathcal{O}_Y$, where $\Delta$ is the diagonal map $Y \to Y \times Y$. We thus have

$$J^0\mathcal{L} = \pi_{1*}(\Delta_*\mathcal{O}_Y \otimes \pi_2^*\mathcal{L}).$$

Using the projection formula, we obtain

$$J^0\mathcal{L} = \pi_{1*}\Delta_*(\mathcal{O}_Y \otimes \Delta^*\pi_2^*\mathcal{L}).$$

By using the categorical identities and the definition of the diagonal, we obtain $\pi_{1*}\Delta_* = (\pi_1\Delta)_* = (\mathrm{id})*$ and $\Delta^*\pi_2^* = (\pi_2\Delta)^* = (\mathrm{id})^*$.




Commutative diagram which illustrates the definition of $\Delta$.

This reduces the expression of $J^0\mathcal{L}$ to

$$J^0\mathcal{L} = (\mathrm{id})_*(\mathcal{O}_Y \otimes (\mathrm{id})^*\mathcal{L}),$$

making it clear that

$$J^0\mathcal{L} = \mathcal{O}_Y \otimes \mathcal{L}.$$

This is a simple computation in the Picard group of $Y$, granting us

$$J^0\mathcal{L} = \mathcal{L}.$$

□

### A.1.2 Proof of Corollary 3.14

*Proof.* We use the inductive principle on $n$, letting the induction hypothesis be the corollary. We saw earlier that $J^0\mathcal{L} = \mathcal{L}$, so the hypothesis holds for $n = 0$.

Suppose the corollary holds for some $n$. We need to show that $J^{n+1}\mathcal{L}$ is a vector bundle of rank $\binom{n+1+k}{k}$. By theorem 3.12 we have the short exact sequence

$$0 \longrightarrow \mathcal{L} \otimes \mathrm{Sym}^{n+1}(\Omega_Y) \longrightarrow J^{n+1}\mathcal{L} \longrightarrow J^n\mathcal{L} \longrightarrow 0$$

and by a well known property of short exact sequences of locally free sheaves, we have that $J^{n+1}\mathcal{L}$ is a vector bundle, since $J^n\mathcal{L}$ is a vector bundle by assumption and $\mathcal{L} \otimes \mathrm{Sym}^{n+1}(\Omega_Y)$ is a vector bundle. The latter have respective rank $\binom{n+k}{k}$ and $\binom{n+k}{k-1}$, and so $J^{n+1}\mathcal{L}$ must have rank $\binom{n+k}{k} + \binom{n+k}{k-1}$, which by recurrence of binomial coefficients equals $\binom{n+1+k}{k}$. This is the desired rank of $J^{n+1}\mathcal{L}$. □

### A.1.3 Computing the Fibres of $\mathcal{L}_{[n]}$

**Claim A.1.** Let $S$ be a smooth projective surface equipped with a line bundle $\mathcal{L}$. Then the fibre of the induced tautological bundle $\mathcal{L}_{[n]}$ at a point $[Z] \in S^{[n]}$ is $H^0(Z, \mathcal{O}_Z)$.

*Proof.* Consider the pullback diagram below.

$$\begin{array}{ccccc} [Z] \times_{S^{[n]}} \mathcal{Z}_n(S) & \overset{\iota}{\hookrightarrow} & \mathcal{Z}_n(S) & \overset{q}{\longrightarrow} & S \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} & & \\ [Z] & \overset{i}{\hookrightarrow} & S^{[n]} & & \end{array}$$

We have that $q^*(\mathcal{L})$ is a line bundle on $\mathcal{Z}_n(S)$. Furthermore, we have that

$$[Z] \times_{S^{[n]}} \mathcal{Z}_n(S) = (\pi \circ \iota)^{-1}([Z]) \cong Z,$$

and that $i^*(p_*(q^*(\mathcal{L}))) = \pi_*(\iota^*(q^*(\mathcal{L})))$, since $p$ is flat. This gives us the set of equalities

$$\begin{aligned} \mathcal{L}|_{[Z]} &= i^*(\mathcal{L}_{[n]}) \\ &= i^*(p_*q^*(\mathcal{L})) \\ &= \pi_*(\iota^*(q^*(\mathcal{L}))) \\ &= \pi_*(\mathcal{L}|_Z) \\ &= H^0(Z, \mathcal{L}|_Z) \cong H^0(Z, \mathcal{O}_Z). \end{aligned}$$

□

### A.1.4 $\mathcal{L}$ is 0-Very Ample $\iff$ $\mathcal{L}$ is Globally Generated

**Claim A.2.** Let $\mathcal{L}$ be a line bundle on a smooth projective variety $X$. Then $\mathcal{L}$ is 0-very ample if and only if $\mathcal{L}$ is globally generated.

*Proof.* Suppose $\mathcal{L}$ is 0-very ample. Then for all points $p \in X$, the map $\phi : H^0(X, \mathcal{L}) \to H^0(p, \mathcal{L}|_p)$ is surjective. Since $H^0(X, \mathcal{L})$ is a finite-dimensional $\mathbb{C}$-vector space, we have that there exists an $N \in \mathbb{N}$ such that

$$H^0(X, \mathcal{L}) = \mathbb{C}^N = (\mathcal{O}_{X,p}/\mathfrak{m}_p)^N = \mathcal{O}^N_{x,p}/\mathfrak{m}_p\mathcal{O}^N_{x\,p}.$$

Recall that $H^0(p, \mathcal{L}|_p) = \mathcal{L}_p/\mathfrak{m}_p\mathcal{L}_p$. Substituting these modules for the domain and codomain gives us that $\phi$ is a surjective map of $\mathbb{C}$-modules $\phi : \mathcal{O}^N_{x,p}/\mathfrak{m}_p\mathcal{O}^N_{x\,p} \to \mathcal{L}_p/\mathfrak{m}_p\mathcal{L}_p$. By Nakayama's lemma, we have that $\phi$ is surjective if and only if it lifts uniquely to a surjective map

$$\psi : \mathcal{O}^N_{X,p} \to \mathcal{L}_p.$$

Since $p$ was chosen arbitrarily, we have that $\mathcal{O}^N_X \to \mathcal{L}$ is surjective on stalks, meaning it is surjective. Hence $\mathcal{L}$ is globally generated. □

## A.2 The ongoing game of tennis in the part of my head that belongs to a mathematicican

*Mensen willen zijn als andere mensen maar anders,*
*onvergelijkbaar, eenmalig, uniek, om nooit te vergeten,*
*uitblinken willen ze graag in schoonheid, in macht of in wijsheid.*
*Maar de goden dulden het niet, en al wie te hoog vliegt*
*vangen ze in een kleverig web van futiele beletsels,*
*door hun onbetekenendheid zo vernietigend voor de*
*sterveling die zich op eigen kracht probeert te verheffen.*
*Vrienden, we weten wat we zijn maar niet wat we worden.*
*Dit zijn de mythen van mensen die in hun overmoed meenden*
*goden te evenaren, en van de val die ze maakten.*

- Imme Dros, *Griekse mythen*

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