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Symmetries on self-similar fractals

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Symmetries on self-similar fractals

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Abstract

This thesis discusses self-similar fractals, more specifically different symmetries on self-similar fractals. We build the fractal space as a collection of compact subsets of a complete metric space. Further, we add IFS theory to give the foundation to what fractals are and how they are constructed. We also use IMS's to represent the fractals. We will investigate the Sierpinski gasket and the Vicsek fractal, two different fractals that is shown to have two different outcomes. We find that the Sierpinski gasket fall under the same symmetry group as the equilateral triangle, however the Vicsek fractal does not fall under the same symmetry group as the square.

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I also want to give my thanks and appreciation to my fellow students for good discussions on difficult topics and for hours playing geoguessr. Last but not least a huge thanks to my family and friends for talking with me about other things than just maths and fractals.

And of course, a thanks to He whom created the fractals.

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Mauritz Angell Mentzoni

Chapter 1

Introduction

Fractals is a relatively new topic in mathematics which has become an addition to how we can model different objects and phenomena in nature. Depending on whether you ask a mathematician or the general public, there are some different connotations associated with the word fractal. Fractal is often connected to what is fractal art, and on social media it is easy to find videos that zoom into a certain fractal. Even though these are fractal, the mathematical discussion is quite different.

We also find fractals in nature and fractals has been shown to be applicable to modelling trees, rivers and coastlines. There is an issue with fractals, which is that fractals are, mathematically, infinitely detailed at arbitrary small spaces, but we live in a world that seems to have a limit in how small things can get (or at least a smallest scale we can measure). So, fractals are not perfect in modelling nature, but the fact that nature seems to be self-similar at different scales makes fractals relevant tools.

To examine and discuss fractals, there are several different mathematical subject we need an understanding for. It is therefore we are going to use some time to lay out some basic theory in group theory and real analysis in the form of metric spaces, we shall also dive into some lesser known topics namely iterated function systems and injective mapping systems shortened to IFS and IMS respectively. When all this theory is accounted for, we begin to examine two established fractals, the Sierpinski gasket and the Vicsek fractal. My contributions to fractals, in the thesis, are the way we can discuss some symmetries on fractals, i.e., the definitions, theorems and proofs in 3. As we already have a an established field that discusses symmetries on geometric shapes, namely abstract algebra and more specifically group theory, we only need to try to map the fractals in such a manner that we can treat them the same as an object we already are familiar with. We manage to map the Sierpinski gasket to the equilateral triangle, but we are on the contrary not able to map the Vicsek fractal to the square. Lastly, we will shortly discuss some underlying reasons and causes for the two different outcomes.

Now, lets get into it.

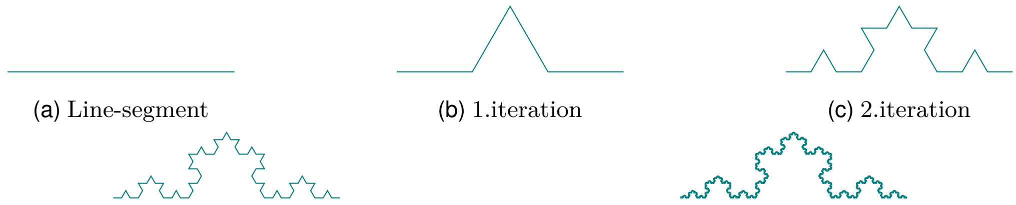
Chapter 2

Space where fractals live

2.1 How we came to understand fractals

There is no one precise mathematical definition of what a fractal is, but there are several features that is found in different fractal structures. It will be useful to have an overview of such features before we dive into the mathematics we use when we discuss fractals.

Starting with a historical perspective, there are several ideas that had to evolve in mathematics before one could properly discuss what today is known as fractals. Leibniz had ideas of recursive self-similarity which is one of the foundational ideas of the structure of a fractal, but it was not until the late 19th century Weierstrass presented something that actually would be a fractal, namely a function that was everywhere continuous and also nowhere differentiable. Later, in early 21st century, Koch came with a more geometrical perspective, and created the Koch snowflake. There were several more contributions before Mandelbrot coined the term "fractal" in 1975 and created computer-generated visualizations of fractals, specifically Julia sets and the Mandelbrot set.
So, with the culmination of the different ideas in place we became ready to describe what fractals actually are, the problem is that fractals are quite different in the way they are structured and there are many different variants. When we now are going to capture what a fractal is, we do not have a strict mathematical definition, but there are different features that are ubiquitous in fractals, and we use these features to check whether an object is in fact a fractal. These features are self-similarity, detailed structure at arbitrary small scales, and some way of describing the irregularity found in fractals.
A good example to illustrate this is the Koch curve, which is a part of the Koch snowflake:



(a) Line-segment (b) 1.iteration (c) 2.iteration (d) 3.iteration (e) 6.iteration

Figure 2.1: Some iterations toward the Koch curve

Here we see an example on how we can construct a fractal: first we create this process, where we take the line-segment, make four copies third the size of the original line-segment and place them such that we get the object we see on the second illustration. Now we do the exact same process, but with the new object illustrated in Figure 2.1b instead of the line-segment in 2.1a. We continue with precisely the same process but with the "newest" object we have evoked. When we iterate this process infinitely, we are left with what we call the Koch curve. The Koch curve is clearly self-similar on different levels, it has a detailed structure at arbitrary small scales, and the process creates the irregularity we observe the curve to have. Even though the features are what specifically defines the Koch curve to be a fractal, the gestalt of them gives reason to define the Koch curve as a fractal.

A good counter-example is the line-segment we start with, even though a line is self-similar, it does not have arbitrary small detailed structure or some kind of irregularity attached to its features, and so it is not a fractal.

The Koch curve is a simplified example of the Koch snowflake, which is a fractal. But there might be useful to look at another fractal before we go further. So, we are going to be considering the Sierpinski gasket, it might be useful to give it a quick description, as we are going to discuss this particular fractal later, and as notation we will be writing the Sierpinski gasket as S_g.

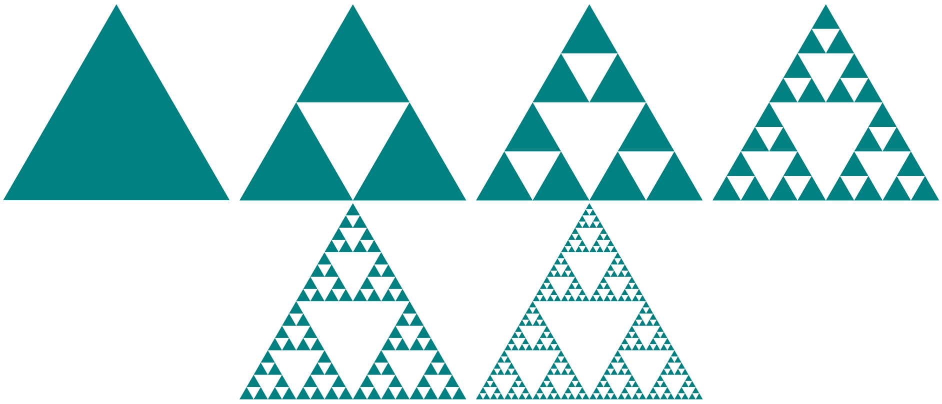


Figure 2.2: 5 first iterations toward the Sierpinski gasket

As we look at the different iterations toward the Sierpinski gasket, S_g, there are several different ways to interpret and model how the figures relate to the objects in the iteration before and after. One interpretation is that you remove an area of the colored triangles that corresponds with a flipped triangle which is a fourth the area but with the same shape, and then we iterate this process.

