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Review and Evaluation of Effective Length Formulas

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ABSTRACT

An overview is given of approximate effective length formulas. The overview is limited to formulas that are relevant for the important class of compression members defined by constant stiffness and constant axial force along the length, and, furthermore, limited to members that are either completely unbraced (with zero shear and free-sway) or fully braced against lateral translation of one end relative to the other. Whereas most approximate effective length formulas have been developed for positive end restraints, buckling modes of compression members in a frame can be associated with both positive and negative end restraints. It is a main concern of the study to identify the applicability of the various approximate formulas for a reasonable wide range of positive and negative restraint combinations. Extensive comparisons of approximate predictions with exact effective length results have been carried out.

KEYWORDS

Stability, Buckling, Columns, Compression Members, Effective Length, Critical Load, Structures, Frames

Content

1 Introduction

For the purpose of determining effective lengths (and critical loads) of compression members for *given* end restraints, there is a number of aids available. These include diagrams and formulas for a great number of special cases in various handbooks etc. In structural design codes and standards, the most typical compression member is that with constant cross-sectional bending stiffness (EI) as well as constant axial force (N) along the length (L). For this important case, aids for unbraced and braced members are also included in some of the relevant structural analysis and design codes and standards.

Buckling modes of compression members in a frame can be associated with both positive and negative end restraints (Bridge and Fraser 1987; Hellesland 1992a, 1992b). A range of realistic buckling modes for such restraints are reviewed and discussed in a companion study (Hellesland 1994). Most of the approximate formulas available have been developed for members with positive end restraints. Even so, these formulas will often be valid, to various extent, for restraint combinations that include negative restraints. The author is not familiar with any evaluation of the applicability of existing approximate formulas in the negative restraint range.

The main objective of the present study is to give an overview of some of the available approximate effective length formulas for "constant stiffness and constant axial force" members, and, further, to establish the accuracy by comparing approximate predictions with exact results. The extent to which the formulas are applicable will be sought identified for a reasonable wide range of both positive and negative restraints. The scope is limited to members that are either completely unbraced (with zero shear and free-sway) or fully braced against lateral translations of one end relative to the other. The most recently published formulas will be presented first.

2 Restraint parameters

Effective lengths are functions of the relative magnitude of rotational restraint stiffnesses k at the two ends and EI/L of the member. Normally, they are expressed in terms of one of the restraint parameters below. Symbols adopted here are not necessarily those used in the original presentations.

Relative restraint stiffness, $\overline{k}$ (= ∞ for fully fixed and 0 for pinned end):

$$\overline{k}_i = \frac{k_i}{(EI/L)} \qquad i = A, B \tag{1}$$

Relative restraint flexibility , G' (= 0 for fully fixed and ∞ for pinned end):

$$G'_i = \frac{(EI/L)}{k_i} = \frac{1}{\overline{k}_i} \qquad i = A, B \tag{2}$$

Scaled, relative restraint flexibility, G (= 0 for fully fixed and ∞ for pinned end) with scaling (or reference) factors b_o= 6 for unbraced members and b_o= 2 for braced members:

$$G_i = b_o \frac{(EI/L)}{k_i} = \frac{b_o}{\overline{k}_i} \qquad i = A, B \tag{3}$$

Degree of rotational fixity, R (= 1 for fully fixed and 0 for pinned end):

$$R_i = \frac{k_i}{k_i + cEI/L} = \frac{1}{1 + c/\overline{k}_i} \qquad i = A, B \tag{4}$$

Degree of rotational flexibility, η (= 0 for fully fixed and 1 for pinned end):

$$\eta_i = \frac{cEI/L}{cEI/L + k_i} = \frac{1}{1 + \overline{k}_i/c} \qquad i = A, B \tag{5}$$

For the c–coefficient in the last two expressions, various values can be found in the literature. The two last expressions are related by $\eta = 1 - R$ provided the same c–value is used in both expressions.

Framed compression members.
The rotational restraint stiffness k_i at end i of a given member, is equal to the sum of the rotational stiffnesses of all members that frame into the considered member end. Thus,

$$k_i = \sum (b\, EI/L)_{all,i} \qquad i = A, B \tag{6}$$

In the summation, that includes all members but the considered member itself, EI, L, and b denote the cross-sectional bending stiffness, length and rotational stiffness coefficient of the members, respectively. The coefficient b is a function of the restraint condition at the far end of the member and of its axial force.

For compression members in the summation, a major problem is to assess the axial force effects on the b–coefficient. In NS 3472 (NSF 1973), a rather common approximate relationship is given for the axial force effect. A more accurate approach, that is iterative and therefore more cumbersome to use, has been proposed by Bridge and Fraser (1986).

Continuous compression members/columns.
In typical frames, the restraining elements are typically the beams (girders). Normally, axial forces (and second order effects of the axial forces) are negligible in the beams. In a frame with continuous columns (e.g., multistorey frame), the restraint at an end of a column can be expressed in several ways. One way is to express it by $k = fk_b$, where f is a factor and k_b, defined by

$$k_b = \sum (b\, EI/L)_b \tag{7}$$

is the rotational stiffness of the beams framing into the joint (with the considered column end). Here, subscript b denotes the beams. Normally, axial force effects in beams can be neglected. The fraction f, that appropriately may be termed *restraint participation factor* (Hellesland 1992a), reflects the interaction between columns above and below the joint ("vertical interaction"). It is very common to assume that k_b partakes in the restraint of the column(s) above and below the joint in proportion to the EI/L-values of the respective columns. Based on this assumption, the restraint participation factor at a column end i becomes

$$f_i = \frac{EI/L}{\sum(EI/L)_i} \qquad i = A, B \tag{8}$$

The restraint stiffness at a column end i can then be written

$$k_i = f_i k_{b,i} = \frac{EI}{L} \cdot \frac{\sum(bEI/L)_{b,i}}{\sum(EI/L)_i} \qquad i = A, B \tag{9}$$

Here, the summations are over all beams (numerator) and columns (denominator) framing into the joint, and EI/L in front of the fraction is for the considered column. Substitution of this restraint stiffness expression into, for example, the G–factor expression above, Eq. 3 gives

$$G_i = \frac{\sum(EI/L)_i}{\sum(mEI/L)_{b,i}} \qquad i = A, B \tag{10}$$

where $m = b/b_o$ is a stiffness modifier (defined by the ratio of the real beam stiffness coefficient, b, and the scaling (reference or datum) value b_o. Such G–factor definitions are well known. In normal practice, m is not included (i.e., it is taken equal to 1.0). However, the expression is more complete and transparent with m included.

Although the vertical interaction reflected by Eqs. 8, 9 or 9 suffers from several shortcomings (Hellesland (1992a), it has been adopted by a great number of codes and standards around the world. These include AISC (1993 Commentary), ACI (1989 Commentary), AS4100 (AS 1990), BS 5950 (BSI 1985), Eurocode 2 (CEN 1991), Eurocode 3 (CEN 1992 Annex), NS 3473 (NSF 1989), and many others. This type of interaction is sometimes referred to as *conventional* or *simple*.

3 Hellesland (1988/1994)

Approximate formulas for effective length factors, β, and inflection point locations, L_A and L_B, derived in detail in Hellesland (1994), are summarized below. One formula is given for the unbraced case, and 5 formulas (giving different results) for the braced case. They are all derived using a restrained cantilever member as the base model. The formulas are expressed in terms of end restraints defined by the so-called *degree of rotational fixity factors*, R, that reflects the degree of which the member ends are fixed against rotation. They are normally between 0 (pinned end) and 1.0 (clamped end), but may also become negative, and even greater than 1.0 in some cases (Hellesland 1994).

