PRICING INTEREST RATE GUARANTEES IN A DEFINED BENEFIT PENSION SYSTEM

PÅL NICOLAI HENRIKSEN, ARNE HOVE, THILO MEYER-BRANDIS, FRANK PROSKE

ABSTRACT. In Norwegian defined benefit pensions, assets corresponding to the premium reserve and premium fund are guaranteed a minimum return of a fixed rate r. This r is the same interest rate used for discounting when calculating the premium reserve. The guarantee is issued by the insurance company to each client. In this paper we aim at pricing an interest rate guarantee which is given by a put option with a stochastic strike depending on events in the membership data. We want to consider a complete and an incomplete asset market model with respect to this put option with an underlying given by the client assets and buffer funds. A risk indifferent pricing principle will be applied in the incomplete case, and results from this will be compared with Black and Scholes prices in the complete case.

1. INTRODUCTION AND BACKGROUND

In Norwegian defined benefit pensions, assets corresponding to the premium reserve and premium fund are guaranteed a minimum return of a fixed rate r. This r is the same interest rate used for discounting when calculating the premium reserve. The guarantee is issued by the insurance company to each client. If the minimum return is not met, the insurance company can use buffer funds to fill in the gap. However, if the buffer funds are empty and the minimum return is still not met, the insurance company has to use its' own equity to fill the gap. Thus, since the company equity is at risk there is a clear risk that needs to be managed. Thus, the insurance company invoices each client to cover this. In this paper we will introduce methods to compute the price of this risk, and we will call it the price of the interest rate guarantee.

Up until 2008 when the new legislation came to effect, see the law at lovdata.no [16], the price of the interest rate guarantee was implicitly included in the total premium issued to each client. The new legislation forces insurance companies to price each risk element they face separately. This is a great challenge in many aspects. For instance; is the price for each risk element derived by the insurance company consistent with market prices, is it possible to split the total risk into separate linearly added risk elements, regarding the interest rate guarantee; is it possible to introduce a mathematical model for the guarantee

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when there are so many different agendas to how the assets should be allocated in the financial market. We will take these challenges into account.

This paper is split into two sections. The first section applies a complete asset model to price the interest rate guarantee. The ease of the complete model makes it possible to adapt the mathematics to the details in Norwegian legislation. Hence, this section resembles real life and the methods should be directly applicable to practitioners. The second section applies an incomplete asset model given by a two dimensional exponential Lévy process. This asset model is known to be more realistic, see e.g. Cont and Tankov [5], and thus the corresponding option price is expected to be more accurate. In this paper we will derive this option price based on a risk indifferent principle, see e.g. An, Øksendal and Proske [1].

Many papers have been written on interest rate guarantees, see. e.g. Schrager and Pelsser [14] with focus on guarantees in unit link insurance under stochastic interest rates, Miltersen and Person [12] with focus on maturity guarantees and multi-period guarantees or Benth and Proske [4] with utility indifference pricing of interest rate guarantees in a asset liability framework. The main contributions of this paper is the proposed solution of the real world problem in Section 2 and the application of the risk indifference pricing principle in Section 3.

2. COMPLETE ASSET MODEL

2.1. **The Norwegian defined benefit system and the guaranteed interest rate.** Defined benefit pensions in Norway gives a guarantee of a certain percent of salary in old age pension at retirement. In addition, one has annuity benefits at possible disablement and possibly also at death for spouse and orphans. To cover these liabilities, actuaries calculate premium reserves, V_t, based on Thiele differential equations. The Thiele equations use a constant discount rate r. Employers and employees pay premiums to insurance companies corresponding to the increase in premium reserves as the liabilities grow. If the payments are greater than what the actuary has calculated or if the insurance policies attain a part of the financial or actuarial result of the insurance company, this may be accumulated in something called a premium fund PF_t. This premium fund may e.g. be used to cover future premiums. Also, parts of the financial result may be allocated to something called an additional reserve AR_t. These additional reserves may be used in situations where the financial return is less than a given minimum. This minimum is called the guaranteed interest rate. The guaranteed interest rate says that V_t and PF_t should be given an interest rate of r. There are a lot of other funds in the Norwegian defined benefit system, but details on these are less relevant to the price of the interest rate guarantee. All of these funds and the premium reserve, the premium fund and additional reserve have corresponding assets. In the new legislation, the client's assets and company equity has to be split into two different portfolios. The strategy of the first portfolio may be determined by each client, and the strategy of the second portfolio may be determined by the owners or the board of the insurance company. Since the insurance company puts it's own equity at stake with the interest rate guarantee, this motivates us to split the assets in our model into two categories; the client assets, S_t^b, and the buffer assets, S_t^b.

The price of the interest rate guarantee introduced in this paper has been derived with special attention given to the following issues: 1) The interest rate guarantee will cover all interest on premium fund, PF_t, and premium reserves, V_t, - also interest on premium reserves due to insurance cases during the guarantee period. This means that the interest rate guarantee will be part of the financial result[1], and not part of the insurance result even though interest is covered for insurance cases. 2) The derived price of the interest rate guarantee will be a Black and Scholes type of price with corresponding perfect hedging. We can not expect that this asset allocation strategy will be perfectly applied in practice, but it is a natural assumption for the mathematical model. 3) Should the level of the model be a description of each policy, each client or the whole insurance company in one go? The last alternative has a great drawback as the price of the interest rate guarantee is invoiced each client, and it would be hard to split one price for the whole insurance company into each client when taking the client's buffer funds, different interest rates, number of members, etc. into account. Prices on an individual level, has the advantage that insurance risk may easily be taken into account through indicator functions of which state each member is in. This is a somehow greater challenge on a client level, as we can not track each policy of the client. However, buffer funds are easily taken into account on a client level. This is important, as this will influence the price of the interest rate guarantee significantly. Also important is the fact that we have to apply simulations to be able to find prices. This is more feasible on a client level as this will reduce calculation time. The decision in this paper has therefore been to price the product on a client level.

2.2. **Model specification.** There are several issues that needs to be taken into account when pricing the interest rate guarantee.

First, we need to determine an expression for what the company actually guarantees, i.e. an expression for the minimum that the return on assets should cover. As mentioned in Section 1 the guarantee covers a fixed interest on the premium reserve (V_t) and the premium fund (PF_t) over the guarantee period $t \in [0, T]$. Strictly speaking, the guarantee (K) can be expressed as

$$(2.1) \qquad K = (V_0 + PF_0)((1+r)^T - 1) + \int_0^T ((1+r)^{T-t} - 1) \, d(V_t + PF_t),$$

where V_t and PF_t are stochastic processes. From Equation (2.1) we see that the interest rate guarantee only will cover the accumulated interest of changes in the liability and not the changes in liability in itself. This may be interpreted as covering the effects on the financial result of the insurance company and leaving out effects on actuarial results and administrational results.

V_t and PF_t could be modelled as Lévy processes, e.g. by diffusion or jump diffusion processes. However, these would be very hard to estimate as the premium reserve is a complex quantity incorporating probabilities for death and disability, probability of being

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[1]In Norwegian accounting the result of an insurance company has to be split into a financial result, an actuarial result and an administrational result

married and expected number of children for all possible ages for men and women. These probabilities may in general be dynamic or even stochastic. The most feasible way of modelling (2.1) is probably through a simulation of the whole membership data. This way one can track each policy in the portfolio and keep account of disabilities, recoveries and deaths with corresponding spouses and orphans for each simulation. This simulation can also be exact in the sense that one does not have to use discrete time point to estimate the integral in (2.1). By using the conditional actuarial probabilities of e.g. death, one can simulate the time of death. Doing the same for disability and assuming independence with death one can use the first occurrence as the next event, and the time of the occurrence as the discrete time point to evaluate the integral.

Second, how should one model the assets. We will split the assets into two categories; the client assets S_t^c and the buffer funds S_t^b. The clients assets, S_t^c, cover the assets corresponding to the premium reserve, the additional reserve and the premium fund. The buffer assets, S_t^b, cover company equity and all buffer funds excluding the additional reserve. This split in assets is motivated by the new insurance law in Norway which gives the clients and the owners of the insurance company the opportunity to decide on their own investment strategies. We will use a two dimensional geometric Brownian motion with constant parameters for modelling the assets in the period $(0, T)$

$$(2.2) \qquad\qquad dS_t^c = \mu^c S_t^c \, dt + \sigma^c S_t^c \, dW_t^c,$$

$$(2.3) \qquad\qquad dS_t^b = \mu^b S_t^b \, dt + \sigma^b S_t^b \, dW_t^b,$$

where W_t^c and W_t^b are correlated Brownian motions with correlation factor τ. $S_{0-}^c = (V_0 + PF_0 + AR_0)$ and S_0^b may easily be found from the beginning of year balance sheet. For the client capital we will assume that a billed premium for the whole period $E[\pi(0, T)]$ is paid at time $t = 0$, i.e. $S_0^c = S_{0-}^c + E[\pi(0, T)]$ will be used as initial value for (2.2). Further, we will assume that benefit payments $\beta(0, T)$ are withdrawn from the assets at time T. This is of course not the case in practice. However, in practice the benefits are probably not paid from the risky assets, but rather from a bank account. Thus, we assume that the pension fund can borrow money from a bank account at a risk free rate, ρ, (alternatively with an additional risk premium) to make benefit payments during the year, and then repay this from the assets at the end of the year. I.e. $S_T^c = S_{T-}^c - \beta(0, T)(1+\rho)^{T/2}$, where S_{T-}^c is given by the initial assets S_0^c and the dynamics (2.2) propagated up to time T. Notice that $\pi(0, T)$ is a stochastic variable in the sense that we do not initially know the true premium during the time period $(0, T)$. The true premium consists of both savings and premiums at insurance events. We know the billed premium $E[\pi(0, T)]$, but the true premium $\pi(0, T)$ is affected by deaths and disabilities which are stochastic.

The parameters in the asset model may be estimated from historical data or determined based on the overall investments strategy of the two groups.