Another way of looking at this is through something we call an iterated function system, IFS for short. We will go through what an IFS is more thoroughly later, but for now we can formulate it like this. We look at a specific object, whichever one of the iterations shown in Figure 2.2, and then we shrink the object, make two more copies, and place the three different copies in each corner to replace original object.

We will also take a look at injective mapping systems or IMS's, this is an alternative that bears a lot of similarities to IFS's and is developed by Klara Hveberg. The point is that there are several ways to attack the issue of fractals, and S_g is relatively "simple" compared to other fractals. We are going to mostly take use of the theory of IFS and IMS to discuss the fractals and their properties. But before we can dive into how the fractals behave and emerge, we need to know where they are. And to find where fractals live, we need some tools and theory.

2.2 Metric spaces

Before we go into the construction of the space where we find fractals, we will do our due diligence and go through some core ideas of metric spaces. The goal is to have all the building blocks necessary so we can construct a space where all elements are compact subsets of a complete metric space, and to build this we need some theory on metric spaces. The source for the different definitions, theorems and proof are mainly from [Lin17]. So, we start with the definition of a metric space:

Definition 2.2.1 (Metric space). A metric space: $(\mathbf{X}, d)$, consists of a space $\mathbf{X}$ with a real-valued function $d : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$, which is interpreted as a measure for distance between two points in $\mathbf{X}$. d is called a metric on $\mathbf{X}$ if the following is obeyed:

- $d(x, y) \geq 0,\ \forall x, y \in \mathbf{X}$, with equality if and only if $x = y$.
- $d(x, y) = d(y, x),\ \forall x, y \in \mathbf{X}$.
- $d(x, z) \leq d(x, y) + d(y, z),\ \forall x, y, z \in \mathbf{X}$.

Probably the most familiar example of a metric space is the 3-dimensional Euclidean space. Now when we look at the definition of a metric considering the Euclidean 3-D space, it is sensible when we say a distance between two points is positive, the distance is the same whether you go from point a to b or the other way, and if you have to go through another point, c, on the way from a to b then the total distance is the same or longer. This do apply for all n-dimensional Euclidean spaces with this metric, i.e. $\mathbb{R}^n,\ \forall\ n \in \mathbb{N}$ with the euclidean metric is a metric space. Another foundational definition for when we are going to discuss metric spaces, are limits.

Definition 2.2.2 (Limits). Let $(\mathbf{X}, d)$ be a metric space, and let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of numbers where all $x_i \in \mathbf{X}$. This sequence, $\{x_n\}_{n=1}^{\infty}$, converges to a point $a \in \mathbf{X}$ if for every $\epsilon > 0,\ \exists\ N$ such that $d(x_n - a) < \epsilon\ \forall\ n \geq N$. We call this the limit of the sequence and write: $\lim_{n\to\infty} x_n = a$.

With the definitions for metric spaces and limits in place, we want to define what a complete metric space is, but then we need to start with Cauchy sequences, as such sequences lays the foundation for how we define what a complete metric space is.

Definition 2.2.3 (Cauchy sequence). We have a metric space $(\mathbf{X}, d)$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points where $x_n \in (\mathbf{X}, d)$. This sequence is a Cauchy sequence if for any given $\epsilon > 0,\ \exists\ N > 0$ such that $d(x_n, x_m) < \epsilon,\ \forall\ n, m > N$.

To summarize the idea: if we have a sequence of infinite numbers, and from a certain point in the sequence, the difference between any set of two numbers is arbitrarily small. Then the sequence is a Cauchy sequence.
And now we use the idea of Cauchy sequences to define a complete metric space.

Definition 2.2.4 (Completeness). A metric space $(\mathbf{X}, d)$ is a complete metric space, if every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbf{X}$ has a limit $x \in \mathbf{X}$.

To have a better intuition of what this actually means, we say that a complete metric space is a metric space that have no "missing point". To illustrate this, we can look at the rational numbers, $\mathbb{Q}$, which has a lot of "holes" in it, by not containing the irrational numbers. As an example, $\sqrt{2}$ is "missing" from $\mathbb{Q}$ in the sense that we can construct a Cauchy sequence of rational numbers that converges to $\sqrt{2}$, but $\sqrt{2}$ is not in $\mathbb{Q}$. And therefore, $\mathbb{Q}$ has "holes" and not complete.
Now we will give a definition for compact metric spaces.

Definition 2.2.5 (Compactness). Let $A \subset \mathbf{X}$ be a subset of $(\mathbf{X}, d)$. A is compact if every sequence $\{x_n\}_{n=1}^{\infty}$, where $x_i \in A$, contains a subsequence that has a limit which is in A.

The notion of a compact metric space is related to the ideas of closed and bounded sets. A compact space does not have "missing end points", i.e. it contains all limits, hence the importance of the limits of the subsequences. As an example, we can look at some subsets of $\mathbb{R}$: namely $(0,1), [0,1), [0,1]$. The first and second sets are not compact, as there are limits of sequences that are not contained in the set itself, for example the sequence $\lim_{n=1}^{\infty} \frac{1}{2^n}$ has a limit in 0, and is not contained in the first set. But where third set here is compact.

Now that we understand different spaces with different characteristics and properties. As we will be mapping these spaces on each other, we need to investigate functions and how we can use them to relate the different spaces. One of the most important and useful ideas connected to functions is continuity, so this is where we start.

Definition 2.2.6 (Pointwise continuity). Let $A \subset \mathbf{X}$ be a subset of $(\mathbf{X}, d)$. A is compact if every sequence $\{x_n\}_{n=1}^{\infty}$, where $x_i \in A$, contains a subsequence that has a limit which is in A.

So, if f is continuous at a point a, we can choose a distance, δ, between x and a, such that the distance between $f(x)$ and $f(a)$ is "as small as we want". We shall expand a bit on the idea of continuity, we have already seen the definition and usefulness of limits and convergence. Next, we shall show that if we have a continuous function, the function will "carry over" the convergent points from one metric space to the other.

Lemma 2.2.7 (Continuity carries convergence). *Given that $f : \mathbf{X} \to \mathbf{Y}$ is a continuous function, and that a sequence $\{x_n\}$ converges to a point a. Then the sequence $\{f(x_n)\}$ converges to the point $f(a)$.*

Proof. What we want to show, given the assumptions, is that the distance between $f(x_n)$ and $f(a)$ is as small as we want.
Since f is a continuous function we know that for any $\epsilon > 0$, $\exists\ \delta > 0$ such that $d_Y(f(x_n), f(a)) < \epsilon$ whenever $d_X(x, a) > \delta$. We already have that x_n converges to a, and so there is an $N \in \mathbb{N}$ such that $d_X(x_n, a) < \delta$ when $n \geq N$. ■

We have established some ideas for continuity and convergence for a single point, and will expand the terminology and idea for functions, not just for single points, but on whole spaces.