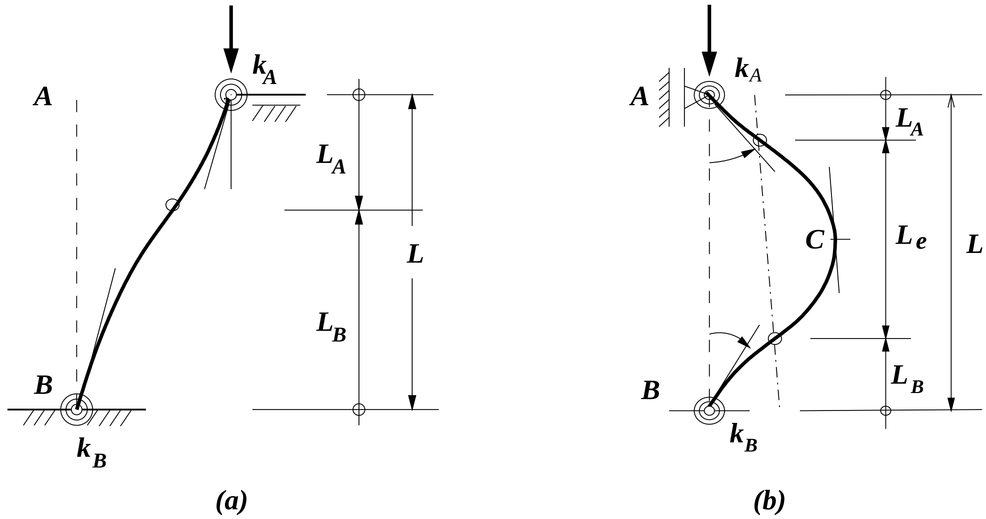

Figure 1: Unbraced and braced member with positive end restraints

Degree of rotational fixity:

$$R_i = \frac{1}{1 + c/\overline{k}_i} \qquad \left(= \frac{1}{1 + \frac{c}{b_o} G_i} \right) \qquad i = A, B \tag{11}$$

Unbraced member, c = 2.4

$$\beta = \frac{2\sqrt{R_A + R_B - R_A R_B}}{R_A + R_B} \qquad ; \quad \frac{L_i}{L} = \frac{R_i}{R_A + R_B} \tag{12}$$

For $1 \geq R \geq 0$, the β–predictions are accurate within about 0 to +2% of exact results. In the wide range defined by $1.25 > R > -0.4$, the accuracy will be within about ± 3%, and generally well within these percentages. In the latter range, some cases with β-predictions in excess of about 4 were excluded. Cases with such effective length factors are believed to be of little practical interest. For details of selected comparisons, see Table A.1 for exact results and Table B.1a and Fig. 2 for predictions compared to exact results. Also included in the figure are results of another β-expression that will be discussed in Section 5.

Comparatively, predictions using Eq. 12 with c= 2.5, Table B.1b, are somewhat greater (within 0 to +3% of exact results) for combinations of positive restraints and somewhat smaller for combinations that include negative restraints. The difference is not too big, but on the overall it is felt that c= 2.4 is to be preferred to c= 2.5. Eq. 12, with c= 2.5, but written in a different form, was adopted by NS 3473–1989 (see Section 6).

The segment lengths, directed from a member end to the inflection point, may be **positive or negative**. A positive value implies that the direction is from the considered end and towards the other end. A negative value implies that the direction is from the considered end and away from the other end. In the

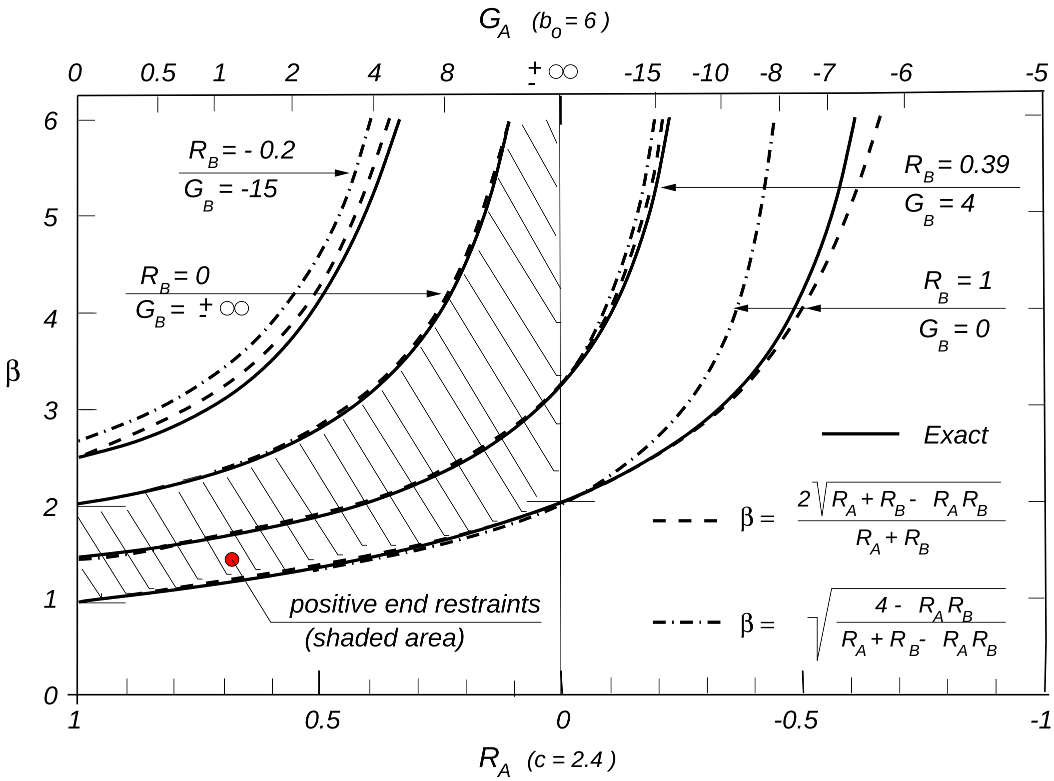

Figure 2: Unbraced effective length factor comparisons – Approximate vs. exact results

latter case the inflection point will be on the outside of the member length (on the theoretical continuation of the buckling curve from the end considered).

Braced member, c = 4.8

$$\beta = \frac{2}{2 + 1.1R_{min} + 0.9R_{max}} \qquad ; \qquad \frac{L_i}{L} = (1-\beta)\frac{R_i}{R_A + R_B} \qquad (13)$$

R_{min} is the algebraically smallest and R_{max} the greatest of R_A and R_B.

For a member with identical restraints at both ends, β can be obtained by applying Eq. 12 to half the member, and can be expressed by $\beta = 1/(1 + R_i)$ $(i = A, B)$. In order to generalize, R_i was above replaced by a weighted mean of the fixity factors ($\beta = 1/(1 + R_{wm})$) to give Eq. 13.

For $1 \geq R \geq 0$, the accuracy of β–predictions is within about –1.5 and +1% of exact results. In the wider range of $1 \geq R > -0.5$, the accuracy is within –1.5 and +5%. Predictions in excess of about β=2 are not believed to be of much practical interest in the sense that such members can probably be treated as flexural restraining members rather than compression members. Further, braced members with $R > 1.0$ is not considered too realistic, and have not been evaluated in any detail. For details of selected comparisons, see Table A.2 for exact results and Table B.2a and Fig. 3 for predictions compared to exact results.

