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Portfolio optimisation in exponential Lévy markets

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Abstract

A multi-variate exponential Lévy market is introduced along with sufficient conditions on existence, uniqueness, and square integrability in the dynamics of processes that represent wealth, both in a self-financing and in a consumption setting. An utility maximisation problem is proposed and sufficient conditions for well-posedness of solutions are stated. In the context of CRRA utility functions, some properties of the set of constant portfolio proportions that have an expected utility above a minimum threshold are analysed.

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A mi Danielita, por cruzar el mar conmigo para seguir un sueño juntos.

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Chapter 1

Introduction and Motivation

There could not be a better way of motivating the importance to study models for portfolio optimisation than the market turmoil that has happened as this text was being written. In fact, during the master's studies that give context to this text, the phrase *"it is a once in a lifetime crisis"* has been often mentioned in the media to refer to events like the covid pandemic, a war in Europe, a potential US-China conflict, and a banking crisis that, as of the delivery of this thesis, caused three banks in the United States to go bankrupt. Credit Swiss, too, had to be forcefully merged by its rival UBS while, oddly, the value of some of its bonds indeed vanished. Just to put a further example, for year end 2022, Norges Bank had -14.1% overall return on its pension fund and such return was still 0.87% higher than its benchmark index, see [Nor23]. All this happening, while inflation has been historically high across the world. Just for reference, the April 2023 Norwegian consumer price index yearly growth was of 6.4%.

Since the seminal work by Bachelier [Bac06] to describe the stock markets, the necessity of having good models that describe the trajectories of assets, has not diminished. In the present text, the log-returns of financial assets will be model by Lévy processes driven diffusions. A good compendium of reasons supporting such a model choice can be found in [CT04, Sections 7.3 and 7.4], however, we briefly enumerate some of the stylised empirical properties of log-returns that are captured with jump diffusion processes and not captured by the classic Black-Scholes model exposed in the seminal paper [BS73]:

- Heavy tails: as originally pointed out by Mandelbrot in [Man63], it is often the case that in short periods of time, one can observe once or more that daily returns on assets' magnitude surpass a threshold that is supposed to be observed not as often under the assumption of normality.

- Distributional asymmetry: usually big daily losses are greater than big daily wins although wins happen more often. This fact is also point out in [Man63]

- Jumps in prize trajectories: although it can sound like harmless mathematical simplification to assume that asset's prices follow a continuous path, a rational investor could have very different behaviour under a discontinuous path market. This assertion has big implications in financial concepts such as leverage and short selling.

- As exposed in [Bar01], when the log-returns of an asset are assumed to follow a standard Brownian motion with stochastic volatility, and the square of the volatility is a generalised inverse Gaussian law type Ornstein-Uhlenbeck process, then the non-instantaneous log-returns of assets can be very well approximated by generalised hyperbolic laws, which are a class of Lévy processes.

1.1 Thesis outline

The thesis would develop as follows:

- Chapter 2 is devoted to give context and state the main results upon which the rest of the thesis would be based on. We will start by introducing Lévy processes and some of their properties in Section 2.1 so that later on, in Section 2.2, we can present results on stochastic differential equations driven by such Lévy processes. Section 2.3 would present results on optimisation problems in the context of general Itô-Lévy diffusions and Section 2.4 would set the context upon which we will seek to optimise. None of the results therein presented are original and most of the theorems and proofs are cited with exception of a corollary that is not specifically stated in any of the main sources consulted. The proof of the Hamilton-Jacobi-Bellman is also shown as it turns to be quite insightful for the development of Section 3.4.

- Chapter 3 is where the thesis itself develops. In Section 3.1 we begin by introducing a multi-variate market model and state the integrabilty condition that will be imposed to the Lévy measures of the processes describing the log-returns of risky assets. Three examples of well known Lévy processes that satisfy such assumptions are also presented. The model herein exposed was established by the author with a lot of inspiration from [ØS19] although later on, after some further research, it turned out to be not too different from other models previously presented in the literature, where the most similarity was found with [EK04].

  Next, Section 3.2 states conditions to be assumed in the dynamics of self-financing portfolios and it is proven that any process satisfying such dynamics would be well-posed. Section 3.3 presents a fixed time horizon portfolio optimisation problem for general utility functions and states sufficient conditions such that a solution can be indeed found. The main emphasis of this section was placed in showing that all the integrability conditions such that the optimisation problem is well-posed are indeed met. The results and proofs herein presented were developed by the author based in results from Chapter 2 and standard tools from probability theory and analysis. An example applied to CRRA (or HARA) utility functions is presented at the end of this section.

  Section 3.4 analyses well-posedness, from a financial perspective, of solutions of the optimisation problem presented in the previous chapter in the context of CRRA utiliy functions. It is shown that the set of constant portfolio proportions whose expected future utility is above a minimum threshold, is a convex set whose boundary conditions are inversely proportional to time to maturity. The supervisor had the original idea on this approach to well-posedness and the analysis itself was developed by the author.

  Finally, Section 3.5 aims to show that the results obtained in Section 3.1, Section 3.2, and Section 3.3 can be easily extended to a consumption setting. The section is mainly based in an an example, in the context of CRRA utility functions, in which the investor presents indifference between consuming wealth or receiving a lump sum after consumption. This example was thought by the author to be original, but turned out to be equivalent to others presented in the literature.

- Chapter 4 presents the concluding remarks and suggests some further lines of research.

Chapter 2

Theoretical background

2.1 Lévy processes

In this section, we will expose some basic theoretical results about Lévy processes. The literature on this topic is rich, however, we will mainly base our exposition on this topic upon [App09], [CT04] and [Sat13].

Definition 2.1.1 (Lévy process). Let $L = \{L(t)\ , t \geq 0\}$ be a stochastic process in $\mathbb{R}^n$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that L is a *Lévy process* if:

1. $L(0) = 0$ $\mathbb{P}$-almost surely.
2. L has *independent increments*, i.e., for each $n \in \mathbb{N}$ and each $0 \leq t_1 < \cdots < t_n < \infty$ the random variables $\{L(t_{j+1}) - L(t_j),\ 1 \leq j \leq n\}$ are independent.
3. L has *stationary increments*, i.e., for all $s, t > 0$, the distribution of $L(s+t) - L(s)$ does not depend on s.
4. L is *stochastically continuous*, i.e., for all $\epsilon > 0$ and for all $t \geq 0$, $\lim_{s \to t} \mathbb{P}\left(|L(s) - L(t)| > \epsilon\right) = 0$

Remark 2.1.2. Note that conditions 1 and 3 above allow us to express condition 4 as $\lim_{s \downarrow 0} \mathbb{P}(|L(s)| > \epsilon) = 0$ as well.

In [Sat13, Section 2.11] it is shown that any Lévy process has a *càdlàg* modification. In the rest of the thesis we will assume to work with such modification. In fact, some sources like [CT04] set such *càdlàg* property as a defining characteristic of Lévy processes.

Definition 2.1.3 (Lévy measure). A Borel measure ν on $\mathbb{R}^n$ is said to be a *Lévy measure* if it satisfies the following conditions:

1. $\nu(\{0\}) = 0$
2. $\int_{\mathbb{R}^n} \min\{|z|^2, 1\} \nu(dz) < \infty$.

Lévy measures contain all the information necessary to understand how often and with which magnituded Lévy processes jump through the following result.

Theorem 2.1.4. *Let $L(t)$ be a Lévy process in $\mathbb{R}^n$ and let $B \in \mathcal{B}(\mathbb{R}^n)$. Then,*

$$\int_B \nu(dz) = \mathbb{E}\left[\#\{t \in [0,1] : \Delta L(t) \neq 0, \Delta L(t) \in B\}\right]. \tag{2.1}$$

Proof. See [CT04, Proposition 3.5]. ■

The definition above characterises Lévy processes with jumps. In fact, any Lévy process can be decomposed into a Brownian motion with drift plus a jumping process. This result is formally given as follows:

Theorem 2.1.5 (Itô-Lévy decomposition). *If $L(t)$ is an n-dimensional Lévy process, then there exist $r \in [0, \infty]$, $b \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times m}$, an m-dimensional Brownian motion $B(t)$, and an independent Poisson random measure on $\mathbb{R}^n \times \mathbb{R}^+$, N, such that:*

$$L(t) = bt + MB(t) + \int_{|z|<r} z\tilde{N}(dz, t) + \int_{|z|\geq r} zN(dz, t). \tag{2.2}$$

Where $\tilde{N}(dz, dt) := N(dz, t) - \nu(dz)t$.

Proof. See [App09, Section 2.4] or [Sat13, Section 4.20]. ∎

Remark 2.1.6. The literature usually refers to the triplet (b, M, ν) as the *characteristic triplet* of the Lévy process L. Such processes can have either of the following sample path characteristics $\mathbb{P}$-almost surely (see [CT04]):

1. It is continuous: this is the case when $\nu = 0$.
2. It is piece-wise straight line: this happens when $M = 0$ and $\int_{|z|<r} \nu(dz) < \infty$. A process with this characteristic is known as a (compensated) *compound Poisson process* and its Lévy measure can be written as follows:

   $$\nu(dz) = \lambda \mathbb{P}_J(dz), \quad \lambda > 0,$$

   where $\mathbb{P}_J(dz)$ is a probability measure. It is called compounded Poisson process because it can be expressed as:

   $$L(t) = \sum_{i \leq N(t)} J_i,$$

   where $N(t) \sim \text{Poi}(\lambda t)$ and $\mathbb{P}_J(dz)$ is the law of the independent copies of the random variable J. It is important to note that any process with a Lévy measure with support bounded away from zero will immediately fall into this category.
3. The process has an infinite amount of jumps in any interval of time when $\int_{|z|<r} \nu(dz) = \infty$. It is also said that it has *infinite jump activity.*
4. It is a process of finite variation: This is the case when $M = 0$ and $\int_{|z|<r} |z|\nu(dz) < \infty$. Conversely, if any of these two conditions is not satisfied, we the process will be of infinite variation.

Note that Theorem 2.1.5 and Remark 2.1.6 do not specify anything about the uniqueness of such decomposition. Indeed, r can be regarded as a "free" (not any Lévy process admits any $r \in [0, \infty]$) variable as long as b in Equation (2.2) is adjusted accordingly. The following two theorems will be the base to setting restrictions in the Lévy measures we will work with and, too, will justify our choice of r for the rest of the text.

Theorem 2.1.7. *Let $L(t)$ be an $\mathbb{R}^n$ dimensional Lévy process. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a non negative function satisfying that for some $k \in \mathbb{R}^+$:*

$$g(x + y) \leq kg(x)g(y) \quad \forall x, y \in \mathbb{R}^n. \tag{2.3}$$

Then,

$$\mathbb{E}\left[g\left(L(t)\right)\right] < \infty \iff \int_{|z|\geq 1} g(z)\nu(dz) < \infty \tag{2.4}$$

Proof. See [Sat13, Theorem 25.3]. ■

As $g(x) = e^x$ is clearly sub-multiplicative, the theorem above will be specially useful in Section 3.1. The next result we will justify our choice of r for the rest of the text.

Theorem 2.1.8. *Let $m \in \mathbb{N}$ and $L(t)$ be an n-dimensional Lévy process. Then*

$$\mathbb{E}[|L(t)|^m] < \infty \iff \int_{|z|\geq 1} |z|^m \nu(dz) < \infty. \tag{2.5}$$

Proof. In [Sat13, Corollary 25.8], the theorem above is presented as a corollary of Theorem 2.1.8 after developing some theory on submultiplicative functions. In [App09, Theorem 2.5.2], the proof relies on the fact that any process with jumps bounded from below has a finite m-th moment. ■

In what follows, we will assume to work with at least square integrable Lévy processes, i.e., processes that fulfil the assumptions of the above theorem if we fix $m = 2$. Consequently, we can choose $r = \infty$ which would let Equation (2.2) to be expressed as:

$$L(t) = bt + MB(t) + \int_{\mathbb{R}^n} z\tilde{N}(dz, t). \tag{2.6}$$

The remaining results will be presented under such assumptions even though they can be easily extended to a more general setting.

Theorem 2.1.9 (Lévy-Khinchine formula for Lévy processes). *Let L be an n-dimensional Lévy process with Itô-Lévy decomposition as in Theorem 2.1.5. Let also $a \in \mathbb{C}^n$ such that*

$$\int_{|z|\geq 1} e^{(\operatorname{Re} a)^\top z} \nu(dz) < \infty. \tag{2.7}$$

If we define the Lévy symbol $\psi_L(a)$ of L as ,

$$\psi_L(a) := b^\top a + \frac{1}{2} a^\top M M^\top a + \int_{\mathbb{R}^n} \left(e^{a^\top z} - 1 - a^\top z \right) \nu(dz), \tag{2.8}$$

then for all $t \geq 0$,

$$\mathbb{E}\left[e^{a^\top L(t)} \right] = e^{t\psi_L(a)}. \tag{2.9}$$

Proof. See [App09, Theorem 1.3.3] for the case when a is imaginary and [Sat13, Theorem 25.17(iii)] to generalise for $a \in \mathbb{C}^n$. ■

One of the most important characteristics of Lévy processes are affine after linear transformations.

Theorem 2.1.10 (Linear transformations of Lévy processes). *Let $L(t)$ be a Lévy process on $\mathbb{R}^n$ with characteristic triplet (b, M, ν). Let $T \in \mathbb{R}^{m\times n}$. Then $Y(t) := TL(t)$ is a Lévy process on $\mathbb{R}^m$ with characteristic triplet (b_Y, M_Y, ν_Y)given by:*

- $b_Y = Tb$,
- $M_Y = TMT^\top$,
- $\nu_Y(U) = \nu(\{x \colon Tx \in U\})$, $\forall U \in \mathcal{B}(\mathbb{R}^n)$.

Proof. See [CT04, Theorem 4.1]. ■

A particular result of the theorem above that we will use in Section 3.1 is that sums of Lévy processes are still Lévy processes.

Corollary 2.1.11. *Let $m, n \in \mathbb{N}$. The sum of m Lévy processes in $\mathbb{R}^n$ is again a Lévy process in $\mathbb{R}^n$.*

Proof. Let $L_i(t)$, $i = 1, \dots, m$ be an n-dimensional Lévy processes. Define $L(t) := (L_1(t)^\top, \dots, L_m(t)^\top)^\top$. Define the matrix T as the $n \times (nm)$ matrix given by:

$$T := \underbrace{\begin{bmatrix} \mathbb{I}_n, & \cdots, & \mathbb{I}_n \end{bmatrix}}_{m \text{ times}}$$

where $\mathbb{I}_n$ is the $n \times n$ identity matrix. Afterwards, apply the previous theorem. ■

If the results above were not convenient enough, there is at least one more kind of transformation that preserves a Lévy process

Theorem 2.1.12. *If $X(t)$ is an univariate Lévy process, $Y(t) := e^{X(t)}$ is also a Lévy process.*

Proof. See [App09, Theorem 5.1.6]. Later we will mention an example and show the resulting characteristic triplets. ■

2.2 Itô-Lévy SDEs and jump diffusions

Now we will present the results of the main stochastic processes that come to our focus. Some of these results can indeed be generalised for a local setting. However, we will hold on to our previously stated assumptions on square integrability.

Definition 2.2.1 (Itô-Lévy SDE). Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete and filtered probability space with such filtration being generated by an m-dimensional Brownian motion $B(t)$ and l 1-dimensional independent compensated Poisson random measures $\tilde{N}_k$ with Lévy measure $\nu = \nu_k$. Let $\alpha \colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\Sigma = \Sigma_{i,j} \colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$, and $\gamma = \gamma_{i,j} \colon [0,T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n \times l}$ be Borel-measurable functions. We define the *Itô-Lévy stochastic differential equation (SDE)* or simply *Itô-Lévy SDE* in $\mathbb{R}^n$ as

$$\begin{aligned} dX(t) &= \alpha\left(t, X(t)\right) dt + \Sigma\left(t, X(t)\right) dB(t) + \sum_{j=1}^{l} \int_{\mathbb{R}^l} \gamma_{\cdot,j}\left(t, X(t^-), z\right) \tilde{N}(dz, dt) \\ X(0) &= x, \quad x \in \mathbb{R}^n, \end{aligned} \tag{2.10}$$

subject to the following condition:

$$\sum_{i=1}^{n} \mathbb{E}\left[\int_0^T \left(|\alpha_i(t,x)| + \sum_{j=1}^{m} \Sigma_{i,j}(t,x)^2 + \sum_{j=1}^{l} \int_{\mathbb{R}} \gamma_{i,j}(t,x,z)^2 \nu_j(dz)\right) dt\right] < \infty, \tag{2.11}$$

which is equivalent as to saying that $\Sigma, \gamma \in L^2\left([0,T] \times \Omega\right)$ and $\alpha \in L^1\left([0,T] \times \Omega\right)$.