Definition 2.2.8 (Continuity). A function $f : \mathbf{X} \to \mathbf{Y}$ is continuous if f is continuous at all points $x \in \mathbf{X}$.

So, if we have a function that is continuous at all points, then we say that the function is continuous. There is one last thing we want to describe before we go ahead with discussing the fractals, and that is the Banach fixed point theorem, which is essential to describing how fractals are constructed. And to understand this theorem, we first need to establish a type of function which we call contractions, which definition and following lemma is found in [Bar93].

Definition 2.2.9 (Contraction). Given the metric space $(\mathbf{X}, d)$, and a contractivity factor $0 \leq s < 1$. A contraction is a function $w : \mathbf{X} \to \mathbf{X}$ such that, $d(w(x), w(y)) \leq s \cdot d(x, y),\ \forall x, y \in \mathbf{X}$.

In other words, all that a contraction do, is to push different points closer to each other. So, when we would look at sets instead of only single points, this would mean that all points would be closer to all other points in the output of the function. And as with convergence, we want to see the the c

Lemma 2.2.10 (Continuous contraction). *Let $w : \mathbf{X} \to \mathbf{X}$ be a contraction on $(\mathbf{X}, d)$, then w is continuous.*

Proof. Given an $\epsilon > 0$, let $s > 0$ be a contraction on $w : \mathbf{X} \to \mathbf{X}$ with $x, y \in \mathbf{X}$. Then we have that $d(w(x), w(y)) \leq s \cdot d(x, y) < \epsilon$ whenever $d(x, y) < \delta$ where $\delta = \frac{\epsilon}{s}$. Hence w is continuous. ■

Now we are ready for Banach's fixed point theorem.

Theorem 2.2.11 (Banach fixed point). *Again we let* $(\mathbf{X}, d)$ *be a non-empty complete metric space, we also let* $w : \mathbf{X} \to \mathbf{X}$ *be a contraction mapping on* $(\mathbf{X}, d)$*. Then* w *has an accompanying unique fixed point* x^**, i.e.* $w(x^*) = x^*$ *where* $x^* \in \mathbf{X}$*. Moreover, if we also start with any point* $x_0 \in \mathbf{X}$ *and have the sequence:*

$$x_0, x_1 = w(x_0), x_2 = w^{\circ 2}(x_0), ..., x_n = w^{\circ n}(x_0), ...$$

which converge to x^**.*

Proof. To prove Banach fixed point theorem, there are two considerations we must address, namely that the function w must have a fixed point and that it cannot have more than one fixed point. We start with the latter:
If w has any two fixed points $x^*, y^* \in \mathbf{X}$ with a contraction factor s on w, then we have:

$$d(x^*, y^*) = d(w(x^*), w(y^*)) \leq s \cdot d(x^*, y^*).$$

Since $0 \leq s < 1$, the only way that this holds is if $d(x^*, y^*) = 0$, and hence that $x^* = y^*$. And as we chose x^*, y^* arbitrarily, if one where to claim that w had more than two fixed points, we could use the same argument showing all claimed point must equal x^* and there can therefore only be one fixed point.

Now we want to show that there must exist such a fixed point. We start with an arbitrary starting point x_0 and with the sequence:

$$x_0, x_1 = w(x_0), x_2 = w^{\circ 2}(x_0), ..., x_n = w^{\circ n}(x_0), ...$$

Here we will first show that the sequence $\{x_n\}$ is a Cauchy sequence, and then given that the sequence is Cauchy, we have a fixed point:
So, we choose to elements of the sequence; x_n, x_{n+k}. We perform a repetition of the triangle inequality:

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+(k-1)}, x_{n+k}) \\ &= d(w^{\circ n}(x_0), w^{\circ n}(x_1)) + d(w^{\circ n+1}(x_0), w^{\circ n+1}(x_1)) + ... \\ &\quad ... + d(w^{\circ n+(k-1)}(x_0), w^{\circ n+(k-1)}(x_1)) \\ &\leq s^n d(x_0, x_1) + s^{n+1} d(x_0, x_1) + ... + s^{n+(k-1)} d(x_0, x_1) \\ &= \tfrac{s^n(1-s^k)}{1-s} d(x_0, x_1) \leq \tfrac{s^n}{1-s} d(x_0, x_1) \end{aligned}$$

Now we have shown that the distance between two points in the given sequence is less or equal to the geometric sum written lastly. We can write this sum arbitrarily small if we choose n large enough. Let $\epsilon > 0$, then $\exists\ N$ such that $\frac{s^N}{1-s}d(x_0, x_1) < \epsilon$. And so, for $n, m = (n+k) \geq N$ we have that:

$$d(x_n, x_m) \leq \tfrac{s^n}{1-s}d(x_0, x_1) < \epsilon$$

And therefore, we have that $\{x_n\}$ is a Cauchy sequence.

Now for showing that there is a fixed point. Since the metric space $(\mathbf{X}, d)$ is complete, the sequence must necessarily converge to a point x^*. We further see from the sequence, that $x_{n+1} = w(x_n)\ \forall n$, and when we consider the limit $n \to \infty$, we have that $x^* = w(x^*)$. Hence x^* is a fixed point for w.

■

And with that, we have all the necessary theory to build our desired space where we can find and discuss fractals.

2.3 Fractal space

To understand what fractals are, we need to get an understanding of their structure, and so we are going to use metric spaces, more specifically compact sub-spaces of a complete metric space, to be the playground for fractals. The following construction of fractal space is found in [Bar93].

So we take a complete metric space, $(\mathbf{X}, d)$, where d is a metric on the space $\mathbf{X}$. Further we want to construct a space which is the collection of compact and non-empty subsets of $\mathbf{X}$, we denote the set for this space: $\mathcal{H}(\mathbf{X})$.

Now we need a way to define the distance in the set $\mathcal{H}(\mathbf{X})$, since every point in $\mathcal{H}(\mathbf{X})$ is a set in itself, we want a metric that is able to capture the distance not just between two points, but two sets. So first we define the distance from a set to a point by choosing the shortest distance between a point from the set to the singular point: we let $A, B \subset \mathcal{H}(\mathbf{X})$, and let $d(x, B) = \min\{d(x, y) \mid y \in B\}$ be the distance between the set B and a point $x \in A$. Now that we have a description for the distance between a point and a set, we define the distance between to sets to be the largest distance between one of the sets to a point in the other: $d(A, B) = \max\{d(x, B) \mid x \in A\}$. The problem with this definition is that this does not work as a metric for $\mathcal{H}(\mathbf{X})$, because it is not symmetric, i.e. $d(A, B) \neq d(B, A)$. Therefore we need to define the metric as the largest distance of these variations, hence we define the metric: $h(A, B) = \max\{d(A, B),\ d(B, A)\}$.

This distance is called the Hausdorff distance, and in Figure 2.3 we see an illustration of the Hausdorff metric. Here we have two sets A and B, where the teal-colored arrow, (left to right), is the maximal distance from a point in A to the set B, and the purple arrow, (right to left), represent the maximal distance from a point in B to the set A. Here it might be more obvious that it is necessary to choose one of the arrows, so that h is symmetric and therefore a metric.