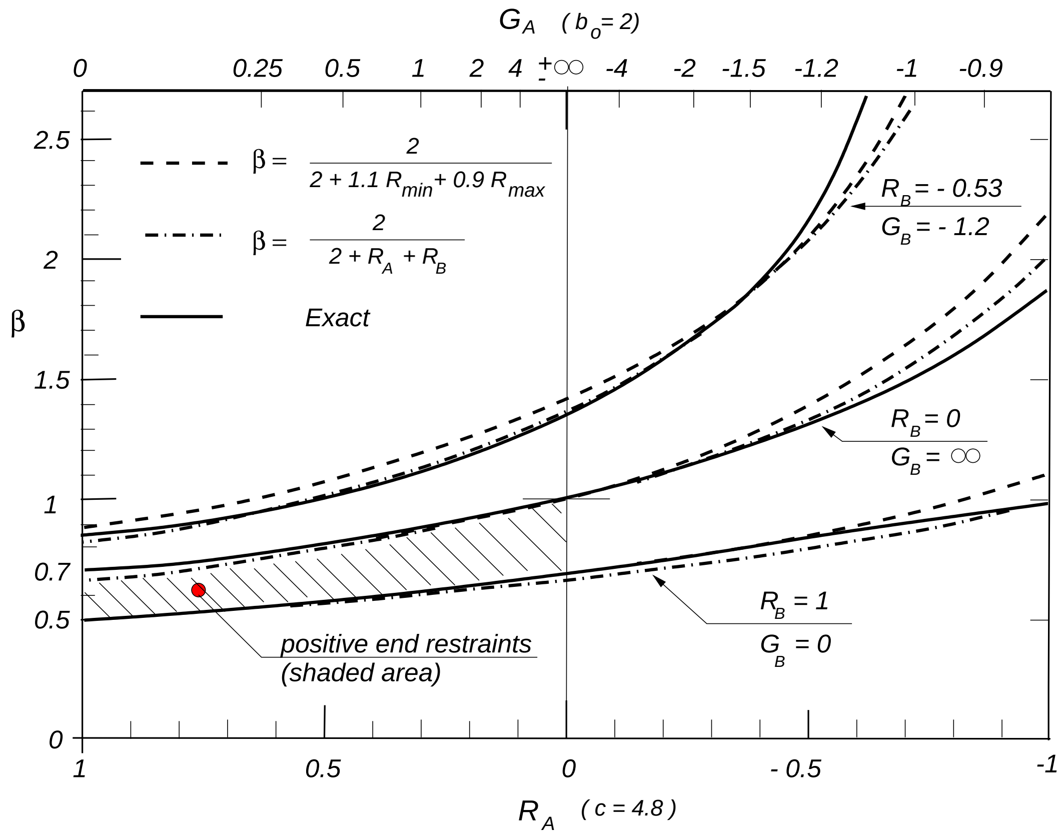

Figure 3: Braced effective length factor comparisons – Approximate vs. exact results (from Hellesland (1994)).

Use of c= 2.5, Table B.2b, are found to give slightly greater predictions than use of c= 2.4 for combinations of positive restraints (within ± 1.5% of exact results) and somewhat smaller for combinations that include negative restraints. The difference is very little, however. On the overall, a slight preference is given to c= 2.4.

The difference in the c–coefficients for the braced (4.8) and the unbraced case (2.4) is physically motivated as the formula for the braced member is based on the same physical model as that used for the unbraced one. Operating with two different c–coefficient is not a major inconvenience. In particular since the end restraint assessment normally will be different in the two cases anyhow.

Alternative formulas, braced member with **c = 4.8**

$$\beta = \frac{1}{1+R_m} = \frac{2}{2+R_A+R_B} \qquad ; \quad \frac{L_i}{L} = (1-\beta)\frac{R_i}{R_A+R_B} \qquad (14)$$

Above, R_m (in the first formulation) is taken as the simple mean of R_A and R_B. Within the restraint limits $0.7 > R > -0.55$, the accuracy of Eq. 14 is generally found to be well within ± 2% of the exact results, and for $0.45 > R > -0.4$ within ± 1%.

For details of selected comparisons, see Table B.3 and Fig. 3. As seen from the table, acceptable accuracy is also obtained in many cases for negative restraints outside the right hand side limits above. Outside the upper (left hand) limits, prediction errors increase to at most -5% (below exact results) in the case of a member clamped (fully fixed) at one end and flexibly restrained at the other. In a practical case, it will normally be difficult to obtain full fixity. In recognition of this, some codes recommend that effective length calculations be carried out with a restraint stiffness that do not exceed an upper limit. For such cases, Eq. 14, which is attractively simple, is most suitable in practical applications.

Instead of replacing R_i with the simple or weighted mean value of the fixity factors at the two ends, $1 + R_i$ may be replaced by the square root of the product $(1 + R_A)(1 + R_B)$. Thus,

$$\beta = \sqrt{\frac{1}{(1 + R_A)(1 + R_B)}} \tag{15}$$

which is also a rather simple and attractive formulation. As far as accuracy is concerned, this formula is comparable to Eq. 13 for positive restraint combinations, where it is accurate within about -1 and +1.5% (Table B.4a). However, for positive/negative combinations the accuracy is not as good. It is generally more conservative. Provided negative restraints are such that $R \geq -0.25$, the error will not exceed about +5%. For negative/negative combinations the accuracy is better, but still not as good as that of Eq. 13.

Predictions using c= 5 are given in Table B.4b. As before, they are found to give slightly greater predictions than use of c= 2.4 for combinations of positive restraints (within about -0.5 and +2% of exact results) and somewhat smaller for combinations that include negative restraints. However, again the difference is rather small. Eq. 15 with c= 5 can be written in the exact same form as Newmark's formula (e.g., Eq. 48, cfr. Section 11).

Alternative formulas, braced member with $\mathbf{c = 2.4}$

Expressed in terms of rotational fixity factors with $c = 2.4$ (2.5), β for a member with identical restraints at both ends can be expresed by $\beta = 1 - 0.5R_i$ ($i = A, B$). In the general case, it has been found that replacement of R_i by the approximate weighted mean defined by $R_{wm} = 0.4R_{min} + 0.58R_{max}$ give reasonable results. Then,

$$\beta = 1 - 0.5R_{wm} = 1 - 0.2R_{min} - 0.29R_{max} \tag{16}$$

where, R_{min} and R_{max} are the algebraically smallest and greatest end restraint factor, respectively.

The accuracy of Eq. 16 is within about -1 and +2% for any combinations of positive end restraints, which is very acceptable. For positive/negative combinations, the accuracy is not as good. It is comparable, but generally slightly more conservative than found by Eq. 15. Provided negative restraints are such that $R \geq -0.25$, the error will not exceed about +5%. For negative/negative combinations the accuracy is better. For details, see Table B.5.

Use of the simple mean of the fixity factors, $R_m = (R_A + R_B)/2$, gives

$$\beta = 1 - 0.5R_m = 1 - 0.25(R_A + R_B) \qquad (17)$$

For positive end restraint combinations, this equation is found to be accurate to within about 0 and +7% of exact results. In this restraint range, the error increases with increasing difference between the end restraints at the two ends. Thus, for a member pinned at one end and clamped at the other, the effective length is overestimated by 7%. For positive/negative and negative/negative combinations the accuracy is acceptable in some cases, but is in general not very good. For details, see Table B.6. A similar formulation was used in NS 3472 (cfr. Section 8).

Finally, Eq. 15 (in which R is defined with c=4.8) can be rewritten in terms of R with c=2.4 as

$$\beta = \sqrt{(1 - 0.5R_A)(1 - 0.5R_B)} = \frac{1}{2}\sqrt{(2 - R_A)(2 - R_B)} \qquad (18)$$

4 Duan, King and Chen (1993)

Duan, King and Chen (1993) derived β–expressions using a best-fit approach in combination with a so-called partial fraction model.

Unbraced member

$$\beta < 2: \qquad \beta = 4 - \frac{1}{1 + 0.2G_A} - \frac{1}{1 + 0.2G_B} - \frac{1}{1 + 0.01G_A G_B} \qquad (19)$$

$$\beta > 2: \qquad \beta = \frac{2\pi a}{0.9 + \sqrt{0.81 + 4ab}} \qquad (20)$$

where

$$a = \frac{G_A G_B}{G_A + G_B} + 3 \quad ; \quad b = \frac{36}{G_A + G_B} + 6 \qquad (21)$$

For positive restraints, the accuracy of these expressions are within about -1 and +2%, which is slightly worse than that of Eq. 12. For many combinations involving negative restraints, the accuracy is somewhat better or comparable to that for Eq. 12. For details of selected predictions, see Table C.1. The limits ($\beta < 2$ or > 2), indicating which expression to use, are not always adequate. In the table locations marked “2)”, predictions could not be obtained with the limitations given. This is a drawback of the expressions. Also, a “double expression” is in itself a drawback, in particular since it is rather complicated in comparisons to alternative expressions for the unbraced case given elsewhere in this report.

Braced member

$$\beta = 1 - \frac{1}{5 + 9G_A} - \frac{1}{5 + 9G_B} - \frac{1}{10 + G_A G_B} \qquad (22)$$

For positive restraints, the accuracy of this expression is within about -1 and +12 % of exact results. The greatest inaccuracy is for cases with one end fully fixed. For such cases, the above expression is inferior to any of the others in this study. However, for positive restraints less than full fixity, the accuracy will be quite acceptable. The expression is not very accurate for negative restraints. See Table C.2 for details of selected comparisons.