Remark 2.2.2. To keep the thesis as self-contained as possible, we have purposely omitted any more general kind of SDE where, for instance, the integrands Σ, γ, need not to be in $L^2\left([0,T] \times \Omega\right)$. Furthermore, some sources like [App09], for instance, define these Itô-Levy SDEs for more general jump measures. In this case, we could relax the notation, and further computations, from $\int_{\mathbb{R}^l} \gamma(z) \tilde{N}(dz, dt)$ to $\sum \int_{\mathbb{R}} \gamma(z) \tilde{N}(dz, dt)$ by letting the support of ν to be the coordinate axis of $\mathbb{R}^n$. From a modelling perspective, this means that none of the jump components can jump at the same time (see [EK04] or [Sat13, Section 12]) or, equivalently, that the Poisson random measures are independent among each others, as stated in Definition 2.2.1.

Similarly to ordinary or partial differential equations, it is also relevant to ask about the existence, uniqueness, and stability of solutions to problems of the form of Equation (2.10). The next theorem states sufficient conditions to have existence and uniqueness in such problems and also gives a sense of stability of the solutions, at least in a stochastic sense.

Theorem 2.2.3 ([ØS19, Theorem 1.19]). *Let $X(t)$ be a stochastic differential equation as in Definition 2.2.1. Let M be a real matrix and define $\|M\|^2 :\ = \sum_{i,j=1}^{k}(MM^\top)_{i,j}$. Assume the following:*

1. *(At most linear growth) there exists a constant $K_1 < \infty$ such that:*

$$|\alpha(t,x)|^2 + \|\Sigma(t,x)\|^2 + \sum_{k=1}^{l}\int_{\mathbb{R}} |\gamma_{\cdot,k}(t,x,z)|^2 \nu_k(dz_k) \le K_1(1+|x|)^2 . \tag{2.12}$$

for all $x \in \mathbb{R}^n$.

2. *(Lipschitz continuity) there exists a constant $K_2 < \infty$ such that:*

$$\begin{aligned} &|\alpha(t,x_1)-\alpha(t,x_2)|^2 + \|\Sigma(t,x_1)-\Sigma(t,x_2)\|^2 \\ &+ \sum_{k=1}^{l}\int_{\mathbb{R}} |\gamma_{\cdot,k}(t,x_1,z)-\gamma_{\cdot,k}(t,x_2,z)|^2 \nu_k(dz_k) \le K_2|x_1-x_2|^2 . \end{aligned} \tag{2.13}$$

for all $x_1, x_2 \in \mathbb{R}^n$.

Then there exists a unique càdlàg adapted solution $X(t)$ such that

$$\mathbb{E}\left[|X(t)|^2\right] < \infty$$

for all t.

Proof. See[App09, Section 6.2] for a proof involving more general types of integrators (not necessarily independent jump measures) and more general types of integrands that do not need to be constrained by the assumption stated in Equation (2.11). ■

Now, let us address the issue of how to find solutions to problems like Equation (2.10).

Theorem 2.2.4 (Itô's Formula for Itô-Lévy SDE's). *Let $X(t)$ be a Itô-Lévy process as in Definition 2.2.1. Let also $f \in C^{1,2}([0,T], \times\mathbb{R}^n)$. Define $Y(t) :\ = f(t,X(t))$. Then*

$$\begin{aligned} dY(t) =& \partial_t f(t,X(t))dt + \sum_{i=1}^{n}\partial_{x_i} f(t,X(t))\left(\alpha_i(t,X(t))dt + \Sigma_{i,\cdot}(t,X(t))dB(t)\right) \\ &+ \frac{1}{2}\sum_{i,j=1}^{n}(\Sigma\Sigma^\top)_{i,j}(t,X(t))\partial_{x_ix_j} f(t,X(t)) \\ &+ \sum_{k=1}^{l}\int_{\mathbb{R}}\Bigg(f(t,X(t^-)+\gamma_{\cdot,k}(t,X(t),z)) - f(t,X(t^-)) \\ &- \sum_{i=1}^{n}\gamma_{i,k}(t,X(t),z)\partial_{x_i} f(t,X(t))\Bigg)\nu_k(dz_k)dt \\ &+ \sum_{k=1}^{l}\int_{\mathbb{R}}\left(f(t,X(t^-)+\gamma_{\cdot,k}(t,X(t^-),z) - f(t,X(t^-))\right)\tilde{N}(dz,dt) \end{aligned} \tag{2.14}$$

Proof. As in Theorem 2.2.3, [App09, Section 6.2] shows a proof for more general Lévy measures and more general integrands. Intuitively, the formula is a chain rule for deterministic functions with a stochastic argument where the stochastic process satisfies Definition 2.2.1. It is very similar to the classical Itô formula but with the addition of the compensated jumps and the effect of compensating such jumps. ■

Remark 2.2.5. Theorem 2.2.4 should be evaluated at its left-limit to preserve the *càdlàg* property of X into Y. To compact the notation, we wrote t instead of t^- in the elements of the formula not related to the jumps and the process X.

Theorem 2.2.4 above has an immediate application to understand the variance (and martingality) of the stochastic integrals that result from Definition 2.2.1.

Theorem 2.2.6 (Itô Isometry). *Let $X(t) \in \mathbb{R}^n$ be as in Definition 2.2.1 such that $X(0) = 0$ and $\alpha = 0$. Then,*

$$\begin{aligned} &\mathbb{E}\left[\left|\int_0^T \Sigma\left(t, X(t)\right) dB(t) + \int_0^T \sum_{j=1}^{l} \int_{\mathbb{R}^l} \gamma_{\cdot,j}\left(t, X(t^-), z\right) \tilde{N}(dz, dt)\right|^2\right] \\ &\quad = \sum_{i=1}^{n} \mathbb{E}\left[\int_0^T \left(\sum_{j=1}^{m} \Sigma_{i,j}(t,x)^2 + \sum_{j=1}^{l} \int_{\mathbb{R}} \gamma_{i,j}(t,x,z)^2 \nu_j(dz)\right) dt\right] \end{aligned} \tag{2.15}$$

Proof. As we have assumed that Equation (2.11) holds, the result above is consequence of applying Itô's formula to $f(t, X(t)) = |X(t)^2|$. However, the isometry is fundamental in understanding the construction of stochstic integrals themselves. For details, see [App09, Lemma 4.2.2] or [Øks13, Lemma 3.1.5]. ■

Next, we will define the main type of processes that we will study for the rest of the thesis:

Definition 2.2.7 (Itô-Lévy Diffusion). Consider a stochastic differential equation of the form of Definition 2.2.1, then it is said to be a *Itô-Lévy diffusion, jump diffusion*, or simply a *diffusion* if it is time homogeneous, i.e., when $\alpha(t,x) = \alpha(x)$, $\Sigma(t,x) = \Sigma(x)$, and $\gamma(t,x,z) = \gamma(x,z)$

Note however that any equation of the form Definition 2.2.1 can be re-written to be an Itô diffusion by defining a new process $\bar{X}(t) := (t, X(t)^\top)^\top$, by letting $\bar{\alpha} := (1, \alpha^\top)^\top$ and adding a zero row on top of Σ and γ to make $\bar{\Sigma}$ and $\bar{\gamma}$, respectively. In the rest of the thesis, we will work with processes satisfying both conditions in Theorem 2.2.3 and which can be written as $\bar{X}(t)$ (although for convenience we will drop the bar).

Theorem 2.2.8. *A jump diffusion is a Markov process.*

Proof. See [App09, Theorem 6.3.1]. ■

Our *best guess* of the future state of an Itô diffusion is, by its Markovianity, a function of its current value. Let us then mention some results relates to such *best guesses.*

Definition 2.2.9 (Generator of a diffusion). Let $X(t)$ be an Itô-Lévy diffusion as in Definition 2.2.7 that complies with the assumptions of Definition 2.2.1. Then the *generator* A of $X(t)$ is defined on functions $f : \mathbb{R}^n \to \mathbb{R}$ by

$$Af(x) = \lim_{t\to 0^+} \frac{\mathbb{E}_x\left[f\left(X(t)\right)\right] - f\left(X(t)\right)}{t} \tag{2.16}$$

where such limit exists and where $\mathbb{E}_x\left[f\left(X(t)\right)\right] = \mathbb{E}\left[f\left(X(t)\right)|X(0)=x\right]$. The set of functions such that the limit exists at x will be denoted as $\mathcal{D}_A(x)$ while $\mathcal{D}_A$ denotes the set of functions for which the limit exists for all $x \in \mathbb{R}^n$ (or $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ according to the context).

Remark 2.2.10. We could have as well written $\mathbb{E}\left[f\left(X(t)\right)|\mathcal{F}_0\right]$ instead of $\mathbb{E}_x\left[f\left(X(t)\right)\right]$ in the definition above, but we will omit this notation to stress on the Markovianity of diffusions and to avoid an excessive use of it.

Let us now give a explicit representation to the diffusion generators just defined.

Theorem 2.2.11 ([ØS19, Theorem 1.22]). *Suppose $g \in C_0^2(\mathbb{R}^n)$, i.e., $f \in C^2(\mathbb{R}^n)$ such that $\lim_{|x|\to\infty} f(x) = 0$. Let $X(t) \in \mathbb{R}^n$ be a jump diffusion. Then $Af(x)$ exists and is given by*

$$\begin{aligned} Af(x) =& \sum_{i=1}^{n} \alpha_i(x)\partial_{x_i} f(x) + \frac{1}{2}\sum_{i,j=1}^{n} (\Sigma\Sigma^\top)_{i,j}\partial_{x_i x_j} f(x) \\ &+ \sum_{k=1}^{l}\int_{\mathbb{R}} \left(f\left(x + \gamma_{\cdot,k}(x,z)\right) - f(x) - Df(x)\cdot\gamma_{\cdot,k}(x,z)\right)\nu_k(dz_k). \end{aligned} \tag{2.17}$$

Proof. Formally, the argument relies on operator theory and on the Lévy-Khinchine decomposition. See [App09, Theorem 3.3.3] or [Sat13, Theorem 31.5]. Heuristically however, one can observe that the formula is the same as Theorem 2.2.4 without the random terms. ∎

Remark 2.2.12. Note that if $\nu = 0$ and $\Sigma = I_n$, then $Af(x)$ will be the the Laplace operator, Δf, or the heat operator, $\Delta f + \partial_t f$, depending if $\alpha = 0$ or $\alpha = e_1$. In the latter case, we would also need that $\bar{X}(t) = (t, X(t)^\top)^\top$ as previously discussed.

The next theorem will be a cornerstone in the rest of the thesis as it gives an expression explicitly to compute *best guesses* of the future state of diffusions.

Theorem 2.2.13 (Dynkin's formula). *Let $X(t) \in \mathbb{R}^n$ be a jump diffusion, let $\mathcal{S} \in \mathbb{R}^n$ be an open set and let $g \in C^2(\mathbb{R}^n)$. Suppose that $\tau < \infty$ such that $\tau \leq \tau_{\mathcal{S}} \colon = \inf\{t > 0; X(t) \notin \mathcal{S}\}$. Furthermore assume the integrability condition:*

$$\mathbb{E}_x\left[|g(X(\tau))| + \int_0^\tau |Ag(X(t))|\,dt\right] < \infty \tag{2.18}$$

Then we can we get the following:

$$\mathbb{E}_x\left[g(X(\tau))\right] = g(x) + \mathbb{E}_x\left[\int_0^\tau Ag(X(t))dt\right]. \tag{2.19}$$

Proof. See [ØS19, Theorem 7.24] for a formal proof. Heuristically, what happens is that $g(X(\tau))$ is expressed using Itô's formula in its integral form and the stochastic integrals vanish because their martingality is guaranteed by the integrability condition given in Equation (2.18). ∎

In what follows, we will do as in [ØS19], and will compute $Af(x)$ by the expression in Equation (2.17) for all f such that the derivatives and integrals involved are well defined (regardless if f is not a function such that $\lim_{|x|\to\infty} f(x) = 0$ as in Theorem 2.2.11) because the Theorem 2.2.13 shows that we can compute $Af(x)$ as in Equation (2.17) as long as Equation (2.18) holds.

Remark 2.2.14. Letting $\tau = t$ in Theorem 2.2.13, it is easy to observe that the process $g(X(t)) - \int_0^t Ag(X(s))ds$ is indeed a martingale.

2.3 Stochastic control

Now, we will focus on optimisation problems that involve taking decisions under uncertainty. Hence, a new stochastic process similar to that in Definition 2.2.1 will be introduced with the caveat of having a new argument for a choice process or *steering wheel* to implement decisions under uncertainty.

Definition 2.3.1 (Control process, [ØS19]). Let $\mathcal{S} \subset \mathbb{R}^n$ and $U \in \mathbb{R}^p$. The process $u(t) = u(t,\omega)\colon [0,\infty)\times\Omega \to U$ is a control process assumed to be càdlàg and adapted. If the process is a function $u(t) = u(Y(t)) : \Omega \to U$, we say that it is a *Markovian control*.

Similarly, we call $Y(t) = Y_u(t)$ a *controlled diffusion* whose dynamics can be explicitly written as:

$$\begin{aligned} dY(t) &= \alpha\left(Y(t), u(t)\right) dt + \Sigma\left(Y(t), u(t)\right) dB(t) + \int_{\mathbb{R}^n} \gamma\left(Y(t^-), u(t^-), z\right) \tilde{N}(dz, dt) \\ Y(0) &= y, \quad y \in \mathbb{R}^n. \end{aligned} \tag{2.20}$$

Now we will give the context in which we care to steer our control, in other words, our *objective function*.

Definition 2.3.2 (Performance Criterion). Let

$$\tau_{\mathcal{S}} = \inf\{t > 0; \quad Y_u(t) \notin \mathcal{S}\}$$

Define $J = J_u(x)$ as

$$J_u(y) = \mathbb{E}_y\left[\int_0^{\tau_{\mathcal{S}}} f\left(Y(t), u(t)\right) dt + g\left(Y(\tau_{\mathcal{S}})\right) \mathbf{1}\{\tau_{\mathcal{S}} < \infty\}\right] \tag{2.21}$$

such that $f\colon \mathcal{S}\times U \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given functions. We say that the process u is *admissible* and write $u \in \mathcal{A}$ if the stochastic differential equation that appears in Definition 2.3.1 has a unique, strong solution $Y(t)$ for all $y \in \mathcal{S}$ and

$$\mathbb{E}\left[\int_0^{\tau_{\mathcal{S}}} f^-\left(Y(t), u(t)\right) dt\right] < \infty \tag{2.22}$$

We would also need that the family $\{g^-\left(X(\tau)\right)\}_{\tau\leq\tau_{\mathcal{S}}}$ is uniformly integrable.

The stochastic control problem is to find the value function $\Phi(x)$ and optimal control $u_* \in \mathcal{A}$ such that:

$$\Phi(y) = \sup_{u\in\mathcal{A}} J_u(y) = J_{u_*}(y). \tag{2.23}$$

Definition 2.3.1 is general in the sense that it requires u to be adapted. However, under some conditions, the optimal adapted control will be the same as the optimal Markovian control. The next theorem will enumerate sufficient conditions needed to be in such a convenient situation:

Theorem 2.3.3 (Equivalence of adapted and Markovian controls). *Let $\Phi_M(y) = \sup\{J_u(y)\colon\ u = u(Y)$ is a Markov control$\}$ and $\Phi_a(y) = \sup\{J_u(y)\colon u = u(t,\omega)$ is an $\mathcal{F}_t$-adapted control$\}$. Suppose there exists an optimal Markov control u_0 for all $y \in \mathcal{S}$ and such that:*

1. *all the boundary points of $\mathcal{S}$, $\partial\mathcal{S}$, are regular with respect to $Y_{u_0}(t)$, i.e., $\mathbb{P}_y(\tau_{\mathcal{S}} = 0) = 1$ for all $y \in \partial\mathcal{S}$, $\tau_{\mathcal{S}} = \inf\{t > 0; \quad Y_u(t) \notin \mathcal{S}\}$.*

2. *$\Phi_M \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$ is bounded in $\mathcal{S}$.*
3. *$\mathbb{E}_y\left[|\Phi_M(Y_\tau)| + \int_0^\tau |A\Phi_M(Y(t))|\, dt\right] < \infty$ for all bounded stopping times $\tau \leq \tau_{\mathcal{S}}$, $y \in \mathcal{S}$.*

Then, $\Phi_M = \Phi_a$ for all $y \in \mathcal{S}$.