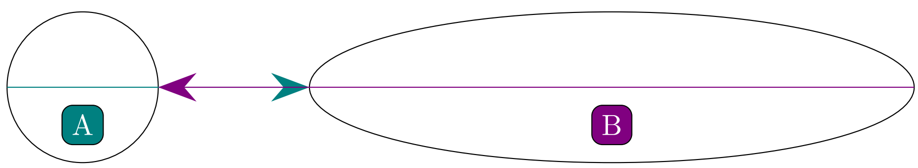


Figure 2.3: Hausdorff distance

For further discussions we will be naming $\mathcal{H}(\mathbf{X})$ as the fractal space. It will also be useful to summarize and formulate the fractal space in mathematical terms, so here we will give it a formal definition accompanied by a proof for h being a metric.

Definition 2.3.1 (Fractal space). Given complete metric space $(\mathbf{X}, d)$, $\mathcal{H}(\mathbf{X})$ denotes the space of non-zero, compact subsets of $\mathbf{X}$. Further we let $A, B \subset \mathcal{H}(\mathbf{X})$, and $d(A, B) = \max\{d(x, B) \mid x \in A\}$. Then we construct a metric for $\mathcal{H}(\mathbf{X})$ such that $h = \max\{d(A, B),\ d(B, A)\}$. And so, we have our space $(\mathcal{H}(\mathbf{X}), h)$.

With the given definition of the space $(\mathcal{H}(\mathbf{X}), h)$, we now want to prove that the Hausdorff metric, h, is indeed a metric for $\mathcal{H}(\mathbf{X})$, and we will further be naming $(\mathcal{H}(\mathbf{X}), h)$ as the fractal space.

Proof. It is quite straight forward to prove, we need to show positivity, symmetry and the triangle inequality. But firstly, let $A, B, C \in \mathcal{H}(\mathbf{X})$

- Non-negative: Obviously $h(A, A) = \max\{d(A, A),\ d(A, A)\} = \max\{d(x, A) \mid x \in A\} = 0$, and as when $h(A, B)$ we get some $d(a, b)$ where $a \in A$ and $b \in B$, h is non-negative.
- Symmetry: As $h = \max\{d(A, B),\ d(B, A)\}$, we obviously have symmetry for h.
- Triangle inequality: Firstly we need to show that $d(A, B) \leq d(A, C) + d(C, B)$. So, for any $a \in A$, $b \in B$:
  $d(a, B) = \min\{d(a, b)\} \leq \min\{d(a, c) + d(c, b)\} = d(a, c) + \min\{d(c, b)\}$ this is $\forall\ c \in C$.
  $d(a, B) \leq \min\{d(a, c)\} + \max\{\min\{d(c, b)\}\} = d(a, C) + d(C, B)$, for some $c \in C$.
  $d(A, B) \leq d(A, C) + d(C, B)$
  We use the same argument for $d(B, A)$, and therefore we have that $h(A, B) = \max\{d(A, B),\ d(B, A)\} \leq \max\{d(B, C), d(C, B)\} + \max\{d(A, C), d(C, A) = h(B, C) + h(A, C)$.

■

Before we continue, it is worth to comment on the non-negative aspect of the proof. We only have non-negativity when A and B are compact sets. To give an example: if $A = [0, 1], B = (0, 1)$ we have that $h(A, B) = 0$ even though $A \neq B$. But for us this is not an issue as we are concentrated on $\mathcal{H}(\mathbf{X})$, and A, B are compact sets. Another thing worth mentioning, is the generality of this space. We have looked at an example in 2.3, and this exhibits the hausdorff metric on the basis that the metric d, that we have from our assumed metric space $(\mathbf{X}, d)$, is the euclidean metric in two dimensional space, i.e., free movement in $\mathbb{R}^2$. But there are no restrictions on the metric d, given that it is a

metric for the space, which also need to be complete.
Now that we have our metric space: $(\mathcal{H}(\mathbf{X}), h)$. There is one attribute of this space that we want to prove, namely that $(\mathcal{H}(\mathbf{X}), h)$ is complete, this is necessary so that we can use Banach fixed point theorem, which is essential in how we construct fractals. However, there is one condition for using Banach fixed point theorem that is not yet attained, and that is that $(\mathcal{H}(\mathbf{X}), h)$ must be complete. We shall write this completeness in mathematical terms, but the proof is found on page 36 in [Bar93].

Theorem 2.3.2 (($\mathcal{H}(\mathbf{X}), h$) is complete). *Let $(\mathbf{X}, d)$ be a complete metric space, then $(\mathcal{H}(\mathbf{X}), h)$ is also a complete metric space. Moreover, let $\{A_n \in \mathcal{H}(\mathbf{X})\}_{n=1}^{\infty}$ be a Cauchy sequence, then we have that $A = \lim_{n\to\infty} A_n \in \mathcal{H}(\mathbf{X})$.*

So, not only do we have that $(\mathcal{H}(\mathbf{X}), h)$ is a complete metric space itself, but also have a set A that is the set that consists of the points that Cauchy sequences in $\mathcal{H}(\mathbf{X})$ converge to. This is important for the proof of $(\mathcal{H}(\mathbf{X}), h)$ being complete. Now we are ready to define iterated function systems, and investigate how they can be a source to structure and describe fractals.

2.4 Iterated function systems - IFS

In the construction the fractals the most common and used tool is an iterated function system, which we will denote as IFS. This creates a foundation where we describe a development of fractalization from a starting geometric object, where the iterated function copies, contracts and moves the original object in a careful manner such that when this operation is repeated infinitely many times, we are left with a fractal. We already have the contraction for general metric spaces $(\mathbf{X}, d)$. And now that we have the new fractal space in place, we want to specify the properties for contraction mappings and show that a contraction for the fractal-space also is continuous, as we already have for general metric spaces. And again, these definitions, lemmas and proofs are found in [Bar93].

Lemma 2.4.1 (Continuous contraction on $(\mathcal{H}(\mathbf{X}), h)$). *Let $w : \mathbf{X} \to \mathbf{X}$ be continuous on $(\mathbf{X}, d)$, then w maps $\mathcal{H}(\mathbf{X})$ onto itself.*

Proof. Let $B \subset \mathcal{H}(\mathbf{X})$, since B is in the fractal-space, we want to show that $w(B)$ also is in the fractal-space, i.e., that $w(B)$ is non-empty and compact.
As B itself is non-empty, $w(B)$ is clearly also non-empty. To show that $w(B)$ is compact we first let $\{y_n = w(x_n)\}$ be an infinite sequence, where all $y_i \in B$. Then $\{x_n\}$ is a sequence where $x_i \in B$. We already have that B is compact, so there is a subsequence $\{x_{m_n}\}$ that converge to a point $x^* \in B$. Since w is continuous, we have that $\{y_{m_n}\} = f(\{x_{m_n}\}$ is a subsequence of $\{y_n\}$ that converge to a point $y^* = f(x^*) \in w(B)$. Hence B is both non-empty and compact, and w maps $\mathcal{H}(\mathbf{X})$ onto itself. ∎

And so, we have been able to show that the function w is both continuous and sends $\mathcal{H}(\mathbf{X})$ to itself, now we are ready to show that w is actually a contraction on $\mathcal{H}(\mathbf{X})$.