5 Eurocode 3 (1992)

Effective length factor expressions given in Annex E of Eurocode 3 (CEN 1992) are summarized below. They are expressed in terms of so-called distribution factors labelled η. The same restraint parameter was used in BS 5950 (BSI 1985), and there labelled k.

Distribution factor (degree of rotational flexibility)

The η–parameters in the code are defined by the sum of I/L of the compression members at a joint (member end) divided by the sum of I/L of all members at the joint. In terms of the symbols in the present report, η may be defined by

$$\eta_i = \frac{\sum(EI/L)_i}{\sum(EI/L)_i + \sum(mEI/L)_b, i} \qquad i = A, B \tag{23}$$

In the code, a beam stiffness corresponding to full fixity at the far end and neglible axial force is taken as the reference (datum) case. This corresponds to taking the previously defined b_o equal to 4, and the previously defined stiffness modifier $m = b/4$. It is described in the text of the code and values are tabulated for other far end conditions and axial force effects. Although not explicitly shown in the code formulation, the coefficient m is included here for the sake of transparency. The above expression implies a vertical interaction defined previously by Eq. 8. Substitution of the restraint stiffness k by Eq. 9, η can then be written in the general form given by Eq. 5 with c=4, i.e.,

$$\eta_i = \frac{4EI/L}{4EI/L + k_i} = \frac{1}{1 + \overline{k}_i/4} \qquad i = A, B \tag{24}$$

Unbraced member ("Sway mode")

For the unbraced member case, the code gives

$$\beta = \sqrt{\frac{1 - 0.2(\eta_A + \eta_B) - 0.12\eta_A\eta_B}{1 - 0.8(\eta_A + \eta_B) + 0.6\eta_A\eta_B}} \tag{25}$$

In order to compare with the formulation given by Hellesland (Eq. 12), it is found that the η–expression above may be expressed in an alternative form in terms of fixity factors by

$$\beta = \sqrt{\frac{4 - R_A R_B}{R_A + R_B + R_A R_B}} \tag{26}$$

where the R–factors are defined with c=**2.4**. Alternatively, it can be expressed in terms of G–factors ($b_o = 6$) by

$$\beta = \sqrt{\frac{1.6 G_A G_B + 4(G_A + G_B) + 7.5}{G_A + G_B + 7.5}} \tag{27}$$

By comparison with the French rules' Eq. 38, it is seen that the Eurocode expression for unbraced members is the same as the French one except for the formulation in terms of different restraint parameters. An expression giving identical results has been developed by Mekonnen (1987).

A closer examination of the Hellesland Eq. 12 and the Eq. 26 above reveals that they become identical for members that are pinned at one end and for members with equal end restraints. The difference in the two expressions for other cases is due to the different displacement assumptions adopted in the derivations of the expressions.

Effective length predictions by both expressions are compared to exact results in Fig. 2 for various combinations of positive and negative end restraints.

Results in the shaded area of the figure are obtained for end restraints that are positive at both ends ($1 > R > 0$ or $0 < \eta < 1$). For such cases, the predictions by the expression above are within about -1 to +2% of the exact results, and by Eq. 12 within about 0 to +2%. Thus, in the positive restraint range, the accuracies of the two approximations are almost identical. In cases involving negative restraints, the predictions by the Eurocode expression are in general not so good. In the limited range defined by $-0.3 < \eta < 1.08$ (corresponding to about $1.15 > R(c = 2.4) > -0.15$), predictions are within about -3 and +5% of exact results when β–predictions greater than 4 are excluded. For details of selected Eurocode comparisons, see Table D.1.

Braced member ("Non sway mode")

For the braced member case, two expressions are given. They are defined by

$$\beta = 0.5 + 0.14(\eta_A + \eta_B) + 0.055(\eta_A + \eta_B)^2 \tag{28}$$

and

$$\beta = \frac{1 + 0.145(\eta_A + \eta_B) - 0.265\eta_A\eta_B}{2 - 0.364(\eta_A + \eta_B) - 0.247\eta_A\eta_B} \tag{29}$$

For positive restraints, $0 < \eta < 1$, the accuracy of the two expressions is comparable. It is within about ± 1.5% and within about -0.5 and +1% for the first and second expression, respectively. For negative/negative restraint combinations, none of the two expressions is particularly good. For positive/negative combinations, the first expression provides the better predictions, and within about -4 and +5% for the range $0 < \eta < 1.7$ ($1 > R(c = 4.8) > -0.5$). For details of selected Eurocode comparisons, see Table D.2 and D.3.

By rewriting the second expression, Eq. 29, in terms of G–factors ($b_o = 2$) by

$$\beta = \frac{3G_AG_B + 1.676(G_A + G_B) + 0.732}{3G_AG_B + 2.394(G_A + G_B) + 1.464} \tag{30}$$

it can be seen that it is similar, but not identical, to the corresponding French rules' Eq. 44 for braced members. The difference is in the coefficients in the two last terms in the numerator and denominator. The French rules' expression is found to be the more accurate of the two over a wider range of restraints. Cfr. Section 10.

6 NS 3473 (1989)

In the predecessors to the Norwegian standard for concrete structures NS 3473 of 1989 (NSF 1989), expressions for β–factors were not given. Such expressions, given in terms of fixity factors, were first included in NS 3473 of 1989 (for unbraced and braced compression members). The β –expression adopted for the unbraced case was proposed for inclusion in the standard by the author in 1988. It can be rewritten in the same form as Eq. 12. A value of c=2.5 was adopted. As seen before (cfr. Tables B.1a and B.1b), use of c=2.5 results in almost the same accuracy as use of c=2.4. For braced compression members, a β–factor given by Burheim (1968) was adopted. A review of the this factor is given separately in Section 9.

7 Mekonnen (1987)

Mekonnen (1987) developed an expression for **unbraced** members that may be rewritten in the exact same form as the French rules' expression for the effective length of unbraced compression members. It can be given by Eq. 38, by Eq. 27 in terms of G–factors, by Eq. 25 in terms of η–factors or by Eq. 26 in terms of R–factors. Mekonnen based his derivation on a first order displacement assumption, and incorporated certain empirical adjustments in order to arrive at the final expression.

8 NS 3472 (1973)

In the first edition of the Norwegian standard for steel structures, NS 3472-1973 (NSF 1973), an effective length factor for fully **braced** members was given by

$$\beta = 1 - s_A - s_B \tag{31}$$

where

$$s_i = 0.25 \, \frac{\sum(\frac{I}{L} \cdot \frac{P_{kd}-P}{P_{kd}})_i}{\frac{I}{L} + \sum(\frac{I}{L} \cdot \frac{P_{kd}-P}{P_{kd}})_i} \qquad i = A, B \tag{32}$$

The I and L outside the summations are for the considered member itself. The summations are over all members framing into the respective end of the member considered (i.e., they include all members at an end except for the considered member itself). The s_i–parameters are not to be entered with values > 0.2. The axial force P, positive in compression, is for tension members to be taken equal to 0. Tension members with forces in excess of 0.95 times their design yield capacity are not to be included in the summations. P_{kd} is the individual members design buckling load. The members framing into the considered end are assumed to be pinned at their far ends. (To account for other far end conditions than pinned ends, the restraint stiffness can be modified).

In order to generalize, Eq. 32 can be rewritten in a more familiar form given by

$$s_i = 0.25 \, \frac{1}{1 + \frac{3EI/L}{k_i}} = 0.25 \, R_i(c = 3) \tag{33}$$

where

$$k_i = \sum \left(\frac{bEI}{L} (1 - \frac{P}{P_{kd}}) \right)_{all,i} \qquad i = A, B \tag{34}$$

is the end restraint stiffness (and not just the stiffness of the restraining beams). This and the other symbols are all defined previously. The summation is over all members that branch into end i (A or B) of the one considered (the member itself is not to be included). The formulation corresponds to that given previously by Eq. 6. Previously, the b–coefficient was considered to include axial force effect. However, rather than include it in b, this effect is above explicitly reflected by the term $(1 - P/P_{kd})$.