2.3. **The put option.** Having determined the strike of the option K through (2.1) and the dynamics of the assets through (2.2) and (2.3), it is straight forward to define the put option. Initially, the interest rate guarantee makes sure that the strike, K, is covered by

the return on the client assets given by

$$(2.4) \qquad S_{T-}^c - S_0^c - \beta(0,T)((1+\rho)^{T/2} - 1),$$

where S_{T-}^c is given by the initial assets S_0^c and the geometric Brownian motion (2.2), and the last term is included because we pay risk free interest on benefit payments for half a year. If the return (2.4) is inadequate to cover the guarantee, the insurance company may use the additional reserve and buffer funds before covering the gap. This means that the strike is reduced and takes the form

$$K - AR_0 - \alpha S_T^b,$$

where $\alpha \in [0,1]$ is a control parameter of how much of the buffer capital one wish to allocate to the interest rate guarantee objective. Thus, we end up with an option with payoff

$$((K - AR_0 - \alpha S_T^b) - (S_{T-}^c - S_0^c - \beta(0,T)((1+\rho)^{T/2} - 1)))^+$$
$$= (K^* - (S_{T-}^c + \alpha S_T^b))^+.$$

This is recognized as a basket option with stochastic strike

$$K^* = K + \beta(0,T)((1+\rho)^{\frac{T}{2}} - 1) + S_0^c - AR_0.$$

Looking at Equations (2.2) and (2.3), we immediately see that the number of Brownian motions is the same as the number of assets. Thus, the asset model is a complete market and we therefore choose to find the price Π_t^T of the option at time t with maturity at time T as a discounted conditional expectation under an equivalent martingale measure Q

$$(2.5) \qquad \Pi_t^T = e^{-\rho(T-t)} E_Q[(K^* - (S_T^c + \alpha S_T^b))^+ | \mathcal{F}_t].$$

Here $\mathcal{F}_t$ is the σ-algebra containing market information up to time t.

Contained in the option price (2.5) we have five stochastic variables; K, π, β, S_T^c, S_T^b. Clearly K and π and β are closely connected, and these should be based on the same simulations. These variables are not tradable. However, the actuarial tables applied in practise usually take into account a insurance market situation. Hence, K, π and β can be regarded as implicitly being under a pricing measure. The two latter variables are tradable, and therefore we need to find a dynamics under which these variables are martingales after discounting with the risk free interest rate ρ. The obvious choice with this complete asset model is the Black and Scholes dynamics given by

$$(2.6) \qquad dS_t^c = \rho S_t^c \, dt + \sigma^c S_t^c \, dB_t^c,$$
$$(2.7) \qquad dS_t^b = \rho S_t^b \, dt + \sigma^b S_t^b \, dB_t^b,$$

where $B_t = [B_t^c, B_t^b]$ is a two dimensional correlated Brownian motion with correlation coefficient τ under the risk free measure Q. Notice that we in total are in an incomplete market situation because of the stochastic strike, premiums and benefit payments. I.e. it is not possible to derive a replicating portfolio based on S_t^c, S_t^b and a risk free bank account to 100% hedge the option payoff.

2.4. **Some results regarding the option value.** There are several approaches of valuing the option (2.5). One possibility is by approximating $S_T^c + S_T^b$ by a new lognormal variable S_T, see e.g. Henriksen [9]. This leads to an European put option with normally distributed strike. A special case of one underlying asset is if the client assets and the buffer assets are managed through the same portfolio. In practice, especially for small pension funds, this is often the case. If we further assume that the stochastic strike may be modelled as a normally distributed variable $K^* \sim N(\mu_K T, \sigma_K \sqrt{T})$, the underlying asset, S_T, and the strike, K^*, may be represented as

$$(2.8) \qquad S_T = S_0 \exp((\rho - \frac{1}{2}\sigma_S^2)T + \sigma_s \sqrt{T}X),$$

$$(2.9) \qquad K^* = \mu_K T + \sigma_K \sqrt{T}Y,$$

under the risk neutral pricing measure Q where X and Y are independent standard normal variables.

Proposition 2.1. *The price of an European put option at time $t = 0$ with normally distributed strike, K^*, given by (2.9) and maturity T is given by*

$$\Pi_0^T = e^{-\rho T} E_Q[(K - S_T)^+]$$

$$= e^{-\rho T}\mu_K T \Phi(X \le d(Y), Y \ge -a) + e^{-\rho T}\frac{\sigma_K \sqrt{T}}{\sqrt{2\pi}} \int_{-a}^{\infty} Y\Phi(d(Y))e^{-\frac{1}{2}Y^2} \, dY$$

$$(2.10) \qquad - S_0 \Phi(X \le d(Y) + \sigma_S \sqrt{T}, Y \ge -a)$$

where

$$d(Y) = \frac{1}{\sigma_S \sqrt{T}}\Big(\ln\big(\frac{\mu_K T + \sigma_K \sqrt{T}Y}{S_0}\big) - (\rho - \frac{1}{2}\sigma_S^2)T\Big), \qquad a = \frac{\mu_K}{\sigma_K}\sqrt{T}.$$

X and Y are independent standard normal variables and Φ is the cumulative bivariate normal distribution.

Proof. The proof is by direct calculation of the expectation

$$(2.11) \qquad \Pi_0^T = \frac{e^{-\rho T}}{2\pi} \int_{-a}^{\infty} \int_{d(Y)}^{\infty} (\mu_K T + \sigma_K \sqrt{T}Y - S_0 e^{(\rho - \frac{1}{2}\sigma_S^2)T + \sigma_s \sqrt{T}X})e^{-\frac{1}{2}X^2 - \frac{1}{2}Y^2} \, dX \, dY$$

$$\square$$

There are no closed form expressions for the integral term or the cumulative distribution $\Phi(X, Y)$ in (2.10). Hence, we must resort to numerical methods, and Gauss-Laguerre quadratures are ideal for these types of estimations, see e.g. Davis and Rabinowitz [6]. We will not go into further details on this matter here.

Because the strike, K, is independent of the assets, both in the case of one underlying asset and two underlying assets, we can use Proposition 2.1 to analyse changes in option

price with changes in μ_K and σ_K. Taking derivatives of Equation (2.10) with respect to the parameters of K^* we find

$$\frac{\partial}{\partial \mu_K} \Pi_0^T = e^{-\rho T} T \Phi(X \leq d(Y), Y \geq -a),$$

$$\frac{\partial}{\partial \sigma_K} \Pi_0^T = \frac{e^{-\rho T}}{\sqrt{2\pi}} \sqrt{T} \int\limits_{-a}^{\infty} Y \Phi(d(Y)) e^{-\frac{1}{2}Y^2} \, dY.$$

The first derivative is clearly positive, and since $d(Y)$ is an increasing function in Y, the second derivative is positive too. Hence the price of the option increases with increasing μ_K and σ_K. For μ_K this is natural because an increase in expected strike will increase the price of the put option. For σ_K this means that the number of policies covered by the client will play a part in the option price because a high number of policies decrease the uncertainty σ_K. For big clients the price of the interest rate guarantee will be smaller for each policy compared to small clients. Notice also that the derivatives are proportional to one term each in the option price (2.10). Hence, changing μ_K or σ_K will only effect these respective terms of the option price.

In general, we do not have one underlying asset, we have two. The option (2.5) may then be valued by Monte Carlo simulations or by numerical solution of a Feynman-Kac PDE. We will evaluate the option price directly through the expectation analogously to Equation (2.10) for the two underlying assets case.

Proposition 2.2. *Assume that the strike, K, is a given constant and that the two underlying assets, S_t^c and S_t^b, are given by (2.6) and (2.7) under a risk neutral measure Q. Then the unique price of the option (2.5) at time $t = 0$ with maturity T can be expressed as:*

$$
\begin{aligned}
\Pi_0^T = & Ke^{-\rho T} \Phi(Y_1 \leq a, Y_2 \leq d(Y_1)) \\
& - S_0^c \Phi(Y_1 \leq a - \sigma_c \sqrt{T}, Y_2 \leq d(Y_1 + \sigma_c \sqrt{T})) \\
& - \alpha S_0^b \Phi(Y_1 \leq a - \sigma_b \tau \sqrt{T}, Y_2 \leq d(Y_1 + \sigma_b \tau \sqrt{T}) - \sigma_b \sqrt{1-\tau^2}\sqrt{T})
\end{aligned}
$$
(2.12)

where

$$
\begin{aligned}
d(Y_1) = & \frac{1}{\sigma_b \sqrt{1-\tau^2}\sqrt{T}} \Big(\ln \big(K - S_0^c e^{\sigma_c \sqrt{T} Y_1 + T(\rho - \frac{1}{2}\sigma_c^2)} \big) \\
& - \ln(\alpha S_0^b) - T(\rho - \frac{1}{2}\sigma_b^2) - \sigma_b \tau \sqrt{T} Y_1 \Big)
\end{aligned}
$$
(2.13)

and

$$a = \frac{1}{\sigma_c \sqrt{T}} \Big(\ln \big(\frac{K}{S_0^c} \big) - (\rho - \frac{1}{2}\sigma_c^2)T \Big).$$

Y_1 and Y_2 are independent standard normal variables and Φ is the cumulative bivariate normal distribution.

Proof. The proof follows by direct calculation based on the fact that S_t^c and S_t^b may be expressed as:

$$S_T^b = S_0^b \exp(\sigma_b \tau \sqrt{T} y_1 + \sigma_b \sqrt{1 - \tau^2} \sqrt{T} y_2 + T(\rho - \frac{1}{2}\sigma_b^2))$$

$$S_t^c = S_0^c \exp(\sigma_c \sqrt{T} y_1 + T(\rho - \frac{1}{2}\sigma_c^2))$$

under Q, where y_1 and y_2 are independent standard normally distributed variables. With a constant strike, K, we are in a complete market setting and the solution is unique. $\square$

The practical idea behind Proposition 2.2 is that we now have a relatively simple expression for the option price (2.5) given the strike K. Hence, we can simulate the strike (2.1), and find option prices based on the closed form expression and the simulated K.

New legislation in Norway suggests that the owners of an insurance company can determine the investment strategy of the buffer capital, while the clients can decide on the strategy of the client capital. This means that ρ is determined by the bank account, μ_c and σ_c by the client and μ_b and σ_b by the owners. The only free parameter to be decided by risk managers in the insurance company is the correlation τ (in addition to the price of the interest rate guarantee). This leaves us with the following lemma.