Proof. See [Øks13, Theorem 11.2.3]. ■

The next theorem is an important result that later on will help us to find the value of the optimal performance criterion.

Theorem 2.3.4 (Hamilton-Jacobi-Bellman for Optimal Control of Jump Diffusions). *Let $\phi_u = \phi$ be a function $\phi\colon \mathcal{S} \to \mathbb{R}$. Then,*

1. *Suppose $\phi \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$ satisfies:*
   (a) *$A\phi(y) + f(y, v) \leq 0$ for all $y \in \mathcal{S}$, $v \in U$.*
   (b) *$\phi(y) = g(y)$ for all $y \in \partial\mathcal{S}$.*
   (c) *$\mathbb{E}_y\left[|\phi(Y(\tau))| + \int_0^\tau |A\phi(Y(t))|\, dt\right] < \infty$ for all $u \in \mathcal{A}$ and all $\tau \in \mathcal{T}$ where $\mathcal{T}$ represents the set of all stopping times $\tau \leq \tau_{\mathcal{S}}$.*
   (d) *$\{\phi^-(Y(\tau))\}_{\tau \leq \tau_{\mathcal{S}}}$ is uniformly integrable for all $u \in \mathcal{A}$ and $y \in \mathcal{S}$.*

   Then:
   $$\phi(y) \geq \Phi(y) \quad \text{for all } y \in \mathcal{S} \tag{2.24}$$
2. *Moreover, suppose that for each $y \in \mathcal{S}$ there exists a $\hat{u}(y) \in U$ such that $\hat{u}(y)$ maximises $u(y) \mapsto A\phi_{u(y)} + f(y, u(y))$ and such that:*
   (a) *$A\phi_{\hat{u}(y)} + f(y, \hat{u}(y)) = 0$ and*
   (b) *$\{\phi^-(Y_{\hat{u}}(\tau))\}_{\tau \leq \tau_{\mathcal{S}}}$ is uniformly integrable.*

   Suppose $u_*(t)\colon = \hat{u}(Y(t^-)) \in \mathcal{A}$. Then u_* is an optimal Markov control and:
   $$\phi(y) = \Phi(y) = J_{u_*}(y) \quad \text{for all } y \in \mathcal{S} \tag{2.25}$$

Proof. At a glance, the wording of the theorem can seem rather confusing. However, it is important to note the the first part of the claim is related to the sufficient conditions to have an optimal performance criterion whereas the second part of it enumerates the necessary conditions to have an optimal performance criterion. In what follows, we will follow the proof given in [Øks13].

1. Let $u \in \mathcal{A}$. For $n = 1, 2, \dots$ put $\tau_n = \min(n, \tau_{\mathcal{S}})$. Then, by Theorem 2.2.13:
   $$\mathbb{E}_y\left[\phi(Y(\tau_n))\right] = \phi(y) + \mathbb{E}_y\left[\int_0^{\tau_n} A\phi_u(Y(t))dt\right].$$
   By assumption *1a)* we get the following inequality:
   $$\mathbb{E}_y\left[\phi(Y(\tau_n))\right] \leq \phi(y) - \mathbb{E}_y\left[\int_0^{\tau_n} f(Y(t), u(t))dt\right]$$
   $$\implies \mathbb{E}_y\left[\phi(Y(\tau_n)) + \int_0^{\tau_n} f(Y(t), u(t))dt\right] \leq \phi(y).$$

Taking n to the limit and subsequently applying Fatou's lemma:

$$\begin{aligned}\phi(y) \geq& \liminf_{n\to\infty} \mathbb{E}_y \left[\phi(Y(\tau_n)) + \int_0^{\tau_n} f(Y(t), u(t))dt\right] \\ \geq& \mathbb{E}_y \left[g\left(Y(\tau_{\mathcal{S}})\right) \mathbf{1}\{\tau_{\mathcal{S}} < \infty\} + \int_0^{\tau_{\mathcal{S}}} f\left(Y(t), u(t)\right) dt\right] = J_u(y)\end{aligned}$$

Where the last step was done by considering assumption *1b)*. Note as well that assumptions *1c)* and *1d)* allowed us to give sense to the integrals above. Since u was arbitrary we conclude that

$$\phi(y) \geq \Phi(y) \quad \text{for all } y \in \mathcal{S}. \tag{2.26}$$

2. Applying the same logic as in the previous steps to $u_*(t)$ we get an equality, i.e.,

$$\phi(y) = J_{u_*}(y) \leq \Phi(y). \tag{2.27}$$

Where the last inequality above is given by definition of Φ. Finally, combining Equation (2.26) and Equation (2.27), we get Equation (2.25).

■

2.4 Utility theory

It is a well known principle in non-life insurance that premiums should be greater than the expected value of the potential losses. See [Sch17, Section 1.10] for a good account of reasons why this holds true. At a first glance, it sounds like a flawed principle because by buying such insurance we indeed can expect to reduce our wealth. However, this assessment turns out to be wrong because it ignores the fact that individuals and entities are *risk averse*. Such risk aversion was initially pointed out by Daniel Bernoulli through the *St. Petersburg paradox*, which says that it is not reasonable for anyone to pay an arbitrary large amount of money to play a game where the expected outcome is ∞. Indeed, this paradox is a reformulation of the fact that the marginal utility of an amount of wealth is decreasing. For instance, a billionaire would gain less satisfaction by a 1 unit increase of wealth than a recently graduated student with no wealth. Risk aversion is then the feature that describes investors' preference of certainty over uncertainty, even if it is at expense of the expected wealth. See [Ang14, Section 2.2] for a further motivation on risk aversion or [Pro15, Section 22.2] to analyse it through a more actuarial perspective.

Mathematically, risk aversion is measured through utility functions, which in the context of this thesis would be functions of an investor's wealth (or consumption).

Definition 2.4.1 (Utility function). Let $\mathcal{W} \subseteq \mathbb{R}$ be an open set that is not bounded above i,e, $\mathcal{W} = (-q, \infty)$, $q \in (-\infty, \infty]$. A *utility function* is defined as $g \in C^2(\mathcal{W})$ such that, for all $W \in \mathcal{W}$:

1. $g'(W) > 0$.
2. $g''(W) < 0$.

With this kind of functions in mind, it is more natural to define risk aversion.

Definition 2.4.2 (Risk aversion). Given an utility function $g(W)$, an investor's *risk aversion*, $a(W)$, is given by:

$$a(W) = -\frac{g''(W)}{g'(W)} = -\frac{d}{dW}\ln g'(W) \tag{2.28}$$

Analogously, an investor's *relative risk aversion* is given by the value $Wa(W)$. Sometimes as well, the reciprocal of the risk aversion, $1/a(W)$, is called the *risk tolerance*.

See [Sch17, Section 2.1] for a thorough justification on the above assumptions on g and an explanation on what a actually describes.

Remark 2.4.3. Let $k_1 \in \mathbb{R}$ and $k_2 > 0$. Note that $a(W)$ defined as in Equation (2.28) would be the same for a function $f(W) := k_1 + k_2 g(W)$. This fact will turn out to be useful later on.

Many sources, like for instance [Mer73] or [Sch17, Section 2.1], usually assume $a(W)$ to be a decreasing function (although it is always positive) as this represents that the wealthier an individual is, the more risk she would be willing to take. In fact, this assumption will motivate the most relevant examples that are to appear in the thesis, specifically, utility functions of the form:

$$g(W) = \frac{W^\eta}{\eta}, \quad \eta < 1, \quad \eta \neq 0. \tag{2.29}$$

Utility functions like the one above are said to have *constant relative risk aversion* (CRRA) or *hyperbolic absolute risk aversion* (HARA). In [Ang14, Section 2.3], it is mentioned that empirical studies show that $\eta \in (-9, 0)$ for individual investors.

Remark 2.4.4. In economics texts this utility function is usually parameterised as $g(W) = W^{1-\eta}/(1-\eta)$, $\eta > 0$, $\eta \neq 1$ so that η can be interpreted as a relative risk aversion parameter. We omit this parameterisation here to compact the notation.

One last characteristic of utility functions that is a direct consequence of Definition 2.4.1 is that they are concave. The following result would therefore turn out to be useful later on.

Theorem 2.4.5 (Jensen's inequality for concave functions). *Let λ be a finite measure on $E \subseteq \mathbb{R}$. Let h be a strictly concave function defined on E. If $\int_E |h(z)|\lambda(dz) < \infty$ then,*

$$h\left(\int_E z\lambda(dz)\right) > \int_E h(z)\lambda(dz). \tag{2.30}$$

Proof. See [CK98, Theorem 7.46] for a measure theoretic proof of this result for convex functions or see [Sch17, Theorem G.6] for a more direct proof tailored for concave functions in the context of expectations. ∎

Chapter 3

Methodology and results

3.1 The market model

Let $m \in \mathbb{N}$ and $m \in \mathbb{N} \cup 0$ and let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete and filtered probability space with such filtration being generated by an m-dimensional Brownian motion $B(t)$ and l 1-dimensional independent compensated Poisson random measures $\tilde{N}_k$ with Lévy measure $\nu = \nu_k$.

Let $S = S(t)$ denote the vector of $n+1$ markets asset's value at time t of which $S_i(0) = s_i$, $i = 0, \ldots, n$ are known quantities. S_0 is a risk-free bank account and S_i, $i = 1, \ldots, n$, represent the n risky assets in the market. Such vector containing only the risky assets will be denoted as $\hat{S}$. As mentioned in Section 2.1, we will assume that the log-returns of a pool of assets are described by a jump diffusion as in Definition 2.2.7. Let then $X = X(t)$ denote the $n+1$ vector of instant log-returns of S at time t such that $X(0) = 0$. In other words, let $S_i(t) = s_i e^{X_i(t)}$ for $i = 0, \ldots, n$. Similarly as with $\hat{S}$, $\hat{X}$ will represent the vector of log-returns for the n risky assets.

.The explicit dynamics of the market log-returns are the following:

$$\begin{aligned}\begin{bmatrix} dX_0(t) \\ dX_1(t) \\ \vdots \\ dX_n(t) \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} dt + \begin{bmatrix} 0 & \cdots & 0 \\ \sigma_{1,1} & \cdots & \sigma_{1,m} \\ \vdots & & \vdots \\ \sigma_{n,1} & \cdots & \sigma_{n,m} \end{bmatrix} \begin{bmatrix} dB_1(t) \\ \vdots \\ dB_m(t) \end{bmatrix} \\ + \int_{\mathbb{R}} \begin{bmatrix} 0 & \cdots & 0 \\ \gamma_{1,1} z & \cdots & \gamma_{1,l} z \\ \vdots & & \vdots \\ \gamma_{n,1} z & \cdots & \gamma_{n,l} z \end{bmatrix} \begin{bmatrix} \tilde{N}_1(dz, dt) \\ \vdots \\ \tilde{N}_l(dz, dt) \end{bmatrix},\end{aligned} \tag{3.1}$$

where $\alpha_0 \in \mathbb{R}$, $\alpha_i > \alpha_0$ and $\alpha_i > 0$ for all $i \in 1, \ldots, n$; $\sigma_{i,j} \in \mathbb{R}$ for all $i \in 1, \ldots, n$, $j \in 1, \ldots, m$; and $\gamma_{i,j} \in [-1, 1]$ for all $i \in 1, \ldots, n$, $j \in 1, \ldots, l$. The short-hand notation for the equation above will then be:

$$\begin{aligned} dX(t) =& \alpha dt + \Sigma dB(t) + \int_{\mathbb{R}} \gamma(z) \tilde{N}(dz, dt) \\ =& \alpha dt + \Sigma dB(t) + \sum_{j=1}^{l} \int_{\mathbb{R}} \gamma_{\cdot,j}(z) \tilde{N}_j(dz, dt). \end{aligned} \tag{3.2}$$

Similarly as with the asset's notation, we will denote the matrices $\hat{\Sigma}$ and $\hat{\gamma}(z) = \hat{\gamma} z$ as the the versions without the first 0 row of Σ and $\gamma(z)$, respectively. Furthermore, we will constraint our model such that $\hat{\Sigma}\hat{\Sigma}^\top$ or $\hat{\gamma}\hat{\gamma}^\top$ are invertible, which in this case is equivalent to being positive definite.

The model above is similar to a multivariate Black-Scholes model with the addition of the independent jumps that can affect either one or several stocks at the same time, depending on the shape of γ. It is also an alternative with several risk sources (although assuming that the market has an influence by at least one Brownian motion) to the model presented in [BKR01] and is also a generalisation of the model proposed in [EK04] because we do not constrain the amount of pure jump Lévy processes, in other words, we do not necessarily assume that $l \leq n$. It is also important to note that $\gamma_{i,j} \in [-1, 1]$ is equivalent to saying $\gamma_{i,j} \in \mathbb{R}$ because Theorem 2.1.10 states that we can re-scale the Lévy measures accordingly. We then choose to give this shape to $\gamma(z)$ to avoid an excessive use of notation when stating integrability conditions for the Lévy measures.

Proposition 3.1.1. *There exists a unique square integrable Lévy process that solves Equation* (3.1).

Proof. Recall that we assume that $\int_{\mathbb{R}^n} |z|^2 \nu(dz) < \infty$, hence, the existence, uniqueness, and square integrability part of the statement are a direct consequence of Theorem 2.2.3 as it is trivial to check Lipschitz continuity and linear growth for α, Σ, and γ as all of these functions are constant mappings with respect to the variable $x = X(t)$.

To prove that $X(t)$ is indeed a Lévy process, let us write Equation (3.2) in integral form:

$$X(t) = \alpha t + \Sigma B(t) + \int_{\mathbb{R}} \gamma(z) \tilde{N}(dz, t).$$

The first two terms above are easily recognisable from the Lévy-Itô decomposition, Theorem 2.1.5. Let us analyse then the term of the Poisson random measure:

$$\begin{aligned}
\int_{\mathbb{R}} \gamma(z) \tilde{N}(dz, t) = \sum_{j=1}^{l} \int_{\mathbb{R}} \gamma_{\cdot,j}(z) \tilde{N}_j(dz, t) &= \sum_{j=1}^{l} \int_{\mathbb{R}} \begin{bmatrix} 0 \\ \gamma_{1,j} z \\ \vdots \\ \gamma_{n,j} z \end{bmatrix} \tilde{N}_j(dz, t) \\
&= \sum_{j=1}^{l} \int_{\mathbb{R}} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \gamma_{1,j} & 0 & \cdots & 0 \\ 0 & \gamma_{2,j} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \gamma_{n,j} \end{bmatrix} \begin{bmatrix} 0 \\ z \\ \vdots \\ z \end{bmatrix} \tilde{N}_j(dz, t) \\
&= \sum_{j=1}^{l} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \gamma_{1,j} & 0 & \cdots & 0 \\ 0 & \gamma_{2,j} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \gamma_{n,j} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \int_{\mathbb{R}} z \tilde{N}_j(dz, t).
\end{aligned}$$

By the equation above becomes clear that we can apply Theorem 2.1.10 and Corollary 2.1.11 with which we can conclude that indeed $X(t)$ is a Lévy process. ■

Now that we have investigated the dynamics of the log-returns of the market, we will investigate the explicit dynamics of the assets, which by Theorem 2.1.12 we already know that are still Lévy processes. Starting with the risk-free asset, it is clear that

$$dS_0(t) = \alpha_0 S_0(t) dt \tag{3.3}$$

The dynamics of the risky assets can be obtained by applying Theorem 2.2.4, Itô's formula, to $S_i(t) = f(X_i(t)) = s_i e^{X_i(t)}$ for $i = 1, \ldots, n$. The calculation is as follows:

$$\begin{aligned} dS_i(t) =& S_i(t^-) \Bigg[\left(\alpha_i + \frac{1}{2} \left(\hat{\Sigma} \hat{\Sigma}^\top \right)_{i,i} + \sum_{j=1}^{l} \int_{\mathbb{R}} \left(e^{\gamma_{i,j} z} - 1 - \gamma_{i,j} z \right) \nu_j(dz) \right) dt \\ & + \hat{\Sigma}_{i,\cdot} dB(t) + \sum_{j=1}^{l} \int_{\mathbb{R}} \left(e^{\gamma_{i,j} z} - 1 \right) \tilde{N}(dz, dt) \Bigg] . \end{aligned} \tag{3.4}$$

Note that the equation above shows us an additional constraint that the Lévy measures of the jumps must satisfy so that the stochastic differential equation above is well defined or, in other words, so that our market makes sense. By virtue of Theorem 2.1.7 and Theorem 2.1.8, we would only need to check that for all $i \in 1, \ldots, n$, $j \in 1, \ldots, l$:

$$\int_{|z| \geq 1} e^{\gamma_{i,j} z} \nu_j(dz) < \infty.$$

Now, to check that $S_i(t) \in L^2(\Omega \times [0,T])$ we could use the condition stated in Equation (2.11). However, because we have proven in Proposition 3.1.1 that $X_i(t)$ is a Lévy process and we know that the explicit form of $S_i(t)$ is $S_i(t) = e^{X_i(t)}$, we can use the Lévy-Khinchine formula as an alternative way to check square integrability. By virtue of Theorem 2.1.9, the additional integrability condition has then to be observed for all $j = 1, \ldots, l$:

$$\int_{|z| \geq 1} e^{2z} \nu_j(dz) < \infty.$$

For modelling purposes, it is convenient to have some flexibility in the choice of the parameters $\gamma_{i,j} \in [-1, 1]$ that descibe how jumps can collectively affect a set of stocks. Therefore, we will assume for the rest of the thesis that whatever Lévy measure we work with, will have to satisfy the following condition:

$$\int_{|z| \geq 1} e^{2|z|} \nu_j(dz) < \infty. \tag{3.5}$$

Remark 3.1.2. It is important to note that any Lévy process with $\mathbb{P}$-almost surely bounded jumps will immediately satisfy Equation (3.5).