Eq. 31 is identical to an expression given in NS 424A, of 1956, which was the predecessor to NS 3472-1973. However, it is significantly different, and significantly improved, compared to NS 424A with respect to the definition of the s_i–parameters (Eq. 32). In the next (1984) edition of NS 3473, the effective length factor formula was replaced by a set of diagrams.

In connection with the introduction of NS 3472-1973, Selberg (1972) provided some background for the given β–factor, but no information was given of the basis or derivation of the factor. However, it is obvious that it is similar to Eq. 17. A closer comparison reveals that they are identical as far as form is concerned.

Differences are due the limitations on s_i-values (0.2) to be entered into Eq. 31 and to the use of different c–values (c=2.4 (or 2.5) in Eq. 17 and c=3 above in Eq. 32). With c=3, predictions become more conservative than with c=2.4 for intermediate restraints, but will still be within about 0 and +7% for positive restraints as found with c=2.4 (cfr. Table B.6).

9 Burheim (1968)

In a study of several aspects of second order analyses, Burheim (1968) also presented expressions for an effective length factor and for inflection point locations (segment lengths) of fully **braced** compression members. They can be given by

$$\beta = \frac{7 - R_m}{7 + 5R_m} \tag{35}$$

$$L_i/L = 0.007R_i(1 + 5\beta) \qquad i = A, B \tag{36}$$

where $R_m = 0.5(R_A + R_B)$ is a mean fixity factor defined with and $\mathbf{c} = \mathbf{4}$. (Instead of R, Burheim used the symbol n). This β–factor was adopted in NS 3473-1989 (cfr. Section 6). Burheim did not give any background for these equations. It appears the β–expression was established by some curve fitting procedure for the symmetrical case, and then extended to the general case through use of the mean fixity factor. The segment lengths and β are related through $\beta L = L - L_A - L_B$.

For positive restraints, $1 \geq R \geq 0$, the accuracy of the β–predictions is generally somewhat unconservative, but still within about -2.5 and +0.5% of exact results. In the wider range of $1 \geq R(c = 4) > -0.65$, the accuracy is within about -1.5 and +5%. This is comparable to the Hellesland Eq. 13 in this range (it should be noted that range above corresponds to $1 \geq R(c = 4.8) > -0.5$). For details of selected comparisons, see Table E.1.

Burheim's Eq. 35 and Hellesland's Eq. 13 give comparable accuracy for cases involving negative restraints. Of the β–expressions considered in this study, they are on the overall the most accurate in this range. Also, they are both attractively simple. For positive restraints, Eq. 13 is more accurate, and to be preferred. An advantage of Eq. 13 is also that it can be derived from a rather simple model.

10 French rules (1966)

In the French design rules for steel structures ("Regles" 1966), effective length and inflection point location expressions were given for both braced and braced members according to Dumonteil (1992). The formulations below are taken from Dumonteil (1992). They are given in terms of relative restraint flexibilities that may be written

$$\rho_i = 3\,\frac{EI/L}{k_i} \qquad i = A, B \tag{37}$$

Unbraced member

$$\beta = \sqrt{\frac{3.2\rho_A\rho_B + 4(\rho_A + \rho_B) + 3.75}{\rho_A + \rho_B + 3.75}} \tag{38}$$

$$\text{If} \qquad \rho_B > \rho_A \; : \qquad \frac{L_A}{L} = \frac{1}{2}\sqrt{\frac{4\rho_B - 2\rho_A + 3.75}{\rho_A + \rho_B + 3.75}} \tag{39}$$

$$\text{If} \qquad \rho_A > \rho_B \; : \qquad \frac{L_B}{L} = \frac{1}{2}\sqrt{\frac{4\rho_A - 2\rho_B + 3.75}{\rho_A + \rho_B + 3.75}} \tag{40}$$

Regarding inflection point locations, L_A is measured from end A and L_B from end B. The β–factor above has been adopted by Eurocode 2 (CEN 1992) in a somewhat different form (cfr. Section 5). Mekonnen (1987) developed a β–expression that may be rewritten in the exact same form as that above (cfr. Section 7).

Braced member

$$\beta = \frac{\rho_A\rho_B + 0.7(\rho_A + \rho_B) + 0.48}{\rho_A\rho_B + \rho_A + \rho_B + 0.96} \tag{41}$$

$$\frac{L_A}{L} = \frac{0.3\rho_B + 0.12}{\rho_A\rho_B + 0.6\rho_A + \rho_B + 0.48} \qquad ; \qquad \frac{L_B}{L} = 1 - \beta - \frac{L_A}{L} \tag{42}$$

In terms of G–factors. The expressions above can be expressed in terms of G–factors (Eq. 3). With b_o=6 and $\rho = 0.5G$ for the unbraced case and with b_o=2 and $\rho = 1.5G$ for the braced case, Dumonteil (1992) rewrote the β–expressions in the forms below.

$$\text{Unbraced case :} \qquad \beta = \sqrt{\frac{1.6G_AG_B + 4(G_A + G_B) + 7.5}{G_A + G_B + 7.5}} \tag{43}$$

$$\text{Braced case} \quad : \qquad \beta = \frac{3G_AG_B + 1.4(G_A + G_B) + 0.64}{3G_AG_B + 2.0(G_A + G_B) + 1.28} \tag{44}$$

Eq. 38 (or Eq. 43, which is the same given before in Section 5 by Eq. 27) is given in terms of fixity factors by the rather simple Eq. 26. Also Eq. 41 (or Eq. 44) can be written in a rather simple form in terms of R–factors.

The accuracy of the unbraced member equation has been discussed before in Section 5 with reference to Fig. 1 and Table D.1.

For positive restraints, the braced member equation, Eq. 41 is accurate to within about -0.5 and +1.5% of exact results. This is about the same as for several of the other braced member equations, including the two Eurocode equations. For cases involving negative restraints, Eq. 41 is on the overall considerably more acurate than the Eurocode equations, but less accurate than the Hellesland and Burheim equations (Eq. 13 and Eq. 35). For details of selected comparisons, see Table F.1.

11 Newmark (1949)

Newmark (1949) gave a critical load coefficient C, denoted coefficient of end fixity, for fully **braced** members. This is the coefficient by which the Euler load of a pin-ended member was to be multiplied with in order to get the critical load of a restrained member. Expressed in terms of the effective length factor, $\beta = 1/\sqrt{C}$, Newmark's expressions become

$$\beta = \sqrt{\frac{\pi^2 + 2\overline{k}_A}{\pi^2 + 4\overline{k}_A} \cdot \frac{\pi^2 + 2\overline{k}_B}{\pi^2 + 4\overline{k}_B}} \approx \sqrt{\frac{1 + 0.2\overline{k}_A}{1 + 0.4\overline{k}_A} \cdot \frac{1 + 0.2\overline{k}_B}{1 + 0.4\overline{k}_B}} \tag{45}$$

The relative restraint stiffness $\overline{k} = k/(EI/L)$ was denoted n by Newmark. The two formulations above give almost identical results. The second formulation, obtained by substituting 10 for π^2 in the first formulation, is the one most commonly used.

Prediction errors are less than 2% according to Newmark. Probably he considered positive restraints only. The largest errors resulted for $\overline{k}_A + \overline{k}_B = 10 - 20$. Newmark also gave a simplification, giving errors less than 1.1% for $\overline{k}_A + \overline{k}_B < 1.0$, defined by

$$\beta = \sqrt{\frac{1}{1 + \frac{2}{\pi^2}(\overline{k}_A + \overline{k}_B)}} \approx \sqrt{\frac{1}{1 + 0.2(\overline{k}_A + \overline{k}_B)}} \tag{46}$$

In terms of G–factors ($b_o = 2$), Eq. 45 (second formulation) can be written

$$\beta = \sqrt{(\frac{G_A + 0.4}{G_A + 0.8})(\frac{G_B + 0.4}{G_B + 0.8})} \tag{47}$$

or

$$\beta = \sqrt{\frac{G_A G_B + 0.4(G_A + G_B) + 0.16}{G_A G_B + 0.8(G_A + G_B) + 0.64}} \tag{48}$$

A closer examination of this formula indicates it can be written exactly in the same form given by Hellesland in Eq. 15 in terms of R–factors with $c = 5$ (or by Eq. 18 in terms of R–factors with $c = 2.5$). Most likely, therefore, Newmarks formula has been derived based on the same type model leading to this equation. Regarding accuracy, reference is made to the previous discussion of Eq. 15 and Table B.4b.

