UNIVERSITY
OF OSLO

Master's thesis

Tropical polypols

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Mathematics
60 ECTS study points

Department of Mathematics
Faculty of Mathematics and Natural Sciences

Spring 2023

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Tropical polypols

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Abstract

In this thesis we use tropical geometry to approach an open problem in real geometry. The problem, first posited by Wachspress, E.L. in [Wac75] and further explored by Kohn. et al. in [Koh+21] has to do with a set of objects called regular rational polypols. Wachspress found that one can assign a unique plane curve to each of these objects, and stated as a conjecture that he believes that this curve lie outside the polypol. Here we define analogous objects in tropical geometry and work towards evaluating a similar question. This is complicated by the fact that the tropical adjoint is not necessarily unique, and by the nature of tropical intersections. We approach the problem by restricting to simple cases, finding all the information we can about those, and then increasing the complexity gradually. In particular we give a complete description of all possible tropical polypols bounded by tropical lines. Then we begin a similar description of tropical polypols bounded by lines and one tropical conic. We also give a proof of our version of the conjecture for tropical polypols of degree four or less.

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Acknowledgements

First of all, I would like to thank my wonderful supervisors, Professor Kristian Ranestad and Doctor Karoline Moe, for their guidance over the course of this project. Thank you Kristian for giving me this interesting, challenging and engaging problem to work on, and for your patience and advice when I've been stuck. Thank you Karoline for helping me find different approaches to problems and for introducing me to algebraic geometry in the first place.

I would also like to acknowledge the other people who have been helpful in the work on this thesis. Thank you to Kris Shaw for advice and discussion on tropical geometry in general, and for recommending [Polymake]. Thank you to all my fellow students in room 1101 for providing a motivating environment, Tex support and distractions in the last two years. Finally, I would like to thank my parents and housemates for their support.

Introduction

In this thesis we will use tropical geometry to explore an open problem from real geometry.

Tropical geometry is a subfield of algebraic geometry studying a type of geometric objects with some unusual properties. Tropical plane curves can, like usual algebraic plane curves, be associated with polynomials, but with a different arithmetic than over usual rings. This leads to, among other things, all tropical curves being piecewise linear.

Polypols are a generalization of polytopes, that are have boundaries given by non-linear hypersurfaces. In [Koh+21] Kohn et.al. presents an overview of some currently known properties of these geometric objects. In particular, the question of the adjoint curves, of polypols in the plane. Among other things, Kohn et.al explores a conjecture posited by E. Wachspress in [Wac75] and [Wac80] concerning whether the adjoint includes points in the interior of the polypol. Wachspress believed that they do not. In [Koh+21] this is proven for some cases, but the question remains open.

In this thesis, we seek to explore a parallel question in tropical geometry. To do this we define tropical polypols and their adjoints. Then we describe what the adjoints of some tropical polypols look like, paying particular attention to the question of whether tropical adjoints are always unique. We then attempt to determine whether they can include points in the interior of the polypol. We give a detailed account of tropical polypols with sides from tropical lines. Then present some general results. And begin a similar description of polypols with sides from lines and conics. Finally, we present a few examples of types of polypols without giving a full description.

The following is a quick outline of the thesis. After the introduction, in Chapter 1 we lie the theoretical framework for tropical plane algebraic geometry in general. Here we give the definitions of some important concepts related to tropical plane curves and their intersections. We also give a brief discussion of uniqueness of tropical lines and conics through given sets of points.

Chapter 2 is the main chapter, where we present the main work done in this thesis. We first introduce the real definitions and some results relevant to the open question. Then we define tropical polypols, and present some general results. The rest of the chapter describe particular cases, and gives examples.

Finally, Chapter 3 contains a brief summary, and discussion of some open problems.

Chapter 1

Tropical plane geometry

This chapter gives some results and definitions for tropical geometry in general that are needed as preliminary information before the main part of the thesis. First we present the underlying arithmetic that govern tropical objects. The next section is concerned with tropical plane curves, giving some general definitions and a description of the types of tropical plane curves that show up in this thesis. The next two sections handle some important properties of plane curves. First we look at intersections of two curves and how this is different from classical geometry. Then the number of general points needed to uniquely determine a tropical curve in the plane. The final section deals with tropical convexity.

1.1 Arithmetic and polynomials

In this section we present the definitions of the tropical semiring and of tropical polynomials that we use in this thesis.

Definition 1.1.1 ([MS15, p. 2]). The *tropical semiring* $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ consists of the real numbers and infinity with two operations, $\oplus$ and $\odot$. Tropical addition is defined as equivalent to taking the minimum of two numbers, multiplication defined as equivalent to taking the real sum of two numbers:

$$\begin{aligned} x \oplus y :=& \min(x, y) \\ x \odot y :=& x + y. \end{aligned}$$

It is worth noting that some texts define tropical addition as the maximum of two numbers, rather than the minimum. All general results hold for both cases.

Definition 1.1.2 ([MS15, p. 5]). A *tropical polynomial* is a finite linear combination of tropical monomials

$$\begin{aligned} f(x_1, \ldots, x_n) =& a_1 \odot x_1^{j_{11}} x_2^{j_{12}} \ldots x_n^{j_{1n}} \oplus a_2 \odot x_1^{j_{21}} x_2^{j_{22}} \ldots x_n^{j_{2n}} \oplus \ldots \\ =& \min(a_1 + j_{11}x_1 + \cdots + j_{1n}x_n, a_2 + j_{21}x_1 + \cdots + j_{2n}x_n, \ldots) \end{aligned}$$

where $a_i \in \mathbb{R}$, $j_{ik} \in \mathbb{N}$.

1.2 Tropical plane curves

In this section we present a definition of tropical plane curves, describe some important properties and give descriptions of tropical lines, tropical conics and tropical cubics.

Definition 1.2.1 ([Vig08, p. 97]). The *tropical projective plane* $\mathbb{TP}^2$ is defined as $\mathbb{R}^3/$ where (x, y, z) (x', y', z') if and only if there is some $k \in \mathbb{R}$ such that $(x', y', z') = (x \odot k, y \odot k, z \odot k)$.

Given a homogeneous tropical polynomial $f(x, y, z) = \sum_{i,j,k:i+j+k=d} a_{i,j,k}x^i y^j z^k$, where $d \in \mathbb{N}$, the *tropical curve* C given by f is the set of points $(x_0, y_0, z_0) \in \mathbb{TP}^2$ such that the minimum of $f(x, y, z)$ is attained at least twice at (x_0, y_0, Z_0) [BS17]. We say that d is the *degree* of C.

To visualize the curves we set $z = 0$ and draw in $\mathbb{R}^2$. Then a tropical curve consists of a set of vertices, and a set of linear branches. The branches are either bounded, in which case they connect two vertices, or unbounded, in which case they only contain one vertex. To each branch, one can assign a multiplicity or *weight* m such that at each vertex, the curve fulfills a balancing condition. Said condition is that at each vertex, the sum of the direction vectors of each adjacent branch, in relation to the vertex, multiplied by the weight of the branch is zero [MS15, p. 12].

In this thesis we are mainly considering curves where all branches are of weight one.

A tropical curve is called *simple* if each vertex is either trivalent or is locally the intersection of two line segments [MS15, p. 33].

One construction that can be used to find properties of tropical curves is the so called *dual subdivision*. A tropical curve C of degree d is dual to a regular subdivision of the triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, d)$ and $(d, 0)$. We say that C is *smooth* if this subdivision consists of d^2 triangles all of area $\frac{1}{2}$ [MS15, p. 33].

Definition 1.2.2 ([MS15, p. 33]). Let C be a simple tropical curve, let $t(C)$ be the number of trivalent vertices, and let $r(C)$ be the number of unbounded branches. The *genus* of C is

$$g(C) = \frac{1}{2}t(C) - \frac{1}{2}r(C) + 1$$

Definition 1.2.3. We say that a curve of genus zero is *rational*.

1.2.1 Tropical lines

A tropical line is given by a polynomial of the form

$$a_1 \odot x \oplus a_2 \odot y \oplus a_3 \odot z.$$

It has one vertex and three branches in the directions north, east and southwest.

Example 1.2.4. Figure 1.1 on the facing page shows an example of a tropical line given by the polynomial

$$x \oplus y \oplus z \tag{1.1}$$

. Notice that it has genus $g = \frac{1}{2} - \frac{3}{2} + 1 = 0$.

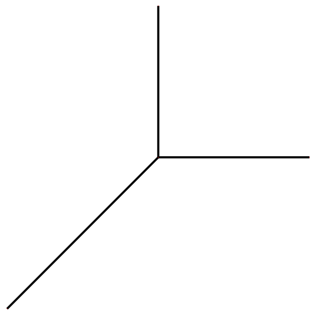


Figure 1.1: Tropical line

1.2.2 Tropical conics

A tropical conic is a tropical curve given by a polynomial on the form

$$a_1 \odot x^2 \oplus a_2 \odot x \odot y \oplus a_3 \odot y^2 \oplus a_4 \odot y \odot z \oplus a_5 \odot z^2 \oplus a_6 \odot x \odot z^2.$$

This can have one or more branches of weight higher than one. If we restrict to conics with all branches having weight one then, they take the form of either two intersecting lines or one type of so called proper conics.

Definition 1.2.5 ([RST05]). A *proper conic* is one where the coefficients fulfill the following inequalities,

$$\begin{aligned} 2a_2 &\leq a_1 + a_3, \\ 2a_4 &\leq a_3 + a_5 \text{ and} \\ 2a_6 &\leq a_1 + a_5. \end{aligned}$$

In [RST05] four different types of proper conics are described. One type, which we will call type 1, has one vertex with only bounded branches. This happens if

$$\begin{aligned} a_2 + a_4 &< a_3 + a_6, \\ a_2 + a_6 &< a_1 + a_4 \text{ and} \\ a_4 + a_6 &< a_2 + a_5. \end{aligned}$$

In the other three types all vertices are on at least one infinite branch. This happens when the conic fulfills the inequalities to be proper, and also either

$$\text{Type 2: } a_2 + a_4 > a_3 + a_6,$$

$$\text{Type 3: } a_2 + a_6 > a_1 + a_4 \text{ or}$$

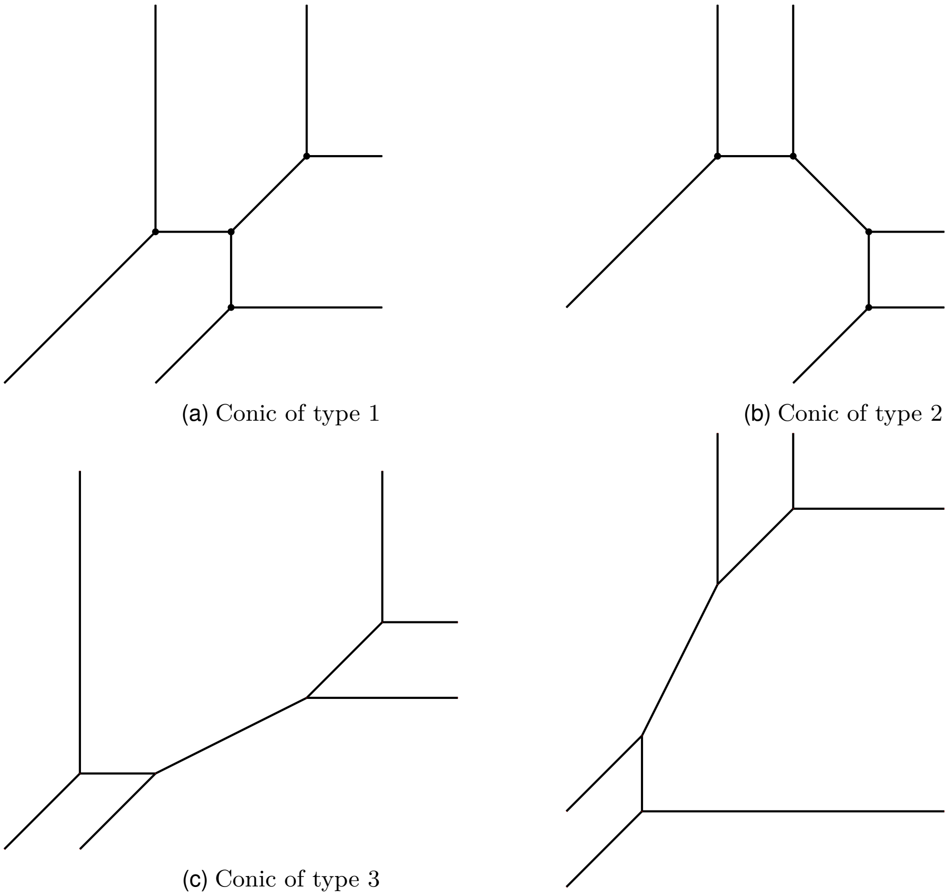
$$\text{Type 4: } a_4 + a_6 > a_2 + a_5.$$

A visualization of all four types is given in Figure 1.2 on the next page.

Notice how all of these have genus $\frac{4}{2} - \frac{6}{2} + 1 = 0$. In fact, proper conics coincide with smooth conics.