Lemma 2.3. *Assume fixed ρ, σ_c and σ_b in Proposition 2.2. Then the option price increases with increasing correlation, τ.*

Proof. We have that our option price (2.12) may be expressed as:

$$\frac{1}{2\pi} \int\limits_{-\infty}^{a} \int\limits_{-\infty}^{d(y_1)} \left(K - S_0^c e^{\sigma_c \tau \sqrt{T} y_1 + \sigma_c \sqrt{1-\tau^2}\sqrt{T} y_2 + T(\rho - \frac{1}{2}\sigma_c^2)} - S_0^b e^{\sigma_b \sqrt{T} y_1 + T(\rho - \frac{1}{2}\sigma_b^2)} \right) e^{-\frac{1}{2}y_1^2 - \frac{1}{2}y_2^2} \, dy_2 \, dy_1$$

where $d(y_1)$ is given by (2.13). To prove the lemma we differentiate this w.r.t. τ using the rule:

$$\frac{\partial}{\partial x} \int\limits_{-\infty}^{g(x)} f(x,y) \, dy = f(x, g(x)) \frac{\partial}{\partial x} g(x) + \int\limits_{-\infty}^{g(x)} \frac{\partial}{\partial x} f(x,y) \, dy.$$

Thus, we end up with

$$\int\limits_{-\infty}^{a} \left(K - S_0^c e^{\sigma_c \tau \sqrt{T} y_1 + \sigma_c \sqrt{1-\tau^2}\sqrt{T} d(y_1) + T(\rho - \frac{1}{2}\sigma_c^2)} - S_0^b e^{\sigma_b \sqrt{T} y_1 + T(\rho - \frac{1}{2}\sigma_b^2)} \right) e^{-\frac{1}{2}d(y_1)^2 - \frac{1}{2}y_2^2} \frac{\partial}{\partial \tau} d(y_1) \, dy_1$$

$$- \int\limits_{-\infty}^{a} \int\limits_{-\infty}^{d(y_1)} S_0^c \left(\sigma_c \sqrt{T} y_1 - \frac{\sigma_c \sqrt{T} \tau y_2}{\sqrt{1-\tau^2}} \right) e^{\sigma_c \tau \sqrt{T} y_1 + \sigma_c \sqrt{1-\tau^2}\sqrt{T} y_2 + T(\rho - \frac{1}{2}\sigma_c^2)} e^{-\frac{1}{2}y_1^2 - \frac{1}{2}y_2^2} \, dy_2 dy_1$$

when constants have been omitted. The first integral term equals zero by construction of $d(y_1)$. Hence, by straight forward substitution and cancellation of constants in the second

integral term, the differential reduces to:

$$\int_{-\infty}^{a-\sigma_c\tau\sqrt{T}} \int_{-\infty}^{d(y_1+\sigma_c\tau\sqrt{T})-\sigma_c\sqrt{1-\tau^2}\sqrt{T}} \left(\frac{\tau}{\sqrt{1-\tau^2}}y_2 - y_1\right)e^{-\frac{1}{2}y_1^2-\frac{1}{2}y_2^2}\, dy_2\, dy_1$$

Hence, we need to prove that this expression is positive for all $\tau \in [-1, 1]$. Define

$$\widetilde{a} = a - \sigma_c\tau\sqrt{T}$$
$$\widetilde{d}(y_1) = d(y_1 + \sigma_c\tau\sqrt{T}) - \sigma_c\sqrt{1-\tau^2}\sqrt{T}.$$

Then the integral may be written as

$$\int_{-\infty}^{\widetilde{a}} -e^{-\frac{1}{2}y_1^2}\frac{\tau}{\sqrt{1-\tau^2}}e^{-\frac{1}{2}(\widetilde{d}(y_1))^2}dy_1 + \int_{-\infty}^{\widetilde{a}} -y_1 e^{-\frac{1}{2}y_1^2}\sqrt{2\pi}\Phi(\widetilde{d}(y_1))dy_1.$$

Let us first treat the case $-1 < \tau \leq 0$. Assume that $\widetilde{d}$ attains the maximum at the point $a^* \in (-\infty, \widetilde{a})$. Then

$$\int_{-\infty}^{\widetilde{a}} \int_{-\infty}^{\widetilde{d}(y_1)} \left(\frac{\tau}{\sqrt{1-\tau^2}}y_2 - y_1\right) e^{-\frac{1}{2}y_1^2}e^{-\frac{1}{2}y_2^2}dy_2 dy_1 = A_1 + A_2 + B,$$

where

$$A_1 = \int_{-\infty}^{a^*} \int_{-\infty}^{\widetilde{d}(y_1)} \frac{\tau}{\sqrt{1-\tau^2}}y_2 e^{-\frac{1}{2}y_1^2}e^{-\frac{1}{2}y_2^2}dy_2 dy_1$$

$$A_2 = \int_{-\infty}^{a^*} \int_{-\infty}^{\widetilde{d}(y_1)} -y_1 e^{-\frac{1}{2}y_1^2}e^{-\frac{1}{2}y_2^2}dy_2 dy_1$$

and

$$B = \int_{a^*}^{\widetilde{a}} \int_{-\infty}^{\widetilde{d}(y_1)} \left(\frac{\tau}{\sqrt{1-\tau^2}}y_2 - y_1\right) e^{-\frac{1}{2}y_1^2}e^{-\frac{1}{2}y_2^2}dy_2 dy_1.$$

We have that

$$A_1 = \int_{-\infty}^{a^*} e^{-\frac{1}{2}y_1^2}\frac{-\tau}{\sqrt{1-\tau^2}}e^{-\frac{1}{2}(\widetilde{d}(y_1))^2}dy_1.$$

Since $\widetilde{d}'(y) \leq \frac{-\tau}{\sqrt{1-\tau^2}}$ for $-1 < \tau < 1$ and $y \in (-\infty, \widetilde{a})$ we find that

(2.14) $$A_1 \geq \int_{-\infty}^{a^*} e^{-\frac{1}{2}y_1^2}\widetilde{d}'(y_1)e^{-\frac{1}{2}(\widetilde{d}(y_1))^2}\, dy_1.$$

On the other hand, since $\widetilde{d}$ is strictly increasing on $(-\infty, a^*)$ we obtain by using Fubini and substitution that

$$A_2 = \int_{-\infty}^{\widetilde{d}(a^*)} e^{-\frac{1}{2}y_2^2} \int_{\widetilde{d}^{-1}(y_2)}^{a^*} -y_1 e^{-\frac{1}{2}y_1^2}dy_1 dy_2$$

$$= \int_{-\infty}^{\widetilde{d}(a^*)} \left(e^{-\frac{1}{2}a^{*2}} - e^{-\frac{1}{2}(\widetilde{d}^{-1}(y_2))^2}\right) e^{-\frac{1}{2}y_2^2}dy_2$$

$$= \quad e^{-\frac{1}{2}a^{*2}} \sqrt{2\pi} \Phi(\widetilde{d}(a^*)) - \int_{-\infty}^{a^*} e^{-\frac{1}{2}y_2^2} \widetilde{d}'(y_2) e^{-\frac{1}{2}(\widetilde{d}(y_2))^2} \, dy_2.$$

So it follows from (2.14) that

$$A_1 + A_2 \geq e^{-\frac{1}{2}a^{*2}} \sqrt{2\pi} \Phi(\widetilde{d}(a^*)) > 0.$$

Using the fact that $\widetilde{d}$ is strictly decreasing on $(a^*, \widetilde{a})$ one can employ the same arguments to conclude that B in (2.14) is non-negative. If $\widetilde{d}$ has no maximum in $(-\infty, \widetilde{a})$ then $\widetilde{d}$ is strictly monotone on $(-\infty, \widetilde{a})$. So the above arguments also apply to this case. So the statement of the lemma is valid for $-1 < \tau \leq 0$.

We turn to the case, when $0 \leq \tau < 1$, which implies that $\widetilde{d}$ is strictly decreasing on $(-\infty, \widetilde{a})$. Also this case can be covered by the above line of reasoning. The proof follows. $\qquad \square$

2.5. **Increasing the maturity of the option.** Many practitioners argue that since the pension liability has a very long duration, the interest rate guarantee given by the put option should have a similarly long time to maturity. The models we have studied so fare are custom made for relatively short time periods like a month, quarter or year. In particular, the notion of premium payments at the beginning of the period and benefit payments at the end of the period would be less good for longer periods. To prolong the maturity of the interest rate guarantee we will adapt the model and the option price to multiperiods.

Let each period be of length T and let the number of periods be given by N. Further, let premiums be paid at the beginning of each period and benefits at the end of each period. Then the assets at time $t = NT$ may be expressed as

$$(2.15) \qquad S_{NT}^c = \sum_{X=0}^{N} s_{XT}^c e^{(\mu_c - \frac{1}{2}\sigma_c^2)(N-X)T + \sigma_c(B_{NT}^c - B_{XT}^c)}$$

$$(2.16) \qquad S_{NT}^b = S_0^b e^{(\mu_b - \frac{1}{2}\sigma_b^2)NT + \sigma_b(\tau B_{NT}^c + \sqrt{1-\tau^2}B_{NT}^b)}$$

where

$$s_0^c = S_{0-}^c + E[\pi(0,T)], \quad s_{NT}^c = -\beta((N-1)T, NT)(1+\rho)^{T/2},$$

$$s_{jT}^c = E[\pi(jT, (j+1)T)|\mathcal{F}_{jT}] - \beta((j-1)T, jT)(1+\rho)^{T/2}, \quad j = 1, \cdots, N-1.$$

The multiperiod strike, K, may still be modelled by simulation of the membership data over several periods. Then the option price at time $t = 0$ may be expressed as

$$(2.17) \qquad \Pi_0^{NT} = e^{-\rho NT} E_Q[(K^* - (S_{NT}^c + \alpha S_{NT}^b))^+]$$

where

$$K^* = K + \sum_{X=1}^{N} \Big(E[\pi((X-1)T, XT)] - \beta((X-1)T, XT) \Big) + S_{0-}^c - AR_0.$$

Taking into account the independence of increments of the Brownian motions and assuming a B&S dynamic on the assets under our pricing measure Q, we end up with an expression for the option price given the premiums and benefits in Proposition 2.4.