12 Concluding remarks

An overview of approximate effective length formulas has been given for compression members with constant stiffness and constant axial force along the length, and that were either completely unbraced or fully braced against lateral translation of one end relative to the other. A more rigorous literature review than carried out here would most likely have revealed additional formulas than those included in the present review. Not least the German literature contains a large number of special cases (e.g., see Petersen (1982)).

Efforts were made to identify the applicability of the various approximate formulas, for a reasonable wide range of positive and negative restraint combinations, by comparisons with exact effective length results. Most of the formulas were found to provide reasonably good predictions (within about 2%) for positive restraints. Not surprisingly, since they were developed primarily for applications in that restraint range. A few also provide reasonable predictions for cases involving negative restraints. If only one formula was to be chosen before any other for each of the unbraced and braced case, it would be Eq. 12 for the unbraced and Eq. 13 for the braced case. On the overall, they are the two formulas that were found to give the better prediction accuracy. Also, they are between the most attractive ones in terms of simplicity of formulation.

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Appendix A –Exact effective length factors

TABLE A.1. Unbraced member – Exact effective length factors (β_{EXACT})

G_B	G_A ($\overline{k}_A$)											
	0 (∞)	0.25 (24)	1 (6)	4 (1.5)	8 (0.75)	$\pm\infty$ (± 0)	-20 (-0.3)	-15 (-0.4)	-10 (-0.6)	-8 (-0.75)	-7 (-0.86)	-6.5 (-0.92)
-1	0.851	0.891	1.000	1.253	1.395	1.677	1.890	1.988	2.257	2.574	2.923	3.235
-0.5	0.919	0.960	1.073	1.346	1.505	1.834	2.098	2.225	2.593	3.070	3.671	4.306
-0.25	0.958	1.000	1.114	1.395	1.562	1.917	2.209	2.353	2.781	3.371	4.187	5.177
0	1.000	1.042	1.157	1.445	1.620	2.000	2.323	2.485	2.984	3.719	4.868	6.590
0.25		1083	1.199	1.494	1.677	2.084	2.438	2.621	3.202	4.128	5.834	9.672
1			1.317	1.634	1.840	2.328	2.793	3.049	3.974	6.051	∞	
4				2.036	2.332	3.179	4.315	5.212	∞			
8					2.724	4.073	7.109	14.85				
∞						∞						

- Due to symmetry, results below the main diagonal are not shown.
- $\overline{k} = k/(EI/L) = 6/G$

TABLE A.2. Braced member – Exact effective length factors (β_{EXACT})

G_B	G_A ($\overline{k}_A$)											
	0 (∞)	0.25 (8)	1 (2)	4 (0.5)	$\pm\infty$ (± 0)	-4 (-0.5)	-2 (-1)	-1.5 (-1.33)	-1.2 (-1.67)	-1 (-2)	-0.8 (-2.5)	-0.6 (-3.33)
0	0.500	0.555	0.626	0.675	0.700	0.730	0.769	0.802	0.844	0.896	1.010	1.452
0.25		0.611	0.688	0.744	0.773	0.809	0.852	0.901	0.955	1.027	1.194	2.13
1			0.774	0.840	0.875	0.921	0.984	1.041	1.116	1.222	1.497	∞
4				0.916	0.956	1.011	1.087	1.158	1.257	1.398	1.824	
$\pm\infty$					1.000	1.060	1.145	1.226	1.338	1.509	2.059	
-4						1.127	1.226	1.321	1.459	1.677	2.503	
-2							1.348	1.469	1.654	1.974	3.796	
-1.5								1.624	1.872	2.352	∞	
-1.2									2.259	3.233		
-1										∞		

- Due to symmetry, results below the main diagonal are not shown.
- $\overline{k} = k/(EI/L) = 2/G$

Appendix B – Effective length factors by Hellesland

TABLE B.1a. Unbraced member – $\beta_{APPROX}/\beta_{\text{EXACT}}$-ratios,
for $\beta_{APPROX} = 2\sqrt{R_A + R_B - R_A R_B}/(R_A + R_B)$ with c=**2.4** (Eq.12).

G_B	G_A (R_A)									
	0 (1)	.25 (.909)	1 (.714)	4 (.385)	$\pm\infty$ ($\pm$ 0)	-20 (-.143)	-15 (-.2)	-10 (-.333)	-8 (-.455)	-7 (-.556)
-0.50	0.967	0.976	0.982	0.976	0.975	0.976	0.976	0.972	0.956	0.925
-0.25	0.988	0.995	0.999	0.991	0.990	0.993	0.993	0.991	0.974	0.931
0	1.000	1.005	1.008	1.000	1.000	1.004	1.006	1.005	0.986	0.924
0.25		1.011	1.014	1.005	1.007	1.014	1.016	1.017	0.993	0.899
1			1.019	1.011	1.017	1.028	1.034	1.040	0.973	1)
4				1.007	1.014	1.044	1.063	1)		
8					1.006	1.062	1.035			

- $R = 1/(1 + 0.4G)$
- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factor is infinite.

TABLE B.1b. Unbraced member – $\beta_{APPROX}/\beta_{\text{EXACT}}$-ratios,
for $\beta_{APPROX} = 2\sqrt{R_A + R_B - R_A R_B}/(R_A + R_B)$ with c=**2.5** (Eq.12).

G_B	G_A (R_A)									
	0 (1)	.25 (.906)	1 (.706)	4 (.375)	$\pm\infty$ ($\pm$ 0)	-20 (-.136)	-15 (-.190)	-10 (-.316)	-8 (-.429)	-7 (-.522)
-0.50	0.962	0.972	0.983	0.979	0.970	0.964	0.960	0.945	0.916	0.870
-0.25	0.986	0.995	1.002	0.996	0.987	0.983	0.980	0.965	0.932	0.872
0	1.000	1.007	1.013	1.007	1.000	0.997	0.994	0.980	0.941	0.859
0.25		1.015	1.021	1.014	1.008	1.008	1.005	0.991	0.945	0.826
1			1.028	1.023	1.023	1.026	1.026	1.010	0.908	1)
4				1.022	1.027	1.046	1.052	1)		
8					1.022	1.057	0.970			

- $R = 1/(1 + 0.417G)$
- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factor is infinite.

TABLE B.2a Braced member – $\beta_{APPROX}/\beta_{\mathrm{EXACT}}$-ratios,
for $\beta_{APPROX} = 2/(2+1.1R_{min}+0.9R_{max})$ with **c=4.8** (Eq.13).

G_B	G_A (R_A)										
	0 (1.0)	0.25 (.625)	1 (.294)	4 (.094)	$\pm\infty$ ($\pm$ 0)	-4 (-.116)	-2 (-.263)	-1.5 (-.385)	-1.2 (-.532)	-1 (-.714)	-0.8 (-1.09)
0	1.000	1.004	0.991	0.986	0.985	0.988	0.996	1.007	1.024	1.056	1.162
0.25		1.007	1.007	1.008	1.010	1.015	1.033	1.038	1.059	1.096	1.225
1			0.998	1.005	1.009	1.016	1.029	1.043	1.067	1.107	1.250
4				0.998	1.003	1.011	1.025	1.039	1.061	1.101	1.233
$\pm\infty$					1.000	1.008	1.021	1.034	1.056	1.091	1.208
-4						1.004	1.016	1.028	1.046	1.075	1.142
-2							1.007	1.016	1.026	1.036	0.928
-1.5								1.001	1.000	0.980	1)
-1.2									0.946	0.841	

- $R = 1/(1+2.4G)$
- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factor is infinite.

TABLE B.2b Braced member – $\beta_{APPROX}/\beta_{\mathrm{EXACT}}$-ratios,
for $\beta_{APPROX} = 2/(2+1.1R_{min}+0.9R_{max})$ with **c=5.0** (Eq.13).