Example 1.2.6. The following is an example of a proper conic with a vertex with only bounded branches, i.e. a conic of type 1,

$$q = 3 \odot x^2 \oplus 2 \odot x \odot y \oplus 3 \odot y^2 \oplus 4 \odot y \odot z \oplus 7 \odot z^2 \oplus 4 \odot x \odot z.$$



(a) Conic of type 1

(b) Conic of type 2

(c) Conic of type 3

(d) Conic of type 4

Figure 1.2: Proper tropical conics

This fulfills the inequalities from the definition, and also

$$\begin{aligned}
a_2 + a_4 &= 2 + 4 = 6 < 7 = 3 + 4 = a_3 + a_6 \\
a_2 + a_6 &= 2 + 4 = 6 < 7 = 3 + 4 = a_1 + a_4 \\
a_4 + a_6 &= 4 + 4 = 8 < 9 = 2 + 7 = a_2 + a_5.
\end{aligned}$$

Hence this is a conic of type 1. This is shown in Figure 1.2a.

Example 1.2.7. The following is a conic of type 2:

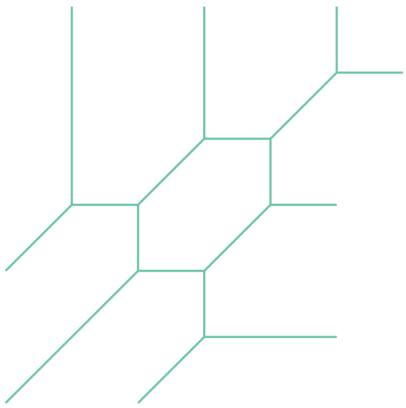
$$p = 12.5 \odot x^2 \oplus 9.5 \odot x \odot y \oplus 7.5 \odot y^2 \oplus 6.5 \odot y \odot z \oplus 9.5 \odot z^2 \oplus 10.5 \odot x \odot z$$

1.2.3 Tropical cubics

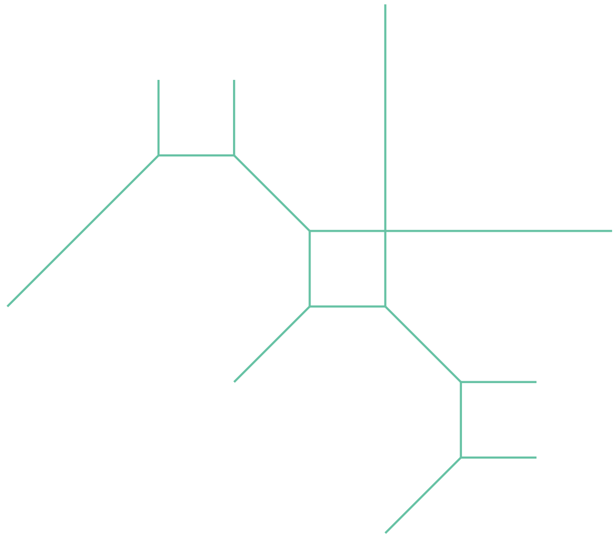
Tropical cubics are tropical plane curves given by polynomials on the form

$$\begin{aligned}
&a_1 \odot x^3 \oplus a_2 \odot x^2 \odot y \oplus a_3 \odot x^2 \odot z \\
\oplus &a_4 \odot x \odot y \odot z \oplus a_5 \odot x \odot y^2 \oplus a_6 \odot x \odot z^2 \\
\oplus &a_7 \odot y^3 \oplus a_8 \odot y^2 \odot z \oplus a_9 \odot y \odot z^2 \oplus a_{10} \odot z^3
\end{aligned}$$

Like tropical conics, simple tropical cubics with branches all of weight one can take many forms. They can take the form of three intersecting tropical lines, one tropical conic and one tropical line or one irreducible curve with or without a singular point.



(a) Smooth tropical cubic



(b) Tropical cubic with one self-intersection

Figure 1.3: Tropical cubics

Example 1.2.8. Figure 1.3a shows an example of a smooth tropical cubic. Note how it has genus $g = \frac{9}{2} - \frac{9}{2} + 1 = 1$, so it is not rational. The dual subdivision has one internal vertex.

Example 1.2.9. Figure 1.3b is an example of a tropical cubic with one singular point. It has genus $\frac{7}{2} - \frac{9}{2} + 1 = 0$ so it is rational. In the dual subdivision, the rectangle dual to the vertex that is not trivalent, but locally the intersection of two branches, has double the area of the triangles dual to the trivalent vertices.

1.3 Intersections

We need to have some intuition about how tropical intersections behave. In this section we present some of the differences between tropical and classical intersections, and present the notion of stable intersections. Which is needed to give a version of Bezout's theorem that holds in tropical geometry.

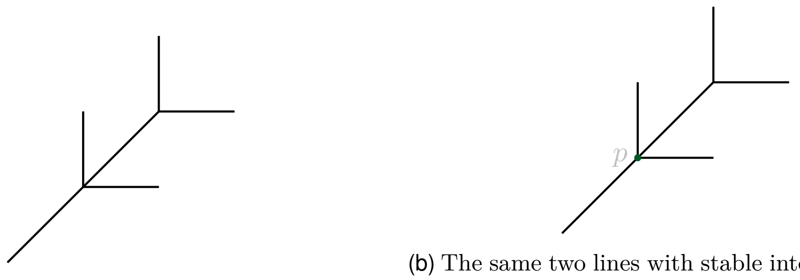
Definition 1.3.1. Given two tropical curves C and D, we say they intersect *finitely* if no vertex of C is on D and vise versa.

Definition 1.3.2 ([Vig08, p. 11]). Let C and D be two tropical curves that intersect finitely at a point p. Assume the edges meeting have weights m_1 and m_2, and direction vectors (u_0, u_1) and (w_0, w_1) respectively. Then the *intersection multiplicity* at p is the absolute value of

$$m_1 m_2 \det\begin{pmatrix} u_0 & u_1 \\ w_0 & w_1 \end{pmatrix}$$

Definition 1.3.3. Given two tropical curves C and D, we say they intersect *transverally* if they intersect finitely, and if all intersections are of multiplicity one.

It is possible for two different irreducible tropical curves to have an infinite number of points in common as in Figure 1.4a. In those cases one can define a finite, well defined set of shared points that function as the intersections. This is described in detail in [RST05].



(a) Two lines with non-finite intersection

(b) The same two lines with stable intersection point highlighted

Figure 1.4: Stable intersection

Definition 1.3.4. Given curves C and D of degrees c and d respectively. Suppose the intersection $C \cap D$ is not transverse. Let $\epsilon \in [0,1]$ be some real number. Let C_ϵ and D_ϵ be translations of C and D, such that C_ϵ and D_ϵ intersect transversely in finitely many points and such that $\lim_{\epsilon \to 0} C_\epsilon \to C$ and $\lim_{\epsilon \to 0} D_\epsilon \to D$. Define the *stable intersection* $C \cap_{st} D$ of C and D as the limit of $C_\epsilon \cap D_\epsilon$ as ϵ goes to zero.

Theorem 1.3.5 ([RST05, p. 303]). *The limit of $C_\epsilon \cap D_\epsilon$ is independent of the choice of perturbations. Hence the stable intersection $C \cap_{st} D$ is well defined.*

Note that in the cases where vertices from one curve is on the other, the stable intersection points coincide with those vertices.

We count points in $C \cap D$ that are not the stable intersection points as having intersection multiplicity zero. Then a tropical version of Bezout's theorem holds for stable intersections.

Theorem 1.3.6 ([Vig08]). *Assume C and D are tropical curves of degrees c and d respectively. Then their stable intersection consists of cd points counted with multiplicities.*

1.4 Uniqueness of curves through points

Classically, in real geometry there is exactly one curve of degree d through any set of $\frac{1}{2}(d^2 + 3d)$ points in general position. The meaning of general position depends on the degree of the curve. The same holds for simple tropical curves [MS15, p. 33]. However, the requirements for a set of points to be in general position are different tropically.

1.4.1 A line between two points

In Euclidean geometry, given two distinct points, there will always be a unique line through them. In tropical geometry, this is not necessarily true. Two points that would fall on the same branch, i.e. they lie on the same vertical, horizontal or diagonal Euclidean line, will lie on an infinite family of lines. There is, however, a way to associate a unique line to any pair of points.

Definition 1.4.1 ([RST05]). Let $A = (a_{ij})$ be a $k \times k$-matrix with entries in $\mathbb{R} \cup \{+\infty\}$. We define the tropical determinant of A as follows:

$$det_t(A) = \bigoplus_{\sigma \in S_k} (a_{1,\sigma_1} \odot \cdots \otimes a_{k,\sigma_k}) = \min_{\sigma \in S_k} (a_{1,\sigma_1} + \cdots + a_{k,\sigma_k}).$$

Here S_k is the symmetry group on k elements.

In his phd thesis, [Vig08], Magnus Vigeland defined a unique line line between two points using the following construction.

Definition 1.4.2 ([Vig08]). Given two points $p = (p_0, p_1, p_2)$ and $q = (q_0, q_1, q_2) \in \mathbb{TP}^2$. Find the vector

$$u = (u_0, u_1, u_2) = (det_t \begin{bmatrix} p_2 & p_3 \\ q_2 & q_3 \end{bmatrix}, det_t \begin{bmatrix} p_1 & p_3 \\ q_1 & q_3 \end{bmatrix}, det_t \begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \end{bmatrix}).$$

We define the *stable join* of p and q as the line with coefficients u_0, u_1 and u_2.

We will apply this to an example.

Example 1.4.3. Let the two points be given by $p = (0, 0, 0)$ and $q = (1, 0, 0)$. Note that since p and q are on the same horizontal Euclidean line, there are infinitely many tropical lines passing through them, as shown by the dashed lines in Figure 1.5. To find their stable join need to find the tropical determinants of the following matrices,

$$\begin{bmatrix} p_2 & p_3 \\ q_2 & q_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} p_1 & p_3 \\ q_1 & q_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Given this find that

$$u = (u_1, u_2, u_3) = (\min\{0 + 0, 0 + 0\}, \min\{0 + 0, 1 + 0\}, \min\{0 + 0, 1 + 0\}) = (0, 0, 0).$$

Hence the line is the given by the polynomial $f = x \oplus y \oplus z$. This line has its vertex at p and passes through q as desired.

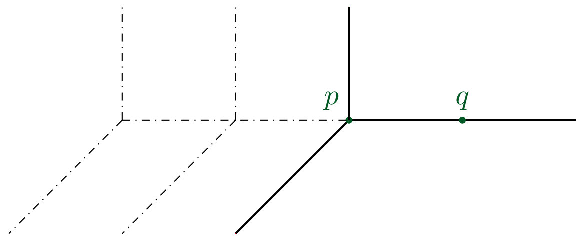


Figure 1.5: Stable join of two points

1.4.2 A conic through five points

In Euclidean geometry we know that for five general points, there is exactly one smooth conic passing through them. When $k \geq 3$ of said points are on the same line, the conic is reducible, i.e. two lines. For $k = 3$ the lines are unique. If $k = 4$, there is a unique line l containing four points and a pencil of lines containing the final point.

In the tropical case, the situation is somewhat different.

Example 1.4.4. In Figure 1.6 there are five points $p_1, \ldots p_5$ that do not determine a unique tropical conic. The points p_1, p_2 and p_3 are colinear. In the classical case, the conic through them would be unique, but reducible. That is, the only conic passing through them would be the two intersecting lines highlighted on the far left. Tropically, as the figure shows, there are at least two smooth conics through the points as well.

Similar to how there can in some cases be infinitely many lines through two points, there can sometimes be infinitely many lines through five points, even if no three or more of them are collinear.

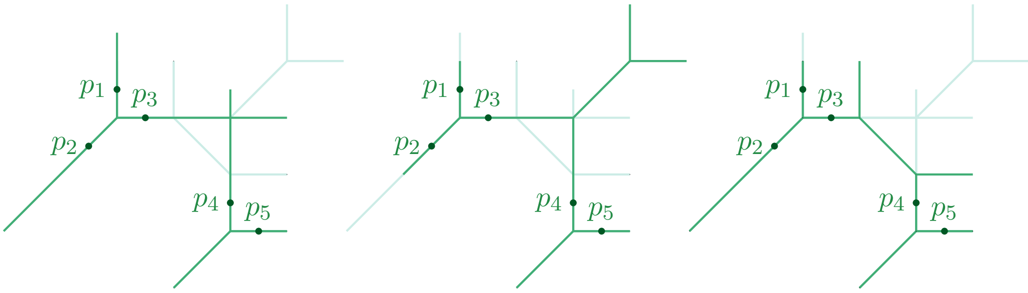
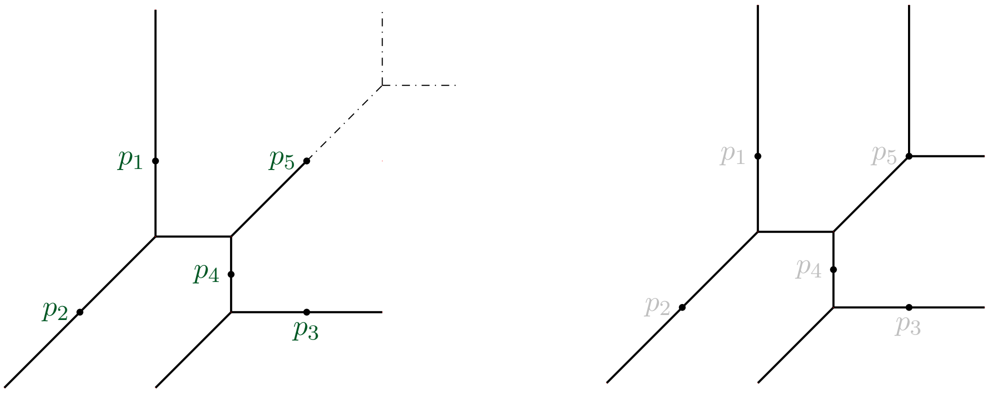


Figure 1.6: Five points that give 3 options for a conic through them



(a) There are potentially infinitely many conics through the five points $v_1 \dots v_5$

(b) Stable conic

Figure 1.7

Example 1.4.5. In Figure 1.7a there is an example of five points,

$$\begin{aligned} p_1 =& (1, 3, 0), \\ p_2 =& (0, 1, 0), \\ p_3 =& (3, 1, 0), \\ p_4 =& (2, 1.5, 0) \text{ and} \\ p_5 =& (3, 3, 0) \end{aligned}$$

such that no three are on a tropical line, yet there are multiple possible conics that pass through all five of them. The fourth vertex could be anywhere along the dashed line.