Proposition 2.4. *The price of a N period interest rate guarantee with fixed strike K, deterministic premiums $\pi(\cdot,\cdot)$, deterministic benefit payments $\beta(\cdot,\cdot)$ and assets S_{NT}^c and S_{NT}^b given by Equation (2.15) and (2.16) under a B&S pricing measure Q is given by*

$$\Pi_0^{NT} = Ke^{-\rho NT}\Phi\big(y_1 \le d(y_2), y_2 \in \mathbb{R}_*^N\big)$$

$$- \sum_{X=0}^{N} s_{XT}^c e^{-\rho XT}\Phi\Big(y_1 \le d(y_2), \begin{bmatrix} y_{21} \\ \vdots \\ y_{2X} \end{bmatrix} \in \mathbb{R}_*^X, \begin{bmatrix} y_{2(X+1)} \\ \vdots \\ y_{2N} \end{bmatrix} \in \mathbb{R}_{**}^{Y-X}\Big)$$

$$(2.18) \qquad - \alpha S_0^b \Phi\big(y_1 \le d(y_2) - \sigma_b\sqrt{NT}\sqrt{1-\tau^2}, y_2 \in \mathbb{R}_{***}^N\big),$$

where

$$d(y_{21},\cdots,y_{2N}) = \frac{1}{\sigma_b\sqrt{1-\tau^2}\sqrt{NT}}\Big(\ln\big(\frac{K - S_{NT}^c(y_{21},\cdots,y_{2N})}{S_0^b}\big) - (\rho - \frac{1}{2}\sigma_b^2)NT - \sigma_b\tau\sum_{X=1}^{N}y_{2X}\Big),$$

and y_1, $y_2 = [y_{21},\cdots,y_{2N}]$ are independent standard normal variables and Φ is the $N+1$ dimensional cumulative normal distribution. Further, $y_2 \in \mathbb{R}_*^N$ is such that $K - S_{NT}^c(y_2) \ge 0$, $y_2 \in \mathbb{R}_{**}^N$ is such that $K - S_{NT}^c(y_2 + \sigma_c\sqrt{T}) \ge 0$ and $y_2 \in \mathbb{R}_{***}^N$ is such that $K - S_{NT}^c(y_2 + \sigma_b\tau\sqrt{T}) \ge 0$.

Proof. By letting y_1 and $y_{21},\cdots,y_{2N}$ be the independent increments of the Brownian motions B^b and B^c, the correlated assets (2.15) and (2.16) may be expressed as

$$(2.19) \qquad S_{NT}^c = \sum_{X=0}^{N} s_{XT}^c e^{(\rho-\frac{1}{2}\sigma_c^2)(N-X)T + \sigma_c\sqrt{T}\sum_{j=X+1}^{N}y_{2j}},$$

$$(2.20) \qquad S_{NT}^b = S_0^b e^{(\rho-\frac{1}{2}\sigma_b^2)NT + \sigma_b\sqrt{T}(\tau\sum_{j=1}^{N}y_{2j} + \sqrt{1-\tau^2}\sqrt{N}y_1)},$$

under Q. The proof follows by direct calculation of the option price represented by the expectation under Q. $\qquad\square$

This proposition will not be used later in the paper. However, we immediately see the similarity to Proposition 2.2, and thus one would think that alot of analysis in one period can be generalized to multi periods, and it is natural to assume that Lemma 2.3 also applies to the multi period situation.

2.6. **Numerical example.** Before we move on to look at some numerical examples, we will make some assumptions in our model that makes analysis of our numerical examples more transparent.

Instead of simulating the whole membership portfolio in our numerical examples, we apply an approximation of the strike, K, given by (2.1). A common way for actuaries to approximate premium reserves is through a discrete Thiele approximation where V_T is

approximated based on V_0, the premium for the period $t \in [0,T]$, guaranteed interest rate r and benefit payments $\beta(0,T)$. If we assume that the premium fund, PF_t, is deterministic, the guarantee (K) may be approximated by
(2.21)
$$K \approx (V_0 + PF_0)((1+r)^T - 1) + (\pi(0,T) + PF_T - PF_0)((1+r)^T - 1) - \beta(0,T)((1+r)^{\frac{T}{2}} - 1).$$

A deterministic premium fund is natural as this fund can be used to pay future premiums. The usage of the fund can be regarded as a strategic choice of the client which is known at time $t = 0$.

The actual premium, $\pi(0,T)$, and the actual benefit payment, $\beta(0,T)$, will be regarded as stochastic variables in (2.21). If we informally think of π as

$$\pi = \text{actual saving amount} + \text{actual cost of insurance cases},$$

it is immediately clear that this is stochastic because we do not initially know the actual saving and actual cost of insurance cases. At the beginning of the period we know $E[\pi(0,T)]$. This is the expected premium, the same amount billed to each client to cover pension accrual and expected insurance cases over the period $(0,T)$. Further, β is stochastic e.g. because some old age retired people may die, some active members may become disabled etc.. From the above, we see that the premium and benefit payment are approximations to the changes in premium reserve

$$\int_0^T ((1+r)^{T-t} - 1)\, dV_t \approx \pi(0,T)((1+r)^T - 1) + \beta(0,T)((1+r)^{\frac{T}{2}} - 1).$$

However, for (2.21) to be a good approximation, we need to put a lot of care into the stochastic variables $\pi(0,T)$ and $\beta(0,T)$. One alternative is to regard these variables as normally distributed. A motivation behind this is that the number of e.g. active members who remain active in the period $[0,T]$ is binomially distributed. Having enough members, the binomial distribution can be approximated by normal distributions. Also, since e.g. an active member who becomes disabled effects both π and β, the two normal distributions are correlated. It is also natural that π and β are dependent on the number of policies covered by the portfolio.

Similarly, the multiperiod strike K may be approximated by:

$$K \approx (V_0 + PF_0)((1+r)^{NT} - 1)$$

$$+ \sum_{X=1}^{N} (\pi((X-1)T, XT) + PF_{XT} - PF_{(X-1)T})((1+r)^{(N-X+1)T} - 1)$$

(2.22)
$$- \sum_{X=1}^{N} \beta((X-1)T, XT)((1+r)^{(N-X+\frac{1}{2})T} - 1).$$

Also here, the most challenging tasks when expanding to multiperiods is the modelling of the premiums $\pi((j-1)T, jT)$ and benefits $\beta((j-1)T, jT)$, $j = 1, \cdots, N$. These are not only dependent on each other but also dependent on membership history. E.g. if a

Parameters	Option prices
$PF_i = \{-10, -5, 0, 5, 10\}$	0.33, 0.40, 0.48, 0.57, 0.66
$TA = \{0, 5, 10\}$	1.36, 0.66, 0.29
$S_0^b = \{5, 10\}$	1.43, 0.66
$\rho = \{0.02, 0.03, 0.04\}$	0.81, 0.66, 0.54
$\sigma_c = \{\frac{0.05}{\sqrt{T}}, \frac{0.10}{\sqrt{T}}, \frac{0.15}{\sqrt{T}}\}$	0.02, 0.66, 2.08
$\sigma_b = \{\frac{0.05}{\sqrt{T}}, \frac{0.10}{\sqrt{T}}, \frac{0.15}{\sqrt{T}}\}$	0.61, 0.66, 0.71
$\tau = \{-0.5, 0, 0.5, 1\}$	0.47, 0.57, 0.66, 0.75
$N = \{1, 3, 5, 10\}$	0.66, 3.26, 5.78, 11.52
$E[\pi((X-1)T, XT)] = \{10, 15\}$	0.66, 0.76
$E[\beta((X-1)T, XT)] = \{5, 10\}$	0.66, 0.66
$\frac{\sigma_\pi}{\sqrt{n}} = \{0.1, 0.15\}$	0.66, 0.66
$\frac{\sigma_\beta}{n} = \{0.05, 0.10\}$	0.66, 0.66

TABLE 1. Prices of the interest rate guarantee based on the basis parameters when changing only one of the parameters at a time.

member has become disabled in the past, this will effect both future premiums and benefit payments.

In our numerical examples we will make some quite crude approximations on the premiums and benefit payments. We will assume that these are independent of each other and of the history, but the model will take into account the number of policies, n, covered by the client. I.e.

$$(2.23) \qquad \pi((j-1)T, jT) = E[\pi(jT, (j+1)T)|\mathcal{F}_0] + \frac{\sigma_\pi}{\sqrt{n}}\epsilon_{\pi,j},$$

$$(2.24) \qquad \beta((j-1)T, jT) = E[\beta(jT, (j+1)T)|\mathcal{F}_0] + \frac{\sigma_\beta}{\sqrt{n}}\epsilon_{\beta,j},$$

for $j = 1, \cdots, N$. Here $\epsilon_{\pi,j}, \epsilon_{\beta,j}$ are independent $N(0,1)$ variables. This model is clearly not good for clients with few policies in the portfolio. But for bigger clients with many policies in the portfolio, the assumption is more reasonable.

In our numerical examples a straight forward Monte Carlo simulation of the premiums $\pi((j-1)T, jT)$, the benefit payments $\beta((j-1)T, jT)$ and the assets S_{jT}^c, S_{jT}^b has been applied for $j = 1, \cdots, N$. We will use the following basis of parameters in our examples

$$V_0 = 100 \quad PF_i = 10 \text{ for all } i, \quad TA = 5, \quad S_{0-}^c = V_0 + PF_0 + TA, \quad S_0^b = 10, \quad r = 0.03,$$

$$\rho = 0.03, \quad \sigma_c = \frac{0.1}{\sqrt{T}}, \quad \sigma_b = \frac{0.1}{\sqrt{T}}, \quad \tau = 0.5, \quad T = 252, \quad N = 1, \quad \frac{\sigma_\pi}{\sqrt{n}} = 0.1, \quad \frac{\sigma_\beta}{\sqrt{n}} = 0.05,$$

$$E[\pi((X-1)T, XT)] = 10 \text{ for all } X, \quad E[\beta((X-1)T, XT)] = 5 \text{ for all } X.$$

Further, we will change one parameter at a time to see how this effects the option price. The option price when using the basis parameters is 0.66, while the other prices when changing one parameter can be found in Table 1.

Looking at the prices in Table 1, there are some issues worth giving extra attention.

The prices increase with increasing PF. This is surprising as one would think of the premium fund as a buffer fond. However, the premium fund is also given an interest rate guarantee. Hence, more money needs to be given interest rate guarantee as the premium fund increases, and this increases the strike and the price of the guarantee.

The prices are extremely sensitive with respect to changes in σ_c. This is partly due to the fact that the risk free rate, ρ, and the guaranteed interest rate, r, are the same. It is also due to the relatively long horizon of the option which leads to a high spread in asset values and the fact that the option is quite fare out of the money.