The following examples will illustrate some pure jump Lévy processes with wide financial applications that satisfy Equation (3.5) in the cases mentioned in Remark 2.1.6 of: finite activity, infinite activity but finite variation, and infinite activity with infinite variation, respectively.

Example 3.1.3 (Compound Poisson process). Let $\mathbb{P}_J(z)$ be the probability density function of the jumps of a compound Poisson process with Lévy measure ν, then it follows by Remark 2.1.6 that

$$\int_{|z| \geq 1} e^{2|z|} \nu(dz) < \infty \iff \int_{|z| \geq 1} e^{2|z|} \mathbb{P}_J(dz) < \infty. \tag{3.6}$$

Particularly, this holds when the jump sizes are normally distributed, which is a process proposed in [Mer76].

In Section 2.1 we cited [Bar01] and mentioned that the non-instantaneous log-returns of some stochastic volatility models can be very well approximated by some Lévy type stochastic processes. The following two examples correspond to such a category and correspond, too, to the class of subordinated Brownian motion processes which, simply put, are classical Brownian motions evaluated at a random time called the subordinator. See [CT04, Section 4.4.2-4.4.3] for further details.

Example 3.1.4 (Variance-gamma process). Assume that $T(t)$ is Gamma process with parameters $a > 0$ and $b > 0$, such that

$$\mathbb{E}\left[e^{cT(t)}\right] = \left(1 - \frac{c}{b}\right)^{-at}, \quad c < b.$$

Let $Y(t) := \sigma B\left(T(t)\right)$ where $\sigma > 0$ and $B(t)$ is a standard Brownian motion. The Lévy measure of $Y(t)$ can be consulted in [CT04, Table 4.5] and is given by:

$$\nu_Y(dz) = \frac{b^2}{a|z|}\exp\left\{-\left(\frac{2b^2}{a\sigma^2}\right)^{\frac{1}{2}}|z|\right\}dz.$$

Hence, Equation (3.5) will hold if

$$\frac{b^2}{a\sigma^2} = \frac{1}{\sigma^2\mathbb{V}[T(1)]} > 2. \tag{3.7}$$

In [CG19] and [MCC98] one can find particular financial examples in which modelling through variance-gamma processes is useful.

Example 3.1.5 (Negative inverse Gaussian process). Similarly as with the previous example, consider a model of the form $Y(t) = \sigma B\left(T(t)\right)$ where $\sigma > 0$ and $B(t)$ is a standard Brownian motion. However, assume this time that $T(t)$ is an inverse Gaussian process as described in [FC78] with parameters $a, b > 0$, such that

$$\mathbb{E}\left[e^{cT(t)}\right] = \exp\left\{\frac{at}{b}\left(1 - \sqrt{1 - \frac{2b^2c}{a}}\right)\right\}, \quad c < \frac{a}{2b^2}.$$

Let us define $B := (\sigma^2\mathbb{V}[T(1)])^{-1/2} = \sqrt{a/b^3}/\sigma$, then the Lévy measure of $Y(t)$ can also be consulted in [CT04, Table 4.5] and is given by:

$$\nu_Y(dz) = \frac{a}{4\pi b^3|z|}\int_0^\infty \exp\left\{-\frac{1}{2}B|z|\left(s + s^{-1}\right)\right\}dsdz$$

By expanding $(s-1)^2$ it is easy to see that, $(s + s^{-1}) \geq 2$ holds for all s positive. Therefore,

$$e^{2|z|}\nu_Y(dz) \leq \frac{a}{4\pi b^3|z|}\int_0^\infty \exp\left\{-\frac{1}{2}(B-2)|z|\left(s + s^{-1}\right)\right\}dsdz.$$

Which shows that a sufficient condition for Equation (3.5) to hold is that

$$B > 2 \iff \frac{1}{\sigma^2\mathbb{V}[T(1)]} > 4.$$

In [Bar97], [Bar01], and [BKR01] it is discussed the importance of this process.

Remark 3.1.6. It is interesting to note that we found stronger limitations on the variance of $T(1)$, the subordinator, for the process with infinite variation than for the process with finite variation. This situation is rather intuitive as evidently we can expect the infinite *variation* process to *vary more* than its finite *variation* counterpart. Therefore, to assure integrability we should not let the latter to *vary* too much by limiting its *variance.*

3.2 The wealth process

Now let us define $\varphi_0(t)$ as the amount of money held in the bank account at time t. The values $\varphi_i(t)$, $i = 1, \dots, n$ will will represent the amount of shares invested in stock i at time t. We will assume that process $(\varphi(t))_{t\in[0,T]}$ is adapted and *càdlàg*. Similarly as with the assets, we define the vector $\varphi(t)$ as $\varphi(t) = (\varphi_0(t), \varphi_1(t), \dots, \varphi_n(t))^\top$ and will let $\hat{\varphi}(t)$ be the vector of the amount of shares invested in the risky assets. Define now the wealth at time t as the univariate stochastic process $W_\varphi(t) := \varphi(t)^\top S(t)$.

3.2.1 Self financing portfolios

Let us assume that the wealth process is self-financing, in other words, that there are no inflows or outflows of money to or from the portfolio. In mathematical terms we have that:

$$dW_\varphi(t) = \varphi(t^-)^\top dS(t). \tag{3.8}$$

Remark 3.2.1. Equation (3.8) is an assumption and not a consequence of Itô's formula.

Usually in the literature, like in [Øks13, Definition 12.1.2] for example, it is defined a set of admissible portfolios Θ_φ as the the self-financing portfolios that are bounded below. Mathematically this means that there is a constant $K = K(\varphi)$, $K \geq 0$ such that:

$$W_\varphi(t) > -K(\varphi), \quad \mathbb{P}\text{-almost surely.}$$

And such an assumption is justified by the fact that an investor must have a limited credit amount. This is indeed true, however, for the purposes of this thesis we will fix a critical credit level $\mathcal{K} \in \mathbb{R}$, for the portfolios we are interested in to allow for a more flexible model. In the context of risk management, for instance, capital (wealth) should stay positive to avoid regulatory interventions in an institution while, in the other hand, a negative wealth can be considered in this model if, for instance, an investor already owns any assets that for any reason cannot be traded but that is willing to risk. In the context of this thesis, we will then require that portfolios satisfy the following condition.

$$W_\varphi(t) > \mathcal{K}, \quad \mathbb{P}\text{-almost surely.}$$

for a fixed $\mathcal{K} < W_\varphi(0)$.

The process φ above can take different values depending of the face value of a share, i.e., it is susceptible to *splits* or dividends that can be re-invested in the same stock, for instance. To avoid this scaling issue and to reduce the dimension we work with, we will rewrite $W\varphi(t)$ in terms of the proportions of the portfolio invested in certain assets. To do so, let us define the vector $\theta = (\theta_1, \dots, \theta_n)^\top$ with i-th components:

$$\theta_i(t) := \frac{\varphi_i(t) S_i(t)}{W_\varphi(t)}, \tag{3.9}$$

Which is adapted and *càdlàg*. Consequently, the proportion of wealth invested in the bank account would be:

$$\theta_0(t) = 1 - \sum_{i=1}^{n} \theta_i(t). \tag{3.10}$$

Although it is important to make clear that θ_0 is not an element of the vector θ, however, θ_0 will represents the proportion on wealth invested in a bank account. Hence, Equation (3.8) can be written as:

$$dW(t) = W(t^-)\sum_{i=0}^{n}\frac{\theta_i(t^-)}{S_i(t)}dS_i(t). \tag{3.11}$$

Using Equation (3.3), Equation (3.4), and dropping the sub-index φ to shorten the notation, we can write Equation (3.11) as:

$$\begin{aligned} dW(t) = W(t^-)\Bigg\{&\left(1-\sum_{i=1}^{n}\theta_i(t^-)\right)\alpha_0 dt \\ &+\sum_{i=1}^{n}\theta_i(t^-)\left[\left(\alpha_i+\frac{1}{2}\left(\hat{\Sigma}\hat{\Sigma}^\top\right)_{i,i}+\sum_{j=1}^{l}\int_{\mathbb{R}}\left(e^{\gamma_{i,j}z}-1-\gamma_{i,j}z\right)\nu_j(dz)\right)dt \right.\\ &\left.+\hat{\Sigma}_{i,\cdot}dB(t)+\sum_{j=1}^{l}\int_{\mathbb{R}}\left(e^{\gamma_{i,j}z}-1\right)\tilde{N}(dz,dt)\right]\Bigg\} \end{aligned} \tag{3.12}$$

Define $\mu = (\mu_1, \ldots, \mu_n)^\top$ as the *risk premia* vector with i-th component given by:

$$\mu_i := (\alpha_i - \alpha_0) + \frac{1}{2}\left(\hat{\Sigma}\hat{\Sigma}^\top\right)_{i,i} + \sum_{j=1}^{l}\int_{\mathbb{R}}\left(e^{\gamma_{i,j}z}-1-\gamma_{i,j}z\right)\nu_j(dz). \tag{3.13}$$

Recall that we have assumed that $\alpha_i > \alpha_0$ and $\alpha_i > 0$ for all $i \in 1, \ldots, n$. Hence $\mu_i > 0$ for all $i \in 1, \ldots, n$ as the mapping $z \mapsto e^z - 1 - z > 0$ for all $z \in \mathbb{R}$.

Define as well the $n \times l$ matrix $\hat{\Gamma}$ with entries:

$$\hat{\Gamma}_{i,j}(z) := e^{\gamma_{i,j}z} - 1 \tag{3.14}$$

Then Equation (3.12) can be written in matrix notation as:

$$dW(t) = W(t^-)\left(\left(\alpha_0 + \theta(t^-)^\top\mu\right)dt + \theta(t^-)^\top\hat{\Sigma}dB(t) + \theta(t^-)^\top\int_{\mathbb{R}}\hat{\Gamma}(z)\tilde{N}(dz,dt)\right). \tag{3.15}$$

If we want the equation above to be an Itô diffusion as in Definition 2.2.7, then the process $(\theta(t))_{t\in[0,T]}$ should be a deterministic function of $(t, W(t))^\top$ which is an assumption that we will consider throughout the rest of the thesis. However, later on we will show that $W(t)$ is good enough for our initial purposes. In what follows, we will denote $W_\theta(t) := W(t)$ were $W(t)$ is the solution to Equation (3.15) parameterised by the asset allocation $\theta(t)$.

A last consideration before moving on is that it is financially reasonable to rule out the possibility of having too big short or levered positions relative to the total value of the wealth. Therefore, we will assume in the rest of the thesis that the process $(\theta(t))_{t\in[0,T]}$ is bounded, i.e., there exists $C > 0$ such that $|\theta(t)| < C$ for all $t \in [0, T]$. In other words we will assume that credit constraints are imposed relative to the amount of wealth, in addition to being to imposed in absolute terms. Hence we define:

Definition 3.2.2 (Reasonable portfolios). The set of *reasonable portfolios* denoted by Θ is the set of all càdlàg and $\mathbb{F}$-adapted stochastic processes in $\mathbb{R}^n$ that satisfy the following conditions:

1. $W_\theta(t) > \mathcal{K}$, $\mathbb{P}$-almost surely, for all $t \in [0,T]$.
2. $|\theta(t)| < C$, $\mathbb{P}$-almost surely, for all $t \in [0,T]$.
3. $\theta(t) = \theta(W(t))$ such that for all $i \in 1,\ldots,n$, $W \mapsto W\theta_i(W)$ is Lipschitz continuous in $(\mathcal{K},\infty)$.

The first two conditions above have been justified so that the wealth process is financially reasonable. The third condition however, appears to be a mere mathematical technicality. Nevertheless, Equation (3.9) gives it financial sense. Heuristically, the third condition above is asking that the amount of wealth invested in a certain asset should not vary too much with respect to changes in total wealth.

Proposition 3.2.3. *Let* $(\theta(t))_{t\in[0,T]} \in \Theta$. *Then there is a unique, square integrable, strong solution to Equation* (3.15).

Proof. The result will follow if we can show that the assumptions in Theorem 2.2.3, Lipschitz continuity and at most linear growth, hold. Usually the former implies the latter because

$$|h(x)|^2 \leq |h(x)+h(0)|^2 + |h(0)|^2 \leq K|x|^2 + |h(0)|^2 \leq \min\{K, |h(0)|^2\}(1+|x|^2),$$

nevertheless in this case we will prove both conditions separately because θ is defined over an open set.

Even though the coefficients in Equation (3.15) appear to be linear in W, in general this is not the case because θ is not necessarily a constant. However, because $|\theta|$ is bounded by the constant C, $\mathbb{P}$-almost surely, the norms in Theorem 2.2.3 can also be bounded as we will now show for the at most linear growth condition:

$$\begin{aligned} &W^2\left(\alpha_0 + \theta^\top \mu\right)^2 + W^2\left|\theta^\top \hat{\Sigma}\right|^2 + W^2 \sum_{j=1}^{l} \int_{\mathbb{R}} \left(\theta^\top \hat{\Gamma}_{\cdot,j}(z)\right)^2 \nu\,(dz) \\ &\qquad < (1+|W|)^2\Bigg(2\alpha_0^2 + 2C^2 n \sum_{i=1}^{n} \mu_i^2 + C^2\left|\mathbf{1}^\top \hat{\Sigma}\right|^2 + C^2 n \sum_{j=1}^{l}\sum_{i=1}^{n} \int_{\mathbb{R}} \hat{\Gamma}_{i,j}(z)^2 \nu_j\,(dz)\Bigg), \end{aligned}$$

where in the inequality we also used the identity $W^2 < (1+|W|)^2$ and a version of Cauchy-Schwartz inequality which in this case is $(\sum_{i=1}^k a_i)^2 \leq k\sum_{i=1}^k a_i^2$. Now it is easy to observe that we only need the nl integrals above to be finite. Let us then fix $i \in 1\ldots,n$ and $j \in 1,\ldots,l$. The explicit form the integral with respect to the Lévy measure will then be:

$$\begin{aligned} \int_{\mathbb{R}} \hat{\Gamma}_{i,j}(z)^2 \nu_k\,(dz_k) &= \int_{\mathbb{R}} \left(e^{\gamma_{i,j} z} - 1\right)^2 \nu_j\,(dz) \\ &= \int_{|z|<1} \left(e^{\gamma_{i,j} z} - 1\right)^2 \nu_j\,(dz) + \int_{|z|\geq 1} \left(e^{\gamma_{i,j} z} - 1\right)^2 \nu_j\,(dz)\,. \end{aligned}$$

The first term of the Taylor expansion of $e^{\gamma_{i,j}z}$ around $z = 0$ is 1, hence, $e^{\gamma_{i,j}z} - 1$ is $\mathcal{O}(z)$ around zero and its square will be $\mathcal{O}(z^2)$ which proves that the integral converges in the region $|z| < 1$. Outside this region, the result follows immediately because,

$$\int_{|z|\geq 1} e^{2\gamma_{i,j} z} \nu_j(dz) \leq \int_{|z|\geq 1} e^{2|z|} \nu_j(dz) < \infty,$$

where the last inequality is given by the fact that $\gamma_{i,j} \in [-1,1]$ and the condition stated in Equation (3.5). Hence, we have proven the at most linear growth condition.