There is a construction similar to the stable join for conics. This method is described in [RST05, pp. 308, 309]. Analogous to the method for finding the conic through five points in $\mathbb{P}^2$, the method uses the tropical determinant of a matrix given by the points. Given five points $p_i = (p_{i1}, p_{i2}, p_{i3})$, $i = 1, \dots, 5$, let M be the matrix with rows $(2p_{i1}, p_{i1} + p_{i2}, 2p_{i2}, p_{i2} + p_{i3}, 2p_{i3}, p_{i1} + p_{i3})$. Then the coefficients a_i of the stable conic are given by the tropical determinants of the 5×5 minors of M. The [Matlab] script in Appendix A can be used to find to find these determinants. Then the *tropical* application in [Polymake] can be used to visualize the curve.

For any five given points in $\mathbb{TP}^2$ the unique stable conic is a proper conic [RST05, Theorem 5.7]. It is stable in the sense of being the limit of conics through perturbations of the five points.

Example 1.4.6. Given the five points from Example 1.4.5 on the preceding page, we find the stable conic using the [Matlab] script. This gives the polynomial

$$f(x,y,z) = 9.5 \odot x^2 \oplus 6.5 \odot x \odot y \oplus 5.5 \odot y^2 \oplus 4.5 \odot y \odot z \oplus 5.5 \odot z^2 \oplus 6.5 \odot x \odot z.$$

Then using [Polymake], we find the conic in Figure 1.7b on the facing page.

1.5 Tropical polygons and tropical convexity

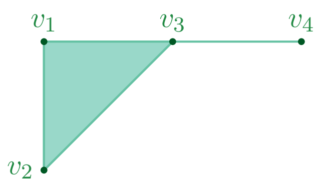
For the sake of this project, it is necessary to have a notion of a convex polytope in the tropical sense. This has been explored in [DS04] and [DY07]. We use their definitions here.

Definition 1.5.1 ([DS04]). A domain A in the plane is *tropically convex* if given any two points x and y in A and any $a, b \in \mathbb{R}$, the point $a \odot x \oplus b \odot y$ is in A.

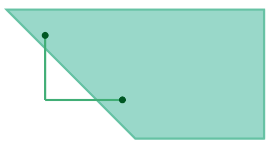
Traditionally, a convex subset of the plane is one where the line segment between any two points in the subset is contained in the subset. Using the following definition of a line segment, the above definition is equivalent to that.

Definition 1.5.2 ([DS04]). Given two points x and y in the plane, the *tropical line segment* $[x, y]$ between them is equal to the set of all points on the form $a \odot x \oplus b \odot y$ where $a, b \in \mathbb{R}$.

The tropical line segment between two points is a subset of the unique line between them.



(a) The convex hull of $v_1, \ldots, v_4$. This is not convex in the Euclidean sense, but it is tropically convex.



(b) This polygon is convex in the Euclidean sense, but not tropically. Figure includes two points in the set and the line segment between them, showing how it is not contained by the set.

Figure 1.8: A tropically convex and non-convex subset of the plane

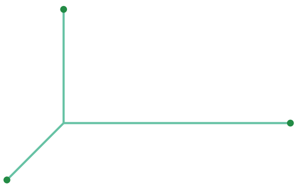
There are many ways to define a polygon.

Definition 1.5.3. A *tropical n-gon* is the tropical convex hull of a set of n points.

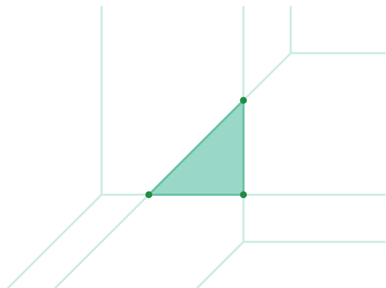
Example 1.5.4. The following are two examples of tropical triangles.

- Figure 1.9a shows the tropical convex hull of three colinear points. This triangle is simply a line segment.
- Figure 1.9b shows the tropical convex hull of the intersections between three tropical lines.

Note that there are no three tropical lines such that Figure 1.9a can be constructed as the convex hull of their intersections.



(a) The tropical convex hull of three colinear points



(b) The tropical convex hull of the intersections of three tropical lines

Figure 1.9: Two tropical triangles

Chapter 2

Polypols and adjoints

This is the main chapter of this thesis. In the first section we begin by presenting the real case. Here we state the relevant definitions for the real version of the open problem. We then present said problem, along with one general result that we will later discuss for the tropical case. That done, we can in the next section, present our definition of tropical polypols and tropical adjoints. Then we present a tropical version of the problem, and discuss some general properties relevant to our approach to answering it. In the third section, we restrict to polypols with sides from tropical lines, and give a classification and some results for them. In the final section, we return to the general case, giving one general result, before spending the rest of the chapter on the case of tropical conics.

2.1 Real polypols and adjoints

To begin discussing the tropical problem we are concerned with here, we first need to give a brief discussion of the real version of the problem. In this section, we give the relevant definitions of real regular rational polypols and their adjoint curve. Then we present Wachspress' conjecture, and some established relevant results. We use language and notation from [Koh+21].

2.1.1 Definitions

Definition 2.1.1 ([Koh+21, p. 4]). Let $C \in \mathbb{P}^2$ be a plane curve with $k \geq 2$ irreducible components $C_1, \ldots, C_k$. Assume there are k points $v_{12} \in C_1 \cap C_2, \ldots, v_{k1} \in C_k \cap C_1$ such that v_{ij} is non-singular on C_i and C_j, and that C_i and C_j intersect transversally at v_{ij}. Then we say that the curves C_i and the points v_{ij} form a *polypol* P.

The curves $C_1, \ldots C_k$ are called the *boundary curves* of P.

The set of points $V(P) = \{v_{ij}\}$ is called the *vertices* of P, and their complement in the singular locus of C is called the *residual points* of C.

We say that P is *rational* if the curves C_i are rational.

Definition 2.1.2 ([Koh+21, Definition 2.9]). A *real polypol* is a polypol with real boundary curves C_i, real vertices $v_{i-1,i} \in C_{i-1} \cap C_i$, and a given choice of segments connecting $v_{i-1,i}$ to $v_{i,i+1}$ in $C_i(\mathbb{R})$, called the *sides* of the polypol, and a closed set $P_{\geq 0}$ whose interior is a union of simply connected sets and whose boundary is the union of the sides of the polypol.

A *quasi-regular polypol* is a real polypol whose sides are non-singular on C_i.

Note that in this thesis we will sometimes refer to the interior and boundary of the polypol, when we mean the interior and boundary of the set $P_{\geq 0}$.

Definition 2.1.3 ([Koh+21, Definition 3.1]). We say that a quasi-regular polypol P is *regular* if all points on the sides of P except the vertices are non-singular on C and C does not intersect the interior of $P_{\geq 0}$.

Definition 2.1.4 ([Wac75, p. 135]). Let γ_i be the multiplicities of the residual points p_i of a regular polypol P of order D. Then the *adjoint curve* A_P of P is the unique curve of order $D-3$ that passes through each residual point with multiplicity at least $\gamma_i - 1$ at each p_i.

2.1.2 The Wachspress conjecture

An important conjecture in the real case is the Wachspress conjecture.

Conjecture 2.1.5 ([Koh+21; Wac75]). *The adjoint curve of a regular rational polypol P does not intersect the interior of $P_{\geq 0}$.*

The question of whether or not this holds remain open.

Here are some results from [Koh+21] that will be useful later.

Proposition 2.1.6 ([Koh+21, Lemma 3.4]). *Let P be a rational polypol defined by boundary curves $C_1, \ldots, C_k$ that intersect transversely. Then the adjoint curve A_P intersects C_i only at the residual points, with intersection multiplicity equal to $2\delta_p$ at each singular point $p \in C_i$ and with intersection multiplicity one at each of the remaining residual points. In particular, if P is regular, then the adjoint curve does not contain any points on the sides of P.*

The final line of the lemma, about regular polypols, warrants some discussion here. In their proof Kohn et.al. makes use of the fact that if C_i is of degree d_i and A_P is of degree $d-3$, then the sum of the intersections at the residual points counted with multiplicity is $d_i(d-3)$. Hence Bezout's theorem indicates that there are no further intersections between C_i and A_P, and so the adjoint does not intersect the boundary of P on the side from C_i. This holds for all i.

This means that in the real case, any connected adjoint curve will have all points outside the interior of the polypol. Any counterexample to Wachpress conjecture, if one exists, would have to be a polypol with an adjoint of degree at least three.

2.2 Plane tropical polypols

This chapter will contain first a section on defining tropical polypols in general. Then a section section on tropical polypols constructed from lines and their adjoint curves. Finally a section on tropical polygons whose constitute curves have degree higher than one.

2.2.1 Definitions and general results

We begin by defining tropical plane polypols. The following definitions are analogous to the definitions of general and real plane polypols given earlier.

Definition 2.2.1. A *tropical polypol* P is a bounded component in $\mathbb{P}^2$ such that there is a tropical curve $C \in \mathbb{TP}^2$ with $k \geq 2$ irreducible, rational, components $C_1, \ldots, C_k$ where the boundary of P coincides with segments from each C_i, and C does not intersect the interior of P. There are k points $v_{12} \in C_1 \cap C_2, \ldots, v_{k1} \in C_k \cap C_1$ on the boundary of P such that v_{ij} is non-singular on C_i and C_j and such that C_i and C_j intersect transversally at v_{ij}. We call the points v_{ij} the *corners* of the polypol and the curves C_i its *boundary curves*. There are segments connecting $v_{i-1,i}$ to $v_{i,i+1}$ in C_i called the *sides* of the tropical polypol. The boundary of P is the union of these segments. All points on the sides of P except the corners are non-singular on C. We say that P is a tropical polypol consisting of the curves $C_1, \ldots C_k$.

Definition 2.2.2. Given a tropical polypol P with boundary given by a tropical curve $C = \cup_{i=0}^{k} C_i$, we call the singular points of C that are not the corners of P the *residual points* of P.

Definition 2.2.3. For a tropical polypol P with boundary curves $C_1, \ldots, C_k$, where C_i is of degree d_i we say that the *degree* of P is

$$d = \sum_{i=1}^{k} d_i.$$

For the polypols considered in this paper, their residual points are either intersections of two curves C_i and C_j or self-intersections of some C_i.

It is worth noting that tropical polypols are not necessarily convex.

Example 2.2.4. Figure 2.1 is an example of a polypol that is not tropically convex.

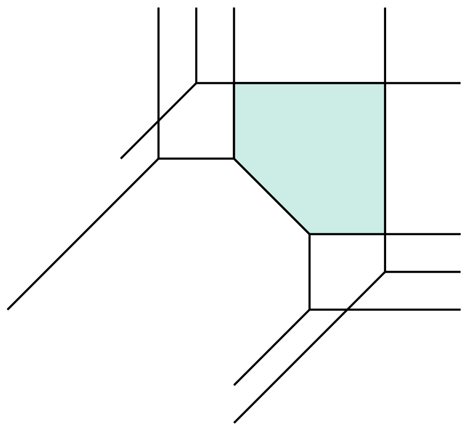


Figure 2.1: A tropical polypol, with boundary curves one conic and two lines

Definition 2.2.5. A *tropical adjoint curve* of a tropical polypol of degree d is a curve of degree $d-3$ passing through all residual points.

In the real case, we know that such a curve is unique [Koh+21]. However, as discussed in Section 1.4 there are times when points that are in general position with regards to defining a real curve, but which have several tropical curves passing through them. Later in this section we will share examples of tropical polypols with multiple possible adjoints.

The main question under discussion in this thesis is the following.