The prices increase with increasing N. Intuitively, one would think that increasing N would decrease prices per period as one could catch up on a shortfall if the time to maturity is long enough. However, with increasing time horizon, one also increases the variance in asset values at maturity. This is illustrated in Figure 1, where we look at the distributions of $S_{NT}^c + S_{NT}^b$ for $N = 1, 3, 5$ years. And as the options are out of the money a high variance increases the price. In addition to this, the strike increases with increasing maturity. For a put option, this leads to higher option prices. Notice also that the option covers asset losses all the way till the assets are worth nothing. For a private commercial insurance company this would probably not be the case, as this company would be bankrupt or taken under public administrative control when liabilities exceeds assets. Taking this into account would reduce option payoffs, which might also decrease the option price per period as time to maturity increases. However, this feature is not covered in this paper.
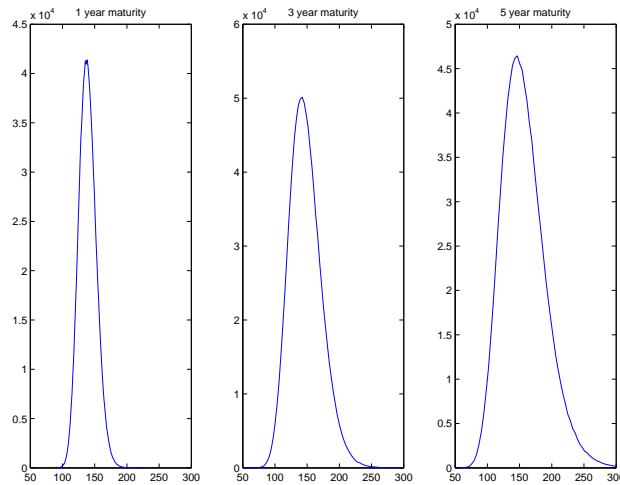

FIGURE 1. Distribution of $S_{NT}^c + S_{NT}^b$ for one, three and five years maturity. The asset values has been propagated based on S_0^c and S_0^b, whereas intermediate payments have not been taken into account.

There is no sensitivity with respect to $E[\beta((X - 1)T, XT)]$. The reason for this is that $r = \rho$ in our calculations. If $r \neq \rho$, we would see a sensitivity, but this would not be significant.

There is no visible sensitivity with respect to $\sigma_\pi/\sqrt{n}$ and $\sigma_\beta/\sqrt{n}$. The reason for this is that the parameters are quite low in the numerical examples. This is natural when there are many policies in the insurance portfolio. However, for smaller portfolios the uncertainty in $\pi(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ would be bigger, and σ_π and σ_β would effect the price of the interest rate guarantee. From the table we see that including expected future premiums $E[\pi((X-1)T, XT)]$ and also expected future benefit payments $E[\beta((X-1)T, XT)]$ in the model is important for the option price. However, including uncertainty in premiums and benefit payments through σ_π and σ_β seems to be less important. Hence, the deterministic assumptions of the strike in Propositions 2.2 and 2.4 is somewhat justified.

In general, the prices are quite sensitive to almost all parameters. This is due to the relatively long time horizon and the option being out of the money. The high sensitivity is an indicator to how difficult pricing of interest rate guarantees in pension insurance is and it underlines the importance of proper estimation of parameters. The high sensitivity is also an indicator to how important proper risk management and hedging is when dealing with interest rate guarantees.

3. Incomplete asset model

In recent years insurance companies in a series of European countries have been committed to finally implement several new initiatives of supervision and politics in their insurance business. These guidelines are known under the notion of "Solvency II" and play a similar role for insurances as the "Basel II" regulations for banks. The main objective of this new framework is roughly speaking to ensure the ability of the insurer to meet its liabilities for all contracts at each time under "appropriate" conditions. More specifically, these guidelines are required to be risk-adjusted and based on market-consistent valuation of the balance sheet of the insurance company. From these perspectives we aim at studying a generalization of the pricing problem of investment guarantees in Section 2.

3.1. Generalized price model for S_t^c, S_t^b in the presence of jumps and partial information. Let us now assume that the the price of the client assets S_t^c and the price of the buffer fund asset S_t^b at time t is described by the following jump processes:

$$dS_t^c = S_{t-}^c \left\{ \mu^c(t)dt + \sigma^c(t)\tau_1 dW_t^{(1)} + \sigma^c(t)\sqrt{1-\tau_1^2}dW_t^{(2)} \right.$$

$$(3.1) \qquad \left. + \int_{\mathbb{R}_0} \gamma_1^c(t,z)\tau_2 \widetilde{N}_1(dt,dz) + \int_{\mathbb{R}_0} \gamma_2^c(t,z)\sqrt{1-\tau_2^2}\widetilde{N}_2(dt,dz) \right\}$$

$$(3.2) \qquad dS_t^b = S_{t-}^b \left\{ \mu^b(t)dt + \sigma^b(t)dW_t^{(1)} + \int_{\mathbb{R}_0} \gamma_1^b(t,z)\widetilde{N}_1(dt,dz) \right\},$$

where $W_t^{(i)}, i = 1, 2$ are independent standard Brownian motions and $\widetilde{N}_i(dt, dz) = N_i(dt, dz) - \nu_i(dz)dt$, $i = 1, 2$, independent compensated Poisson random measures with Lévy measures ν_i on $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$, $i = 1, 2$. Further $\tau_i \in [0, 1]$, $i = 1, 2$ are correlation parameters and $\mu^c(t), \mu^b(t), \sigma_t^c(t), \sigma_t^b(t), \gamma_1^b(t, z), \gamma_i^c(t, z), i = 1, 2$ are predictable processes with respect to

the filtration $\mathcal{F}_t$ generated by $W_t^{(i)}$, $i = 1, 2$ and the Poisson random measures $N_i(dt, dz)$, $i = 1, 2$. We require that

$$\gamma_1^b(t, z), \gamma_1^c(t, z), \gamma_2^c(t, z) > -1 \text{ a.e.}$$

and

$$\int_0^T \Big\{ |\mu^c(t)| + |\mu^b(t)| + (\sigma^c(t))^2 + (\sigma^b(t))^2$$

$$\int_{\mathbb{R}_0} (\log(1 + \gamma_1^c(t, z))^2 + \log(1 + \gamma_1^b(t, z))^2 \nu_1(dz)$$

$$+ \int_{\mathbb{R}_0} \log(1 + \gamma_2^c(t, z))^2 \nu_2(dz) \Big\} dt$$

$$< \infty \text{ a.e.}$$

hold.

The model (3.1) and (3.2), which is a generalization of (2.2) and (2.3) of Section 2, has the advantage that it allows for sudden price jumps whose frequency and intensity are determined by the Lévy measures ν_i, $i = 1, 2$.

Regarding the strike of our put option, we will in this section regard this as deterministic. The motivation for this is based on the numerical results in Section 2.6 where we sow that the variation in the strike had little effect on the option price.

Now, consider a portfolio composed of a risk-free asset S_t^0 (e.g. bank account) and the assets S_t^c, S_t^b . Then the portfolio value at time $t \geq 0$ is given by

$$(3.3) \qquad X_x^{(\pi)}(t) = \pi^0(t)S_t^0 + \pi^c(t)S_t^c + \pi^b(t)S_t^b,$$

where x is the initial wealth and $\pi^0(t)$, $\pi^c(t)$, $\pi^b(t)$ the amounts of money invested in S_t^0, S_t^c, S_t^b at time t, respectively. For simplicity suppose that

$$(3.4) \qquad\qquad\qquad\qquad S_t^0 \equiv 1.$$

Then $X_x^{(\pi)}(t)$ satisfies

$$dX_x^{(\pi)}(t) = \pi^c(t)S_{t-}^c \Big\{ \mu^c(t)dt + \sigma^c(t)\tau_1 dW_t^{(1)} + \sigma^c(t)\sqrt{1 - \tau_1^2} dW_t^{(2)}$$

$$+ \int_{\mathbb{R}_0} \gamma_1^c(t, z)\tau_2 \widetilde{N}_1(dt, dz) + \int_{\mathbb{R}_0} \gamma_2^c(t, z)\sqrt{1 - \tau_2^2} \widetilde{N}_2(dt, dz) \Big\}$$

$$\pi^b(t)S_{t-}^b \Big\{ \mu^b(t)dt + \sigma^b(t)\tau_1 dW_t^{(1)} + \int_{\mathbb{R}_0} \gamma_1^b(t, z)\tau_2 \widetilde{N}_1(dt, dz) \Big\},$$

$$(3.5) \quad X_x^{(\pi)}(0) = x > 0.$$

The portfolio strategy $\pi(t) = (\pi^0(t), \pi^c(t), \pi^b(t))$ is called *self-financing* if the cumulative cost

$$(3.6) \qquad C(t) := X_x^{(\pi)}(t) - \int_0^t \pi^c(u^-)dS_u^c - \int_0^t \pi^b(u^-)dS_u^b$$

equals a constant for all $t \geq 0$. In the sequel we denote by $\mathcal{P}$ the collection of all self-financing strategies such that

$$X_x^{(\pi)}(t) \geq c$$

for some constant c and $0 \leq t \leq T$.

From now on we shall also allow for the case that the trading strategies of the investor are only based on partial market information, that is that $\pi(t)$ is $\mathcal{G}_t$−predictable for a sub-filtration $\mathcal{G}_t \subset \mathcal{F}_t$, $0 \leq t \leq T$. An example of such a filtration is a trader whose market market information is subject to a delay, that is

$$\mathcal{G}_t = \mathcal{F}_{(t-\delta(t))^+},$$

where $\delta(t) \geq 0$ is a function.

We call a $\mathcal{G}_t$−predictable portfolio strategy $\pi \in \mathcal{P}$ *admissible* if there is a strong solution to (3.5) such that

$$\gamma_1^b(t,z), \gamma_1^c(t,z), \gamma_2^c(t,z) > -1 \text{ a.e.}$$

and

$$\int_0^t \left\{ |\pi^c(t)S_t^c\mu^c(t)| + |\pi^b(t)S_t^b\mu^b(t)| + (\pi^c(t)S_t^c\sigma^c(t))^2 + (\pi^b(t)S_t^b\sigma^b(t))^2 \right.$$

$$\int_{\mathbb{R}_0} (\pi^c(t)S_t^c(\gamma_1^c(t,z))^2 + (\pi^b(t)S_t^b\gamma_1^b(t,z))^2\nu_1(dz)$$

$$+ \left. \int_{\mathbb{R}_0} (\pi^c(t)S_t^c\gamma_2^c(t,z))^2\nu_2(dz) \right\} dt$$

(3.7) $\quad < \quad \infty$ a.e.