Now, let us check for Lipschitz continuity. This step will be more notationally heavy and thus we will split it into parts. Also, to avoid confusions, we will write $\theta = \theta(W)$. In all cases, let $W, V \in (\mathcal{K}, \infty)$.

For the drift term, observe that:

$$\begin{aligned}(W(\alpha_0 + \theta(W)^\top \mu) - V(\alpha_0 + \theta(V)^\top \mu))^2 &\leq 2\alpha_0^2 (W - V)^2 + 2|W\theta(W)^\top \mu - V\theta(V)^\top \mu|^2 \\ &\leq 2\alpha_0^2 (W - V)^2 + 2|\mu|^2 |W\theta(W) - V\theta(V)|^2 \\ &\leq 2\alpha_0^2 (W - V)^2 + 2|\mu|^2 K|W - V|^2.\end{aligned}$$

Where in the last step we used Lipschitz continuity and before (several times) Cauchy-Schwartz inequality. Now, for the Brownian motion part:

$$|W\theta(W)^\top \hat{\Sigma} - V\theta(V)^\top \hat{\Sigma}|^2 = |(W\theta(W) - V\theta(V))^\top \hat{\Sigma}|^2 \leq nm \max_{i \leq n, j \leq m} \left\{ \hat{\Sigma}_{i,j} \right\}^2 K|W - V|^2.$$

And finally, for the jump part, let us pick an arbitrary j in $1, \ldots, l$:

$$\begin{aligned}&\left| W \int_{\mathbb{R}} \theta(W)^\top \hat{\Gamma}_{\cdot,j}(z) \nu\,(dz) - V \int_{\mathbb{R}} \theta(V)^\top \hat{\Gamma}_{\cdot,j}(z) \nu\,(dz) \right|^2 \\ &\qquad \leq |W\theta(W) - V\theta(V)|^2 \left| \int_{\mathbb{R}} \hat{\Gamma}_{\cdot,j}(z) \nu\,(dz) \right|^2 \\ &\qquad \leq K|W - V| \left| \int_{\mathbb{R}} \hat{\Gamma}_{\cdot,j}(z) \nu\,(dz) \right|^2.\end{aligned}$$

Where the convergence of the vector of integrals has been proven in a previous step. ■

As a direct consequence of the proposition above, we can conclude that if any $\theta \in \Theta$ is to be interpreted as a control process like in Definition 2.3.1, such control will be immediately admissible in the sense of Definition 2.3.2. In fact, this is what we will do next.

3.3 Optimising wealth

Assume now that an investor holds a positive amount of money at time 0 denoted by w, $w > 0$. Furthermore, assume the investor has the possibility of investing such quantity in a market with stock dynamics described by Equation (3.4). Let us assume as well that the investor cannot deposit or withdraw any money from its portfolio until a specific time T. Furthermore, assume there are no transaction costs associated to the purchase or sale of assets. In mathematical terms, we are assuming that the value of such portfolio would be described by the dynamics exposed in Equation (3.15).

Let the investor be risk averse with a utility function describing her preferences, $g : (q, \infty) \to \mathbb{R}$, $q < \mathcal{K}$, to be given by Definition 2.4.1. The strict inequality $q < \mathcal{K}$ is not a minor assumption as it allows us to bound $g(W(t))$, and $g'(W(t))$ $\mathbb{P}$-almost surely. Furthermore. under this assumptions, we can consider without any loss of generality that the utility function g is non-negative. In fact, if we let $-K_1 = \min\{g(-\mathcal{K}), 0\}$, then the function $h(W(t)) = g(W(t)) + 2K_1$ will be non-negative and concave and, according to Definition 2.4.2, the new function h will give the same risk aversion as g.

A natural question for the investor is how to define $\theta_*(t) \in \Theta$ such that she can maximise her future expected utility. In terms of Definition 2.3.2, the investor wishes to maximise the performance criterion:

$$J_\theta(w) = \mathbb{E}_w \left[g(W_\theta(T)) \right] \tag{3.16}$$

At a first glance one could think about using the Lévy-Khinchine formula in the expression above and then equate its gradient with respect to the vector θ to zero. However, $\theta(t)$ is a process. In order to investigate it we can use Theorem 2.2.13, provided that the integrability condition therein assumed indeed holds, to characterise the equation above as follows:

$$\mathbb{E}_w\left[g(W_\theta(T))\right] = g(w) + \mathbb{E}_w\left[\int_0^T Ag(W_\theta(t))dt\right]. \tag{3.17}$$

The generator $Ag(W_\theta(t))$ can be obtained by applying Equation (2.17) to the dynamics given by Equation (3.15) and is given as follows:

$$\begin{aligned} Ag(W) = &\left(\alpha_0 + \theta^\top \mu\right) W g'(W) + \frac{1}{2}\theta^\top \hat{\Sigma}\hat{\Sigma}^\top \theta W^2 g''(W) \\ &+ \sum_{j=1}^{l} \int_{\mathbb{R}} \left[g\left(W\left(1+\theta^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right) - g(W) - \theta^\top \hat{\Gamma}_{\cdot,j}(z) W g'(W)\right] \nu_j(dz), \end{aligned} \tag{3.18}$$

Where in the equation above, we purposely wrote $W(t)$ as W and $\theta(t)$ as θ to compact the notation. Later in the text we will keep doing so when the context allows.

Remark 3.3.1. The integrands in the l integrals in Equation (3.18) are non-positive for all $z \in \mathbb{R}$. Indeed, recall that g is strictly increasing and g' is strictly decreasing. Denote $h := W\theta^\top \hat{\Gamma}_{\cdot,j}(z)$ then the result follows by the mean value theorem, because if h is positive then

$$g'(W) > \frac{g(W+h) - g(W)}{h},$$

and if h is negative,

$$g'(W) < \frac{g(W+h) - g(W)}{h}.$$

The case when $h = 0$ can be solved by direct substitution where we get the value 0.

If we could maximise Equation (3.18) above as a function of θ for all t then we would maximise Equation (3.17). Let us then compute the gradient of $Ag(W)$ with respect to θ:

$$\begin{aligned} D_\theta Ag(W) = &\mu W g'(W) + \hat{\Sigma}\hat{\Sigma}^\top \theta W^2 g''(W) \\ &+ \sum_{j=1}^{l} \int_{\mathbb{R}} \left(g'\left(W\left(1+\theta^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right) - W g'(W)\right) \hat{\Gamma}_{\cdot,j}(z) \nu_j(dz), \end{aligned} \tag{3.19}$$

assuming that the integrals and derivatives above exist. Later on we will verify such assumptions. Letting $D_\theta Ag(W) = 0$ to get the critical points we can possibly find an optimal portfolio allocation.

Note that any $\theta_*(t) \in \Theta$ satisfying $D_\theta Ag(W)|_{\theta_*} = 0$ will be a global maximum as for all $\theta \in \Theta$,

$$\mathcal{J}\left[D_\theta Ag(W_\theta)\right] = \hat{\Sigma}\hat{\Sigma}^\top W^2 g''(W) + \sum_{j=1}^{l} \int_{\mathbb{R}} g''\left(W\left(1+\theta^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right) \hat{\Gamma}_{\cdot,j}(z) \hat{\Gamma}_{\cdot,j}(z)^\top \nu_j(dz), \tag{3.20}$$

is strictly negative definite because $\hat{\Sigma}\hat{\Sigma}^\top$ is strictly positive definite and all $\hat{\Gamma}_{\cdot,j}(z)\hat{\Gamma}_{\cdot,j}(z)^\top$, $j = 1, \ldots, l$ are positive semidefinite matrices and g'' is, by assumption, negative in all its domain. Hence, as long as the integrals and the derivatives are well defined, we can conclude that the mapping $\theta \mapsto Ag(W_\theta)$ is concave. Thus, if there is a $\theta_* \in \Theta$ satisfying $D_\theta Ag(W)|_{\theta_*} = 0$, it would be unique.

Definition 3.3.2. The *optimal portfolio proportion* or *optimal portfolio allocation* is $\theta_* \in \Theta$ that satisfies the following system of integral equations:

$$\begin{aligned} D_\theta Ag(W)\Big|_{\theta_*} =&\mu W g'(W) + \hat{\Sigma}\hat{\Sigma}^\top \theta_* W^2 g''(W) \\ &+ \sum_{j=1}^{l} \int_{\mathbb{R}} \left(g'\left(W\left(1+\theta_*^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right) - W g'(W) \right) \begin{bmatrix} e^{\gamma_{1,j}z}-1 \\ \vdots \\ e^{\gamma_{n,j}z}-1 \end{bmatrix} \nu_j(dz) = 0. \end{aligned} \tag{3.21}$$

From the definition above, a direct relationship between the regularity of the utility function and of the amount of money invested in different assets can be established in the continuous case.

Proposition 3.3.3. *Let $l = 0$. Then, condition 3) in Definition 3.2.2 holds for θ_* if and only if the risk tolerance (see Definition 2.4.2) is Lipschitz continuous.*

Proof. Solving for θ_* in Equation (3.21) we get:

$$\theta_*(W) = -\left(\hat{\Sigma}\hat{\Sigma}^\top\right)^{-1} \mu \frac{g'(W)}{W g''(W)} = -\left(\hat{\Sigma}\hat{\Sigma}^\top\right)^{-1} \mu \frac{1}{W a(W)},$$

Which immediately yields that:

$$W\theta_*(W) = -\left(\hat{\Sigma}\hat{\Sigma}^\top\right)^{-1} \mu \frac{1}{W a(W)}$$

■

Heuristically, the proposition above states that if the risk tolerance is *well behaved*, then the resulting wealth process with dynamics given by Equation (3.15) are well posed.

However nice the result above can be, it must be considered that not all θ that satisfies Equation (3.21) and condition *3)* in Definition 3.2.2 can be automatically considered as a reasonable strategy.

Example 3.3.4. Let $g(W) = -\exp\{-W\}$. In this case, the risk tolerance is constant and equal to 1. Therefore, in a market with continuous dynamics (where $l = 0$), θ that satisfies Equation (3.21) will satisfy condition *3)*. Particularily, we will get:

$$\theta = \left(\hat{\Sigma}\hat{\Sigma}^\top\right)^{-1} \mu \frac{1}{W}. \tag{3.22}$$

However, if $\mathcal{K} > 0$ and W, are too small, θ above can violate condition *2)* of Definition 3.2.2.

Let us get back to study the case with jumps. In this case, it is not as straight forward to get to natural and interpretable conclusions as in Proposition 3.3.3. To see if a control satisfying Definition 3.3.2 is indeed in Θ, an analysis to each specific case should be considered. A first natural step is to check if the integral terms in Equation (3.18), Equation (3.19), and, Equation (3.20) which we will respectively denote as A_I, A'_I, and A''_I, are well defined. To this end, let us present an auxiliary result.

Lemma 3.3.5. *Assume that $\theta \in \Theta$ and let $W_\theta(t_0^-) > \mathcal{K}$. Define also the set $\mathbb{W}_{t_0}$ as*

$$\mathbb{W}_{t_0} := \left\{ z \in \mathbb{R} : W_\theta(t_0^-) \left(1 + \theta(t_0^-)^\top \hat{\Gamma}_{\cdot,j}(z)\right) \le \mathcal{K} \right\},$$

then:

$$\int_{\mathbb{W}_{t_0}} \nu_j(dz) = 0, \tag{3.23}$$

for all $t_0 \in (0,T)$ and for all $j \in 1,\ldots,l$.

Proof. Let us proceed by contrapositive and assume that

$$\int_{\mathbb{W}_{t_0}} \nu_j(dz) > 0, \quad j \in 1\ldots,l.$$

By Theorem 2.1.4, we know that W can jump in $[0,T]$ with positive probability. Equation (3.15) gives us that:

$$W(t_0) = W(t_0^-)\left(1 + \int_{\mathbb{R}} \theta(t_0^-)^\top \hat{\Gamma}(z) N(dz, dt_0)\right).$$

As we have assumed that the Poisson random measures are independent among them, any jump above can be attributed to only one of such Poisson random measures which without any loss of generality we will assume to be the j-th one. Hence the equation above is equivalent to the following,

$$W(t_0) = W(t_0^-)\left(1 + \int_{\mathbb{R}} \theta(t_0^-)^\top \hat{\Gamma}_{\cdot,j}(z) N_j(dz, dt_0)\right).$$

By observing that $\theta(t_0^-)^\top \hat{\Gamma}_{\cdot,j}(0) = 0$ and that the mapping $z \mapsto \theta(t_0^-)^\top \hat{\Gamma}_{\cdot,j}(z)$ is continuous we can conclude that the region $\mathbb{W}_{t_0}$ is bounded away from 0. Thus, for all $\epsilon > 0$ such that $\mathbb{W}_{t_0} \in \mathbb{R} \cap (-\epsilon, \epsilon)^c$,

$$\mathbb{P}\left(\Delta W(t_0) \le \mathcal{K} - W(t_0^-) \middle| W(t_0) - W(t_0^-) \ge \epsilon\right) \propto \int_{\mathbb{W}_{t_0}} \nu_j(dz) > 0.$$

Which implies that $W(t_0) \le \mathcal{K}$ with positive probability. Hence, $\theta \notin \Theta$, which by contrapositive proves the claim. ■

A consequence of this lemma is that if ν_j has support in $\mathbb{R}$ (except in 0 of course), then, $\mathbb{W}_{t_0} = \emptyset$ for all $t_0 \in (0,T)$. Now we can check if our integrals are well defined.

Proposition 3.3.6. *For fixed $t \in [0,T]$, $\theta(t) \in \Theta$, and $W > \mathcal{K}$, the integral A_I is well defined.*

Proof. To compact the notation, let us define for all $j = 1,\ldots,l$ the function $G_j(z;W) := g(W(1 + \theta^\top \hat{\Gamma}_{\cdot,j}(z)))$. Then, we can write A_{Ij} ,$j = 1,\ldots,l$, as:

$$\begin{aligned} A_{Ij} &:= \int_{\mathbb{R}} \left[G_j(z;W) - g(W) - \theta^\top \hat{\Gamma}_{\cdot,j}(z) W g'(W)\right] \nu_j(dz) \\ &= \int_{|z|<1} \left[G_j(z;W) - g(W) - \theta^\top \hat{\Gamma}_{\cdot,j}(z) W g'(W)\right] \nu_j(dz) \\ &\quad + \int_{|z|\ge 1} \left[G_j(z;W) - g(W) - \theta^\top \hat{\Gamma}_{\cdot,j}(z) W g'(W)\right] \nu_j(dz). \end{aligned} \tag{3.24}$$

Let us denote A_s and A_B the first and second parts, respectively, of the right hand side of the above equation. Note as well that by Lemma 3.3.5, $W(1+\theta^\top\hat{\Gamma}_{\cdot,j}(z)) > \mathcal{K}$ for all z such that $\nu_j(dz) > 0$.

The first two terms of the Taylor expansion of the function $G_j(z;W)$ around 0 are $g(W)$ and $Wg'(W)\sum_{i=1}^n \theta_i\gamma_{i,j}z$ which is $\mathcal{O}(z)$, however, $Wg'(W)\sum_{i=1}^n \theta_i\left(e^{\gamma_{i,j}z}-1\right)$ is also $\mathcal{O}(z)$. Hence we can conclude that the integrand in A_s is $\mathcal{O}(z^2)$ around 0 and because the integrator is a Lévy measure we have proven that the integral A_s converges.

Regarding A_B, we will apply the triangle inequality and verify that each integral term converges individually. Recall that Equation (3.5) holds, hence, by virtue of θ being bounded, the integral of $\theta^\top\hat{\Gamma}_{\cdot,j}(z)Wg'(W)$ will converge. The term $g(W)$ is constant with respect to z and hence it will converge as well. To prove that the term that integrates $G(z;W)$ converges it suffices to recall that g is non-negative in $(\mathcal{K},\infty)$, hence we can apply Jensen's inequality, Theorem 2.4.5, which yields:

$$\int_{|z|\geq 1} G(z;W)\nu_j(dz) \leq g\left(\int_{|z|\geq 1} W(1+\theta^\top\hat{\Gamma}_{\cdot,j}(z))\nu_j(dz)\right) < \infty \tag{3.25}$$

■

The proposition above guarantees that the generator $Ag(W(t))$ is well defined for all t. However, to make sure that we can indeed apply Equation (3.17) we need to check that Equation (2.18) stated in Theorem 2.2.13 holds. We will now confirm that this is the case.