Question 2.2.6. *Is there a tropical polypol that has at least one adjoint curve that intersects the interior of the polypol?*

The rest of this section seeks to give some intuition on what an example of such a polypol would look like. This is in some ways equivalent to searching for a counterexample to a tropical version of Wachspress' conjecture.

We know from Proposition 2.1.6 that the real adjoint curve never intersects the sides a real polypol. In [Koh+21] Koh et. al. uses this lemma to prove the conjecture for polypols of total degree lower than 6. Since polypols of degree less than six have adjoints of degree less than 3, and so the adjoints are connected, and cannot have points on the interior without intersecting the sides. The proof of Proposition 2.1.6 utilizes Bezout's theorem. We know that Bezout's theorem holds in tropical geometry, and that all tropical curves are connected. Thus, one might think that this is sufficient to show that the conjecture holds in the tropical case. However, Bezout's theorem only holds tropically when utilizing the theory of stable intersections. It is fully possible for a curve A to pass through a given point r on another curve C_i yet have the stable intersection point, i.e. the point counted with intersection multiplicity more than 0, not be r. Meanwhile the proof of Proposition 2.1.6 given in [Koh+21] requires that the adjoint has intersection points with each boundary curve at the residual points. This means that Proposition 2.1.6 does not necessarily hold tropically, and the question remains open.

Example 2.2.7. Figure 2.2 shows why Proposition 2.1.6 does not hold in the tropical case. The cubic B passes through all residual points on the boundary conic C_1 and has the point multiplicity of an adjoint curve at those points, but it also intersects the side of the polypol in one point from C_1.

This is not a counterexample to Wachpress conjecture. The curve B is not the adjoint of this polypol. In fact it fails to pass through one of the intersection points between C_2 and C_3. The actual adjoint curve lies outside the polypol here.

We can derive, from the theory of stable intersections, as well as general intersection theory, some intuition about what a counterexample to a tropical version of Wachpress conjecture might look like. For one, since the stable intersection point will lie on the vertex of one of the curves, we can infer that the polypol needs to have curve vertices on its sides. In addition, one or more of the residual points must be near those vertices.

Lemma 2.2.8. *Let C_1 and C_2 be two tropical curves that intersect at a point r, such that $r \in C_1 \cap_{st} C_2$, and let A be a third tropical curve such that $r \in A$. Then the intersection multiplicity at r for the stable intersection $A \cap_{st} C_i$ is at least one for at least one of the C_i.*

Proof. Assume for contradiction that the intersection multiplicity is zero for both C_i. Then r must be on a double branch in $A \cup C_i$ for both C_i. These two double branches can either have the same direction, or different directions.

If they have different directions, then since r is on both of them, r must be a one-point intersection between C_1 and C_2 and a self-intersection of A. In this case the stable intersection multiplicity at r is at least one for both $A \cap_{st} C_i$, which contradicts the assumption that it be zero for both.

If the double branches have the same direction, then r must be on a non-finite connected component of $C_1 \cap C_2$, it cannot be a vertex in either C_i as it would then have non-zero stable intersection multiplicity in $A \cap_{st} C_i$ for that C_i. Therefore the stable intersection multiplicity of C_1 and C_2 at r must be zero, which contradicts the starting conditions.

Hence the intersection multiplicity at r for the stable intersection $A \cap_{st} C_i$ is at least one for at least one of the C_i. ∎

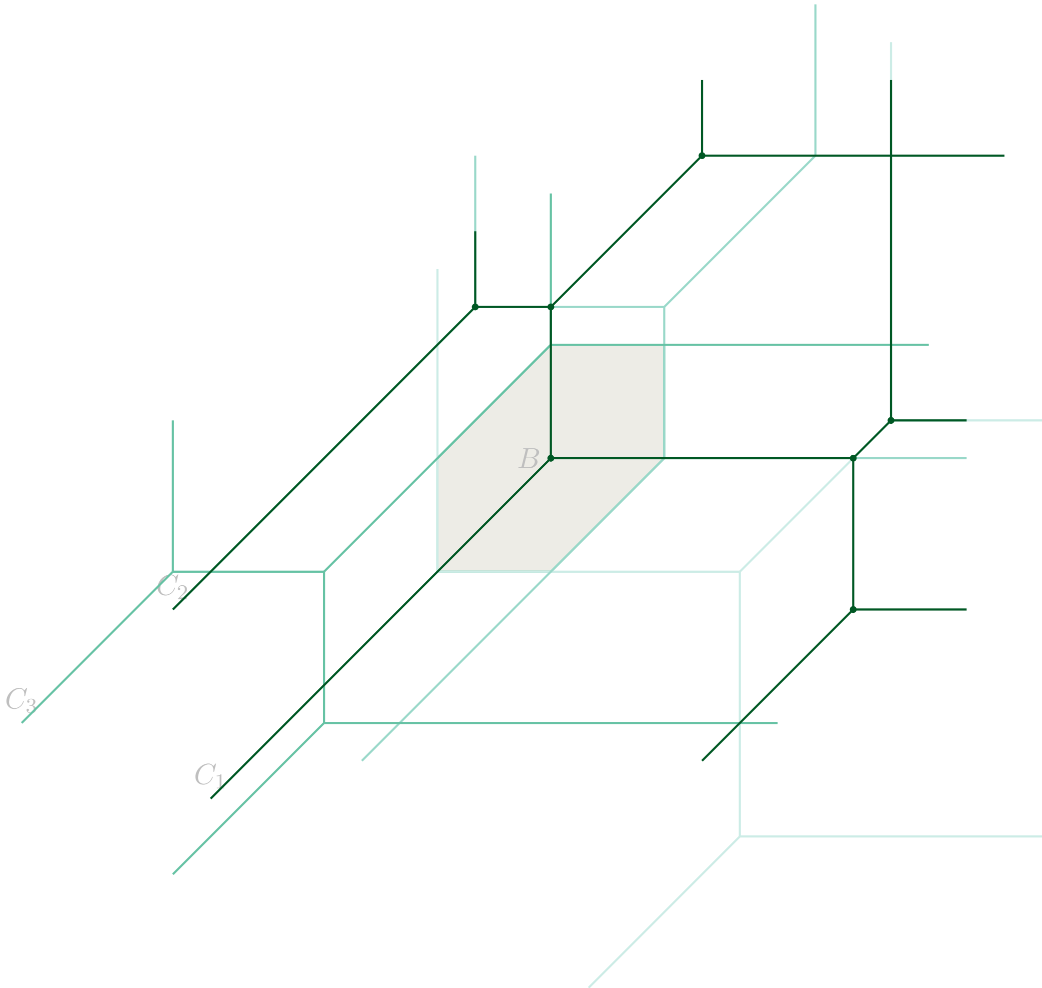
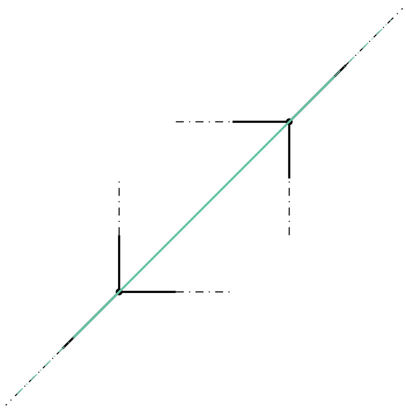
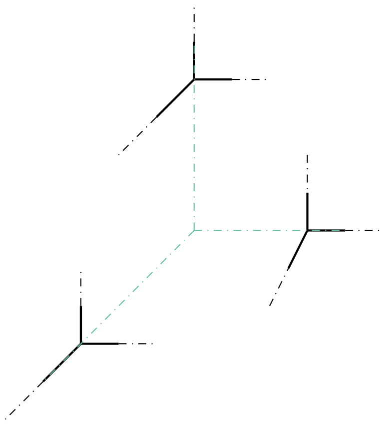


Figure 2.2: Tropical polypol consisting of three conics and a tropical cubic that passes through all residual points on one of them, and the interior of the polypol



(a) One branch of the adjoint intersects the interior



(b) One vertex of the adjoint is on the interior

Figure 2.3: Possible counterexample configurations.

2.3 Tropical polypols consisting of lines and their adjoint curves

In this section, we restrict to the case of tropical polypols bounded by tropical lines. We find that each of these polypols can be associated to an Eucledean polygon, and that there is up to scaling a finite number of such polygons. Using this, we create a classification of tropical polypols consisting of lines that can be used to find all relevant information about the polypols. We first explain the classification, and then evaluate possible polypols in order of number of tropical lines.

Tropical polypols consisting of lines will be a subset of tropical convex polygons.

Definition 2.3.1. Given n tropical lines in the plane, their complement will contain multiple connected components. If one or more of these components are bounded, we say that the tropical polypol given by the lines is the bounded convex component such that one segment from each of the n lines are on the boundaries of the component.

Note that not every constellation of n lines defines a polypol. There are cases where no convex component have segments from all lines on its boundary as well as cases where no bounded component exists. To avoid the latter case, a restriction, which reflects the definition of polypols in general, is that at least n intersections must be distinct and finite. In this thesis we assume that all intersections between the lines are finite and distinct.

2.3.1 Classification of convex figures made from tropical lines

Proposition 2.3.2. *Eucledean convex polygons that can be constructed using tropical lines have at most six sides.*

Proof. It is not possible for three or more parallel lies to be the sides of a convex polygon. Since tropical lines have edges in the same 3 directions, all convex figures constructed from tropical lines have sides parallel to either the x-axis, the y-axis or the diagonal. Therefore, since tropical lines only have branches in three directions, a polygon constructed using tropical lines can have at most three pairs of parallel sides. So they can have at most six sides. ■

We classify a polygon as a cycle of corners, starting in the corner furthest to the southwest. Each corner is given a label based on the types of edges that goes into and out of it, and on whether or not it is the vertex of a line. Edges are labeled according to which direction they are in relation to the vertex of the line. A vertical edge is labeled N (north), a horizontal edge E (east) and a diagonal S (southwest). So a corner of type NE is one where a vertical and horizontal edge meets, and the interior of the cycle is northeast from the corner.

Lemma 2.3.3. *There are, up to scaling, finitely many convex polygons that can be constructed using tropical lines.*

Proof. A convex polygon that can be constructed using tropical lines has $n = 1, \dots 6$ sides. For each side, there is a finite choice of directions that side can have. It has to be either type N, E or S. A polygon can have at most two of any one type of edge. A finite choice of finite choices gives a finite total number of combinations. ■

Proposition 2.3.4. *There are up to scaling 18 Euclidean convex polygons that can be constructed with sides that are segments of the branches of tropical lines. There are*

- *2 with 3 sides*
- *9 with 4 sides*
- *6 with 5 sides*
- *1 with 6 sides.*

These are shown in Figure 2.4.

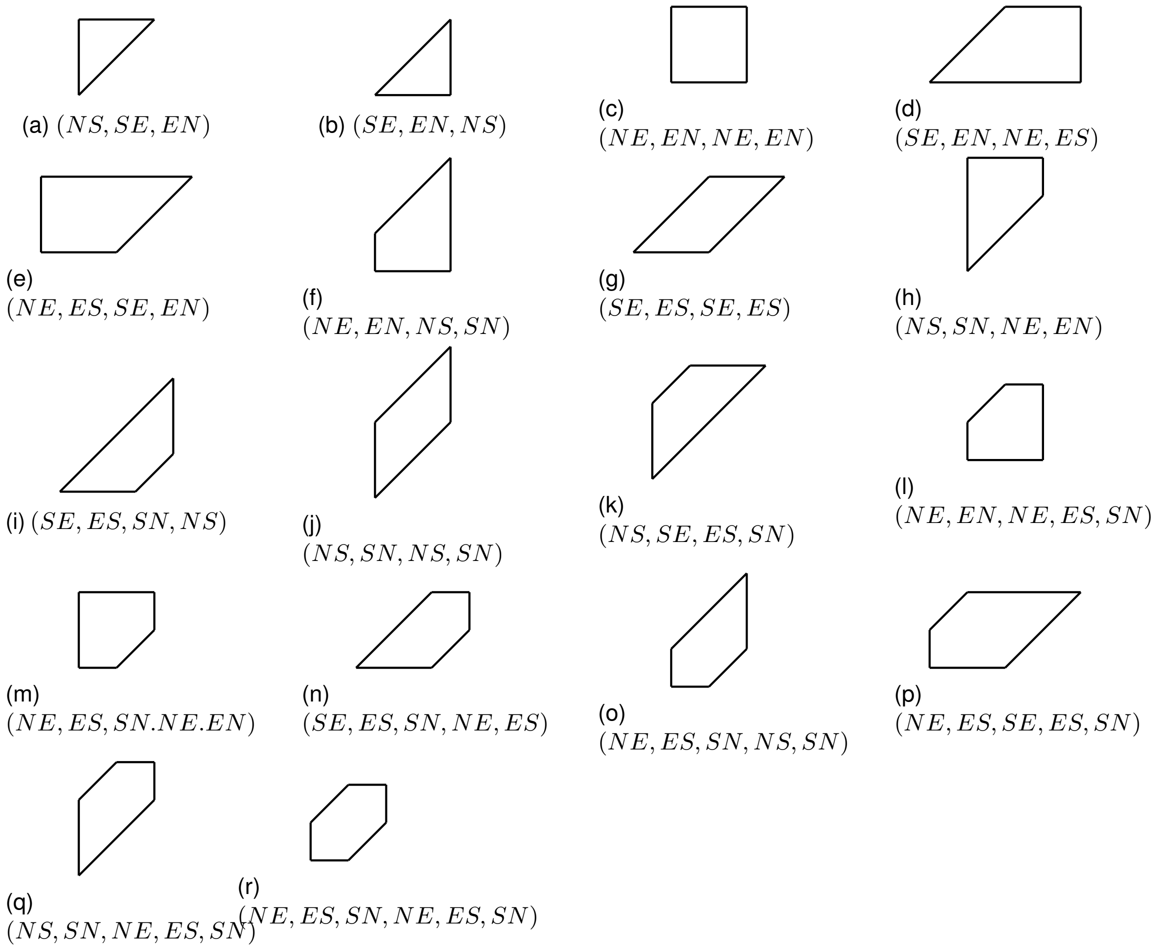


Figure 2.4: The possible polygons whose sides are the branches of tropical lines

Proof. This number is found via a brute force method where every possible combination of 3, 4, 5 or 6 edges in the stated directions was listed, the ones that could not give valid polygons ruled out, and the rest checked to see if it could form a completed cycle giving a convex polygon that could actually be realized with tropical lines. For the cycle to be complete we know that if the first corner is of type $A-$, then the final corner must be of type $-A$. We also do not accept corners of type AA. Every corner must be a shared point on two edges in different directions. Hence if one corner is of type $-A$, then the next must be of type $A-$. Since a cycle starts in the lower left corner, and is traversed clockwise, the first corner cannot be of type $-N$, since that would mean that either the next corner is further south or the final corner further west. It also cannot be of type $E-$, since that would also mean that a different corner is further west.