We denote by Π the set of admissible strategies.

The life company is interested to determine a "fair" price of an investment guarantee of the form

(3.8) $$G = g(S_t^c, S_t^b),$$

where g is a pay-off function.

In a complete market the insurer is able to replicate the claim G, that is

$$X_x^{(\pi)}(T) = G$$

for an arbitrage-free hedging strategy $(\pi^0(t), \pi^c(t), \pi^b(t))$ at terminal time T. In this case it is reasonable to define the fair price of G by

$$p = E_Q[G],$$

where Q is the unique risk neutral measure on $(\Omega, \mathcal{F}, P)$.

In an incomplete market as given by (3.1) and (3.2) perfect hedging of claims is in general not possible and one has to resort to other pricing methods. One approach is e.g. utility indifference pricing, which is based on expected utility maximization. See e.g. Grasselli and Hurd [8], Hodges and Neuberger [10], Takino [15], Benth and Proske [4] and the references therein.

A similar concept introduced later on in the literature is referred to as risk indifference pricing. See e.g. Xu [17], Barrieu and El Karoui [3], Klöppel and Schweizer [11], Øksendal and Sulem [13], An, Øksendal and Proske [1]. Here the price determined by the latter approach is risk-adjusted in the sense that it reflects the risk-tolerance of the issuer of the claim, which is measured by a convex risk measure (e.g. expected shortfall). The resulting risk-indifference price is a risk-minimizing price which makes the insurance company be indifferent to the investment strategies of either entering the market on its own or entering the market after having issued the claim. We want to discuss this approach in the next section.

3.2. **Risk indifference pricing with respect to S_t^c, S_t^b.** In this Section we want to analyze prices of investment guarantees based on the maximum principle approach to risk indifference pricing as developed in An, Øksendal and Proske [1]. This non-Markovian framework also admits the study of prices under the constraint that the insurance company has only limited access to market information.

In what follows we adopt the framework of An, Øksendal and Proske [1] to our setting, i.e. model (3.1), (3.2). For convenience we provide a self contained treatment of this machinery to our basket option setting.

In order give a precise statement of our pricing problem let us pass in review the definition of a convex risk measure (see e.g. Föllmer and Schied [7], Artzner, Delbaen, Eber and Heath [2]): Denote by $\mathbb{F}$ the space of all equivalence classes of random variables $X : \Omega \longrightarrow \mathbb{R}$. A function

$$\rho : \mathbb{F} \longrightarrow \mathbb{R} \cup \{\infty\}$$

is called *convex risk measure* if the following axioms are fulfilled:

Axiom 1 (convexity).

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y), \ 0 < \lambda < 1.$$

The interpretation of this axiom is the following: If λ represents the fraction invested in the financial position X and $(1 - \lambda)$ the remaining part for the alternative investment, then the diversified portfolio $\lambda X + (1 - \lambda)Y$ should not increase the risk.

Axiom 2 (monotonicity). $X \leq Y$ *implies*

$$\rho(X) \geq \rho(Y).$$

Axiom 3 (translation invariance).

$$\rho(X + m) = \rho(X) - m, \ m \in \mathbb{R}.$$

If $\rho(X)$ is interpreted as a capital requirement which makes the financial position $X + \rho(X)$ acceptable then Axiom 3 is a natural assumption.

Using the above concept of risk measure we consider the following two investment plans:

The insurer issues the claim $G = g(S_t^c, S_t^b)$ and enters the market with the collected premium p. Then the minimal risk of the insurer's financial position is given by

(3.9) $$\Phi_G(x + p) = \inf_{\pi \in \mathcal{P}} \rho(X_x^{(\pi)}(T) - G).$$

The other strategy is to enter the market without issuing the claim. In this case the minimal risk involved for the insurer amounts to

$$(3.10) \qquad \Phi_0(x) = \inf_{\pi \in \mathcal{P}} \rho(X_x^{(\pi)}(T))$$

The insurer's *risk indifference price*, $p = p_{risk}^{seller}$, is defined to be the solution of the equation

$$(3.11) \qquad \Phi_G(x + p) = \Phi_0(x).$$

In order to determine the risk indifference price we need the following characterization of convex risk measures which is due to Föllmer and Schied [7]:

Theorem 3.1. *A function* $\rho : \mathbb{F} \longrightarrow \mathbb{R}$ *is a convex risk measure iff there is a set* $\mathcal{L}$ *of measures* $Q << P$ *on* $\mathcal{F}_T$ *and a convex "penalty" function* $\zeta : \mathcal{L} \longrightarrow \mathbb{R}$ *with* $\inf_{Q \in \mathcal{L}} \zeta(Q) = 0$ *such that*

$$\rho(X) = \sup_{Q \in \mathcal{L}} \{E_Q[-X] - \zeta(Q)\}$$

for all $X \in \mathbb{F}$.

Using this result we see that the risk indifference price p_{risk}^{seller} in (3.11) can be obtained by solving the stochastic differential games

$$(3.12) \qquad \Phi_G(x + p) = \inf_{\pi \in \mathcal{P}} \left(\sup_{Q \in \mathcal{L}} \left\{ E_Q\left[-X_{x+p}^{(\pi)}(T) + G\right] - \zeta(Q) \right\} \right)$$

and

$$(3.13) \qquad \Phi_0(x) = \inf_{\pi \in \mathcal{P}} \left(\sup_{Q \in \mathcal{L}} \left\{ E_Q\left[-X_x^{(\pi)}(T)\right] - \zeta(Q) \right\} \right)$$

In the sequel we want to consider convex risk measures which are specified as follows:

In Theorem 3.1 we choose $\mathcal{L}$ to be a parametrized family of measures given by

$$Q_\theta(d\omega) = K_\theta(T)P(d\omega) \text{ on } \mathcal{F}_T,$$

where $\theta(t, z) = (\theta_0(t), \theta_1(t), \theta_2(t, z), \theta_3(t, z))$ are $\mathcal{F}_t$−predictable processes such that

$$dK_\theta(t) = K_\theta(t^-) \left[\theta_0(t)dW_t^{(1)} + \theta_1(t)dW_t^{(2)} \right.$$

$$\left. + \int_{\mathbb{R}_0} \theta_2(t, z)\widetilde{N}_1(dt, dz) + \int_{\mathbb{R}_0} \theta_3(t, z)\widetilde{N}_2(dt, dz) \right],$$

$$(3.14) \qquad K_\theta(0) = k > 0.$$

Here we require that

$$\theta_2(t, z), \theta_3(t, z) > -1 \text{ a.e.}$$

and

$$(3.15) \qquad \int_0^T \left\{ (\theta_0(t))^2 + (\theta_1(t))^2 \right.$$

$$\left. \int_{\mathbb{R}_0} (\theta_2(t, z))^2 \nu_1(dz) + \int_{\mathbb{R}_0} (\theta_3(t, z))^2 \nu_2(dz) \right\} dt$$

$$< \quad \infty \text{ a.e.}$$

The class Θ of *admissible* controls $\theta(t,z) = (\theta_0(t), \theta_1(t), \theta_2(t,z), \theta_3(t,z))$ consists of all $\mathcal{G}_t-$predictable processes $\theta(t)$ such that (3.15) holds and

(3.16) $$E[K_\theta(T)] = K_\theta(0) = k > 0$$

hold.

We define $\mathcal{L}$ as

$$\mathcal{L} = \{Q_\theta : \theta \in \mathbb{M}\},$$

where

(3.17) $$\mathbb{M} = \{\theta \in \Theta : E\left[(M_i\theta)(t)\,|\,\mathcal{G}_t\right] = 0, i = 1, 2 \text{ a.e. for all } t\}$$

for

$$(M_1\theta)(t)$$
$$= -\mu^c(t) + \sigma^c(t)\theta_0(t)\tau_1 + \sigma^c(t)\theta_1(t)\sqrt{1 - \tau_1^2}$$
$$+ \int_{\mathbb{R}_0} \gamma_1^c(t,z)\theta_2(t,z)\tau_2\nu_1(dz) + \int_{\mathbb{R}_0} \gamma_2^c(t,z)\theta_3(t,z)\sqrt{1 - \tau_2^2}\nu_2(dz)$$

and

$$(M_2\theta)(t) = -\mu^b(t) + \sigma^b(t)\theta_0(t) + \int_{\mathbb{R}_0} \gamma_1^b(t,z)\theta_2(t,z)\nu_1(dz).$$

Further we assume that the penalty function in Theorem 3.1 is given by

$$\zeta(Q_\theta)$$
$$= E\left[\int_0^T \left\{\int_{\mathbb{R}_0} \lambda_1(t, \theta_0(t, \widetilde{Y}(t)), \theta_1(t, \widetilde{Y}(t)), \theta_2(t, \widetilde{Y}(t), z), \widetilde{Y}(t), z)\nu_1(dz)\right.\right.$$
$$\left. \int_{\mathbb{R}_0} \lambda_2(t, \theta_0(t, \widetilde{Y}(t)), \theta_1(t, \widetilde{Y}(t)), \theta_3(t, \widetilde{Y}(t), z), \widetilde{Y}(t), z)\nu_2(dz)\right\} dt$$

(3.18) $$\left. + h(\widetilde{Y}(t))\right],$$

where $\widetilde{Y}(t)$ is given by

$$d\widetilde{Y}(t) = \begin{pmatrix} dY_1(t) \\ dY_2(t) \\ dY_3(t) \end{pmatrix} = \begin{pmatrix} dK_\theta(t) \\ dS_t^c \\ dS_t^b \end{pmatrix}$$
$$\widetilde{Y}(0) = \widetilde{y} = (k, s_1, s_2)$$

and where λ_1, λ_2, h are convex functions with $\lambda_1, \lambda_2 \in C^1((0,T) \times \mathbb{R}^2 \times \mathbb{R}_0)$ and $h \in C^1(\mathbb{R})$.