Lemma 2.4.2 (Contraction on $(\mathcal{H}(\mathbf{X}), h)$). *As $w : \mathbf{X} \to \mathbf{X}$ is a contraction on $(\mathbf{X}, d)$, with contractivity factor s. Then $w : \mathcal{H}(\mathbf{X}) \to \mathcal{H}(\mathbf{X})$ is also a contraction on $(\mathcal{H}(\mathbf{X}), h)$ defined by $w(B) = \{w(x) : x \in B\}, \ \forall \ B \in \mathcal{H}(\mathbf{X})$.*

Proof. Here we need to show that the contraction actually happens with the contractivity factor s, and we do this straightforward:
Let $A, B \in \mathcal{H}(\mathbf{X})$, and let $a \in A$ and $b \in B$

- $d(w(A), w(B)) = \max\{\min\{d(w(a), w(b)\}\}$
  $\leq \max\{\min\{s \cdot d(x, y)\}\} = s \cdot d(A, B)$
- Also; $d(w(B), w(A)) \leq s \cdot d(B, A)$
- Hence; $h(w(a), w(B)) \leq \max\{s \cdot d(A, B), s \cdot d(B, A)\} \leq s \cdot d(A, B)$

Therefore we have that w is a contraction on $(\mathcal{H}(\mathbf{X}), h)$. ■

Further we want a way to use a system of contractions so that we can design an IFS we can use to analyze fractals.

Definition 2.4.3 (IFS). Let $\{w_n : n = 1, 2, ..., N\}$ be contractions on $(\mathcal{H}(\mathbf{X}), h)$, with corresponding contractivity factors $\{s_n\}$ where $0 \leq s_n < 1$, $\forall n = 1, 2, ..., N$. Then $W(B) = \bigcup_{n=1}^{n} w_n(B)$, $B \in \mathcal{H}(\mathbf{X})$, such that W is an IFS operating on $\mathcal{H}(\mathbf{X})$.

Finally, we need the IFS determine where the IFS "takes us", so we want to use a fixed point to affirm this. Then we need Banach fixed point theorem, we have stated that the space $(\mathcal{H}(\mathbf{X}), h)$ is complete, and this is necessary for the theorem to be applied.

Theorem 2.4.4. *Let $A \in \mathcal{H}(\mathbf{X})$ and W be an IFS on $\mathcal{H}(\mathbf{X})$. A is a unique fixed point, if $A = W(A) = \bigcup_{n=1}^{N} w_n(A)$, where $A = \lim_{n\to\infty} W^{(n)}(B)$, for any $B \in \mathcal{H}(\mathbf{X})$. And we then call A the attractor to the IFS.*

2.5 Injective mapping systems - IMS

As an alternative to the IFS structure of fractals, Hveberg's theory of IMS's brings a wider generality and detaches the systems from the underlying geometric restrictions present in the IFS's, and the theory that is built here is collected from Hveberg's dissitation [Hve05]. It also contains the "n-cell"-notation which will be useful for discussing different fractals.

To structure an IMS we start with a set A and a finite set $\Psi = \{\psi_1, ..., \psi_m\}$ where $\psi_i : A \to A$ are injective mappings. And so we construct the composition of different injective mappings, $\psi_{i_1} \circ \psi_{i_2} \circ ... \circ \psi_{i_n}$, where each individual mapping is from the set Ψ, we denote the composition as $\psi_{i_1 i_2 ... i_n}$. If we were to use the composition mapping on a set A: $\psi_{i_1 i_2 ... i_n}(A)$ we call this an n-cell, which we denote $A_{i_1 i_2 ... i_n}$. Here, the index is called the address of the cell $A_{i_1 i_2 ... i_n}$. With this notation and terminology, we are ready to formulate the axioms formulated by Hveberg [Hve05]:

- (1) Self similarity: $A = \bigcup_{i=1}^{N} A_i$
- (2) Completeness: given a sequence $i_1, i_2, ...$ of integers such that $i_j \in \{1, 2, ..., N\}$ for each j, then $\bigcap_{n=1}^{\infty} A_{i_1 i_2 ... i_n}$ contains precisely one point.

Further we have a definition from these two axioms:

Definition 2.5.1 (IMS). Given a set A and a finite set $\Psi = \{\psi_1, ..., \psi_m\}$ of injective mappings such that $\psi_i : A \to A$, where $m \geq 2$, is called an injective mapping system (IMS) if it satisfies axiom (1). If the IMS also satisfies axiom (2), then it is a complete IMS.

2.6 Groups and symmetry groups

Groups and symmetry groups are mathematical objects that are found and discussed in abstract algebra, which main goal is to study algebraic structures, i.e., groups. The historical sources to the creation of abstract algebra lies in three different mathematical fields, number theory, algebraic equations, and geometry. It is the latter that is the primary interest for us when we begin to discuss fractals, we will be connecting some ideas from fractals to groups, but the definitions and theorem are gathered from [Ter19]. If we remember the Sierpinski fractal in Fig.2, we see that the origin for the fractal is the equilateral triangle, and this has a specific description in terms of group theory that we will investigate. But we need to lay some groundwork before we discuss the dihedral group, D_3, so here is a short intro to group theory.

2.6.1 Defining group and symmetry group

Definition 2.6.1 (Group). Generally speaking, a group G is a non-empty set of elements with one operation working on the elements. The operation is a function from ordered pairs from $G \times G$ to G, i.e., $a, b \in G \times G$ are sent to a unique element $a \circ b \in G$, such that the operation has:

- Associativity: $a \circ (b \circ c) = (a \circ b) \circ c,\ \forall\ a, b, c \in G$
- Identity: $\exists\ e$ such that; $a \circ e = e \circ a = a,\ \forall\ a \in G$
- Inverses: given any $a \in G,\ \exists\ a^{-1} \in G$ such that; $a \circ a^{-1} = a^{-1} \circ a = e$

In addition to these criteria for being a group, the group can be Abelian or commutative if for all $a, b \in G$ then $a \circ b = b \circ a$. Also we have that $a, b \in G \Rightarrow a \circ b \in G$ is called closure.

For further discussion there are some terminology and definitions:

Definition 2.6.2 (Subgroup). Let H be a subset of G, $H \subset G$. If H is itself a group under the same operation as G, then H is a subgroup of G. H is further a proper subgroup of G if $H \neq G$ nor $H \neq \{e\}$ where e is the identity element of G.

Definition 2.6.3 (Order). Given a finite group G, the order of G is the number of elements in G, we denote this $|G|$.

The order of a group is then a way of defining the size of the group.

Definition 2.6.4 (Isomorphism). Given the groups G, G' and a function $F : G \to G'$. The function F is an isomorphism if and only if F is bijective and preserve the group operations, i.e., $F(a \circ b) = F(a) \circ F(b)$. We formulate that G and G' are isomorphic, with the accompanying notation: $G \cong G'$.