- Since one cannot get a closed cycle with 3 edges if two of them are parallel, the total number of possible combinations of edges is the number of permutations of 3 objects without repetition. That is 6 possibilities. Two of them have $-N$ as the first corner. Out of the remaining four, one has ES as the first corner. So only 3 possibilities need to be evaluated. (NS, SE, EN) and (SE, EN, NS) are both complete cycles that can be constructed using tropical lines. On the other hand (NE, ES, SN) is not. Therefore there are exactly two cycles.
- The first corner has to be of type $-E$ or $-S$. Since all corners must be pairs of different edges, and for each following corner one edge is determined by previous choice, there are two choices for each corner. Hence there are $2^4 = 16$ options to evaluate. Out of those, there are four where the final corner has the same exiting edge as the first. Meaning that either the cycle is not closed or it does not have four distinct edges. Of the remaining 12 options, 3 have $-E$ as the final corner, meaning the first corner would have to be of type $E-$. These are in fact each similar to one of the other nine. The final nine options are all convex cycles with four distinct edges that can be constructed using tropical lines.
- By the same argument as above there are $2^5 = 32$ options to evaluate. Out of those, there are 12 where the fifth corner is the same type as the first, so they cannot form a complete cycle with five distinct sides. Among the remaining 24, 5 have the fifth corner of type $-E$, and can be disregarded, as they are all similar to other options. When drawing the remaining 19 possibilities, find that only 6 can be constructed as convex polygons bounded by tropical lines.
- To get a six-sided cycle, it needs to have two of each type of side. Since the parallel sides face each other, there must be a twice-repeating sequence of three sides in the same order. Excluding the options where the first corner is of type $E-$ or $-N$, there are 3 possible sequences. By drawing, find that only one of them gives a convex polygon with six sides.

■

When some corners are vertexes, we count the polygons that are the same shape, but have vertexes on different corners, as distinct.

2.3.2 Three lines

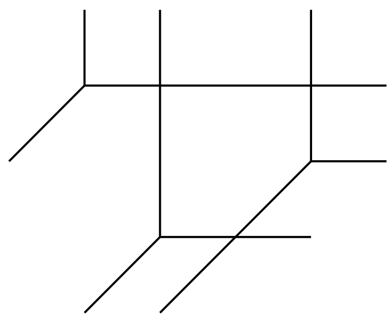
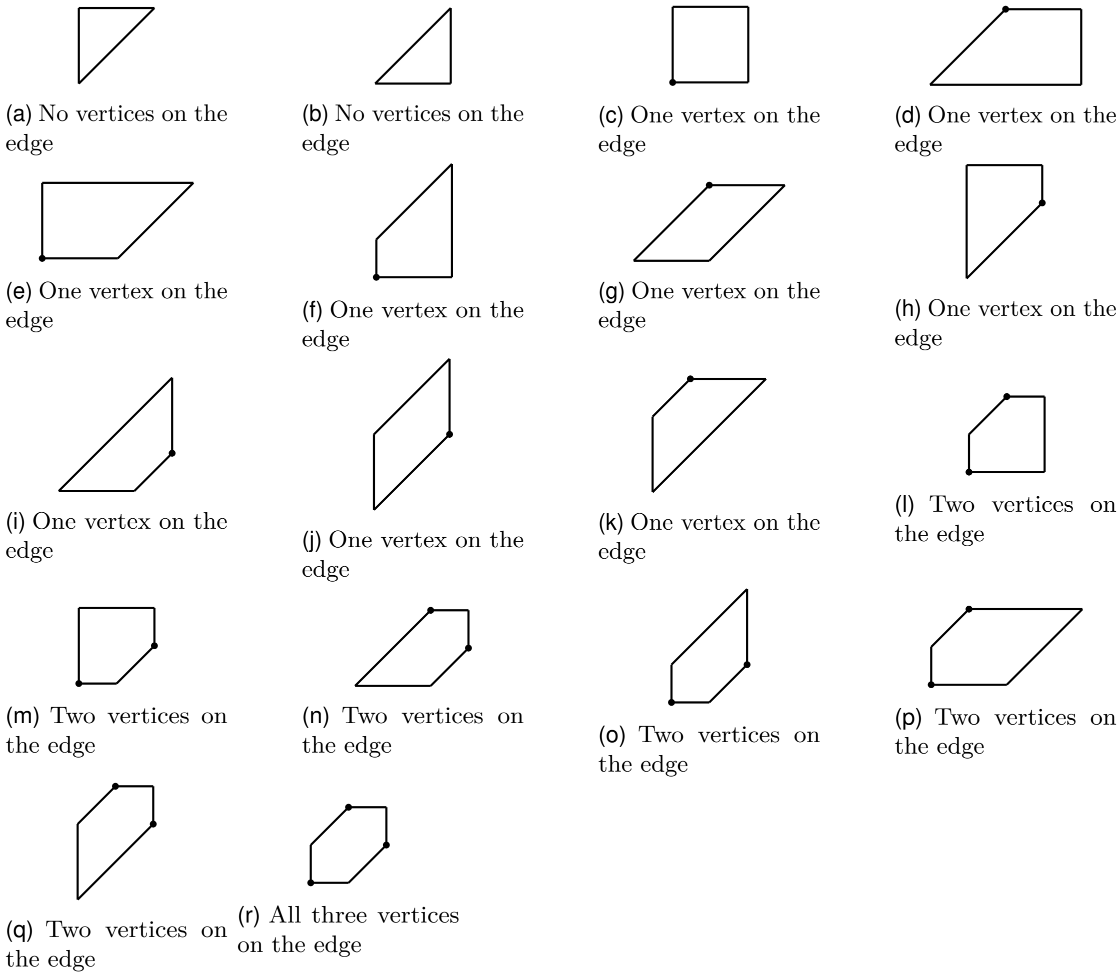


Figure 2.5: Tropical polypol bounded by three tropical lines

With intuition from the Euclidean case, one might assume that the only convex polygons that can be constructed using 3 lines will be the two triangles. This would not



(a) No vertices on the edge
(b) No vertices on the edge
(c) One vertex on the edge
(d) One vertex on the edge
(e) One vertex on the edge
(f) One vertex on the edge
(g) One vertex on the edge
(h) One vertex on the edge
(i) One vertex on the edge
(j) One vertex on the edge
(k) One vertex on the edge
(l) Two vertices on the edge
(m) Two vertices on the edge
(n) Two vertices on the edge
(o) Two vertices on the edge
(p) Two vertices on the edge
(q) Two vertices on the edge
(r) All three vertices on the edge

Figure 2.6: The possible polygons made with three tropical lines

be correct. Since some corners can be nodes, each line can contribute two sides. We therefore have the following.

Proposition 2.3.5. *All 18 polygons from Figure 2.4 occur as polypols consisting of three lines.*

These are given in Figure 2.6.

For three lines all three intersections will be on the edges of the polygon. This means there are no residual points. Hence there is no adjoint curve in this case.

Example 2.3.6. Figure 2.5 shows one example of a tropical polypol consisting of three lines. The polypol is of type (NE, ES, SN, NE, EN). Since it has total degree 3, there is no adjoint curve.

2.3.3 Four lines

Given four lines L_1, L_2, L_3 and L_4 such that all six intersections between them are unique (i.e. no two intersections are in the same point), want to know when there is a well-defined polypol with these lines as its boundary curves.

First we will try the following definition.

Definition 2.3.7. Given four lines L_1, L_2, L_3 and L_4 such that all six intersections between them are unique, their complement in the plane will include multiple finite connected components. The polypol defined by the lines will be the one that has points from all four lines on its boundary.

The problem with this definition is that there is not always such a component.

Example 2.3.8. Figure 2.7 is an example of four tropical lines that do not form a well defined tropical polypol.

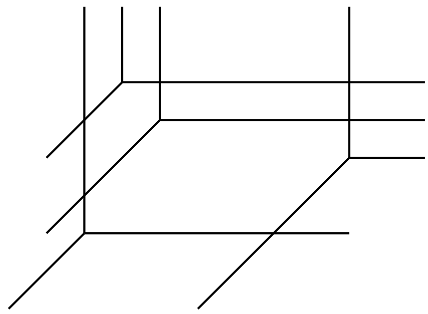


Figure 2.7: Four tropical lines that does not give a well defined tropical polypol

The following is a discussion of the conditions necessary to determine whether four lines with all transversal intersections form a tropical polypol or not.

Focusing on 3 of the lines, there will be a bounded component, call this T, defined by them. The fourth line L_4 will divide the plane into 3 components. Still assuming all intersections are unique, there are 3 possible situations:

1. There are parts of T in all 3 components. This happens when the vertex of L_4 is in the interior of T.
2. All of T is contained in one component.
3. There are parts of T in two of the components, but not in the third.

Lemma 2.3.9. *In situation 1 there is no well-defined polypol with segments from all four lines on the boundary.*

Proof. All 3 intersections between L_4 and the other 3 lines will be on the boundary of T, since the node is necessarily in the interior of T and the branches crosses over to the exterior. Hence the complement of the lines in the plane will have 3 bounded connected components. Call these A, B and C. Assume that the intersection p_{12} between L_1 og L_2 is on the boundary of A. Then, when moving along the boundary of T, the intersection p_{14} will be between this point and p_{13}. Similarily p_{24} will be between p_{12} and p_{23}. Hence, no points from L_3 will be on the boundary of A. The same argument can be made for why there are no points from L_2 on the boundary of B or from L_1 on the boundary of C. Hence there is no convex polygon with segments from all four lines on its boundary. ■

Lemma 2.3.10. *In situation 3, there is exactly one well-defined polypol.*

Proof. Since T is in two different components, in the complement of all the lines T itself will be split into two components A and B. One component will include two of the points p_{12}, p_{13}, p_{23} on its boundary. Hence, it will have points from all four lines on its boundary. It remains to show that this is the only polypol given by the lines. First, assume the component containing two corners from T is A. Need to prove that the boundary of B cannot have segments from all four lines. Any choice of two of the points p_{12}, p_{13}, p_{23} will have one line in common, while the third point will not be on that line. We may without loss of generality assume this line is L_3, that is p_{12} is on the boundary of B. Now traversin the boundary of B from p_{12}, one will meet p_{14} or p_{24} before any points from L_3. Hence no points from L_3 is on the boundary of B. ■

Remark 2.3.11. In situation 2 a relabeling of the lines will give either situation 1 or situation 3.

The above can be summarized as follows.

Proposition 2.3.12. *Four tropical lines with all distinct transversal intersections are the boundary curves of a well defined tropical polypol if and only if no one of the lines has its vertex in the interior of the tropical triangle defined by the other three.*

Looking back at the list of possible polygons. It is not possible to construct (NS, SE, EN) and (SN, NE, ES) with four tropical lines, since each edge in the polygon must be a section of a branch from one of the lines and these only have 3 edges. However all the other polygons can be constructed. In polygons with more than four edges, there is a choice of which corner or corners are nodes. This choice changes the position of the residual points, and we therefore choose to consider them to be different. In total there are 24 polygons with different corners. They are listed in Figure 2.8.

Example 2.3.13. Figure 2.9 is an example of a polypol consisting of four tropical lines. The polypol is shaded in green. It is an Euclidean square. All corners are intersections between the lines. The residual points are marked and the adjoint indicated.