Further we assume that

$$E\left[\int_0^T \left\{\int_{\mathbb{R}_0} \left|\lambda_1(t, \theta_0(t, \widetilde{Y}(t)), \theta_1(t, \widetilde{Y}(t)), \theta_2(t, \widetilde{Y}(t), z), \widetilde{Y}(t), z)\right|\nu_1(dz)\right.\right.$$

$$\int_{\mathbb{R}_0} \left| \lambda_2(t, \theta_0(t, \widetilde{Y}(t)), \theta_1(t, \widetilde{Y}(t)), \theta_3(t, \widetilde{Y}(t), z), \widetilde{Y}(t), z) \right| \nu_2(dz) \bigg\} dt$$
$$+ \left| h(\widetilde{Y}(T)) \right| \Bigg] < \infty.$$

Next we want to determine the risk indifference price p_{risk}^{seller} by solving the following stochastic control problem:

$$(3.19) \qquad\qquad \Psi_G^{\mathcal{G}}(t, \widetilde{y}) := \sup_{\theta \in \mathbb{M}} J_0^\theta(t, \widetilde{y}) = J_0^{\check\theta}(t, \widetilde{y})$$

for a $\check\theta \in \mathbb{M}$, where the performance functional $J_0^\theta(t, \widetilde{y})$ is defined as

$$(3.20) \qquad
\begin{aligned}
& J_0^\theta(t, \widetilde{y}) \\
= {}& E^y \left[-\int_t^T \Lambda(\theta(u, \widetilde{Y}(u)) du - h(\widetilde{Y}(T)) + K_\theta(T) g(S_T^c, S_T^b) \right],
\end{aligned}$$

where

$$
\begin{aligned}
\Lambda(\theta) &= \Lambda(\theta(t, \widetilde{y})) \\
&= \int_{\mathbb{R}_0} \lambda_1(t, \theta_0(t, \widetilde{Y}(t)), \theta_1(t, \widetilde{Y}(t)), \theta_2(t, \widetilde{Y}(t), z), \widetilde{Y}(t), z) \nu_1(dz) \\
&\quad + \int_{\mathbb{R}_0} \lambda_2(t, \theta_0(t, \widetilde{Y}(t)), \theta_1(t, \widetilde{Y}(t)), \theta_3(t, \widetilde{Y}(t), z), \widetilde{Y}(t), z) \nu_2(dz).
\end{aligned}
$$

We use the stochastic maximum principle approach to tackle this problem. To this end we introduce the *Hamiltonian*

$$H : [0, T] \times \mathbb{R}^3 \times \mathbb{R}^4 \times \mathbb{R}^3 \times \mathbb{R}^{3 \times 2} \times \mathcal{R} \longrightarrow \mathbb{R}$$

by

$$
\begin{aligned}
& H(t, k, s_1, s_2, \theta, p, q, r(\cdot, z)) \\
= {}& -\Lambda(t, \widetilde{Y}(t)) + \mu^c(t) s_1 p_2 + \mu^b(t) s_2 p_3 + \theta_0(t) k q_1 + \theta_1(t) k q_2 \\
& + \sigma^c(t) \tau_1 s_1 q_3 + \sigma^c(t) \sqrt{1 - \tau_1^2} s_1 q_4 + \sigma^b(t) \tau_1 s_2 q_5 \\
& + \int_{\mathbb{R}_0} k \theta_2(t, z) r_1(\cdot, z) v_1(dz) + \int_{\mathbb{R}_0} k \theta_3(t, z) r_2(\cdot, z) v_2(dz) \\
& \int_{\mathbb{R}_0} s_1 \gamma_1^c(t, z) \tau_2 r_3(\cdot, z) v_1(dz) + \int_{\mathbb{R}_0} s_1 \gamma_2^c(t, z) \sqrt{1 - \tau_2^2} r_4(\cdot, z) v_2(dz) \\
& + \int_{\mathbb{R}_0} s_2 \gamma_1^b(t, z) \tau_2 r_5(\cdot, z) v_1(dz),
\end{aligned}
$$
$$(3.21)$$

where $\mathcal{R}$ denotes the class of measurable functions $r : [0, T] \times \mathbb{R}_0 \longrightarrow \mathbb{R}^5$ such that the integrals in (3.21) exist.

In order to solve the optimization problems (3.12) and (3.13) we need to introduce another Hamiltonian $\widetilde{H}$. Suppose that $\widetilde{H}$ is differentiable with respect to k, s_1, s_2. Then

the corresponding adjoint equations in the unknown adapted processes $p(t), q(t), r(t, z)$ are given by the backward stochastic differential equations (BSDE's)

$$
\begin{aligned}
dp_1(t) & \\
= & \left\{ -\theta_0(t)q_1(t) - \theta_1(t)q_2(t) - \int_{\mathbb{R}_0} \theta_2(t, z)r_1(t, z)v_1(dz) - \int_{\mathbb{R}_0} \theta_3(t, z)r_2(t, z)v_2(dz) \right\} dt \\
& q_1(t)dW_t^{(1)} + q_2(t)dW_t^{(2)} + \int_{\mathbb{R}_0} r_1(t^-, z)\widetilde{N}_1(dt, dz) + \int_{\mathbb{R}_0} r_2(t^-, z)\widetilde{N}_2(dt, dz),
\end{aligned}
$$

$$
(3.22)\quad p_1(T) = -\frac{\partial h}{\partial k}(\widetilde{Y}(T)) + g(S_T^c, S_T^b),
$$

$$
\begin{aligned}
dp_2(t) & \\
= & \left\{ -\mu^c(t)p_2(t) - \sigma^c(t)\tau_1 q_3(t) - \sigma^c(t)\sqrt{1 - \tau_1^2}q_4(t) \right. \\
& \left. - \int_{\mathbb{R}_0} \gamma_1^c(t, z)\tau_2 r_3(t^-, z)v_1(dz) - \int_{\mathbb{R}_0} \gamma_2^c(t, z)\sqrt{1 - \tau_2^2}r_4(t^-, z)v_2(dz) \right\} dt \\
& + q_3(t)dW_t^{(1)} + q_4(t)dW_t^{(2)} + \int_{\mathbb{R}_0} r_3(t^-, z)\widetilde{N}_1(dt, dz) + \int_{\mathbb{R}_0} r_4(t^-, z)\widetilde{N}_2(dt, dz),
\end{aligned}
$$

$$
(3.23)\quad p_2(T) = -\frac{\partial h}{\partial s_1}(\widetilde{Y}(T)) + K_\theta(T)\frac{\partial g}{\partial s_1}(S_T^c, S_T^b),
$$

$$
\begin{aligned}
dp_3(t) & \\
= & \left\{ -\mu^b(t)p_3(t) - \sigma^b(t)\tau_1 q_5(t) - \int_{\mathbb{R}_0} \gamma_1^b(t, z)\tau_2 r_5(t^-, z)v_1(dz) \right\} dt \\
& q_5(t)dW_t^{(1)} + \int_{\mathbb{R}_0} r_5(t^-, z)\widetilde{N}_1(dt, dz),
\end{aligned}
$$

$$
(3.24)\quad p_3(T) = -\frac{\partial h}{\partial s_2}(\widetilde{Y}(T)) + K_\theta(T)\frac{\partial g}{\partial s_2}(S_T^c, S_T^b).
$$

Finally, let us define

$$
\begin{aligned}
& \widetilde{H}(t, k, s_1, s_2, x, \theta, \pi, z) \\
= & H(t, k, s_1, s_2, \theta, \widetilde{p}(t), \widetilde{q}(t), \widetilde{r}(\cdot, z)) \\
& - s_1\pi_1 K_\theta(t)\left(\mu^c(t) + 2\theta_0(t)\sigma^c(t)\tau_1 + 2\theta_1(t)\sigma_t^c(t)\sqrt{1 - \tau_1^2} \right. \\
& \left. + 2\int_{\mathbb{R}_0} \theta_2(t, z)\gamma_1^c(t, z)\tau_2\nu_1(dz) + 2\int_{\mathbb{R}_0} \theta_3(t, z)\gamma_2^c(t, z)\sqrt{1 - \tau_2^2}\nu_2(dz) \right) \\
& + s_2\pi_2 K_\theta(t)\left(\mu^b(t) + 2\theta_0(t)\sigma^b(t)\tau_1 + 2\int_{\mathbb{R}_0} \theta_2(t, z)\gamma_1^c(t, z)\tau_2\nu_1(dz) \right).
\end{aligned}
$$

Using this definition we can state the following result, which can be found in An, Øksendal and Proske [1]:

Theorem 3.2. *Assume that there exists a solution $\widetilde{p}(t), \widetilde{q}(t), \widetilde{r}(t, z)$ of the adjoint equations (3.22)-(3.24) for $\theta \in \Theta$. Further require that*

$$\theta \longmapsto H(t, \widetilde{Y}(t), \theta, \widetilde{p}(t), \widetilde{q}(t), \widetilde{r}(t, z))$$

is concave. Suppose that for all $\pi \in \mathbb{R}^2$ the function

$$\theta \longmapsto E\left[\widetilde{H}(t, k, s_1, s_2, x, \theta, \pi, z)\, \mathcal{G}_t\right], \theta \in \Theta$$

has a maximum point at $\widehat{\theta} = \widehat{\theta}(\pi)$. In addition assume that

$$\pi \longmapsto E\left[\widetilde{H}(t, k, s_1, s_2, x, \widehat{\theta}(\pi), \pi, z)\, \mathcal{G}_t\right]$$

attains a minimum. Then the risk indifference price for the contingent claim G, p_{risk}^{seller} is given by

$$p_{risk}^{seller} = k^{-1}(\Psi_G^{\mathcal{G}}(t, \widetilde{y}) - \Psi_0^{\mathcal{G}}(t, \widetilde{y})),$$

where $\Psi_H^{\mathcal{G}}(t, \widetilde{y})$ is the value function in (3.19) with respect to a claim H. In particular, if $k = 1$ in (3.16) holds, we get the representation

$$(3.25) \qquad p_{risk}^{seller} = \sup_{Q \in \mathcal{L}} \left\{ E_Q[G] - \zeta(Q) \right\} - \sup_{Q \in \mathcal{L}} \left\{ -\zeta(Q) \right\}.$$

So if the penalty function $\zeta \equiv 0$ in (3.25), then

$$(3.26) \qquad p_{risk}^{seller} = \sup_{Q \in \mathcal{L}} E_Q[G].$$

If in addition the trader has unlimited access to market information, that is $\mathcal{G}_t = \mathcal{F}_t$ for all t, then

$$p_{risk}^{seller} = \rho_0(-G),$$

where ρ_0 is a monetary risk measure given by

$$(3.27) \qquad \rho_0(X) = \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}_0\}$$

for the set of acceptable positions

$$\mathcal{A}_0 := \{X \in L^\infty : \ X \text{ can be hedged without additional costs}\}.$$

See p. 204 in Föllmer and Schied [7]. This is a financial risk measure which provides a fair price p_{risk}^{seller} that captures the "worst scenario".