Proposition 3.3.7. *Fix* $(\theta(t))_{t\in[0,T]} \in \Theta$, *and assume that* g'' *is bounded below in* $(\mathcal{K},\infty)$, *then Equation* (2.18) *for* $g(W_\theta(T))$ *will hold.*

Proof. For convenience we will drop the subindex θ in $W_\theta(t)$. Let us rewrite the left hand side of Equation (2.18) in the context of $g(W(T))$ and denote it as E, recall as well that, as pointed out in Remark 3.3.1, the integrands are non-positive. Applying the triangle inequality and Tonelli's theorem to change the order of the expectations and the integrals with respect to time we get:

$$\begin{aligned}
E &:= \mathbb{E}_x\left[g(W(T)) + \int_0^T |Ag(W(t))|\,dt\right] \\
&\leq \mathbb{E}_w\left[g(W(T))\right] + \int_0^T \mathbb{E}_w\left[\left|\left(\alpha_0 + \theta(t)^\top\mu\right)W(t)g'(W(t))\right|\right]dt \\
&\quad + \frac{1}{2}\int_0^T \mathbb{E}_w\left[\left|\theta(t)^\top\hat{\Sigma}\hat{\Sigma}^\top\theta(t)W(t)^2 g''(W(t))\right|\right]dt \\
&\quad - \int_0^T \sum_{j=1}^l \mathbb{E}_w\Bigg[\int_{\mathbb{R}} g\left(W(t)\left(1+\theta(t)^\top\hat{\Gamma}_{\cdot,j}(z)\right)\right) \\
&\qquad\qquad - g(W(t)) - \theta(t)^\top\hat{\Gamma}_{\cdot,j}(z)W(t)g'(W(t))\nu_j(dz)\Bigg]dt.
\end{aligned}$$

Recall that C the upper bound on $|\theta(t)|$ for all $t\in[0,T]$, let then $\hat{C}$, be an n-dimensional vector in which $C_i = C$ for $i = 1,\ldots,n$, and denote K_1 as the lower bound in g'' hence we get:

$$\begin{aligned} E \leq & g\left(\mathbb{E}_w\left[W(T)\right]\right) + \left(\alpha_0 + \hat{C}^\top \mu\right) g'(\mathcal{K}) \int_0^T \mathbb{E}_w\left[|W(t)|\right] dt \\ & + \frac{1}{2}\hat{C}^\top \hat{\Sigma}\hat{\Sigma}^\top \hat{C}|K_1| \int_0^T \mathbb{E}_w\left[W(t)^2\right] dt \\ & - \int_0^T \sum_{j=1}^{l} \mathbb{E}_w\Bigg[\int_{\mathbb{R}} g\left(W(t)\left(1+\theta(t)^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right) \\ & \qquad - g(W(t)) - \theta(t)^\top \hat{\Gamma}_{\cdot,j}(z) W(t) g'(W(t)) \nu_j(dz)\Bigg] dt. \end{aligned}$$

We know by Proposition 3.2.3 that $W(t)$ is square integrable, thus, the first three terms of the inequality above converge. Let us denote such common bound as K_3. To verify that the l integral terms above converge, we can apply again Tonelli's theorem to change the order of integration of the expectations and the integrals with respect the Lévy measure:

$$\begin{aligned} E \leq K_3 - \int_0^T \sum_{j=1}^{l} \int_{\mathbb{R}} \mathbb{E}_w\Bigg[& g\left(W(t)\left(1+\theta(t)^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right) \\ & - g(W(t)) - \theta(t)^\top \hat{\Gamma}_{\cdot,j}(z) W(t) g'(W(t))\Bigg] \nu_j(dz) dt. \end{aligned} \tag{3.26}$$

Let us focus in the j-th term of the sum above. Define the vector $\bar{C}(z)$ with elements $\bar{C}_i(z) = C\text{sign}(z)\text{sign}(\gamma_{i,j})$, hence we will get that $\bar{C}(z)^\top \hat{\Gamma}_{\cdot,j}(z) \geq \theta(t)^\top \hat{\Gamma}_{\cdot,j}(z)$ for all $z \in \mathbb{R}$ although clearly $\bar{C} \notin \Theta$. By doing this we can verify the following:

$$\begin{aligned} H_j(z) &:= -\mathbb{E}_w\left[g\left(W(t)\left(1+\theta(t)^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right) - g(W(t)) - \theta(t)^\top \hat{\Gamma}_{\cdot,j}(z) W(t) g'(W(t))\right] \\ &\leq g\left(\mathbb{E}\left[W(t)\right]\left(1+\bar{C}(z)^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right) + g\left(\mathbb{E}\left[W(t)\right]\right) + g'(\mathcal{K})\bar{C}(z)^\top \hat{\Gamma}_{\cdot,j}(z)\mathbb{E}[W(t)] \end{aligned}$$

Where $W(t)$ is still of the form $W(t) = W_\theta(t)$. The equation above implies, through a very similar argument to that in Proposition 3.3.6, that

$$\int_{|z|\geq 1} H_j(z)\nu_j(dz) < \infty.$$

Now we are only missing to show that $\int_{|z|<1} H_j(z)\nu_j(dz) < \infty$. To verify this, we can use Lemma 3.3.5 to obtain that:

$$\begin{aligned} \frac{d}{dz} g\left(W(t)\left(1+\theta(t)^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right) =& g'\left(W(t)\left(1+\theta(t)^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right) W(t) \sum_{i=1}^{n} \theta_i \gamma_{i,j} \\ \leq & g'(\mathcal{K}) W(t) \sum_{i=1}^{n} \bar{C}_i(z) e^{\gamma_{i,j} z}, \ \mathbb{P}\text{-almost surely.} \end{aligned}$$

The right hand side of the inequality above is $\mathbb{P}$-integrable. Hence, we can apply dominated convergence theorem so that:

$$\frac{d}{dz}\mathbb{E}_w\left[g\left(W\left(1+\theta(t)^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right)\right] = \mathbb{E}_w\left[\frac{d}{dz} g\left(W\left(1+\theta(t)^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right)\right],$$

which will allow us to do a Taylor analysis to the mapping $z \mapsto \mathbb{E}_w[G_j(z; W(t))]$ similar to that in Proposition 3.3.6, concluding that indeed the integrals converge in the region $|z| < 1$. ■

Now, let us make sure that with the same assumptions as before, the gradient of the generator of $g(W_\theta(t))$ actually makes sense.

Proposition 3.3.8. *For fixed $t \in [0,T]$, $\theta(t) \in \Theta$, and $W > \mathcal{K}$, the integral A'_I is well defined.*

Proof. Let us focus in one of the entries of one of the summands, i.e., let $i \in 1,\dots,n$, $j \in 1,\dots,l$, i.e.,

$$\begin{aligned} A'_{Ii,j} &= \int_{\mathbb{R}} \left(g'\left(W\left(1+\theta^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right) - Wf'(W)\right)\left(e^{\gamma_{i,j}z}-1\right)\nu_j(dz) \\ &= \int_{|z|<1} \left[g'\left(W\left(1+\theta^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right) - Wg'(W)\right]\left(e^{\gamma_{i,j}z}-1\right)\nu_j(dz) \\ &\quad + \int_{|z|\geq 1} \left[g'\left(W\left(1+\theta^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right) - Wg'(W)\right]\left(e^{\gamma_{i,j}z}-1\right)\nu_j(dz) \end{aligned} \tag{3.27}$$

Let us too define as before A'_s and A'_B, respectively, as the first and second integrals of the above equation. As $(e^{\gamma_{i,j}}-1)$ is integrable with respect to $\nu_j(dz)\mathbf{1}(|z|\geq 1)$ and, by Lemma 3.3.5, $\left|g'\left(W\left(1+\theta^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right)\right|$ is bounded, so it is clear that A'_B exists. To prove that A'_s is well defined, note that close to zero $(e^{\gamma_{i,j}}-1)$ is $\mathcal{O}(z)$. Note also that $g'\left(W\left(1+\theta^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right) - Wg'(W)$ is $\mathcal{O}(z)$. The product of the two aformentioned functions around zero will be $\mathcal{O}(z^2)$ which proves that the integral converges. ■

Finally, to justify the claim that the mapping $\theta \mapsto AW_\theta$ is concave we have the following result.

Proposition 3.3.9. *Assume that g'' is bounded below in $(\mathcal{K},\infty)$. Then, For fixed $t \in [0,T]$, $\theta(t) \in \Theta$, and $W > \mathcal{K}$, the integral A''_I is well defined.*

Proof. Let us focus in one of its entries and one of the integrals of A''_I. Let $i,k \in 1,\dots,n$, $j \in 1,\dots,l$. Then,

$$A''_{Iij} = \int_{\mathbb{R}} g''\left(W\left(1+\theta^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right)\hat{\Gamma}_{\cdot,j}(z)\hat{\Gamma}_{\cdot,j}(z)^\top \nu_j(dz).$$

Once more, Lemma 3.3.5 guarantees that $\left|g''\left(W\left(1+\theta^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right)\right|$ is bounded. Hence, it suffices to show that each of the entries of the j-th summand converges, i.e., we need to verify that:

$$\int_{\mathbb{R}} (e^{\gamma_{i,j}z}-1)(e^{\gamma_{k,j}z}-1)\nu_j(dz) < \infty.$$

As the converge of the integral has already been shown in the proof of Proposition 3.2.3. ■

As a summary, let us explicitly state the assumptions under which Proposition 3.3.6, Proposition 3.3.7, Proposition 3.3.8, and Proposition 3.3.9 hold:

1. Absolute credit condition: for some $\mathcal{K} \in \mathbb{R}$, $(W_\theta(t))_{t\in[0,T]} > \mathcal{K}$, $\mathbb{P}$-almost surely.
2. Relative credit condition: for some $C > 0$, $(|\theta(t)|)_{t\in[0,T]} \leq C$, $\mathbb{P}$-almost surely.
3. Stable positions in assets: for all $i \in 1,\dots,n$, $W\theta(W)$ is Lipschitz continuous.
4. Utility well-posedness: the utility function that describes the investor's risk aversion is given by Definition 2.4.1 and is such that $g : (q,\infty) \to \mathbb{R}$ and $q < \mathcal{K}$.

5. Bounded risk aversion: there exists $K_1 < 0$ such that $g'' > K_1$, in $(\mathcal{K}, \infty)$.

Observe too, that the set Θ can be defined as a set satisfying the conditions above. Let us illustrate the results obtained so far with an example.

Example 3.3.10 (Inspired by [Øks13] Example 11.2.5). Let $g(W) = W^\eta$ where $\eta \in (0,1)$. Observe that in this case g satisfies the conditions stated in Definition 2.4.1 as $g'(W) = \eta W^{\eta-1} > 0$ and $g''(W) = \eta(\eta-1)W^{\eta-2} < 0$. Then Equation (3.18) is:

$$\begin{aligned} AW^\eta = &\left(\alpha_0 + \theta^\top \mu\right) \eta W^\eta + \frac{1}{2}\theta^\top \hat{\Sigma}\hat{\Sigma}^\top \theta \eta(\eta-1) W^\eta \\ &+ \sum_{j=1}^{l} \int_{\mathbb{R}} \left(\left(1 + \theta^\top \hat{\Gamma}_{\cdot,j}(z)\right)^\eta - 1 - \eta\theta^\top \hat{\Gamma}(z)_{,j}\right) \nu_j(dz) W^\eta. \end{aligned} \tag{3.28}$$

To meet the assumptions in Proposition 3.3.6, the equation above yields a constraint in Θ as we need for all $j = 1, \ldots, l$ that

$$1 + \theta^\top \hat{\Gamma}_{\cdot,j}(z) = 1 + \sum_{i=1}^{n} \theta_i (e^{\gamma_{i,j} z} - 1) > 0, \quad z \in \mathbb{R}. \tag{3.29}$$

If we let $\Theta = \{\theta \in [0,1]^n; \theta^\top 1 < 1\}$, i.e., we do not allow for any short-selling or any leverage in portfolio and, furthermore, that some proportion of wealth, $\theta_0 = 1 - \sum_{i=1}^n \theta_i$, should be kept in the bank account, then the conditions of Proposition 3.3.6 will indeed be met. Note as well that this definition of Θ will also bound the term $(1+\theta^\top \hat{\Gamma}_{\cdot,j}(z))^{\eta-1}$ by $\theta_0^{\eta-1}$. Such inequality also an assumption in Proposition 3.3.9.

In the literature, for instance [EK04] and [BKR01] the same possibility set Θ is stated. Note however that the condition $\{\theta(t)^\top 1 < 1\}$ could be relaxed by limiting the support of the Lévy measure or by imposing further constraints to the- the matrix γ.

Continuing with the example, Equation (3.19) would be:

$$\begin{aligned} D_\theta A W_\theta^\eta = &\mu \eta W^\eta + \hat{\Sigma}\hat{\Sigma}^\top \theta (\eta-1)\eta W^\eta \\ &+ \sum_{j=1}^{l} \int_{\mathbb{R}} \left(\left(1 + \theta^\top \hat{\Gamma}_{\cdot,j}(z)\right)^{\eta-1} - 1\right) \hat{\Gamma}_{\cdot,j}(z) \nu_j(dz) \eta W^\eta. \end{aligned} \tag{3.30}$$

Setting the gradient above equal to 0, the common term ηW^η will vanish, leaving us with the following system of integral equations for the optimal portfolio proportion θ_*:

$$\mu + (\eta-1)\hat{\Sigma}\hat{\Sigma}^\top \theta_* + \sum_{j=1}^{l} \int_{\mathbb{R}} \left(\left(1 + \theta_*^\top \hat{\Gamma}_{\cdot,j}(z)\right)^{\eta-1} - 1\right) \hat{\Gamma}_{\cdot,j}(z) \nu_j(dz) = 0. \tag{3.31}$$

If there is a solution in Θ for the above system, then such solution would be unique as concavity is guaranteed by Proposition 3.3.9 because $(1 + \theta^\top \hat{\Gamma}_{\cdot,j}(z))^{\eta-2}$ is bounded by $\theta_0^{\eta-2}$. Furthermore, such solution would be time and wealth homogeneous. In other words, for a utility function of the form $g(W) = W^\eta$, $\eta \in (0,1)$ the optimal portfolio proportion is constant regardless of the wealth amount and the time t it is being evaluated at. Such a result is consistent with the findings in [ØS19, Example 5.2] where slightly different market dynamics are assumed.

As a *sanity check*, if we let $n = 1$, $m = 1$,and $l = 0$ then the optimal proportion of the portfolio that should be invested in the risky asset would be

$$\theta_* = \frac{\alpha_1 - \alpha_0 + \sigma^2/2}{(1-\eta)\sigma^2}, \tag{3.32}$$

which is exactly the result obtained in [Øks13, Example 11.2.5] as the geometric Brownian motion model therein presented is equivalent to assuming under our market model a drift term in the risky asset of $\alpha_1 - \sigma^2/2$.

Remark 3.3.11. In Example 3.3.10 we started assuming that $q = \mathcal{K} = 0$, which *a priori* contradicts our *utility well-posedness condition*, however, by virtue of not allowing any short sells nor leverage in the portfolio, and by keeping an amount of money in the bank account , we get that if $\alpha_0 > 0$ then we can pick $\mathcal{K}$ such that $0 < \mathcal{K} < w\theta_0$, and if $\alpha_0 < 0$, we can choose $0 < \mathcal{K} < w\theta_0 \exp\{\alpha_0 T\} > 0$.

3.4 What about stability?

Assume now that we have found $\theta_* \in \Theta$ satisfying Equation (3.21). We have proven that such θ_* would be a unique portfolio maximising the future expected utility. However, one can also wonder about how *stable* such a portfolio proportion actually is. For instance, if the investor's risk aversion changes, how much would the optimal θ_* should change? Even more, as pointed out in [DS21] or [NN18], there can be uncertainty around which utility function can best represent the investor's risk aversion. Indeed, there are several ways to define stability. In this thesis, we will try to answer this question in a more pragmatic way.

Let us assume that an investor would be comfortable or is willing to accept a future utility slightly below the optimum output. Mathematically, if g is non-negative and $\varepsilon > 0$ is small, then the investor would be indifferent about strategies such that their performance criterion $J_\theta(w)$ is greater or equal than $(1-\varepsilon)J_{\theta_*}(w)$. Analogously, if g is non-positive, we would look at the strategies such that $J_\theta(w)$ is greater or equal than $(1+\varepsilon)J_{\theta_*}(w)$. In what follows, we will focus on utility functions with constant relative risk aversion like those in Equation (2.29), i.e., of the form

$$g(W) = \frac{W^\eta}{\eta}, \quad \eta < 1, \quad \eta \neq 0,$$

to which the utility function used in Example 3.3.10 belongs.