Four lines will have six intersections. Four of said intersections will be on the boundary of the polypol. Then the final two intersections will be outside the polypol. For two general points, there will be a unique line going through them.

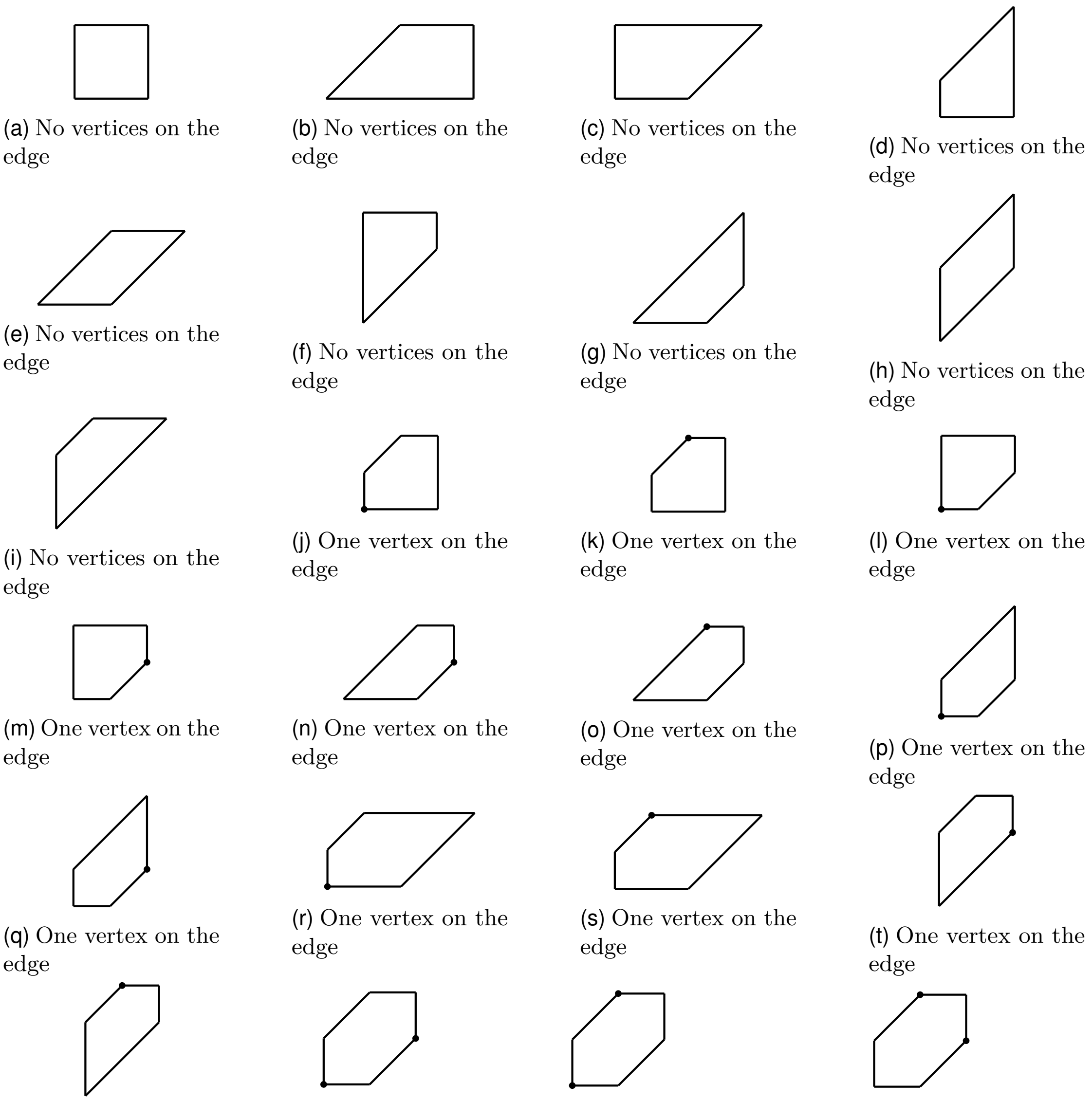
Proposition 2.3.14. *The adjoint A_P of a polypol consisting of four lines $C_1, \dots, C_4$ will never intersect the interior of the polypol.*

Proof. In Section 2.2.1 it is established that the adjoint can only intersect the sides of the polypol at the vertices of the boundary curves. Using the catalog in Figure 2.8 we see that there are only three possible figures with two vertices on the edge of the polypol. These are the last three, the Euclidean hexagons. All three of them are such that the line between the vertices is unique, since the vertices are not on the same branch of any line, and does not intersect the interior of the polypol. ■

It is in fact possible to determine a lot about where the adjoint of a tropical square lies in the plane based on which type of polygon it is.

To know where the line between two points can be it is necessary to know where the points can lie. Since there are only 24 polygons, it is possible to find where the outside intersections must be for each option.

Note that multiple different sets of four tropical lines can give rise to the same polygon. Hence for each possible polygon there are multiple possibilities for where the outside intersections can be located. However, the polygon does give a limited area for the residual points to be in.



(a) No vertices on the edge
(b) No vertices on the edge
(c) No vertices on the edge
(d) No vertices on the edge
(e) No vertices on the edge
(f) No vertices on the edge
(g) No vertices on the edge
(h) No vertices on the edge
(i) No vertices on the edge
(j) One vertex on the edge
(k) One vertex on the edge
(l) One vertex on the edge
(m) One vertex on the edge
(n) One vertex on the edge
(o) One vertex on the edge
(p) One vertex on the edge
(q) One vertex on the edge
(r) One vertex on the edge
(s) One vertex on the edge
(t) One vertex on the edge
(u) One vertex on the edge
(v) Two vertices on the edge
(w) Two vertices on the edge
(x) Two vertices on the edge

Figure 2.8: The possible polygons made with four tropical lines

Example 2.3.15. Let's have a look at (NE, EN, NE, EN) i.e. the square with two horizontal and two vertical edges constructed from four tropical lines $L_1, \dots L_4$. The horizontal edges comes from the branches in the eastward direction from two of the lines. So those two lines have nodes to the west of the polygon. Call these lines L_1 and L_2. The vertical edges comes from the branches in northern direction from the other two lines, which must therefore have nodes south of the polygon. Call these lines L_3 and L_4.

Since the intersections L_1 have with L_3 and L_4 are the corners of the polygon, the intersection r_{12} between L_1 and L_2 must be one of the outside intersections. For a similar reason, the other one is r_{34}.

Any line whose eastward branch passes through v_{13} and v_{14} could take the place of L_1 and give rise to the same polygon. Hence one could say that the node of L_1 could be in any position from the point a infinitesimally close to the corner of the polygon to infinitely far to the west. The same is true for L_2. However there are some boundaries

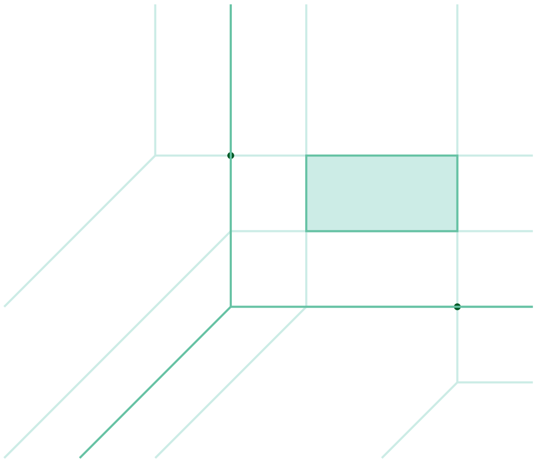


Figure 2.9: Tropical polypol consisting of four tropical lines, with residual points and adjoint curve indicated

to where r_{12} can lie. For one, it cannot be on the northward branch of L_1 since the eastern and southwestern branches of L_2 are south of the node of L_1. So the region A that r_{12} can be in is bounded by the (Euclidean) line extending the northern horizontal edge of the polygon. If r_{12} lies on this line, then it must be an intersection between the eastern branch of L_1 and the northern branch of L_2. Similarly, it cannot be on the southwestward branch of L_2, hence A is also bounded by the Euclidean line extending the other horizontal edge of the polygon. If the intersection r_{12} lies on this line, then it is an intersection of the horizontal branch of L_2 and the diagonal branch from L_1. If r_{12} lies between these two boundary lines, it is an intersection between the northern branch from L_2 and the southwestern branch of L_1, hence A is bounded by the diagonal passing through a. However, A is not bounded to the west.

A similar argument gives the area B to the south of the polypol, bounded on three sides by the vertical lines extending the sides of the polypol and one diagonal, where the intersection r_{34} can lie. Figure 2.10 shows the areas A and B.

Now since r_{12} is always on or above the line extending the southern horizontal edge of P while r_{34} is always below it, and since r_{34} is always on or to the right of the line extending the western vertical edge of P while r_{12} is always to the left of it, we can know that r_{12} and r_{34} are never on the same branch of a line. Hence the adjoint line A_P is always unique, and in fact r_{12} is always on the northern branch of A_P while r_{34} is always on the eastern branch of A_P.

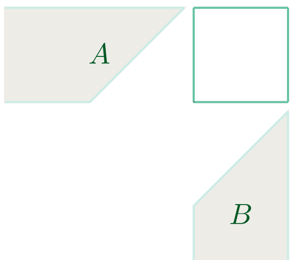
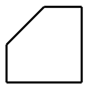


Figure 2.10: Possible positions of residual points of a tropical polypol bounded by four lines described in Example 2.3.15

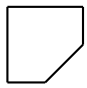
Similar constructions can be made for all 24 possible polygons. For each polygon, each residual point will be somewhere in a predetermined region which is either an area bounded on three sides and unbounded in the fourth like in Example 2.3.15 on page 24 or a bounded line segment. This is determined by whether perturbation of the vertexes of both lines or only of one changes the location of the residual point. In all cases, the adjoint will be unique and lie outside the polypol.

2.3.4 Five lines

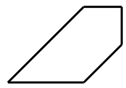
There are nine possible polygons consisting of five lines. As above, we are choosing to view the hexagons with different corners as nodes as different hexagons. All nine polygons are listed in Figure 2.11.



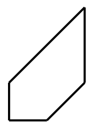
(a) No vertices on the edge



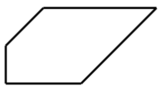
(b) No vertices on the edge



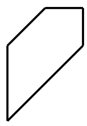
(c) No vertices on the edge



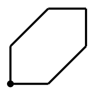
(d) No vertices on the edge



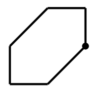
(e) No vertices on the edge



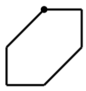
(f) No vertices on the edge



(g) One vertex on the edge



(h) One vertex on the edge



(i) One vertex on the edge

Figure 2.11: The possible polygons made with Five tropical lines

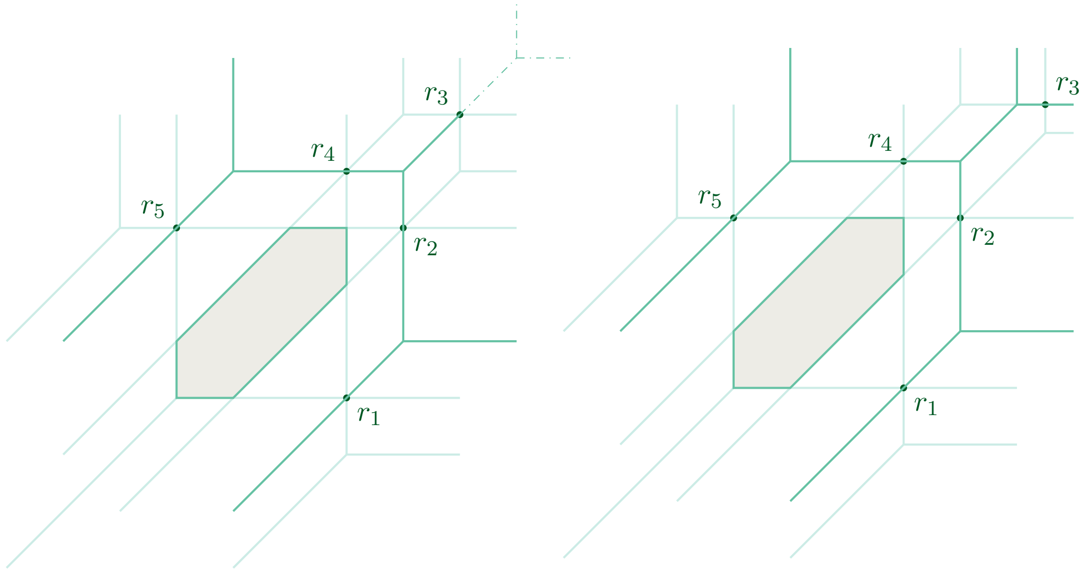
Five lines have a total of 10 intersections, of which five will be on the edge of the polygon, and five will be the residual points.

The adjoint A will be a conic through these five points. As discussed in Section 1.4, there are some cases where five points in the plane does not define a unique conic. It is possible for such points to be the residual points of a tropical polypol.