3.3. Numerical examples.
Also in these numerical examples we will make some simplifying assumptions.

First, let all the parameters in the asset model (3.1) and (3.2) be constants. Let also the jump parts of the assets and the Radon Nikodym derivative (3.14) be given by Poisson processes. The last assumption is quite crude, but it still allows us to investigate the impact of the jumps on the option price.

Further, we have chosen to estimate the option price p_{risk}^{seller} in (3.26) by using a constant parametric form on the Radon Nikodym derivative (3.14). I.e. choose the admissible controls $\theta(t, z) = (\theta_0, \theta_1, \theta_2, \theta_3) \in \mathbb{R}^4$ such that

$$
\begin{aligned}
dK_\theta(t) &= K_\theta(t^-) \left[\theta_0 dW_t^{(1)} + \theta_1 dW_t^{(2)} \right. \\
&\quad \left. + \int_{\mathbb{R}_0} \theta_2 \widetilde{N}_1(dt, dz) + \int_{\mathbb{R}_0} \theta_3 \widetilde{N}_2(dt, dz) \right], \\
K_\theta(0) &= k > 0
\end{aligned}
$$

is a martingale. Further we will restrict θ to only taking positive values. Thus, we estimate the price of the interest rate guarantee (3.26) by

$$(3.28) \qquad \hat{p}_{risk}^{seller} = \max_{\theta \in \mathbb{R}_+^4} E[K_\theta(1)G].$$

The motivation behind this is that it simplifies the derivation of the option price because we do not have to solve the BSDEs (3.22)- (3.24). Also, a constant price of risk is easy to comprehend and is often used in practise.

Numerically, we find the estimated prices (3.28) by using a Nelder-Mead algorithm to maximize in θ under the constraints

$$
\begin{aligned}
0 &= -\mu^c + \sigma^c \theta_0 \tau_1 + \sigma^c \theta_1 \sqrt{1 - \tau_1^2} \\
&\quad + \int_{\mathbb{R}_0} \gamma_1^c \theta_2 \tau_2 \nu_1(dz) + \int_{\mathbb{R}_0} \gamma_2^c \theta_3 \sqrt{1 - \tau_2^2} \nu_2(dz), \\
0 &= -\mu^b + \sigma^b \theta_0 + \int_{\mathbb{R}_0} \gamma_1^b \theta_2 \nu_1(dz),
\end{aligned}
$$

which makes sure that $K_\theta(1)S_1^c$ and $K_\theta(1)S_1^b$ are martingales. In each iteration of the Nelder-Mead algorithm $E[K_\theta(1)g(S_1^c, S_1^b)]$ is estimated by Monte Carlo simulations. This does of course introduce some uncertainty, but with enough simulations (we have used 10^7 simulations) the algorithm converges nicely.

In our examples we let the jumps be Poisson processes with intensity λ_i and jump size γ_1^b and γ_i^c, $i = 1, 2$. Further we let the initial parameters be given by

$$
\begin{aligned}
K &= 103, \quad S_0^c = 100, \quad S_0^b = 10, \quad \mu_c = 0.06, \quad \mu_b = 0.07, \\
&\sigma_c = 0.10, \quad \sigma_b = 0.15, \quad \rho = 0, \quad r = 0.03, \\
&\tau_1 = 0.5, \quad \tau_2 = 0.3, \quad \lambda_1 = 0.5, \quad \lambda_2 = 0.3, \\
&T = 1, \quad \gamma_1^c = 0.04, \quad \gamma_2^c = 0.04, \quad \gamma_1^b = 0.06.
\end{aligned}
$$

Option prices when adjusting one parameter at the time is given in Table 2.

From the table we see that the option prices from the risk indifference method is higher than the B & S prices. This is not surprising as the risk indifference price also incorporates jumps. An interesting result is that the risk indifference price is very close to the B & S price for small jumps in S^c. This means that the maximization over different risk measures has little effect on the price in this case.

Parameters	Risk indifference price	B & S price
$\tau_2 = \{-0.3, 0, 0.3, 0.6\}$	$\{1.63, 1.64, 1.67, 1.68\}$	1.58
$\lambda_1 = \{0.5, 2.0\}$	$\{1.67, 1.72\}$	1.58
$\lambda_2 = \{0.3, 1.2\}$	$\{1.67, 1.81\}$	1.58
$\gamma_1^c = \gamma_2^c = \{0.01, 0.04, 0.08\}$	$\{1.60, 1.67, 1.84\}$	1.58
$\gamma_1^b = \{0.01, 0.06, 0.10\}$	$\{1.65, 1.67, 1.68\}$	1.58

TABLE 2. Option prices based on the risk indifference principle (3.28) in an incomplete market and Black and Scholes prices in a complete asset market. The prices are based on the initial parameters when changing one parameter at a time.

Another observation is that the jumps in S_t^c has greater effect than the jumps in S_t^b. This is not surprising as the client assets makes out the main part of the total assets.

The correlation in jumps, τ_2, has little effect on the price. Still we see that the price increases with increasing correlation. This corresponds to results found in Section 2 about the correlation in Brownian motion.

An overall observation is that the sensitivity in option prices seems to be quite small. The risk indifference pricing principle seems to give stable option prices. This is good, especially since the principle in itself is nonlinear.

Notice that these examples only include positive jumps in asset values. For a put option one would expect that this reduces the option price. However, the jumps also affect the risk neutral measure Q. A framework with general Lévy measures could lead to quite different results. This has not been studied in this paper.

4. Conclusions and future work

In Section 2 we have found a model with corresponding prices which is easy for practitioners to implement. Our numerical examples show that the setup gives robust and reasonable prices.

In Section 3 we have derived a risk indifference pricing method to value options in an incomplete market. The setup is general and should be applicable in many similar situations. In our numerical examples we see that the option prices seem to be stable under our risk indifference principle, and we also see that the prices are somewhat higher than under a complete market setup.

On the modelling side in Section 2, there is in particular one issue that could be given attention in future work. A minimum barrier for the asset values, both the buffer assets and the client assets, could be included. This barrier would represent the point where the insurance company would be bust or taken under administrative control. Including this is not trivial as the time of reaching the barrier would be stochastic.

In Section 3 future work could include incorporating a more sophisticated Lévy measure in our numerical example. Further we could use other risk measures than (3.27), e.g. expected shortfall. This would probably change our setup quite a bit, and would require some research.

References

[1] Ta Thi Kieu An, Øksendal, B. and Proske, F. (2008). A Maximum Principle Approach to Risk Indifference Pricing with Partial Information, *Journal of Applied Mathematics and Stochastic Analysis*, vol. 2008.

[2] Artzner, P., Delbaen, F., Eber, J. M., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance* 4, pp 203-228.

[3] Barrieu, P. and El Karoui, N. (2004). Optimal derivatives design under dynamic risk measures. *Mathematics of Finance*, vol. 351 of Contemporary Mathematics, pp. 13-25.

[4] Benth, F. E. and Proske, F. (2009). Utility indifference pricing of interest-rate guarantees. *International Journal of Theoretical and Applied Finance*, 12(1), pp. 63-82.

[5] Cont, R. and Tankov, P. (2004). Financial Modelling With Jump Processes, *Chapman & Hall*.

[6] Davis, P. J. & Rabinowitz, P. (1975). Methods of Numerical Integration. *Academic Press*

[7] Föllmer, H. and Schied, A. (2004). Stochastic Finance: An Introduction in Discrete Time. *De Gruyter studies in mathematics*, 27, 2. ed.

[8] Grasselli, M. R. and Hurd, T. R. (2005). Indifference pricing and hedging in stochastic volatility models. *Tech. Rep., McMaster University*

[9] Henriksen, P. N. (2008). Lognormal moment matching and pricing of basket options. *E-Print*, University of Oslo

[10] Hodges, S. D. and Neuberger, A. (1989). Optimal replication of contingent claims under transaction costs. *Review of Futures Markets*, vol. 8, no. 2, pp. 222-239.

[11] Klöppel, S. and Schweizer, M. (2007). Dynamic indifference valuation via convex risk measures, *Mathematical Finance*, vol. 17, no. 4, pp. 599-627.

[12] Miltersen, K. R. and Persson, S. A. (1999). Pricing rate of return guarantees in a Heath-Jarrow-Morton framework. *Insurance: Mathematics and Economics*, 25, pp. 307–325.

[13] Øksendal, B. and Sulem, A. (2007). Risk indifference pricing in jump diffusion markets. *Preprint, University of Oslo* 17/2007. Accepted for publication in Mathematical Finance.

[14] David F. Schrager and Antoon A. J. Pelsser (2004). Pricing Rate of Return Guarantees in Regular Premium Unit Linked Insurance. *Insurance: Mathematics and Economics*, 35, pp. 369–398.

[15] Takino, K. (2007). Utility indifference pricing in an incomplete market model with incomplete information. *Tech. Rep., Osaka University, Graduate School of Economics and Osaka School of International Public Policy OSIPP*.

[16] www.lovdata.no (in Norwegian)
LOV 2005-06-10 nr 44: Lov om forsikringsselskaper, pensjonsforetak og deres virksomhet mv.
FOR 2006-06-30 nr 869: Forskrift til forsikringsloven (livsforsikring mv.).

[17] Xu, M. (2006). Risk measure pricing and hedging in incomplete markets. *Annals of Finance*, vol. 2, no. 1, pp. 51-71.

(Pål Nicolai Henriksen), Department of Mathematics, University of Oslo, P.O. Box 1053, Blindern, N–0316 Oslo, Norway
E-mail address: paalnhe@math.uio.no

(Arne Hove), The Norwegian Public Service Pension Fund, Slemdalsveien 37, Postboks 5364 Majorstuen, N–0304 Oslo, Norway
E-mail address: arne.hove@spk.no

(Thilo Meyer-Brandis), Department of Mathematics, University of Munich, Theresienstr. 39, D-80333 Mnchen, Germany
E-mail address: meyerbr@math.uio.no

(Frank Proske), Department of Mathematics, University of Oslo, P.O. Box 1053, Blindern, N–0316 Oslo, Norway

E-mail address: proske@math.uio.no