Remark 3.4.1. Note that when $\eta < 0$, the utility function $g(W) = W^\eta/\eta$ is bounded. Hence, by Theorem 2.3.3, we know that any optimal control that we find will indeed outperform any other adapted control.

We will limit, too, the question about optimal strategies to constant portfolio proportions. Later on we will give a deeper justification on these constraints apart from the fact that this kind of strategies immediately implies that $W\theta(W) = W\theta$ is Lipschitz.

Definition 3.4.2 (ε-sub-optimal portfolio proportions). Let $0 < \varepsilon < 1$. We define the *ε-sub-optimal region*, $\Theta_\varepsilon \in \mathbb{R}^n$, as the set:

$$\Theta_\varepsilon := \begin{cases} \{\theta \in \mathbb{R}^n : J_\theta(w) \geq (1-\varepsilon)J_{\theta_*}(w)\} \text{ if } \eta > 0 \\ \{\theta \in \mathbb{R}^n : J_\theta(w) \geq (1+\varepsilon)J_{\theta_*}(w)\} \text{ if } \eta < 0 \end{cases}, \tag{3.33}$$

The elements of Θ_ε would be called *ε-sub-optimal portfolio proportions* or *ε-sub-optimal strategies.*

With this definition in mind, our stability concerns would be addressed by asking, if ε is reasonably small, would the set Θ_ε also be reasonably small? Would it have a reasonable shape?

We shall later address the size of Θ_ε, however, inspired by the results obtained in Equation (3.20) and Proposition 3.3.9, were we saw that any θ_* would be unique because the mapping $\theta \mapsto AW_\theta$ is concave, we can educatedly guess that the set Θ_ε is convex, and such a guess would not be wrong.

Proposition 3.4.3. *The set Θ_ε is convex.*

Proof. Assume that $\theta_a, \theta_b \in \Theta_\varepsilon$ and that $\xi \in [0,1]$. The result will follow if we show that $\theta_c := \xi\theta_a + (1-\xi)\theta_b \in \Theta_\varepsilon$. Indeed, the explicit dynamics of the wealth process are given in Equation (3.15) and for θ_c can be expressed as follows:

$$\begin{aligned}
dW_{\theta_c}(t) = &\left(\alpha_0 + \theta_c^\top \mu\right) W(t)dt + \theta_c^\top \hat{\Sigma} W(t)dB(t) + \theta_c^\top \int_{\mathbb{R}} \hat{\Gamma}(z)W(t)\tilde{N}(dz,t) \\
=&\xi \left(\left(\alpha_0 + \theta_a^\top \mu\right) W(t)dt + \theta_a^\top \hat{\Sigma} W(t)dB(t) + \theta_a^\top \int_{\mathbb{R}} \hat{\Gamma}(z)W(t)\tilde{N}(dz,t) \right) \\
&+ (1-\xi) \left(\left(\alpha_0 + \theta_b^\top \mu\right) W(t)dt + \theta_b^\top \hat{\Sigma} W(t)dB(t) + \theta_b^\top \int_{\mathbb{R}} \hat{\Gamma}(z)W(t)\tilde{N}(dz,t) \right).
\end{aligned}$$

Furthermore, the initial wealth can be expressed as $w = \xi w + (1-\xi)w$, thus, we can conclude that $W_{\theta_c}(t) = \xi W_{\theta_a}(t) + (1-\xi)W_{\theta_b}(t)$. The result now follows because the utility function g is concave. Indeed,

$$\begin{aligned}
\mathbb{E}_w\left[g(W_{\theta_c}(T))\right] =&\mathbb{E}_w\left[g(\xi W_{\theta_a}(T) + (1-\xi)W_{\theta_b}(T))\right] \\
\geq&\xi\mathbb{E}_w\left[g(W_{\theta_a}(T))\right] + (1-\xi)\mathbb{E}\left[g(W_{\theta_b}(T))\right] \\
\geq&\begin{cases} (1-\varepsilon)J_{\theta_*}(w) \text{ if } \eta > 0 \\ (1+\varepsilon)J_{\theta_*}(w) \text{ if } \eta < 0 \end{cases},
\end{aligned}$$

where the last inequality is given by Definition 3.4.2. ■

In a financial context, it is desirable that the set Θ_ε is convex because any *slightly* suboptimal strategy can be somehow connected to the best one. Furthermore, we will not get into a situation where there is indifference between a too conservative portfolio and a very risky portfolio without being indifferent to invest in any middle point.

Let us now investigate the boundaries of Θ_ε.

3.4.1 Evaluating strategies

In order to say something about Θ_ε, it would be useful to compute the value of the performance criterion $J_{\theta_*}(w)$. We could then follow Section 3.3 and substitute θ_* in Equation (3.17) to obtain $J_{\theta_*}(w)$. However, we could also consider applying Theorem 2.3.4. Indeed, let us define $\Phi(W,t) := J_{\theta_*}(W,t) = \mathbb{E}_{W(t)}\left[g(T)\right]$ such that $\Phi(w,0) = J_{\theta_*}(w)$ as we have been referring to. We also would need that $\Phi(W,T) = g(W)$. Then, by Theorem 2.3.4, we would have that:

$$\begin{aligned}
A\Phi(W,t) =&\partial_t\Phi(W,t) + \left(\alpha_0 + \theta^\top\mu\right) W\partial_W\Phi(W,t) + \frac{1}{2}\theta^\top\hat{\Sigma}\hat{\Sigma}^\top\theta W^2\partial_W^2\Phi(W,t) \\
&+ \sum_{j=1}^{l}\int_{\mathbb{R}} \left(\Phi\left(W\left(1+\theta^\top\hat{\Gamma}_{\cdot,j}(z)\right),t\right) - \Phi(W,t) - \theta^\top\hat{\Gamma}_{\cdot,j}(z)W\partial_W\Phi(W,t)\right)\nu_j(dz) \\
=&0.
\end{aligned} \tag{3.34}$$

The equation above is a (highly) non-linear partial integro-differential equation that seems pretty hard to solve. However, it is almost the same to Equation (3.18). It is so similar, that it sparks the hope of finding a solution of the form $\Phi(W,t) = g(W)\kappa(t)$ for some function κ. Under such assumption Equation (3.34) would be:

$$\begin{aligned}A\Phi(W,t) =& g(W)\kappa'(t) + \left(\alpha_0 + \theta^\top \mu\right) W g'(W)\kappa(t) + \frac{1}{2}\theta^\top \hat{\Sigma}\hat{\Sigma}^\top \theta W^2 g''(W)\kappa(t) \\ &+ \sum_{j=1}^{l} \int_{\mathbb{R}} \left(g\left(W\left(1+\theta^\top \hat{\Gamma}_{\cdot,j}(z)\right)\right) - g(W) - \theta^\top \hat{\Gamma}_{\cdot,j}(z) W g'(W)\right) \nu_j(dz)\kappa(t) \\ =& 0.\end{aligned}$$

Our guess would be true would if we would be able to characterise $\kappa(t)$ in the equation above without saying anything about W, i.e. we would need $Wg'(W)/g(W)$, $W^2 g''(W)/g(W)$ to become constants with respect to W, and for all $A \in \mathbb{R}$ where $g(AW)$ is well defined, we would need $g(AW)/g(W)$ to also not depend on W. This is the reason why we have chosen to work with utility functions with constant relative risk aversion, as described in Equation (2.29). As mentioned in Section 2.4, it is desirable to have decreasing risk aversion and indeed this family of utility functions have such a characteristic. This is also a good argument to further investigate utility functions of this type beyond the mathematical convenience just pointed out.

Proposition 3.4.4. *Let $g(W) = W^\eta/\eta$, $\eta < 1$, $\eta \neq 0$, then $J_{\theta_*}(w)$ is given by:*

$$J_{\theta_*}(w) = \Phi(w,0) = \frac{w^\eta}{\eta} e^{\eta R_{\theta_*} T}, \tag{3.35}$$

where R_{θ_*} is a constant given by:

$$\begin{aligned}R_{\theta_*} =& \left(\alpha_0 + \theta_*^\top \mu\right) + \frac{1}{2}\theta_*^\top \hat{\Sigma}\hat{\Sigma}^\top \theta_* (\eta - 1) \\ &+ \sum_{j=1}^{l} \int_{\mathbb{R}} \left(\frac{1}{\eta}\left(1 + \theta_*^\top \hat{\Gamma}_{\cdot,j}(z)\right)^\eta - \frac{1}{\eta} - \theta_*^\top \hat{\Gamma}_{\cdot,j}(z)\right) \nu_j(dz).\end{aligned} \tag{3.36}$$

and θ_* is given by Equation (3.31).

Proof. Let us follow Theorem 2.3.4 and, as we have just pointed out, quite educatedly guess that $\Phi(W,t) = W^\eta \kappa(t)/\eta$ for some function $\kappa : [0,T] \to (0,\infty)$ where $\kappa(T) = 1$.

The value θ_* that maximises the mapping $\theta \mapsto A\Phi(W_\theta, t)$ can be obtained by repeating the steps that lead us to Equation (3.31) for $g(W) = W^\eta/\eta$ instead of $g(W) = W^\eta$ and by doing so it can be concluded that that the optimal strategy is still given by Equation (3.31). This should come at no surprise as Example 3.3.10 has the same performance criterion as we have in this proposition, multiplied by a constant. Hence the maximisers should be the same.

Following the guess $\Phi(W,t) = W^\eta \kappa(t)/\eta$, Equation (3.34) would be:

$$\begin{aligned}A\Phi(W,t) =& \frac{W^\eta}{\eta}\kappa'(t) + \left(\alpha_0 + \theta_*^\top \mu\right) W W^{\eta-1}\kappa(t) + \frac{1}{2}\theta_*^\top \hat{\Sigma}\hat{\Sigma}^\top \theta_* W^2 (\eta - 1) W^{\eta-2}\kappa(t) \\ &+ \sum_{j=1}^{l} \int_{\mathbb{R}} \left(\frac{W^\eta}{\eta}\left(1 + \theta_*^\top \hat{\Gamma}_{\cdot,j}(z)\right)^\eta - \frac{W^\eta}{\eta} - \theta_*^\top \hat{\Gamma}_{\cdot,j}(z) W W^{\eta-1}\right) \nu_j(dz)\kappa(t) \\ =& \frac{W^\eta}{\eta}\kappa'(t) + W^\eta R_{\theta_*}\kappa(t) = 0.\end{aligned}$$

Where R_{θ_*} gathers all the common terms to $W^\eta \kappa(t)$. The common term W^η vanishes, hence we get the ordinary differential equation:

$$\begin{cases} \kappa'(t) = -\eta R_{\theta_*} \kappa(t) \\ \kappa(T) = 1. \end{cases}$$

Which is solved by:

$$\kappa(t) = e^{\eta R_{\theta_*}(T-t)}.$$

And thus, we get the equation:

$$\Phi(W,t) = \frac{W^\eta}{\eta} e^{\eta R_{\theta_*}(T-t)} \tag{3.37}$$

Because $A\Phi(W,t) = 0$, it is obvious that $\mathbb{E}\left[|A\Phi(W,t)|\right] = 0 < \infty$, hence, we only need to check that the family $(\Phi(W,t))_{t\in[0,T]}$ is uniformly integrable. However, this is guaranteed by Proposition 3.3.7. Furthermore, Proposition 3.3.6 and Proposition 3.3.8 guarantee that all the integrals converge and Proposition 3.3.9 guarantees that the generator is indeed maximised. This completes the proof because $J_{\theta_*}(w) = \Phi(w,0)$. ■

Equation (3.35) from the result above shows us that regardless of the sign of the parameter η, it is *desirable* that $R_{\theta_*} > 0$, as this would mean we can expect the future utility on our wealth to be greater than today's. However, this is not necessarily the case. In fact, the two second terms of of Equation (3.36) are always non-positive (see Remark 3.3.1). Having $R_{\theta_*} \leq 0$ is therefore possible, specially if we specify our model for log-returns (Equation (3.2)) adjusted for inflation. In fact, we can interpret R_{θ_*} as a *risk aversion adjusted risk premia* of the portfolio.

Remark 3.4.5. Equation (3.37) obtained in the proof of Proposition 3.4.4 implies through Dynkin's formula that for $t < T$:

$$\mathbb{E}_{w(t)}\left[\Phi(W_{\theta_*}(T),T)\right] = \Phi(w,t) + \mathbb{E}_{w(t)}\left[\int_t^T A\Phi(W_{\theta_*}(s),s)ds\right] = \Phi(w,t), \tag{3.38}$$

which allows us to say that the discounted utility with an optimal investment strategy is a $\mathbb{P}$-martingale.

So far, we have used Theorem 2.3.4, the Hamilton-Jacobi-Bellman equation, to find the value of the optimal future expected wealth subject to an utility function. However, most of the principles in which such a result is based on, still apply to suboptimal strategies. Mathematically, we are saying that if we define a process $(\phi(W_\theta,t))_{t\in[0.T]}$ such that $\phi(W_\theta,T) = G(W_\theta(T)) = W_\theta^\eta/\eta$ then by Dynkin's formula:

$$\mathbb{E}_w\left[\phi(W_\theta,T)\right] = \mathbb{E}_w\left[g(W_\theta(T))\right] = \phi(w,0) + \mathbb{E}_w\left[\int_0^T A\phi(W_\theta(t),t)dt\right], \tag{3.39}$$

or equivalently:

$$\phi(w,0) = \mathbb{E}_w\left[g(W_\theta(T)) - \int_0^T A\phi(W_\theta(t),t)dt\right]. \tag{3.40}$$

If we want to have $\phi(w,0) = J_\theta(w)$ we then would need that:

$$\mathbb{E}_w\left[\int_0^T A\phi(W_\theta(t),t)dt\right] = 0. \tag{3.41}$$

However, by means of T being arbitrary, or equivalently because we want to preserve that $\phi(W(s),s) = \mathbb{E}_{W(s)}\left[g(W_\theta(T)) - \int_s^T A\phi(W_\theta(t),t)dt\right] = \mathbb{E}_{W(s)}[g(W_\theta(T))]$ for all $s < T$, and for all values of $W_\theta(s)$, we should also set:

$$A\phi_\theta(t,W) = 0. \tag{3.42}$$

Were we have introuduced the subindex θ in ϕ to explicitly state that ϕ is a function of time and wealth parametrized by the investment strategy. Equation (3.42) above also implies that Remark 3.4.5 still applies for suboptimal portfolios.

The reasoning above leeds to the following result.

Proposition 3.4.6. *Let* $(\theta(t))_{t\in[0,T]} = \theta(0) \in \Theta$, $g(W) = W^\eta/\eta$, $\eta < 1$, $\eta \neq 0$. *Then* $J_\theta(w)$ *is given by:*

$$J_\theta(w) = \phi_\theta(w,0) = \frac{w^\eta}{\eta} e^{\eta R_\theta T}, \tag{3.43}$$

where R_θ *is a constant given by:*

$$\begin{aligned} R_\theta = &\left(\alpha_0 + \theta^\top \mu\right) + \frac{1}{2}\theta^\top \hat{\Sigma}\hat{\Sigma}^\top \theta(\eta - 1) \\ &+ \sum_{j=1}^{l} \int_{\mathbb{R}} \left(\frac{1}{\eta}\left(1 + \theta^\top \hat{\Gamma}_{\cdot,j}(z)\right)^\eta - \frac{1}{\eta} - \theta^\top \hat{\Gamma}_{\cdot,j}(z)\right) \nu_j(dz)\kappa(t). \end{aligned} \tag{3.44}$$

Proof. The result follows the arguments given from Equation (3.39) to Equation (3.42) and, because we are restricting θ to be constant, the details are analogous to those in Proposition 3.4.4. ∎

Let us get back to characterise the set Θ_ε. We already know that it is a convex set, so we will investigate its boundary and denote its elements (with some abuse of notation) as θ_∂. By Equation (3.37) and Equation (3.43) we get the following:

$$R_{\theta_\partial} = \begin{cases} \frac{\ln(1-\varepsilon)}{\eta T} + R_{\theta_*} & \text{if } \eta > 0 \\ \frac{\ln(1+\varepsilon)}{\eta T} + R_{\theta_*} & \text{if } \eta < 0. \end{cases} \tag{3.45}$$

As in choosing the optimal strategy, Equation (3.45) above shows that determining ε-sub-optimal strategies is still independent of the wealth level, however, $R_{\theta_*} - R_{\theta_\partial}$ is inversely proportional to the time to maturity T, which does not appear in determining optimal strategies. This makes financial sense because we are investing in a (continuously) compounded way and the further we invest in a sub-optimal manner, we can expect to affect the most our future utility, i.e., such a magnitude is inversely proportional to time to maturity.