Example 2.3.16. Figure 2.12 shows two tropical polypols with boundary curves five lines. They both have the same cycle, $(V_{NE}, ES, SN, NE, ES, SN)$, but one of the lines is different. Figure 2.12a has multiple possible adjoints. In fact, its residual points, $r_1, \dots r_5$ are in the same position as the points $p_1, \dots p_5$ in Example 1.4.5. Like in that example, the stable conic through these five points is the proper conic of type 1 that has its fourth vertex at r_3. Figure 2.12b is only slightly different. One of the lines has its vertex slightly further to the northeast, which moves r_3 to the east. This set of residual points have one unique conic passing through them. So the adjoint in Figure 2.12b is unique.

Proposition 2.3.17. *For any polypol P with five lines as its boundary curves, its adjoint curve A_P does not intersect the interior of P.*

Proof. No possible polypol constructed from five tropical lines have more than one trivalent vertex on its boundary. Hence, since an adjoint that intersects the interior would intersect the boundary in two points that are trivalent curve vertices, no adjoint can intersect the interior of a polypol bounded by five tropical lines. ■



(b) Unique adjoint

(a) Non-unique adjoint

Figure 2.12: Tropical polypols bounded by five lines

As in the case of tropical squares, each pentagon comes with a set of possible regions for each intersection point to lie in. Like for squares, these regions are either unbounded areas or bounded line segments. From these possible regions, one can determine that all adjoints of tropical polypols with sides from five lines are either the union of two tropical lines or proper conics of type 1. As illustrated by both polypols in Figure 2.12 being of type $(V_{NE}, ES, SN, NE, ES, SN)$, the information given by the underlying polygon alone is not sufficient to determine whether or not the adjoint is unique.

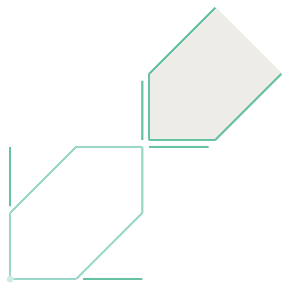


Figure 2.13: Possible positions of residual points of a tropical polypol bounded by five lines

2.3.5 Six lines

There is only one possible cycle that can be constructed using six lines. It is an Euclidean hexagon and it has no vertices on the edge of the polygon.

A tropical polypol consisting of six lines have exactly nine residual points. The adjoint curve is a cubic.

Example 2.3.18. Figure 2.14 shows a polypol consisting of six lines with the nine residual points and adjoint curve indicated. Note that the polypol is an Euclidean hexagon, and the adjoint is a cubic. The adjoint in this example has one self intersection, and lies entirely outside the polypol.

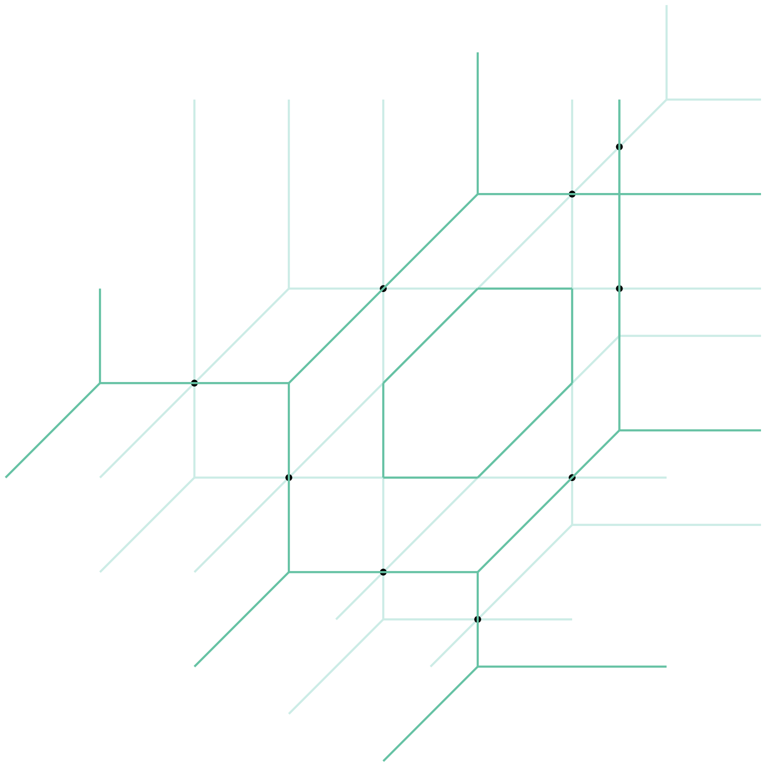


Figure 2.14: Tropical polypol bounded by six lines

Proposition 2.3.19. *There is no polypol with six lines as its boundary curves that has an adjoint that intersects the interior of the polypol.*

Proof. There is no polypol with six lines as its boundary curves that has the vertexes of any of the lines on its boundary. ■

Like for four and five lines, for each of the nine residual points of a tropical polypol consisting of six lines it is possible to determine a subset of the plane where it is possible for that residual point to be. In particular, six of the residual points reside on bounded line segments that extend the sides of the polypol. Note that there is necessarily a gap between the side of the polypol and the residual point, otherwise it would coincide with the corner. The final three residual points have unbounded areas as their possible location regions.

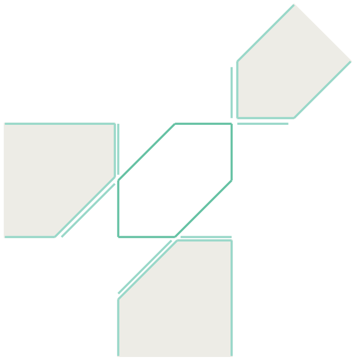


Figure 2.15: Possible positions of residual points of polypols bounded by six lines

2.4 Polypols with boundary curves of higher degree

Now that we know that the adjoint curves of polypols with linear boundary curves do not intersect the interior, we want to investigate the situation for polypols whose boundary curves are of higher degree than one. To do so, we will first look at what can be proven generally, then we will attempt to create a classification and catalog of possible cycles like the one we made for polypols consisting of tropical lines.

Using what we know about the possible counterexamples, and stable intersections, we can show that for polypols of degree four, the adjoint lies outside the polypol.

Proposition 2.4.1. *A tropical adjoint curve of degree one cannot intersect the interior of the polypol.*

Proof. Assume, for contradiction, that there is a tropical polypol P with a linear adjoint, A_P, that intersects the interior of P.

We know that A_P has degree 1 if and only if the polypol had degree $d = 1 + 3 = 4$. There are four possible combinations of curves that add up to degree four. The boundary curves are either four lines, one conic and two lines, two conics or one cubic and one line.

To have points on the interior of the polypol, the adjoint line needs to intersect the sides in at least two points. As established in Section 2.2.1, these points must be vertices of two of the boundary curves, sayC_1 and C_2. Then C_1 and C_2 must be such that the residual points, r_1 and r_2, lies on the third branch, not included in the side of the polypol, adjacent to each of the vertices on the boundary. Call these branches l_1 and l_2. Since the adjoint line must pass through both residual points and the intersections with one of the curves in a neighborhood of the point must be non-finite, either the vertex of A_P must lie on one of the curves, say C_1, or one vertex of C_1, in addition to the vertex of C_1 that lies on the boundary of P, must lie on A_P. In either case, A_P and C_1 have two stable intersection points on l_1. Since two lines can only have one stable intersection, C_1 cannot be a line. There are now three possibilities:

- C_1 is a cubic, in which case C_2 is a line,
- C_1 is a conic and C_2 is also a conic, or
- C_1 is a conic and C_2 is a line.

If C_1 is a cubic, then we know, since it is rational that either of r_1 or r_2 is a self intersection of C_1 and the other point is an intersection between C_1 and C_2. If $r_1 \in l_1$ is an intersection between C_1 and C_2, then by Lemma 2.2.8, C_2 and A_P have a stable intersection point at r_1, but they also have one at the vertex of C_2. It is not possible for two lines to have two stable intersection points. Hence, this situation cannot occur. On the other hand, if r_1 is the singular point of C_1, then, still by Lemma 2.2.8, C_1 and A_P have a stable intersection point at r_1 and one at r_2 in addition to the two at the vertices on l_1. A line and a cubic cannot have more than three stable intersections points, so this is also not possible.

If C_1 is a conic, then regardless of whether C_2 is a conic or a line, r_2 is an intersection between C_1 and C_2. By Lemma 2.2.8 C_1 and A_P have a stable intersection point at r_2 in addition to the two in l_1. A conic and a line can only have two stable intersection points. Therefore this is not possible.

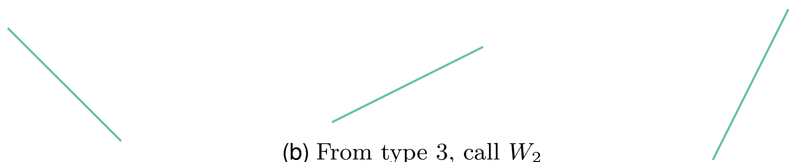
Since situations where A_P intersects the interior of P are impossible, then A_P cannot intersect the interior of P. ■

2.4.1 Conics and lines

In this section tropical polypols consisting of one conic and some number of lines will be discussed.

Classification

Recall from Section 1.2 that there are four types of proper tropical conics. Three of these included branches in directions other than the three directions of the branches of tropical lines. This means that when constructing a classification and catalog of possible Euclidean polygons with sides from conics like the one for lines in Section 2.3, we need to account for three additional directions. One to the southeast, perpendicular to direction S from before and two with different slopes.



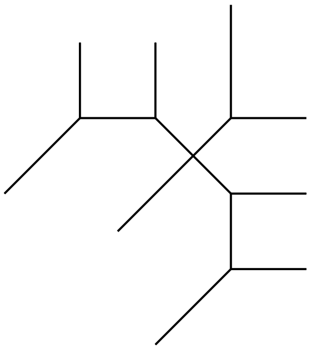
(a) From type 2, call W_1 (b) From type 3, call W_2 (c) From type 4, call W_3

Figure 2.16: Possible sides from conics

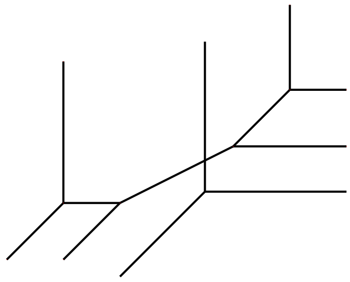
For each type of conic, a polypol consisting of one such conic and any number of lines, can have up to seven Euclidean sides. A polypol with boundary curves consisting of an arbitrary number of conics can have up to 12 Euclidean sides. The requirement that all corners of a polypol must be transversal intersections does however give a useful restriction on the types of cycles a polypol with sides from conics can look like. Some types of corners can only come from tangencies, and can therefore not be on the sides of a polypol. Figure 2.17 show what these tangencies can look like. Expanding on the notation from Section 2.3.1 on page 18, we call the sides from conics of type 2, type 3 and type 4 W_1, W_2 and W_3 respectively. Then the corners that cannot be on the edges of a polypol are of the types, SW_1, W_1S, NW_2, W_2N, EW_3 and O_3E.

Proposition 2.4.2. *All cycles from Figure 2.4 can be constructed using one proper tropical conic and multiple lines.*

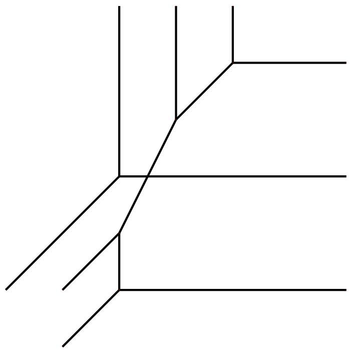
Proof. All proper tropical conics have branches in all the directions that tropical lines do. Hence every type of corner needed to construct the cycles can be constructed as an



(a) Conic of type two with tangent



(b) Conic of type three with tangent



(c) Conic of type four with tangent

Figure 2.17: Conics and tangents

intersection between a tropical conic and a tropical line. Therefore any of the cycles can be constructed with at least one side being a segment from a tropical conic. ■

Restricting to proper conics of type 2, we can find how many convex Euclidean polygons can be constructed with one side being of type W_1.

Then there are altogether 30 such polygons, where the maximum number of sides is seven:

- two cycles with three sides,
- eight cycles with four sides,
- ten cycles with five sides,
- eight cycles with six sides,
- and two cycles with seven sides.

We expect that the numbers would be the same for conics of type 3 and type 4, but we have not verified this claim.

One conic and one line

By Bezout's theorem, a tropical line and a tropical conic will intersect either in two points or in one point twice. If the line is a tangent to the conic, there is no polypol. The total degree of the polypol is 3. Hence there is no adjoint curve.

Remark 2.4.3. All polygons from Figure 2.4 can be constructed using one conic and one line.

This can easily be shown by going through the list and attempting the construction for each cycle. Some of the polygons can be constructed in several ways, while others only from one particular part of one type of tropical conic.