G_B	G_A (R_A)										
	0 (1.0)	0.25 (.615)	1 (.286)	4 (.091)	$\pm\infty$ ($\pm$ 0)	-4 (-.111)	-2 (-.250)	-1.5 (-.364)	-1.2 (-.500)	-1 (-.667)	-0.8 (-1.0)
0	1.000	1.007	0.994	0.988	0.985	0.986	0.991	0.998	1.008	1.030	1.100
0.25		1.013	1.014	1.013	1.013	1.017	1.030	1.031	1.045	1.070	1.152
1			1.005	1.010	1.013	1.017	1.025	1.035	1.050	1.074	1.155
4				1.001	1.005	1.010	1.018	1.027	1.039	1.061	1.117
$\pm\infty$					1.000	1.005	1.013	1.020	1.031	1.046	1.079
-4						0.998	1.003	1.009	1.015	1.022	0.999
-2							0.989	0.990	0.987	0.973	0.781
-1.5								0.968	0.952	0.905	1)
-1.2									0.885	0.757	

- $R = 1/(1+2.5G)$
- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factor is infinite.

TABLE B.3 Braced member – $\beta_{APPROX}/\beta_{\mathrm{EXACT}}$-ratios, for $\beta_{APPROX} = 1/(1+R_m)$ with **c=4.8** (Eq.14).

G_B	G_A (R_A)										
	0 (1)	0.25 (.625)	1 (.294)	4 (.094)	$\pm\infty$ ($\pm$ 0)	-4 (-.116)	-2 (-.263)	-1.5 (-.385)	-1.2 (-.532)	-1 (-.714)	-0.8 (-1.09)
0	1.000	0.994	0.970	0.958	0.952	0.950	0.950	0.953	0.960	0.977	1.035
0.25		1.007	0.996	0.989	0.986	0.985	0.994	0.991	1.001	1.019	1.089
1			0.998	0.997	0.996	0.997	1.001	1.006	1.017	1.036	1.107
4				0.998	0.999	1.000	1.005	1.010	1.018	1.037	1.088
$\pm\infty$					1.000	1.002	1.006	1.010	1.018	1.031	1.064
-4						1.004	1.007	1.010	1.014	1.020	1.003
-2							1.007	1.007	1.004	0.991	0.811
-1.5								1.001	0.986	0.944	1)
-1.2									0.946	0.821	

- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factor is infinite.

TABLE B.4a Braced member – $\beta_{APPROX}/\beta_{\mathrm{EXACT}}$-ratios, for $\beta_{APPROX} = 1/\sqrt{(1+R_A)(1+R_B)}$ with **c=4.8** (Eq.15).

G_B	G_A (R_A)										
	0 (1)	0.25 (.625)	1 (.294)	4 (.094)	$\pm\infty$ ($\pm$ 0)	-4 (-.116)	-2 (-.263)	-1.5 (-.385)	-1.2 (-.532)	-1 (-.714)	-0.8 (-1.09)
0	1.000	0.999	0.993	1.001	1.010	1.030	1.071	1.124	1.225	1.476	im.)
0.25		1.007	1.002	1.008	1.015	1.031	1.073	1.110	1.201	1.429	im.)
1			0.998	1.000	1.005	1.015	1.041	1.076	1.151	1.346	im.)
4				0.998	1.000	1.006	1.024	1.052	1.112	1.279	im.)
$\pm\infty$					1.000	1.004	1.017	1.040	1.092	1.240	im.)
-4						1.004	1.011	1.027	1.066	1.187	im.)
-2							1.007	1.011	1.029	1.104	im.)
-1.5								1.001	0.995	1.014	1)
-1.2									0.946	0.846	

- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact factor is infinite. • im.) Approximate factor is imaginary (root of neg. number).
- Identical results are obtained with $\beta_{APPROX} = 0.5\sqrt{(2-R_A)(2-R_B)}$ for **c=2.4** (Eq.18)

TABLE B.4b Braced member – $\beta_{APPROX}/\beta_{\text{EXACT}}$-ratios,
for $\beta_{APPROX} = 1/\sqrt{(1+R_A)(1+R_B)}$ with **c=5.0** (Eq.15, or Eq.48).

G_B	G_A (R_A)										
	0 (1.0)	0.25 (.615)	1 (.286)	4 (.091)	$\pm\infty$ ($\pm$ 0)	-4 (-.111)	-2 (-.250)	-1.5 (-.364)	-1.2 (-.500)	-1 (-.667)	-0.8 (-1.0)
0	1.000	1.002	0.996	1.003	1.010	1.027	1.062	1.105	1.185	1.367	im.)
0.25		1.013	1.009	1.012	1.018	1.032	1.066	1.095	1.165	1.327	im.)
1			1.005	1.005	1.008	1.016	1.035	1.061	1.118	1.250	im.)
4				1.001	1.001	1.004	1.017	1.036	1.077	1.186	im.)
$\pm\infty$					1.000	1.001	1.008	1.022	1.057	1.148	im.)
-4						0.998	0.999	1.007	1.028	1.095	im.)
-2							0.989	0.985	0.987	1.013	im.)
-1.5								0.968	0.947	0.923	1)
-1.2									0.885	0.758	

- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact factor is infinite. • im.) Approximate factor is imaginary (root of neg. number).
- Identical results are obtained with $\beta_{APPROX} = 0.5\sqrt{(2-R_A)(2-R_B)}$ for **c=2.5** (Eq.18)

TABLE B.5 Braced member – $\beta_{APPROX}/\beta_{\text{EXACT}}$-ratios,
for $\beta = 1 - 0.2(R_{min} + 1.45R_{max})$ with **c=2.4** (Eq.16).

G_B	G_A (R_A)										
	0 (1.0)	0.25 (.769)	1 (.455)	4 (.172)	$\pm\infty$ ($\pm$ 0)	-4 (-.263)	-2 (-.714)	-1.5 (-1.25)	-1.2 (-2.27)	-1 (-5.00)	-0.8 (-25.0)
0	1.020	1.002	0.989	1.001	1.014	1.045	1.109	1.197	1.380	1.909	neg.)
0.25		1.020	0.997	0.998	1.005	1.025	1.080	1.140	1.290	1.730	neg.)
1			1.004	0.992	0.992	1.000	1.028	1.074	1.185	1.529	neg.)
4				1.000	0.994	0.992	1.005	1.036	1.117	1.395	neg.)
$\pm\infty$					1.000	0.993	0.998	1.020	1.087	1.325	neg.)
-4						1.002	0.994	1.004	1.049	1.238	neg.)
-2							1.002	0.992	1.005	1.118	neg.)
-1.5								0.993	0.971	1.005	1)
-1.2									0.936	0.823	

- Due to symmetry, results below the main diagonal are not shown. • $R = 1/(1+1.2G)$.
- 1) Exact effective length factor is infinite. • neg.) Approximate factor is negative.

TABLE B.6 Braced member – $\beta_{APPROX}/\beta_{\text{EXACT}}$-ratios,
for $\beta = 1 - 0.25(R_A + R_B)$ with **c=2.4** (Eq.17)

G_B	G_A (R_A)										
	0 (1.0)	0.25 (.769)	1 (.455)	4 (.172)	$\pm\infty$ ($\pm$ 0)	-4 (-.263)	-2 (-.714)	-1.5 (-1.25)	-1.2 (-2.27)	-1 (-5.00)	-0.8 (-25.0)
0	1.000	1.005	1.017	1.047	1.071	1.118	1.208	1.325	1.562	2.232	neg.)
0.25		1.007	1.009	1.028	1.045	1.080	1.158	1.283	1.441	2.004	neg.)
1			0.998	1.004	1.013	1.034	1.082	1.152	1.303	1.748	neg.)
4				0.998	1.001	1.012	1.045	1.096	1.213	1.579	neg.)
$\pm\infty$					1.000	1.005	1.029	1.071	1.172	1.491	neg.)
-4						1.004	1.015	1.043	1.200	1.381	neg.)
-2							1.007	1.015	1.056	1.230	neg.)
-1.5								1.001	1.005	1.089	1)
-1.2									0.946	0.872	

- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factor is infinite. • neg.) Approximate factor is negative.

Appendix C –Effective length factors by Duan et al.