Having two groups to be isomorphic is a quite strong claim, it is the same as saying that the two groups are mathematically the same. Because of the bijection and preserving of operation, the groups need to be of the same order (size) and structure respectively. With all of these new tools and terms to discuss groups, we are ready to examine some more specific types of groups.

2.6.2 Symmetric groups

There are some more specific groups that will be interesting for us to have a closer look at, namely symmetry groups. Several of the fractals we are going to examine take have a source in an n-polygon, for example the Sierpinski gasket has a base in the equilateral triangle. And so, the definition of a symmetric group:

Definition 2.6.5 (Symmetric groups). The symmetric group S_n, where $n \in \mathbf{Z}^+$, is the group of bijective functions from $\{1, 2, ..., n\}$ onto itself where the operation is composition of functions, also $n \geq 2$. A given element in S_n is a permutation of the integers $1, 2, ..., n$.

Let σ be the permutation on symmetric groups, then we have that a permutation, σ, is uniquely determined by the input:

$$\sigma = \begin{pmatrix} 1 & 2 & ... & n \\ \sigma(1) & \sigma(2) & ... & \sigma(n) \end{pmatrix}$$

S_n is an important group, as it is a tool for describing all finite groups with a given order. This is shown by Cayley's theorem, for the proof for this theorem see page 91 in [Ter19]:

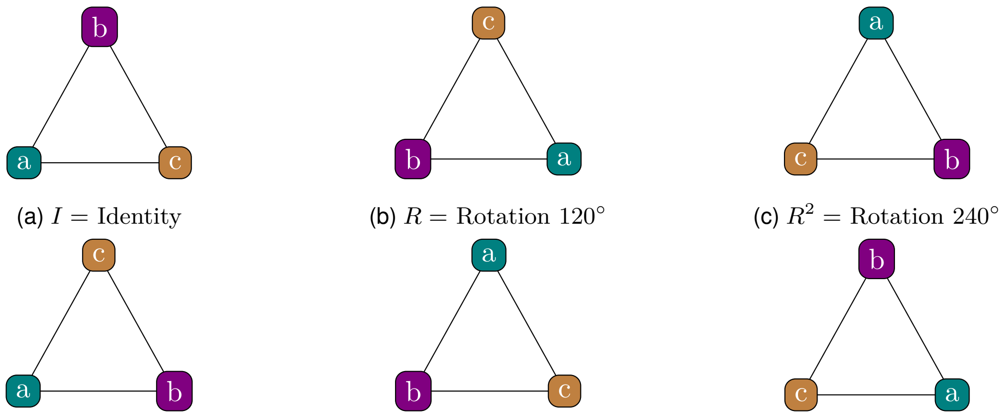
Theorem 2.6.6. *If G is a finite group of order n, then G is isomorphic to a subgroup of S_n, the group of permutations of n objects.*

2.6.3 Dihedral group

To be even more specific to our situation, there is reason to comment on the dihedral group, let us start with the definition:

Definition 2.6.7 (Dihedral group). For $n \in \mathbf{Z}^+$ where $n \geq 3$, the dihedral group is the group describing possible positions of a rigid regular n-gon, i.e., the different ways one can rotate and flip a regular n-gon.

Now we can closer examine the symmetries of different geometrical objects, when we look at the dihedral group, we see that it is related to the symmetric group. Firstly, we can look at D_3 and S_3:



(a) I = Identity (b) R = Rotation 120° (c) R^2 = Rotation 240°

(d) F = Flip (e) FR (f) FR^2

Figure 2.4: Permutations of D_3

Here in Figure 2.4, we observe all the permutations for the equilateral triangle, i.e., D_3 the dihedral group for $n = 3$. When we compare D_3 and S_3 we can see that this is a special case for the relation between D_n and S_n, because D_3 is isomorphic to S_3. But when we examine at the relation for when $n \geq 3$, D_n and S_n are not isomorphic, rather D_n is a proper subgroup of S_n. This distinction can be explained by the sizes of the groups: for the dihedral group, the size is $2 \cdot n$, while for the symmetric group the size is $n!$. Hence, the size of the different groups is the same only for $n = 3$, as D_n is only defined for $n \geq 3$.

Further we will prove that D_n is a subgroup of S_n:

Proof. We let $\{a_1, a_2, ..., a_n\}$ denote the vertices of an n-gon. Then we have that the cyclic permutation $\sigma_1 = (a_1, a_2, ..., a_n) \in S_n$. We also choose any vertex, for example a_1 and draw a line through the center of the n-gon such that we get a reflection of the n-gon, i.e., the permutation $\sigma_2 = (a_2 a_n)(a_3 a_{n-1})...$ of order two. This permutation is also be in S_n. Now D_n is generated by $\sigma_1, \sigma_2 \in S_n$ which shows that $D_n \leq S_n$. ■

Chapter 3

Symmetries on fractals

3.1 Considering symmetries on fractals

We have built up a lot of background theory so that we can mathematically discuss fractals, what they are, where they are and how they behave. This is all necessary to have a good intuition for what we will discuss, but the center of attention will be how fractals behave, more specifically we shall be considering symmetries on fractals. And then it is useful to pick up where we left of on symmetry groups, a permutation σ is uniquely determined by the input from the group that we start with. So, when we look at symmetry groups, the σ is the bijective operation that maps the set of elements onto itself. Now that we shall talk about fractals, we already have the starting set, the fractal itself, and we shall take a closer look at what happens when we send a fractal onto itself, in the same manner that the a permutation does in a symmetry group. From now on we say that such a permutation, that sends a fractal onto itself, is a symmetry, and denote it T (we will also call T a map as well). Another restriction we must discuss for T, is how it maps different parts of the fractal. As we will be using the IMS-structure to describe the fractal, a symmetry T will be mapping n-cells. And then it is important to stress that T sends n-cells to other n-cells, and refrain from sending an n-cell to another level. Put in mathematical terms, T cannot send an n-cell to a k-cell where $n \neq k$.
A problem with describing different permutations/symmetries on a fractal, is that the fractal has infinitely many points. One of the well-known attributes of the Sierpinski gasket, S_g, is that the area of the surface converge toward zero, and the circumference diverge up to infinity. For us, it is the points that are the vertices on the different triangles, n-cells, that are of interest. They are what make up the fractal, and therefore the points that a symmetry must map. And a potential solution to this problem, is the idea that we can fix a set. We can first give the general definition:

Definition 3.1.1 (Fixing a set). Let $B \subset A$ and a symmetry/map $T : A \to A$ such that $T(b) = b, \ \forall \ b \in B$. B fixes the set A if the symmetry T is necessarily the identity map.