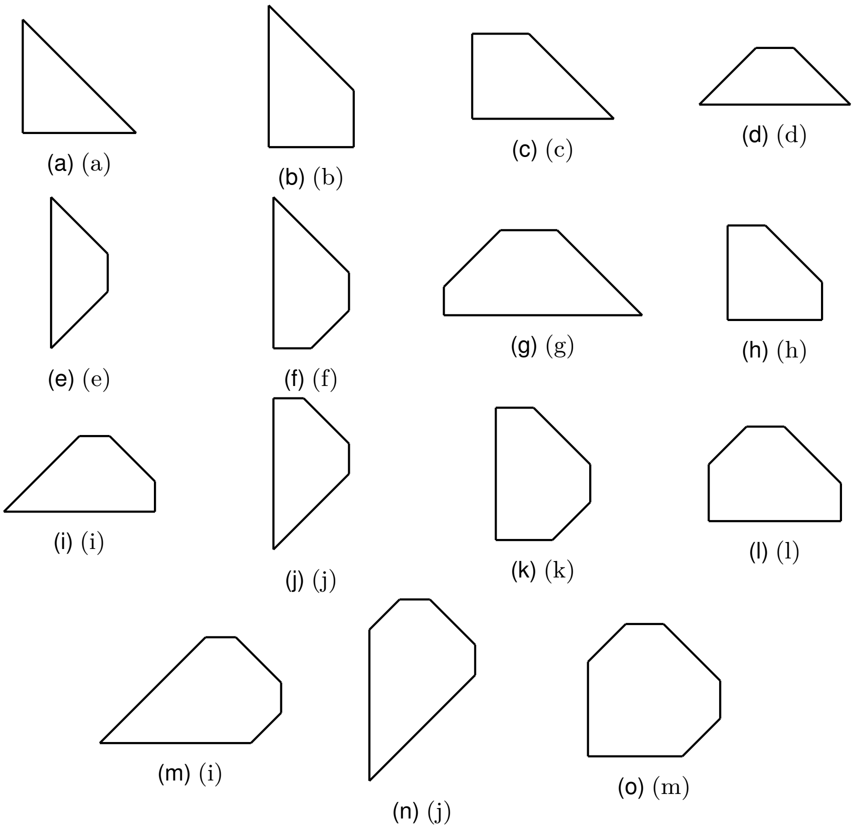


Figure 2.18: The fifteen possible polygons that can be constructed with sides from one tropical conic of type 2 and one tropical line

Remark 2.4.4. For each of the three conics that have the branches in Figure 2.16 there are an additional fifteen polygons that can be constructed, one triangle, four squares, five pentagons, four hexagons and one heptagon. The ones that can be constructed using tropical conics of type 2 are given in Figure 2.18.

One can show that the list in Figure 2.18 is complete by fixing one conic of type 2 and trying different lines. Then we will have either a tangent, a polypol with cycle from Figure 2.4 or one of these fifteen cycles.

Example 2.4.5. Figure 2.19 shows a polypol consisting of one conic of type 2 and one line.

One conic and multiple lines

We return to discussing tropical polypols bounded by one conic and some lines in more general terms.

Note that, like how three lines is sufficient to construct every convex polygon in Figure 2.4 on page 19, two lines and one conic of type 2 is sufficient to construct all 30 possible convex polygons listed in Section 2.4.1. However, there is a choice of which of the Euclidean corners are trivalent curve vertices. Creating a classification that can be used to determine the position of the adjoint requires evaluating more than 30 figures.

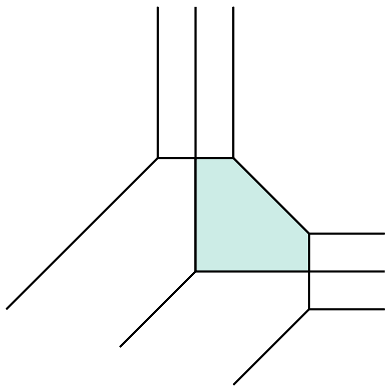


Figure 2.19: Example of a tropical polypol bounded by one conic of type 2 and one line

The largest cycle that can be constructed using one tropical proper conic and a set of tropical lines have seven sides, out of which one must come from the conic. Hence the highest possible number of lines is six lines. A tropical polypol with boundary curves one conic and six lines has degree 8, and its adjoint has degree 5. Notably, such a polypol cannot have any of the curves' trivalent vertices on its boundary. Since each side is from a different curve, all corners must be intersections between the curves. Therefore, the adjoint lies outside the polypol.

In fact, the number of possible vertices on the boundary of the polypol goes down when the number of lines goes up. In the case where the conic is of type 2, the possible seven-sided polypol with one conic and one line has five trivalent vertices on its boundary. Both the possible seven-sided cycles with two lines have at most four. We already know from Proposition 2.4.1 that the adjoint does not intersect the edge in this case. For one conic and three lines the number of trivalent vertices is three, for four lines it is two and for five lines only one. Hence a possible counterexample to Question 2.2.6 consisting of one conic and lines would necessarily have either three or four lines.

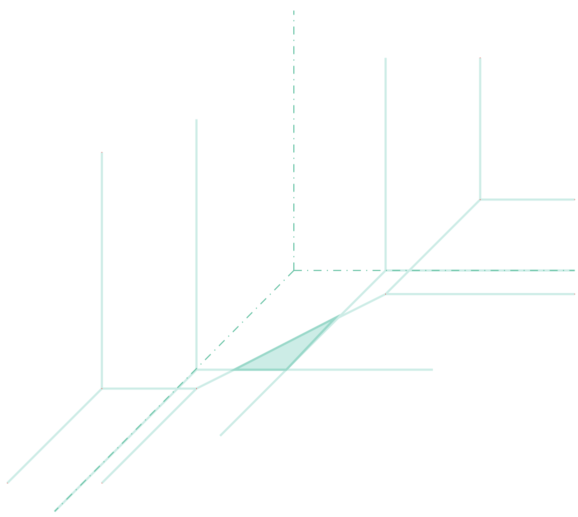


Figure 2.20: Example of a tropical polypol bounded by one conic of type 3 and two lines

Example 2.4.6. Figure 2.20 shows an example of a conic with sides from one conic and two lines. It represents one example of a polypol that cannot be constructed from lines alone, nor from one conic and one line. The adjoint is indicated with a dashed line.

2.4.2 Three conics

In the real case, the first case where Conjecture 2.1.5 has not been proven is for polypols bounded by three conics. The problem is still open for these polypols in the tropical case as well. In this section we therefore explore an example of a polypol consisting of three conics. The boundary curve is a not necessarily unique cubic.

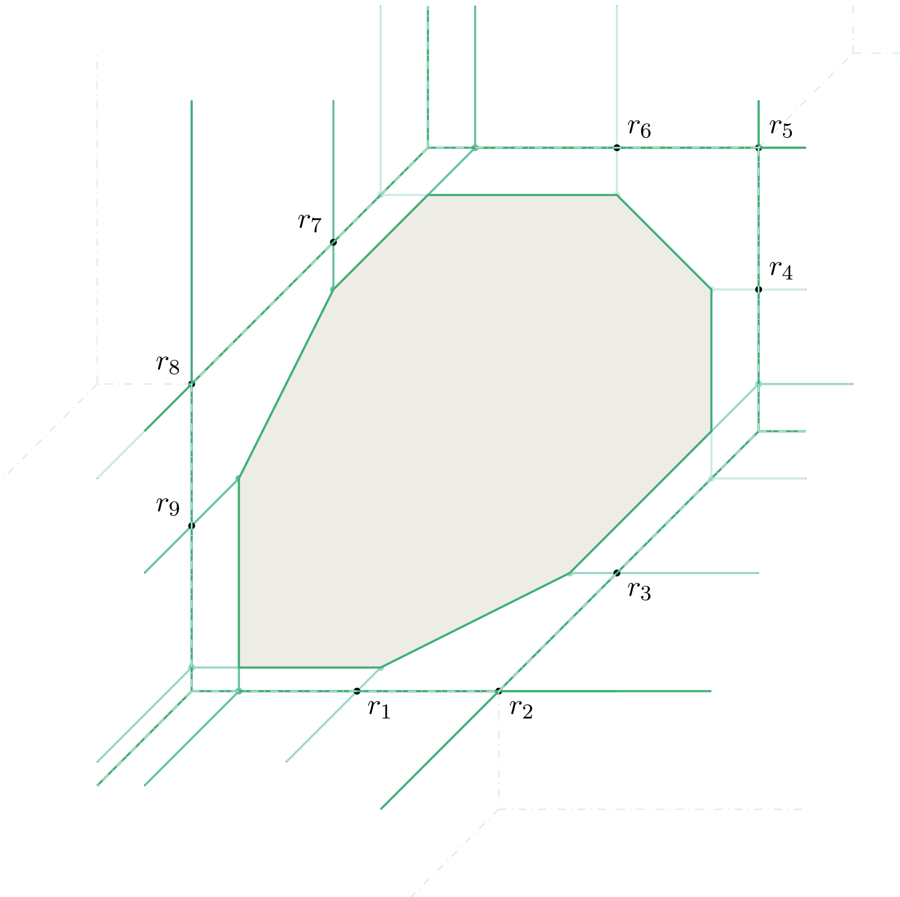


Figure 2.21: Example of a polypol with sides from three different conics, with residual points and adjoint curves indicated

Example 2.4.7. Figure 2.21 shows an example of a polypol consisting of three conics of different types. The adjoint of this polypol is a curve of degree three, but it is notably not unique. Since all nine residual points $r_1, \dots r_9$ lie along the cycle of the adjoint cubic, any of the points r_2, r_5 or r_8 have the potential to be either a vertex or a self-intersection. Hence the adjoint could be of the form of three lines intersecting at those points (shown on the figure as solid color lines), three variations of one line and one conic, three variations of a nodal cubic or a smooth cubic (shown in figure as dotted lines).

Chapter 3

Conclusion

In this thesis we have given a definition of tropical polypols and tropical adjoints. Using that definition, we have given a full description of tropical polypols consisting of lines. They have been shown to occur as a finite number of types, where relevant properties of their adjoints are determined by type. With this we have proven a tropical version of Wachspress' conjecture for polypols whose boundary curves are lines. Using some specific examples, we have discussed cases where the adjoints of such polypols are or are not unique. We have also begun the work of a similar handling of tropical polypols with sides from tropical conics as well as tropical lines. In this we have gotten as far as determining that a counterexample to the conjecture with boundary curves one conic and some lines would need to have three or four lines.

More generally, we have found that no tropical polypol of degree four can have the adjoint intersecting the interior. Additionally, Lemma 2.2.8 gives a result about the intersections of multiple tropical curves that might be useful in finding properties of tropical polypols that have not been found here.

Finally, we have given some examples of tropical polypols with more or higher degree boundary curves than the ones we have discussed in detail.

There are some questions that arise naturally from the things we have looked at in this thesis, that we have not given an answer to. The following are the most important open problems.

It is possible to expand the classification for polypols consisting of tropical lines given here to polypols with boundary curves of higher degree. Since there are finitely many possible edges for each degree, there will also be finitely many possible cycles.

Since this thesis explores tropical objects from tropical definitions, rather than as tropicalizations of real objects, the question of correspondence between the objects described here and real equivalents is open.

Over the course of this project, a lot of time was spent trying and failing to find an argument that, similar to Proposition 2.1.6 in the real case, utilizes Bezout's theorem to show that the adjoint cannot intersect the sides of the polypol. It still seems entirely possible that such an argument might exist.

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Appendix A

The First Appendix

The following is the [Matlab] code used in Section 1.4 on page 8.

Matlab Code

```
v = [1,2,3,4,5];
P = perms(v);

q1 = [0,3,1];
q2 = [0,1,0];
q3 = [0,1,3];
q4 = [0,1.5,2];
q5 = [0,3,3];

sums = zeros(1,120);
a = zeros(1,6);

%order x^2+xy+y^2+yz+z^2+xz

M = [2*q1(1) q1(1)+q1(2) 2*q1(2) q1(2)+q1(3) 2*q1(3) q1(1)+q1
    (3); 2*q2(1) q2(1)+q2(2) 2*q2(2) q2(2)+q2(3) 2*q2(3) q2(1)+
    q2(3);2*q3(1) q3(1)+q3(2) 2*q3(2) q3(2)+q3(3) 2*q3(3) q3(1)+
    q3(3);2*q4(1) q4(1)+q4(2) 2*q4(2) q4(2)+q4(3) 2*q4(3) q4(1)+
    q4(3);2*q5(1) q5(1)+q5(2) 2*q5(2) q5(2)+q5(3) 2*q5(3) q5(1)
    +q5(3)];

M1 = M(:,2:6);
M2 = M(:,[1 3:6]);
M3 = M(:,[1:2 4:6]);
M4 = M(:,[1:3 5:6]);
M5 = M(:,[1:4 6]);
M6 = M(:,1:5);

a(1)=tropdet(M1,P,sums);
a(2)=tropdet(M2,P,sums);
a(3)=tropdet(M3,P,sums);
a(4)=tropdet(M4,P,sums);
```

```
a(5)=tropdet(M5,P,sums);
a(6)=tropdet(M6,P,sums);

function td = tropdet(Matrix,Permutations,sumarray)
    for index=1:120
        temp = Permutations(index, :);
        sum = 0;
        for j = 1:5
            sum = sum + Matrix(temp(j),j);
        end
        sumarray(index) = sum;
    end
    td=min(sumarray);
end
```