TABLE C.1 Unbraced member – $\beta_{APPROX}/\beta_{\text{EXACT}}$-ratios,
for β_{APPROX} by Eqs. 19 and 20.

G_B	G_A									
	0	.25	1	4	$\pm\infty$	-20	-15	-10	-8	-7
-0.5	0.967	0.974	0.979	0.975	2)	1.004	1.008	1.007	0.992	0.963
-0.25	0.989	0.994	0.998	0.990	2)	1.007	1.008	1.002	0.980	0.942
0	1.000	1.005	1.008	1.000	1.000	1.007	1.007	0.996	0.968	0.915
0.25		1.012	1.015	1.005	1.002	1.007	1.005	0.990	0.954	0.881
1			1.020	1.010	1.005	1.001	0.996	0.969	0.901	1)
4				1.001	0.996	0.980	0.967	1)		
8					0.988	0.967	0.921			

- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factor is infinite.
- 2) Predictions not obtained (Eqs. 19 and 20 exclude each other).

TABLE C.2 Braced member – $\beta_{APPROX}/\beta_{\mathrm{EXACT}}$-ratios, for β_{APPROX} by Eq.22.

G_B	G_A 0	0.25	1	4	$\pm\infty$	−4	−2	−1.5	−1.2	−1	−0.8
0	1.000	1.013	1.004	1.001	1.000	1.003	1.010	1.020	1.034	1.060	1.143
0.25		1.023	1.007	1.004	1.115	0.968	0.979	0.972	0.975	0.983	1.017
1			0.990	0.991	1.061	0.862	0.895	0.892	0.885	0.874	0.851
4				0.997	1.021	1.162	0.508	0.728	0.760	0.758	0.704
$\pm\infty$					1.000	0.974	0.941	0.912	0.876	0.828	0.706
−4						0.910	0.859	0.823	0.779	0.722	0.564
−2							0.803	0.761	0.707	0.630	0.381
−1.5								0.710	0.644	0.545	1)
−1.2									0.557	0.412	

- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factor is infinite.

Appendix D – Effective length factors by Eurocode 3

TABLE D.1 Unbraced member – $\beta_{APPROX}/\beta_{\mathrm{EXACT}}$-ratios,
for β_{APPROX} by Eq. 25 (or Eqs. 26, 27, 38).

G_B	G_A (η_A) 0 (0)	0.25 (.143)	1 (.400)	4 (.727)	$\pm\infty$ (1.0)	−20 (1.081)	−15 (1.111)	−10 (1.176)	−8 (1.231)	−7 (1.273)
−0.5	0.965	0.971	0.972	0.958	0.975	1.011	1.035	1.146	1.460	im.)
−0.25	0.988	0.993	0.993	0.977	0.990	1.026	1.052	1.178	1.618	im.)
0	1.000	1.005	1.005	0.989	1.000	1.037	1.065	1.208	1.882	im.)
0.25		1.011	1.013	0.998	1.007	1.045	1.074	1.241	2.503	im.)
1			1.019	1.008	1.017	1.058	1.095	1.371	im.)	1)
4				1.007	1.014	1.080	1.181	1)		
8					1.006	1.142	im.)			

- $\eta = 1/(1 + (1.5/G))$
- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact factor is infinite. • im.) Approximate factor is imaginary (root of neg. number).

TABLE D.2 Braced member – $\beta_{APPROX}/\beta_{\mathrm{EXACT}}$-ratios, for β_{APPROX} by Eq.28.

G_B	G_A (η_A)										
	0 (0)	0.25 (.333)	1 (.667)	4 (.889)	$\pm\infty$ (1.0)	-4 (1.143)	-2 (1.333)	-1.5 (1.5)	-1.2 (1.714)	-1 (2.0)	-0.8 (2.667)
0	1.000	0.996	0.987	0.990	0.993	1.003	1.020	1.040	1.068	1.116	1.252
0.25		1.011	1.010	1.013	1.015	1.022	1.040	1.045	1.065	1.097	1.185
1			1.014	1.013	1.013	1.014	1.016	1.020	1.026	1.035	1.054
4				1.007	1.005	1.001	0.996	0.992	0.984	0.975	0.928
$\pm\infty$					1.000	0.993	0.984	0.974	0.961	0.938	0.851
-4						0.983	0.966	0.949	0.925	0.884	0.732
-2							0.938	0.911	0.869	0.799	0.511
-1.5								0.871	0.811	0.707	1)
-1.2									0.720	0.550	

- $\eta = 1/(1 + (0.5/G))$
- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factor is infinite.

TABLE D.3 Braced member – $\beta_{APPROX}/\beta_{\mathrm{EXACT}}$-ratios, for β_{APPROX} by Eq.29.

G_B	G_A (η_A)										
	0 (0)	0.25 (.333)	1 (.667)	4 (.889)	$\pm\infty$ (1.0)	-4 (1.143)	-2 (1.333)	-1.5 (1.5)	-1.2 (1.714)	-1 (2.0)	-0.8 (2.667)
0	1.000	1.005	0.997	0.998	1.000	1.008	1.020	1.044	1.075	1.132	1.334
0.25		1.010	0.998	0.997	0.998	1.005	1.028	1.040	1.077	1.147	1.459
1			0.989	0.988	0.991	0.999	1.018	1.045	1.098	1.208	1.946
4				0.989	0.994	1.005	1.032	1.070	1.146	1.331	4.04
$\pm\infty$					1.000	1.004	1.047	1.094	1.193	1.449	60
-4						1.034	1.079	1.144	1.289	1.739	neg.)
-2							1.151	1.263	1.550	3.07	neg.)
-1.5								1.466	2.15	neg.)	1)
-1.2									12.2	neg.)	

- $\eta = 1/(1 + (0.5/G))$
- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact factor is infinite. • neg.) Approximate factor is negative.

Appendix E – Effective length factors by Burheim

TABLE E.1 Braced member – $\beta_{APPROX}/\beta_{\text{EXACT}}$-ratios,
for $\beta = (7 - R_m)/(7 + 5R_m)$ with **c=4** (Eq.35).

G_B	G_A (R_A)										
	0 (1.0)	0.25 (.667)	1 (.333)	4 (.111)	$\pm\infty$ ($\pm$ 0)	-4 (-.143)	-2 (-.333)	-1.5 (-.5)	-1.2 (-.714)	-1 (-1.0)	-0.8 (-1.667)
0	1.000	0.995	0.979	0.976	0.977	0.985	1.000	1.020	1.053	1.116	1.361
0.25		1.003	0.994	0.993	0.995	1.002	1.024	1.035	1.069	1.132	1.396
1			0.993	0.995	0.997	1.003	1.016	1.034	1.065	1.125	1.397
4				0.995	0.998	1.003	1.015	1.031	1.058	1.115	1.371
$\pm\infty$					1.000	1.004	1.015	1.028	1.055	1.104	1.343
-4						1.008	1.016	1.028	1.048	1.090	1.275
-2							1.020	1.027	1.038	1.059	1.054
-1.5								1.026	1.025	1.014	1)
-1.2									0.996	0.895	

- $R = 1/(1 + 2G)$
- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factor is infinite.

Appendix F –Effective length factors by French rules

TABLE F.1 Braced member – $\beta_{APPROX}/\beta_{\text{EXACT}}$-ratios, for β_{APPROX} by Eq.41 (or Eq.44).

G_B	G_A										
	0	0.25	1	4	$\pm\infty$	-4	-2	-1.5	-1.2	-1	-0.8
0	1.000	1.002	0.994	0.996	1.000	1.011	1.033	1.058	1.100	1.178	1.485
0.25		1.013	1.008	1.009	1.011	1.020	1.044	1.058	1.095	1.164	1.256
1			1.005	1.004	1.006	1.010	1.022	1.038	1.067	1.123	1.373
4				1.001	1.001	1.003	1.010	1.021	1.041	1.085	1.280
$\pm\infty$					1.000	1.000	1.004	1.011	1.028	1.060	1.214
-4						0.998	0.998	1.000	1.007	1.025	1.099
-2							0.989	0.984	0.977	0.966	0.835
-1.5								0.968	0.943	0.894	1)
-1.2									0.885	0.748	

- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factor is infinite.