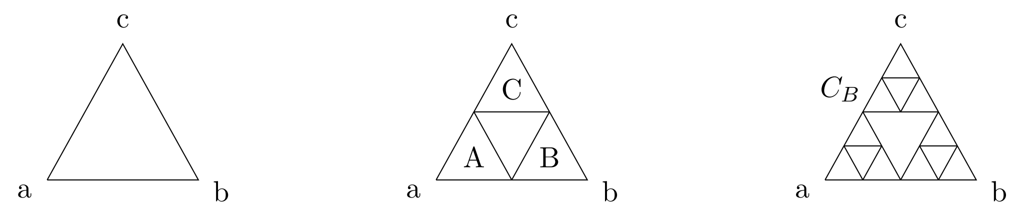
The idea is that we can hold specific parts of a set in place, and then the rest of the set will necessarily be held in place as well. This is quite powerful, as we only need to specify a few finite parts of a fractal and we can predict what happens with the fractal as a whole.
We shall also define another descriptive word for further discussion, connected points:

Definition 3.1.2 (Connection point). Let A, B be two n-cells on a fractal, also let k be a point in the fractal. The point is a connected point between the n-cells A and B if $k \in A \cup B$, i.e., k is the union of different n-cells. We say that k connects A and B, also that A and B are connected in k.

Now, to further get a better understanding of what fixing a set means, and what consequences this may bring it is best to actually check this on a fractal. It is with good reason we have used the Sierpinski gasket, S_g, as an example of a fractal earlier, namely that the symmetries are relatively easy to understand, especially compared to other fractals. As well as there actually is a set that will fix the rest of the fractal. And so, we will therefore start exactly here.

3.2 The Sierpinski gasket

Remember back to Figure 2.2, here we saw the development, through iterations, of the figure toward becoming S_g, a fractal. Looking under, at Figure 3.1, we want to name the different parts of the fractal. This is not necessarily trivial, so we use the ideas and notation provided by the IMS-theory, namely n-cells.



(a) Equilateral triangle (b) 1-cells of S_g (c) 2-cells of S_g

Figure 3.1: The three first iterations of S_g. We name the vertices a, b, c, and the 1-cells A, B, C.

To clarify the notation, we will be using some different variants of the n-cell notation, as this will make it somewhat easier to formulate. Firstly, we can reiterate the n-cell notation: $A_{i_1 i_2 \ldots i_n}$ is an n-cell. But in the proofs and discussions we call each cell with a name, as it is more explicit toward which specific n-cell we are talking about at different times. So we follow the denotation from Figure 3.1, where 1-cells are denoted A, B, C. For the 2-cells on S_g we will denote each cell, $A_A, A_B, ..., C_B, C_C$, where C_B denote the 2-cell on the bottom left of the 2-cells that are under the 1-cell A.
Further we want to combine the structure of the n-cells to how we discuss the idea of mapping the fractal (or have a symmetry on the fractal). We are given a symmetry $T : A \to A$, also as we represent the fractal in terms of an IMS, and so it is sending n-cells to n-cells. Concretely, for S_g let us look at Figure 3.1, if we have $T : S_g \to S_g$, we specifically have that the 1-cells of S_g form the dihedral group. And A, B, C, are sent to some permutation of themselves.
Now we are ready to examine S_g a bit closer, we are going to see that there is a set that fixes S_g, we will prove this, and look at what consequences this will have for how we can describe S_g. A quick reminder for the upcoming theorem, that the points a, b are those found in Figure 3.1.

Theorem 3.2.1 (Fixing S_g). *Let $T : S_g \to S_g$ be a continuous and bijective map that sends n-cells to n-cells. The symmetry holds two points: $T(a) = a$ and $T(b) = b$, where $B = \{a, b\}$.*
If $T = id$, then B fixes A.

Proof. To reiterate, we have that $T : S_g \to S_g$ is continuous, bijective, and sends n-cells to n-cells, and that $T(a) = a$ and $T(b) = b$. With these restrictions we want to show that the map T actually is the identity map on S_g.
On S_{g_1} we have that A, B, C on Figure 3.1 overlap with each other on exactly one point, these points also happen to be the vertices of the cells, these are connected points. Therefore, when T is restricted so that a, b are held in place (or fixed) and the 1-cells A, B also must meet in a shared point, i.e. the point that connect A and B. T must send this connected to itself, hence also with the connected points between A, C and B, C, i.e., all connected point must necessarily be sent to itself by T. And since T is bijective we now know that not only are the 1-cells sent to themselves, but with the fixing of the connected points on this cell level we also have that the 2-cells are necessarily sent to themselves. To show this, we look at the connected points between 2-cells.
Now, we have that T is the identity map for S_{g_1}, we look at S_{g_2} (vertices on the 2-cells on S_g). Let $k_1 = A \cup B$. Now we use the same argument for each triangle one "cell-level" down, where we use the situation we now have in the 1-cell A: from assumption, and argument in the previous paragraph we respectively have that a and k_1 are sent to themselves. We have an identical situation as with the 1-cells, but now with the 2-cells A_A, A_B, A_C, that are "under" the 1-cell A. Since it is the same situation, we use the exact same argument for fixing the connected points that are of the different unions of a pair of 2-cells: $A_A \cup A_B, A_A \cup A_C$... We iterate this argument for the 2-cells that are "under" the three different 1-cells. Hence T sends all points in S_{g_2} to themselves, and T is the identity map also for the 2-cells.
This way of continuing the argument from n-cells to $(n + 1)$-cells, as shown by the example from 1-cells to 2-cells, leads to that T must send all connected points, which lies in the different unions between cells on the same level, to themselves. We also need to address others points that are not connected points, i.e., points that are not a union between to n-cells at any level. All these other points can be defined by being the limit of a sequence of connected points. Since we have that T is continuous, these limit-points will be sent to themselves by T. And so, T sends a, b, all connected points, and all points in between these connected points to themselves, hence we also have that $T(c) = c$.
Therefore, when $T : S_g \to S_g$ is a continuous, bijective map on S_g, where $T(a) = a$ and $T(b) = b$, T is necessarily the identity map on S_g. And so, we can say that $\{a, b\}$ fixes the Sierpinski gasket. ■

Now that we have proven that we can fix S_g not just with a subset of S_g but a set that is also contained in the equilateral triangle, if we only were interested in the

"0-cell" that constitutes the equilateral triangle. One of the reasons we are especially interested in the vertices, is the way that group theory talks about symmetries, and has this connection between different orientations of geometrical objects and groups, more specifically symmetry groups. A direct consequence of $\{a, b\}$ fixing S_g is that all the corners are held in place, and then as the rest of the fractal is held in place, we can see that S_g potentially can fall under the same symmetry group that of the equilateral triangle, namely the dihedral group D_3. To show this, we need to show that S_g indeed has no other symmetry than the symmetries it inherits from the equilateral triangle.