Hence, we can conclude that the optimal strategy given by Definition 3.3.2 is stable in the sense that Θ_ε has a reasonable shape as we have shown that it is convex, and that the magnitude of the values that belong to it depends in a reasonable way upon financially reasonable parameters.

Remark 3.4.7. Equation (3.45) can seem to be wrong because if we calculate the risk aversion $a = -g''/g'$ as in Definition 2.4.2, we would get that $a(W) = (1-\eta)/W$ which is a function that increases as η decreases. In other words, the lower η is, the more risk averse an investor is. Therefore, we should expect R_{θ_∂} to be closer to R_{θ_*} the smaller η is. However $\eta \mapsto R_{\theta_\partial}(\eta)$ given by Equation (3.45) is not only non-monotone but singular at $\eta = 0$ (a parameter value that we have ruled out but to which we can get arbitrarily close to). This analysis however is wrong. As pointed out in the introduction of [DS21], "*utility has only relative, no nominal, meaning in the sense that it allows for ordering of different strategies but the value function in itself is not directly comparable with other value functions based on other preferences*". In the context of our problem, a value ε should not be *reasonably small* for two investors with different risk aversions. In the context of Equation (3.45) ε is, in reality, $\varepsilon(\eta)$.

3.5 Extension to optimal consumption

Let us continue to work with the framework of the previous section. This time however, we will modify the wealth process so that the investor can dispose some of the wealth for consumption purposes, i.e., we will drop the assumption stated in Equation (3.15) and instead we will consider that

$$dW(t) = W(t^-)\left(\left(\alpha_0 + \theta(t^-)^\top \mu\right) dt + \theta(t^-)^\top \left(\hat{\Sigma} dB(t) + \int_{\mathbb{R}} \hat{\Gamma}(z)\tilde{N}(dz, dt)\right)\right) - \rho(t^-)dt. \tag{3.46}$$

Where ρ now represents consumption from the portfolio and $W(t) = W_{\theta,\rho}(t)$ is still a controlled diffusion where the controls are θ and ρ. Let us define the reasonable consumption strategies in an analogous way to Definition 3.2.2.

Definition 3.5.1 (Reasonable consumption). The set of *reasonable consumption strategies* denoted by $\mathcal{R}$ is the set of all càdlàg and $\mathbb{F}$-adapted stochastic processes in $\mathbb{R}$ that satisfy the following conditions:

1. $r < \rho(t) < W(t) - \mathcal{K}$, $\mathbb{P}$-almost surely, for all $t \in [0, T]$ and some $0 < r \leq \mathcal{K}$.
2. $\rho(t) = \rho(W(t))$ such that $W \mapsto \rho(W)$ is Lipschitz continuous in $(\mathcal{K}, \infty)$.

The purpose of the bounds imposed to $\rho \in \mathcal{R}$ are very natural. The lower bound guarantees that we can always evaluate $g(\rho) = \rho^\eta/\eta$ and that such value is bounded below. The upper bound avoids reaching the critical threshold to wealth $\mathcal{K}$. Lipschitz continuity allows us to have the next result.

Corollary 3.5.2. *Fix* $(\theta(t))_{t\in[0,T]} \in \Theta$ *and* $(\rho(t))_{t\in[0,T]} \in \mathcal{R}$. *Then Equation* (3.46) *has a unique, strong, square integrable solution.*

Proof. Equation (3.46) is almost the same as Equation (3.15) but with different drift. Hence the analysis for the integrands of the stochastic terms in Proposition 3.2.3 remains the same. Proving Lipschitz continuity and at most linear growth follows almost the same steps as Proposition 3.2.3. ■

In this new context, we will assume that the investor wishes to continuously consume her wealth up to a point in time in which she will enjoy a lump sum. To this end, let us define $\delta > 0$, a discounting factor that will allow us to compare different consumption utilities at different points in time. Let us also define $\xi > 0$, a constant that will allow to tune the preferences of discounted wealth over discounted consumption (this is not inconsistent with Remark 3.4.7 because we are not comparing preferences of the same object). In this new setting our performance criterion is the following:

$$J_{\theta,\rho}(w) = \mathbb{E}\left[\int_0^T \frac{\rho(s)^\eta}{\eta} e^{-\delta s} ds + \xi \frac{W_\theta(T)^\eta}{\eta} e^{-\delta T}\right]. \tag{3.47}$$

We therefore want to find $J_{\theta_*,\rho_*}(w)$, the maximiser of Equation (3.47), and state sufficient conditions such that it is well defined. Using the theory developed in Section 3.3 and with arguments similar to those in Section 3.4.1, we will still guess (as educatedly as before) that $\Phi(W,t) = \kappa(t)W^\eta/\eta$. By Theorem 2.3.4 we would need that:

$$\begin{aligned}
A\Phi(W_\theta,t) + g(\rho)e^{-\delta t} = & \frac{W^\eta}{\eta}\kappa'(t) + \left(\left(\alpha_0 + \theta^\top\mu\right)W - \rho\right)W^{\eta-1}\kappa(t) \\
& + \frac{1}{2}\theta^\top\hat\Sigma\hat\Sigma^\top\theta W^2(\eta-1)W^{\eta-2}\kappa(t) \\
& + \sum_{j=1}^{l}\int_{\mathbb{R}}\left(\frac{W^\eta}{\eta}\left(1+\theta^\top\hat\Gamma_{\cdot,j}(z)\right)^\eta - \frac{W^\eta}{\eta} - \theta^\top\hat\Gamma_{\cdot,j}(z)WW^{\eta-1}\right)\nu_j(dz)\kappa(t) \\
& + \frac{\rho^\eta}{\eta}e^{-\delta t} = 0.
\end{aligned} \tag{3.48}$$

The value that maximises $\theta \mapsto A\Phi(W_\theta,t) + g(\rho)$ above is still given by Equation (3.31). Let us find the maximiser for $\rho \mapsto A\Phi(W_\theta,t) + g(\rho)$:

$$-W^{\eta-1}\kappa(t) + \rho_*^{\eta-1}e^{-\delta t} = 0 \iff \rho_* = W\left(\kappa(t)e^{\delta t}\right)^{1/(\eta-1)} \tag{3.49}$$

Taking $\partial_\rho^2(A\Phi(W_\theta,t)+g(\rho))$, we will see that ρ_* is a global maximum, hence, we found the right ρ. Observe too that, similarly as before, the common term W^η vanishes, leaving us with the following:

$$\begin{aligned}
A\Phi(W_{\theta_*,\rho_*},t) + \frac{\rho_*^\eta}{\eta}e^{-\delta t} = & \frac{\kappa'(t)}{\eta} + \left(\left(\alpha_0 + \theta_*^\top\mu\right) - \left(\kappa(t)e^{\delta t}\right)^{1/(\eta-1)}\right)\kappa(t) \\
& + \frac{1}{2}\theta_*^\top\hat\Sigma\hat\Sigma^\top\theta_*(\eta-1)\kappa(t) \\
& + \sum_{j=1}^{l}\int_{\mathbb{R}}\left(\frac{1}{\eta}\left(1+\theta_*^\top\hat\Gamma_{\cdot,j}(z)\right)^\eta - \frac{1}{\eta} - \theta_*^\top\hat\Gamma_{\cdot,j}(z)\right)\nu_j(dz)\kappa(t) \\
& + \frac{\left(\kappa(t)e^{\delta t}\right)^{\eta/(\eta-1)}}{\eta}e^{-\delta t} = 0.
\end{aligned} \tag{3.50}$$

With our previous knowledge and the equation above, we can guess that $\kappa(t) = Ke^{-\delta t}$ for some constant K. Substituting we get:

$$\begin{aligned}
A\Phi(W_{\theta_*,\rho_*},t) + \frac{\rho_*^\eta}{\eta}e^{-\delta t} = & \frac{-\delta Ke^{-\delta t}}{\eta} + \left(\left(\alpha_0 + \theta_*^\top\mu\right)K - K^{\eta/(\eta-1)}\right)e^{-\delta t} \\
& + \frac{1}{2}\theta_*^\top\hat\Sigma\hat\Sigma^\top\theta_*(\eta-1)Ke^{-\delta t} \\
& + \sum_{j=1}^{l}\int_{\mathbb{R}}\left(\frac{1}{\eta}\left(1+\theta_*^\top\hat\Gamma_{\cdot,j}(z)\right)^\eta - \frac{1}{\eta} - \theta_*^\top\hat\Gamma_{\cdot,j}(z)\right)\nu_j(dz)Ke^{-\delta t} \\
& + \frac{K^{\eta/(\eta-1)}}{\eta}e^{-\delta t} = 0.
\end{aligned} \tag{3.51}$$

The common term $Ke^{-\delta}$ will vanish, hence, we can solve for K and get:

$$\begin{aligned} K =&\Bigg(\frac{\eta}{\eta-1}\bigg(-\frac{\delta}{\eta}+\Big(\alpha_0+\theta_*^\top\mu\Big)+\frac{1}{2}\theta_*^\top\hat{\Sigma}\hat{\Sigma}^\top\theta_*(\eta-1) \\ &+\sum_{j=1}^{l}\int_{\mathbb{R}}\Big(\frac{1}{\eta}\Big(1+\theta_*^\top\hat{\Gamma}_{\cdot,j}(z)\Big)^\eta-\frac{1}{\eta}-\theta_*^\top\hat{\Gamma}_{\cdot,j}(z)\Big)\,\nu_j(dz)\bigg)\Bigg)^{\eta-1}. \end{aligned} \tag{3.52}$$

Where Proposition 3.3.6 guarantees that the integral terms in K will converge if θ_* satisfies the credit conditions stated in Definition 3.2.2 (Lipschitz continuity of $W\theta_*$ is immediately guaranteed because θ_* is constant). Hence, we have (almost) proven for $\xi = K$ that:

$$J_{\theta_*,\rho_*}(w)=\Phi(w,0)=K\frac{w^\eta}{\eta}=\left(\frac{\eta}{\eta-1}\left(R_{\theta_*}-\frac{\delta}{\eta}\right)\right)^{\eta-1}\frac{w^\eta}{\eta}. \tag{3.53}$$

Where R_{θ_*} is given in Equation (3.36).

By the way we have found a solution, it is difficult to determine a general method to maximise Equation (3.47) for a general $\xi > 0$ as the step between Equation (3.50) and Equation (3.51) is too restrictive. However, the constant K is interesting enough to be studied because it makes the value of the optimised performance criterion to be independent from time to maturity as T does not appear in Equation (3.53).In other words, an investor with such a preference, is indifferent about receiving the lump sum tomorrow or in a thousand years. Nevertheless, this assertion is only valid if we check that the integrability conditions in Theorem 2.3.4 are satisfied because, if that is not the case, any investor would be indifferent between two undetermined amounts of money. Before checking integrabilty conditions, let us observe that $R_{\theta_*}-\delta/\eta$ in Equation (3.53) should be negative so that $K > 0$. This will indeed suffice to prove integrabilty in our framework.

Proposition 3.5.3. *Let $\theta_* \in \Theta$, $\rho_* \in \mathcal{R}$ and let $\delta > R_{\theta_*}\eta$. Then Equation* (3.53) *is well posed.*

Proof. Choose an arbitrary $T \in [0,\infty)$ and recall that g is bounded below and increasing. Recall also that, by Definition 3.5.1, $0 < r < \rho < W$, hence,

$$\begin{aligned} \mathbb{E}_w\left[\int_0^T |A\Phi(W_{\theta_*,\rho_*}(t),t)|\,dt\right] &= \mathbb{E}\left[\int_0^T\left|\frac{\rho_*(t)^\eta}{\eta}\right|e^{-\delta t}dt\right] \\ &\le -2\min\left\{\frac{r^\eta}{\eta},0\right\}\int_0^T e^{-\delta t}dt+\mathbb{E}_w\left[\int_0^T\left|\frac{W_{\theta_*,\rho_*}(t)^\eta}{\eta}\right|e^{-\delta t}dt\right] \\ &= K_1+\int_0^T\mathbb{E}_w\left[\left|\frac{W_{\theta_*,\rho_*}(t)^\eta}{\eta}\right|\right]e^{-\delta t}dt \\ &\le K_1+\int_0^T\mathbb{E}_w\left[\left|\frac{W_{\theta_*}(t)^\eta}{\eta}\right|\right]e^{-\delta t}dt \\ &= K_1+\left|\frac{w^\eta}{\eta}\right|\int_0^T e^{(\eta R_{\theta_*}-\delta)t}dt<\infty. \end{aligned}$$

Where $K_1 = -2\min\{r^\eta/\eta, 0\}\int_0^T e^{-\delta t}dt$, and $W_{\theta_*}(t)$ is a portfolio with the same investment strategy as W_{θ_*,ρ_*} but without consumption. The last line is given by Proposition 3.4.4. ■

Remark 3.5.4. Because the claim above was proved for all T, even if $T = \infty$, solving this problem turns out to be analogous to solving the problem of perpetual consumption presented in [Mer71] or [ØS19, Example 5.2].

Chapter 4

Concluding remarks

In this thesis we defined a multi-variate Lévy process driven market model and stated conditions such that the model itself and some optimisation problems that naturally arise in the context of portfolio management, are well defined and can be solved. A general conclusion to this thesis is that, under relatively mild conditions that are quite reasonable in a financial context, we can find optimal portfolio proportions for CRRA utility functions and, furthermore, the ground is settled to extend these conclusions to other type of risk aversions. The financial implications of this thesis' results ultimately depend upon what we believe the actual structure of the market is or, in other words, depend (as any mathematical model) upon what it is assumed.

4.1 The importance of assumptions

The model herein presented is quite flexible because it is very suitable for factor analysis and, according to how we specify it, different financially meaningful conclusions can be drawn. Let us for a moment get back to Example 3.3.10 at the end of Section 3.3 where we solved the problem of finding the optimal investment strategy for a fixed time horizon portfolio in a very general setting. In that case we concluded that not having any levered nor short positions guaranteed that the problem wouldn't be miss-specified and that expectations could be evaluated. If this were to be always true, why do banks or private equity firms even exist? Are they doomed to fail?

It is not crystal clear that any model can answer any of these questions, quite the opposite, somehow a (hopefully well supported) opinion on these issues is what should drive any model choice. More concretely, in the context of the Example 3.3.10, if an investor with risk preferences described by a CRRA utility function truly believes that there is a systemic risk that can negatively impact all the risky assets at once with an unbounded impact (which in the context of the thesis is equivalent to saying that $\gamma_{\cdot,j} > 0$ with a Lévy measure supported in $(-\infty, r)$ for some $r > 0$), then the answer is yes, the value of the wealth of levered institutions is undetermined for such an investor. On the other hand, if such an investor with CRRA preferences does believe that the market operates in a continuous manner, then any constant proportion strategy is allowed. Perhaps an investor believes there exists systemic risk but with a bounded impact on log-returns, then questions like leverage can be opened to discussion again. Which of these is the most reasonable assumption? Answering this question is left as an exercise to the reader.

Ironies aside, this example illustrates and motivates why it is relevant to further investigate models with jumps, because introducing jumps completely changes the strategies a rational investor may follow. Specially in a world that changes as fast as ours, it is imperative to have models in which any challenges to the what is believed in the *status quo* can have an answer.

4.2 Suggestions for further research

It would be rather ironic to talk about assumptions as in the previous section without pointing out what can be considered to be unrealistic or incomplete in our model. On this regard, the first issue that can be mentioned is that the model here presented does not account for transaction costs. Definitely it would be a great contribution to develop an analogous analysis to the one presented herein in which market friction is accounted for.

Furthermore, it is also clear that not everything has been done under the frictionless market assumption, let alone in the concrete context of this thesis. For instance, the most natural next step (according to the author) to this thesis would be to find an analogous result for the general Lévy model to Proposition 3.3.3, where we found that in the purely Brownian motion case, the regularity of the utility function is analogous to the regularity of the amount of money invested in every asset when the investment strategy is optimal. Another natural step in the direction of this thesis would be to generalise the results here presented for the case when the Poisson random measures are defined over $\mathbb{R}^n$, i.e., to change the structure of the Lévy measures we work with from $\sum_{i=1}^{l} \int_{\mathbb{R}} \nu(dz)$ to $\int_{\mathbb{R}^n} \nu(dz)$. This would allow to account for more general dependence structures in the log-returns of the risky assets.

Many other interesting suggestions for further research can be proposed because, as in any topic that pertains to human knowledge, more questions arise after some study. Indeed, any piece of research accomplishes its purpose, not by answering its questions but by swapping the latter for better ones. I hope you enjoyed reading this thesis as much as I enjoyed writing it.

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