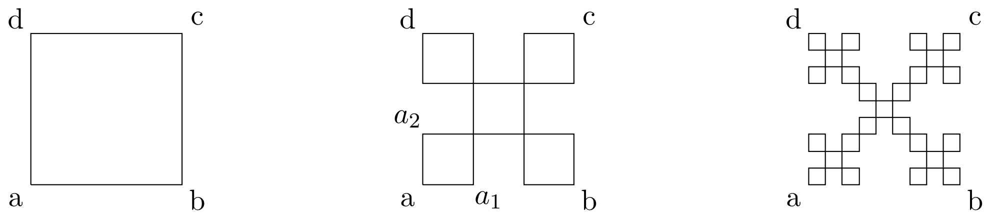
Theorem 3.2.2 (Conserving symmetries, S_g). *All symmetries found in S_g are inherited from the equilateral triangle, i.e., we have a conservation of symmetries.*

Proof. When we look at what symmetries S_g have, we will first look at what symmetries the equilateral triangle have and if S_g inherit these, also check if S_g has any other symmetries that the equilateral triangle does not.
The equilateral triangle have rotational symmetry, where it can be rotated by $k \cdot 120°$ where $k = 0, 1, 2$. It also have "mirror" symmetries through symmetry-lines, where we rotate the triangle 180° around this line, we have three such lines, all crossing a given corner and the midpoint on the opposite line-segment on the triangle.
Now we shall look at whether S_g inherit these symmetries. First we have that S_g has a sort of rigidity as it can be fixed by $\{a, b\}$ such that it can inherit the rotational and mirror symmetries: if we define a symmetry $T : S_g \to S_g$, but we now send the corners differently, for example: $T(a) = b$ and $T(b) = c$. Then, following the same argument from the proof for fixing S_g, the rest of the fractal will follow in what will look like a rotation of 120°. Hence, T can send the corners $\{a, b, c\}$ to any corner position, where the different possibilities correspond to the different rotational and mirror symmetries on the equilateral triangle. It is also the same rigidity of T that allows only for these different iterations of the position of the corners, and therefore there cannot be any other "new" symmetries on S_g that it does not inherit from the equilateral triangle. ∎

Indeed, S_g have no "extra" symmetries, and can therefore be described under D_3. A natural next step is to try to explore the potential reasons for why S_g is fixable, this is a more complicated question, and we obviously need more than one fractal to try to understand what the underlying causes are to be able to fix a fractal. And so, we shall discuss the Vicsek fractal, and examine whether its fixable and what consequences follow.

3.3 The Vicsek fractal

The Vicsek fractal is a fractal named after the mathematician Tamas Vicsek, and similarly to S_g take a starting point in a familiar geometric shape, the square. We will be denoting the Vicsek fractal as V_f. There are to standard versions of the Vicsek fractal, V_f, both start with the square and we further divide the square into nine equal squares and we always keep the square in the middle. The two constructions differ in whether we choose to either remove or keep the corner-squares. Here we will discuss V_f where we keep the corners.



(a) Square (b) 1-cells of V_f (c) 2-cells of V_f

Figure 3.2: The three first iterations of V_f

To construct a similar situation as with S_g, we here let $B = \{a, b, c, d\}$, where the elements are the vertices of the square. Further we would want to check if this subset fix the whole set, i.e. the fractal V_f. And for V_f we will have the result that this set B does not fix the fractal, even further we do not have any set that fixes V_f, that is not unless $B = V_f$. To formally state our claim: Let $T : V_f \to V_f$ be a continuous and bijective symmetry such that $T(a) = a, T(b) = b, T(c) = c$ and $T(d) = d$. Even with these assumptions B will not fix V_f.

Proof. In this case it is only necessary to give an example of a symmetry that is not the identity. First look at Figure 3.2b, and the points a_1 and a_2 that are the vertices of one of the 1-cells. Given that all points in B are held in place, we can argue that all the 1-cells cannot swap places with other 1-cells. The 1-cell in the center must also be held exactly in place, because all the vertices are fixed. Here we take use of the same idea that the vertices of the center 1-cell are the union of the center 1-cell with each other 1-cell respective to each corner. So actually, all individual 1-cells are send to themselves, but the problem arises when we look at what configuration we can have each 1-cell in. Take another look at a_1 and a_2, even though the 1-cell to the bottom left is send back to itself, there are two potential outcomes: $T(a_1) = a_1, T(a_2) = a_2$ or $T(a_1) = a_2, T(a_2) = a_1$.

This is the case for each 1-cell, except for the 1-cell in the center. So the map T is not necessarily the identity map, but have extra symmetries in each "corner-cell". So we see that V_f does not have the same rigidity that we find on S_g, even if we were to have a continuous and bijective map $T : V_f \to V_f$ such that all corners are sent to themselves, T will not necessarily be the identity map. ■

Then we can ask the question: is there a subset of V_f such that it fixes the fractal? And there is a subset that fixes V_f, but this subset must be the set that contains all connections points between n-cells on every level. The reason for this is that for any subset of points we choose to try to fix V_f we can always go to the next cell level and find the same situation, that we can "twist" the corner-cells. More formally: for any subset $B_n \subset V_f$ with elements corresponding to vertices on n-cells, we only need to look at the $(n+1)$-cells, specifically the corner-cells. These will be able to "twist" such that T can send the two vertices that are not connected points to either themselves or each other. And so, there are subset of V_f that fixes the fractal, but that subset does not help us in defining an interesting symmetry group on V_f.

This have consequences for the conservation of symmetries from the starting object, the square, onto V_f. The fractal has more symmetries, already after the first iteration it has 15 extra symmetries for each symmetry inherited from the square. We get a total of 16 symmetries as each of the corner-cells have two possible outcomes. And as we have four corner-cells we have number of symmetries: 4^2. This is only for the 1-cells, and this number will obviously tend toward infinity as we want to cover the whole fractal. This also means if we were to appreciate the symmetries on V_f as a symmetry group, the order of this group would also be infinite.

3.4 Conserving symmetries

As mentioned in the end of the chapter discussing the Sierpinski gasket, it is quite interesting to discuss the underlying causes for why we are able to fix certain fractals, and some not. We have discussed and examined two different fractals, with two different outcomes, so here we can look into the differences and how they might matter.

The Sierpinski gasket is fixable, and when we look at the way that we have constructed the argument that proved it was fixable, we see that at each cell level, both in the case for S_g and V_f, are the n-cells are sent to themselves, it is only that for V_f there is a possibility for different symmetries on each cell, except for the center cell. And it is exactly here that there is a crucial difference, the n-cells themselves are connected to other n-cells differently. On S_g we see that all n-cells have their vertices either already held in place, or that it is a connected point that connects that n-cell to another. On the other hand, at V_f the n-cells, not including the center n-cell, we do not have the same outcome as there is still are two points that are still in a sense "free", even though there are two point either is already held in place or is a connected point. Another necessary observation is that the n-cells are only connected to other n-cells by two connected point, and these connected point lie on a symmetry-line for the individual cell, so the structure. Again we see that the exception for this is the n-cell at the center, and therefore it is only the corner-cells that brings additional symmetries to V_f. As we see rigidity arise when n-cells are connected to several other n-cells, we also see why this is a cause for the rigidity of fractals, and is an essential factor in why symmetries are conserved in certain fractals, and not in others.

Chapter 4

Concluding thoughts

We already have a mathematical structure for discussing geometric shapes, especially for simpler shapes like the regular polygons that fit nicely into symmetry groups as the dihedral groups. The fact that we can show that S_g falls under the same symmetry group as the equilateral triangle, demonstrates the potential for simplifying fractals. Making them more manageable, regardless of the arbitrary small details they possess. However V_f demonstrated that this is not necessarily the case. Therefore, some requirements to the structure of a fractal is necessary for it to be described with some known symmetry group, obviously ignoring some symmetry group of arbitrarily large order.

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